Ch-8 Z-Transform

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]\cdot z^{-n}$$

$$z = re^{j\omega}$$

Magn. $= |z| = r$, $\angle z = \omega = \Omega$

Now, For $|r| = 1$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j\omega n} = \text{DTFT}$$

$$\text{Z.T.} = \text{DTFT}\left[x[n]\cdot r^{-n}\right]$$

if $|r| = 1$

Z.T. = DTFT (calculated at unit circle radius $r=1$)

L.T.

$x(t)$

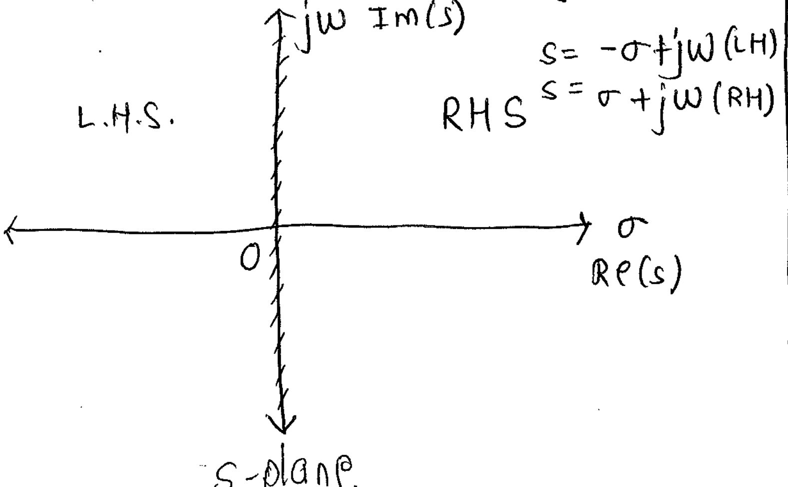
$$X[s] = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt \quad , s = \sigma + j\omega$$

$$X[s] = \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t}\cdot e^{-j\omega t}\,dt$$

if $\sigma = 0$

$$X[s] = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\,dt = \text{CTFT}$$

LT = CTFT if LT calculated on $j\omega$ axis



s-plane

Z.T.

$x[n]$

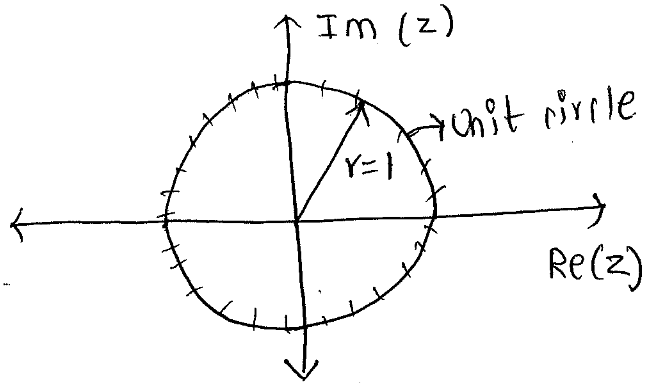
$$X[z] = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \quad , z = re^{j\omega}$$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]\cdot r^{-n}\cdot e^{-j\omega n}$$

if $r = 1$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j\omega n} = \text{DTFT}$$

Z.T. = D.T.F.T. calculated on unit circle



NOTE:

+ve part of $j\omega$ axis is corresponds to upper half of unit circle. [ω varies from 0 to π]

-ve part of $j\omega$ axis is corresponds to lower half of unit circle [ω varies from π to 2π]

Relation between s-plane and z-plane

$$x(t) = e^{st} \quad ; \quad x[n] = z^n$$

↓ sampling

$t = nT_s$

$$x[nT_s] = e^{snT_s}$$

$$x[n] = (e^{sT_s})^n = z^n$$

$$\boxed{z = e^{sT_s}}$$

$$\ln z = sT_s$$

$$\boxed{s = \frac{1}{T_s} \ln z}$$

Region of convergence for z-Transform

-The range of values of z for which if $\sum_{n=-\infty}^{\infty} |x(n) \cdot r^{-n}| < \infty$ then $|X(z)| < \infty$

*Properties of ROC:

1) The ROC of $X[z]$ consist of ring in z plane centered about origin.

2) The ROC doesn't contains any pole

3) If $x[n]$ is of finite duration then ROC is entire z-plane except possibly $z=0$ and/or $z=\infty$

4) If $x[n]$ is two sided and if the circle $|z|=r_0$ is in the ROC then the ROC will consist of ring in the z-plane that includes the circle $|z|=r_0$

5) If the z-transform $X[z]$ of $x[n]$ is rational and $x[n]$ is right sided then the ROC is the region in z-plane outside the outermost pole. i.e. outside the circle of radius equal to the largest magnitude of the poles of $X(z)$. Further more if $x[n]$ is causal (i.e. right sided) and equal to zero(0) for $n<0$ then the ROC also includes $z=\infty$

6) If z-Transform $X[z]$ of $x[n]$ is rational and $x[n]$ is left sided then the ROC is the region in z-plane inside the innermost pole. i.e. inside the circle of radius equal to smallest magnitude of poles of $X(z)$.

Q:- Find Z-Transform of following signals:-

(i) $x[n] = \{ \underset{\uparrow}{1}, 1, 1 \}$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= 1 \cdot 1 + 1 \cdot z^{-1} + 1 \cdot z^{-2}$$

$$= 1 + z^{-1} + z^{-2} = 1 + \frac{1}{z} + \frac{1}{z^2}$$; ROC: entire z-plane except $z = 0$

(ii) $x[n] = \{ 1, 1, \underset{\uparrow}{1} \}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= 1 \cdot z^2 + 1 \cdot z^1 + 1$$

$$X(z) = z^2 + z + 1$$; ROC: entire z-plane except $\infty = z$

(iii) $x[n] = \{ 1, \underset{\uparrow}{1}, 1 \}$

$$X(z) = z + 1 + z^{-1} = z + 1 + \frac{1}{z}$$; ROC: entire z-plane except $z = 0$, or ∞

*Z-Transform of some standard signals

t	n
$u(t) \to$ RHS	$u[n] \to$ RHS
$u(-t) \to$ LHS	$u[-n-1] \to$ LHS

In laplace we have seen

$$e^{-at} \cdot u(t) \xleftrightarrow{L.T.} \frac{1}{s+a} \text{ ; ROC } \sigma > -a$$

① $x[n] = a^n u[n]$, $|a|<1$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]\cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

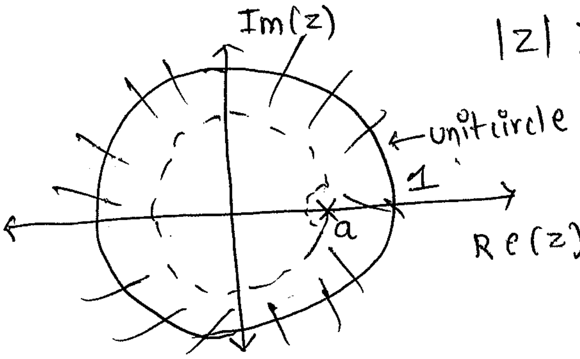
$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= 1 + (az^{-1})^1 + (az^{-1})^2 + \dots$$

$$X[z] = \frac{1}{1-az^{-1}} \quad ; \quad |az^{-1}|<1$$

$|a| < |z|$

$|z| > |a|$: ROC



z-plane

stable

② $x[n] = -a^n \cdot u[-n-1]$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} (az^{-1})^n$$

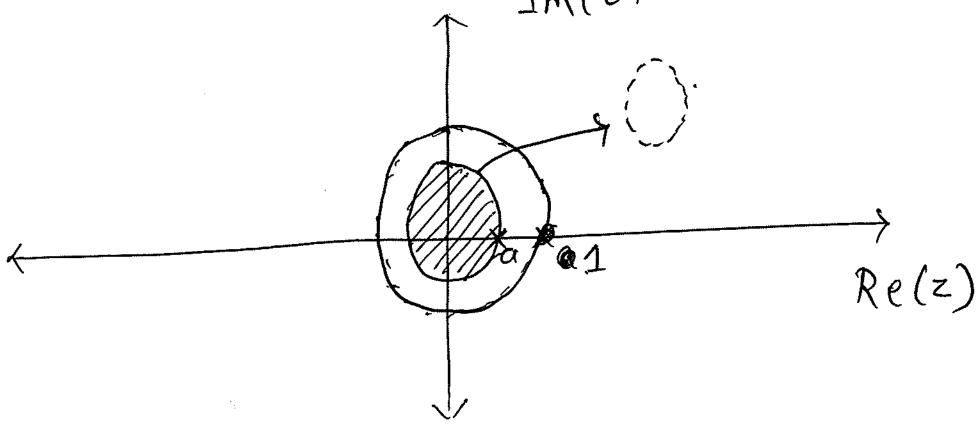
$$= -\left((a^{-1}z)^1 + (a^{-1}z)^2 + (a^{-1}z)^3 + \cdots\right)$$

$$= -a^{-1}z\left[1 + (a^{-1}z)^1 + (a^{-1}z)^2 + \cdots\right]$$

$$= \frac{-a^{-1}z}{1-a^{-1}z} < \infty \quad \text{if} \quad |a^{-1}z| < 1$$

$$= \frac{a^{-1}z}{a^{-1}z-1}$$

$$= \frac{1}{1-az^{-1}} < \infty \quad \text{if} \quad |z| < |a| : \text{ROC}$$



unstable bcz. $|z| < 1$

③ $u[n]$

$$X[z] = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} u[n]\cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + \cdots$$

$$= \frac{1}{1-z^{-1}} \; ; \; \text{ROC: } |z^{-1}| < 1$$

$$|z| > 1$$

OR

$$\lim_{a\to 1} a^n u(n) \xleftrightarrow{Z.T.} \lim_{a\to 1} \frac{1}{1-az^{-1}} \; ;$$

ROC: $|z| > |a|$

$$u[n] \xleftrightarrow{Z.T.} \frac{1}{1-z^{-1}} \; ; \; \text{ROC: } |z| > 1$$

④ $-u[-n-1] \xleftrightarrow{Z.T.} \dfrac{1}{1-z^{-1}}$; $|z|<1$: ROC

⑤ $u[-n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{0} z^{-n}$$

$$= 1 + z^{+1} + z^{2} + \cdots$$

$$= \frac{1}{1-z} \; ; \; |z|<1$$

$$X(z) = \frac{z^{-1}}{z^{-1}-1} = \frac{-z^{-1}}{1-z^{-1}} \; ; \; |z|<1$$

⑥ $x[n] = a^{|n|}$, $0<a<1$

$$= a^{n} \; ; \; n \geq 0 = a^{n} u[n]$$

$$= a^{-n} \; ; \; n<0 = a^{-n} \cdot u[-n-1]$$

[In L.T. $x(t) = e^{-|a|t} \xleftrightarrow{L.T.} \dfrac{2a}{s^2+a^2}$]

$$x[n] = a^{|n|} = a^{n} u[n] + a^{n} \cdot u[-n-1]$$

[$-a^{n} u[-n-1] \xleftrightarrow{Z.T.} \dfrac{1}{1-az^{-1}}$; $|z|<|a|$

$(a^{-1})^{n} u[-n-1] \xleftrightarrow{Z.T.} \dfrac{-1}{1-a^{-1}z^{-1}}$; $|z|<|a^{-1}|$]

$$X(z) = \underbrace{\frac{1}{1-az^{-1}}}_{\substack{\updownarrow \text{ROC} \\ |z|>|a|}} - \underbrace{\frac{1}{1-a^{-1}z^{-1}}}_{\substack{\updownarrow \text{ROC} \\ |z|<\left|\frac{1}{a}\right|}}$$

ROC :- $\boxed{|a| < |z| < \left|\frac{1}{a}\right|}$

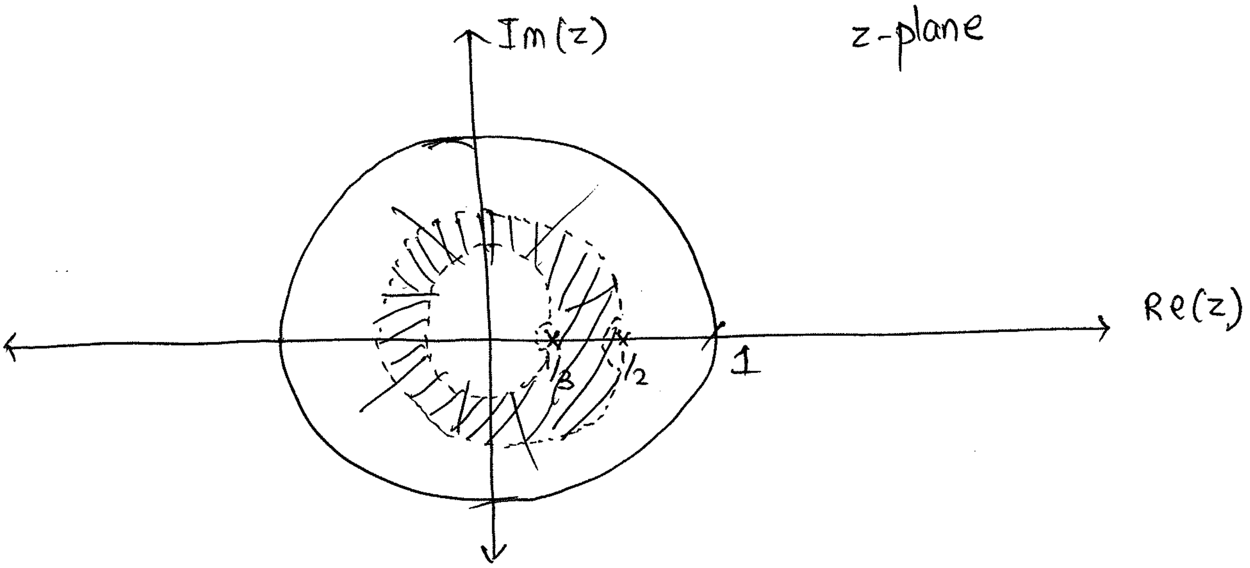
If $a = 1/2$

$$\frac{1}{2} < |z| < 2$$

Q:- $x[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$

$$X(z) = \frac{1}{1-\frac{1}{3}z^{-1}} - (-1)\frac{1}{1-\frac{1}{2}z^{-1}}$$

$|z| > 1/3$ $\quad$ $|z| < 1/2$



$$\frac{1}{3} < |z| < \frac{1}{2}$$

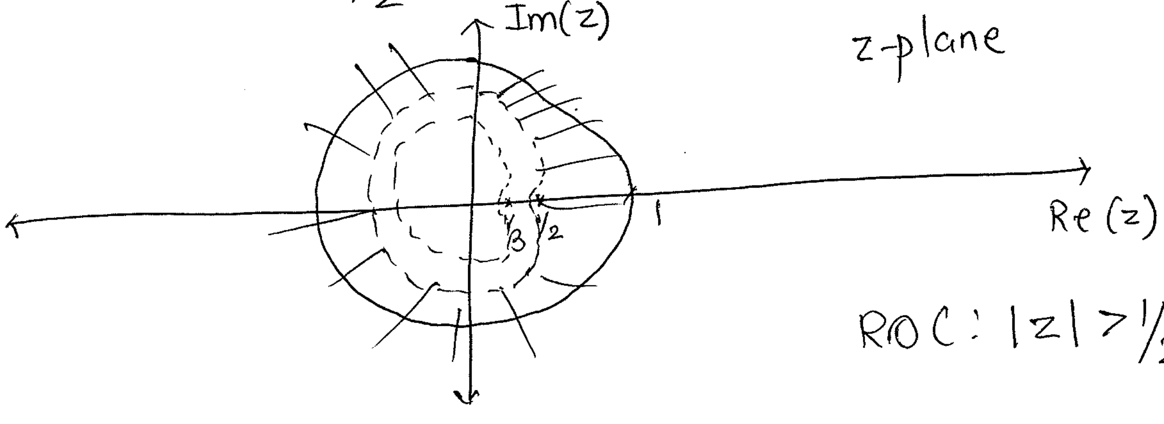
Q:- Find z-transform of following signals:-

$x[n] = \cos \omega_0 n \, u(n)$

(1) $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}}$$

$|z| > 1/2$ $\quad$ $|z| > 1/3$



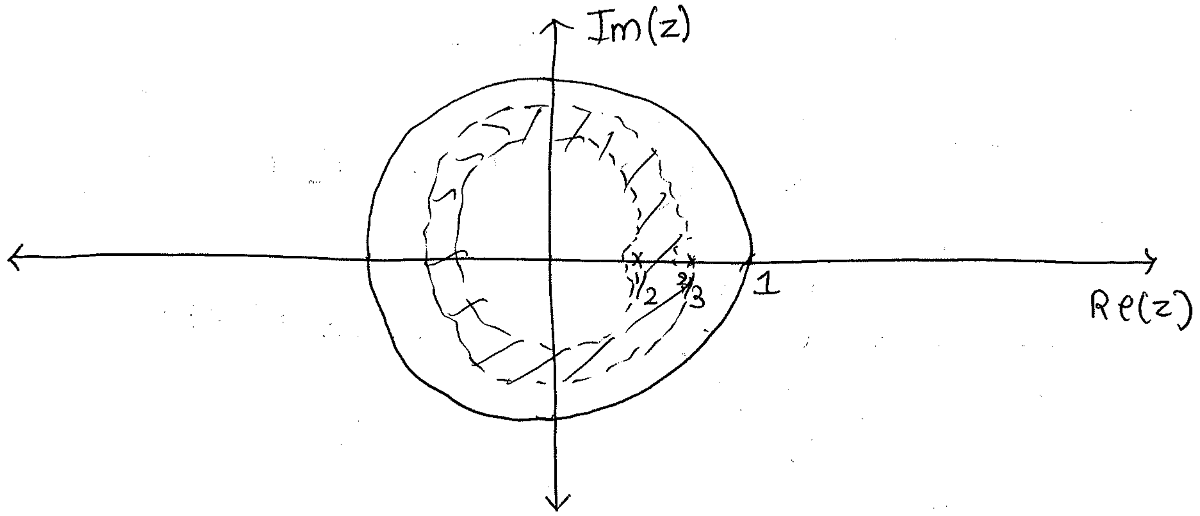
ROC: $|z| > 1/2$

(2) $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{2}{3}\right)^n \cdot u[-n-1]$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} + (-1)\frac{1}{1-\frac{2}{3}z^{-1}}$$

$$\downarrow$$

$|z| > 1/2 \qquad |z| < 2/3$



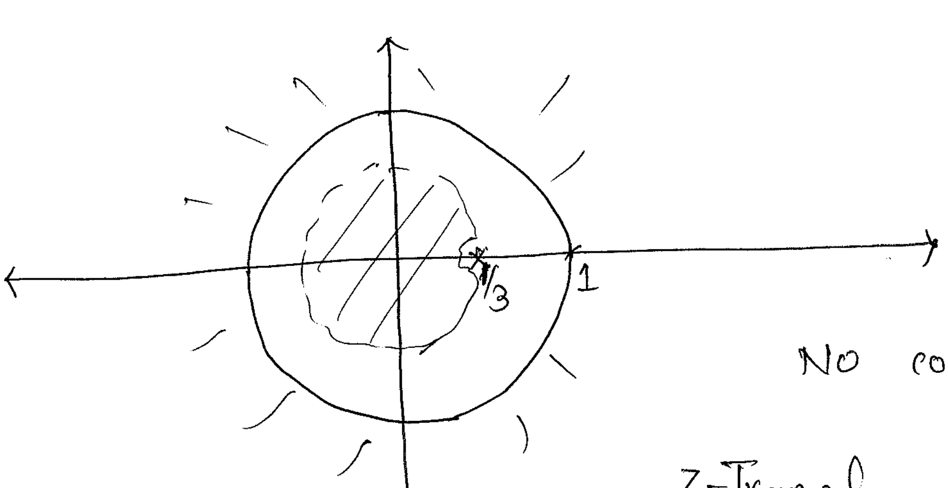
<u>ROC:</u> $\cancel{2/3 < |z| < 1/2}$ $\qquad 1/2 < |z| < \frac{2}{3}$

(3) $x[n] = \cancel{\left(\frac{1}{3}\right)} u[n] + \left(\frac{1}{3}\right)^n u[-n-1]$

$$= \frac{1}{1-z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}}$$

$\downarrow \qquad \downarrow$

$|z| > 1 \qquad |z| < 1/3$



No common ROC

Z-Transform is not defined

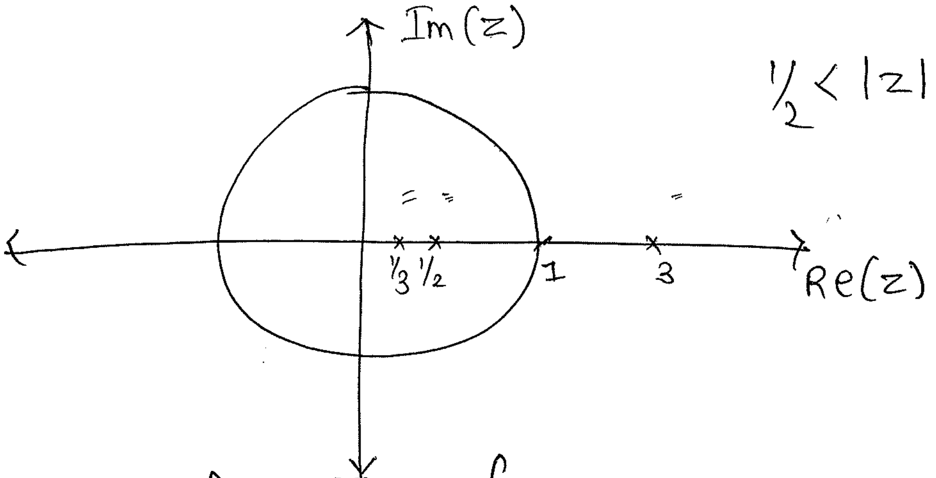
(4) If $x[n] = \left(\frac{1}{3}\right)^{|n|} - \left(\frac{1}{2}\right)^n u[n]$ then the region of convergence of it's Z-Transform in z-plane will be

→ $x[n] = \left(\frac{1}{3}\right)^{|n|} - \left(\frac{1}{2}\right)^n u[n]$

$\left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{3}\right)^{-n} u(-n-1)$ $\quad |z| > 1/2$

$|z| > 1/3 \quad |z| < |1/3|^{-1} \quad |z| > 1/2$

$|z| < 3$



$1/2 < |z| < 3$

Property of Z-Transform:-

(1) Linearity

$x_1[n] \xleftrightarrow{Z.T.} X_1[z]$; ROC : R_1

$x_2[n] \xleftrightarrow{Z.T.} X_2[z]$; ROC : R_2

$\alpha x_1[n] + \beta x_2[n] \xleftrightarrow{Z.T.} \alpha X_1[z] + \beta X_2[z]$; ROC : $R_1 \cap R_2$

(2) Time shifting

$x[n-n_0] \xleftrightarrow{Z.T.} z^{-n_0} \cdot X[z]$

(3) Time scaling in z-domain

$a^n x[n] \xleftrightarrow{Z.T.} X\left[\frac{z}{a}\right]$; ROC : $R|a|$

$X[a^{-1}z]$

(4) Time scaling

$$x\left[\frac{n}{m}\right] \xleftrightarrow{Z.T.} X[z^m] \quad ; ROC: R^{1/m}$$

(5) Time reversal

$$x[-n] \xleftrightarrow{Z.T.} X[z^{-1}] \quad ; ROC: R' = 1/R$$

(6) Differentiation in z-domain

$$n \cdot x[n] \xleftrightarrow{Z.T.} -z\frac{dX(z)}{dz}$$

$$(n)^k \cdot x[n] \xleftrightarrow{Z.T.} (-z)^k \frac{d^k X(z)}{dz^k}$$

(7) Convolution

$$x_1(n) * x_2[n] \xleftrightarrow{Z.T.} X_1[z] X_2[z]$$

$$x_1[n] * x_2[-n] \xleftrightarrow{Z.T.} X_1[z] \cdot X_1[z^{-1}]$$

(8) Conjugate

$$x^*[n] \xleftrightarrow{Z.T.} X^*[z^*]$$

(9) Accumulation

$$\sum_{n=-\infty}^{K} x[n] \xleftrightarrow{Z.T.} \frac{X[z]}{1-z^{-1}}$$

(10) First difference

$$x[n] - x[n-1] \xleftrightarrow{Z.T.} X(z) - z^{-1}X(z) = (1-z^{-1})X(z)$$

Q:- Find Z-Transform

(i) $x[n] = \sin \omega_0 n \cdot u[n]$

$$= \cancel{\frac{e^{j\omega_0} + e^{j\omega_0}}{2j}} \quad \left(\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j} \right) u[n]$$

$$= \frac{\overset{a}{e^{j\omega_0 n}} \cdot u[n]}{2j} - \frac{e^{-j\omega_0 n} u[n]}{2j} \qquad \left[\because a^n u[n] \xleftrightarrow{Z.T.} \frac{1}{1-az^{-1}} \right]$$

$$\cancel{X(z) = \sum_{n=-\infty}^{\infty} e^{j\omega_0} \cdot u[n] z^{-n}} \qquad = \frac{1}{2j} \left[\frac{1}{1-e^{j\omega_0} z^{-1}} - \frac{1}{1-e^{-j\omega_0} z^{-1}} \right]$$

$$\cancel{= \sum_{n=0}^{\infty} e^{j\omega_0} \cdot z^{-n}} \qquad = \frac{z^{-1} \sin \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}} \; ; |z| > 1$$

$$\cancel{= e^{j\omega_0} \left(\frac{1}{1-z^{-1}} \right)}$$

(ii) $x[n] = \cos \omega_0 n \cdot u[n]$

$$= \left(\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} \right) u[n]$$

$$= \frac{1 - z^{-1} \cos \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}} \; ; |z| > 1$$

Q:- Given $x[n] = a^n \cdot u[n]$

Find Z-transform of following signals

(1) $g[n] = n \cdot x[n]$

(2) $g[n] = [n+1]\, x[n+1]$

(3) $g[n] = x[n] * x[n]$

Sol:- (1) $x[n] = a^n \cdot u[n]$

$$X(z) = \frac{1}{1-az^{-1}}$$

$$g[n] = n \cdot x[n]$$

$$g[z] = -z \frac{dX(z)}{dz}$$

$$= -z \frac{d}{dz} \frac{1}{1-az^{-1}}$$

$$= -z \frac{d}{dz} \frac{z}{z-a}$$

$$g[z] = -z \left[\frac{(z-a) \cdot 1 - z(1)}{(z-a)^2} \right] = \frac{+za}{(z-a)^2}$$

$$g[z] = \frac{az^{-1}}{(1-az^{-1})^2} \quad ; \text{ ROC: } |z| > |a|$$

(2) $g[n] = [n+1]\, x[n+1]$

Suppose $g_2[n]$ and from sum (1)

$$g_2[n] = g_1[n+1]$$

$$= z^{1} \cdot G_1(z)$$

$$= z^{1} \cdot \frac{az^{-1}}{(1-az^{-1})^2} = \frac{a}{(1-az^{-1})^2}$$

(3) $g_3[n] = x[n] * x[n]$

$$\cancel{G[z] = X[z] \cdot X[z]}$$

$$\cancel{= \frac{az^{-1}}{(1-az^{-1})^2} \cdot \frac{az^{-1}}{(1-az^{-1})^2}}$$

$$g_3[n] = a^n u[n] * a^n u[n] \xleftrightarrow{Z.T.} \frac{1}{(1-az^{-1})^2}$$

$$(n+1)a^n u[n] \xleftrightarrow{Z.T.} \frac{1}{(1-az^{-1})^2}$$

(4) $g_4[n] = a^n \cdot u\left[\frac{n}{2}\right]$

$$g_4[n] = (a^2)^{n/2} u[n/2]$$

$$(a^2)^n \cdot u[n] \xleftrightarrow{Z.T.} \frac{1}{1-a^2 z^{-1}}$$

$$G_1(z) \xleftrightarrow{Z.T.} \frac{1}{1-a^2 z^{-2}}$$

(5) $g_5[n] = u\left[\frac{n}{2}\right]$

$$G_5[z] \xleftrightarrow{Z.T.} \frac{1}{1-z^{-2}}$$

(6) $g_6[n] = u\left[\frac{n-1}{2}\right]$

$$u[n] \xleftrightarrow{Z.T.} \frac{1}{1-z^{-1}}$$

$$u\left[\frac{n}{2}\right] \xleftrightarrow{Z.T.} \frac{1}{1-z^{-2}}$$

$$u\left[\frac{1}{2}(n-1)\right] \xleftrightarrow{Z.T.} \frac{z^{-1}}{1-z^{-2}}$$

* Inverse Z-Transform

Format:- $a^n u[n] \xleftrightarrow{Z.T.} \frac{1}{1-az^{-1}}$

Q:- Consider $X[z] = \frac{z^2}{(z-1)(z-0.5)}$. Evaluate $x[n]$ if ROC's

are (i) ROC: $|z|>1$

(ii) ROC: $|z|<0.5$

(iii) ROC: $0.5<|z|<1$

Sol^n:- $X(z) = \frac{z^2}{(z-1)(z-0.5)}$

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)}$$

$$= \frac{A}{z-1} + \frac{B}{z-0.5}$$

$$A = 2, \quad B = -1$$

$$= \frac{2}{z-1} - \frac{1}{z-0.5}$$

$$\frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-1/2}$$

$$\frac{X(z)}{z} = \frac{1}{\cancel{z}}\left[\frac{2}{1-z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}\right]$$

$$X(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}$$

① $x[n] = 2u[n] - \left(\frac{1}{2}\right)^n u[n]$ causal & unstable

② $x[n] = -2u[-n-1] + \left(\frac{1}{2}\right)^n u[-n-1]$ Non-causal & unstable

③ $x[n] = -2u[-n+1] - \left(\frac{1}{2}\right)^n u[n]$ Non-causal & unstable

Q:- $X(z) = \dfrac{1}{(1+z^{-1})(1-z^{-1})^2}$. If $x(n)$ is causal then determine inverse Z-transform

$$X(z) = \frac{A}{1-(-1)(z)^{-1}} + \frac{B}{1-z^{-1}} + \frac{C}{(1-z^{-1})^2}$$

$$x[n] = A \cdot (-1)^n u[n] + B u[n] + C[n+1] \cdot u[n]$$

COMPLEX FUNCTIONS

- Complex number is defined as ordered pair of real no. x and y and it is denoted by z

$$z = x + iy \quad \text{——— (1)}$$

Real part $Re(z)$

Imaginary part $Im(z)$

$i^2 = -1 \quad \therefore i = \sqrt{-1}$

$i^3 = i^2 \cdot i = -i$

$i^4 = 1$

In general,

$i^{4n} = 1$

$i^{4n+1} = i$

$i^{4n+2} = -1$

$i^{4n+3} = -i$

$\bar{z} = \overline{x+iy}$

$\bar{z} = x - iy$ ——— (2)

Add (1) & (2)

$z + \bar{z} = x + \cancel{iy} + x - \cancel{iy}$

$x = \dfrac{z+\bar{z}}{2}$

$$\boxed{Re(z) = \frac{z+\bar{z}}{2}}$$

(1) − (2)

$z - \bar{z} = x + iy - x + iy$

$y = \dfrac{z-\bar{z}}{2i}$

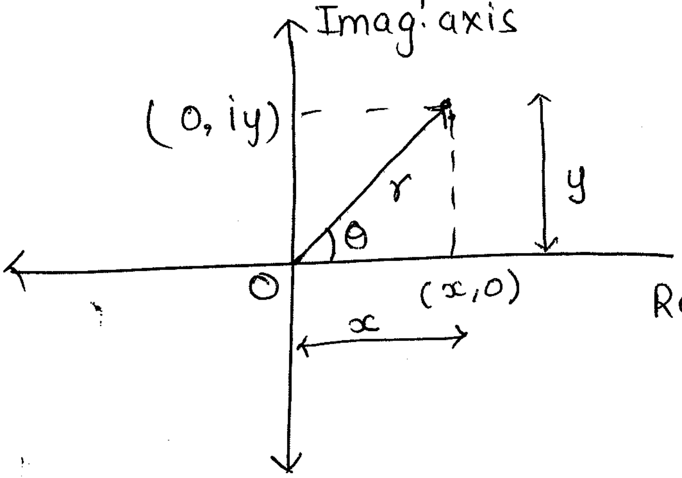
$$\boxed{Im(z) = \frac{z-\bar{z}}{2i}}$$

$$z\cdot\bar{z} = (x+iy)(x-iy)$$

$$= x^2 - xiy + iyx - i^2y^2$$

$$\boxed{z\cdot\bar{z} = x^2+y^2}$$

Geometrical representation of complex numbers



complex-plane
OR
Argand plane
OR Gaussian plane

Polar form of complex no.

$$z = x+iy$$

$$= r\cos\theta + ir\sin\theta$$

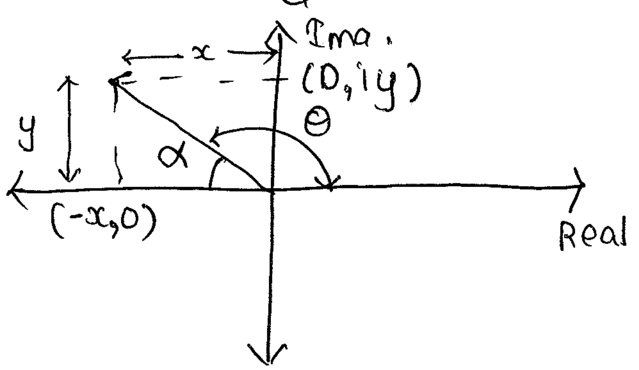
$$= r(\cos\theta + i\sin\theta)$$

$z = r\angle\theta$ Polar form of C.N.

$r = \sqrt{x^2+y^2}$
$\theta = \tan^{-1}(y/x)$

argument

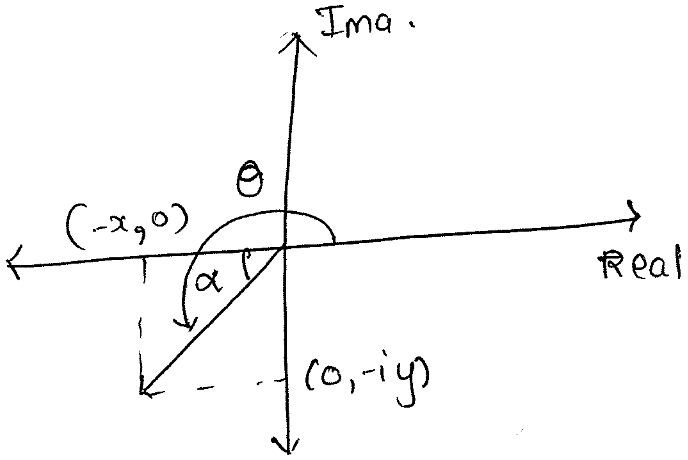
2nd Quadrant



$$\theta = \pi - \tan^{-1}\left|\frac{y}{x}\right| \quad (AC)$$

$$\theta = -\pi - \tan^{-1}\left|\frac{y}{x}\right| \quad (CC)$$

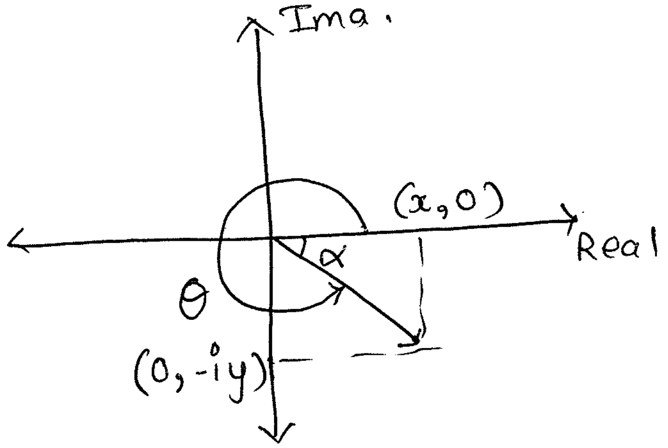
3rd Quadrant



$$\theta = \pi + \tan^{-1}\left|\frac{y}{x}\right| \quad (AC)$$

$$\theta = -\pi + \tan^{-1}\left|\frac{y}{x}\right| \quad (C)$$

4th Quadrant



$$\theta = \frac{3\pi}{2} + \tan^{-1}\left|\frac{y}{x}\right| \quad (AC)$$

$$\theta = -\tan^{-1}\left|\frac{y}{x}\right| \quad (C)$$

* De-Moiver's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

where n is rational no.

*Roots of complex no:-

$$z^{1/n} = [r(\cos\theta + i\sin\theta)]^{1/n}$$

$$= r^{1/n}[\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)]^{1/n}$$

$$\boxed{z^{1/n} = r^{1/n}\left[\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right]}$$

where $k = 0, 1, 2, \ldots, n-1$

n distinct roots

For $n = 3$

Take $k = 0, 1, 2$

Exponential form of complex no.

As we know,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{(Maclaurian series)}$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta \longrightarrow (1)$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \longrightarrow (2)$$

(1) + (2)

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

(1) − (2)

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

$$\boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

$$\because Z = r(\cos\theta + i\sin\theta)$$

$$\therefore \boxed{Z = re^{i\theta}} \rightarrow \text{Exponential form of CN}$$

Q:- Find the values of

(1) $e^{i\pi} = \cos\pi + i\sin\pi = -1$

(2) $e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = i$

(3) $e^{i2\pi n} = \cos 2\pi n + i\sin 2\pi n$

$= 1$

*Logarithm of complex number

$z = r(\cos\theta + i\sin\theta)$

$= r(\cos(2k\pi+\theta) + i\sin(2k\pi+\theta))$; k is integer

$z = re^{i(2k\pi+\theta)}$

Take ln on both sides

$\ln z = \ln r + i(2k\pi+\theta)\ln e$

$$\boxed{\ln z = \ln r + i(2k\pi+\theta)}$$

Put k=0 for principle log

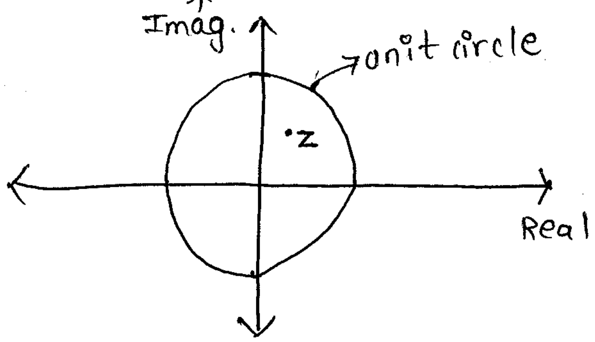
$\ln z = \ln r + i\theta$

$$\boxed{\ln z = \ln(x^2+y^2)^{1/2} + i\tan^{-1}\left|\frac{y}{x}\right|}$$

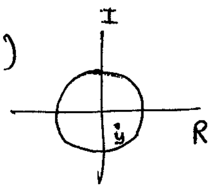
2017 -2m

Q:- A point z has been plotted in complex plane as shown. Then $y = \frac{1}{z}$ representes

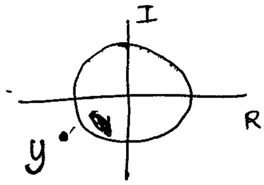
EE



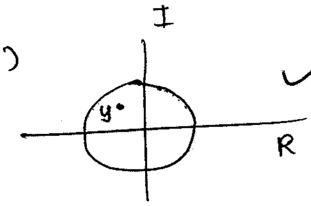
(A)



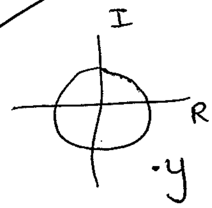
(B)



(C)



✓(D)



Sol^n:

$$|z| < 1$$

$$|y| = \frac{1}{|z|} = \frac{1}{|z<1|} = >1$$

$$|y| > 1$$

OR

$$z = a+ib \qquad a>0,\ b>0$$

$$0 < \sqrt{a^2+b^2} < 1$$

$$\frac{1}{\sqrt{a^2+b^2}} > 1$$

$$y = \frac{1}{z} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib}$$

$$= \frac{a-ib}{a^2+b^2}$$

$$y = \frac{1}{z} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2} \quad (4^{th} \text{ quadrant})$$

$$|y| = \sqrt{\left(\frac{a}{a^2+b^2}\right)^2 + \left(\frac{b}{a^2+b^2}\right)^2} = \sqrt{\frac{\cancel{a^2+b^2}}{(a^2+b^2)^{\cancel{2}}}} = \frac{1}{\sqrt{a^2+b^2}} > 1$$

Q:- The value i^i is

(a) $e^{-\pi/2}$ (b) $e^{\pi/2}$ (c) i (d) 1

Sol:- $y = i^i$

$$y = \left(e^{i\pi/2}\right)^i$$

$$y = e^{i^2 \pi/2} = e^{-\pi/2}$$

Q:- $\sin \log i^i$

$$\sin \log e^{-\pi/2}$$

$$\sin(-\pi/2) = -\sin \pi/2 = -1$$

Q:- The modulus of complex no. $\left(\frac{3+4i}{1-2i}\right)$

$$\left|\frac{3+4i}{1-2i}\right| = \frac{\sqrt{3^2+4^2}}{\sqrt{1^2+2^2}} = \sqrt{\frac{25}{5}} = \sqrt{5}$$

ME 2015

Q:- $Z_1 = 5+5\sqrt{3}i$, $Z_2 = \frac{2}{\sqrt{3}} + 2i$

The argument of $\frac{Z_1}{Z_2}$ = $\underline{0^\circ}$ (in degrees)

$$\frac{Z_1}{Z_2} = \frac{r_1 \angle \theta_1}{r_2 \angle \theta_2}$$

$$= \frac{r_1}{r} \angle \theta_1 - \theta_2$$

$$\tan^{-1}\left|\frac{\cancel{5}\sqrt{3}}{\cancel{5}}\right| - \tan^{-1}\left|\frac{\cancel{2}\sqrt{3}}{\cancel{2}}\right|$$

$$60^\circ - 60^\circ = 0^\circ$$

NOTE:-

$$Z_1 \pm Z_2 = x_1 \pm x_2 + i(y_1 \pm y_2)$$

$$Z_1 Z_2 = r_1 \angle \theta_1 \cdot r_2 \angle \theta_2$$

$$= r_1 \cdot r_2 \angle \theta_1 + \theta_2$$

$$\frac{Z_1}{Z_2} = \frac{r_1 \angle \theta_1}{r_2 \angle \theta_2}$$

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \angle \theta_1 - \theta_2$$

CE
2015

Q:- Find the value of I where $I = \int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x}\, dx$

$$I = \int_0^{\pi/2} \frac{(\cos x + i\sin x)^2}{\cos^2 x - i^2 \sin^2 x}$$

$$= \int_0^{\pi/2} \frac{\cos^2 x - \sin^2 x + 2i\sin x\cos x}{1}$$

$$= \int_0^{\pi/2} \cos^2 x + \underbrace{i\sin 2x}_{\text{odd}} - \sin^2 x$$

$\propto$.

$$= \int_0^{\pi/2} \frac{1+\cos 2x}{2} - \left[\frac{1-\cos 2x}{2}\right]$$

$$= \int_0^{\pi/2} \cos 2x \, dx$$

$$= \left[\frac{-\sin 2x}{2}\right]_0^{\pi/2}$$

$$=$$

OR

$$I = \int_0^{\pi/2} \frac{e^{ix}}{e^{-ix}}\, dx$$

$$= \int_0^{\pi/2} e^{2ix}\, dx$$

$$= \left[\frac{e^{2ix}}{2i}\right]_0^{\pi/2}$$

$$= \frac{-1}{2i} - \frac{1}{2i}$$

$$= \frac{-2}{2i} \times \frac{i}{i}$$

$$I = \frac{-i}{i^2} = \frac{-i}{-1} = i$$

$$\boxed{I = i}$$

Q:- The square root of $-i$ are ____.

(a) $i, -i$ (b) $\cos(-\pi/4) + i\sin(\pi/4), \cos(3\pi/4) + i\sin(3\pi/4)$ ✓

(c) $\cos(\pi/4) + i\sin(3\pi/4), \cos(\frac{3\pi}{4}) + i\sin(\pi/4)$

(d) $\cos(3\pi/4) + i\sin(\frac{-3\pi}{4}), \cos(\frac{-3\pi}{4}) + i\sin(\frac{3\pi}{4})$

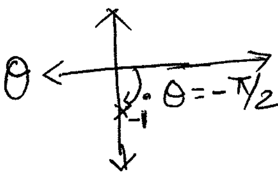
Soln:- (b) $\left| 1/\sqrt{2} + i\, 1/\sqrt{2} \right| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$

$\sqrt{-i} = (-i)^{1/2}$

$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right] \qquad k = 0, 1, \ldots, n-1$$

Here $n = 2$

$\therefore$ For $k = 0$ & $k = 1$



$$z^{1/2} = 1^{1/2}\left[\cos\left(\frac{-\pi/2}{2}\right) + i\sin\left(\frac{-\pi/2}{2}\right)\right] \qquad k = 0$$

$$\& \quad 1^{1/2}\left[\cos\left(\frac{2\pi - \pi/2}{2}\right) + i\sin\left(\frac{2\pi - \pi/2}{2}\right)\right]$$

$$\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} \qquad k = 1$$

ANS: (B).

14

Q:- Let S be the set of points in complex plane corresponding to unit circle $[S = [z : |z| = 1]]$. Consider the function $f(z) = z\bar{z}$ where $\bar{z}$ denotes complex conjugate of z. The $f(z)$ maps S to which of the following complex plane

EE-1m

(A) unit circle (b) Horizontal axis line segment from origin to $(1,0)$ point.

(C) The point $(1,0)$

(D) The entire horizontal axis.

Sol^n:- $z \cdot \bar{z} = |z|^2 = 1$

$= 1 + 0i \quad (1, 0) - (C)$

NOTE:-

- If $z = x+iy$ and $z_0 = x_0+iy_0$ are two complex no. in complex plane then the distance between z and z_0 is given by $|z-z_0|$

- $|z| = r$ represents eq^n of circle having center $(0,0)$ with radius $= r$.

- $|z-z_0| = r$ represents eq^n of circle having center (x_0, y_0) with radius r.

* Complex Variables.

- Introduction (complex funct^n, limit, Continuity, Differ.)
- Analytic function
- Complex Integration (Cauchy-Integral Formula)
- Zeros and types of singular points
- Residue theory [Residue Theorem]

Complex Variables: If x and y are two real variable then variable of the form $z = x+iy$ is called complex variable.

Complex Function: If $z = x+iy$ is a complex variable then the function $w = f(z)$ is known as complex function.

$$w = f(z) = \underset{\substack{\uparrow \\ \text{Real part} \\ \text{of } f(z)}}{u} + i\underset{\substack{\uparrow \\ \text{Img. part} \\ \text{of } f(z)}}{v}$$

- For eg:- $z = x + iy$

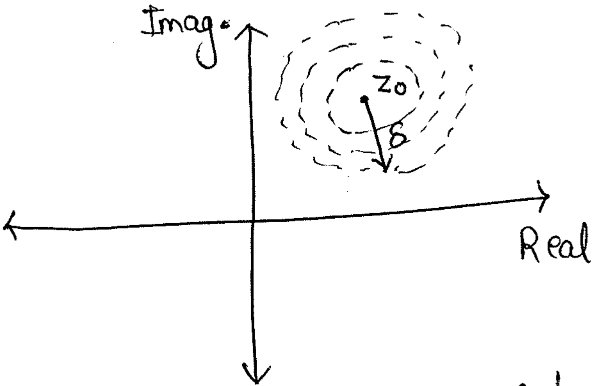
$$f(z) = z^2 + z$$
$$= (x+iy)^2 + x + iy$$
$$= x^2 + 2ixy - y^2 + x + iy$$

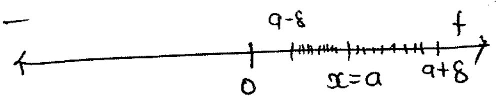
$$f(z) = \underbrace{(x^2 - y^2 + x)}_{u} + i\underbrace{(2xy + y)}_{v}$$

*Complex Calculus

- Neighbourhood of point z_0

The set of all points within the circle having center at z_0 but not on the circle is called δ neighbourhood of point z_0.





δ neighbourhood

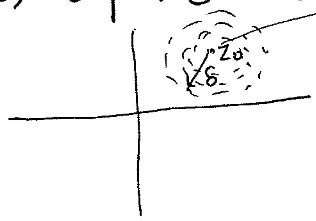
$x \in (a-\delta, a+\delta)$

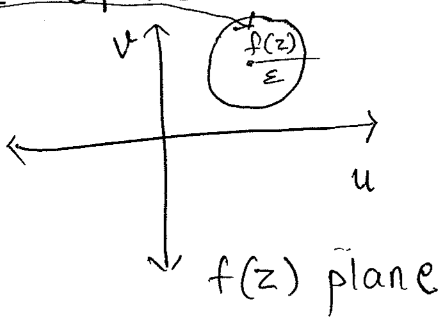
$\delta > 0$

Limit of complex function

- A function $w = f(z)$ is said to have a limit l as z approaches to a point z_0 if for every real ε we can find a positive real δ such that

$$|f(z) - l| < \varepsilon \text{ for } |z - z_0| < \delta$$





$$\boxed{\lim_{z \to z_0} f(z) = l}$$

*Continuity of f(z)

- A function $w = f(z)$ is said to be continuous at $z = z_0$ if

$$\boxed{\lim_{z \to z_0} f(x) = f(z_0)}$$

*Derivative of function

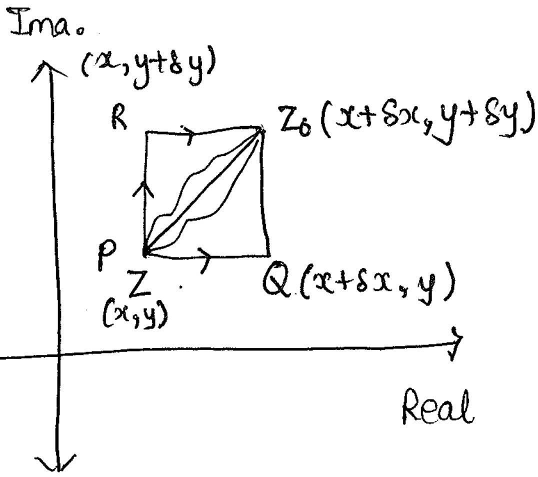
- Let a function $f(z)$ be a single value function defined over domain D then function $f(z)$ is said to be differentiable at $z = z_0$, if

$$f'(z) = \boxed{\lim_{\Delta z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists}}$$

Let $z - z_0 = \Delta z$

$$z - z_0 \to 0 \Rightarrow \Delta z \to 0$$

$$\boxed{\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad \text{exists}}$$



PROOF:-

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{[u(x+\delta x, y+\delta y) + i(v(x+\delta x, y+\delta y)] - [u(x,y) + iv(x,y)]}{\delta x + i\delta y}$$

$\Delta z \to 0$

$\delta x \to 0$, $\delta y \to 0$

Assume δx is whole real $\delta x \to 0$, $\delta y = 0$

$$= \lim_{\delta x \to 0} \frac{[u(x+\delta x, y) + i\,v(x+\delta x, y)] - [u(x,y) + i\,v(x,y)]}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[\frac{u(x+\delta x, y) - u(x,y)}{\delta x}\right] + i\left[\frac{v(x+\delta x, y) - v(x,y)}{\delta x}\right]$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{—— (1)}$$

Assume Δz is pure imaginary

δy, $\delta x = 0$

$$= \lim_{\delta y \to 0} \frac{[u(x, y+\delta y) + i\,v(x, y+\delta y)] - [u(x,y) + i\,v(x,y)]}{i\delta y}$$

$$\lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + \cancel{i}\,\frac{v(x, y+\delta y) - v(x,y)}{\cancel{i}\delta y}$$

$$\lim_{\delta y \to 0} \frac{1}{i}\left[\frac{u(x, y+\delta y) - u(x,y)}{\delta y}\right] + \frac{v(x, y+\delta y) - v(x,y)}{\delta y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{—— (2)}$$

For $f'(z)$, $\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ should exist

$\therefore$ Eq^n (1) = Eq^n (2)

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \qquad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

cauchy Riemann Eq^n

$\therefore$ Necessary & sufficient cond^n for $f(z)$ to be differentiable

$U_x = V_y$ & $U_y = -V_x$

Q:- $f(z) = \bar{z}$. State whether the function is differentiable or not.

$z = x + iy$

$\bar{z} = \underset{u}{x} - i\underset{v}{y}$; $u = x$ & $v = -y$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$1 = -1$ ✗ $\quad 0 = 0$ ✓

not differentiable

$\boxed{\text{CONJUGATE function are not differentiable}}$

* Analytic function OR Holomorphic OR Regular

- A function $f(z)$ of a complex variable z is said to be analytic at a point z_0 if it is not only differentiable at point z_0 but also at every point in some neighbourhood of z_0.


→ Analytic function

$\cdot z_0$ → Not analytic


→ Not analytic

- A function which is analytic everywhere in complex plane is called entire function.

- As derivative of polynomial exists at every point, a polynomial of any degree is an entire function.

* Condition for analytic function

Suppose $f(z) = u(x,y) + iv(x,y)$ is continuous in some neighbourhood of point $z = x+iy$ and is differentiable at z then the first order partial derivative should exists and it should satisfies Cauchy Riemann equation at point z then $f(z)$ is analytic function.

Eg:- $f(x) = |z|^2$

$$= x^2 + y^2$$

$$\frac{\partial u}{\partial x} = 2x \quad , \quad \frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial v}{\partial x} = 0 \quad , \quad \frac{\partial v}{\partial y} = 0$$

$\therefore$ CR eq^n are not satisfied

$\therefore f(z) = |z|^2$ is not analytic

At $z = 0$, CR eq^n satisfied

$\therefore f(z)$ is diff. at $z = 0$

But not analytic.

NOTE:- e^z, $\sin z$, $\cos z$, $\sinh x$, $\cosh x$ and every polynomial are everywhere defined, continuous, diff. and analytic

Cauchy Riemann's Equation in polar form

① $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ & $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$

Q:- $f(x) = (x+ay)+i(bx+cy)$ is analytic. Then which of following is true?

(a) $a=1, b=1, c=1$

(b) $a=-1, b=2, c=1$

(c) $a=-1, b=1, c=1$

(d) $a=-1, b=1, c=1$

$u = x+ay$, $v = bx+cy$

$\frac{\partial u}{\partial x} = 1$ $\quad \frac{\partial v}{\partial y} = c$

$\frac{\partial u}{\partial y} = a$ $\quad \frac{\partial v}{\partial x} = b$

$\boxed{c=1}$ $\quad \boxed{a=-b}$

Q:- Find value of p so that $f(z) = \underset{u}{r^2\cos 2\theta} + i\,\underset{v}{r^2 \sin p\theta}$ is analytic Find p

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial r}{\partial \theta}$ $\qquad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$

$2r\cos 2\theta = \frac{1}{\cancel{r}}\,\cancel{r^2}\cos p\theta\; p$ $\qquad 2r\sin p\theta = \frac{+1}{\cancel{r}}\,\cancel{r^2}\sin 2\theta\,\cancel{2}$

$\boxed{p=2}$ $\qquad \boxed{p=2}$

Construction of analytic function

Milne Thomson Method

Steps:- (1) Given $u(x,y)$ or $v(x,y)$

Find u_x & u_y or v_x & v_y

(2) $f'(z) = u_x + i v_x$ or $v_y - i u_y$

Replace 'x' by z and 'y' by 0

$f'(z) = g(z)$

(3) Integrate both sides

$f(z) = \int g(z) + c$

Q:- If $u(x,y) = x^2 - y^2 - x + 4$ is a real part of analytic function then find analytic function $f(z)$.

$u(x,y) = x^2 - y^2 - x + 4$

$U_x = \frac{\partial u}{\partial x} = 2x - 1$

$U_y = \frac{\partial u}{\partial y} = -2y$

$f'(z) = 2x - 1$.

$f'(z) = 2z - 1$

$f(z) = \int (2z - 1)$

$\boxed{f(z) = z^2 - z + c}$

$f(z) = U_x + i V_x$

$f(z) = U_x - i U_y$ $[V_x = -U_y]$

(It sastifies Cauchy-Riemann)

$f(z) = 2x - 1 - i(-2y)$

$f(z) = 2x - 1 + i2y$

$f(z) = u(x,y) + i v(x,y)$ is Analytic function

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Diff. w.r.t y

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \quad \text{—(1)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Diff. w.r.t x

$$-\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{—(2)}$$

From eq^n (1) and (2)

$$\frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\Rightarrow \boxed{v_{xx} + v_{yy} = 0} \longrightarrow \text{Laplacian eq}^n$$

- A real value function $\phi(x,y)$ that has continuous 2nd order partial ~~different~~ derivative in domain D and satisfies laplace eq^n i.e. $\phi_{xx} + \phi_{yy} = 0$ then $\phi(x,y)$ is known as harmonic function.

- If $f(z) = u + iv$ where u and v are the function of x and y is analytic in domain D then u and v will satisfy laplace equation. Therefore u and v are called harmonic function.

$$f(z) = u(x,y) + i v(x,y) \longrightarrow \text{A.F.}$$

$f(z)$ ↑ complex potential function

$u(x,y)$ ↑ velocity function

$v(x,y)$ ↑ stream function

Method for constructing u_{xy} or v_{xy}

1. Assume $u(x, y)$ is given

$f(z)$ is analytic function

$v(x,y) = ?$

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy$$

$$dV = \underbrace{-\frac{\partial u}{\partial y}}_{M} dx + \underbrace{\frac{\partial u}{\partial x}}_{N} dy \longrightarrow \text{Exact differential eq}^n$$

$$\left[u_x = v_y \quad , \quad u_y = -v_x \right]$$

$$M dx + N dy .$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \quad , \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

Sol^n is

$$\boxed{\int M dx + \int \begin{pmatrix} \text{terms in N} \\ \text{free from } x \end{pmatrix} dy = c}$$

$$\boxed{\int -u_y \, dx + \int \begin{pmatrix} \text{terms in } u_x \\ \text{free from } x \end{pmatrix} dy = c}$$

Q:- If $f(z) = x^3 - 3xy^2 + i\,v(x,y)$ and $f(z)$ is analytic function then find stream function.

→ $dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy$

$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$

$U_x = 3x^2 - 3y^2$

$U_y = -6xy$

$\int -U_y\, dx + \int \left(\begin{matrix}\text{terms } U_x \\ \text{not containing } x\end{matrix}\right) dy = c$

$+\int 6xy\, dx + \int (-3y^2)\, dy = c$

$3yx^2 - y^3 = c$

$v(x,y) = 3yx^2 - y^3 + c'$

Complex Integration (IMP).

(i) Complex line integral

- If a function $f(x)$ is defined at curve c from point a to point b then evaluation of integral of $f(z)$ along any curve c or any path c is called line integral of $f(z)$ and it is denoted by $\int_C f(z)dz$ where c is path of integration.

Suppose, $f(z) = u + iv$

$dz = dx + idy$

$$\int_C (u+iv)(dx+idy)$$

$$= \int_C (udx - vdy) + i(vdx + udy)$$

Eg:- Evaluate $\int_0^{1+i} z\,dz$ along the curve i) $y = x$ ii) $y = x^2$

Sol: i) $\int_0^{1+i} (x+iy)(dx+idy)$

$$= \int_0^{1+i} (xdx - ydy) + i(ydx + xdy)$$

Along $y = x$

$dy = dx$

$$= \int_0^1 (\cancel{xdx} - \cancel{xdx}) + i(xdx + xdx)$$

$$= \int_0^1 2xi\,dx = i[x^2]_0^1 = \underline{i}$$

ii) $= \int_0^{1+i} (x+iy)(dx+idy)$

$= \int_0^{1+i} (x\,dx - y\,dy) + i(y\,dx + x\,dy)$

Along $y = x^2$

$dy = 2x\,dx$

$= \int_0^1 (x\,dx - x^2\, 2x\,dx) + i(x^2 dx + x\, 2x\,dx)$

$= \int_0^1 (x - 2x^3)dx + i(3x^2 dx)$

$= \left[\frac{x^2}{2} - \frac{\cancel{2} x^4}{\cancel{4}2}\right]_0^1 + i\left[x^3\right]_0^1$

$= \frac{1}{2} - \frac{1}{2} + i \cdot 1$

$= \underline{i}$

Using NOTE:-

$f(z) = z$ is analytic

$\therefore \int_0^{1+i} z\,dz = \left[\frac{z^2}{2}\right]_0^{1+i}$

$= \frac{(1+i)^2 - 0}{2}$

$= i$

$\therefore$ It is independent of path

Q:- $\int_{z=0}^{z=1+i} (x^2 - iy)\,dz$ → not analytic

i) $y = x$

ii) $y = x^2$

$\int_0^{1+i} (x^2 - iy)(dx + idy)$

$\int_0^{1+i} (x^2 dx + y\,dy) + i(x^2 dy - y\,dx)$

(i) Along $y = x$

$dy = dx$

$$= \int_0^1 (x^2dx + x\,dx) + i\,(x^2dx - x\,dx)$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 + i\left[\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{2} + i\left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{5}{6} - \frac{i}{6}$$

(ii) $y = x^2$

$dy = 2x\,dx$

$$= \int_0^1 (x^2dx + x^2dx \cdot 2x) + i\,(x^2 \cdot 2x\,dx - x^2dx)$$

$$= \left(\frac{x^3}{3} + \frac{2x^4}{4}\right)_0^1 + i\left(\frac{2x^4}{4} - \frac{x^3}{3}\right)_0^1$$

$$= \frac{1}{3} + \frac{1}{2} + i\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{5}{6} + \frac{i}{6}$$

<u>NOTE:</u> If the integral function $f(z)$ is not analytic then value of integral depends on path but not on end points of path.

If the integral function is analytic then value of integral depends on end points on path and not on path.

*Parametric equations of circle

- Parametric form of circle $|z-z_0| = r$ is $z = z_0 + re^{i\theta}$

- Parametric form of circle $|z| = r$ is $z = re^{i\theta}$

Q:- $\int_C \frac{3z+5}{z}\, dz$ along curve $|z| = 5$

$|z| = r \Rightarrow z = re^{i\theta}$

$|z| = 5 \quad z = 5e^{i\theta}$

$dz = 5ie^{i\theta}\, d\theta$

$$= \int_0^{2\pi} \frac{3\cdot 5e^{i\theta}+5}{\cancel{5}\cancel{e^{i\theta}}} \cancel{5}i\cancel{e^{i\theta}}\, d\theta$$

$$= \int_0^{2\pi} 15ie^{i\theta} + 5i\, d\theta$$

$$= \left[15\cancel{i}\frac{e^{i\theta}}{\cancel{i}} + 5i\theta\right]_0^{2\pi}$$

$$= 15e^{i2\pi} + 5i(2\pi) - [15+0]$$

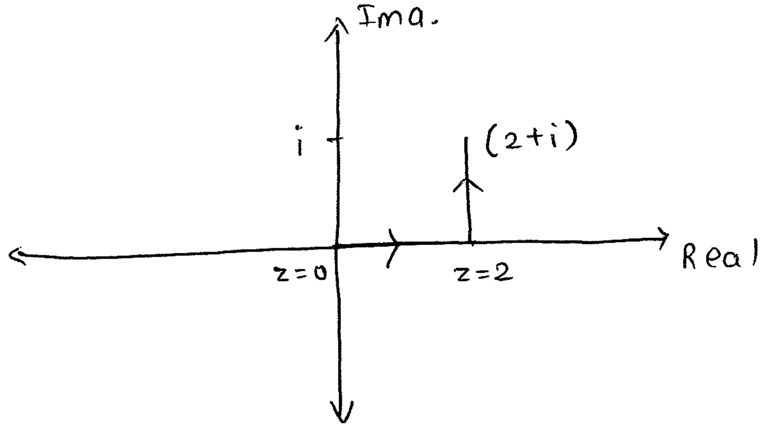
$$= 15(\cos 2\pi + i\sin 2\pi) + 10\pi i - 15$$

$$= \cancel{15} + 10\pi i - \cancel{15}$$

$$= \underline{10\pi i}$$

Q:- $\int_0^{2+i} (\bar{z})^2\, dz$ along real axis from $z=0$ to $z=2$

and then $z=2$ to $z=2+i$



I_1 : $z=0$ to $z=2$

$$I_1 = \int_0^{2+i} (x - iy)^2 \, dz$$

$$I_1 = \int_0^{2+i} (x^2 - 2xiy - y^2)(dx + i\,dy)$$

But $y=0$ for Path 1

$$I_1 = \int_0^{2} x^2 \, dx$$

$$I_1 = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

I_2 : $z=2$ to $z=2+i$

$$I_2 = \int_0^{2+i} (x^2 - 2xiy - y^2)(dx + i\,dy)$$

Put $x=2$
$dx = 0$

$$= \int_0^1 (4 - 4iy - y^2)(i\,dy)$$

$$= \int_0^1 (4i + 4y - y^2 i)\, dy \Rightarrow \left[4iy + 2y^2 - \frac{y^3}{3} i\right]_0^1 = 4i + 2 - \frac{i}{3} = \frac{11i}{3} + 2$$

$$I = I_1 + I_2$$

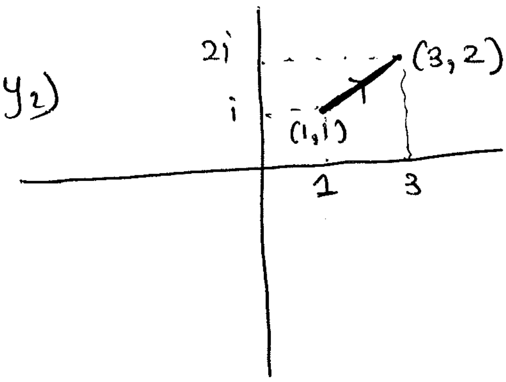
$$= \frac{8}{3} + \frac{11}{3} i + 2$$

$$I = \frac{14}{3} + \frac{11}{3} i$$

Q:- $\int_c Re(z)$ where c is the shortest distance path joining

$z = 1+i$ to $z = 3+2i$

→ If two points (x_1, y_1) & (x_2, y_2)



$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

$$\frac{x-1}{3-1} = \frac{y-1}{2-1}$$

$$x - 1 = 2y - 2$$

$$x - 2y = -1$$

$$dx - 2dy = 0$$

$$dx = 2dy$$

$$dy = \frac{dx}{2}$$

$$I = \int_1^3 x\,(dx + i\,dy)$$

$$= \int_1^3 x\left(dx + i\frac{dx}{2}\right)$$

$$= \int_1^3 x\,dx + i\frac{x}{2}\,dx$$

$$= \left[\frac{x^2}{2} + \frac{x^2}{4} i\right]_1^3$$

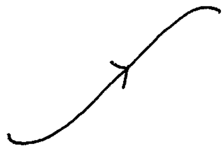
$$= \frac{9}{2} + \frac{9}{4} i - \frac{1}{2} - \frac{i}{4}$$

$$I = 4 + 2i$$

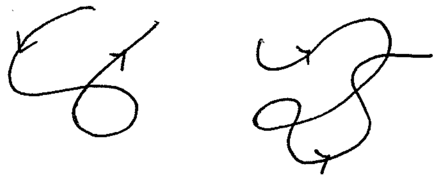
* Cauchy's Integral Theorem

Types of curves

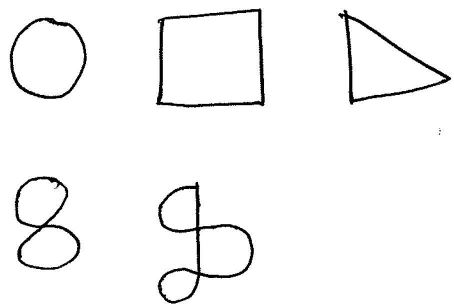
① Simple curve



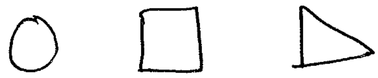
② Multiple curve



③ Closed curve

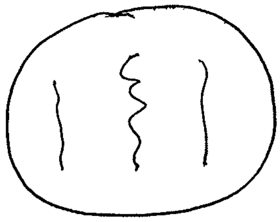


④ Simple closed curve

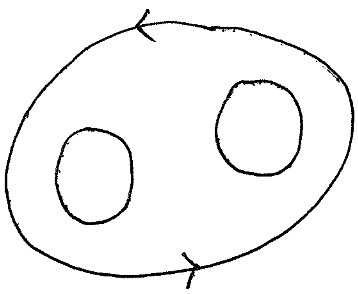


Types of Region

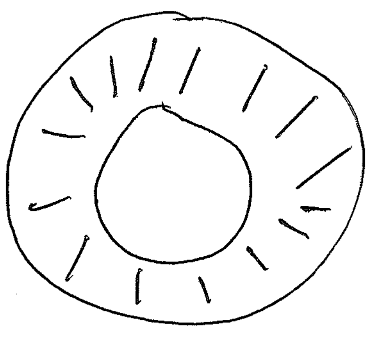
① Simply connected region



② Multiple connected region



③ Region betn concentric circle or ring shape



Statement:-

Let $f(z)$ is an analytic function within an on simple closed curve c then

$$\int_C f(z)\,dz = \underline{\quad 0 \quad}.$$

Proof:-

According to Green's Theorem:-

$$\oint (M\,dx + N\,dy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

Now,

$$\int_C (u\,dx - v\,dy) + i(v\,dx + u\,dy)$$

$$= \oint_C (\overset{M}{u}\,dx - \overset{N}{v}\,dy) + i \oint (\overset{M}{v}\,dx + \overset{N}{u}\,dy) \qquad \text{for simple closed path}$$

$$= \iint \left(\cancelto{0}{-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}} \right) dx\,dy + i \iint \left(\cancelto{0}{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}} \right) dx\,dy$$

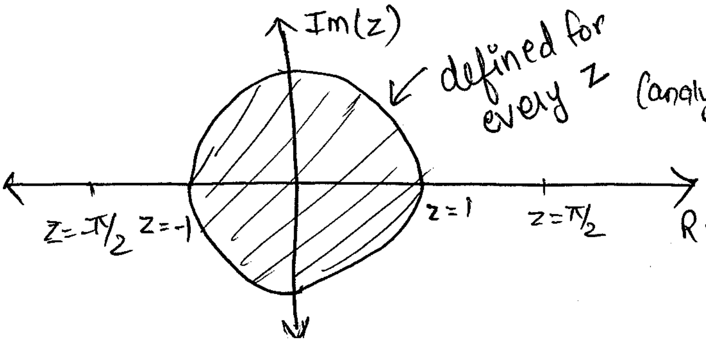
$$= 0 \qquad \left[\because \text{It is analytic} \quad u_x = v_y \;\&\; v_x = -u_y \right]$$

Q:- $\int_C \sec z\,dz$ where $c: |z| = 1$

$$\sec z = \frac{1}{\cos z}$$

Not defined at $\cos z = 0$, $z = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$-\frac{\pi}{2}, -\frac{3\pi}{2}, \ldots$



defined for every z (analytic fcn)

$$\therefore \int_C \sec z\,dz = 0$$

By Cauchy Integral

Q:- $\int_c \frac{1}{z^2} dz \qquad c: |z| = 1$

At $z = 0$

function is not defined. So it is not analytic
∴ Cauchy integral will not be applicable.

$\cancel{\int \frac{1}{(x+iy)}} \cancel{(dx+idy)}$

$r = 1$

$z = re^{i\theta}$

$dz = re^{i\theta} i\, d\theta$

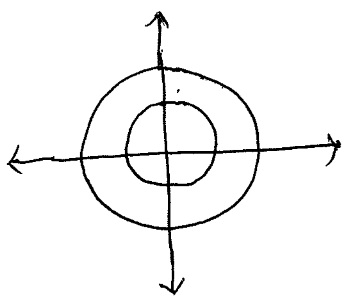
$= \int_0^{2\pi} \frac{1}{(re^{i\theta})^{\cancel{2}}} \cdot \cancel{re^{i\theta}}\, i\, d\theta$

$= \int_0^{2\pi} \frac{i}{r} e^{-i\theta} d\theta$

$= \frac{\cancel{i}}{r} \left[\frac{e^{-i\theta}}{-\cancel{i}} \right]_0^{2\pi}$

$= \left[e^{-i\theta} \right]_{2\pi}^{0} = 1 - 1 = 0$

Q:- $\int_c \frac{1}{z} dz \qquad \frac{1}{2} < |z| < \frac{3}{2}$



$0.5 < |z| < 1.5$

~~It~~ is ~~analytic function so by Cauchy~~

It is not simple closed curve. Therefore Cauchy integral will be not applicable

$z = re^{i\theta}$

$dz = re^{i\theta} i\, d\theta$

$$= \int_0^{2\pi} \frac{1}{\cancel{re^{i\theta}}} \cdot \cancel{re^{i\theta}}\, i\, d\theta$$

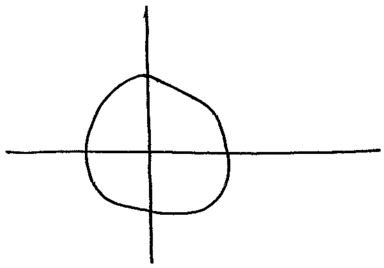
$$= \int_0^{2\pi} i\, d\theta$$

$$I = [i\theta]_0^{2\pi} = \underline{2\pi i}$$

Q:- $\int_c \bar{z}\, dz$ $\quad$ $c: |z| = 1$

$z\bar{z} = 1$

$\bar{z} = \frac{1}{z}$



From Above sum $\int_c \frac{1}{z} dz = \underline{\underline{2\pi i}}$

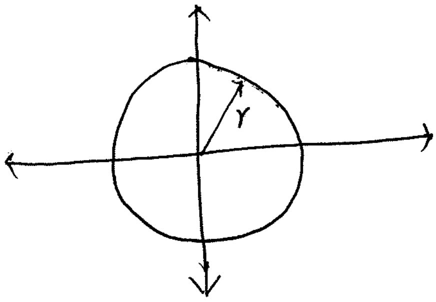
* Cauchys integral formula

Let $f(z)$ is analytic within an on simple closed curve c and $z = a$ is any point inside the curve c then

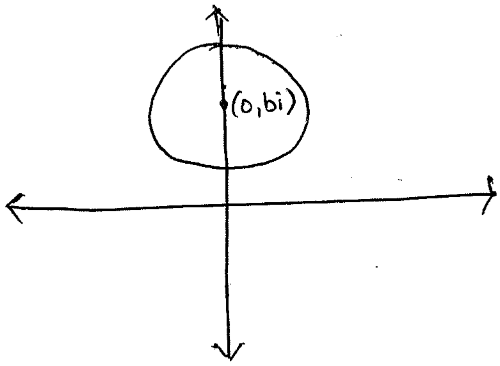
$$\int_c \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\boxed{\int \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(a)}$$

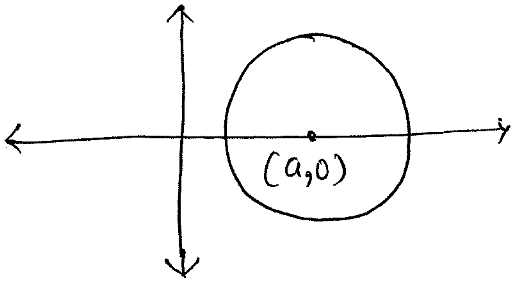
① $|z| = r$



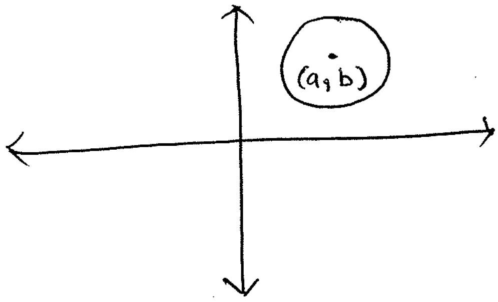
③ $|z - bi| = r$



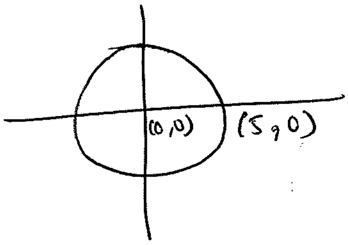
② $|z - a| = r$



④ $|z - a - bi| = r$



Q:- $\int_c \frac{3z+5}{z} \, dz$ $c: |z| = 5$



$\int_c \frac{\overset{f(z)}{3z+5}}{z-0} \, dz$

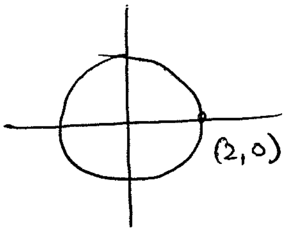
$= 2\pi i f(a)$; $f(z) = 3z+5$

$f(a) = 3z+5$

$= 2\pi i (5)$ $f(0) = 5$

$\underline{\underline{I = 10\pi i}}$

Q:- $\int_c \frac{z^2 - z + 1}{z - 1} \, dz$ $c: |z| = 2$



$= 2\pi i f(a)$

$= 2\pi i (1 - 1 + 1)$

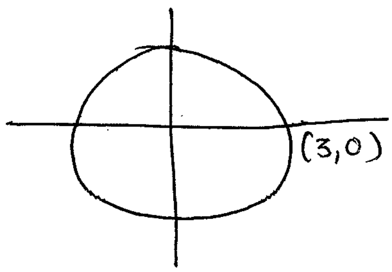
$\underline{I = 2\pi i}$

Q:- If $|z| = 0.5$
Then given ques. is analytic (using Cauchy)
$I = 0$.

Q:- $\displaystyle\int_c \frac{e^{2z}}{(z-1)(z-2)}$; $c: |z| = 3$

$$\cancel{\oint_c \frac{-e^{2}}{z-1} + \frac{e^{4}}{z-2}}$$

$$\int_c e^{2z}\left[\frac{-1}{z-1} + \frac{1}{z-2}\right]$$



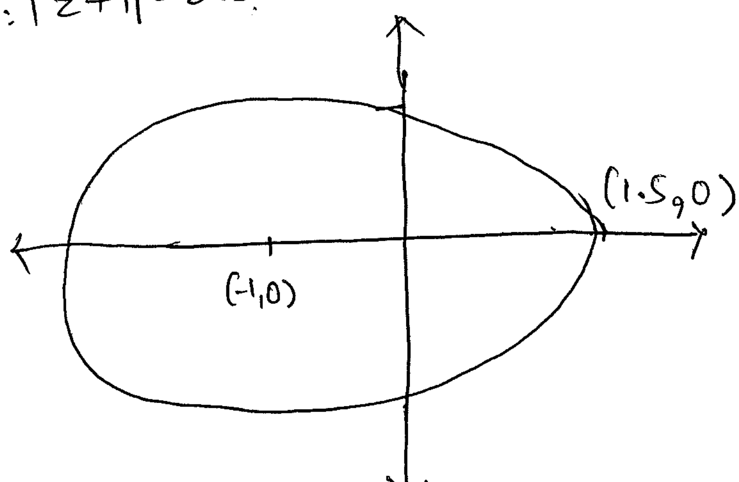
$$\int_c \frac{-e^{2z}}{(z-1)} + \frac{e^{2z}}{(z-2)}$$

$$= 2\pi i f(1) + 2\pi i f(2)$$

$$= 2\pi i(-e^{2}) + 2\pi i e^{4}$$

$$I = 2\pi i(e^{4} - e^{2})$$

Q:- $\displaystyle\int_c \frac{e^{2z}}{(z-1)(z-2)}\,dz$ $c: |z+1| = 2.5$



$$= \int_c e^{2z}\left[\frac{-1}{z-1} + \frac{1}{z-2}\right]$$

$$= \int_c \frac{-e^{2z}}{z-1} + \underbrace{\frac{e^{2z}}{z-2}}_{\text{Analytic}}$$

$$= \cancel{2\pi i f(1) + 2\pi i f(2) = 2\pi i(-e^{2}) + 2\pi i e^{4}}$$

$$= \int_c \frac{-e^{2z}}{z-1} + 0$$

$$= 2\pi i f(1)$$

$$I = 2\pi i (-e^2)$$

OR

$$\int_c \frac{\frac{e^{2z}}{z-2}}{z-1} \rightarrow f(z)$$

$$2\pi i f(1)$$

$$2\pi i \frac{e^2}{-1}$$

$$I = -2\pi i e^2$$

Q:- $\int_c \frac{1}{z^2 e^z} dz \quad ; \quad c : |z| = 1$

$$\int_c \frac{e^{-z}}{z^2} dz \quad f(z)$$

$$\int_c \frac{e^{-z}}{(z-0)^2} dz \qquad \left[\therefore \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(a) \right]$$

$$= \frac{2\pi i}{(2-1)!} f'(0)$$

$f(z) = e^{-z}$

$f'(z) = -e^{-z}$

$f'(0) = -1$

$$= 2\pi i \left[-e^{-z} \big|_{z=0} \right]$$

$$I = 2\pi i (-1) = -2\pi i$$

Q:- In the following integral the contour c encloses $2\pi i$ and $-2\pi i$ the value of integral

$$\frac{-1}{2\pi}\oint_c \frac{\sin z}{(z-2\pi i)^3}\,dz = \underline{\qquad}.$$

$$\frac{-1}{\cancel{2\pi}}\;\frac{\cancel{2\pi} i}{2!}\,f''(2\pi i)$$

$$f(z)=\sin z$$

$$f'(z)=\cos z$$

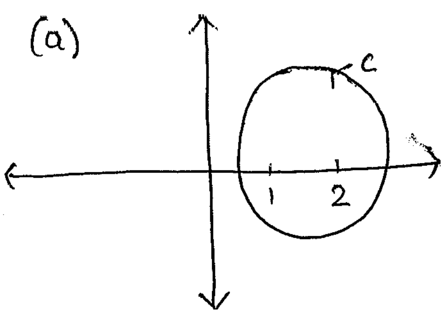
$$= \frac{+i\sin(2\pi i)}{2}$$

$$f''(z) = -\sin z$$

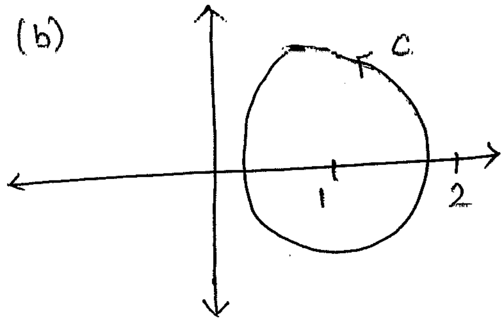
$$f''(2\pi i) = -\sin(2\pi i)$$

Q:- The value of $\oint_c \frac{3z-5}{(z-1)(z-2)}\,dz = 4\pi i$ along the closed path c then the correct path for c is __.

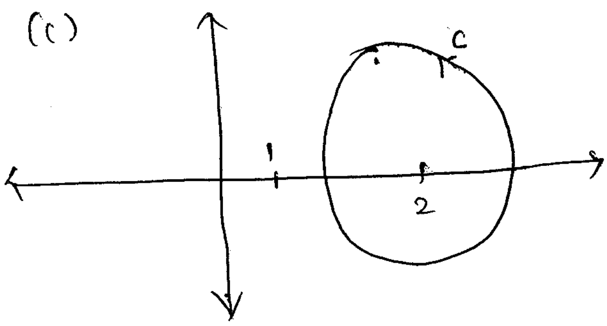
(a)



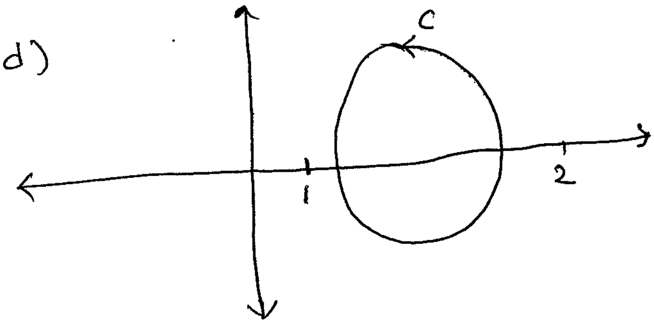
(b)



(c)



(d)

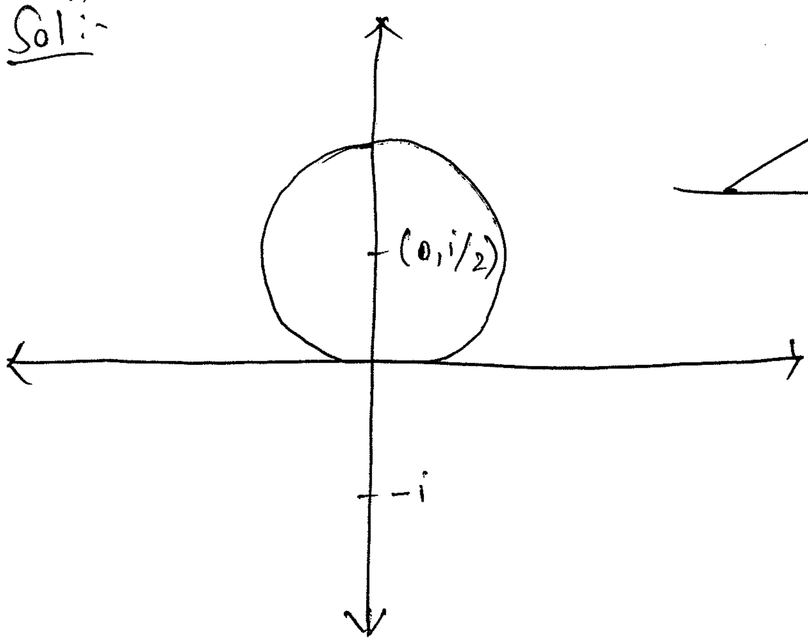


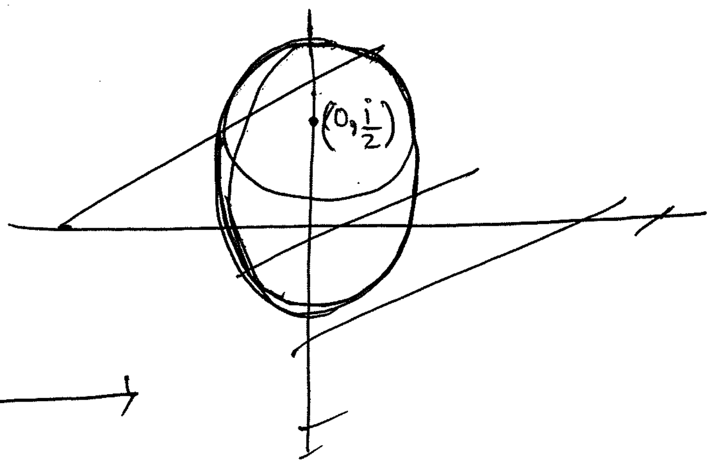
$$\int_c \frac{3z-5}{(z-1)(z-2)}\,dz = 4\pi i$$

$$\int_c \left(\frac{2}{z-1} + \frac{1}{z-2}\right) dz$$

Q :- $\int \frac{dz}{1+z^2}$ where c is $\left|z-\frac{i}{2}\right| = 1$

Sol^n :-





$1+z^2 = (z+i)(z-i)$

$$\int \frac{dz}{(z-i)(z+i)}$$

$$\int \frac{\frac{1}{z+i}}{z-i}\, dz$$

$= 2\pi i\, f(a)$

$= 2\pi i\, f(i)$

$= 2\pi i \frac{1}{2i}$

$= \pi$

Q:- $\int_c \frac{z^2+8}{0.5z-1.5i}\,dz$ $\quad c: x^2+y^2=16$

$r=4$

$2\int_c \frac{z^2+8}{z-3i}$

$2\ 2\pi i\, f(3i)$

$2\cdot 2\pi i(-1)$

$= \underline{-4\pi i}$

Q:- $\int_c \frac{\cos 2\pi z}{(2z-1)(z-3)}\,dz$ $\quad c: |z|=1$

$z=\frac{1}{2},\ z=3$

$\int_c \frac{\frac{\cos 2\pi z}{z-3}}{2(z-\frac{1}{2})}\,dz$

$f(z)=\frac{\cos 2\pi z}{z-3}$

$f(\frac{1}{2})=\frac{\cos 2\pi\cdot\frac{1}{2}}{\frac{1}{2}-3} = \frac{+2}{5}$

$= \frac{1}{\cancel{2}}\cancel{2}\pi i\, f(\frac{1}{2})$

$= \pi i\,\frac{2}{5} = \frac{2\pi i}{5}$

Q:- $\int_c \frac{1+f(z)}{z}\,dz$ where $f(z)=c_0+c_1z^{-1}$

$c: |z|=1$

$\int_c \frac{1+c_0+c_1z^{-1}}{z-0}\,dz = \int_c \frac{z+zc_0+c_1}{z^2}\,dz$

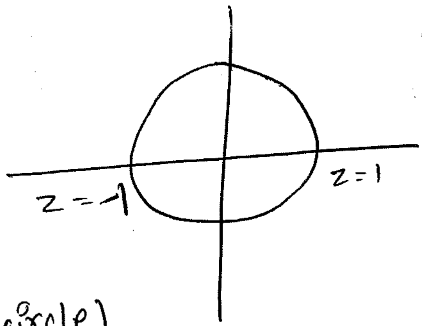
$$= \frac{2\pi i\, f'(0)}{(2-1)!}$$

$f(z) = z + z c_0 + c_1$

$f'(z) = 1 + c_0$

$$I = 2\pi i\,(1 + c_0)$$

Q:- $\int_c \frac{z+4}{z^2+2z+5}\, dz \qquad c: |z| = 1$



$z^2 + 2z + 1 = -4$

$(z+1)^2 = -4$

$z + 1 = \pm 2i$

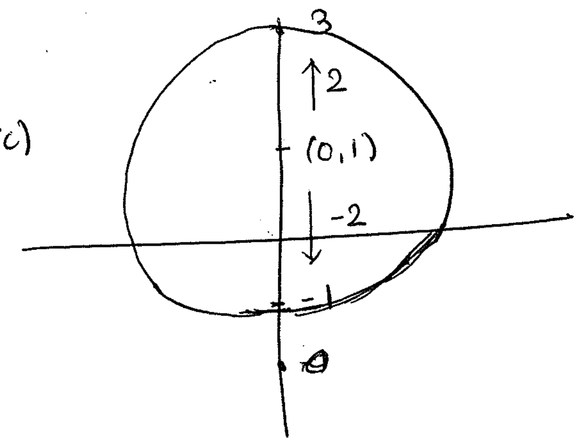
$z = -1 \pm 2i$ (both outside circle)

$|z| = \sqrt{1^2 + 2^2} = \sqrt{5} > 1$

not applied

It is analytic

Q:- $\int_c \frac{z^2-4}{z^2+4}\, dz \qquad c: |z-i| = 2$



$\int_c \frac{z^2-4}{(z+2i)(z-2i)}\, dz$

outside circle -analytic

inside (not analytic)

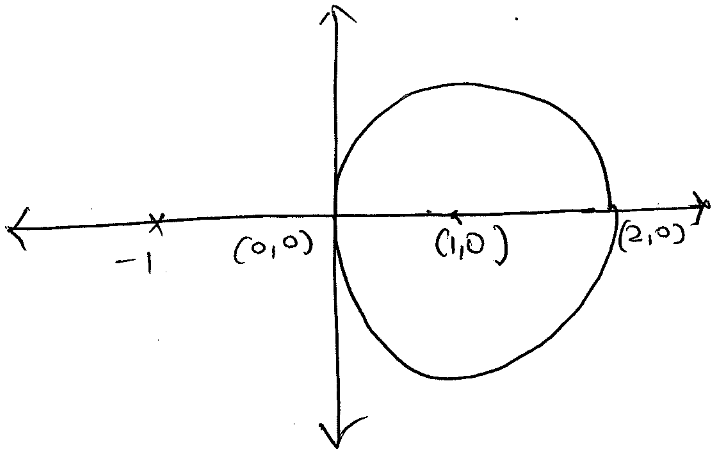
$\int_c \frac{\frac{z^2-4}{z+2i}}{z-2i}\, dz$

$2\pi i\, f(2i)$

$2\pi i\, \frac{-4-4}{2i+2i} = 2\pi \cancel{i}\, \frac{(-\cancel{8}^{\,2})}{4\cancel{i}} = -4\pi$

Q:- $\int_C \frac{z^2}{z^2-1}\,dz$ $\quad$ $c: |z-1| = 1$

$$= \int \frac{z^2}{(z-1)(z+1)}\,dz$$



$$= \int \frac{\frac{z^2}{z+1}}{z-1}\,dz$$

$$= 2\pi i f(1)$$

$$= \cancel{2}\pi i \,\frac{1}{\cancel{2}}$$

$$= \underline{\pi i}$$

Zeros and types of singular points

Zeros of analytic function

-If the function $f(z)$ is analytic at z_0 and $f(z_0) = 0$ then point z_0 is called zero of $f(z)$

Eg:-(1) $f(z) = (z-8)^{10}$

$f(z) = 0$ when $z = 8$

$z = 8$ is zero of $f(z)$

(2) $f(z) = (z-10)^{1/2}$

$f(z) = 0$ when $z = 10$

$z = 10$ is zero of $f(z)$

$$f'(z) = \frac{1}{2\sqrt{z-10}}$$

At $z = 10$ $f'(z)$ doesn't exists $\therefore f(z)$ is not analytic

* Order of zeros of analytic function

- If $f(z)$ is analytic function at z_0 and $f(z_0)=0$

$f(z_0)=0$

$f'(z_0)=0$

$\vdots$

$f^{m-1}(z_0)=0$ but $f^{m}(z_0)\neq 0$ them m is called order of zero of AF.

Eg:- $f(z)=(z-4)^2$

$f(z)=0$ at $z=4$

$f'(z)=2(z-4)=0$

$f''(z)=2\neq 0$

$\therefore m=2$ $z=4$ is zero of order 2

Q:- $f(z)=\sin z$

$f(z)=0$, $\sin z=0$

$z=n\pi$

$f'(z)=\cos z\neq 0$ $z=n\pi$

$m=1$

$z=n\pi$ is zero of order 1

simple zero

Singular point

If a function $f(z)$ is not defined or not differentiable or not analytic at any point z_0 then z_0 is called singular point of $f(z)$.

Q:- $f(z) = \dfrac{z+4}{z-11}$

$z-11=0 \Rightarrow z=11$

$f(z)$ is not defined at $z=11$

$z=11$ is singular point

Q:- $f(z) = \sqrt{z-4}$

$f'(z) = \dfrac{1}{2\sqrt{z-4}}$

$f'(z)$ does not exists at $z=4$

$\therefore z=4$ is singular point

* Taylor Series

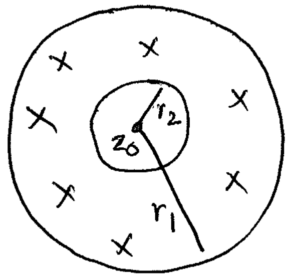
- Let $f(z)$ is analytic function inside a circle with center a, for any point z inside the circle $f(z)$ is

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \ldots\ldots$$

Laurent Series

If $f(z)$ is analytic function at every point within a ring shaped region bounded by two concentric circles c_1, c_2 having center at z_0 with radius r_1 and r_2 $(r_2 < r_1)$ then for every point z it can be expressed as power series i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$



where $$a_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz$$

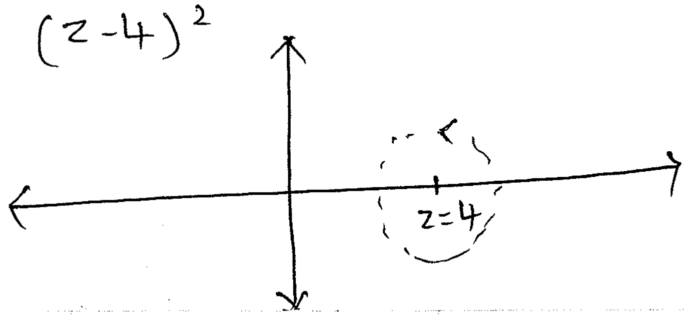
$$b_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{-n+1}} dz$$

* Types of singular point

1. Isolated singular point

- If z_0 is singular point of $f(z)$ and $f(z)$ is analytic at every point except z_0 in atleast one neighbourhood of point z_0 then singular point z_0 is called isolated singular point.

Eg:- $$\frac{(z+8)^{10}}{(z-4)^2} = f(z)$$



(2) $f(z) = \frac{1}{\sin z}$

$\sin z = 0 \Rightarrow z = n\pi$

Then neighbourhood of $z = n\pi$ will be defined at atleast one point

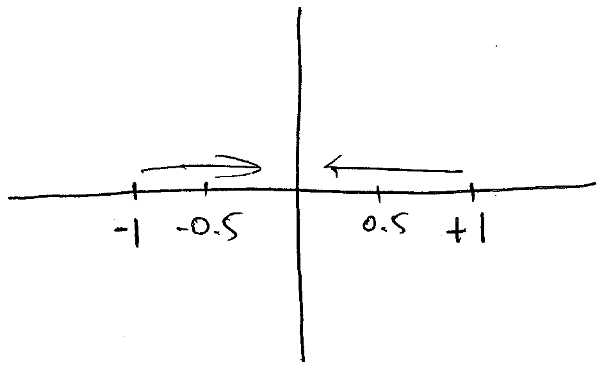
So $n\pi$ is isolated S.P.

(3) $f(z) = \dfrac{1}{\sin(\pi/z)}$

$\sin \pi/z = 0$

$\dfrac{\pi}{z} = n\pi$

$z = 1/n$



$n = \pm 1 \Rightarrow z = \pm 1$

$n = \pm 2 \Rightarrow z \rightarrow \pm 0.5$

No region is there in bet^n -1 to 1 in which our $f(z)$ is defined.

$\therefore$ Here there is no isolated singular point.

* Types of isolated singular point

Acc. to Laurent series

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{Analytic part}} + \underbrace{\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}}_{\text{Principle part}}$$

① Removable singular point

If principle part of Laurent series expansion of $f(z)$ about $z=z_0$ doesn't exists then the singular point z_0 is called removable singular point.

Eg:- (1) $f(z) = \frac{\sin z}{z}$

At $z=0$, singular point

About $z=0$, Taylor series

$$f(z) = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

$z=0$ is removable singular point

② Poles (of order n^{th})

If principle part of Laurent series expansion of $f(z)$ about $z=z_0$ contains finite no. of terms [say m terms] then singular point z_0 is called poles of order m.

Eg:- $\frac{e^z}{z^2} = f(z)$

$z=0$ is I.S.P.

$$f(z) = \frac{1}{z^2}\left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right]$$

$$= \underbrace{z^{-2} + z^{-1}}_{2} + \frac{1}{2!} + \frac{z}{3!} + \cdots$$ $z=0$, is pole of order 2

③ Essential singular point

- If the principle part of Laurent's series expansion of $f(z)$ about $z=z_0$ contains infinite no. of terms then singular point z_0 is called ~~removable~~ essential singular point.

Eg:- $\sin\left(\frac{3}{z-4}\right)$

$z=4$ is singular point

$$f(z) = \frac{3}{z-4} - \frac{\left(\frac{3}{z-4}\right)^3}{3!} + \cdots\cdots$$

$$= 3(z-4)^{-1} - \frac{27}{3!}(z-4)^{-3} + \cdots\cdots\cdots (z-4)^{-\infty}$$

$\therefore$ $z=4$ is essential singular point

* Residue Theory

- Residues of $f(z)$: If z_0 is an isolated s.P. of $f(z)$ then co-efficient of $\frac{1}{(z-z_0)}$ in Laurent's series expansion of $f(z)$ about $z=z_0$ is called residue of $f(z)$ at $z=z_0$

Symbollically :- $\underset{z=z_0}{\text{Res}}\, f(z)$

Co-efficient of $(z-z_0)^{-1}$ in Laurent's series expansion

$$b_1 = \frac{1}{2\pi i}\oint f(z)\,dz$$

* Methods to find Residue:-

① Removable singular point

If z_0 is removable singular point of $f(z)$ then residue of $f(z)$: $\underset{z=z_0}{\text{Re}}\, f(z) = 0 = b_1$

② Poles

i) If $f(z)$ is $\frac{p(z)}{q(z)}$ has a simple pole at $z=z_0$ then residue of $f(z)$ is given by

$$\underset{z=z_0}{\text{Res}}\, f(z) = \lim_{z\to z_0}\left[(z-z_0)f(z)\right]$$

ii) If $f(z) = \frac{\phi(z)}{\psi(z)}$ (not polynomials) has simple pole at $z=z_0$

$$\underset{z=z_0}{\text{Res}}\, f(z) = \frac{\phi(z_0)}{\psi'(z_0)} \quad [\psi'(z_0)\neq 0]$$

iii) If $f(z) = \frac{\phi(z)}{z-z_0}$ has simple pole at $z=z_0$

$$\underset{z=z_0}{\text{Res}}\, f(z) = \phi(z_0)$$

iv) If $f(z)$ has poles at z_0 of order k then

$$\underset{z=z_0}{\text{Res}}\, f(z) = \frac{1}{(k-1)!}\lim_{z\to z_0}\frac{d^{k-1}}{dz^{k-1}}\left((z-z_0)^k f(z)\right)$$

③ Essential point

If z_0 is an essential s.P. of $f(z)$ then

then expand $f(z)$ as Laurent's series expansion and

find co-efficient of $\frac{1}{z-z_0}$ which is known as

$$\operatorname*{Res}_{z=z_0} f(z)$$

* Algorithm to find Residue

(1) $f(z) = \frac{N_r}{D_r}$

(2) Find S.P.

zeros of D_r function (say m order)

(3) Put all S.P. in N_r

n: order of zeros of N_r

- $N_r = 0$
  - Poles $[m > n]$ → Again 4 cases
  - RSP $[m \leq n]$ → by Laurent's series
- $N_r \neq 0$ → Pole → Again 4 cases

Q:- $f(z) = \frac{z}{z^2+16}$

① $f(z) = \frac{z}{z^2+16}$

② $z^2+16 = 0$

$z^2 = -16$

$z = \pm 4i$ are singular point

$f'(z) = 2z \neq 0 \Rightarrow m = 1$ (order)

③ Put $z = \pm 4i$ in Nr

Nr $\neq 0$

$z = \pm 4i$ is pole of order $m=1$

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} [(z-z_0) f(z)]$$

$$\operatorname*{Res}_{z=4i} f(z) = \lim_{z \to 4i} \cancel{(z-4i)} \frac{z}{\cancel{(z-4i)}(z+4i)}$$

$$= \frac{4i}{8i} = \frac{1}{2}$$

$$\operatorname*{Res}_{z=-4i} f(z) = \lim_{z \to -4i} \cancel{(z+4i)} \cdot \frac{z}{(z-4i)\cancel{(z+4i)}}$$

$$= \frac{-4i}{-8i} = \frac{1}{2}$$

Q:- $f(z) = \dfrac{\cos z}{z-\pi}$

① $f(z) = \dfrac{\cos z}{z-\pi}$

② $z - \pi = 0$

$z = \pi$ are singular point
is zero of Dr

$\cancel{f'(z)} = \cancel{\dfrac{(z-\pi)(-\sin z) - \cos z}{(z-\pi)^2}}$

$\therefore m = 1$

$f'(z) = 1 \neq 0$

ii) Put $z = \pi$ in Nr

$\cos \pi = -1 \neq 0$

$N_r \neq 0$

iii) $z = \pi$ is pole

$$\operatorname*{Res}_{z \to z_0} f(z) = \lim_{z \to z_0} [(z - z_0) f(z)]$$

$$= \lim_{z \to \pi} \cancel{(z-\pi)} \frac{\cos z}{\cancel{z-\pi}}$$

$$= \cos \pi = \underline{-1}$$

Q:- $f(z) = \cot z$

$$f(z) = \frac{\cos z}{\sin z}$$

$\sin z = 0$

$z = \pm n\pi$ is ~~pole~~ zero of Dr

Order $= 1$

Put $z = \pm n\pi$ in Nr

$\cos n\pi = -1 \neq 0$

$N_r \neq 0$

$z = n\pi$ is pole

Zeros of numerator

$\cos z = 0$

$z = (2n+1)\pi/2$

So $N_r = 1$

R.S.P.

$$\cancel{\underset{z \to z_0}{Res} f(z) = \lim_{z \to n\pi} (z - n\pi) \cdot \frac{\cos z}{\sin z}}$$

$\therefore$ Residue $= 0$

Q:- $f(z) = \dfrac{\sin z}{z - 2\pi}$

$z - 2\pi = 0$

$z = 2\pi$

So RSP then

Residue $= 0$

$m = 1$ (order)

$N_r = 0$ for $z_0 = 2\pi$

Order $= 1$

zeros of $N_r = \sin z |_{z=2\pi} = 0$

$z = \pm n\pi$

Q:- $\frac{\sin z}{z^2} = f(z)$

① $z=0$ is zero of Dr

$f(z) = z^2$, $f(0) = 0$

$f'(z) = 2z$, $f'(0) = 0$

$f''(z) = 2$, $f''(0) \neq 0$

$m=2 \rightarrow$ order of Dr

Put $z=0$ in Nr

$\sin 0 = 0$

$Nr = 0$

$n = 1$

$m > n$

$\therefore$ Poles of order $(m-n) = 2-1 = 1$

$$\underset{z \to 0}{\text{Res}}\, f(z) = \lim_{z \to 0} (z-0)\frac{\sin z}{z^2}$$

$$= 1$$

Q:- $f(z) = \frac{1-\cos z}{z}$

① $z=0$ is zero of Dr

$f(z) = z$

$f'(z) = 1 \neq 0$ $m=1 \rightarrow$ order of Dr

Put $z = 0$ in Nr

$Nr = 1 - 1 = 0$

$f(z) = 1 - \cos z$

$f'(z) = +\sin z = 0$

$f''(z) = +\cos z \neq 0$

$n = 2$

$m < n$

R.S.P.

Residue $= 0$

* Cauchy's Residue Theorem

- If a function $f(z)$ is analytic at every point within and on simple closed curve c except at finite number of isolated singular point

i.e. $z_1, z_2, \ldots\ldots z_n$ within c then

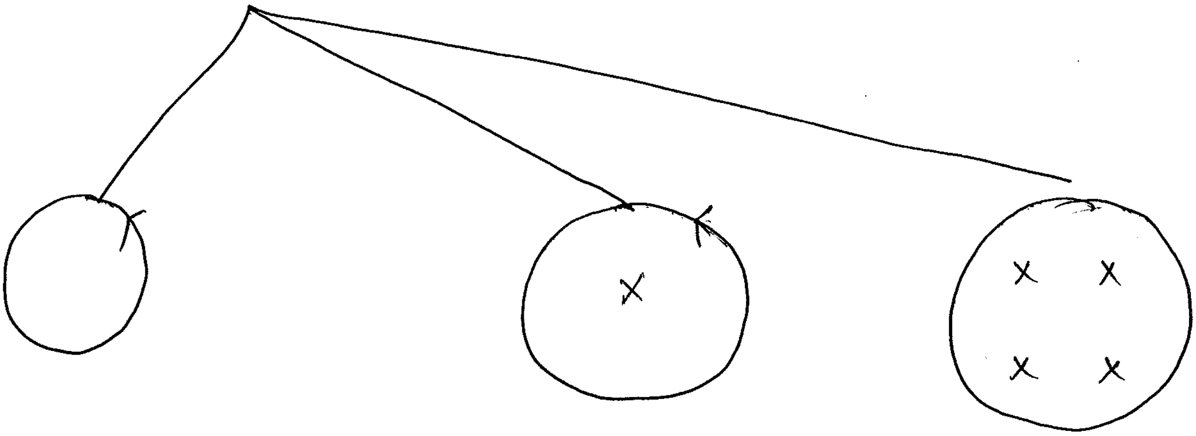
$$\oint f(z)\,dz = 2\pi i \left[\sum_{j=1}^{n} R_j \right]$$

where $R_j = \underset{z = z_j}{\text{Res}}$ of $f(z)$ where $j = 1, \ldots\ldots n$

Algorithm of CRT

① $\oint f(z)\,dz$

② Find S.P.



By CT,
$\oint f(z)dz = 0$

By CIF
$\oint \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$

By CRT
Go to step ③

③ Identify types of S.P.

④ Find residue at that S.P.

⑤ $\oint f(z)\,dz = 2\pi i$ (sum of Residues)

Q:- $\int \frac{z}{(z-1)(z-2)^2}\,dz$ $\quad c: |z-2| = \frac{1}{2}$. Solve using Cauchy Residue theorem.

Soln:- singular points

$z=1,\ z=2$ but in curve there is only one S.P.

$z=2$

$z=2$ is zeros of D_r of $m=2$

$N_r \neq 0$

$z=2$ pole order $m=2$

$$\underset{z=2}{\text{Res}}\, f(z) = \frac{1}{(2-1)!} \lim_{z\to 2} \frac{d}{dz} \cancel{(z-2)^2} \cdot \frac{z}{(z-1)\cancel{(z-2)^2}}$$

$$= \frac{1}{1} \lim_{z\to 2} \frac{(z-1(1)-z(1))}{(z-1)^2}$$

$$= -1$$

$$\oint f(z)\, dz = 2\pi i(-1) = -2\pi i$$