5 Probability and Statistics

1 Probability Fundamentals

.1 Definitions

Sample Space and Event: Consider an experiment whose outcome is not predictable with certainty. Such an periment is called a **random experiment**. However, although the outcome of the experiment will not be known in vance, let us suppose that the set of all possible outcomes is known. This set of all possible outcomes of an periment is known as the **sample space** of experiment and is denoted by S. Some examples follow.

1. If the outcome of an experiment consist in the determination of the sex of a newborn child, then $S = \{g, b\}$ where the outcome g means that the child is a girl and b is the boy.
2. If the outcome of an experiment consist of what comes up on a single dice, then $S = \{1, 2, 3, 4, 5, 6\}$.
3. If the outcome of an experiment is the order of finish in a race among the 7 horses having post positions 1, 2, 3, 4, 5, 6, 7; then S = {all 7! permutations of the (1, 2, 3, 4, 5, 6, 7)}.

The outcome (2, 3, 1, 6, 5, 4, 7) means, for instances, that the number 2 horse comes in first, then the mber 3 horse, then the number 1 horse, and so on.

Any subset E of the sample space is known as **Event**. That is, an event is a set consisting of me or all of the possible outcomes of the experiment. For example, in the throw of a single dice = {1, 2, 3, 4, 5, 6} and some possible events are

$$E_1 = \{1, 2, 3\}$$
$$E_2 = \{3, 4\}$$
$$E_3 = \{1, 4, 6\} \text{ etc.}$$

If the outcome of the experiment is contained in E, then we say that E has occurred. Always $E \subseteq S$.

Since E & S are sets, theorems of set theory may be effectively used to represent and solve probability oblems which are more complicated.

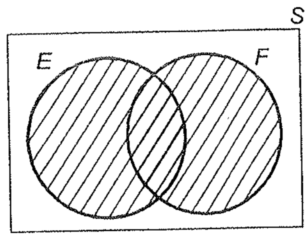
Example: If by throwing a dice, the outcome is 3, then events E_1 and E_2 are said to hare occurred.

In the child example – (i) If $E_1 = \{g\}$, then E_1 is the event that the child is a girl.

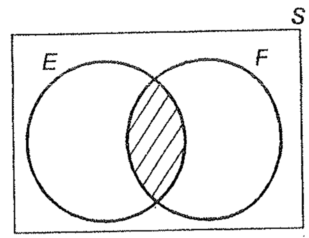
Similarly, if $E_2 = \{b\}$, then E_2 is the event that the child is a boy. These are examples of **Simple** events. **mpound** events may consist of more than one outcome. Such as $E = \{1, 3, 5\}$ for an experiment of throwing a e. We say event E has happened if the dice comes up 1 or 3 or 5.

For any two events E and F of a sample space S, we define the new event $E \cup F$ to consists of outcomes that are either in E or in F or in both E and F. That is, the event $E \cup F$ will occur if her E or F or both occurs. For instances, in the dice example (i) if event $E = \{1, 2\}$ and $F = \{3, 4\}$, then $E \cup F$ {1, 2, 3, 4}.

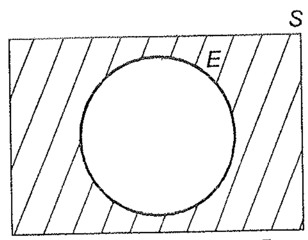
That is $E \cup F$ would be another event consisting of 1 or 2 or 3 or 4. The event $E \cup F$ is called **union** of event and the event F. Similarly, for any two events E and F we may also define the new event $\cap F$, called **intersection** of E and F, to consists of all outcomes that are common to both E and F.



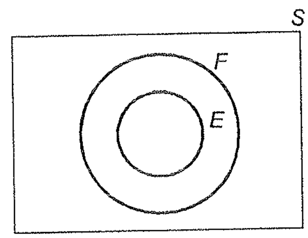
(a) Shaded region : $E \cup F$



(b) Shaded region : $E \cap F$



(c) Shaded region : E^C



(d) $E \subset F$

5.1.2 Types of Events

5.1.2.1 Complementary Event

The event E^C is called complementary event for the event E. It consists of all outcomes not in E, but in S. For example, in a dice throw, if E = {Even nos} = {2, 4, 6} then E^C = {Odd nos} = {1, 3, 5}.

5.1.2.2 Equally Likely Events

Two events E and F are equally likely iff

$$p(E) = p(F)$$

For example, $E = \{1, 2, 3\}$

$F = \{4, 5, 6\}$

are equally likely, since $p(E) = p(F) = 1/2$.

5.1.2.3 Mutually Exclusive Events

Two events E and F are mutually exclusive, if $E \cap F = \phi$ i.e. $p(E \cap F) = 0$. In other words, if E occurs, F cannot occur and if F occurs, then E cannot occur (i.e. both cannot occur together).

5.1.2.4 Collectively Exhaustive Events

Two events E and F are collectively exhaustive, if $E \cup F = S$ i.e. together E and F include all possible outcomes, $p(E \cup F) = p(S) = 1$.

5.1.2.5 Independent Events

Two events E and F are independent iff

$$p(E \cap F) = p(E) * p(F)$$

Also $$p(E \mid F) = p(E) \text{ and } p(F \mid E) = p(F).$$

Whenever E and F are independent. i.e. when two events E and F are independent, the conditional probability becomes same as marginal probability. i.e. probability E is not affected by whether F has happened or not, and vice-versa i.e., when E is independent of F, then F is also independent of E.

5.1.3 DeMorgan's Law

1. $\left(\bigcup_{i=1}^{n} E_i\right)^C = \bigcap_{i=1}^{n} E_i^C$

2. $\left(\bigcap_{i=1}^{n} E_i\right)^C = \bigcup_{i=1}^{n} E_i^C$

Example: $(E_1 \cup E_2)^C = E_1^C \cap E_2^C$

$(E_1 \cap E_2)^C = E_1^C \cup E_2^C$

Note that $E_1^C \cap E_2^C$ is the event **neither E_1 nor E_2**.

$E_1 \cup E_2$ is the event **either E_1 or E_2 (or both)**.

Demorgan's law is often used to find the probability of neither E_1 nor E_2.

i.e. $p(E_1^C \cap E_2^C) = p[(E_1 \cup E_2)^C] = 1 - p(E_1 \cup E_2)$.

5.1.4 Approaches to Probability

There are 2 approaches to quantifying probability of an Event E.

1. Classical Approach:

$$P(E) = \frac{n(E)}{n(S)} = \frac{|E|}{|S|}$$

i.e. the ratio of number of ways an event can happen to the number of ways sample space can happen, is the probability of the event. Classical approach assumes that all outcomes are equally likely.

Example 1.

If out all possible jumbles of the word "BIRD", a random word is picked, what is the probability, that this word will start with a "B".

Solution:

$$p(E) = \frac{n(E)}{n(S)}$$

In this problem $n(S)$ = all possible jumbles of BIRD = 4!

$n(E)$ = those jumbles starting with "B" = 3!

So, $$p(E) = \frac{n(E)}{n(S)} = \frac{3!}{4!} = \frac{1}{4}$$

Example 2.

From the following table find the probability of obtaining "A" grade in this exam.

Grade	A	B	C	D
No. of Students	10	20	30	40

Solution:

N = total no of students = 100

By frequency approach,

$$p(A\text{ grade}) = \frac{n(A\text{ grade})}{N} = \frac{10}{100} = 0.1$$

5.1.5 Axioms of Probability

Consider an experiment whose sample space is S. For each event E of the sample space S we assume that a number $P(E)$ is defined and satisfies the following three axioms.

Axiom-1: $0 \le P(E) \le 1$

Axiom-2: $P(S) = 1$

Axiom-3: For any sequence of mutually exclusive events E_1, E_2, ...(that is, events for which $E_i \cap E_j = \phi$ when $i \neq j$