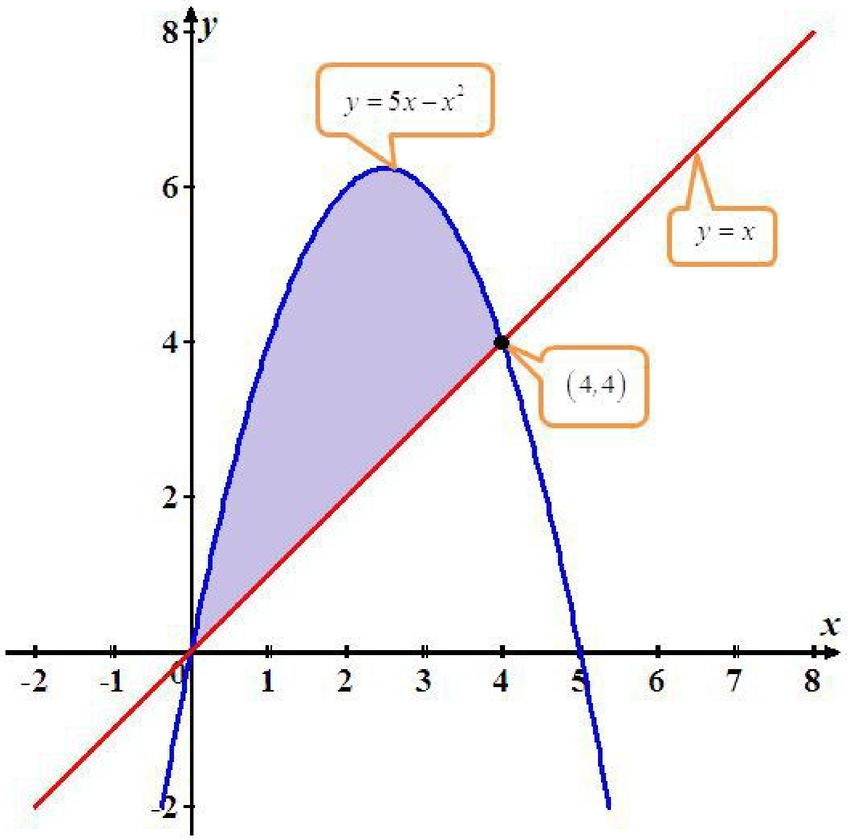
Exercise 5.1

Answer 1E.

Consider the following figure:



To find the area of the shaded region, use the parameters in the figure.

First, find the points of intersections of the parabola and the line by solving the equations, $y = 5x - x^2$ and $y = x$ simultaneously.

This gives $x = 5x - x^2$,

$x^2 - 4x = 0$.

$x(x-4) = 0$,

$x = 0$ or 4.

Therefore, the required region between lines $x = 0$ and $x = 4$.

The points of intersections are $(0,0)$ and $(4,4)$.

The area A of the region bounded by the curves $y=f(x), y=g(x)$ and the lines, $x=a,\ x=b$, where the curves, f and g are continuous and $f(x)\ge g(x)$ for all x in $[a,b]$ is as follows:

$$A=\int_a^b [f(x)-g(x)]dx.$$

Here, the area of the shaded region bounded by the curves $y=5x-x^2$ and $y=x$.

The region between $x=0$ and $x=4$, where the functions $y=5x-x^2$ and $y=x$ are continuous and $5x-x^2\ge x$ for all x in $[0,4]$ is as follows:

$$A=\int_0^4 \left[5x-x^2-x\right]dx$$

$$=\int_0^4 \left[4x-x^2\right]dx$$

$$=\left[4^2\frac{x^2}{2}-\frac{x^3}{3}\right]_0^4$$

$$=\left[2x^2-\frac{x^3}{3}\right]_0^4$$

$$=\left[\left(2(4)^2-\frac{(4)^3}{3}\right)-\left(2(0)^2-\frac{(0)^3}{3}\right)\right]$$

$$=\left[\left(32-\frac{64}{3}\right)-(0)\right]$$

$$=32-\frac{64}{3}$$

$$=\frac{96-64}{3}$$

$$=\boxed{\frac{32}{3}}$$

Hence, the area of the shaded region in the figure is $\boxed{\frac{32}{3}}$.

Answer 2E.

Given curves are $y=x^2-4x$ and $y=2x$ and from the graph the point of intersection of the curves are $(0,0)$ and $(6,12)$.

The upper boundary curve is $y=2x$ and the lower boundary curve is $y=x^2-4x$. So, we use the area formula with $f(x)=2x, g(x)=x^2-4x, a=0$ and $b=6$.

Therefore area, $A=\int_0^6\left[2x-\left(x^2-4x\right)\right]dx$

$$=\int_0^6\left(2x-x^2+4x\right)dx$$

$$=\int_0^6\left(6x-x^2\right)dx$$

$$=\left[6\frac{x^2}{2}-\frac{x^3}{3}\right]_0^6$$

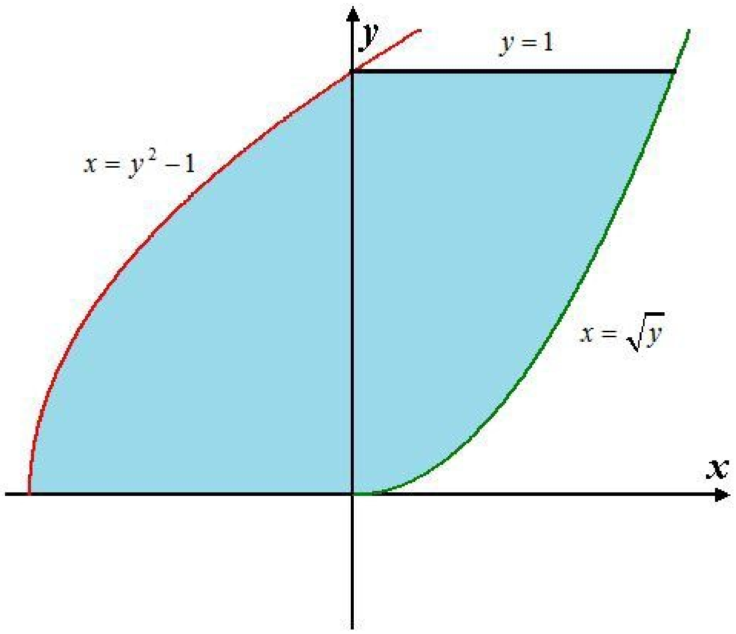
$$=3\times36-36\times2$$

$$=36\text{ square units}$$

Therefore $\boxed{\text{Area}=36\text{ square units}}$

Answer 3E.

Consider the graph



Required to find the area of the shaded region

Observe that the shaded region lies between $y=0$ to $y=1$

And $\sqrt{y}\ge y^2-1$ for $0\le y\le1$

Then the area of the shaded region is

$$A=\int_0^1\left[y^{\frac{1}{2}}-\left(y^2-1\right)\right]dy$$

$$=\int_0^1\left[y^{\frac{1}{2}}-y^2+1\right]dy$$

$$=\left[\frac{2}{3}\cdot y^{\frac{3}{2}}-\frac{y^3}{3}+y\right]_0^1$$

$$=\left[\frac{2}{3}-\frac{1}{3}+1-0\right]$$

$$=\frac{4}{3}$$

Therefore, area of the shaded region is $\boxed{\frac{4}{3}}$.

Answer 4E.

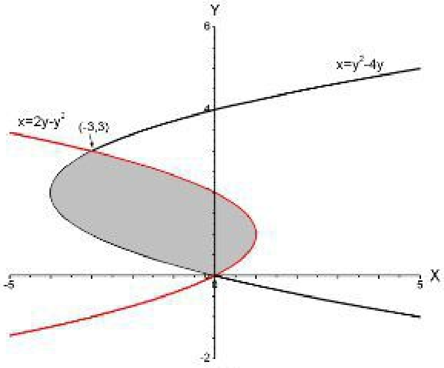


Fig.1

Since in the interval $0 \le y \le 3$, the curve $x = 2y - y^2$ is above the curve $x = y^2 - 4y$

So area of shaded region

$$A = \int_0^3 \left[\left(2y - y^2\right) - \left(y^2 - 4y\right)\right] dy$$

Or

$$A = \int_0^3 \left(6y - 2y^2\right) dy$$

$$= \left[3y^2 - \frac{2}{3}y^3\right]_0^3 \qquad \text{[By FTC - 2]}$$

$$= \left[3\left(3^2\right) - \frac{2}{3}\left(3^3\right)\right] = 27 - 18$$

Or $\boxed{A = 9}$

Answer 5E.

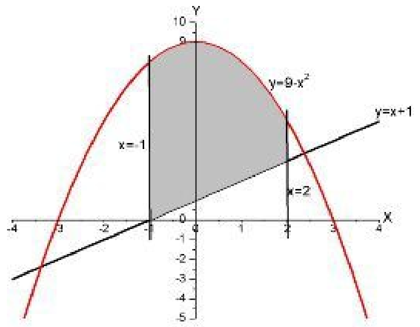


Fig.1

The shaded region is bounded by the curves $y = x + 1$, $y = 9 - x^2$, $x = -1$ and $x = 2$

Area of the shaded region is

$$A = \int_{-1}^{2} \left\{\left(9 - x^2\right) - \left(x + 1\right)\right\} dx$$

$$= \int_{-1}^{2} \left(8 - x^2 - x\right) dx$$

$$= \left\{8x - \frac{x^3}{3} - \frac{x^2}{2}\right\}_{-1}^{2}$$

$$= \left(16 - \frac{8}{3} - 2\right) - \left(-8 + \frac{1}{3} - \frac{1}{2}\right)$$

$$= 24 - 3 - \frac{3}{2}$$

$$= \boxed{19.5}$$

Answer 6E.

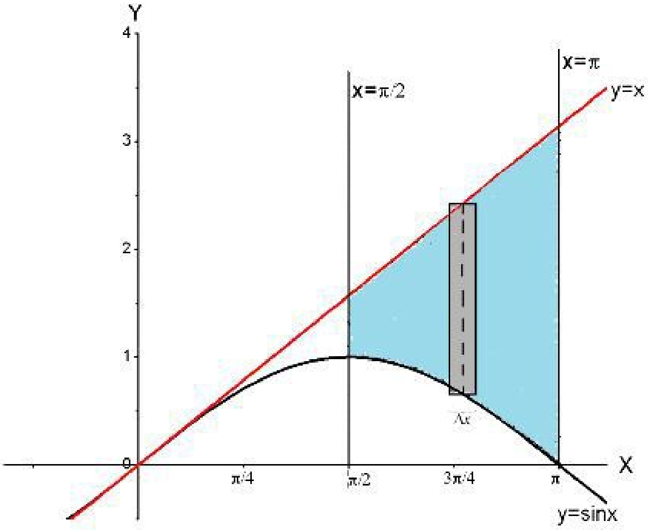


Fig.1

We have the curves $y = \sin x,\ y = x,\ \ x = \frac{\pi}{2},\ x = \pi$

We have to find the area enclosed by the given curves, which is shows in figure by shaded region

Area of shaded region $= \int_{\pi/2}^{\pi} [x - \sin x]dx$

Or $\quad A = \int_{\pi/2}^{\pi} x dx - \int_{\pi/2}^{\pi} \sin x dx$

$A = \left[\frac{x^2}{2}\right]_{\pi/2}^{\pi} + [\cos x]_{\pi/2}^{\pi}$ [By Fundamental Theorem of calculus part 2]

$A = \left[\frac{\pi^2}{2} - \frac{\pi^2}{8}\right] + \left[\cos \pi - \cos \frac{\pi}{2}\right]$

$A = \frac{3\pi^2}{8} + [-1 - 0]$

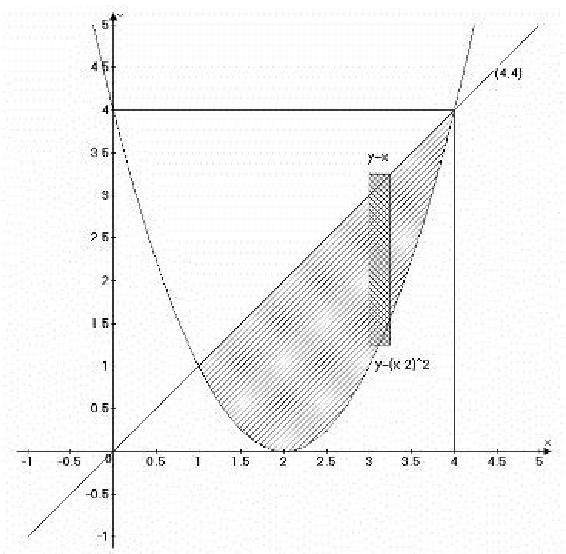
Or $\quad \boxed{A = \frac{3\pi^2}{8} - 1}$

Answer 7E.

Given curves are $y = (x-2)^2, y = x$

The graph of the curves is

In the following graph y=(x-2)[2] represented by $y = (x-2)^2$



Solving the equation $y=(x-2)^2$, $y=x$

i.e $\quad x=(x-2)^2$

$\Rightarrow \quad x=x^2-4x+4$

$\Rightarrow \quad x^2-5x+4=0$

$\Rightarrow \quad x^2-4x-x+4=0$

$\Rightarrow \quad x(x-4)-1(x-4)=0$

$\Rightarrow \quad (x-1)(x-4)=0$

$\Rightarrow \quad x=1,4$

Therefore point of intersection of curves are $(1,1),(4,4)$.

The top and bottom boundaries the $y_T=x$ and $y_B=(x-2)^2$

The area of a typical rectangle is

$$(y_T-y_B)\Delta x=\left[x-(x-2)^2\right]\Delta x$$

And the region lies between $x=1$ and $x=4$. So, the total area is

$$A=\int_1^4\left[x-(x-2)^2\right]dx$$

$$=\int_1^4\left[x-x^2+4x-4\right]dx$$

$$=\int_1^4\left[-x^2+5x-4\right]dx$$

$$=-\frac{x^3}{3}+5\cdot\frac{x^2}{2}-4x\Bigg]_1^4$$

$$=-\frac{64}{3}+5\cdot\frac{16}{2}-16+\frac{1}{3}-\frac{5}{2}+4$$

$$=-\frac{63}{2}+\frac{75}{2}-12$$

$$=-21+\frac{75}{2}-12$$

$$=-33+\frac{75}{2}$$

$$=\frac{9}{2}$$

Area, $A=\frac{9}{2}$ square units

Answer 8E.

Consider the functions:

$y=x^2-2x$

$y=x+4$

To find the point of intersection of the two curves, solve the given equations. Solve them as follows.

$x^2-2x=x+4$

$x^2-2x-x-4=0$

$x^2-3x-4=0$

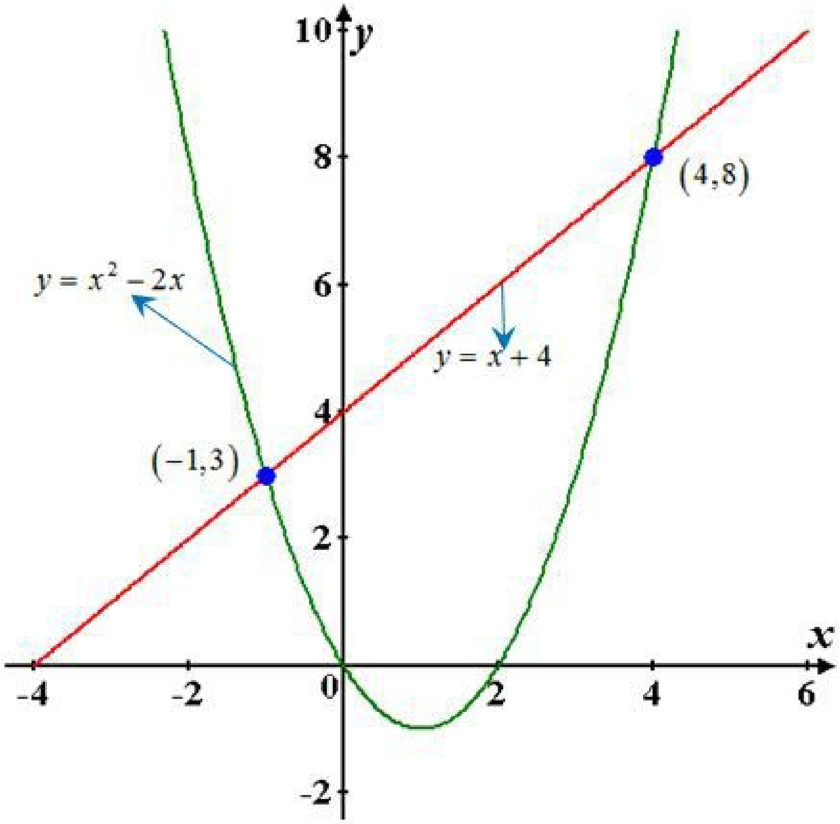
$(x-4)(x+1)=0$

This implies that $x=4$ or $x=-1$.

Therefore, the bounds of the region are $x=4$ and $x=-1$.

Then, the limits of integration to find the area are $a=-1,\ b=4$.

Sketch the region enclosed by these two curves which is shown as follows.



From the figure, observe that the line $y = x + 4$ is the upper boundary and the parabola $y = x^2 - 2x$ is the lower boundary.

Therefore, $f(x) = x + 4, g(x) = x^2 - 2x$

To sketch the typical rectangles, find the height and the width of the rectangle.

The height of the rectangle is the difference of the given equations.

Height $= f(x) - g(x)$

$= x + 4 - x^2 + 2x$

$= 4 + 3x - x^2$

And the width of the rectangle is calculated as follows:

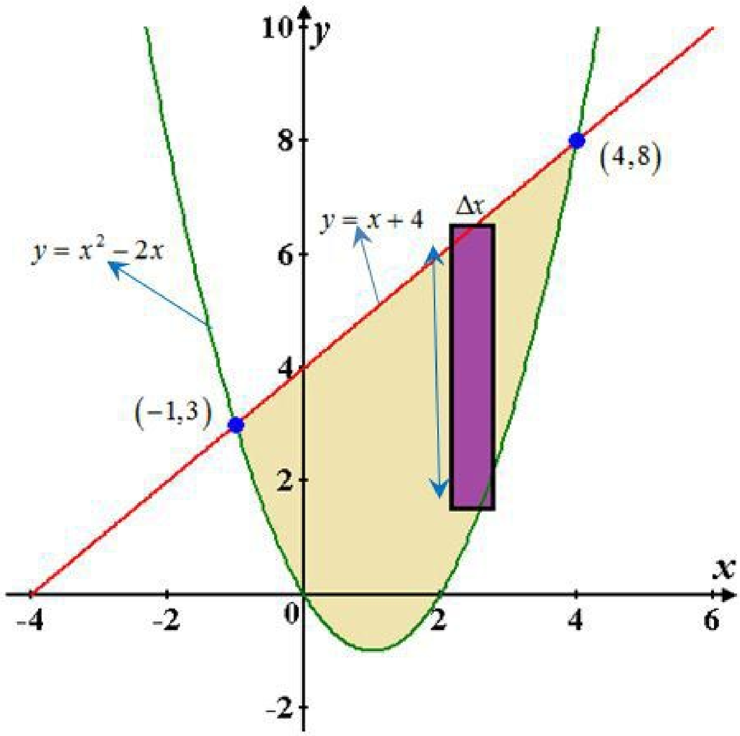
$$\Delta x = \frac{b-a}{n}$$

$$= \frac{4+1}{n}$$

$$= \frac{5}{n}$$

Now, sketch the typical rectangle representing the height and width of the rectangle.

The graph is as follows.



The area of the region is bounded by the curves $f(x) = x+4, g(x) = x^2 - 2x$ and the lines $x = 4$ and $x = -1$. Then the area enclosed by these curves is given by the following:

$$\begin{aligned}\int_a^b \left[f(x) - g(x)\right]dx &= \int_{-1}^{4} (x+4) - \left(x^2 - 2x\right)dx \\ &= \int_{-1}^{4} 4 + 3x - x^2 dx \\ &= \left[4x + \frac{3x^2}{2} - \frac{x^3}{3}\right]_{-1}^{4} \\ &= \left(16 + 24 - \frac{64}{3}\right) - \left(-4 + \frac{3}{2} + \frac{1}{3}\right)\end{aligned}$$

$$= \frac{125}{6}$$

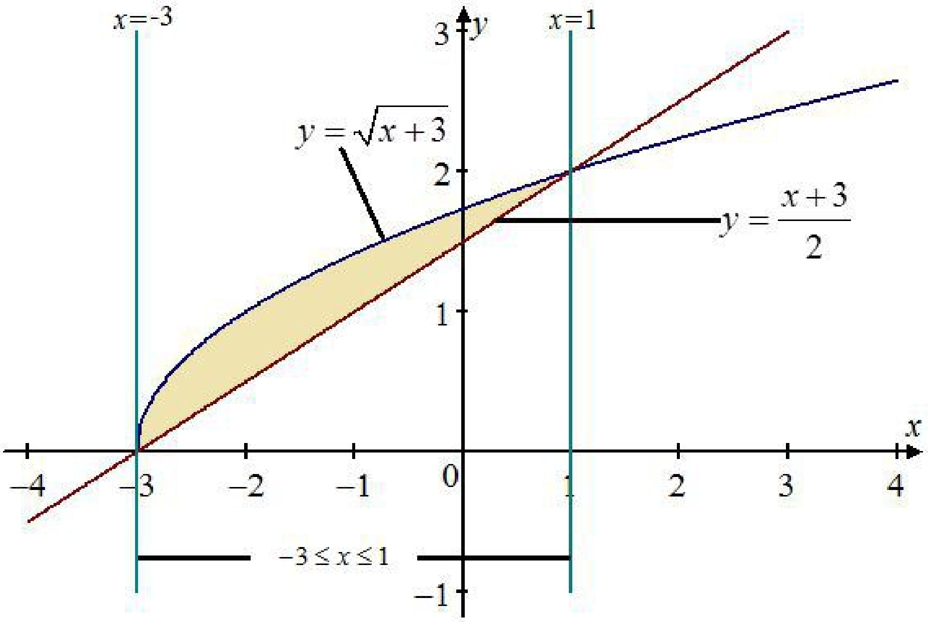
Hence, the area of the region is $\boxed{\approx 20.833}$.

Answer 9E.

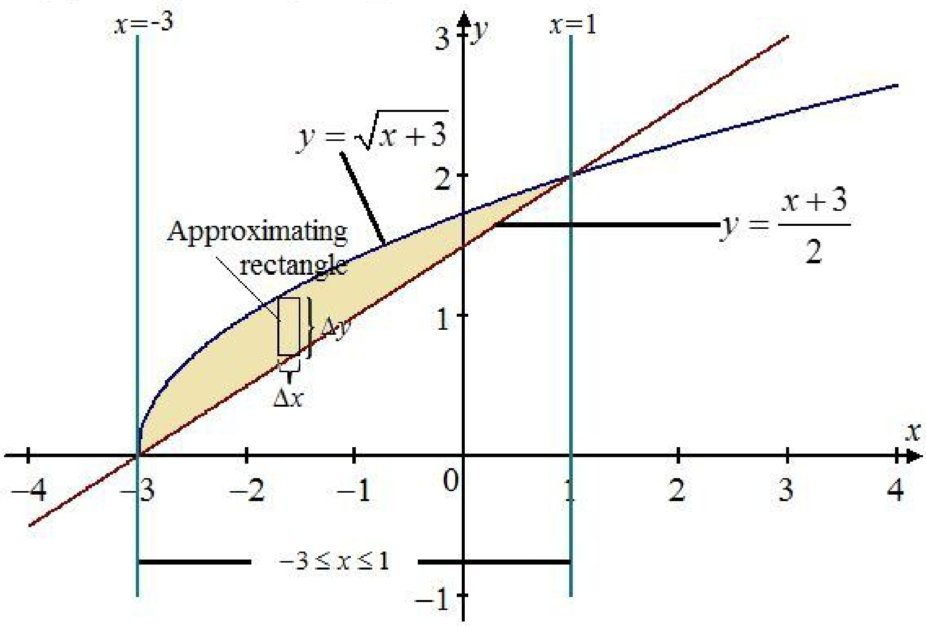
Consider the curves,

$$y=\sqrt{x+3},\ y=\frac{x+3}{2} \quad \text{......} \ (1)$$

Sketch the graph of the curves.



The graph with approximating rectangle is



Recall that,

The area between the curves $y=f(x)$, $y=g(x)$ and between $x=a$ and $x=b$ is

$A=\int_a^b |f(x)-g(x)|dx$ ...... (2)

To find the area by integration, the bounds are to be known. So, first find the bounds.

To find the bounds for the region find the point of intersection of these curves.

$$(x+3)=\frac{(x+3)^2}{4}$$
$$4(x+3)=(x^2+9+6x)$$
$$4x+12=x^2+9+6x$$
$$x^2+2x-3=0$$
$$x^2+3x-x-3=0$$
$$(x+3)(x-1)=0$$

$x=-3$ or $x=1$

Find the area of the region bounded by the curves $y=\sqrt{x+3}, y=\frac{x+3}{2}$ and between $x=-3$ to $x=1$.

To find the area of shaded region use (2).

From the graph it is clear that $\frac{x+3}{2}\le\sqrt{x+3}$ as x varies from $x=-3$ to $x=1$.

So the Area of the required region (shaded region) is

A $=\int_{-3}^{1}\left(\sqrt{x+3}-\frac{x+3}{2}\right)dx$

Use substitution to simplify the integral.

Substitute $x+3=u$

Differentiate on each side give $dx=dt$

And also change the limits.

$x=-3\Rightarrow u=0$
$x=1\Rightarrow u=4$

Therefore,

$A=\int_0^4 u^{\frac{1}{2}}dx-\frac{1}{2}\int_0^4 udx$ Use the substitutions

$=\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4-\frac{1}{2}\left[\frac{u^2}{2}\right]_0^4$ Use $\int x^n dx=\frac{x^{n+1}}{n+1}$

$=\frac{2}{3}\left[4^{\frac{3}{2}}-0\right]-\frac{1}{4}\left[4^2-0\right]$ Apply the limits

Continue the above steps.

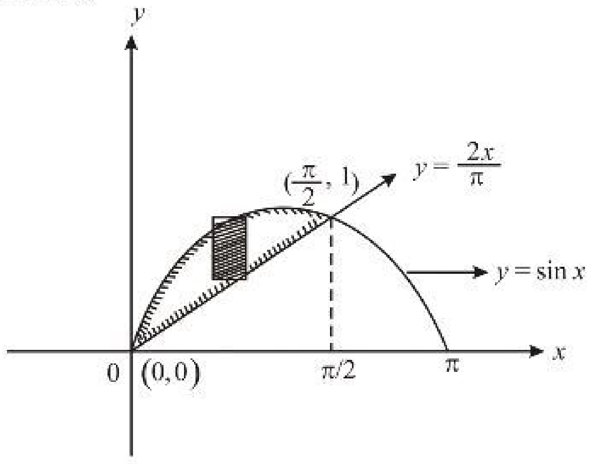
$$A=\frac{2}{3}[8]-\frac{1}{4}[16]$$
$$=\frac{16}{3}-4$$
$$=\frac{4}{3}$$

Thus, the area of the region bounded by the curves given by (1) is $\boxed{\frac{4}{3}}$.

Answer 10E.

Given curves are $y = \sin x, y = \frac{2x}{\pi}, x \geq 0$

The graph of the curves is



We first find the points of intersection of the curve and straight line simultaneously. This given $x = 0, \pi/2$ and hence the points are $(0,0), (\pi/2, 1)$.

The top and bottom boundaries are

$$y_T = \sin x, y_B = \frac{2x}{\pi}$$

The area of a typical rectangle is

$$(y_T - y_B)\Delta x = \left(\sin x - \frac{2x}{\pi}\right)\Delta x$$

And the region lies between $x = 0$ and $x = \pi/2$. So, the total area

$$A = \int_0^{\pi/2} \left[\sin x - \frac{2x}{\pi}\right] dx$$

$$= \left[-\cos x - \frac{2}{\pi} \cdot \frac{x^2}{2}\right]_0^{\pi/2}$$

$$= -\cos\frac{\pi}{2} - \frac{1}{\pi} \times \frac{\pi^2}{4} + 1$$

$$= 1 - \frac{\pi}{4}$$

Hence $\boxed{\text{Area}, A = \left(1 - \frac{\pi}{4}\right) \text{square units}}$

Answer 11E.

Consider the two curves

$x = 1 - y^2$, $x = y^2 - 1$

Now, sketch the region enclosed by the above curves.

First find the intersection points.

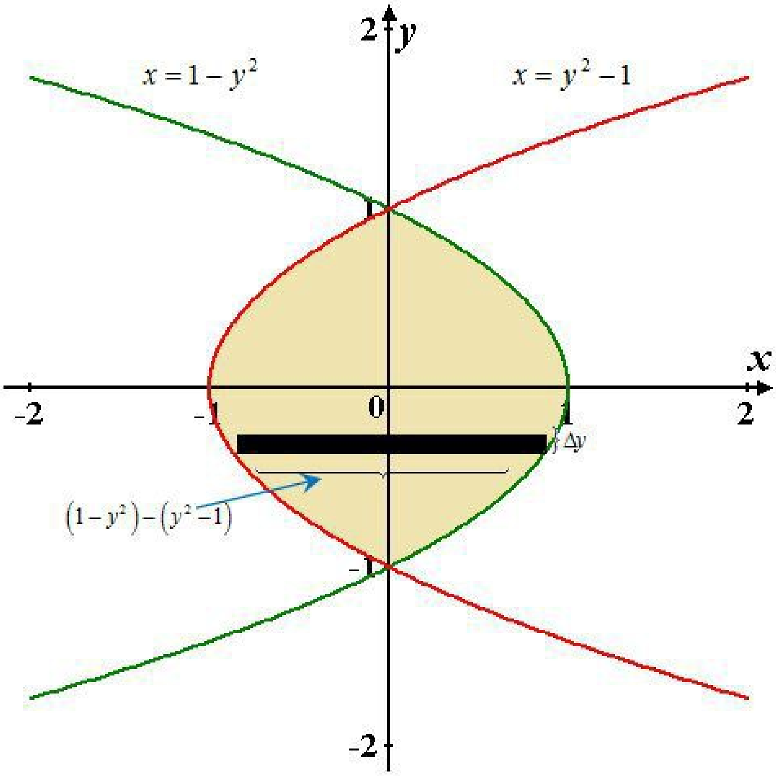
To do so, set the two equations equal to each other and solve.

$$1 - y^2 = y^2 - 1$$
$$2y^2 = 2$$
$$y = \pm 1$$

The points of intersection are $(0,1)$ and $(0,-1)$.

Now, sketch the region enclosed by the above curves.

And then draw a typical approximating rectangle as shows below



In above graph, a typical approximating rectangle is drawn.

Its height is Δy

And its width is $1-y^2-(y^2-1)$, that is the distance between the right most curve and the left most curve on the interval $[-1,1]$

Now, find the area of the region

The area between two curves can be described as the integral of the rightmost curve minus the leftmost curve over the appropriate interval.

From the graph, observe that, on the interval $-1 < y < 1$, $x = 1 - y^2$ is to the right of $x = y^2 - 1$

.

From figure see that it will be easier to integrate with respect to *y* for finding the area of the region. Since both the equations of the curves are in terms of *y*.

Thus, the area of the bounded region is

$$\begin{aligned}\int_{-1}^{1}\left(\left(1-y^2\right)-\left(y^2-1\right)\right)dy &= \int_{-1}^{1}\left(2-2y^2\right)dy \\ &= \left[2y-\frac{2y^3}{3}\right]_{-1}^{1} \\ &= 2-\frac{2}{3}-\left[-2+\frac{2}{3}\right] \\ &= \frac{8}{3}\end{aligned}$$

Therefore, the area bounded by the region is $\boxed{\frac{8}{3}}$

Answer 12E.

First we find the points of intersection of the curve $4x + y^2 = 12$ *and* $x = y$

Rewrite the equation $x = 3 - \frac{y^2}{4}$ *and* $x = y$

So these curves will intersect when

$$3 - \frac{y^2}{4} = y$$

Or $\quad y^2 + 4y - 12 = 0$

Or $\quad y^2 + 6y - 2y - 12 = 0$

Or $\quad y(y+6) - 2(y+6) = 0$

Or $\quad (y+6)(y-2) = 0$

Or $\quad$ y = -6 or y = 2

So these curve intersect at y = -6 and y = 2

We sketch the curves $x = 3 - \frac{y^2}{4}$ *and* $x = y$

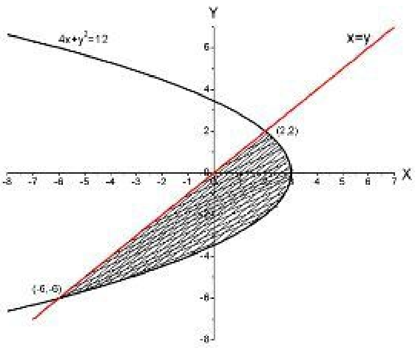


Fig.1

We have to find the area of shaded region

Area of shaded region

$$= \int_{-6}^{2} \left[\left(3 - \frac{y^2}{4}\right) - y\right] dy$$

Or $\quad A = \int_{-6}^{2} \left(3 - \frac{y^2}{4} - y\right) dy$

$$= \left[3y - \frac{y^3}{12} - \frac{y^2}{2}\right]_{-6}^{2}$$

$$= \left[3(2) - \frac{(2)^3}{12} - \frac{(2)^2}{2}\right] - \left[3(-6) - \frac{(-6)^3}{12} - \frac{(-6)^2}{2}\right]$$

$$= \left[6 - \frac{8}{12} - \frac{4}{2}\right] - \left[-18 + \frac{216}{12} - \frac{36}{2}\right]$$

$$= \left[\frac{10}{3}\right] + 18$$

Or $\quad \boxed{A = \frac{64}{3}}$

Answer 13E.

First we find the points of intersection of the two curves $y = 12 - x^2$ and $y = x^2 - 6$. These are given by

$$12 - x^2 = x^2 - 6$$

I.e. $\quad x^2 = 9 \;\; or \;\; x = \pm 3$

When $\quad x = \pm 3, \; y = 3$

So the curves intersect at point (-3, 3) and (3, 3)

Intersection of the curves with x and y axis are given by

$$(0,12) \; and \; \left(\pm 2\sqrt{3}, 0\right)$$

And $\quad (0,-6) \; and \; \left(\pm\sqrt{6}, 0\right)$

We sketch the curves and mark the area which is required to be calculated

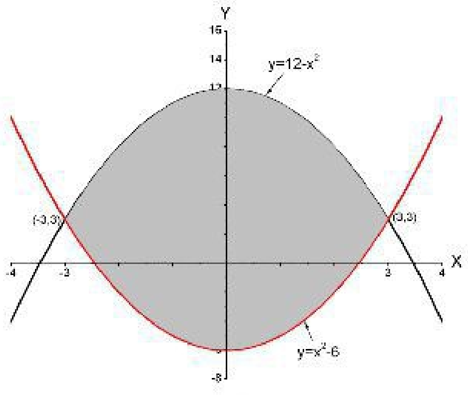


Fig.1

Total area between the two curves is shown in the figure by shaded region. If we denote this area by A_1 and A_2 where A_1 and A_2 are the area for the two curves between x = -3 and x = 3, then the required area is the algebraic sum of the area A_1 and A_2.

So we proceed to find the two areas A_1 and A_2

$$\begin{aligned}\text{Area}\quad A_1 &= \int_{-3}^{3}\left(12-x^2\right)dx\\ &= 12x-\frac{1}{3}x^3\Big]_{-3}^{3}\\ &= 12(3-(-3))-\frac{1}{3}\left(3^3-(-3)^3\right)\\ &= 54\end{aligned}$$

$$\begin{aligned}\text{Area}\quad A_2 &= \int_{-3}^{3}\left(x^2-6\right)dx\\ &= \frac{x^3}{3}-6x\Big]_{-3}^{3}\\ &= 9-18-\left(-9+18\right)\\ &= -18\end{aligned}$$

Hence the required area

$$\begin{aligned}&= A_1+|A_2|\\ &= 54+18\\ &= \boxed{72 \text{ units}}\end{aligned}$$

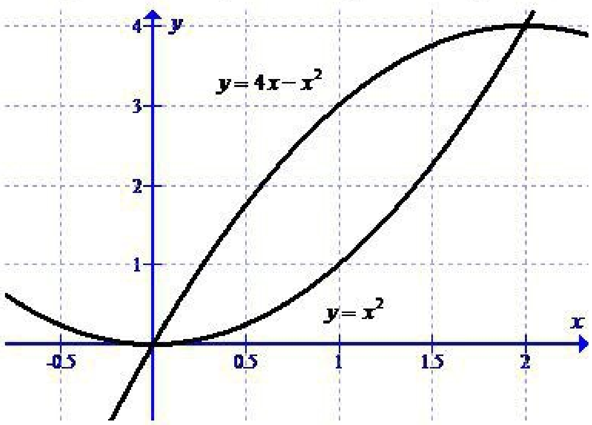
Answer 14E.

Consider the functions,

$y = x^2$ and $y = 4x - x^2$

The objective is to sketch the region enclosed by the given curves and find its area.

The region enclosed by the curves $y = x^2$ and $y = 4x - x^2$ is shown below.



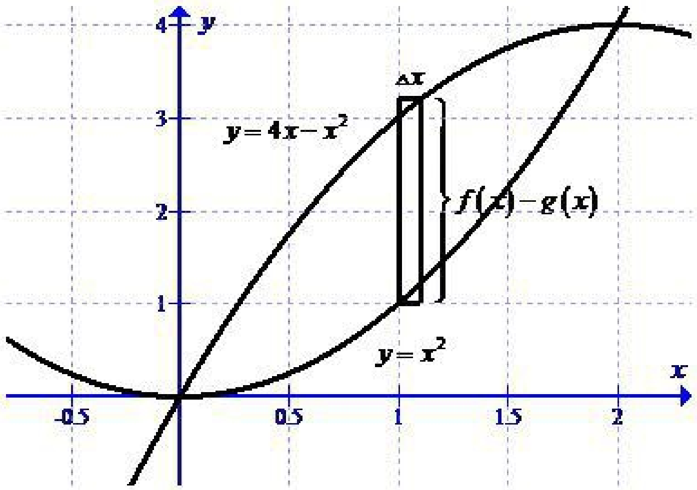
The integration has to be done with the respect to x because the curves $y = x^2$ and $y = 4x - x^2$ are functions of x.

Consider

$$y = 4x - x^2 = f(x)$$

$$y = x^2 = g(x)$$

A typical approximation rectangle is shown below:



The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)|\,dx$$

Here, the curves $y = x^2$ and $y = 4x - x^2$ intersect each other at points $(0,0)$ and $(2,4)$, so the area of the region bounded by the curves $y = x^2$ and $y = 4x - x^2$ between $x = 0$ and $x = 2$ is,

$$A = \int_0^2 |4x - x^2 - x^2|\,dx$$

$$= \left|2x^2 - \frac{2x^3}{3}\right|_0^2$$

$$= \left|\left(2(2)^2 - \frac{2(2)^3}{3}\right)\right|$$

$$= \boxed{\frac{8}{3}}.$$

Answer 15E.

Given curves area $y = \sec^2 x$ and $y = 8\cos x \frac{-\pi}{3} \le x \le \frac{x}{3}$

The points of intersection curves when $\sec^2 x = 8\cos x$

$$\sec^2 x = \frac{8}{\sec x}$$

$$\Rightarrow \sec^3 x = 8$$

$$\Rightarrow \sec x = 2$$

$$x = \frac{\pi}{3} \text{ and } \frac{-\pi}{3}$$

Therefore, the required area is

$$A = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left|8\cos x - \sec^2 x\right| dx$$

$$= 2\int_0^{\frac{\pi}{3}} \left(8\cos x - \sec^2 x\right) dx \qquad \text{Here } f(x) = 8\cos x - \sec^2 x \text{ is an even function}$$

$$= 2\left[8\sin x - \tan x\right]_0^{\frac{\pi}{3}}$$

$$= 2\left[8\sin\frac{\pi}{3} - \tan\frac{\pi}{3} - 8\sin 0 + \tan 0\right]$$

$$= 2\left[8\cdot\frac{\sqrt{3}}{2} - \sqrt{3}\right]$$

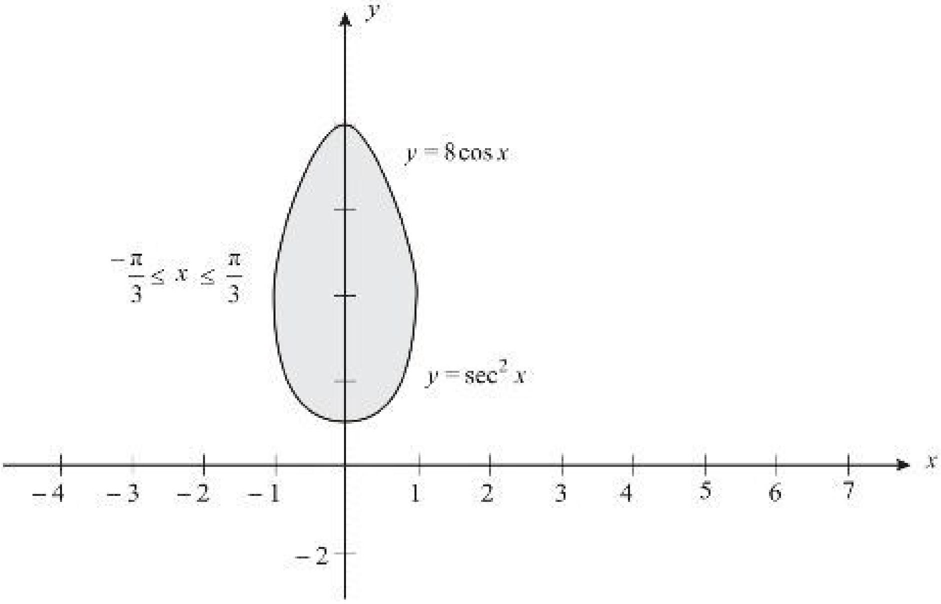
$$= 2\left[4\sqrt{3} - \sqrt{3}\right]$$

$$= 2\left[3\sqrt{3}\right]$$

$$= 6\sqrt{3}$$

$$\Rightarrow A = 6\sqrt{3}$$

So, the required area is $6\sqrt{3}$



Answer 16E.

We are given that two equations of the curves $y = \cos x$ and $y = 2 - \cos x \quad 0 \le x \le 2\pi$

We first find the points of intersection of the curves $y = \cos x$ and $y = 2 - \cos x$

This gives $y = \cos x = 2 - \cos x$

$\Rightarrow 2\cos x - 2 = 0$

$\Rightarrow \cos x - 1 = 0$

$\Rightarrow \cos x = 1$

$\Rightarrow x = 0$ and $x = 2\pi$

The required region is given by A

$$A = \int_0^{2\pi} (2 - \cos x - \cos x)\,dx$$

$$= \int_0^{2\pi} (2 - 2\cos x)\,dx$$

$$= 2\int_0^{2\pi} (1 - \cos x)\,dx$$

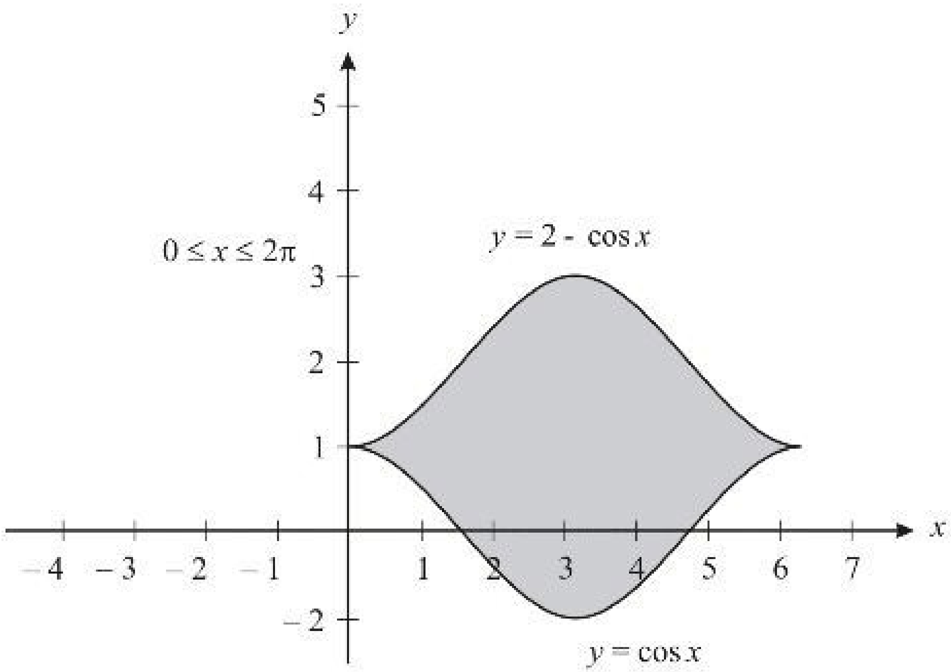
$$= 2[x - \sin x]_0^{2\pi}$$

$$= 2[2\pi - \sin 2\pi - 0 + \sin 0]$$

$$= 2[2\pi]$$

$$= 4\pi$$

So, the required region is 4π



Answer 17E.

We are given that two equations of the curves $x = 2y^2$ and $x = 4 + y^2$

The points of intersection curves when $2y^2 = 4 + y^2$

$\Rightarrow y^2 = 4$

$\Rightarrow y \pm 2$

When $y = 2 \Rightarrow x = 8$

When $y = -2 \Rightarrow x = 8$

Therefore, the points of intersection are $(8, 2)$ and $(8, -2)$

For the required area we must integrate between appropriate y-values, $y=-2$ and $y=2$

Thus

$$A=\int_{-2}^{2}\left(4+y^2-2y^2\right)dy$$

$$=\int_{-2}^{2}\left(4-y^2\right)dy \qquad \text{Here } f(y)=4+y^2-2y^2 \text{ is an even function}$$

$$=2\left[4y-\frac{y^3}{3}\right]_0^2$$

$$=2\left[4(2)-\frac{(2)^3}{3}\right]$$

$$=2\left[8-\frac{8}{3}\right]$$

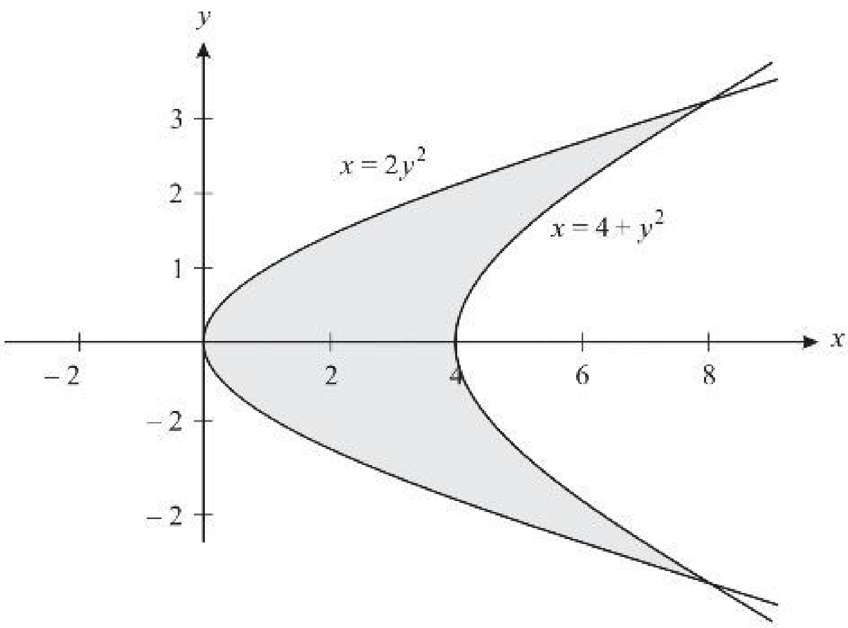
$$=2\left[\frac{24-8}{3}\right]$$

$$=2\left[\frac{16}{3}\right]$$

$$=\frac{32}{3}$$

$$\Rightarrow A=\frac{32}{3}$$

So, the required area is $\frac{32}{3}$

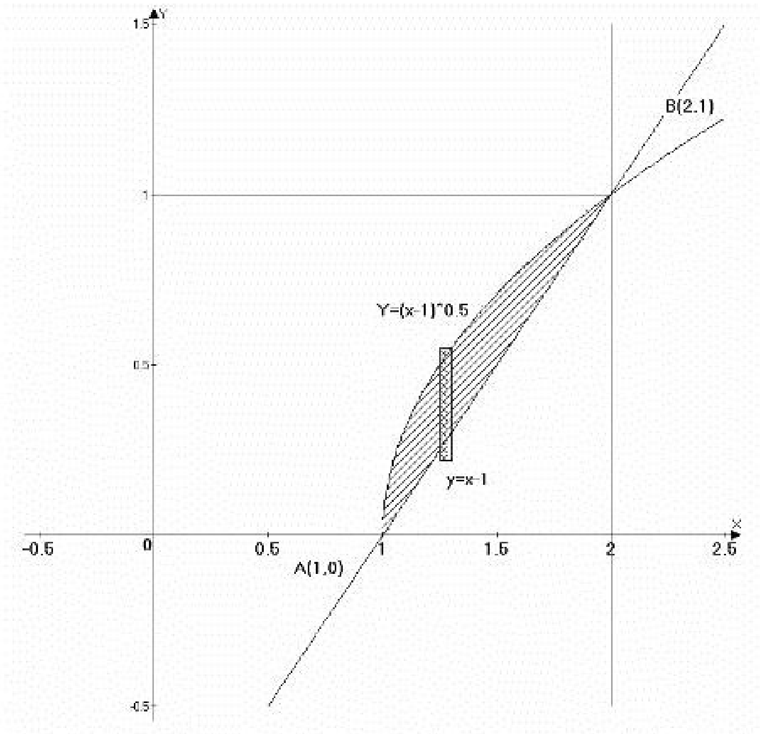


Answer 18E.

Given curves are $y=\sqrt{x-1}, x-y=1$

The graph of the curves is

In the graph $y=(x-1)$ represents $y=\sqrt{x-1}$



We first find the point of intersection of the curves $y=\sqrt{x-1}$, and $x-y=1$ simultaneously.

Therefore $\quad x-1=\sqrt{x-1}$

$\Rightarrow \quad (x-1)^2 = x-1$

$\Rightarrow \quad x^2-2x+1=x-1$

$\Rightarrow \quad x^2-3x+2=0$

$\Rightarrow \quad x^2-2x-x+2=0$

$\Rightarrow \quad x(x-2)-(x-2)=0$

$\Rightarrow \quad (x-1)(x-2)=0$

$\Rightarrow \quad x=1,2$

Therefore points of intersections are $(1,0),(2,1)$

The top and bottom boundaries are $y_T=\sqrt{x-1}$ and $y_B=x-1$
Hence, the area of a typical rectangle is

$$(y_T-y_B)\Delta x=\left[\sqrt{x-1}-(x-1)\right]\Delta x$$

And the region lies between $x=1$ and $x=2$. so, the area is

$$A=\int_1^2\left[\sqrt{x-1}-x+1\right]dx$$

$$=\left[\frac{2}{3}(x-1)^{3/2}-\frac{x^2}{2}+x\right]_1^2$$

$$=\frac{2}{3}-\frac{4}{2}+2+\frac{1}{2}-1$$

$$=\frac{2}{3}+\frac{1}{2}-1$$

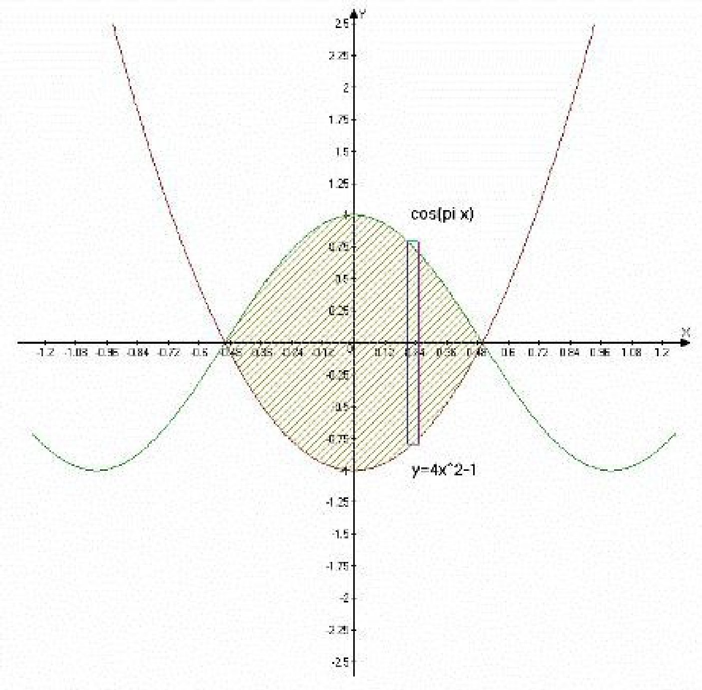
$$=\frac{1}{6}$$

Hence $\boxed{\text{Area, } A=\frac{1}{6}\text{ square units}}$

Answer 19E.

Given curves are $y=\cos\pi x, y=4x^2-1$

The graph of the curves is



In the graph cos (pi x) represents $y=\cos(\pi x)$ and $y=4x^2-1$represents y = 4x^2-1.

We first find the point of intersection of the curves $y=\cos\pi x$ and $y=4x^2-1$.

Solving these equations simultaneously, the points of intersections are $\left(-\frac{1}{2},0\right)$ and $\left(\frac{1}{2},0\right)$

The top and bottom boundaries are

$$y_T=\cos\pi x \text{ and } y_B=4x^2-1$$

And the area of a typical rectangle is

$$\left(y_T-y_B\right)\Delta x=\left[\cos\pi x-\left(4x^2-1\right)\right]\Delta x$$
$$=\left[\cos\pi x-4x^2+1\right]\Delta x$$

And the area lies between $x=\frac{-1}{2}$ and $x=\frac{1}{2}$.

Since the area is symmetric about y-axis

$$\text{Total area, } A=2\cdot\int_0^{1/2}\left[\cos\pi x-4x^2+1\right]dx$$
$$=2\left[\frac{\sin\pi x}{\pi}-\frac{4x^3}{3}+x\right]_0^{1/2}$$
$$=2\left[\frac{1}{\pi}-\frac{1}{6}+\frac{1}{2}\right]$$
$$=2\left[\frac{1}{\pi}+\frac{1}{3}\right]$$

Hence $\boxed{\text{Area, } A=2\left[\frac{1}{\pi}+\frac{1}{3}\right]\text{square units}}$

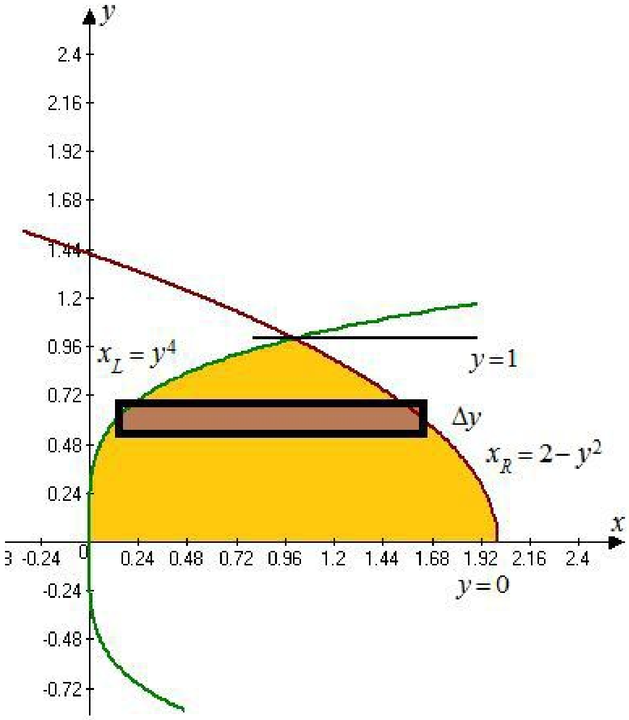
Answer 20E.

Consider the curves are $x=y^4$ and $y=\sqrt{2-x}, y=0$

The above two curves intersect at the point $(1,1)$

The region between the curves is bounded by the curves $y=0$ and $y=1$

Figure showing the area bounded by the curves:



Now

$x = 2 - y^2 \geq x = y^4$, for $0 \leq y \leq 1$

Write x_R for the right boundary and x_L left boundary.

Hence the area is

$$A = \int_c^d (x_R - x_L)dy$$

From the figure the left and right boundary curves are

$x_L = y^4$ and $x_R = 2 - y^2$

Integrate between the appropriate $y-$ values, $y=0$ and $y=1$. Thus

$$A = \int_0^1 (x_R - x_L)dy$$

$$A = \int_0^1 \left[\left(2 - y^2\right) - y^4\right]dy$$

$$= \int_0^1 \left(2 - y^2 - y^4\right)dy$$

$$= 2[y]_0^1 - \frac{1}{3}\left[y^3\right]_0^1 - \frac{1}{5}\left[y^5\right]_0^1$$

$$= 2(1-0) - \frac{1}{3}(1-0) - \frac{1}{5}(1-0)$$

$$= 2 - \frac{1}{3} - \frac{1}{5}$$

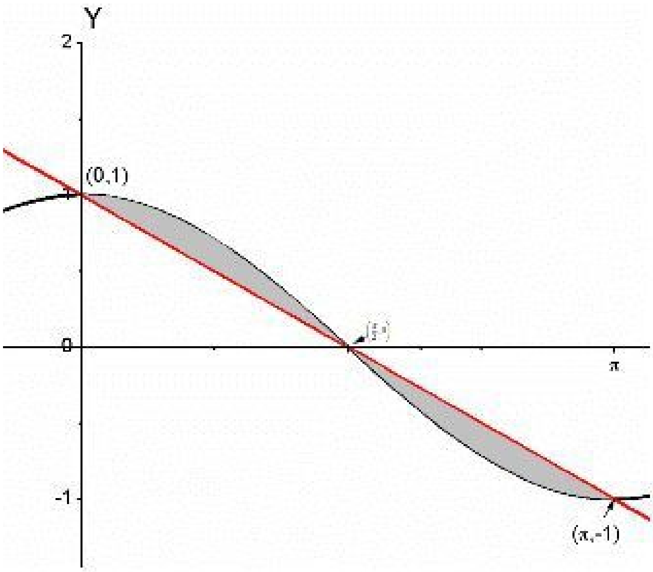
$$= \frac{22}{15}$$

Thus area bounded by the given curves is $\boxed{\frac{22}{15}}$.

Answer 21E.

First we find the point of intersection of the curves $y = \cos x$ *and* $y = 1 - \frac{2x}{\pi}$

For this we sketch the curves $y = \cos x$ *and* $y = 1 - \frac{2x}{\pi}$



We see that points of intersection are $(0,1), \left(\frac{\pi}{2}, 0\right)$ *and* $(\pi, -1)$

Since $y = \cos x$ is symmetric about x – axis and $y = 1 - \frac{2x}{\pi}$ is a straight line passing through the point $(\pi/2, 0)$. So area enclosed by these curve above the x – axis will be same in magnitude as the area below the x – axis enclosed by these curves.

So total area of shaded region = 2 × Area of the region on $\left[0, \frac{\pi}{2}\right]$

Or $$A = 2\int_0^{\pi/2}\left[\cos x - \left(1 - \frac{2x}{\pi}\right)\right]dx$$

$$= 2\int_0^{\pi/2}\left[\cos x - 1 + \frac{2x}{\pi}\right]dx$$

$$= 2\left[\sin x - x + \frac{x^2}{\pi}\right]_0^{\pi/2}$$

$$= 2\left[\sin\frac{\pi}{2} - \frac{\pi}{2} + \frac{\pi}{4} - \sin 0 + 0 - 0\right]$$

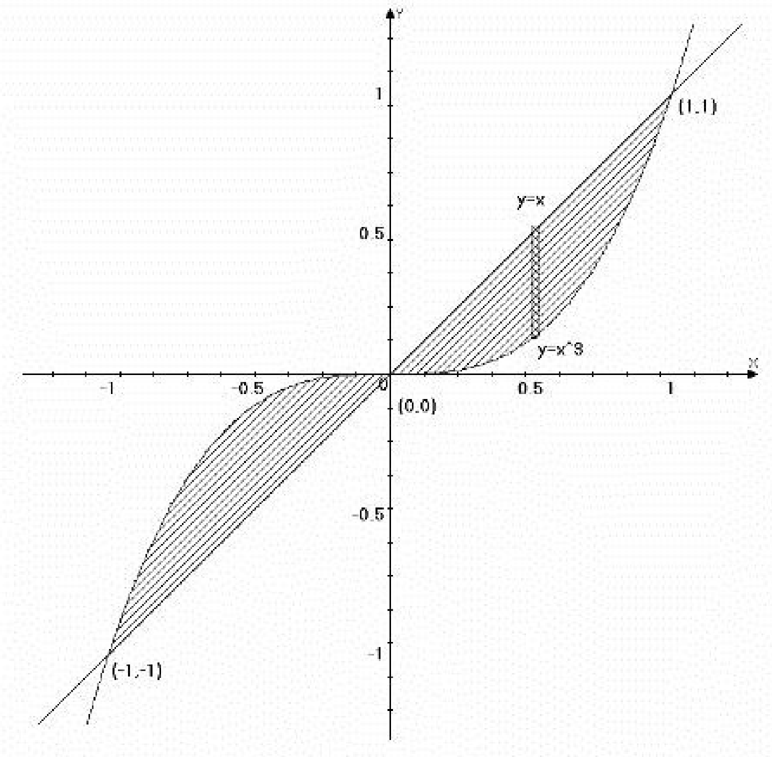
$$= 2\left[1 - \frac{\pi}{4}\right]$$

Or $$\boxed{A = 2 - \frac{\pi}{2}}$$

Answer 22E.

Given curves are $y = x^3$ and $y = x$.

The graph of the curves is



In the graph $y = x^3$ represents y =x^3

Solving the given curves $y = x^3$ and $y = x$ simultaneously

$$x = x^3 \Rightarrow \left(x^2 - 1\right)x = 0$$

$$\Rightarrow \quad x = 0, 1, -1$$

Therefore point of intersection of the curves is $(0,0), (-1,-1), (1,1)$.

Since the area is symmetric about the origion, total area $= 2\times$ area of the region in the first quadrant.

In the first quadrant, the top and bottom boundaries are

$y_T = x$ And $y_B = x^3$

The area of a typical rectangle is

$(y_T - y_B)\Delta x = \left(x - x^3\right)\Delta x$ and

The region lies between $x = 0$ and $x = 1$

Therefore total area,

$$A = 2\int_0^1 \left(x - x^3\right)dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= 2\left[\frac{1}{2} - \frac{1}{4}\right]$$

$$= 2\times\frac{1}{4}$$

$$= \frac{1}{2}$$

Therefore $\boxed{\text{Total area}, A = \frac{1}{2}\text{ square units}}$

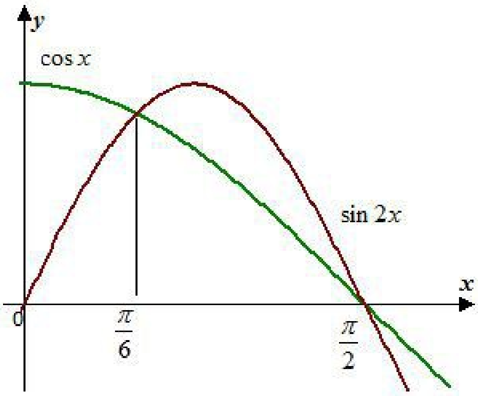
Answer 23E.

Consider the following functions:

$$y=\cos x, y=\sin 2x, x=0, x=\frac{\pi}{2}$$

The objective is to sketch the region enclosed by the curves and then find the area .

The graph of the region between the curves is as shown below:



To find the area between the curves, use the following formula.

$$\text{The area } A=\int_a^b |f(x)-g(x)|dx$$

$$\begin{aligned} A&=\int_0^{\pi/2}|f(x)-g(x)|dx \\ &=\int_0^{\pi/6}|f(x)-g(x)|dx+\int_{\pi/6}^{\pi/2}|f(x)-g(x)|dx \quad \text{......(1)} \end{aligned}$$

Now, simplify each integral is as follows:

$$\begin{aligned} \int_0^{\pi/6}|f(x)-g(x)|dx &= \int_0^{\pi/6}|\cos x-\sin 2x|dx \\ &=\left(\sin x+\frac{\cos 2x}{2}\right)_0^{\pi/6} \\ &=\sin(\pi/6)+\frac{1}{2}\cos(\pi/3)-\frac{1}{2} \\ &=\frac{1}{4} \end{aligned}$$

Compute the second integral as follows:

$$\begin{aligned} \int_{\pi/6}^{\pi/2}|f(x)-g(x)|dx &= \int_{\pi/6}^{\pi/2}|\sin 2x-\cos x|dx \\ &=\left(-\frac{\cos 2x}{2}-\sin x\right)_{\pi/6}^{\pi/2} \\ &=\frac{1}{2}-1+\frac{1}{2}\cos(\pi/3)+\frac{1}{2} \\ &=\frac{1}{4} \end{aligned}$$

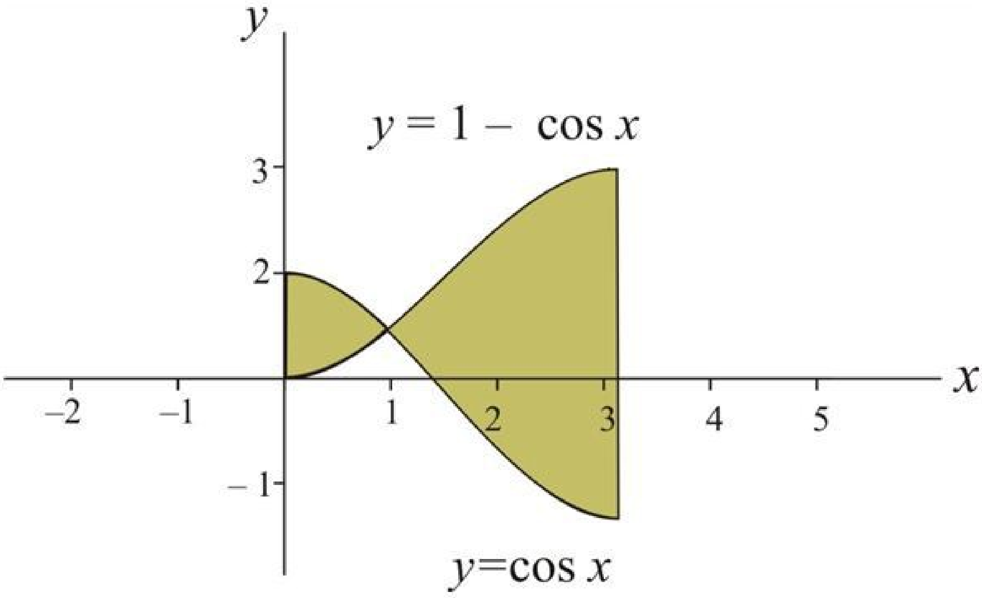
Substitute these values in (1)

$$\begin{aligned} A&=\int_0^{\pi/6}|f(x)-g(x)|dx+\int_{\pi/6}^{\pi/2}|f(x)-g(x)|dx \\ &=\frac{1}{4}+\frac{1}{4} \\ &=\frac{1}{2} \end{aligned}$$

Therefore, the area of the region between the curves is $\boxed{\frac{1}{2}}$.

Answer 24E.

Sketch the following two curves $y = \cos x$ and $y = 1 - \cos x$ as shown below:



Now we can integrate with respect to x by subtracting the top function from the bottom one but first we must find out where the intersection points are by substitution.

$$\cos x = 1 - \cos x$$
$$2\cos x = 1$$
$$\cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}$$

The region is sketched in figure below, observe that

$\cos x \geq 1 - \cos x$ When $0 \leq x \leq \frac{\pi}{3}$

$1 - \cos x \geq \cos x$ When $\frac{\pi}{3} \leq x \leq \pi$

Find the area of region.

$$\begin{aligned}
A &= \int_0^{\pi} \left|\cos x - (1 - \cos x)\right| dx \\
&= \int_0^{\pi/3} \left(\cos x - (1 - \cos x)\right) dx + \int_{\pi/3}^{\pi} \left((1 - \cos x) - \cos x\right) dx \\
&= \int_0^{\pi/3} (2\cos x - 1) dx + \int_{\pi/3}^{\pi} (1 - 2\cos x) dx \\
&= \left[2\sin x - x\right]_0^{\pi/3} + \left[x - 2\sin x\right]_{\pi/3}^{\pi} \\
&= \left(2\sin\frac{\pi}{3} - \frac{\pi}{3}\right) + \left(\pi - 2\sin\pi - \frac{\pi}{3} + 2\sin\frac{\pi}{3}\right) \\
&= 2\cdot\frac{\sqrt{3}}{2} - \frac{\pi}{3} + \pi - 0 - \frac{\pi}{3} + 2\cdot\frac{\sqrt{3}}{2} \\
&= 2\sqrt{3} + \frac{\pi}{3}
\end{aligned}$$

Therefore, the required area is $\boxed{\frac{\pi}{3} + 2\sqrt{3}}$.

Answer 25E.

First we find the point of intersection of the curves $y=\sqrt{x}$ and $y=\frac{x}{2}$

These curves will intersect if

$$\sqrt{x}=\frac{x}{2}$$

Or $\quad x=\frac{x^2}{4}$

Or $\quad \frac{x^2}{4}-x=0$

Or $\quad x\left(\frac{x}{4}-1\right)=0$

Or $\quad x=0$ or $x=4$

So these curves intersect each other at $x=0$ and $x=4$

Now we sketch the curve $y=\sqrt{x}$ and $y=\frac{x}{2}$ and $x=9$ in figure 1

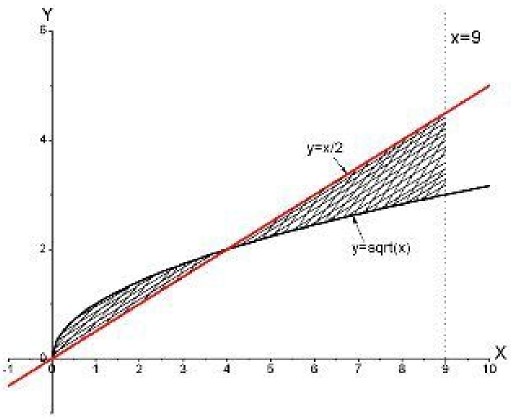


Fig.1

The area is shown by shaded region in the figure1
So the area of shaded region is

$$A=\int_0^4\left[\sqrt{x}-\frac{x}{2}\right]dx+\int_4^9\left[\frac{x}{2}-\sqrt{x}\right]dx$$

$$=\left[\frac{2}{3}x^{3/2}-\frac{x^2}{4}\right]_0^4+\left[\frac{x^2}{4}-\frac{2}{3}x^{3/2}\right]_4^9 \qquad \text{[By FTC - 2]}$$

$$=\left[\frac{2}{3}(4)^{3/2}-\frac{16}{4}\right]+\left[\frac{81}{4}-\frac{2}{3}(9)^{3/2}\right]-\left[\frac{4^2}{4}-\frac{2}{3}.(4)^{3/2}\right]$$

$$=\left[\frac{16}{3}-4\right]+\left[\frac{81}{4}-18\right]-\left[4-\frac{16}{3}\right]$$

$$=\frac{81}{4}+\frac{32}{3}-26=\frac{59}{12}$$

Or $\quad \boxed{A=\frac{59}{12}}$

Answer 26E.

First we find the points of intersections of the curves $y=|x|$ and $y=x^2-2$

We know by definition of modulus when $x<0$ $\;y=-x$

And $y=-x$ and $y=x^2-2$ will intersect each other when

$$x^2-2=-x$$

Or $\quad x^2+x-2=0$

Or $\quad (x+2)(x-1)=0$

Or $\quad x=-2$ for $x<0$

Similarly when $x \geq 0$, $y = x$

Then $y = x$ and $y = x^2 - 2$ will intersect when

$$x^2 - 2 = x$$

Or $$x^2 - x - 2 = 0$$

Or $$(x-2)(x+1) = 0$$

Or $$x = 2 \quad \text{for } x \geq 0$$

So the curves $y = |x|$ and $y = x^2 - 2$ will intersect each other at x = -2 and 2.

We sketch the curves $y = |x|$ and $y = x^2 - 2$

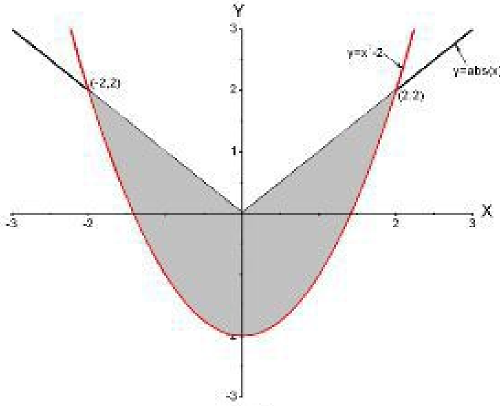


Fig.1

We see that $y = |x|$ and $y = x^2 - 2$ are even function so area enclosed by these curves is symmetric in magnitude about $y -$ axis

So Area of shaded region

A = 2× Area of the region on [0, 2] enclosed by the curves

$$A = 2\int_0^2 \left[|x| - (x^2 - 2)\right] dx$$

Since $x \geq 0$ so $|x| = x$

Then $$A = 2\int_0^2 (x - x^2 + 2)\, dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_0^2 \qquad \text{[By FTC - 2]}$$

$$= 2\left[\frac{4}{2} - \frac{8}{3} + 4\right]$$

$$= 2 \times \frac{20}{6}$$

$$= \frac{20}{3}$$

Or $$\boxed{A = \frac{20}{3}}$$

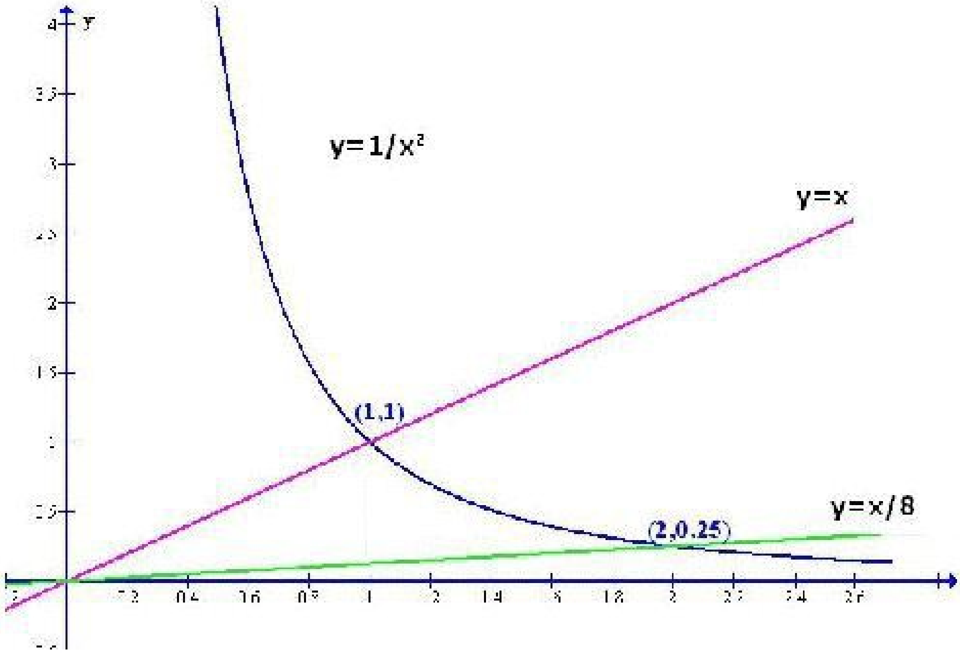
Answer 27E.

First, we need to sketch the graph of

$$y = \frac{1}{x^2}$$

$$y = x$$

$$y = \frac{1}{8}x$$



$A = \int_a^b \left[f_{top} - f_{bottom} \right] dx$

On the interval $0 \le x \le 1 \Rightarrow f_{top} = x; f_{bottom} = \frac{1}{8}x$

On the interval $1 \le x \le 2 \Rightarrow f_{top} = \frac{1}{x^2}; f_{bottom} = \frac{1}{8}x$

Therefore

Area of region would be

$A = \int_0^1 \left(x - \frac{1}{8}x \right) dx + \int_1^2 \left(\frac{1}{x^2} - \frac{1}{8}x \right) dx$

$\Rightarrow$

$A = \int_0^1 \left(\frac{7}{8}x \right) dx + \int_1^2 \left(\frac{1}{x^2} \right) dx - \int_1^2 \left(\frac{1}{8}x \right) dx$

$\Rightarrow$

$A = \left[\frac{7x^2}{16} \right]_0^1 + \left[-\frac{1}{x} \right]_1^2 - \left[\frac{1}{16}x^2 \right]_1^2$

$\Rightarrow$

$A = \frac{7}{16} - \frac{1}{2} + 1 - \frac{1}{4} + \frac{1}{16} = \frac{3}{4}$

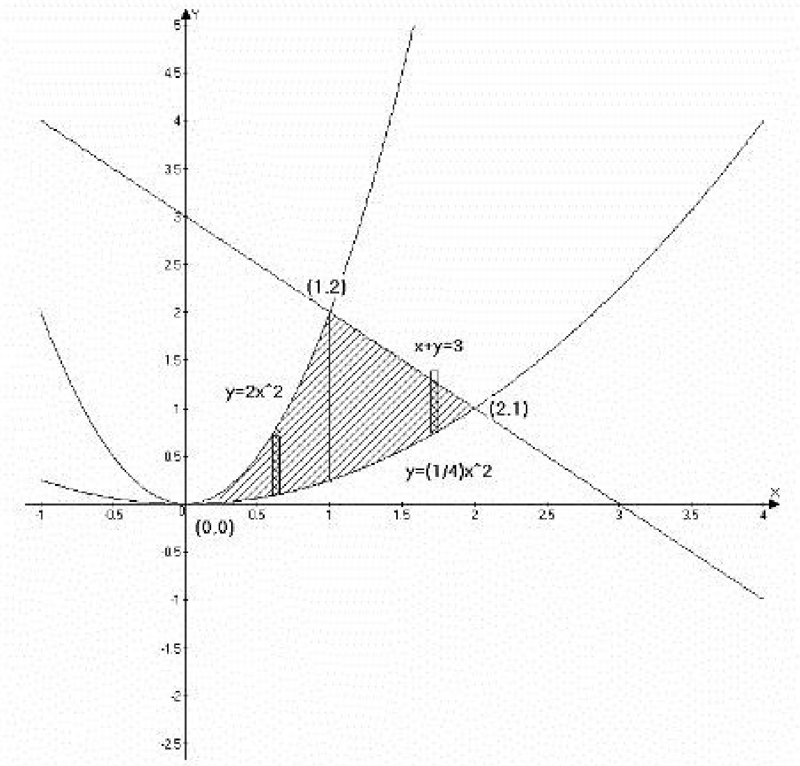
$\Rightarrow$

$A = \frac{3}{4}$

Answer 28E.

Given curves are $y = \frac{1}{4}x^2, y = 2x^2, x + y = 3, x \ge 0.$

The graph of the curves is



In the graph y=(1/4)x^2 represents $y = x^2/4$ and y =2x^2 represents $y = 2x^2$.
Here we want to find the area of OAB. the point of intersection of the curves is

$$O(0,0), A(2,1), B(1,2)$$

Draw a line parallel to y-axis passes through $B(1,2)$ then this line intersects $y = \frac{x^2}{4}$ at C.

The area of OAB = Area of OBC + Area of ABC.
In the region OBC, top and bottom boundaries are

$$y_T = 2x^2 \text{ and } y_B = \frac{x^2}{4}$$

The area of a typical rectangle is

$$(y_T - y_B)\Delta x = \left(2x^2 - \frac{x^2}{4}\right)\Delta x$$
$$= \frac{7x^2}{4}\Delta x$$

And the region lies between $x = 0$ and $x = 1$.

$$\text{Area of } OBC = \int_0^1 \frac{7x^2}{4} dx$$
$$= \frac{7}{4} \cdot \frac{x^3}{3}\Big|_0^1$$
$$= \frac{7}{12}$$

In the region ABC, the top and bottom boundaries are

$$y_T = 3 - x \text{ And } y_B = \frac{x^2}{4}$$

The area of a typical rectangle is

$$(y_T - y_B)\Delta x = \left(3 - x - \frac{x^2}{4}\right)\Delta x$$

And the region lies between $x = 1$ and $x = 2$

$$\text{Area of } ABC = \int_1^2 \left[3 - x - \frac{x^2}{4}\right] dx$$
$$= \left[3x - \frac{x^2}{2} - \frac{x^3}{12}\right]_1^2$$
$$= \left(6 - 2 - \frac{2}{3}\right) - \left(3 - \frac{1}{2} - \frac{1}{12}\right)$$
$$= \frac{11}{12}$$

Hence total area, area of ABC = area of OBC + area of ABC

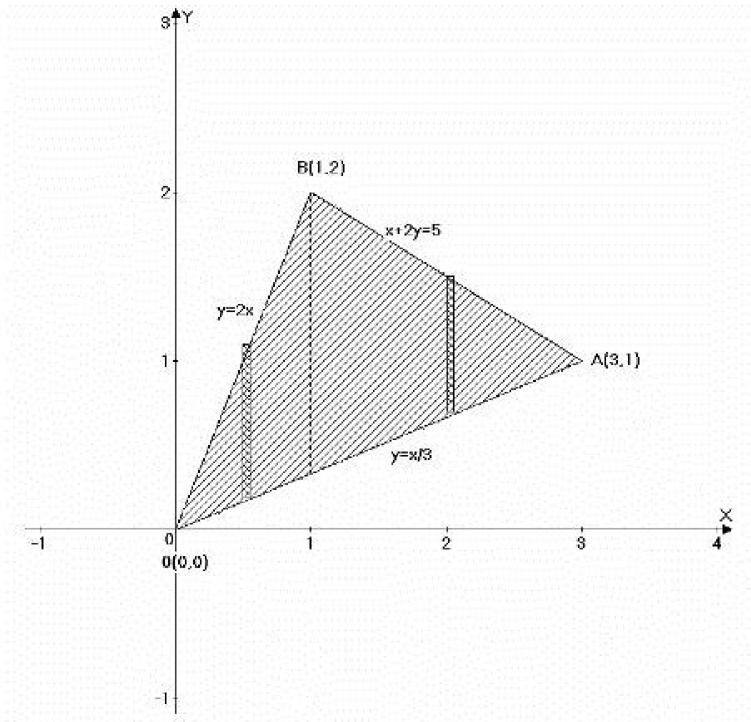
$$= \frac{7}{12} + \frac{11}{12}$$
$$= \frac{18}{12}$$
$$= \frac{3}{2}$$

$$\boxed{\text{Therefore, Area} = \frac{3}{2} \text{ square units}}$$

Answer 29E.

Given vertices are $(0,0), (3,1), (1,2)$

The graph of the triangle is



Equation of the line OB is $y = 2x$

Equation of the line OA is $y = \frac{x}{3}$

Equation of the line AB is $x + 2y = 5$

Here we want to find the area of OAB

Draw a line parallel to y-axis passes through the point $B(1,2)$.

Then this line intersects the line $y = \frac{x}{3}$ at C.

Therefore area of OAB = Area of OBC + Area of ABC.

In the region OBC, top and bottom boundaries are

$$y_T = 2x \text{ and } y_B = \frac{x}{3}$$

The area of a typical rectangle is

$$(y_T - y_B)\Delta x = \left(2x - \frac{x}{3}\right)\Delta x$$
$$= \frac{5x}{3}\Delta x$$

And the region lies between $x = 0$ and $x = 1$.

$$\text{Area of } OBC = \int_0^1 \frac{5x}{3}\cdot dx$$
$$= \frac{5}{3}\cdot\frac{x^2}{2}\Big]_0^1$$
$$= \frac{5}{3}\left(\frac{1}{2}\right)$$
$$= \frac{5}{6}$$

In the region ABC, the top and bottom boundaries are

$$y_T = \frac{5-x}{2} \text{ and } y_B = \frac{x}{3}$$

The area of a typical rectangle is

$$(y_T - y_B)\Delta x = \left(\frac{5-x}{2} - \frac{x}{3}\right)\Delta x$$
$$= \left(\frac{15-5x}{6}\right)\Delta x$$

And the region lies between $x = 1$ and $x = 3$.

$$\text{Area of } ABC = \int_1^3 \left(\frac{15-5x}{6}\right)\Delta x$$
$$= \frac{1}{6}\left[15x - 5\cdot\frac{x^2}{2}\right]_1^3$$
$$= \frac{1}{6}\left[45 - \frac{45}{2} - 15 + \frac{5}{2}\right]$$
$$= \frac{10}{6}$$

Therefore hence area of OAB = Area of OBC + Area of ABC

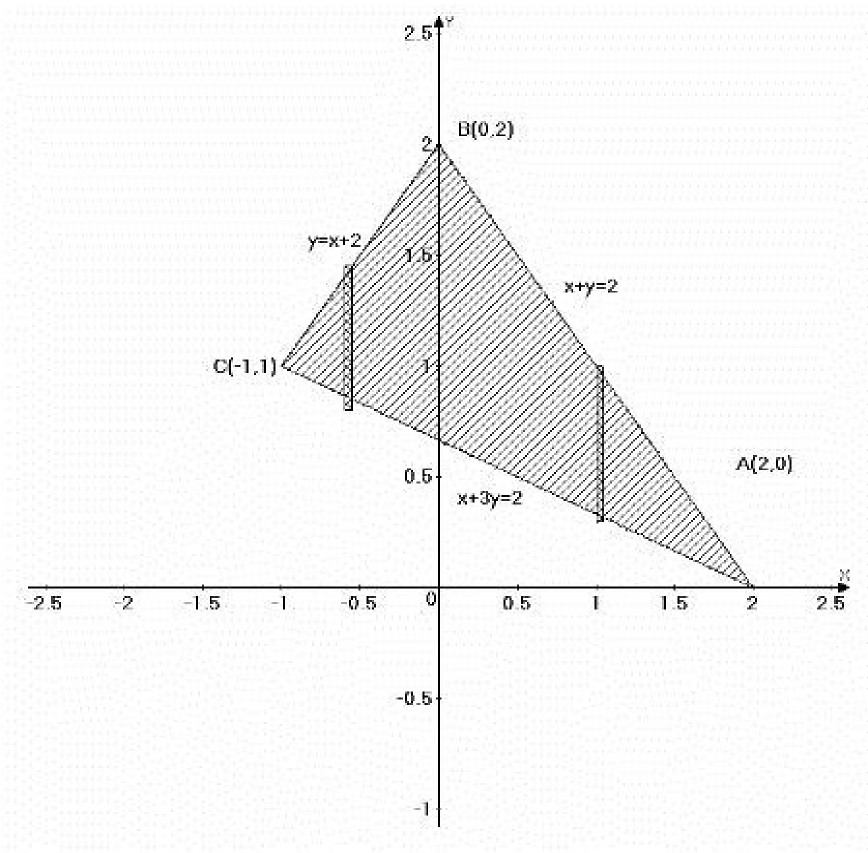
$$= \frac{5}{6} + \frac{10}{6}$$
$$= \frac{15}{6}$$
$$= \frac{5}{2}$$

$$\boxed{\text{Therefore, area} = \frac{5}{2}\text{ square units}}$$

Answer 30E.

Let the given vertices be $A(2,0), B(0,2), C(-1,1)$

Then the graph of the triangle is



Equation of the line AB is $x+y=2$,

Equation of the line AC is $x+3y=2$

Equation of the line BC is $y=x+2$

Here we want to find the area of triangle ABC.

Let D be the point of intersection of y-axis and $x+3y=2$

Then the line BD divides the region ABC into two sub regions BCD and ABD.

Therefore area of $\Delta ABC =$ area of $\Delta BCD +$ Area of ΔABD.

In the region ΔBCD, top and bottom boundaries are

$$y_T = x+2 \text{ And } y_B = \frac{2-x}{3}$$

The area of a typical is

$$(y_T - y_B)\Delta x = \left(x+2-\frac{2-x}{3}\right)\Delta x$$

$$= \left(\frac{4x+4}{3}\right)\Delta x$$

And the region lies between $x=-1$ and $x=0$.

$$\text{Area of } \Delta BCD = \int_{x=-1}^{0}\left(\frac{4x+4}{3}\right)dx$$

$$= \frac{1}{3}\left[4\cdot\frac{x^2}{2}+4\cdot x\right]_{-1}^{0}$$

$$= -\frac{1}{3}(2-4)$$

$$= \frac{2}{3}$$

In the region ABD, the top and bottom boundaries are

$$y_T = 2 - x \text{ And } y_B = \frac{2-x}{3}$$

The area of typical rectangle is

$$(y_T - y_B)\Delta x = \left[2 - x - \frac{2-x}{3}\right]\Delta x$$

$$= \frac{4-2x}{3}\cdot \Delta x$$

And the region lies between $x = 0$ and $x = 2$

$$\text{Area of } \Delta ABD = \int_{x=0}^{2} \left(\frac{4-2x}{3}\right) dx$$

$$= \frac{1}{3}\left[4x - 2\cdot\frac{x^2}{2}\right]_0^2$$

$$= \frac{1}{3}[8-4]$$

$$= \frac{4}{3}$$

Hence area of ΔABC = Area of ΔBCD + Area of ΔABD

$$= \frac{2}{3} + \frac{4}{3}$$

$$= \frac{6}{3}$$

$$= 2$$

i.e, $\boxed{\text{Area} = 2 \text{ square units}}$

Answer 31E.

Consider the integral $\int_0^{\frac{\pi}{2}} |\sin x - \cos 2x| dx$.

The objective is to find the integral and interpret it as area of the region.

First find the intersection point of curve by solving $\sin x = \cos 2x$.

$\sin x = \cos 2x$ Given equation

$\sin x - \cos 2x = \cos 2x - \cos 2x$ Subtract $\cos 2x$ from both sides

$\sin x - \cos 2x = 0$ Combine like terms

$\sin x - (1 - 2\sin^2 x) = 0$ Use $\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x + \sin x - 1 = 0$ Distribute -1

$2\sin^2 x + 2\sin x - \sin x - 1 = 0$ Break middle term

$(2\sin^2 x + 2\sin x) + (-\sin x - 1) = 0$ Group pairs

$2\sin x(\sin x + 1) - 1(\sin x + 1) = 0$ Factor each pair

$(2\sin x - 1)(\sin x + 1) = 0$ $ab - bc = (a-c)b$

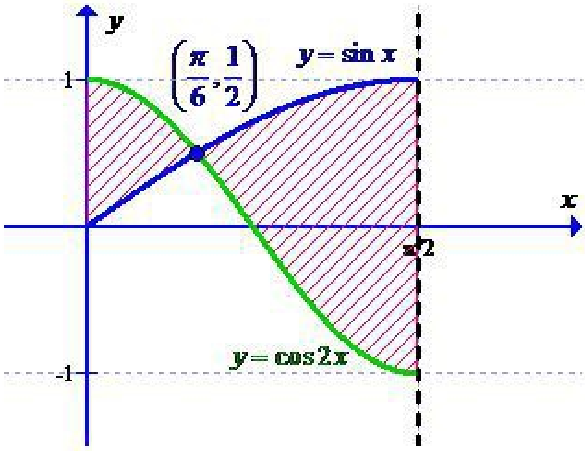
$\sin x = -1, \frac{1}{2}$ Zero product property

$x = -\frac{\pi}{2}, \frac{\pi}{6}$ Simplify

Since $x = -\frac{\pi}{2}$ is not in domain $\left[0, \frac{\pi}{2}\right]$, discard this value.

Therefore, the two curves intersect at $\left(\frac{\pi}{6}, \frac{1}{2}\right)$.

The region between curves $y = \sin x$ and $y = \cos 2x$ is as shown below:



From the above graph, it can observed that in interval $\left[0, \frac{\pi}{6}\right]$ the curve $y = \cos 2x$ is above the $y = \sin x$ and in interval $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ the curve $y = \cos 2x$ is below the $y = \sin x$.

The above integral represents area of the region.

So, split the integral as follows:

$$\int_0^{\frac{\pi}{2}} |\sin x - \cos 2x| dx = \int_0^{\frac{\pi}{6}} (\cos 2x - \sin x) dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin x - \cos 2x) dx$$

Now evaluate the integral as follows:

$$\int_0^{\frac{\pi}{2}} |\sin x - \cos 2x| dx = \left[\frac{1}{2}\sin 2x + \cos x\right]_0^{\frac{\pi}{6}} + \left[-\cos x - \frac{1}{2}\sin 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(\frac{1}{2}\sin\frac{2\pi}{6} + \cos\frac{\pi}{6}\right) - \left(\frac{1}{2}\sin 0 + \cos 0\right) + \left(-\cos\frac{\pi}{2} - \frac{1}{2}\sin\pi\right)$$
$$- \left(-\cos\frac{\pi}{6} - \frac{1}{2}\sin 2\cdot\frac{\pi}{6}\right)$$
$$= \left(\frac{1}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) - (0+1) + (-0-0) - \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)$$
$$= \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}\right) - 1 + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}$$
$$= \frac{3\sqrt{3}}{2} - 1$$

Therefore, the area of the region between the curves is $\boxed{\frac{3\sqrt{3}}{2} - 1}$.

Answer 32E.

We have to evaluate $\int_0^4 \left|\sqrt{x+2} - x\right| dx$

Since $f(x) = \sqrt{x+2} - x$ when $\sqrt{x+2} - x \geq 0$, it is true when $x \leq 2$

So $\quad f(x) = \sqrt{x+2} - x$ when $0 \leq x \leq 2$

And $\quad f(x) = x - \sqrt{x+2}$ when $2 < x \leq 4$

We can write

$$\int_0^4 \left|\sqrt{x+2}-x\right|dx = \int_0^2 \left|\sqrt{x+2}-x\right|dx + \int_2^4 \left|\sqrt{x+2}-x\right|dx$$

$$= \int_0^2 \left(\sqrt{x+2}-x\right)dx + \int_2^4 \left(x-\sqrt{x+2}\right)dx$$

$$= \int_0^2 \sqrt{x+2}\,dx - \int_0^2 x\,dx + \int_2^4 x\,dx - \int_2^4 \sqrt{x+2}\,dx$$

$$= \left[\frac{2}{3}(x+2)^{3/2}\right]_0^2 - \left[\frac{x^2}{2}\right]_0^2 + \left[\frac{x^2}{2}\right]_2^4 - \left[\frac{2}{3}(x+2)^{3/2}\right]_2^4$$

[By chain rule and FTC -2]

$$= \left[\frac{2}{3}(4)^{3/2} - \frac{2}{3}(2)^{3/2}\right] - [2] + [8-2] - \left[\frac{2}{3}(6)^{3/2} - \frac{2}{3}(4)^{3/2}\right]$$

$$= \frac{16}{3} - \frac{4\sqrt{2}}{3} - 2 + 6 - \frac{12}{3}\sqrt{6} + \frac{16}{3}$$

Or $$\boxed{\int_0^4 \left|\sqrt{x+2}-x\right|dx = \frac{4}{3}\left(11-\sqrt{2}-3\sqrt{6}\right)}$$

Since in the interval [0, 4]

$$\int_0^4 \left|\sqrt{x+2}-x\right|dx = \int_0^2 \left[\left(\sqrt{x+2}\right)-x\right]dx + \int_2^4 \left[x-\left(\sqrt{x+2}\right)\right]dx$$

So we can sketch the curve $y = x$ and $y = \sqrt{x+2}$ in the interval [0, 4]

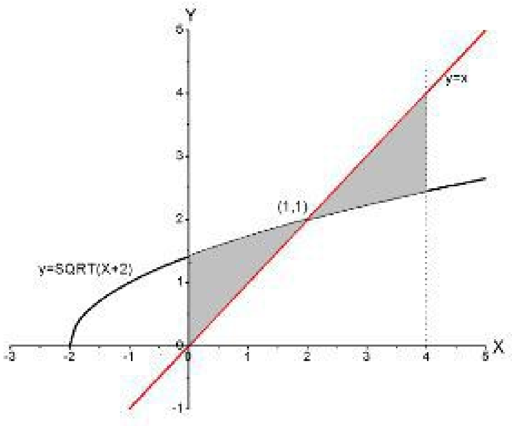


Fig.1

Before x = 2 the curve $\sqrt{x+2} > x$ so area of the region in the interval [0, 2] enclosed by these curves

$$= \int_0^2 \left(\sqrt{x+2}-x\right)dx$$

And after $x = 2$ the curve $\sqrt{x+2} < x$ so area of the region in the interval [2, 4] enclosed by these curves

$$= \int_2^4 \left(x-\sqrt{x+2}\right)dx$$

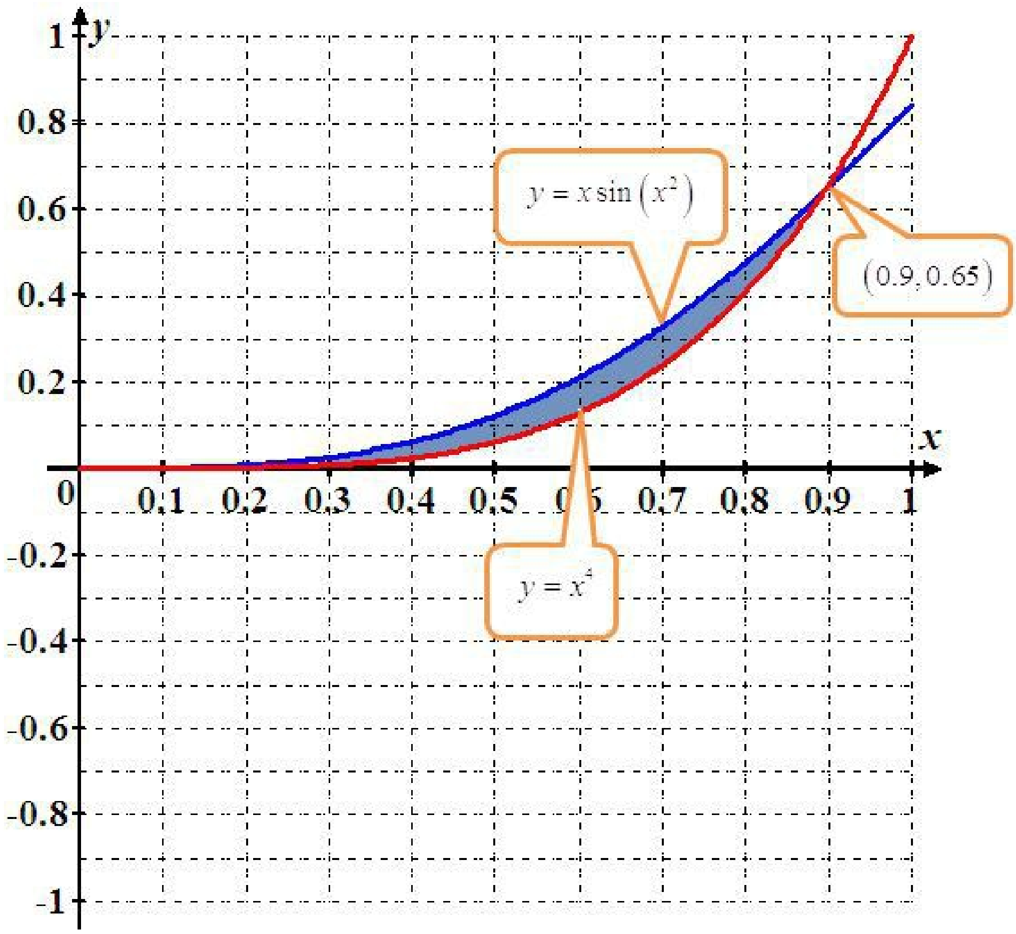
So total area $= \int_0^2 \left(\sqrt{x+2}-x\right)dx + \int_2^4 -\left(\sqrt{x+2}-x\right)dx = \int_0^4 \left|\sqrt{x+2}-x\right|dx$

Answer 33E.

Consider the following functions:

$y = x\sin(x^2)$, $y = x^4$, and $x \geq 0$.

The region enclosed by the curves, $y = x\sin(x^2)$ and $y = x^4$ is as shown below:



From the graph, the x – coordinates of the point of intersection of the curves, $y = x\sin(x^2)$ and $y = x^4$ are $x = 0$, and $x = 0.9$.

The area A of the region bounded by the curves, $y = f(x), y = g(x)$ and the lines $x = a,\ x = b$, where the curves f and g are continuous and $f(x) \geq g(x)$ for all x in $[a,b]$ is,

$$A = \int_a^b [f(x) - g(x)]dx.$$

Here, the area of the shaded region is bounded by the curves $y = x\sin(x^2)$ and $y = x^4$.

The area of the region bounded by the curves $y = x^4$ and $y = x\sin(x^2)$ between $x = 0$ and $x = 0.9$ is as follows:

$$A = \int_0^{0.9} \left| x\sin(x^2) - x^4 \right| dx$$

$$= \int_0^{0.9} x\sin(x^2)\,dx - \int_0^{0.9} x^4 dx \quad ......(1)$$

To solve $\int_0^{0.9} x\sin(x^2)\,dx$, use the substitution method.

Set $x^2 = t \Rightarrow 2xdx = dt$

If $x = 0$, then $t = 0$ and

If $x = 0.9$, then $t = 0.81$

Therefore,

$$\begin{aligned}
\int_0^{0.9} x\sin(x^2)\,dx &= \frac{1}{2}\int_0^{0.9} 2x\sin(x^2)\,dx \\
&= \frac{1}{2}\int_0^{0.81} \sin(t)\,dt \\
&= \frac{1}{2}\left[-\cos t\right]_0^{0.81} \\
&= 0.5\left[-\cos(0.81) + \cos(0)\right] \\
&= 0.5\left[-0.6894984330 + 1\right] \\
&\approx 0.1552507835
\end{aligned}$$

Substitute $\int_0^{0.9} x\sin(x^2)\,dx = 0.1552507835$ into the equation (1) as follows:

$$\begin{aligned}
\int_0^{0.9} x\sin(x^2)\,dx - \int_0^{0.9} x^4 dx &= 0.1552507835 - \left[\frac{x^5}{5}\right]_0^{0.9} \\
&= 0.1552507835 - \left[\frac{(0.9)^5}{5} - \frac{(0)^5}{5}\right] \\
&\approx 0.1552507835 - 0.118098 \\
&\approx 0.0371527835
\end{aligned}$$

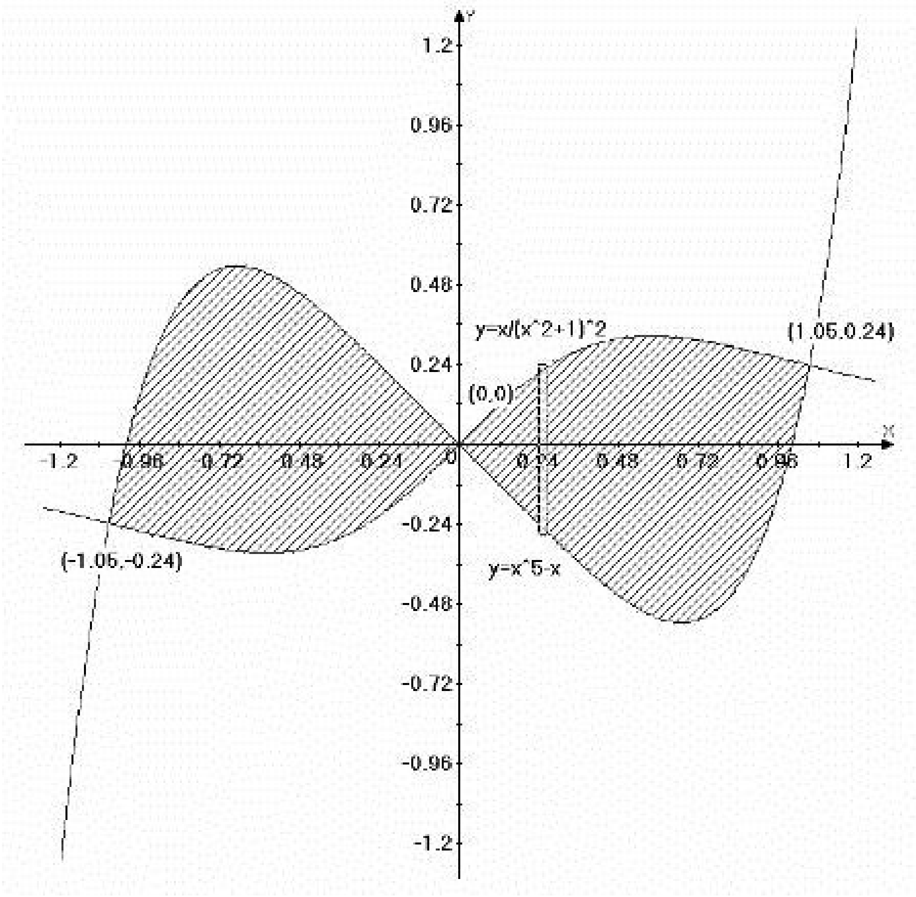
≈ 0.04

Hence, the area of the shaded region in the graph is $\boxed{0.04}$.

Answer 34E.

Given curves are $y = \dfrac{x}{(x^2+1)^2}$ and $y = x^5 - x$ and $x \geq 0$

The graph of the curves is



In the graph y=x/ ((x^2+1) ^2 denotes $y = x/(x^2+1)^2$ and y=x^5 –x denotes $y=x^5-x$

Solving the given equations simultaneously,

$$x^5 - x = \frac{x}{\left(x^2+1\right)^2}$$

$$\Rightarrow \quad \left(x^2+1\right)^2 x\left(x^4-1\right) = x$$

$$\Rightarrow \quad x = 0, 1, -1$$

Therefore points of intersections are $\left(-1, \frac{-1}{4}\right), (0,0), (1, 1/4)$

Since $x \geq 0$ we considerably $(0,0)$ and $(1,1/4)$.

We want to find the area in first and second quadrants.

The top and bottom boundaries are $y_T = \frac{x}{\left(x^2+1\right)^2}$ and $y_B = x^5 - x$

The area of a typical rectangle is

$$\left(y_T - y_B\right)\Delta x = \left[\frac{x}{\left(x^2+1\right)^2} - \left(x^5 - x\right)\right]\Delta x$$

$$= \left[\frac{x}{\left(x^2+1\right)^2} - x^5 + x\right] dx$$

And the region lies between $x = 0$ and $x = 1$.

So, Area, $A=\int_{x=0}^{1}\left[\frac{x}{\left(x^2+1\right)^2}-x^5+x\right]dx$

$$=\int_{x=0}^{1}\frac{x}{\left(x^2+1\right)^2}dx-\int_{x=0}^{1}x^5dx+\int_{x=0}^{1}x\cdot dx$$

$$=\frac{1}{2}\int_{x=0}^{1}\frac{2x}{\left(x^2+1\right)^2}dx-\left(\frac{x^6}{6}\right)_0^1+\left(\frac{x^2}{2}\right)_0^1$$

$$=\frac{1}{2}\left[-\frac{1}{1+x^2}\right]_0^1-\left(\frac{1}{6}\right)+\frac{1}{2}$$

$$=\frac{1}{2}\left[-\frac{1}{2}+1\right]-\frac{1}{6}+\frac{1}{2}$$

$$=\frac{1}{4}-\frac{1}{6}+\frac{1}{2}$$

$$=\frac{3}{4}-\frac{1}{6}$$

$$=\frac{7}{12}$$

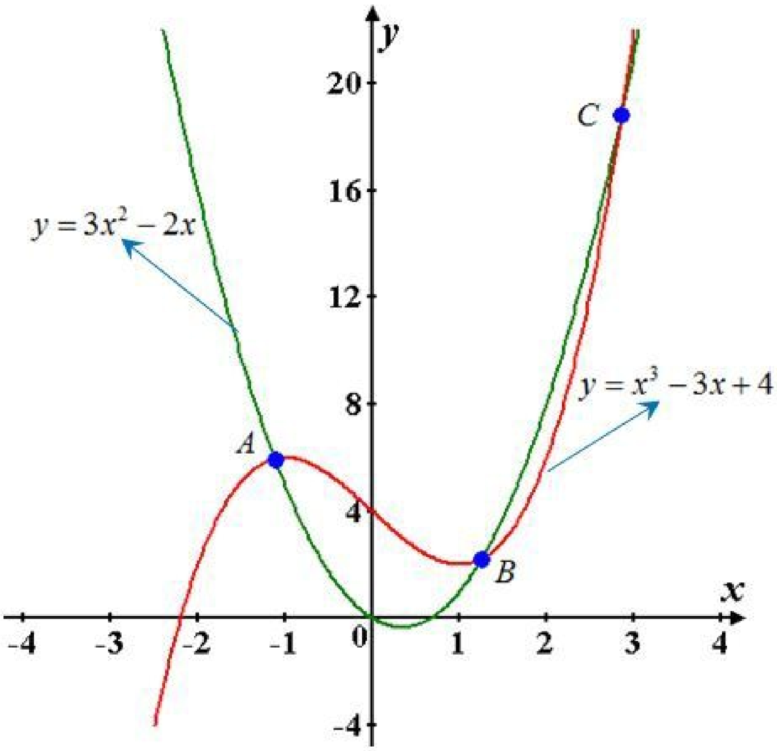
Hence, $\boxed{\text{Area}=\frac{7}{12}\text{ square units}}$

Answer 35E.

Consider the curves $y=3x^2-2x$ and $y=x^3-3x+4$

Use a graph to find approximate *x*-coordinates of the points of intersection of the given curves.

Graph of the two curves



Observe that there are three points of intersection.

With a coordinate finder tool approximate the *x*-coordinates of the points *A*, *B*, and *C*. obtain -1.11, 1.25, and 2.86 as approximate values of the points *A*, *B*, and *C* respectively. To estimate the area between the graphs integrate the upper function minus the lower function.

Between the points *A* and *B*, $y=x^3-3x+4$ is the upper function while between the points *B* and *C*, $y=3x^2-2x$ is the upper function.

Area between A and B

$$\text{Area} = \int_{-1.11}^{1.25} (\text{Upper curve} - \text{lower curve})\,dx$$
$$= \int_{-1.11}^{1.25} \left[x^3 - 3x + 4 - \left(3x^2 - 2x\right)\right] dx$$
$$= \int_{-1.11}^{1.25} \left(x^3 - 3x^2 - x + 4\right) dx$$
$$= \left[\frac{x^4}{4} - x^3 - \frac{x^2}{2} + 4x\right]_{-1.11}^{1.25}$$
$$= \left(\frac{(1.25)^4}{4} - (1.25)^3 - \frac{(1.25)^2}{2} + 4(1.25)\right) - \left(\frac{(-1.11)^4}{4} - (-1.11)^3 - \frac{(-1.11)^2}{2} + 4(-1.11)\right)$$
$$= 6.1848$$

Area between B and C

$$\text{Area} = \int_{1.25}^{2.86} (\text{Upper curve} - \text{lower curve})\,dx$$
$$= \int_{1.25}^{2.86} \left[3x^2 - 2x - \left(x^3 - 3x + 4\right)\right] dx$$
$$= \int_{1.25}^{2.86} \left(3x^2 - 2x - x^3 + 3x - 4\right) dx$$
$$= \left[x^3 - \frac{x^4}{4} + \frac{x^2}{2} - 4x\right]_{1.25}^{2.86}$$
$$= \left(-\frac{(2.86)^4}{4} + (2.86)^3 + \frac{(2.86)^2}{2} - 4(2.86)\right) - \left(-\frac{(1.25)^4}{4} + (1.25)^3 + \frac{(1.25)^2}{2} - 4(1.25)\right)$$
$$= 2.1929$$

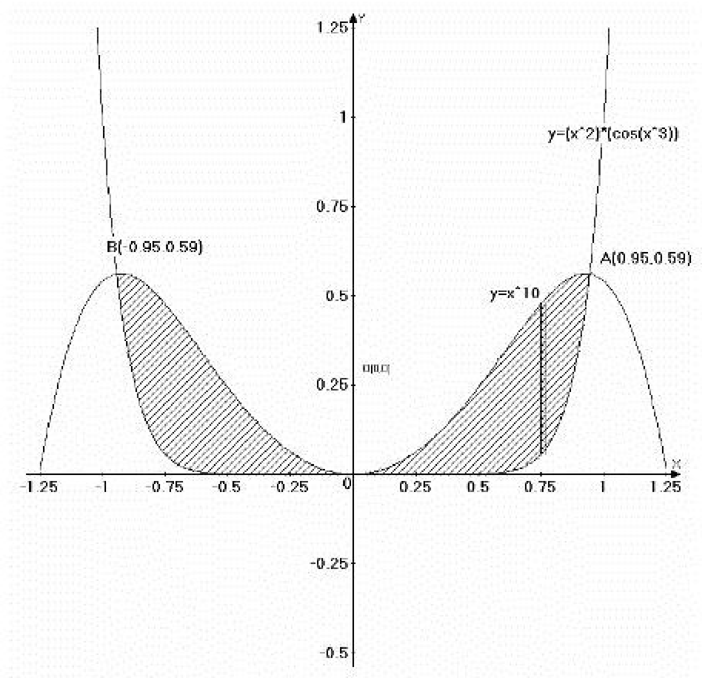
Total area=sum of the two areas

$$= 6.1848 + 2.1929$$
$$= \boxed{8.38}$$

Answer 36E.

Given curves are $y = x^2 \cos\left(x^3\right)$ and $y = x^{10}$

The graph of the curves is



In the graph y=x^10 represents $y=x^{10}$ and y=(x^2)*cos(x^3) represents $y=x^2\cos x^3$

Solving the given equations,

$$x = \pm 0.95$$

Therefore points of intersections are $A(0.95, 0.59), B(-0.95, 0.59)$.

Here we want to find the area of the shaded region. Since the region is symmetric about y-axis, total area is equal to $2\times$area of the region in the first quadrant.

The top and bottom boundaries are $y_T = x^2\cos x^3$ and $y_b = x^{10}$.

The area of a typical rectangle is

$$(y_T - y_B)\Delta x = \left[x^2\cos x^3 - x^{10}\right]\Delta x$$

And the region lies between $x = 0$ and $x = 0.95$

So total area $A = 2\int_0^{0.95}\left[x^2\cos\left(x^3\right) - x^{10}\right]dx$

$$= 2\left[\frac{1}{3}\int_0^{0.95} 3x^2\cos\left(c^3\right)dx - \int_0^{0.95} x^{10}dx\right]$$

$$= 2\left[\frac{1}{3}\left(\sin\left(x^3\right)\right)_0^{0.95} - \left(\frac{x^{11}}{11}\right)_0^{0.095}\right]$$

$$= 2\left[\frac{1}{3}(0.7561) - \frac{1}{11}(0.95)^{11}\right]$$

$$= 0.1$$

Therefore $\boxed{\text{Area} = 0.1\,\text{square units}}$

So total area $A = 2\int_0^{0.95}\left[x^2\cos\left(x^3\right) - x^{10}\right]dx$

$$= 2\left[\frac{1}{3}\int_0^{0.95} 3x^2\cos\left(c^3\right)dx - \int_0^{0.95} x^{10}dx\right]$$

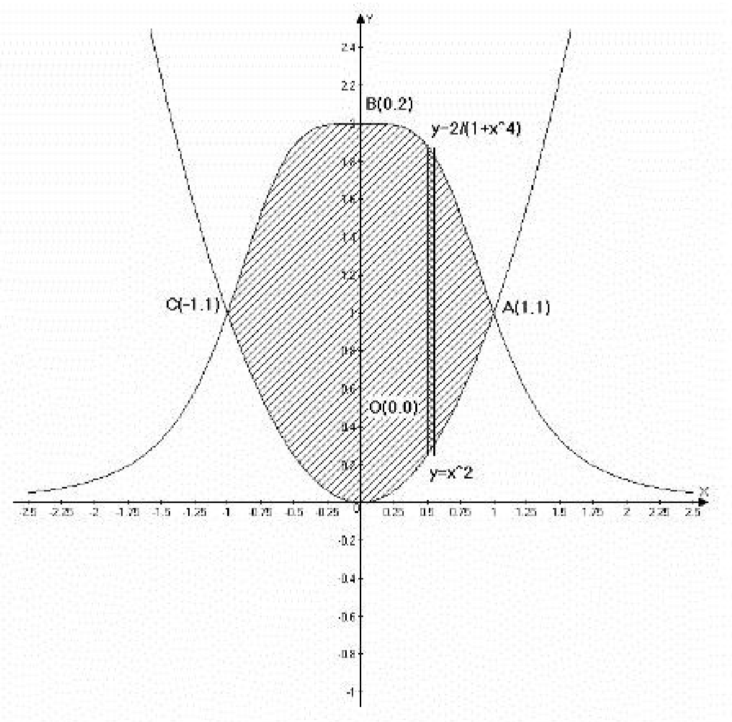
$$= 2\left[\frac{1}{3}\left(\sin\left(x^3\right)\right)_0^{0.95} - \left(\frac{x^{11}}{11}\right)_0^{0.095}\right]$$

$$= 2\left[\frac{1}{3}(0.7561) - \frac{1}{11}(0.95)^{11}\right]$$

$$= 0.1$$

Therefore $\boxed{\text{Area} = 0.1\,\text{square units}}$

Answer 37E.



In the graph y=2/(1+x^4) represents $y=2/(1+x^4)$ and y=x^2 represents $y=x^2$

Here we want find the area of the shaded region.
The point of intersection of the curves are $(1,1),(-1,1)$.
In the shaded region, top and bottom boundaries are

$$y_T=\frac{2}{1+x^4} \text{ And } y_B=x^2$$

The area of a typical rectangle is

$$(y_T-y_B)\Delta x=\left(\frac{2}{1+x^4}-x^2\right)\Delta x$$

And the region lies between $x=-1$ and $x=1$

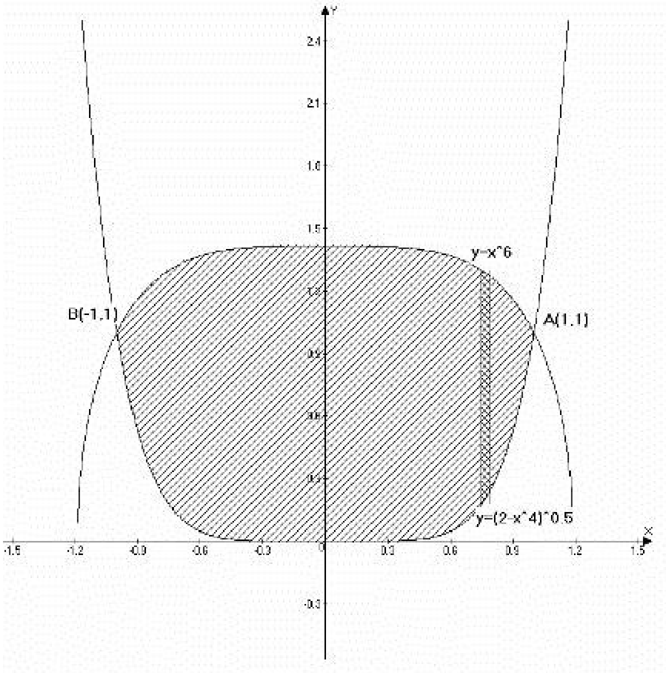
Area of the shaded region $=\int_{x=-1}^{1}\left[\frac{2}{1+x^4}-x^2\right]dx$

$= 2.8012253$ using CAS

Therefore $\boxed{\text{Area} = 2.8012253}$

Answer 38E.

Given curves are $y=x^6, y=\sqrt{2-x^4}$
The graph of the given curves is



In the graph y=x^6 denote $y=x^6$ and y=(2-x^4)^0.5 denotes $y=\sqrt{2-x^4}$

Here we want to find the area of the shaded region.
The points of intersection of the curves are $(-1,1)$ and $(1,1)$.
In the shaded region, the top and bottom boundaries are

$$y_T=\sqrt{2-x^4} \text{ And } y_B=x^6$$

The area of a typical rectangle is

$$(y_T-y_B)\Delta x=\left(\sqrt{2-x^4}-x^6\right)\Delta x$$

And the region lies between $x=-1$ and $x=1$

Area of the shaded region $=\int_{x=-1}^{1}\left[\sqrt{2-x^4}-x^6\right]dx$

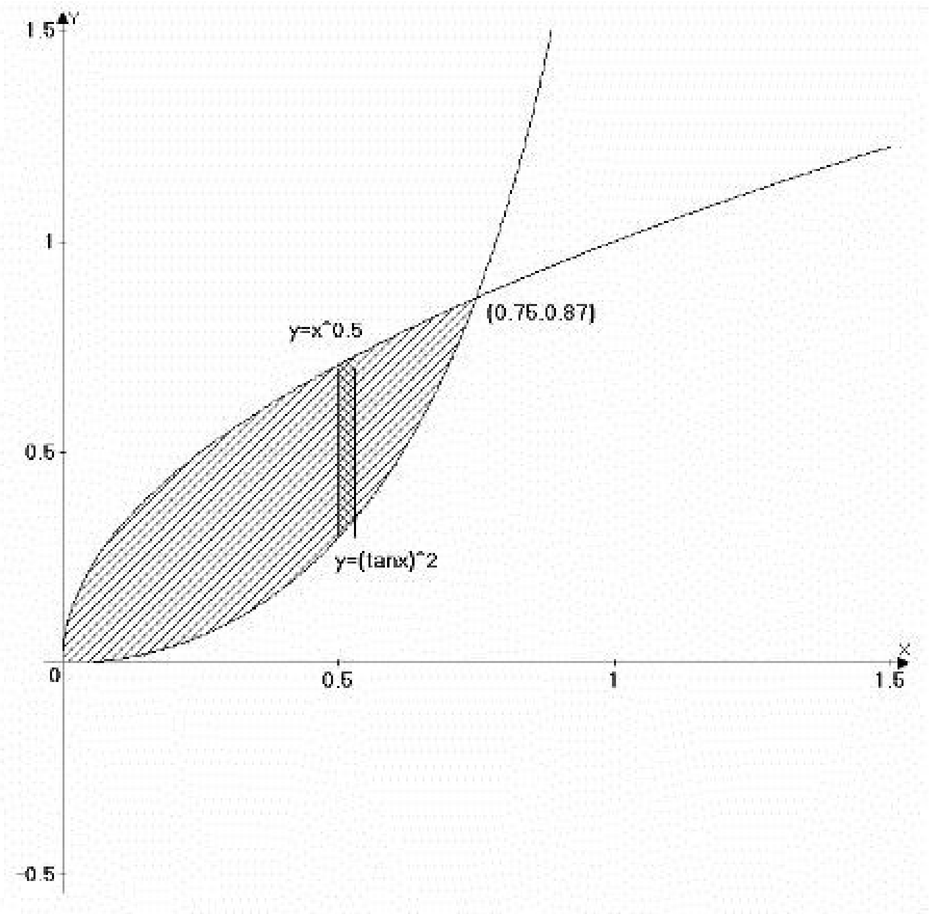
$= 2.3891939$ (By using CAS)

$\boxed{\text{Area of the region between the given curves}, A = 2.3891939}$

Answer 39E.

Given curves are $y = \tan^2 x, y = \sqrt{x}$

The graph of the curves is



In the graph y=(tan x)^2 denotes $y = \tan^2 x$ and y=x^0.5 denotes $y = \sqrt{x}$

Here we want to find the area of the shaded region.

The point of intersection is $(0.75, 0.87)$

In the shaded region, top and bottom boundaries are

$$y_T = \sqrt{x} \text{ and } y_B = \tan^2 x.$$

The area of a typical rectangle is

$$(y_T - y_B)\Delta x = \left(\sqrt{x} - \tan^2 x\right)\Delta x$$

And the region lies between $x = 0$ and $x = 0.75$

$$\text{Area of the shaded region } = \int_0^{0.75} \left[\sqrt{x} - \tan^2 x\right] dx$$

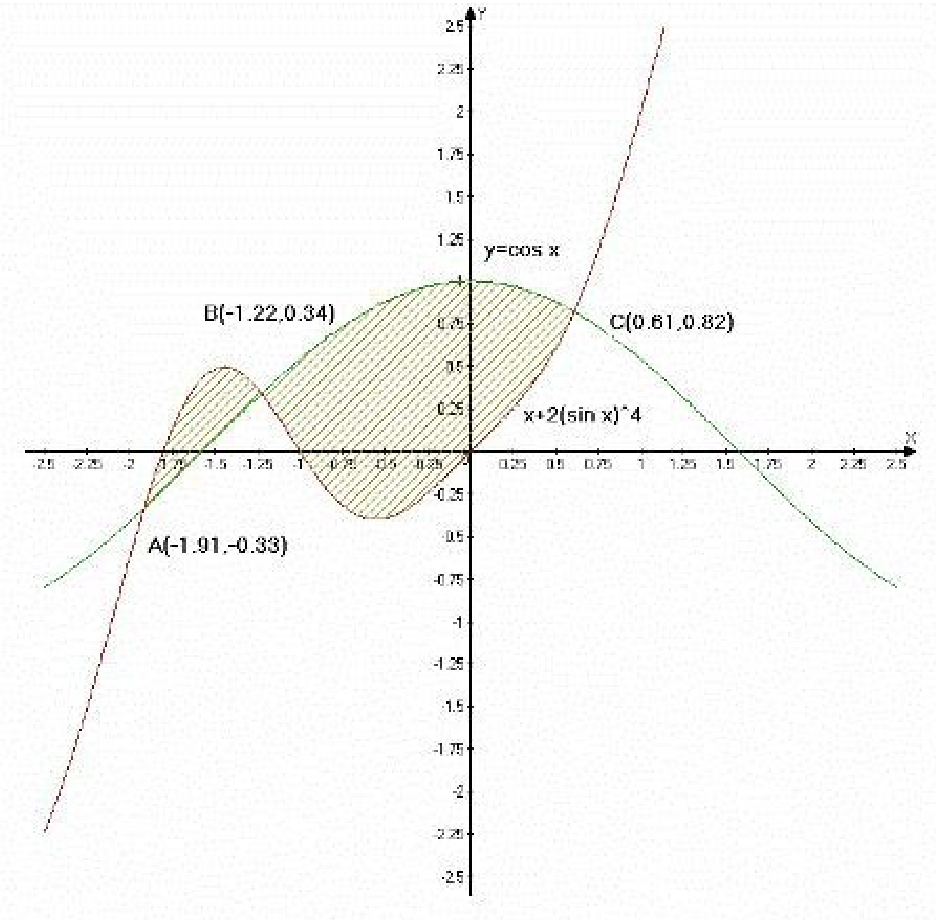
$$\approx 0.251416 (\text{using CAS})$$

Therefore $\boxed{\text{Area of the region between curves is } A = 0.251416}$

Answer 40E.

Given curves are $y = \cos x, y = x + 2\sin^4 x$

The graph of the curves is



In the graph y=x+2(sinx)^4 represents $y = x + 2\sin^4 x$

Here we want to find the area of the shaded region.

The point of intersections of the curves are $(-1.91, -0.33), (-1.22, 0.34)$ and $(0.61, 0.82)$.

In the first region (between $x = -1.91$ and $x = -1.22$), the top and bottom boundaries are

$$y_T = x + 2\sin^4 x \text{ and } y_B = \cos x$$

The area of typical rectangle is $(y_T - y_B)\Delta x = \left[\left(x + 2\sin^4 x\right) - \cos x\right]\Delta x$ and the region lies between $x = -1.91$ and $x = -1.22$

In the second region (between $x = -1.22$ and $x = 0.61$) the top and bottom boundaries are

$$y_T = \cos x \text{ and } y_B = x + 2\sin^4 x$$

The area of a typical rectangle is $(y_T - y_B)\Delta x$

$$= \left[\cos x - \left(x + 2\sin^4 x\right)\right]\Delta x$$

And the region lies between $x = -1.22$ and $x = 0.61$

Therefore the area of the shaded region

$$A = \int_{-1.91}^{-1.22} \left[x + 2\sin^4 x - \cos x\right]dx + \int_{-1.22}^{0.61} \left[\cos x - x - 2\sin^4 x\right]dx$$

$$= 0.1929524 + 1.5111803 \text{ (using CAS)}$$

$$= 1.7041326$$

Therefore $\boxed{\text{Area} = 1.7041326}$

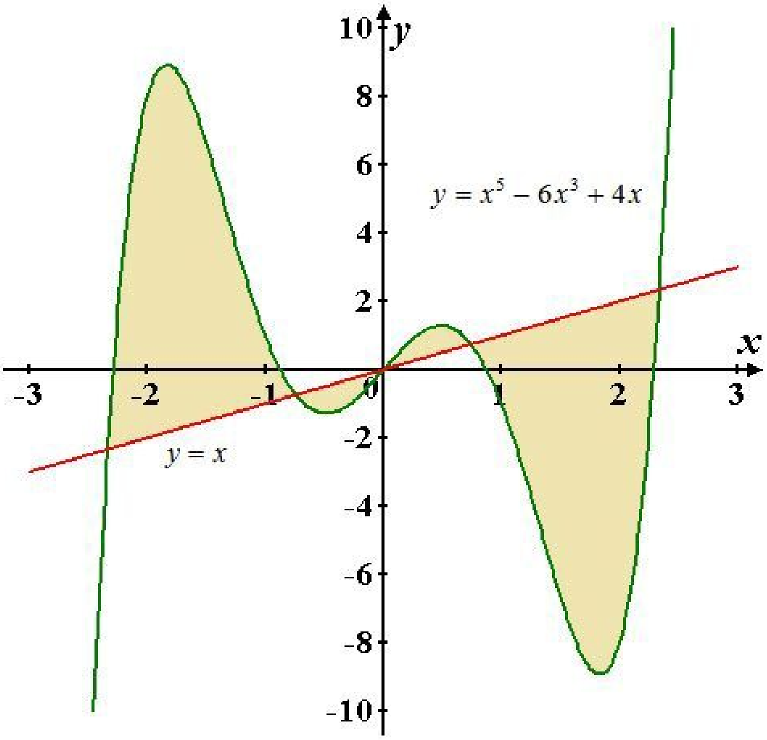
Answer 41E.

Consider the curves.

$y = x^5 - 6x^3 + 4x$ and $y = x$

Use a computer algebra system to find the exact area enclosed by the curves.

Consider the graph of the two curves.



From the above graph, observe that $y = x^5 - 6x^3 + 4x$ encloses a four-part region symmetric about the origin.

Since $y = x^5 - 6x^3 + 4x$ and $y = x$ are odd functions of *x*.

The curves intersect at values of *x*.

Find the point of intersection of the curves.

$$\begin{aligned} x^5 - 6x^3 + 4x &= x \\ x^5 - 6x^3 + 3x &= 0 \\ x\left(x^4 - 6x^2 + 3\right) &= 0 \\ x = 0 \quad \text{or} \quad x^4 - 6x^2 + 3 &= 0 \end{aligned}$$

Now, factor the $x^4 - 6x^2 + 3 = 0$

This the quadratic equation in terms of x^2.

Use the formula for $x^2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here, $a = 1, b = -6$ and $c = 3$

Substitute the above values in the above formula.

$$x^2 = \frac{6 \pm \sqrt{36 - 12}}{2}$$
$$x^2 = 3 \pm \sqrt{6}$$
$$x = \pm\sqrt{3 \pm \sqrt{6}}$$

Therefore, the roots are as follows:

$x = \sqrt{3+\sqrt{6}}, x = \sqrt{3-\sqrt{6}}$ and $x = -\sqrt{3+\sqrt{6}}, x = -\sqrt{3-\sqrt{6}}$

Observe from the graph that the enclosed region is symmetric about the origin.

So, the area of the shaded region is twice the area of the region on one side of x – axis.

Observe that $y = x^5 - 6x^3 + 4x$ is above $y = x$ between 0, $\sqrt{3-\sqrt{6}}$ and in the reverse order between $\sqrt{3-\sqrt{6}}$ and $\sqrt{3+\sqrt{6}}$.

Therefore, the exact area is as follows:

$$A = 2\left[\int_0^{\sqrt{3-\sqrt{6}}} \left(x^5 - 6x^3 + 3x\right)dx + \int_{\sqrt{3-\sqrt{6}}}^{\sqrt{3+\sqrt{6}}} \left(-x^5 + 6x^3 - 3x\right)dx\right]$$

$$= 2\left[\frac{x^6}{6} - \frac{3}{2}x^4 + \frac{3}{2}x^2\right]_0^{\sqrt{3-\sqrt{6}}} + 2\left[-\frac{x^6}{6} + \frac{3}{2}x^4 - \frac{3}{2}x^2\right]_{\sqrt{3-\sqrt{6}}}^{\sqrt{3+\sqrt{6}}}$$

Using computer algebra system, the exact area is $A = \boxed{12\sqrt{6} - 9}$

Answer 42E.

Consider the inequalities $x - 2y^2 \ge 0, 1 - x - |y| \ge 0$

Sketch the region in the xy-plane defined by the inequalities and find its area.

Now

$$x - 2y^2 \ge 0$$
$$x \ge 2y^2 \qquad \text{......(1)}$$

And

$$1 - x - |y| \ge 0$$
$$x \le 1 - |y| \qquad \text{......(2)}$$

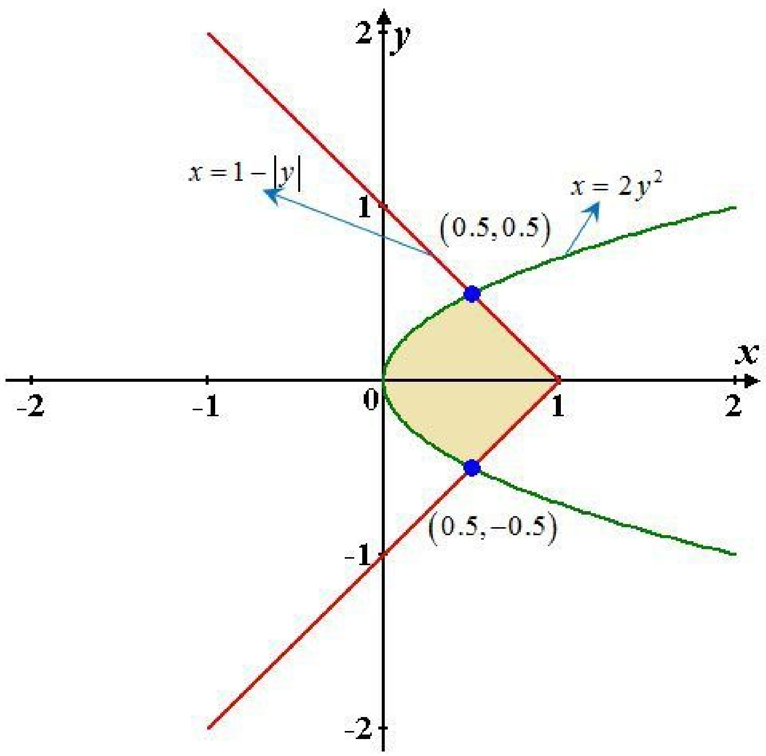
Combing these two inequalities to get

$$2y^2 \le x \le 1 - |y|$$

So sketch the region bounded by the curves $x = 2y^2$ and $x = 1 - |y|$

Since both the functions are even functions then the region will be symmetric about x-axis.

Graph of the two curves



First find the points of intersection of the two curves

$$2y^2 = 1 - |y|$$

Since in the first quadrant then take

$$2y^2 = 1 - y$$
$$2y^2 + y - 1 = 0$$
$$2y^2 + 2y - y - 1 = 0$$
$$(y+1)(2y-1) = 0$$

$y = 1/2$, since $y > 0$

Thus one point of intersection is $\left(\frac{1}{2}, \frac{1}{2}\right)$

Since in the fourth quadrant then take

$$2y^2 = 1 + y$$
$$2y^2 - y - 1 = 0$$
$$2y^2 - 2y + y - 1 = 0$$
$$(y-1)(2y+1) = 0$$

$y = -1/2$, since $y < 0$

And the other point of intersection is $\left(\frac{1}{2}, -\frac{1}{2}\right)$

Now the area is

$$A = 2\int_0^{1/2}\left[1 - |y| - 2y^2\right]dy$$
$$= 2\int_0^{1/2}\left[1 - y - 2y^2\right]dy \qquad [\text{since } y > 0]$$
$$= 2\left[y - \frac{1}{2}y^2 - \frac{2}{3}y^3\right]_0^{1/2}$$
$$= 2\left[\frac{1}{2} - \frac{1}{8} - \frac{2}{24}\right]$$
$$= 2\left[\frac{7}{24}\right]$$
$$= \boxed{\frac{7}{12}}$$

Answer 43E.

First we change the unit of velocity in feet/s

$$1 \text{ Miles per hour } = \frac{5280}{3600}\text{feet/sec}$$
$$= \frac{88}{60} = \frac{22}{15}\text{feet/sec}$$

Now we make new table for velocities.

$t(s)$	v_1 (feet/s)	v_2 (feet/s)
0	0	0
1	440/15	484/15
2	704/15	814/15
3	1012/15	1144/15
4	1188/15	1342/15
5	1364/15	1562/15
6	1518/15	1760/15
7	1650/15	1892/15
8	1782/15	2046/15
9	1892/15	2156/15
10	1980/15	2244/15

We have the time interval [0, 10]
If we divide this time interval in to 5 sub intervals, then n = 5 and width of the sub interval is $\Delta t = \frac{10-0}{5} = 2$
And sub intervals are [0, 2], [2, 4], [4, 6], [6, 8] and [8, 10]
Then mid points of the sub intervals are 1, 3, 5, 7, and 9

By the mid point rule, total distance traveled by first person after 10 seconds

$$\int_0^{10} (v_1) dt \approx \Delta t \{v_1(1)+v_1(3)+v_1(5)+v_1(7)+v_1(9)\}$$
$$\approx 2.\left\{\frac{440}{15}+\frac{1012}{15}+\frac{1364}{15}+\frac{1650}{15}+\frac{1892}{15}\right\}$$
$$\approx \frac{12716}{15}$$

Similarly total distance traveled by second person after 10 seconds

$$\int_0^{10} (v_2) dt \approx \Delta t \{v_2(1)+v_2(3)+v_2(5)+v_2(7)+v_2(9)\}$$
$$\approx 2.\left\{\frac{484}{15}+\frac{1144}{15}+\frac{1562}{15}+\frac{1892}{15}+\frac{2156}{15}\right\}$$
$$\approx \frac{14476}{15}$$

The difference between the distances traveled by both persons is

$$\int_0^{10} (v_2) dt - \int_0^{10} v_1 dt \approx \frac{14476}{15} - \frac{12716}{15}$$
$$\approx \frac{1760}{15} \text{ feet}$$
$$\approx \frac{352}{3} \text{ feet}$$

Or $\boxed{\approx 117\frac{1}{3}\text{feet}}$ so second person travels $117\frac{1}{3}$ feet more than Chris

Answer 44E.

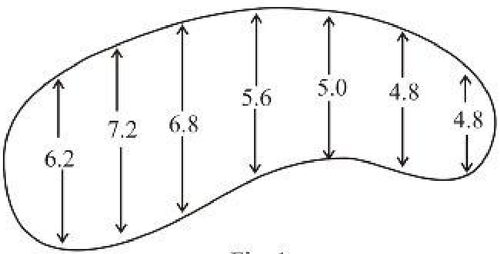


Fig. 1

First we make a table of the widths (in meters) of the swimming pool at 2 meters intervals as

(meter)(intervals)	Widths of a swimming pool (meters)
0	0
2	6.2
4	7.2
6	6.8
8	5.6
10	5.0
12	4.8
14	4.8
16	0

Now we have the interval [0, 16]
If we divide this interval into 4 sub intervals, so n = 4 and width of the sub interval

$$= \frac{16}{4} = 4 = \Delta L \text{ (Let)}$$

So sub intervals are [0, 4], [4, 8], [8, 12] and [12, 16]
And mid points are 2, 6, 10 and 14.

Then by mid point rule, we can estimate the area of pool as

$$A = \int_0^{16} w_I d \approx \Delta L\{w_2 + w_6 + w_{10} + w_{14}\}$$
$$\approx 4\{6.2 + 6.8 + 5.0 + 4.8\}$$

Or $\boxed{A \approx 91.2\,\text{m}^2}$

Answer 45E.

Measurements of the thickness of the wing, in centimeters, at 20-centimeter intervals are

5.8, 20.3, 26.7, 29.0, 27.6, 27.3, 23.8, 20.5, 15.1, 8.7, and 2.8.

Use the midpoint rule to estimate the area of the wing's cross-section.

Midpoint rule: $\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x = \Delta x\left[f(\bar{x}_1) + \ldots + f(\bar{x}_n)\right]$

Where $\Delta x = \dfrac{b-a}{n}$ and

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$$
$$= \text{midpointof}\,[x_{i-1}, x_i]$$

Since the problem does not state the number of subintervals or the function describing the shape of the wing, to select midpoints based on the data given in the problem. This means use every other value as a midpoint, namely 20.3, 29, 27.3, 20.5, 8.7.

From this easily deduce the subintervals as $n = 5$.

First find the Δx

$$\Delta x = \frac{b-a}{n}$$
$$= \frac{200-0}{5}$$
$$= 40$$

Use the midpoint formula to find the area.

$$A \approx \Delta x\left(f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + \ldots + f(\bar{x}_n)\right)$$
$$A \approx 40\left(f(20) + f(60) + f(100) + f(140) + f(180)\right)$$
$$= 40(20.3 + 29 + 27.3 + 20.5 + 8.7)$$
$$= 40(105.8)$$
$$= \boxed{4232\ \text{cm}^2}$$

Answer 46E.

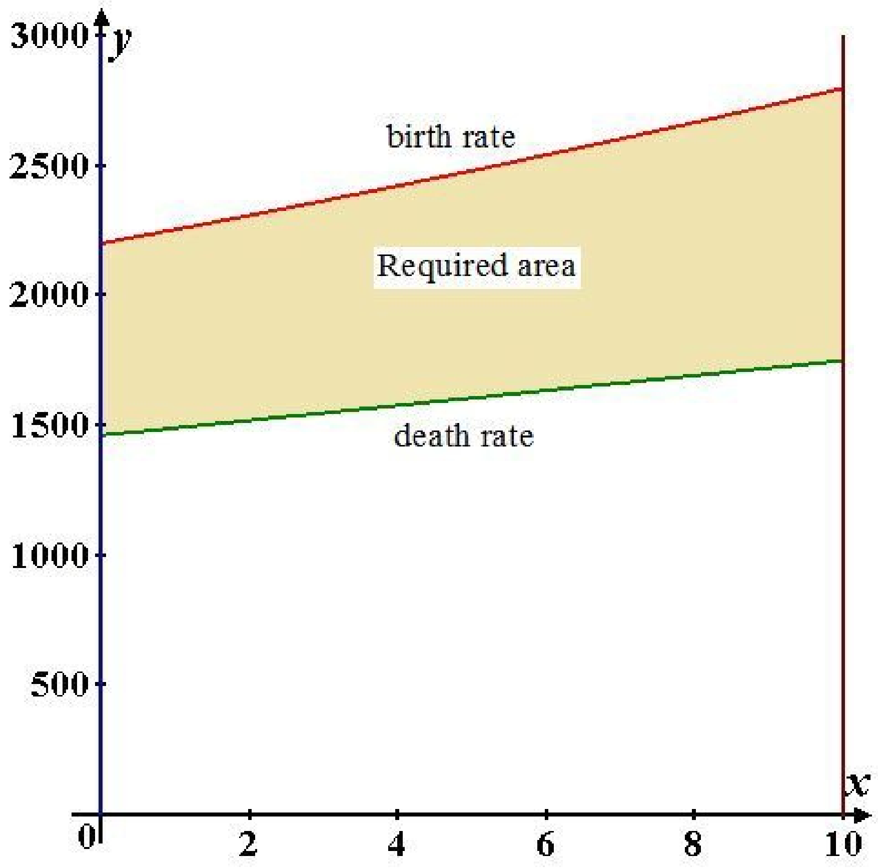
Consider the data

The birth rate of the population is $b(t) = 2200 + 52.3t + 0.74t^2$

The death rate of the population is $d(t) = 1460 + 28.8t$ people for year.

Required to find the area between these curves for $0 \le t \le 10$.

Graph of the area required



The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where *f* and *g* are continuous and $f(x) \geq g(x)$ for all *x* in $[a,b]$, is A

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Here $f(x)$ is birth rate and $g(x)$ is death rate

And $a = 0$, $b = 10$

Thus area

$$A = \int_0^{10} [0.74t^2 + 52.3t + 2200 - 28.8t - 1460]\, dt$$

$$= \int_0^{10} [0.74t^2 + 23.5t + 740]\, dt$$

$$= \left[0.74\frac{t^3}{3} + 23.5\frac{t^2}{2} + 740t\right]_0^{10}$$

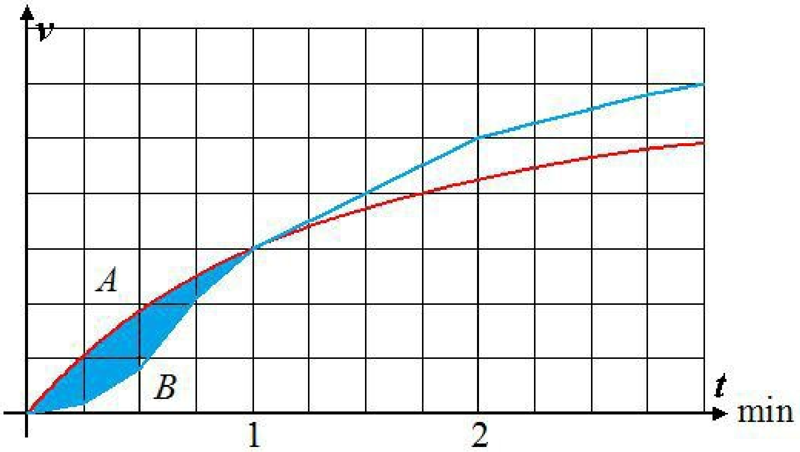
$$= \left[0.74\frac{(10)^3}{3} + 23.5\frac{(10)^2}{2} + 740(10) - 0\right]$$

$$\approx 8821.67$$

Therefore, the area represent the $\boxed{\approx 8821.67}$ persons alive.

Answer 47E.

Consider the graph:



The area under A between $t=0$ and $t=x$ is $\int_0^x V_A(t)\,dt = S_A(x)$

Here, $V_A(t)$ is the velocity of car A and S_A is its displacement.

Similarly, the area under curve B between $t=0$ and $t=x$ is $\int_0^x V_B(t)\,dt = S_B(x)$.

(a)

Car A is ahead of car B after one minute, because the distance traveled in a velocity time graph is determined by finding the area under the curve within the time limits. From the graph, see that area under the curve A is greater than area under B between $t=0$ to $t=1$.

(b)

The area of the shaded region has numerical value, $S_A(1)-S_B(1)$

The shaded region represents the distance by which car C is ahead of car B after 1 minute.

(c)

After two minutes, car B is travelling faster than car A and has gained some ground, but the area under curve A from $t=0$ to $t=2$ is still greater than the corresponding area for curve B, so car A is still ahead.

In other words, still car A is ahead of car B after two minutes, because area under A is slightly greater than area under B, within the limits $t=0$ to $t=2$

(d)

From the graph, it appears that the area between curves A and B for $0 \le t \le 1$ (when car A is going faster), which corresponds to the distance by which car A is ahead, seems to be about 3 squares.

Therefore, the cars will be side by side at the time *x*, where the area between the curves for $1 \le t \le x$ (when car B going faster) is the same as the area for $0 \le t \le 1$.

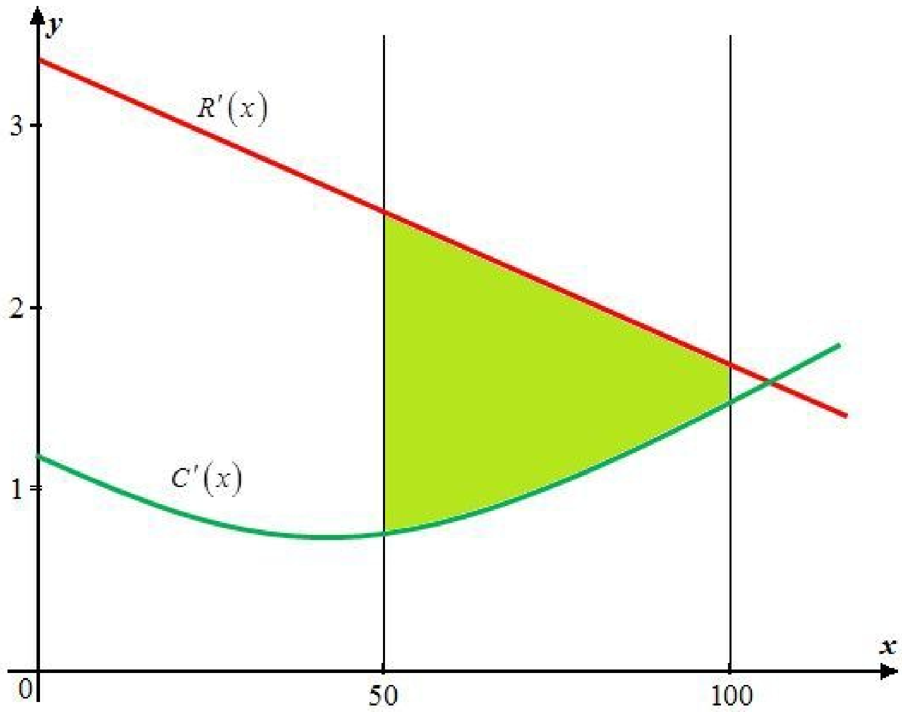
From the graph, it appears that the time is $x \approx 2.2$

So, the cars are side by side when $\boxed{t \approx 2.2 \text{ minutes}}$

In other words, the cars will be side by side, when the areas under both curves from $t=0$ to $t=x$ will be the same. From the figure, we see that between $t=0$ and $t=2.2$ minutes (approx), the areas under both the curves will be equal, so both the cars will be side by side when $\boxed{t=2.2}$ minutes (approx) .

Answer 48E.

Consider the graph of marginal revenue function R' and the marginal cost function C' for a manufacturer.



Recall the marginal profit function is,

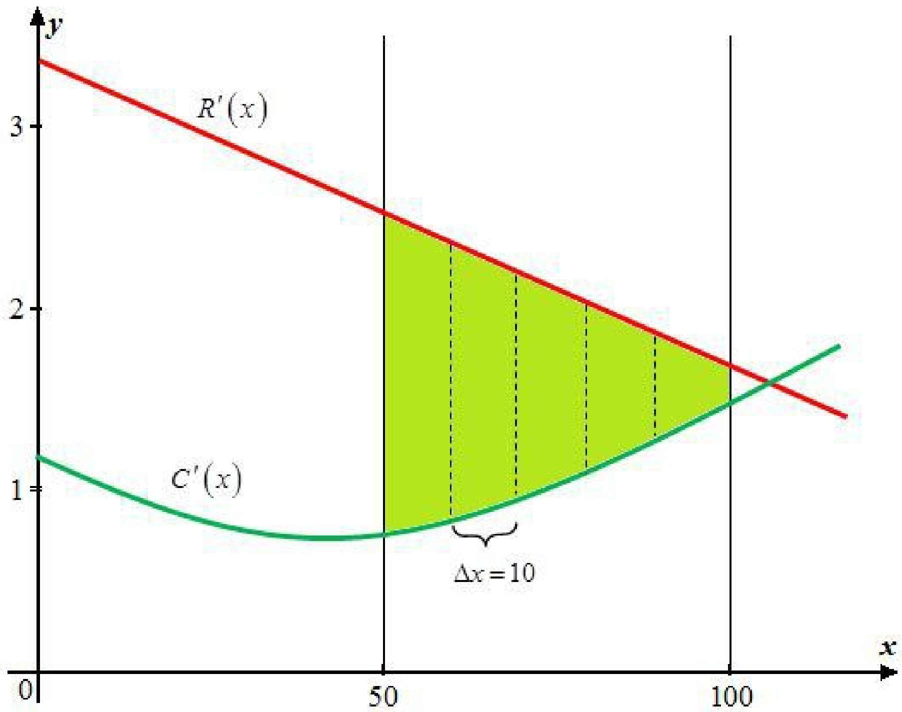
$P'(x) = R'(x) - C'(x)$

So, here the shaded region represents the difference between $R'(x)$ and $C'(x)$.

In the boundaries $x = 50$ to $x = 100$.

Use the mid-point rule to estimate the value of this quantity:

Divide the region into five strips of width $\Delta x = 10$.



The area of shaded region is:

$$= \Delta x \begin{pmatrix} [R'(55) - C'(55)] + [R'(65) - C'(65)] + \\ [R'(75) - C'(75)] + [R'(85) - C'(85)] + [R'(95) - C'(95)] \end{pmatrix}$$

$$\approx 10 \begin{pmatrix} [2.4 - 0.85] + [2.2 - 0.9] + \\ [2 - 1] + [1.8 - 1.1] + [1.7 - 1.2] \end{pmatrix}$$ Observing the graph.

$$\approx 10(1.55 + 1.55 + 1 + 0.7 + 0.5)$$
$$= 10(5.3)$$
$$= 53$$

This is the marginal profit value.

Therefore, the marginal profit is 53 dollors.

Answer 49E.

Since we have $y^2 = x^2(x+3)$

Then $y = x\sqrt{x+3}$ *and* $y = -x\sqrt{x+3}$

Now we sketch both the curves

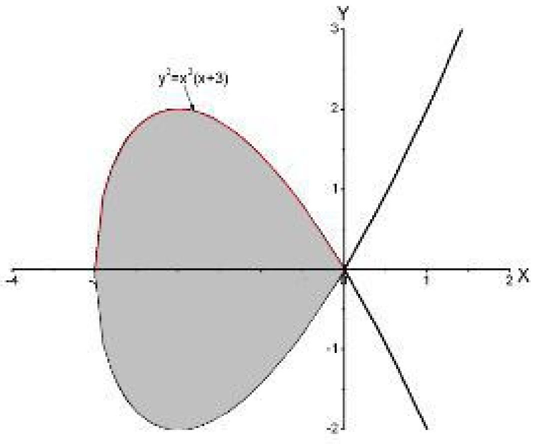


Fig.1

We see that these curves make a loop in the closed interval [-3, 0]

Combination of these two curves is the graph of $y^2 = x^2(x+3)$

Since both the graphs are the mirror mage of each other about x-axis

So total area of the loop is $= 2 \times$ area of region under the

Curve $y = -x\sqrt{x+3}$ on [-3, 0]

Or $A = 2 \times \int_{-3}^{0} -x\sqrt{x+3}dx$

Or $A = -2\int_{-3}^{0} x\sqrt{x+3}dx$

Let $x + 3 = t^2$ *so* $x = (t^2 - 3)$

Then $dx = 2t\,dt$

And when x = -3, t = 0

And x = 0, $t = \sqrt{3}$

Then $-2\int_{-3}^{0} x\sqrt{x+3}dx = -2\int_{0}^{\sqrt{3}} \left(t^2-3\right).t.2t\,dt$

$$= -4\int_{0}^{\sqrt{3}} \left(t^4-3t^2\right)dt$$

$$= -4\left[\frac{t^5}{5}-t^3\right]_0^{\sqrt{3}} \qquad \text{[By FTC - 2]}$$

$$= -4\left[\frac{\left(\sqrt{3}\right)^5}{5}-\left(\sqrt{3}\right)^3\right]$$

$$= -4\left[\frac{9\sqrt{3}}{5}-3\sqrt{3}\right]$$

$$= -4\left[\frac{9\sqrt{3}-15\sqrt{3}}{5}\right]$$

$$= \frac{24\sqrt{3}}{5}$$

So the area of loop formed by the curve $y^2 = x^2\left(x+3\right)$

$$\boxed{is = \frac{24\sqrt{3}}{5}}$$

Answer 50E.

Equation of the parabola is $y = x^2$

Slope of the tangent at any point of the parabola $= \frac{dy}{dx} = 2x$

So slope of the tangent at (1, 1) is $\frac{dy}{dx} = 2$

So equation of tangent line is (which is passing though (1, 1))

$$\left(y-1\right) = 2.\left(x-1\right)$$

Or $\quad y-1 = 2x-2$

Or $\quad \boxed{y = 2x-1}$

Now we sketch the curves $y = x^2$ and $y = 2x-1$

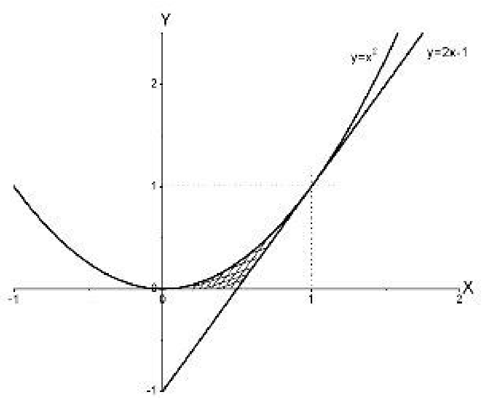


Fig.1

We have to find the area of shaded region

Since line $y = 2x-1$ and curve $y = x^2$ intersect at $x = 1$ and curve $y = x^2$ touches the x – axis at $x = 0$

So we will find the area under the limits $x = 0$ and $x = 1$ or in closed interval [0, 1]

x – intercept of $y = 2x-1$ is $x = \frac{1}{2}$

From figure it is clear that area bounded by the curve $y = x^2$, tangent at (1, 1), $y = 2x-1$ and x – axis is

A = Area of the region under curves $y = x^2$ from $x = 0$ to 1

- Area of the region under the line $y = 2x-1$ from x =1/2 to $x = 1$

So $A=\int_0^1 x^2dx-\int_{1/2}^1(2x-1)\,dx$

$=\left[\frac{x^3}{3}\right]_0^1-\left[x^2-x\right]_{1/2}^1$ [By FTC - 2]

$=\frac{1}{3}-\left[1-1-\frac{1}{4}+\frac{1}{2}\right]$

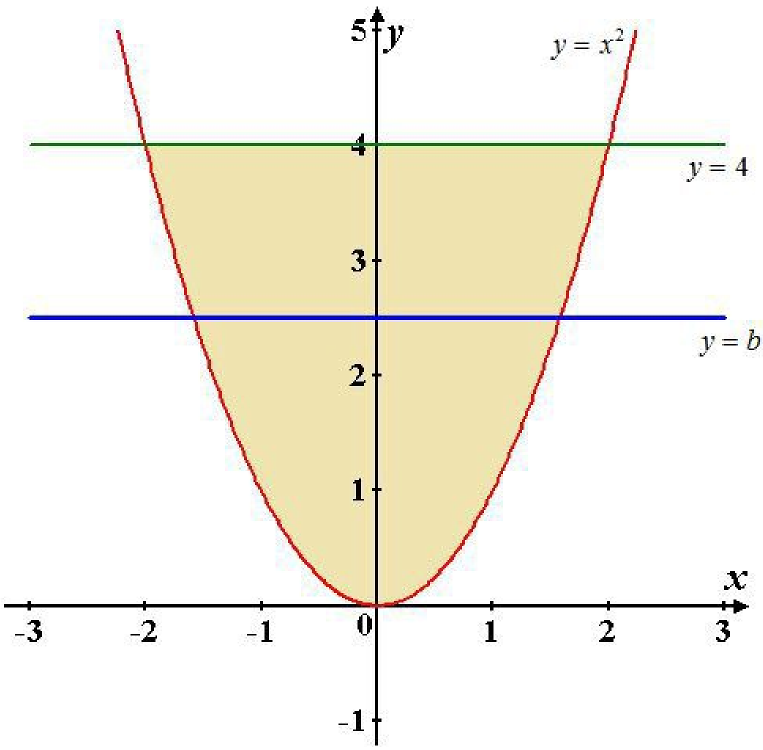
$=\frac{1}{3}-\left[\frac{1}{4}\right]$

Or $\boxed{A=\frac{1}{12}}$ this is the area of required region

Answer 51E.

Consider the curves $y=x^2$ and $y=4$

Find the number b such that the line $y=b$ divides the region bounded by the above curves into two regions with equal area.



The points of intersection of the curves

$x^2=4$

$x=\pm 2$

Intersection points are $(-2,4)$ and $(2,4)$

Here upper curve is $y=4$ and lower curve is $y=x^2$

Now the area of the region bounded by these curves is

$A=\int_{-2}^{2}\left(4-x^2\right)dx$

$=\left[4x-\frac{1}{3}x^3\right]_{-2}^{2}$

$=\left[4(2)-\frac{1}{3}(2)^3\right]-\left[4(-2)-\frac{1}{3}(-2)^3\right]$

$=\left[8-\frac{8}{3}\right]-\left[-8+\frac{8}{3}\right]$

$=\left[16-\frac{16}{3}\right]$

$=\frac{32}{3}$

The points of intersection of the curves $y=x^2$ and $y=b$ are $\left(\pm\sqrt{b},b\right)$

Now the area of the region bounded by these curves is

$$A=\int_{-\sqrt{b}}^{\sqrt{b}}\left(b-x^2\right)dx$$
$$=\left[bx-\frac{1}{3}x^3\right]_{-\sqrt{b}}^{\sqrt{b}}$$
$$=\left[b\left(\sqrt{b}\right)-\frac{1}{3}\left(\sqrt{b}\right)^3\right]-\left[b\left(-\sqrt{b}\right)-\frac{1}{3}\left(-\sqrt{b}\right)^3\right]$$
$$=\left[b\sqrt{b}-\frac{1}{3}b\sqrt{b}\right]-\left[-b\sqrt{b}+\frac{1}{3}b\sqrt{b}\right]$$
$$=\frac{4}{3}b\sqrt{b}$$

Now since the area of the region bounded by the curves $y=x^2$ and $y=b$ divides the area of the region bounded by the curves $y=x^2$ and $y=4$ into two regions with equal area.

Then

$$\frac{4}{3}b\sqrt{b}=\frac{1}{2}\left(\frac{32}{3}\right)$$
$$b\sqrt{b}=4$$
$$b^{3/2}=4$$
$$b=4^{2/3}$$

Therefore, $\boxed{b=4^{2/3}}$

Answer 52E.

(A) First we sketch the curve $y=\frac{1}{x^2}$

Let x = a is any point of x – axis, such that the line x = a bisects the area under the curve $y=\frac{1}{x^2}$, $1\le x\le 4$

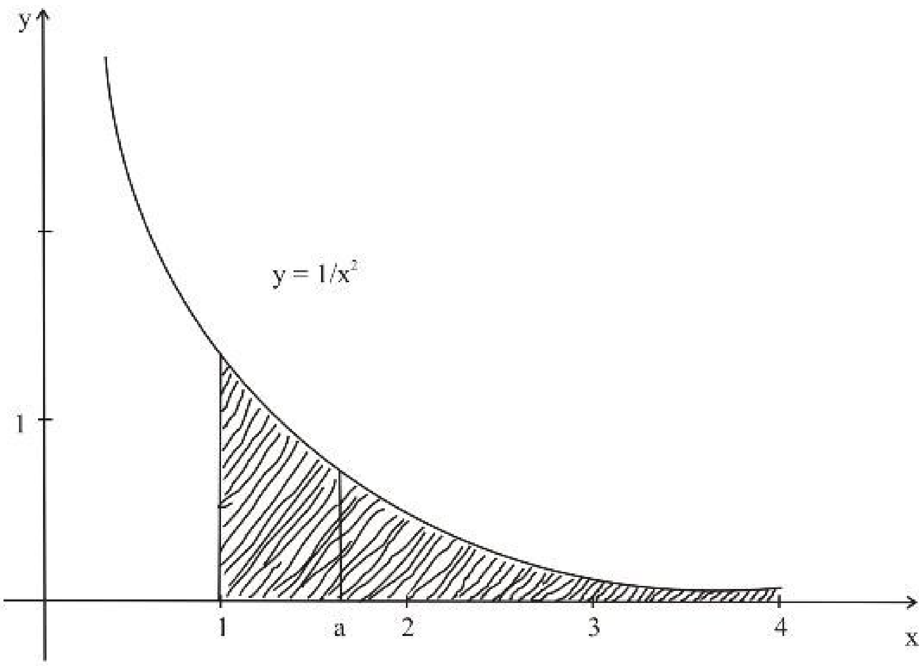


Fig. 1

Area of the region bounded by the curves $y=\frac{1}{x^2}$ and x = 1 & x = 4 is

$$A=\int_1^4 \frac{1}{x^2}dx$$

Or $\quad A=\left[-\frac{1}{x}\right]_1^4$ [By FTC - 2]

Or $\quad A=\left[-\frac{1}{4}+1\right]=\frac{3}{4}$

Since according to our problem, x = a bisects the area A

So $\quad \int_1^a \frac{1}{x^2}dx=\int_a^4 \frac{1}{x^2}dx=\frac{A}{2}$

Or $\quad \left[-\frac{1}{x}\right]_1^a=\left[-\frac{1}{x}\right]_a^4=\frac{3}{8}$ [By FTC - 2]

Or $\quad \left[-\frac{1}{a}+1\right]=\left[-\frac{1}{4}+\frac{1}{a}\right]=\frac{3}{8}$

Or $\quad 4(a-1)=4-a$

Or $\quad 5a=8$

Or $\quad \boxed{a=\frac{8}{5}=1.6}$

(B) The area enclosed between x = 1 and x = 4 and the curve has been obtained in part (a) and is equal to $\frac{3}{4}$.

We have now to find the value of b such that the line y = b bisects this area i.e. each of the two areas PRS and RSABQ in the figure below is equal to $\frac{3}{8}$

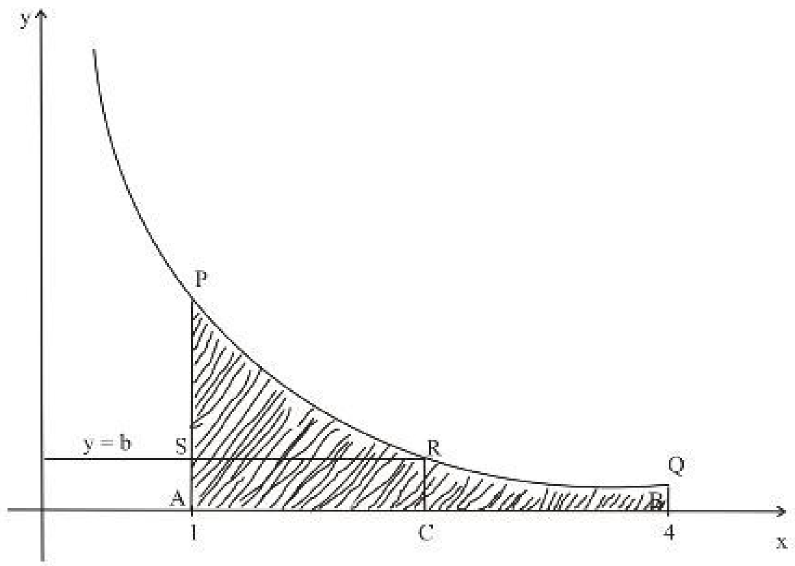


Fig. 2

We have $y=\frac{1}{x^2}$ or $x^2=\frac{1}{y}=\frac{1}{b}$

So for y = b, we have $x^2=\frac{1}{b}$ or $x=\frac{1}{\sqrt{b}}$ [x is +ve]

Area PSR = Area PACR – Area ASRC

Or $\quad \frac{3}{8}=\int_1^{1/\sqrt{b}} y\,dx - AC.AS \qquad$ [Area of rectangle]

$$=\int_1^{1/\sqrt{b}} \frac{1}{x^2}dx-\left(\frac{1}{\sqrt{b}}-1\right).b$$

$$=\left[-\frac{1}{x}\right]_1^{\frac{1}{\sqrt{b}}}-\left(\sqrt{b}-b\right)$$

$$=1-\sqrt{b}-\sqrt{b}+b$$

$$\frac{3}{8}=1-2\sqrt{b}+b$$

Or $\quad \left(1-\sqrt{b}\right)^2=\frac{3}{8}=0.375$

Or $\quad 1-\sqrt{b}=0.612$

Or $\quad \sqrt{b}=1-0.612=0.388$

Or $\quad \boxed{b=0.15}$

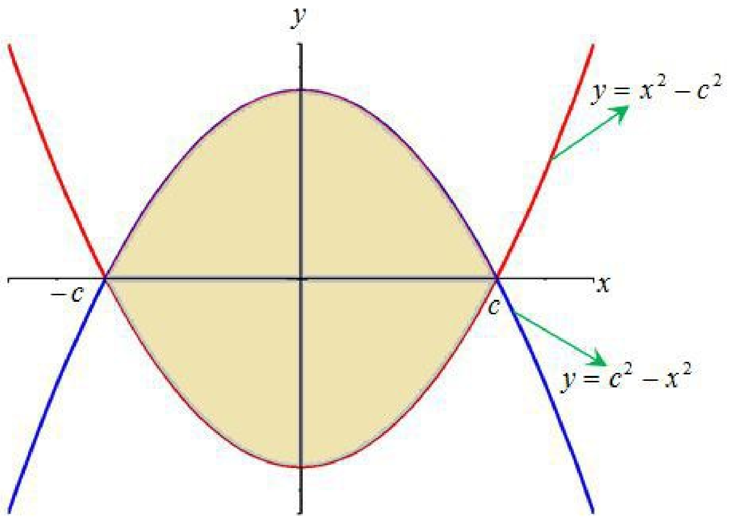
Answer 53E.

Consider the parabolas

$y=x^2-c^2$ and $y=c^2-x^2$

Required to find the value of c such that the area of the region enclosed by the parabolas is 576.

Graph of the parabolas



For finding the points of intersection to equating the parabolas

$$x^2-c^2=c^2-x^2$$
$$2x^2-2c^2=0$$
$$x^2=c^2$$
$$x=\pm c$$

Here upper curve is $y=c^2-x^2$ and lower curve is $y=x^2-c^2$

Then the area is

$$A=\int_{-c}^{c}\left[\left(c^2-x^2\right)-\left(x^2-c^2\right)\right]dx$$

$$=\int_{-c}^{c}\left(2c^2-2x^2\right)dx$$

$$=\left[2c^2x-\frac{2x^3}{3}\right]_{-c}^{c}$$

$$=2c^3+2c^3-\frac{2}{3}c^3-\frac{2}{3}c^3$$

$$=\frac{8}{3}c^3$$

Since the area is 576.

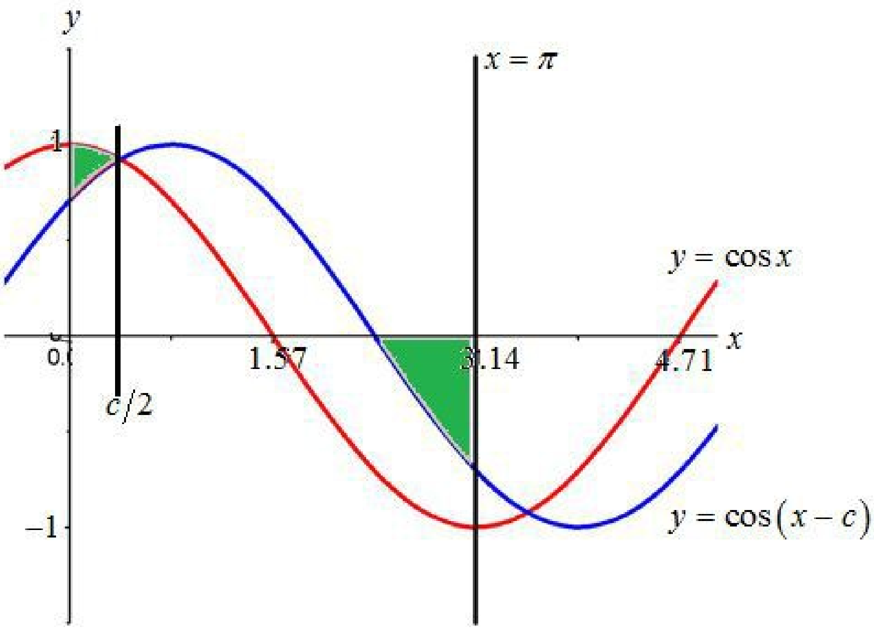
Then

$$\frac{8c^3}{3} = 576$$
$$c^3 = 216$$
$$c = \left(6^3\right)^{\frac{1}{3}}$$
$$c = 6$$

This value of ' c ' can be $\boxed{\pm 6}$, since the region is symmetric about y-axis.

Answer 54E.

Consider the graph



For the points of intersection of the curves $y = \cos x$ and $y = \cos(x - c)$

To equating the two curves

$$\cos x = \cos(x - c)$$
$$x = 2n\pi \pm (x - c)$$

Put $n = 0$, since from figure see that the point of intersection is before $\pi/2$

So $x = \pm(x - c)$

Or $x = -(x - c)$ [since by positive sign get $c = 0$ and $0 < c < \pi/2$]

Thus $x = c/2$

Now the area of the region bounded by the curves $y = \cos x$, $y = \cos(x - c)$ and $x = 0$ is

$$A_1 = \int_0^{c/2} \left[\cos x - \cos(x - c)\right] dx$$
$$= \left[\sin x - \sin(x - c)\right]_0^{c/2}$$
$$= \left[\sin(c/2) - \sin(c/2 - c) - 0 + \sin(-c)\right]$$
$$= \left[\sin(c/2) + \sin(c/2) - \sin(c)\right]$$
$$= \left[2\sin(c/2) - \sin(c)\right]$$

Now for getting the x-intercept of the curve $y=\cos(x-c)$

To equating y is equal to zero

$$\cos(x-c)=0$$
$$(x-c)=2n\pi \pm \pi/2$$

But the first x intercept of the curve $y=\cos(x-c)$ lies between $\pi/2$ and π

Thus $(x-c)=\pi/2$ [taking $n=0$ and positive sign]

Or $x=\pi/2+c$

Now the area of the region bounded by the curves $y=\cos(x-c)$, $y=0$ and $x=\pi$ is

$$A_2=\int_{\pi/2+c}^{\pi}\left[-\cos(x-c)\right]dx$$
$$=\left[-\sin(x-c)\right]_{\pi/2+c}^{\pi}$$
$$=\left[-\sin(\pi-c)+\sin(\pi/2+c-c)\right]$$
$$=\left[1-\sin(\pi-c)\right]$$
$$=\left[1-\sin(c)\right] \qquad \left[\text{Recollect}\sin(\pi-\theta)=\sin\theta\right]$$

Since $A_1=A_2$

Then

$$\left[2\sin(c/2)-\sin(c)\right]=1-\sin(c)$$
$$2\sin(c/2)=1$$
$$\sin(c/2)=1/2$$
$$(c/2)=\pi/6$$
$$c=\pi/3$$

Therefore, the value of $\boxed{c=\frac{\pi}{3}}$.

Answer 55E.

First we sketch the curves $y=\dfrac{1}{x}, y=\dfrac{1}{x^2}, x=2$

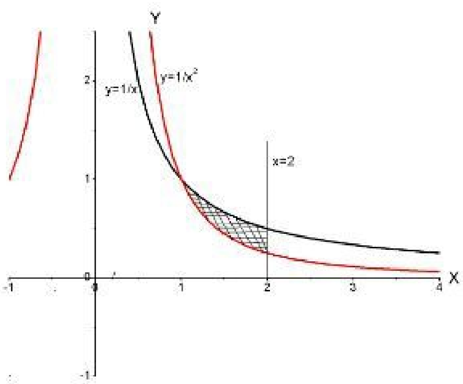


Fig.1

We see that $\dfrac{1}{x^2}<\dfrac{1}{x}$ in the interval $[1, 2]$

So total area enclosed by $y=\dfrac{1}{x}, y=\dfrac{1}{x^2}$ and $x=2$ is

$$A=\int_1^2\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$$

Or $\quad A=\int_1^2\frac{1}{x}dx-\int_1^2\frac{1}{x^2}dx$

Since anti derivatives of $\frac{1}{x}$ and $\frac{1}{x^2}$ are $\ln x$ and $-\frac{1}{x}$ respectively then by fundamental theorem part 2

$$A = [\ln x]_1^2 - \left[-\frac{1}{x}\right]_1^2$$

Or $$A = [\ln 2 - \ln 1] - \left[-\frac{1}{2} + 1\right]$$

Or $$A \approx [\ln 2 - 0] - \left[\frac{1}{2}\right]$$

Or $$\boxed{A = \ln 2 - \frac{1}{2}}$$

Or $$\boxed{A \approx 0.193147}$$

Answer 56E.

First we sketch the curves $y = \sin x, y = e^x, x = 0$ *and* $x = \frac{\pi}{2}$

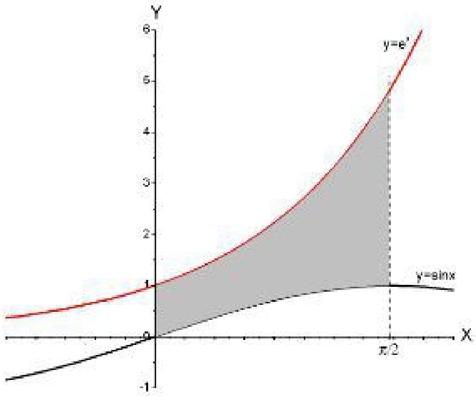


Fig.1

We have to find the area of shaded region enclosed by the curves

$$y = \sin x, y = e^x, x = 0 \text{ and } x = \frac{\pi}{2}$$

We see that from figure $\sin x < e^x$ in the interval $\left[0, \frac{\pi}{2}\right]$

So the area enclosed by these curves

$$A = \int_0^{\frac{\pi}{2}} \left(e^x - \sin x\right) dx$$

$$= \int_0^{\frac{\pi}{2}} e^x dx - \int_0^{\frac{\pi}{2}} \sin x dx$$

Since anti derivatives of e^x and $\sin x$ are e^x and $(-\cos x)$ respectively

So by fundamental theorem part -2

$$A = \left[e^x\right]_0^{\pi/2} - \left[-\cos x\right]_0^{\pi/2}$$

Or $$A = \left[e^{\pi/2} - e^0\right] - \left[-\cos\frac{\pi}{2} + \cos 0\right]$$

Since $e^0 = 1$, $\cos\frac{\pi}{2} = 0$ and $\cos 0 = 1$

So $$A = \left[e^{\pi/2} - 1\right] - [-0 + 1]$$

Or $$A = e^{\pi/2} - 1 - 1$$

Or $$\boxed{A = e^{\pi/2} - 2}$$

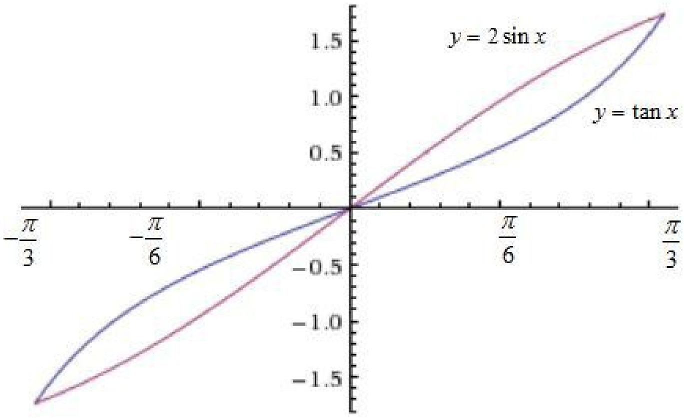
This is the area enclosed by $y = e^x$ and $y = \sin x$ in the interval $\left[0, \frac{\pi}{2}\right]$

Answer 57E.

Consider the curves.

$y = \tan x, \quad y = 2\sin x, \qquad -\pi/3 \le x \le \pi/3$

Sketch the region bounded by the curves and find the area of the region.



The graph clearly shows that from $-\frac{\pi}{3}$ to 0, the curve $y = \tan x$, is upper curve and $y = 2\sin x$ is lower curve and $\tan x \ge 2\sin x$.

From 0 to $\frac{\pi}{3}$, the curve $y = 2\sin x$ is upper curve and $y = \tan x$ is lower curve and $2\sin x \ge \tan x$.

So,the area of region is calculated as follows:

$$A = \int_{-\frac{\pi}{3}}^{0} (\tan x - 2\sin x)\,dx + \int_{0}^{\frac{\pi}{3}} (2\sin x - \tan x)\,dx$$

$$= \left[\ln|\sec x| + 2\cos x\right]_{-\frac{\pi}{3}}^{0} + \left[-2\cos x - \ln|\sec x|\right]_{0}^{\frac{\pi}{3}}$$

$$= \left[\ln|\sec(0)| + 2\cos(0) - \ln\left|\sec\left(-\frac{\pi}{3}\right)\right| - 2\cos\left(-\frac{\pi}{3}\right)\right] +$$

$$\left[-2\cos\left(\frac{\pi}{3}\right) - \ln\left|\sec\left(\frac{\pi}{3}\right)\right| + 2\cos(0) + \ln|\sec(0)|\right]$$

$$= \left[\ln 1 + 2 - \ln 2 - 2\cdot\frac{1}{2}\right] + \left[-2\cdot\frac{1}{2} - \ln 2 + 2 + \ln 1\right]$$

$$= 2 - \ln 2 - 1 - 1 - \ln 2 + 2$$

$$= 2 - 2\ln 2$$

Therefore, the area is $\boxed{2 - 2\ln 2}$.

Answer 58E.

Line $y = mx$ and the curve $y = \frac{x}{(x^2+1)}$ will enclose a region when these curves intersect each other at least 2 points.

So we will find the points of intersection of these curves

These curves will intersect each other when

$$mx = \frac{x}{(x^2+1)}$$

Or $\quad mx^3 + mx - x = 0$

Or $\quad x(mx^2 + m - 1) = 0$

Or $\quad x = 0 \ or \ x^2 = \frac{1-m}{m}$

Or $\quad x = 0 \ or \ x = \pm\sqrt{\frac{1-m}{m}}$ $\qquad$ which is defined when $0 < m < 1$

So $y = mx$ and $y = \frac{x}{(x^2+1)}$ will enclose a region when $\boxed{0 < m < 1}$, as for m = 1, there is first one point of intersection i.e. (0, 0).

Since $y = mx$ and $y = \frac{x}{(x^2+1)}$ are odd function

So area enclosed by these curves will be symmetric about the origin. We can see in following figure also

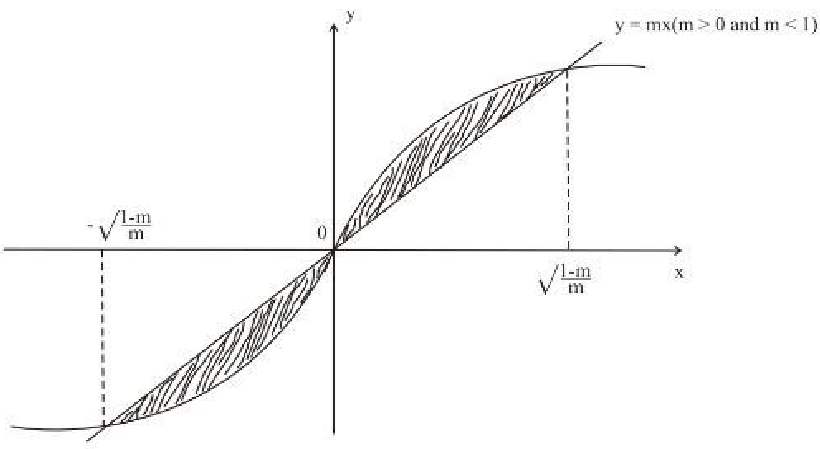


Fig. 1

So the area enclosed by $y = mx$ and $y = \frac{x}{(x^2+1)}$ for 0 < m < 1 is

$$A = 2\int_0^{\sqrt{\frac{1-m}{m}}} \left(\frac{x}{(x^2+1)} - mx\right)dx$$

$$= 2\int_0^{\sqrt{\frac{1-m}{m}}} \frac{x}{(x^2+1)}dx - 2\int_0^{\sqrt{\frac{1-m}{m}}} mxdx$$

First we evaluate $\int_0^{\sqrt{\frac{1-m}{m}}} \frac{x}{(x^2+1)}dx$

Let $x^2+1 = t$ *then* $2xdx = dt$ *or* $xdx = \frac{1}{2}dx$

And when x = 0, t = 1

And when $x = \sqrt{\frac{1-m}{m}}$, $t = \frac{1-m}{m} + 1$

Or $t = \frac{1-m+m}{m}$ *or* $t = \frac{1}{m}$

So $\int_0^{\sqrt{\frac{1-m}{m}}} \frac{x}{(x^2+1)}dx = \frac{1}{2}\int_1^{1/m} \frac{dt}{t}$

Since an anti derivative of $\frac{1}{t}$ is $\ln t$ then by fundamental theorem part 2

$$\frac{1}{2}\int_1^{1/m} \frac{dt}{t} = \frac{1}{2}\left[\ln t\right]_1^{1/m}$$

Or $\frac{1}{2}\int_1^{1/m} \frac{dt}{t} = \frac{1}{2}\left[\ln\frac{1}{m} - \ln 1\right]$

$$= \frac{1}{2}\ln\frac{1}{m} \qquad [\ln 1 = 0]$$

So $\int_0^{\sqrt{\frac{1-m}{m}}} \frac{x}{(x^2+1)}dx = \frac{1}{2}\ln\frac{1}{m}$

Now we evaluate $\int_0^{\sqrt{\frac{1-m}{m}}} mxdx$

$$\int_0^{\sqrt{\frac{1-m}{m}}} mx\,dx = m\int_0^{\sqrt{\frac{1-m}{m}}} x\,dx$$

$$= m\left[\frac{x^2}{2}\right]_0^{\sqrt{\frac{1-m}{m}}} \qquad \text{[By FTC - 2]}$$

$$= \frac{m}{2}.\frac{(1-m)}{m}$$

$$= \frac{1-m}{2}$$

Then area

$$A = 2.\frac{1}{2}\ln\frac{1}{m} - 2.\left(\frac{1-m}{2}\right)$$

Or $A = \ln\frac{1}{m} - (1-m)$

Or $A = m + \ln\frac{1}{m} - 1$

Or $\boxed{A = m - \ln m - 1}$ $\qquad \left[\text{since } \ln m^{-1} = -\ln m\right]$

Where $0 < m < 1$

This is the area enclosed by $y = mx$ *and* $y = \frac{x}{x^2+1}$ when $\boxed{0 < m < 1}$ in the first quadrant and third quadrant