Exercise 8.5

Answer 1E.

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(a)

Here, $f(x)$ denote the probability density function for the life time of a manufactures highest quality car tire. The variable x is measured in miles.

The probability density function for the life time of a manufactures highest quality car tire $\mu = 40{,}000$ and $\sigma = 30{,}000$ miles is defined as,

$\int_{30,000}^{40,000} f(x)\,dx$.

Hence, the integral $\int_{30,000}^{40,000} f(x)\,dx$ is the probability density function for the life time of a manufactures highest quality car tire.

(b)

Here, the probability density function for the life time of a manufactures highest quality car tire which have life time more than or equal to 25,000 miles is denoted as,

$\int_{25,000}^{\infty} f(x)\,dx$.

Hence, the integral $\int_{25,000}^{\infty} f(x)\,dx$ is the probability density function for the life time of a manufactures highest quality car tire which have a life time more than or equal to 25,000 miles.

Answer 2E.

(A) The probability that we drive to school in less than 15 minutes is $\int_0^{15} f(t)\,dt$

(B) The probability that we drive to school in more than half an hour is $\int_{30}^{\infty} f(t)\,dt$

Answer 3E.

If $\int_{-\infty}^{\infty} f(x)\,dx = 1$, then $f(x)$ is a probability density function.

Keeping this in view, we are given that $f(x) = 30x^2(1-x)^2$ for $0 \le x \le 1$ and
$f(x) = 0$ for all other values of x.

Consider $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$ keeping the interval in view.

$$= 0 + \int_0^1 30x^2(1-x)^2\,dx + 0$$

$$= 30\int_0^1 x^2\left(1+x^2-2x\right)dx$$

$$= 30\int_0^1 \left(x^2+x^4-2x^3\right)dx$$

$$= 30\left[\frac{x^3}{3}+\frac{x^5}{5}-2\frac{x^4}{4}\right]_0^1$$

$$= 30\left[\frac{1}{3}+\frac{1}{5}-\frac{1}{2}\right]$$

$$= 30\left[\frac{10+6-15}{30}\right]$$

$$= 1$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$\therefore$ f is a probability density function

(b)

$$p\left(X \le \frac{1}{3}\right) = \int_{-\infty}^{1/3} f(x)dx$$

$$= \int_{-\infty}^{0} f(x)dx + \int_{0}^{1/3} f(x)dx$$

$$= \int_0^{1/3} 30x^2(1-x)^2\,dx$$

$$= 30\int_0^{1/3}\left(x^2+x^4-2x^3\right)dx$$

$$= 30\left[\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^4}{2}\right]_0^{1/3}$$

$$= 30\left[\frac{1}{81}+\frac{1}{5}\cdot\frac{1}{35}-\frac{1}{2}\cdot\frac{1}{34}\right]$$

$$= \frac{17}{81}$$

Therefore $p\left(X \le \frac{1}{3}\right) = \frac{17}{81}$

≈ 0.2098

Answer 4E.

Consider the function,

$$f(x)=\begin{cases} xe^{-x}, & x\ge 0 \\ 0, & x<0 \end{cases}$$

(a)

The objective is to verify f is a probability density function or not.

To verify that *f* is probability density function, it is enough to show that $\int_{-\infty}^{\infty} f(x)dx=1$

consider $\int_{-\infty}^{\infty} f(x)dx$

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{0} f(x)dx+\int_{0}^{\infty} f(x)dx \\ &= 0+\int_{0}^{\infty} xe^{-x}dx \\ &= \lim_{t\to\infty}\int_{0}^{t} xe^{-x}dx\end{aligned}$$

Now find the integration by using the following formula.

$$\int udv = uv-\int vdu$$

$$\text{Here } u=x,\ dv=e^{-x}dx$$
$$du=dx,\ v=-e^{-x}$$

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \lim_{t\to\infty}\left[x\left(-e^{-x}\right)-\int\left(-e^{-x}\right)dx\right] \\ &= \lim_{t\to\infty}\left[x\left(-e^{-x}\right)-\left(e^{-x}\right)\right]_0^t \\ &= \lim_{t\to\infty}\left[-te^{-t}-e^{-t}+1\right]\end{aligned}$$

Here, if $t\to\infty$ then $e^{-t}\to 0$

So

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \lim_{t\to\infty}\left(-te^{-t}\right)-\lim_{t\to\infty}\left(e^{-t}\right)+\lim_{t\to\infty}(1) \\ &= 0-0+1 \\ &= 1\end{aligned}$$

Therefore, the value of the integral is,

$$\int_{-\infty}^{\infty} f(x)dx=\boxed{1}$$

Hence, the function $f(x)$ is the probability density function.

(b)

The objective is to find the value of $P(1 \le X \le 2)$

$$P(1 \le X \le 2) = \int_1^2 f(x)\,dx$$
$$= \int_1^2 xe^{-x}dx$$
$$= \left[x(-e^{-x}) - (e^{-x})\right]_1^2$$
$$= \left[(-2e^{-2} - e^{-2}) - (-e^{-1} - e^{-1})\right]$$
$$= \left(-\frac{2}{e^2} - \frac{1}{e^2}\right) - \left(-\frac{1}{e} - \frac{1}{e}\right)$$
$$= -\frac{3}{e^2} + \frac{2}{e}$$
$$= \frac{2}{e} - \frac{3}{e^2}$$

Therefore, the value is,

$$P(1 \le X \le 2) = \boxed{\frac{2}{e} - \frac{3}{e^2}}.$$

Answer 5E.

Given that $f(x) = \dfrac{c}{1+x^2}$

(a)

We have to find that for what values of c is f a probability density function.

A function f is a probability density function if the following condition is satisfied.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Therefore

$$\int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx = 1$$
$$\int_{-\infty}^{\infty} \frac{c}{1+x^2}dx = 1$$
$$c\left(\tan^{-1} x\Big|_{-\infty}^{\infty}\right) = 1$$
$$c\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 1$$
$$\boxed{c = \frac{1}{\pi}}$$

b)

Using the above value $c = \dfrac{1}{\pi}$, we have to find $P(-1 < X < 1)$.

This is done by evaluating the probability density function from -1 to 1:

$$= \int_{-1}^{1} f(x)\,dx$$
$$== \int_{-1}^{1} \frac{c}{1+x^2}dx$$
$$= \int_{-1}^{1} \frac{1/\pi}{1+x^2}dx$$
$$= \frac{1}{\pi}\left(\tan^{-1}(x)\Big|_{-1}^{1}\right)$$
$$= \frac{1}{\pi}\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$
$$= \boxed{\frac{1}{2}}$$

Answer 6E.

A function f is called a probability density function if it satisfies:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

The probability of a function f is defined as shown below:

$$P(a < X < b) = \int_{a}^{b} f(x)\,dx$$

The mean of a function is defined as shown below:

$$\mu = \int_{-\infty}^{\infty} xf(x)\,dx$$

Consider the function shown below:

$$f(x) = \begin{cases} 0, & x < 0 \\ k\left(3x - x^2\right), & 0 \le x \le 3 \\ 0, & x > 3 \end{cases}$$

a.

Evaluate the integral shown below:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx \\ &= \int_{-\infty}^{0} 0dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} 0dx \\ &= 0 + \int_{0}^{3} k\left(3x - x^2\right)dx + 0 \\ &= k\int_{0}^{3} 3xdx - k\int_{0}^{3} x^2dx \end{aligned}$$

Evaluate the integral further:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \frac{3k}{2}x^2\Big|_0^3 - \frac{k}{3}x^3\Big|_0^3 \\ &= \frac{3k}{2}\left(3^2 - 0^2\right) - \frac{k}{3}\left(3^3 - 0^3\right) \\ &= \frac{27k}{2} - \frac{27k}{3} \\ &= \frac{27k}{6} \end{aligned}$$

Determine the value of k:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\frac{27k}{6} = 1$$

$$k = \frac{6}{27}$$

Hence, the value of k is $\boxed{k = \frac{6}{27}}$.

b.

Substitute the value of *k* in the function:

$$f(x)=\begin{cases}0, & x<0\\ \frac{6}{27}\left(3x-x^2\right), & 0\le x\le 3\\ 0, & x>3\end{cases}$$

Evaluate the probability of the function as shown below:

$$\begin{aligned}P(X>1)&=\int_1^{\infty} f(x)\,dx\\ &=\frac{6}{27}\int_1^3\left(3x-x^2\right)dx+\frac{6}{27}\int_3^{\infty}0\,dx\\ &=\frac{6}{27}\left[3\frac{x^2}{2}-\frac{x^3}{3}\right]_1^3+0\\ &=\frac{6}{27}\left[\left(\frac{27}{2}-\frac{27}{3}\right)-\left(\frac{3}{2}-\frac{1}{3}\right)\right]\end{aligned}$$

Continue further to obtain the value:

$$\begin{aligned}P(X>1)&=\frac{6}{27}\left[\frac{27}{2}-\frac{27}{3}-\frac{3}{2}+\frac{1}{3}\right]\\ &=\frac{6}{27}\left[\frac{24}{2}-9+\frac{1}{3}\right]\\ &=\frac{20}{27}\end{aligned}$$

Hence, the value of the probability is $\boxed{P(X>1)=\frac{20}{27}}$.

c.

Use the formula of mean to obtain the required value:

$$\begin{aligned}\mu&=\int_{-\infty}^{\infty} xf(x)\,dx\\ &=\int_{-\infty}^{0} xf(x)\,dx+\int_0^3 x\frac{6}{27}\left(3x-x^2\right)dx+\int_3^{\infty} xf(x)\,dx\\ &=0+\frac{6}{27}\int_0^3\left(3x^2-x^3\right)dx+0\\ &=\frac{6}{27}\left(3\times\frac{x^3}{3}-\frac{x^4}{4}\right)_0^3\end{aligned}$$

Continue further to evaluate the integral:

$$\begin{aligned}\mu&=\frac{6}{27}\left(x^3-\frac{x^4}{4}\right)_0^3\\ &=\frac{6}{27}\left[\left(3^3-\frac{3^4}{4}\right)-\left(0^3-\frac{0^4}{4}\right)\right]\\ &=\frac{6}{27}\left(27-\frac{81}{4}\right)\\ &=\frac{3}{2}\end{aligned}$$

So, the mean is 1.5.

Hence, the final value of the mean is $\boxed{\mu=1.5}$.

Answer 7E.

(A) We have the function $f(x)=\begin{cases}0.1 & if\ \ 0\le x\le 10\\ 0 & if\ \ x<0\ or\ x>10\end{cases}$

We see that in the interval $0\le x\le 10$, $f(x)\ge 0$

And for $x < 0$ or $x > 10$, $f(x) = 0$

So $f(x)\ge 0$ for all x

Now we have to check $\int_{-\infty}^{\infty} f(x)dx=1$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)dx=\int_0^{10}0.1dx$$
$$=0.1[x]_0^{10}$$
$$=1$$

So $f(x)$ is a probability density function

(B) The value of mean will be the mid point of the interval, which is 5

Now we calculate the mean

$$\mu=\int_{-\infty}^{\infty} x\,f(x)\,dx=\int_0^{10}0.1x\,dx$$
$$=0.1\left[\frac{x^2}{2}\right]_0^{10}$$
$$\Rightarrow \mu=0.1[50]$$
$$=5$$
$$\boxed{\mu=5}$$

Answer 8E.

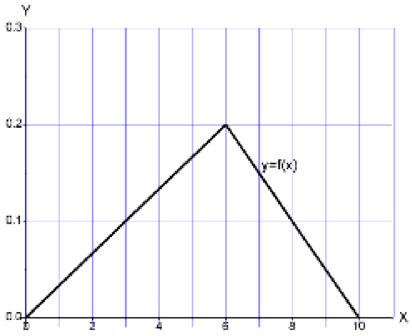

Fig.1

(A) Here we see from the graph that $f(x)\ge 0$ for all x

Now we check $\int_{-\infty}^{\infty} f(x)dx=1$

$$\int_{-\infty}^{\infty} f(x)\,dx=\int_0^{10} f(x)dx\ = \text{Area of the region from } x=0 \text{ to } x=10$$

This is the area of a triangle

So $$\int_{-\infty}^{\infty} f(x)dx\ =\frac{1}{2}\times 10\times 0.2$$
$$=1$$

So $f(x)$ is a probability density function

(B) (1) $$P(x<3)=\int_0^3 f(x)dx$$

$$= \text{Area of the triangle from } x=0 \text{ to } x=3$$
$$=\frac{1}{2}\times 3\times 0.1$$
$$\boxed{P(x<3)=0.15}$$

(2) We have to find $P(3 \le x \le 8)$

$$P(3 \le x \le 8) = \int_3^8 f(x)\,dx$$
$$= \int_0^8 f(x)\,dx - \int_0^3 f(x)\,dx$$
$$= \int_0^{10} f(x)\,dx - \int_8^{10} f(x)\,dx - \int_0^3 f(x)\,dx$$
$$= 1 - \frac{1}{2} \times 0.1 \times 2 - 0.15$$
$$= 1 - 0.25$$
$$= 0.75$$
$$\boxed{P(3 \le x \le 8) = 0.75}$$

(C) Mean $\int_{-\infty}^{\infty} x\ f(x)\,dx = \int_0^{10} x\ f(x)\,dx$

We divide the interval [0, 10] in to 10 subintervals

Then $\Delta x = \frac{10-0}{10}$
$= 1$
$\Rightarrow \Delta x = 1$

Then by Simpson's rule we can approximate the value of mean

$$\mu = \int\ x\,f(x)\,dx \approx \frac{\Delta x}{3}\left[x_o f(x_o) + 4x_1 f(x_1) + 2f(x_2)\,x_2 + 4x_3 f(x_3) + \ldots\right.$$
$$\left.\ldots\ldots\ldots + 4x_9 f(x_9) + x_{10} f(x_{10})\right]$$
$$\approx \frac{1}{3}\left[0 + 4.(1).(0) + 2.(2)(0.04) + 4.(3).(0.08) + 2.(4).(0.12) + 4.(5).(0.16)\right.$$
$$\left.+2.(6).(0.2) + 4.(7).(0.15) + 2.(8).(0.1) + 4.(9).(0.05) + 10.(0)\right]$$
$$\approx \frac{1}{3}\left[0.16 + 0.96 + 0.96 + 3.2 + 2.4 + 4.2 + 1.6 + 1.8\right]$$

Or $\boxed{\mu \approx 5.09}$

Or $\boxed{\mu = 5}$

Answer 9E.

We have the probability density function
$$f(t) = \begin{cases} 0 & \text{if } \ t < 0 \\ 0.2e^{-t/5} & \text{if } t \ge 0 \end{cases}$$

If m is the median waiting time for a phone call to the company then
$$\int_m^{\infty} f(t)\,dt = \frac{1}{2}$$
$$\Rightarrow \int_m^{\infty} 0.2e^{-t/5}dt = \frac{1}{2}$$
$$\Rightarrow \lim_{x\to\infty} \int_m^x 0.2e^{-t/5}dt = \frac{1}{2}$$
$$\Rightarrow \lim_{x\to\infty} \left[-5 \times 0.2 \times e^{-t/5}\right]_m^x = \frac{1}{2}$$

Therefore
$$\Rightarrow \lim_{x\to\infty} \left[-e^{-x/5} + e^{-m/5}\right] = \frac{1}{2}$$
$$\Rightarrow e^{-m/5} - \lim_{x\to\infty} e^{-x/5} = \frac{1}{2}$$
$$\Rightarrow e^{-m/5} = 1/2 \qquad \left[\lim_{x\to\infty} e^{-x/5} = 0\right]$$
$$\Rightarrow \frac{-m}{5} = \ln(1/2)$$
$$\Rightarrow \frac{-m}{5} = -\ln 2 \Rightarrow \boxed{m = 5\ln 2}\ \text{min}$$
$$\Rightarrow \boxed{m \approx 3.47}\ \text{min}$$

Answer 10E.

(A) We have exponential density function

$$f(t)=\begin{cases}0 & if \quad t<0\\ \mu^{-1}e^{-t/\mu} & if \quad t\ge 0\end{cases}$$

We have $\mu=1000$

Then $$f(t)=\begin{cases}0 & if \quad t<0\\ (0.001)e^{-0.001t} & if \quad t\ge 0\end{cases}$$

(1) We have to find $P(t\le 200)$

$$\Rightarrow P(t<200)=\int_0^{200}(0.001)e^{-0.001t}dt$$
$$=\left[\frac{0.001}{-0.001}e^{-0.001t}\right]_0^{200}$$
$$=\left[-e^{(0.001\times 200)}+e^{0}\right]$$
$$=1-e^{-0.2}\approx 0.18$$

$\boxed{P(t\le 200)\approx 0.18}$

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(2) We have to find $P(t>800)$

$$P(t>800)=\int_{800}^{\infty}(0.001)e^{-0.001t}dt$$
$$=\lim_{x\to\infty}\left[-e^{-0.001t}\right]_{800}^{x}$$
$$=\lim_{x\to\infty}\left[-e^{-(0.001)x}+e^{-0.8}\right]$$
$$=e^{-0.8} \qquad \left[\lim_{x\to\infty}e^{-(0.001)x}=0\right]$$

$\boxed{P(t>800)\approx 0.45}$

(B) If median is m, then $\int_m^{\infty}f(t)\,dt=\frac{1}{2}$

$$\Rightarrow \int_m^{\infty}(0.001)e^{-0.001t}dt=\frac{1}{2}$$
$$\Rightarrow \lim_{x\to\infty}\int_m^{x}(0.001)e^{-0.001t}dt=\frac{1}{2}$$
$$\Rightarrow \lim_{x\to\infty}\left[-e^{-0.001t}\right]_m^{x}=\frac{1}{2}$$

Therefore

$$\Rightarrow \lim_{x\to\infty}\left[e^{-0.001m}-e^{-0.001x}\right]=\frac{1}{2}$$
$$\Rightarrow e^{-0.001m}=\frac{1}{2}$$
$$\Rightarrow -0.001m=\ln\left(\frac{1}{2}\right)$$
$$\Rightarrow -0.001m=-\ln 2$$
$$\Rightarrow 0.001m=\ln 2$$
$$\Rightarrow m=\frac{1}{0.001}\ln 2$$
$$\Rightarrow \boxed{m=1000\ln 2}\approx 693.15 \text{ hours}$$

Answer 11E.

(A) We have $\mu=2.5$minutes

Thus probability density function as

$$f(t)=\begin{cases}0 & if \quad t<0\\ \frac{1}{2.5}e^{-t/2.5} & if \quad t\ge 0\end{cases}$$

Then probability that a customer has to wait for more than 4 minutes is

$$P(t>4)=\int_4^\infty \frac{1}{2.5}e^{-t/2.5}dt$$
$$=\lim_{x\to\infty}\int_4^x \frac{1}{2.5}e^{-t/2.5}dt$$
$$=\lim_{x\to\infty}\left[-e^{-t/2.5}\right]_4^x$$
$$=\lim_{x\to\infty}\left[-e^{-x/2.5}+e^{-4/2.5}\right]$$
$$=e^{-4/2.5}\boxed{\approx 0.20}$$

(B) Probability that a customer is served with in first 2 minutes is

$$P(t\le 2)=\int_0^2 \frac{1}{2.5}e^{-t/2.5}dt$$
$$=\left[-e^{-t/2.5}\right]_0^2$$
$$=\left[-e^{-2/2.5}+e^{o}\right]=1-e^{-0.8}$$
$$\approx 0.55$$
$$\boxed{P(t\le 2)\approx 0.55}$$

(C) Let the customer has been given free hamburgers after ‘a’ minutes

Then $\int_a^\infty \frac{1}{2.5}e^{-t/2.5}dt=0.02$

$$\Rightarrow \lim_{x\to\infty}\left[-e^{-t/2.5}\right]_a^x=0.02$$
$$\Rightarrow \lim_{x\to\infty}\left[e^{-x/2.5}+e^{-a/2.5}\right]=0.02$$
$$\Rightarrow e^{-a/2.5}=0.02$$
$$\Rightarrow \frac{-a}{2.5}=\ln(0.02)$$
$$\Rightarrow a=-2.6\times\ln(0.02)$$
$$\Rightarrow a\approx 9.78\text{ minutes}=\boxed{10\text{ minutes}}$$

Thus the advertisement should say that if the customer is not served with in 10 minutes, he get a free hamburger.

Answer 12E.

(A) We have $\mu=69$ and $\sigma=2.8$

Then probability density function is $f(x)=\frac{1}{2.8\sqrt{2\pi}}e^{-(x-69)^2/2(2.8)^2}$

And the probability that an adult male chosen at random is between 65 inches and 73 inches tall is

$$P(65\le x\le 73)=\int_{65}^{73}\frac{1}{2.8\sqrt{2\pi}}e^{-(x-69)^2/2(2.8)^2}dx$$

Now we use Simpson’s rule to evaluate this integral interval is $[65, 73]$

Taking $n=8\Rightarrow \Delta x=(73-65)/8=1$

Subintervals are $[65,66]$, $[66,67]$, $[67,68]$, $[68,69]$, $[69,70]$, $[70,71]$, $[71,72]$, $[72,73]$

Then by Simpson’s rule

$$P(65\le x\le 73)\approx \frac{1}{2.8\sqrt{2\pi}}\cdot\frac{1}{3}\left[e^{-(65-69)^2/2(2.8)^2}+4e^{-(66-69)^2/2(2.8)^2}+2e^{-(67-69)^2/2(2.8)^2}\right.$$
$$\left.\ldots\ldots+4e^{-(72-69)^2/2(2.8)^2}+e^{-(73-69)^2/2(2.8)^2}\right]$$
$$\approx 0.8469$$
$$\Rightarrow \boxed{P(69\le x\le 73)=85\%}$$

(B) The probability that the adult male population is more than 6feet $=6\times 12=72$inches tall (1 feet = 12 inch)

$$P(x>72)=1-\int_0^{72}\frac{1}{2.8\sqrt{2\pi}}e^{-(x-69)^2/2(2.8)^2}dx$$

For evaluating this integral we use Simpson's rule with $n = 36$

Then $\Delta x = \frac{72-0}{36} = 2$

Then subintervals are $[0,2],[2,4], ,\ldots\ldots\ldots\ldots,[70,72]$

Let $g(x) = e^{-(x-69)^2/2(2.8)^2}$

Then by Simpson's rule

$$P(0 \le x \le 72) = \frac{1}{2.8\sqrt{2\pi}}\frac{2}{3}\left[g(0)+4g(2)+2g(4)+4g(6)+\ldots..+4g(70)+g(72)\right]$$

$$\approx 0.858$$

Then $P(x > 72) \approx 1 - 0.858$

$\Rightarrow \boxed{P(x > 72) \approx 14.2\%}$

Answer 13E.

We have $\mu = 9.4$ and $\sigma = 4.2$

Then probability density function is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/\left(2(4.2)^2\right)}$

$$f(x) = \frac{1}{4.2\sqrt{2\pi}} e^{-(x-9.4)^2/\left(2(4.2)^2\right)}$$

The probability of households throw out at least 10lb of paper a week is

$$P(x \ge 10) = \int_{10}^{\infty} f(x)\,dx$$

This is an improper integral so we can approximate it by the integral 10 to 100 because more than 100lb of paper extremely rare.

Then $p(x > 10) = \int_{10}^{100} f(x)dx$

Where $f(x) = \frac{1}{4.2\sqrt{2\pi}} e^{-(x-9.4)^2/2(4.2)^2}$

MIDPOINT RULE:

$$\int_a^b f(x)\,dx \approx M_n \approx \Delta x\left[f(\overline{x}_1)+2f(\overline{x}_2)+2f(\overline{x}_3)+\cdots+f(\overline{x}_n)\right]$$

where $\Delta x = \frac{b-a}{n}$ and $\overline{x}_i = \frac{1}{2}(x_{i-1}+x_i) =$ midpoint of $[x_{i-1}x_i]$

We can use mid point rule for approximating this integral taking $n = 45$

$$\Delta x = \frac{100-10}{45} = 2$$

Then subintervals are $[10,12], [12,14], [14,16]\ ,\ldots\ldots\ldots\ldots,[98,100]$

And mid points are 11, 13, 15, 17, 19, 21, 23,.....................99

Then $P(x \ge 10) \approx 10 \times \begin{bmatrix} f(11)+f(13)+f(15)+f(17)+f(19)+f(21) \\ \ldots\ldots\ldots\ldots+f(97)+f(99) \end{bmatrix}$

$$\approx 0.44$$

$\Rightarrow \boxed{P(x \ge 10) \approx 44\%}$

Answer 15E.

Consider the speeds of vehicles on a highway with speed limit 100 km/h are normally distributed with mean 112 km/h and standard deviation 8 km/h.

(a)

Find the probability that a randomly chosen vehicle is travelling at a legal speed.

The probability density function is, $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

The legal speed is, 0 to 100 km/h, mean $\mu = 112$, standard deviation $\sigma = 8$.

To find the probability that a randomly chosen vehicle is travelling at a legal speed, find

$\int_0^{100} f(x)\,dx$.

Substitute mean $\mu = 112$, standard deviation $\sigma = 8$.

Therefore,

$$\begin{aligned}\int_0^{100} f(x)\,dx &= \int_0^{100} \frac{1}{8\sqrt{2\pi}} e^{-\frac{(x-112)^2}{2(8)^2}}\,dx \\ &= \int_0^{100} \frac{1}{8\sqrt{2\pi}} e^{-\frac{(x-112)^2}{128}}\,dx \\ &\approx 0.06681\end{aligned}$$

Thus, the probability that a randomly chosen vehicle is travelling at a legal speed is,

$\boxed{0.06681}$.

(b)

If police are instructed to ticket motorists driving 125 km/h or more, find the percentage of motorists are targeted.

To find the percentage of motorists is targeted, find the probability of $\int_{125}^{\infty} f(x)\,dx$.

Therefore,

$$\begin{aligned}\int_{125}^{\infty} f(x)\,dx &= \int_{125}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2(\sigma)^2}}\,dx \\ &= \int_{125}^{\infty} \frac{1}{8\sqrt{2\pi}} e^{-\frac{(x-112)^2}{2(8)^2}}\,dx \quad \text{Substitute } \sigma = 8, \mu = 112 \\ &= \int_{125}^{\infty} \frac{1}{8\sqrt{2\pi}} e^{-\frac{(x-112)^2}{128}}\,dx \\ &\approx 0.0521\end{aligned}$$

The percentage is, $(0.0521)\times 100 = \boxed{5.21\%}$.

Answer 16E.

The objective is to show that the probability distribution function for a normally distributed random variable has inflection points at $x = \mu \pm \sigma$.

Consider the normal distribution function,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Where $\mu > 0$ and $\sigma > 0$ called the standard deviation.

Finding the derivative $f'(x)$ we obtain

$$f'(x) = -\frac{(x-\mu)}{\sigma^3\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Thus, $x = \mu$ is the only critical number. Finding the second derivative we obtain

$$f''(x) = \frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left(\frac{(x-\mu)^2}{\sigma^2} - 1 \right)$$

Hence, $f''(\mu) = -\frac{1}{\sigma^3\sqrt{2\pi}} < 0$ so that $x = \mu$ is a maximum.

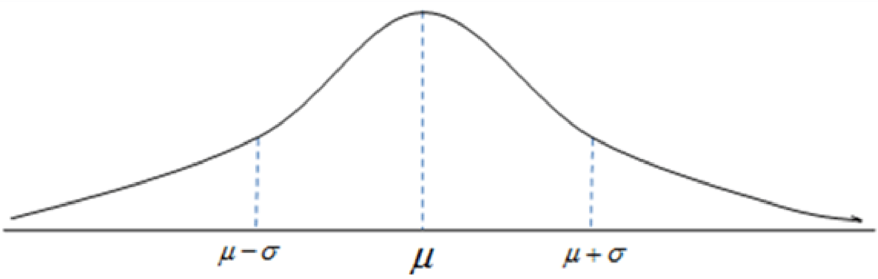
Solving the equation $f''(x) = 0$ we find $x = \mu \pm \sigma$.

If $\mu - \sigma < x < \mu + \sigma$ then $|x - \mu| < \sigma$ and therefore $\frac{(x-\mu)^2}{\sigma^2} < 1$ so that $f''(x) < 0$ and the graph of f is concave down.

Similarly, we see that $f''(x) > 0$ for $x < \mu - \sigma$ or $x > \mu + \sigma$ so that the graph of $f(x)$ is concave up.

It follows that $f(x)$ has points of inflection at $x = \mu \pm \sigma$.

The graph is shown below:



From the above figure, it is shown that the probability distribution function for a normally distributed random variable has inflection points at $x = \mu \pm \sigma$.

Answer 17E.

Two standard deviations of the mean are $\mu - 2\sigma$ and $\mu + 2\sigma$

Then $P = \int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx$

Let $\frac{x-\mu}{\sigma} = t$

$\Rightarrow \frac{1}{\sigma} dx = dt$

And when $x = \mu - 2\sigma$, $\Rightarrow t = -2$ and when $x = \mu + 2\sigma$, $t = 2$

Then $P = \int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$

Let $f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$

Interval is $= [-2, 2]$

Taking $n = 8$, $\Delta x = \frac{2+2}{8} = \frac{1}{2}$

Then by Simpson's rule

$$P = \frac{\Delta x}{3}\left[\begin{array}{r} f(-2)+4f(-1.5)+2f(-1)+4f(-0.5)+2f(0)+4f(0.5)+2f(1) \\ +4f(1.5)+f(2) \end{array}\right]$$

$$= \frac{1}{6}\left[\begin{array}{r} f(-2)+4f(-1.5)+2f(-1)+4f(-0.5)+2f(0)+4f(0.5) \\ +2f(1)+4f(1.5)+f(2) \end{array}\right]$$

$$\Rightarrow \boxed{P \approx 0.9545}$$

This is the probability that the random variable lies within two standard deviation of the mean.

Answer 18E.

Consider the standard deviation for a Random variable with probability density function f and μ is defined by

$$\sigma = \left[\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx\right]^{\frac{1}{2}}$$

Need to find a standard deviation for an exponential density function with mean μ.

Write the probability density function as,

$$f(x) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{\mu} e^{-\frac{x}{\mu}} & \text{if } t \geq 0 \end{cases}$$

That is $f(x) = \begin{cases} 0 & \text{if } -\infty < x < 0 \\ \frac{1}{\mu} e^{-\frac{x}{\mu}} & \text{if } \ 0 \leq x < \infty \end{cases}$

Thus, the standard deviation

$$\sigma = \left[\int_{-\infty}^{0} (x-\mu)^2 0\,dx\right]^{\frac{1}{2}} + \left[\int_{0}^{\infty} (x-\mu)^2 \frac{1}{\mu} e^{-\frac{x}{\mu}} dx\right]^{\frac{1}{2}}$$

$$\sigma = 0 + \frac{1}{\sqrt{\mu}}\left[\int_{0}^{\infty} (x-\mu)^2 e^{-\frac{x}{\mu}} dx\right]^{\frac{1}{2}}$$

$$\sigma = \frac{1}{\sqrt{\mu}}\left[\int_{0}^{\infty} (x-\mu)^2 e^{-\frac{x}{\mu}} dx\right]^{\frac{1}{2}}$$

Squaring on both sides,

$$\sigma^2 = \mu^2 \int_{0}^{\infty} \left(\frac{x}{\mu} - 1\right)^2 e^{-\frac{x}{\mu}} dx$$

$$= \mu \lim_{x \to \infty} \int_{0}^{x} \left(\frac{x}{\mu} - 1\right)^2 e^{-\frac{x}{\mu}} \frac{1}{\mu} dx$$

Setting $\frac{x}{\mu}=t \Rightarrow \frac{1}{\mu}dx = dt$

$$\sigma^2 = \mu \lim_{x\to\infty} \int_0^x (t-1)^2 e^{-t} dt$$

$$= \mu \lim_{x\to\infty} \left[(t-1)^2 \int e^{-t}dt - \int \frac{d(t-1)^2}{dt} \left(\int e^{-t}dt \right) dt \right]$$

$$= \mu \lim_{x\to\infty} \left[\left[-(t-1)^2 e^{-t} \right]_0^x + \int 2(t-1)e^{-t}dt \right]$$

$$= \mu \lim_{x\to\infty} \left[-(x-1)^2 e^{-x} + (0-1)^2 e^{-0} + 2(t-1)\int e^{-t}dt - \int \frac{d(t-1)}{dt}\left(\int e^{-t}dt \right) dt \right]$$

$$= \mu \lim_{x\to\infty} \left[-(x-1)^2 e^{-x} + 1 - \left[2(t-1)e^{-t} \right]_0^x + \int e^{-t}dt \right]$$

$$= \mu \lim_{x\to\infty} \left[-(x-1)^2 e^{-x} + 1 - 2(x-1)e^{-x} + 2(0-1)e^{-0} - \left[e^{-t} \right]_0^x \right]$$

$$= \mu \lim_{x\to\infty} \left[-(x-1)^2 e^{-x} + 1 - 2(x-1)e^{-x} + 2 - \left[e^{-x} - e^{-0} \right] \right]$$

$$= \mu \lim_{x\to\infty} \left[-(x-1)^2 e^{-x} + 1 - 2(x-1)e^{-x} + 2 - e^{-x} + 1 \right]$$

$$= \mu \left[-(\infty-1)^2 e^{-\infty} - 2(\infty-1)e^{-\infty} - e^{-\infty} + 4 \right]$$

$$= \mu \left[-(\infty-1)^2 (0) - 2(\infty-1)(0) - 0 + 4 \right]$$

$$= 4\mu$$

$$\sigma^2 = 4\mu$$

Hence, the standard deviation for an exponential density function is $\boxed{\sigma = 16\mu^2}$.

Answer 19E.

The probability density function is

$$p(r) = \frac{4}{a_0^3} r^2 e^{-\frac{2r}{a_0}}, \qquad r \geq 0$$

where a_0 is the Bohr radius $\left(a_0 \approx 5.59 \times 10^{-11} \text{ m}\right)$.

Consider the integral

$$P(r) = \int_0^r \frac{4}{a_0^3} s^2 e^{-\frac{2s}{a_0}} ds$$

This integral gives the probability that the electron will be found within the sphere of radius r meters centered at the nucleus.

(a)

Need to verify that $p(r)$ is a probability density function.

$$\int_{-\infty}^{\infty} p(r)\,dr = \int_{0}^{\infty}\left[\frac{4}{a_0^{3}} r^2 e^{-\frac{2r}{a_0}}\right]dr \qquad [\text{For } r \geq 0]$$

$$= \frac{4}{a_0^{3}}\int_{0}^{\infty}\left(r^2 e^{-\frac{2r}{a_0}}\right)dr$$

$$= \frac{4}{a_0^{3}}\left[-\frac{1}{4}a_0\left(a_0^{2}+2a_0r+2r^2\right)e^{-\frac{2r}{a_0}}\right]_0^{\infty}$$

$$= \frac{4}{a_0^{3}}\left[-\frac{1}{4}a_0\left(a_0^{2}+2a_0r+2r^2\right)e^{-\infty}+\frac{1}{4}a_0\left(a_0^{2}+2a_0r+2r^2\right)e^{0}\right]$$

$$= \frac{4}{a_0^{3}}\left[-\frac{1}{4}a_0\left(a_0^{2}+2a_0r+2r^2\right)(0)+\frac{1}{4}a_0\left(a_0^{2}+2a_0(0)+2(0)^2\right)(1)\right]$$

$$= \frac{4}{a_0^{3}}\left[\frac{1}{4}a_0\left(a_0^{2}\right)\right]$$

$$= 1$$

Therefore, $\int_{-\infty}^{\infty} p(r)\,dr = 1$, so $p(r)$ is a probability density function.

(b) Find the limit of $p(r)$ as r tends to infinity.

$$\lim_{r\to\infty} p(r) = \lim_{r\to\infty}\left(\frac{4}{a_0^{3}} r^2 e^{-\frac{2r}{a_0}}\right)$$

$$= 0 \qquad \left[\text{Since as } r\to\infty, e^{-r}\to 0, e^{-\frac{2r}{a_0}}\to 0, r^2 e^{-\frac{2r}{a_0}}\to 0\right]$$

Consider $p(r) = \frac{4}{a_0^{3}} r^2 e^{-\frac{2r}{a_0}}$.

Differentiate with respect to r, get

$$p'(r) = \frac{d}{dr}\left(\frac{4}{a_0^{3}} r^2 e^{-\frac{2r}{a_0}}\right)$$

$$= \frac{4}{a_0^{3}}\left(2re^{-\frac{2r}{a_0}} - \frac{2}{a_0} r^2 e^{-\frac{2r}{a_0}}\right)$$

Equate $p'(r)$ to zero and solve for the variable x, get

$$\frac{4}{a_0^3}\left(2re^{-\frac{2r}{a_0}}-\frac{2}{k}r^2e^{-\frac{2r}{a_0}}\right)=0$$

$$2re^{-\frac{2r}{a_0}}-\frac{2}{k}r^2e^{-\frac{2r}{a_0}}=0$$

$$re^{-\frac{2r}{a_0}}\left(2-\frac{2}{a_0}r\right)=0$$

$$r\left(2-\frac{2}{a_0}r\right)=0$$

$$\text{either } r=0 \text{ or } 2-\frac{2}{a_0}r=0$$

$$\text{either } r=0 \text{ or } r=a_0$$

When $r=0,$

$$p(r)=\frac{4}{a_0^3}r^2e^{-\frac{2r}{a_0}}$$

$$p(0)=\frac{4}{a_0^3}(0)^2e^{-\frac{2(0)}{a_0}}$$

$$=0$$

When $r=a_0,$

$$p(r)=\frac{4}{a_0^3}r^2e^{-\frac{2r}{a_0}}$$

$$p(a_0)=\frac{4}{a_0^3}(a_0)^2e^{-\frac{2(a_0)}{a_0}}$$

$$=\frac{4}{a_0}e^{-2}$$

Therefore $p(r)$ have its maximum value when $r=a_0\approx 5.59\times 10^{-11}\,\text{m}.$

(c)

Let $p(r)=\frac{4}{\left(5.59\times10^{-11}\right)^3}r^2e^{-\frac{2r}{5.59\times10^{-11}}}$.

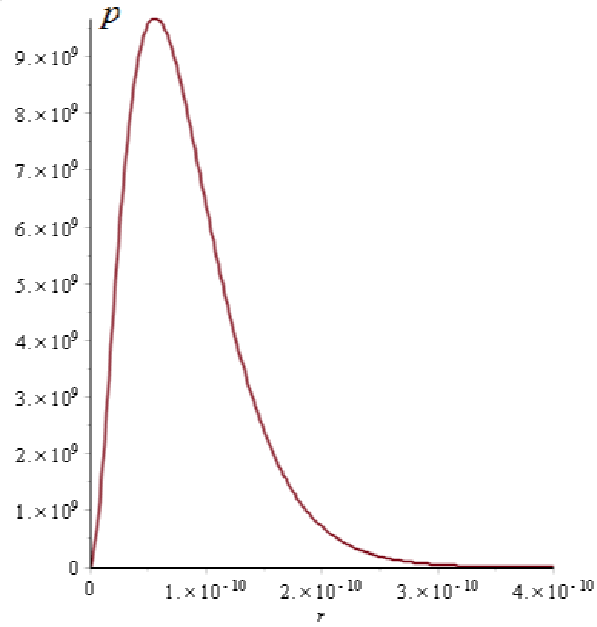
Use Maple to graph the function $p(r)$.

The graph of the density function $p(r)$ is shown below:

Keystrokes:

plot((4*r/(5.59*10^(-11))^3*r)*exp(-2*r/(5.59*10^(-11))), r = 0 .. 4*10^(-10));

Maple result:



(d)

The probability that the electron will be within the sphere of radius r meters centered at the nucleus is

$$P(r)=\int_0^r \frac{4}{a_0^3}s^2e^{-\frac{2s}{a_0}}ds$$

$$P(r)=\int_0^r \frac{4}{\left(5.59\times10^{-11}\right)^3}s^2e^{-\frac{2s}{5.59\times10^{-11}}}ds$$

When the radius of the sphere $r=4a_0\approx4\left(5.59\times10^{-11}\right)$, the probability is

$$P(r)=\int_0^{4\left(5.59\times10^{-11}\right)} \frac{4}{\left(5.59\times10^{-11}\right)^3}s^2e^{-\frac{2s}{5.59\times10^{-11}}}ds$$

$$=\frac{4}{\left(5.59\times10^{-11}\right)^3}\int_0^{4\left(5.59\times10^{-11}\right)} s^2e^{-\frac{2s}{5.59\times10^{-11}}}ds$$

Use Maple to solve this integral

Keystrokes:

4*(int(s^2*exp(-2*s/(5.59*10^(-11))), s = 0 .. (4*5.59)*10^(-11)))/(5.59*10^(-11))^3;

Maple result:

$$> \frac{4}{\left(5.59\cdot10^{-11}\right)^3}\cdot\int_0^{4\cdot5.59\cdot10^{-11}} s^2\cdot e^{-\frac{2\cdot s}{5.59\cdot10^{-11}}}\,ds$$

0.9862460327

Therefore the probability that the electron will be within the sphere of radius $r=4a_0$ meters centered at the nucleus is $\approx\boxed{0.986}$.

(e)

Mean distance of the electron from the nucleus in the ground state of the hydrogen atom is

$$\int_{-\infty}^{\infty} r \cdot p(r)\,dr = \int_{0}^{\infty} rp(r)\,dr = \int_{0}^{\infty} r\left[\frac{4}{a_0^{\ 3}} r^2 e^{-\frac{2r}{a_0}}\right] dr \qquad [\text{For } r \geq 0]$$

$$= \frac{4}{a_0^{\ 3}} \int_{0}^{\infty} \left(r^3 e^{-\frac{2r}{a_0}}\right) dr$$

Continuation to the above

$$= \frac{4}{a_0^{\ 3}} \left[-\frac{1}{8} a_0 \left(3a_0^{\ 3} + 6a_0^{\ 2} r + 6a_0 r^2 + 4r^3\right) e^{-\frac{2r}{a_0}}\right]_0^{\infty}$$

$$= \frac{4}{a_0^{\ 3}} \left[\begin{array}{l} -\frac{1}{8} a_0 \left(3a_0^{\ 3} + 6a_0^{\ 2} r + 6a_0 r^2 + 4x^3\right) e^{-\infty} \\ +\frac{1}{8} a_0 \left(3a_0^{\ 3} + 6a_0^{\ 2}(0) + 6a_0 (0)^2 + 4(0)^3\right) e^{-0} \end{array}\right]$$

$$= \frac{4}{a_0^{\ 3}} \left[\begin{array}{l} -\frac{1}{8} a_0 \left(3a_0^{\ 3} + 6a_0^{\ 2} r + 6a_0 r^2 + 4x^3\right)(0) \\ +\frac{1}{8} a_0 \left(3a_0^{\ 3} + 6a_0^{\ 2}(0) + 6a_0 (0)^2 + 4(0)^3\right)(1) \end{array}\right]$$

$$= \frac{4}{a_0^{\ 3}} \left[0 + \frac{1}{8} a_0 \left(3a_0^{\ 3}\right)(1)\right]$$

$$= \frac{3}{2} a_0$$

Therefore, the mean distance of the electron from the nucleus in the ground state of the hydrogen atom is $\boxed{\frac{3}{2} a_0}$.