Circles

Exercise-11.1

Question 1:

A and B are the points on ⊙(O, r). $\overline{AB}$ is not a diameter of the circle. Prove that the tangents to the circle at A and B are not parallel.

Solution :

Given : A and B are the points on ⊙(O, r). $\overline{AB}$ is not a diameter of the circle.

To prove: Tangents to the circle at A and B are not parallel.

Proof: Let us try using the method of contradiction.

Suppose l_1 and l_2 are parallel tangents to the circle having centre at O drawn at the points A and B.

$\therefore \overline{OA} \perp l_1$ and $\overline{OB} \perp l_2$

Consider $\overline{OA}$ and $\overline{OB}$ perpendicular respectively to l_1 and l_2 and O is a common point.

$\therefore A - O - B \qquad (\because l_1$ and l_2 are the parallel lines.)

Hence, $\overline{AB}$ is a diameter, which contradicts with our assumption.

$\therefore$ Our assumption is wrong.

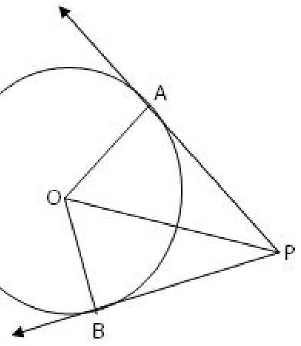
$\therefore l_1$ and l_2 are the intersecting lines.

Hence, tangents to the circle at A and B are not parallel.

Question 2:

A, B are the points on ⊙(O, r) such that tangents at A and B intersect in P. Prove that $\overrightarrow{OP}$ is the bisector of ∠AOB and $\overrightarrow{PO}$ is the bisector of ∠APB.

Solution :



Given: A and B are the points on $\odot(O, r)$ such that the tangents at A and B intersect in P.

To prove: i) $\overrightarrow{OP}$ is the bisector of $\angle AOB$

ii) $\overrightarrow{PO}$ is the bisector of $\angle APB$.

Proof: In $\odot(O, r)$, $\overleftrightarrow{AP}$ is a tangent at A and $\overleftrightarrow{BP}$ is the tangent at B.

$\therefore$ $m\angle OAP = m\angle OBP = 90°$

Considering the correspondence $OAP \leftrightarrow OBP$,

$\overline{OA} \cong \overline{OB}$ (both are radius)

$\overline{OP} \cong \overline{OP}$ (common segment)

$\angle OAP \cong \angle OBP$

Hence, $OAP \leftrightarrow OBP$ is a congruence. (by R.H.S. theorem)

$\therefore$ $m\angle APO = m\angle BPO$

Also $m\angle AOP = m\angle BOP$

We know that, O is in the interior part of $\angle APB$ and P is in the interior part of $\angle AOB$.

$\therefore$ $\overrightarrow{OP}$ is the bisector of $\angle AOB$ and

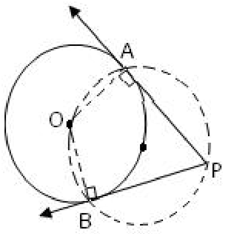
$\overrightarrow{PO}$ is the bisector of $\angle APB$.

Question 3:

A, B are the points on $\odot(O, r)$ such that tangents at A and B to the circle intersect in P.

Show that the circle with $\overrightarrow{OP}$ as a diameter passes through A and B.

Solution :



Given : A and B are the points on $\odot(O,\ r)$ such that tangents at A and B to the circle intersect in P.

To prove: The circle with $\overline{OP}$ as diameter passes through A and B.

Proof: We know that a tangent drawn to a circle is perpendicular to the radius drawn from the point of contact.

$\therefore\ \overline{OA} \perp \overline{AP}$ and $\overline{OB} \perp \overline{PB}$

$\therefore\ m\angle OAP = 90^\circ$ and $m\angle OBP = 90^\circ$

$\therefore\ m\angle OAP + m\angle OBP = 180^\circ \qquad \dots\ \dots(1)$

For □OAPB,

$m\angle OAP + mAPB + m\angle AOB + mOBP = 360^\circ$

$\therefore\ (m\angle APB + m\angle AOB) + (m\angle OAP + m\angle OBP) = 360^\circ$

$m\angle APB + m\angle AOB + 180^\circ = 360^\circ \qquad (\because \text{By}(1))$

$\therefore\ m\angle APB + m\angle AOB = 180^\circ \qquad \dots\ \dots(2)$

By (1) and (2), we get,

□OAPB is a cyclic.

Here $\overline{OP}$ makes a right angle at A.

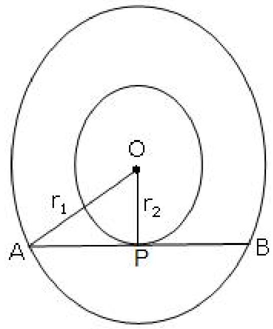
Then $\overline{OP}$ is a diameter.

Hence, the circle with $\overline{OP}$ as a diameter passes through A and B.

Question 4:

$\odot(O,\ r_1)$ and $\odot(O,\ r_2)$ are such that $r_1 > r_2$. Chord $\overrightarrow{AB}$ of $\odot(O,\ r_1)$ touches $\odot(O,\ r_2)$. Find AB in terms of r_1 and r_2 .

Solution :



Here $\odot(O, r_1)$ and $\odot(O, r_2)$ are such that $r_1 > r_2$.
So the circles are concentric.
Let chord $\overline{AB}$ of $\odot(O, r_1)$ touches $\odot(O, r_2)$ at P.
$\therefore$ $\overline{AB}$ is tangent to $\odot(O, r_2)$.
$\therefore$ $\overline{OP} \perp \overline{AB}$
Also, $P \in \overline{AB}$
P is the foot of the perpendicular drawn from centre O on the chord $\overline{AB}$ of $\odot(O, r_1)$.
$\therefore$ P is the midpoint of $\overline{AB}$.
$\therefore$ $AB = 2AP$ ... ...(1)
In right angled ΔOPA,
$OA^2 = AP^2 + OP^2$
$\therefore$ $r_1^2 = AP^2 + r_2^2$
$\therefore$ $AP^2 = r_1^2 - r_2^2$
$\therefore$ $AP = \sqrt{r_1^2 - r_2^2}$
But, $AB = 2AP$ (From (1))
$\therefore$ $AB = 2\sqrt{r_1^2 - r_2^2}$

Question 5:

In example 4, if $r_1 = 41$ and $r_2 = 9$, find AB.

Solution :

Using the formula derived in example 4,
Taking $r_1 = 41$ and $r_2 = 9$, we get,
The length of the chord,
$AB = 2\sqrt{r_1^2 - r_2^2}$
$= 2\sqrt{(41)^2 - (9)^2}$
$= 2\sqrt{1681 - 81}$
$= 2\sqrt{1600}$
$= 2 \times 40$
$= 80$
$\therefore$ $AB = 80$.

Exercise-11.2

Question 1:

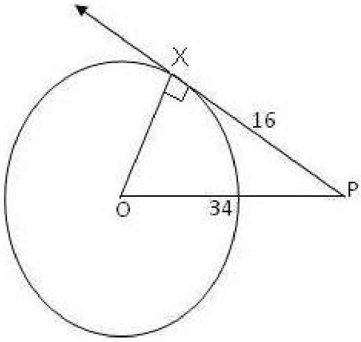
P is the point in the exterior of $\odot(O, r)$ and the tangents from P to the circle touch the circle at X and Y.

1. Find OP, if r = 12, XP = 5
2. Find $m\angle XPO$, if $m\angle XOY = 110$

3. Find r, if OP = 25 and PY = 24
4. Find m∠XOP, if m∠XPO = 80

Question 1(1):

Solution :



$\overline{OX}$ is the radius of the circle and $\overleftrightarrow{PX}$ is a tangent.

$\therefore \overline{OX} \perp \overleftrightarrow{PX}$

Now, r = 12 = OX and XP = 5

For right angled ΔPXO,

$OP^2 = PX^2 + OX^2$

$\therefore OP^2 = (5)^2 + (12)^2$

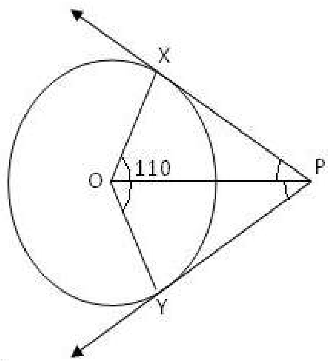
$= 25 + 144$

$\therefore OP^2 = 169$

$\therefore OP = 13$

Question 1(2):

Solution :



Here ∠XPY are ∠XOY are supplementary angles.

$\therefore m\angle XPY + m\angle XOY = 180^\circ$

$\therefore m\angle XPY + 110 = 180^\circ$

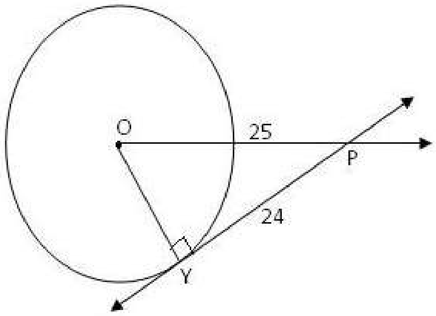
$\therefore m\angle XPY = 70$

Also, $\overrightarrow{OP}$ is a bisector of ∠XPY.

$\therefore m\angle XPO = \frac{1}{2} m\angle XPY = \frac{1}{2}(70) = 35^\circ$

Question 1(3):

Solution :



$\overline{OY}$ is the radius of the circle and $\overleftrightarrow{PY}$ is a tangent.

$\therefore \overline{OY} \perp \overleftrightarrow{PY}$

Now, $OY = r$,

Given is $OP = 25$ and $PY = 24$.

In right angled ΔPYO,

$OP^2 = OY^2 + PY^2$

$\therefore (25)^2 = r^2 + (24)^2$

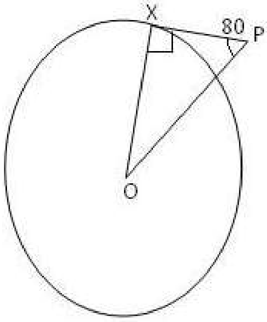
$\therefore r^2 = 625 - 576$

$\therefore r^2 = 49$

$\therefore r = 7$

Question 1(4):

Solution :



Here $\overline{OX}$ is the radius of the circle and $\overleftrightarrow{PX}$ is a tangent.

$\therefore \overline{OX} \perp \overleftrightarrow{PX}$

In right angled ΔPXO,

Also, $\angle XOP$ and $\angle XPO$ are complementary angles.

$\therefore m\angle XOP + m\angle XPO = 90^\circ$

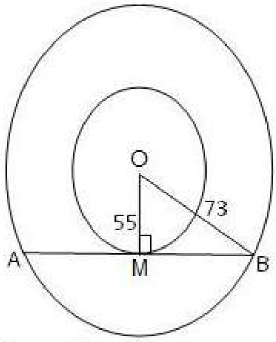
$\therefore m\angle XOP + 80^\circ = 90^\circ$

$\therefore m\angle XOP = 10^\circ$

Question 2:

Two concentric circles having radii 73 and 55 are given. The chord of the circle with larger radius touches the circle with smaller radius. Find the length of the chord.

Solution :



Here OB = radius of a larger circle = 73 and PM = radius of a smaller circle = 55

$\overline{OM} \perp \overline{AB}$ ($\because \overleftrightarrow{AB}$ is a tangent.)

Now, we find the length $\overline{MB}$.

In ΔOMB, $\angle OMB$ is a right angle.

$\therefore OB^2 = OM^2 + MB^2$

$\therefore MB^2 = OB^2 - OM^2$

$= (73)^2 - (55)^2$

$= (73+55)(73-55)$

$= (128)(18)$

$= 64 \times 36$

$= 8^2 \times 6^2$

$= (48)^2$

$\therefore MB = 48$

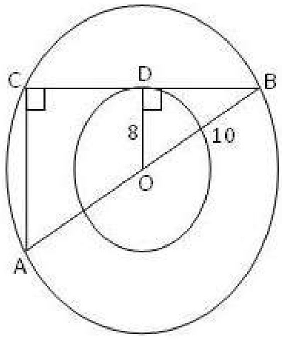
Now, AB = 2MB = 2(48) = 96

The length of the chord is 96.

Question 3:

$\overline{AB}$ is a diameter of ⊙(O, 10). A tangent is drawn from B to ⊙(O, 8) which touches ⊙(O, 8) at D. $\overleftrightarrow{BD}$ intersects ⊙(O, 10) in C. Find AC.

Solution :



In $\odot(O, 10)$, $OA = OB = 10 =$ radius

$\therefore AB = 20$ ($\because$ Diameter)

In $\odot(O, 8)$, $OD = 8 =$ radius

$\overline{OD} \perp \overleftrightarrow{BD}$ ($\because \overleftrightarrow{BD}$ is a tangent.)

Also, $\overline{AB}$ is a diameter.

$\therefore$ $m\angle ACB = 90$ ($\because$ Angle incribed in a semicircle)

$\therefore m\angle ODB \cong m\angle ACB = 90$

$\angle DBO \cong \angle CBA$

$\therefore$ The correspondence $ODB \leftrightarrow ACB$ is a similarity. (AA theorem)

Then, $\frac{AC}{OD} = \frac{AB}{OB}$

$\therefore \frac{AC}{8} = \frac{20}{10}$

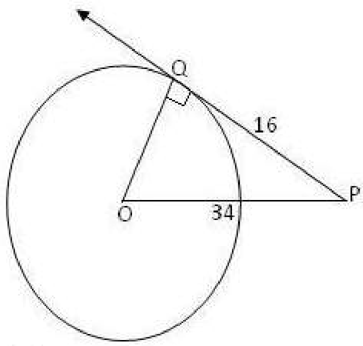
$\therefore AC = \frac{20 \times 8}{10}$

$\therefore AC = 16$

Question 4:

P is in the exterior of a circle at distance 34 from the centre O. A line through P touches the circle at Q. PQ = 16, find the diameter of the circle.

Solution :



Here it is given that, $OP = 34$, $PQ = 16$ and $\overline{OQ}$ is a radius of a circle.

As $\overleftrightarrow{PQ}$ is a tangent to the circle, $\overline{OQ} \perp \overleftrightarrow{PQ}$

In right angled ΔOQP,

$OP^2 = PQ^2 + OQ^2$

$\therefore (34)^2 = (16)^2 + OQ^2$

$\therefore OQ^2 = 1156 - 256$

$\therefore OQ^2 = 900$

$\therefore OQ = 30$

The diameter of the circle $= 2r = 2 \times OQ = 2 \times 30 = 60$

Question 5:

In figure 11.24, two tangents are drawn to a circle from a point A which is in the exterior of the circle. The points of contact of the tangents are P and Q as shown in the figure. A line l touches the circle at R and intersects $\overline{AP}$ and $\overline{AQ}$ in B and C respectively. If AB = c, BC = a, CA = b, then prove that

1. AP + AQ = a + b + c
2. AB + BR = AC + CR = AP = AQ = $\frac{a+b+c}{2}$

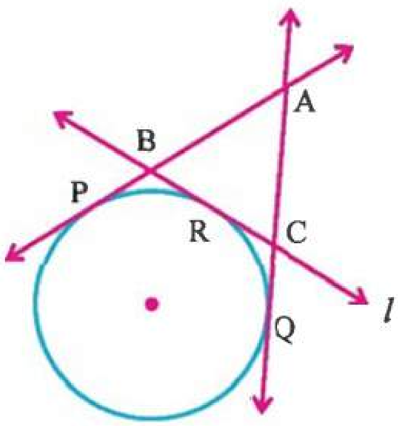
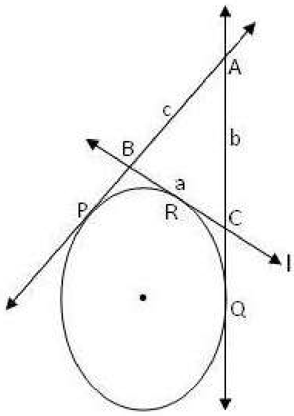


Figure 11.24

Solution :



Given: $\overleftrightarrow{AP}$ and $\overleftrightarrow{AQ}$ are the tangent to the circle.
A line l touches the circle at R and it intersects $\overline{AP}$ in B and $\overline{AQ}$ in C.
Also AB = c, BC = a and CA = b.

To prove:
1. AP + AQ = a+b+c

2. $AB + BR = AC + CR = AP = AQ = \dfrac{a+b+c}{2}$

Proof: Here $\overleftrightarrow{AP}$, $\overleftrightarrow{AQ}$ and l are tangents to the circle.
It is given that a = BC, b = CA and c = AB.
We know that, AP = AQ, BP = BR and CQ = CR (by thm.)... ...(1)

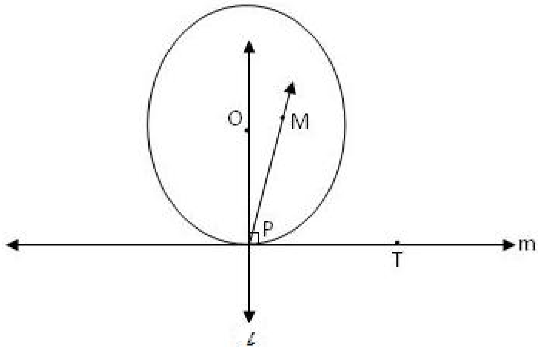
$1)\ AP + AQ = (AB + BP) + (AC + CQ) \quad (\because A-B-P \text{ and } A-C-Q)$
$=(AB + BR)+(AC + CR) \quad (\text{From } (1))$
$= AB + AC + (BR + CR)$
$= AB + AC + BC \quad (\because B-R-C)$
$= c + b + a$
$= a + b + c$
$AP + AQ = a + b + c \quad \ldots \ \ldots(2)$

$2)\ AB + BR = AB + BP \quad (\text{From } (1))$
$= AP \quad (\because A-B-P)$
$= AQ \quad (\because \text{by}(1))$
$= AC + CQ \quad (\because A\text{-}C\text{-}Q)$
$= AC + CR \ (\text{From } (1))$
Thus $AB + BR = AC + CR = AP = AQ \quad \ldots \ \ldots(3)$
$(\because \text{by}(1))$
Using result (2),
$AP + AQ = a + b + c$
$\therefore AQ + AQ = a + b + c \quad (\because \text{From } (1))$
$\therefore 2AQ = a + b + c$
$\therefore AQ = \dfrac{a+b+c}{2}$
Hence, from result (3),
$AB + BR = AC + CR = AP = AQ = \dfrac{a+b+c}{2}$

Question 6:

Prove that the perpendicular drawn to a tangent to the circle at the point of contact of the tangent passes through the centre of the circle.

Solution :



Given : m is a tangent to the circle having centre at O. m touches the circle at P. l is a perpendicular line m from P.

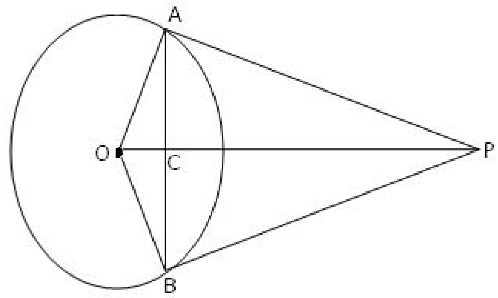
To prove: l passes through O i.e., $O \in l$.

Proof: If $O \notin l$ then we can find such $M \notin l$ that O and M are in the same half-plane of m.
$T \in m$ is a point distinct from P.
$\therefore m\angle MPT = 90°$ ($\because m \perp M$) and
$m\angle OPT = 90°$
M and O are the points of the same half-plane so this is impossible.
$\therefore$ Our assumption is wrong.
$\therefore O \in l$
Therefore, the perpendicular drawn to a tangent to the circle at the point of contact of the tangent passes through the centre of the circle.

Question 7:

Tangents from P, a point in the exterior of ⊙(O, r) touch the circle at A and B. Prove that $\overline{BD} \perp \overline{AC}$ and $\overline{OP}$ bisects $\overline{AB}$.

Solution :



Given : $\overline{PA}$ and $\overline{PB}$ are tangents to the circle drawn from point P which is in the exterior of $\odot(O, r)$. A and B are on the circle.

To prove: $\overline{OP} \perp \overline{AB}$ and $\overline{OP}$ bisects $\overline{AB}$.

Proof : Here $\overline{PA}$ and $\overline{PB}$ are tangents to the circle drawn from an exterior point P.

$\therefore$ $\overline{OP}$ intersect $\overline{AB}$ at C.

Also points A and B lie on the circle.

$\therefore PA = PB$ (by thm 11.3)

$OP = OP$ (common)

$OA = OB$ (radii of a circle)

$\therefore \Delta OAP \cong \Delta OBP$ (SSS theorem)

Then, $m\angle AOP \cong m\angle BOP$

$\therefore m\angle AOC \cong m\angle BOC$ $(\because C \in \overline{OP})$

Also, $OA = OB$ $(\because$ radii $)$

$OC = OC$ (common)

$\therefore \Delta AOC \cong \Delta BOC$ (SAS theorem)

$\therefore AC = BC$ and $m\angle ACO = m\angle BCO = 90°$

Now, $C \in \overline{OP}$,

$\therefore$ $\overline{OP}$ bisects $\overline{AB}$ $(\because AC = BC)$

Also, $\overline{AC} \perp \overline{OC}$ and $\overline{BC} \perp \overline{OC}$

$\therefore \overline{OP} \perp \overline{AB}$ $(\because A - C - B)$

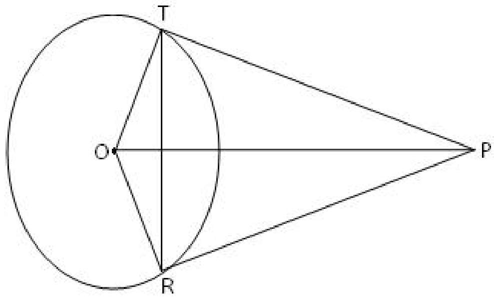
$\therefore$ $\overleftrightarrow{OP}$ is a perpendicular bisector of $\overline{AB}$.

$\therefore \overline{OP} \perp \overline{AB}$ and $\overline{OP}$ bisects $\overline{AB}$.

Question 8:

$\overleftrightarrow{PT}$ and $\overleftrightarrow{PR}$ are the tangents drawn to $\odot$(O, r) from point P lying in the exterior of the circle and T and R are their points of contact respectively. Prove that m∠TPR = 2m∠OTR.

Solution :



Given : $\overrightarrow{PT}$ and $\overrightarrow{PR}$ are the tangents drawn to $\odot(O,\ r)$ from point P lying in the exterior of the circle.

To prove: $m\angle TPR = 2m\angle OTR$

Proof: We know that $\overline{PT} \cong \overline{PR}$ $(\because$ by thm.)

$m\angle PTR = m\angle PRT$ $\left(\begin{array}{l}\because \text{angles opp. to congruent} \\ \text{sides are equal.}\end{array}\right)$

In ΔPTR,

$m\angle PTR + m\angle PRT + m\angle TPR = 180$

$\therefore m\angle PTR + m\angle PTR + m\angle TPR = 180$

$\therefore 2m\angle PTR + m\angle TPR = 180$

$\therefore 2m\angle PTR = 180 - m\angle TPR$

$\therefore m\angle PTR = 90 - \frac{1}{2}m\angle TPR$

$\therefore \frac{1}{2}m\angle TPR = 90 - \frac{1}{2}m\angle TPR$ $\quad \ldots \ldots(1)$

Next, $\overline{OT} \perp \overline{PT}$

$\therefore m\angle OTP = 90^\circ$

$\therefore m\angle OTR + m\angle PTR = 90^\circ$

$\therefore m\angle OTR = 90^\circ - m\angle PTR$ $\quad \ldots \ldots(2)$

From (1) and (2), we get

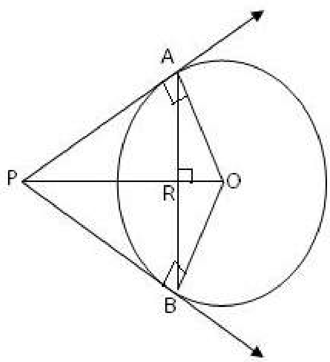
$\frac{1}{2}m\angle TPR = m\angle OTR$

$\therefore m\angle TPR = 2m\angle OTR$

Question 9:

$\overline{AB}$ is a chord of $\odot$(O, 5) such that AB = 8. Tangents at A and B to the circle intersect in P. Find PA.

Solution :



Let, PR $= x$.

AB $= 8$. $\therefore$ AR $=$ BR $= 4$ $\quad (\because \overleftrightarrow{OP}$ is perpendicular bisector of $\overline{AB}.)$

For ΔORA, $m\angle R = 90^\circ$

$OA^2 = OR^2 + AR^2$

$\therefore OR^2 = OA^2 - AR^2 = (5)^2 - (4)^2 = 25 - 16 = 9$

$\therefore OR = 3$

Now, $OP = PR + RO = x + 3$

For ΔARP, $m\angle R = 90^\circ$

$PA^2 = AR^2 + PR^2$

$= (4)^2 + x^2 \qquad \ldots \ldots (1)$

For ΔOAP, $m\angle A = 90^\circ$

$PA^2 = OP^2 - OA^2$

$= (x + 3)^2 - (5)^2 \qquad \ldots \ldots (2)$

From (1) and (2),

$(4)^2 + x^2 = (x + 3)^2 - (5)^2$

$\therefore 16 + x^2 = x^2 + 6x + 9 - 25$

$\therefore 6x = 32$

$\therefore x = \dfrac{16}{3}$

Now, $PA^2 = (4)^2 + x^2 \qquad (\because$ From (1))

$= 16 + \left(\dfrac{16}{3}\right)^2$

$= 16 + \dfrac{256}{9}$

$= \dfrac{144 + 256}{9}$

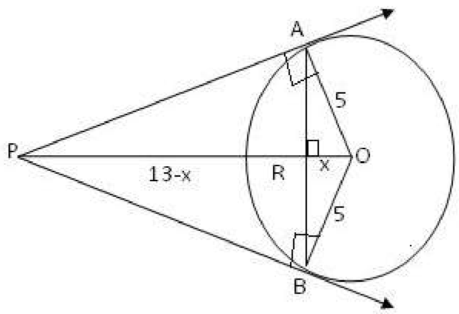
$= \dfrac{400}{9}$

$\therefore PA = \dfrac{20}{3}$

Question 10:

P lies in the exterior of ⊙(O, 5) such that OP = 13. Two tangents are drawn to the circle which touch the circle in A and B. Find AB.

Solution :



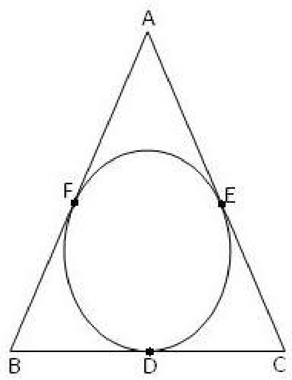
Let, OR $= x$.
For ΔOAP, $m\angle A = 90^\circ$
$OP^2 = AP^2 + OA^2$
$\therefore (13)^2 = AP^2 + (5)^2$
$\therefore AP^2 = 169 - 25 = 144$
$\therefore AP = 12$
For ΔORA, $m\angle R = 90^\circ$
$AR^2 = OA^2 - OR^2$
$= (5)^2 - x^2 \quad \ldots \ldots (1)$
For ΔARP, $m\angle R = 90^\circ$
$AR^2 = AP^2 - PR^2$
$= (12)^2 - (13 - x)^2 \quad \ldots \ldots (2)$
Form (1) and (2), we get
$(5)^2 - x^2 = (12)^2 - (13 - x)^2$
$\therefore 25 - x^2 = 144 - 169 + 26x - x^2$
$\therefore 26x = 50$
$\therefore x = \frac{25}{13}$
Now, $AR^2 = (5)^2 - x^2 \quad (\because \text{by } (1))$
$= 25 - \frac{(25)^2}{(13)^2}$
$= 25 - \frac{625}{169}$
$= \frac{4225 - 625}{169} = \frac{3600}{169}$
$\therefore AR = \frac{60}{13}$
But, $AB = 2AR$
$\therefore AB = 2\left(\frac{60}{13}\right)$
$\therefore AB = \frac{120}{13} = 9.23$

Exercise-11

Question 1:

A circle touches the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of ΔABC at points D, E, F respectively. BD = x, CE = y, AF = z. Prove that the area of ΔABC = $\sqrt{xyz(x+y+z)}$.

Solution :



Given : A circle touches the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of ΔABC at points D, E, F respectively. BD=x, CE=y, AF=z.

To prove: Area of the $\Delta ABC = \sqrt{xyz(x+y+z)}$

Proof : A circle touches the sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ of ΔABC at points D, E, F respectively.

$\therefore$ $\overleftrightarrow{BD}$ and $\overleftrightarrow{BF}$ are tangents drawn from B and D and F are the points of contact.

$\therefore BD = BF = x$ $\qquad$ ($\because$ by thm.)

Similarly, for tangents $\overleftrightarrow{CE}$ and $\overleftrightarrow{CD}$ drawn from C and tangents $\overleftrightarrow{AE}$ and $\overleftrightarrow{AF}$ drawn from A, we get,

$CE = CD = y,$

$AF = AE = z$

The sides of ΔABC,

$AB = c = AF + BF = z+x \quad \ldots \quad \ldots (1)$

$BC = a = BD + DC = x+y$ and $\quad \ldots \quad \ldots (2)$

$CA = b = CE + AE = y+z \quad \ldots \quad \ldots (3)$

In ΔABC, $2s = AB+BC+CA$

$$= (z+x)+(x+y)+(y+z)$$

$$= 2(x+y+z)$$

$\therefore s = x+y+z \quad \ldots \quad \ldots (4)$

Now, area of ΔABC,

$$\Delta = s\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{\begin{array}{l}(x+y+z)(x+y+z-(x+y))\\(x+y+z-(y+z))\\(x+y+z-(z+x))\end{array}} \qquad \left(\begin{array}{l}\because \text{Subs. the values from}\\(1),(2),(3)\text{ and }(4)\end{array}\right)$$

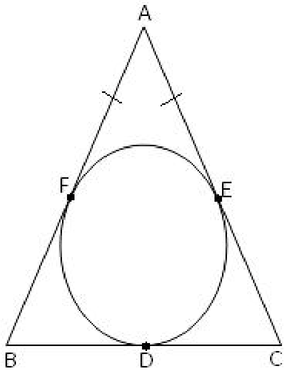
$$= \sqrt{(x+y+z)(x)(y)(z)}$$

$$= \sqrt{xyz(x+y+z)}$$

Question 2:

ΔABC is an isosceles triangle in which $\overline{AB} \cong \overline{AC}$. A circle touching all the three sides of ΔABC touches $\overline{BC}$ at D. Prove that D is the mid-point of $\overline{BC}$.

Solution :



Given : ΔABC is an isosceles triangle in which $\overline{AB} \cong \overline{AC}$.

A circle touching all the three sides of ΔABC touches $\overline{BC}$ at D.

To prove: D is the mid-point of $\overline{BC}$.

Proof : A circle touches all sides of the triangle, so the lengths of the tangents drawn from A, B, and C are equal.

$\therefore$ AE = AF, BD = BF and CD = CE $(\because$ by thm.$)$ ... ...(1)

Now, AB = AC

$\therefore$ AB − AF = AC − AF (Substracting AF from both the sides)

$\therefore$ AB − AF = AC − AF $(\because$ by (1)$)$

$\therefore$ BF = CE $(\because A-F-B$ and $A-E-C)$

$\therefore$ BD = CD $(\because$ by (1)$)$

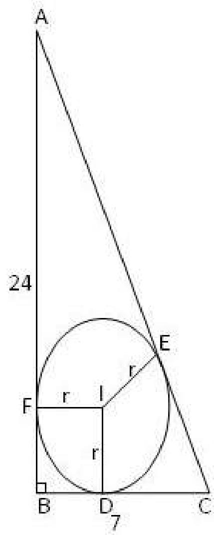
$\because$ B − D − C and BD = CD,

D is the midpoint of $\overline{BC}$.

Question 3:

$\angle B$ is a right angle in ΔABC. If AB = 24, BC = 7, then find the radius of the circle which touches all the three sides of ΔABC.

Solution :



Let the radius and the center of the circle touching all three sides of a triangle be r and I respectively.

As shown in the figure,

ID = IE = IF = r

In ΔABC, $m\angle B = 90^\circ$.

Also, $\overline{ID} \perp \overline{BC}$ and $\overline{AB} \perp \overline{BC}$

$\therefore \overline{ID} \parallel \overline{AB}$

$\therefore \overline{ID} \parallel \overline{FB} \quad (\because A-F-B)$

Similarly, $\overline{IF} \parallel \overline{BD}$

$\therefore$ ▭IFBD is a parallelogram.

$\therefore$ ID = FB = r and BD = IF = r

$\therefore$ ▭^{m}IFBD is a rhombus.

Also, $\angle B$ is a right angle.

$\therefore$ ▭IFBD is a square,

Next, by Pythagoras theorem, we get,

$AB^2 + BC^2 = AC^2$

$\therefore AC^2 = (24)^2 + (7)^2 = 576 + 49 = 625$

$\therefore AC = 25$

$\therefore AB + BC + AC = 24 + 7 + 25$

$\therefore AF + FB + BD + DC + AC = 56$

$\therefore AE + r + r + CE + AC = 56 \quad (\because AF = AE,\ DC = CE)$

$\therefore 2r + (AE + CE) + AC = 56$

$\therefore 2r + 2AC = 56 \quad (\because A-E-C)$

$\therefore 2r + 2(25) = 56$

$\therefore r + 25 = 28$

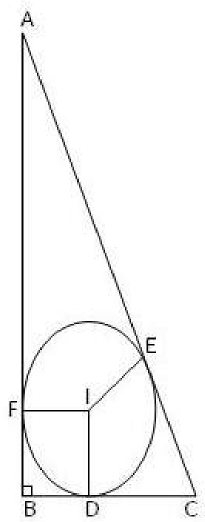
$\therefore r = 3$

The radius of a circle is 3.

Question 4:

A circle touches all the three sides of a right angled ΔABC in which $\angle B$ is right angle. Prove that the radius of the circle is $\frac{AB+BC+AC}{2}$.

Solution :



Given : A circle touches all the three sides of a right angled ΔABC in which $\angle B$ is a right angle.

To prove: radius of a circle $= \dfrac{AB + BC - AC}{2}$.

Proof : Let the radius be r and the centre of the circle touching all the sides of a triangle be I. (Incentre of triangle)

As shown in the figure,

$ID = IE = IF = r$

ΔABC is a right angled triangle, $m\angle B = 90°$.

Also, $\overline{ID} \perp \overline{BC}$ and $\overline{AB} \perp \overline{BC}$

$\therefore \overline{ID} \,||\, \overline{AB}$

$\therefore \overline{ID} \,||\, \overline{FB} \quad (\because A-F-B)$

Similarly, $\overline{IF} \,||\, \overline{BD}$

$\therefore \square IFBD$ is a parallelogram.

$\therefore FB = ID = r$ and $BD = IF = r \quad \ldots \quad \ldots (1)$

$\therefore \square^m IFBD$ is a rhombus.

Also, $\angle B$ is a right angle.

$\therefore \square IFBD$ is a square,

Now, $AE = AF \quad (\because$ by thm.$)$

$\therefore AE = AB - FB \quad (\because A-F-B)$

$\therefore AE = AB - r \quad (\text{by } (1)) \quad \ldots \quad \ldots (2)$

and $CE = CD \quad (\text{theorem } 11.3)$

$\therefore CE = BC - BD$

$CE = BC - r \quad (\text{by}(1)) \quad \ldots \quad \ldots (3)$

Now, $AC = AE + CE$

$AB = AB - r + BC - r \quad (\text{by } (2) \text{ and } (3))$

$AC = AB + BC - 2r$

$\therefore 2r = AB + BC - AC$

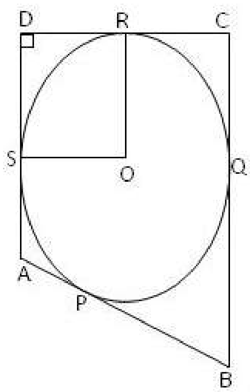
$\therefore r = \dfrac{AB + BC - AC}{2}$

Hence, the radius of a circle is $\dfrac{AB + BC - AC}{2}$.

Question 5:

In $\square$ABCD, m∠D = 90. A circle with centre O and radius r touches its sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ in P, Q, R and S respectively. If BC = 40, CD = 30 and BP = 25, then find the radius of the circle.

Solution :



We know that tangents drawn to a circle are perpendicular to the radius of the circle.

∴ m∠ORD = m∠OSD = 90°

m∠D = 90° (Given)

Also, OR = OS = radius

∴ □ORDS is a square. The tangents drawn to a circle from a point in the exterior of the circle are congruent.

∴ BP = BQ, CQ = CR and DR = DS (by thm.)

Now, BP = BQ

∴ BQ = 25 (∵ BP = 25)

∴ BC – CQ = 25 (∵ BC = 40)

∴ CQ = 15

∴ CR = 15 (∵ CQ=CR)

∴ CD – DR = 15 (∵ C – R – D)

∴ 30 – DR = 15 (∵ given CD = 30)

∴ DR = 15

But □ORDS is a square.

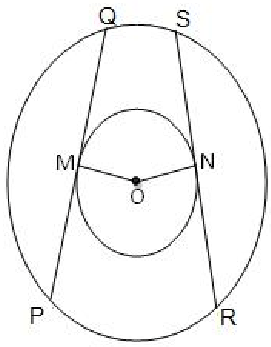
∴ OR = DR = 15

Thus, the radius of a circle is OR=15.

Question 6:

Two concentric circles are given. Prove that all chords of the circle with larger radius which touch the circle with smaller radius are congruent.

Solution :



Given : In two concentric circles, two chords $\overline{PQ}$ and $\overline{RS}$ of the circle with larger radius touch the cirlce with smaller radius.

To prove: $\overline{PQ} \cong \overline{RS}$

Proof : Let the chords $\overline{PQ}$ and $\overline{RS}$ be the chords of the circle with larger radius and it touches the circle with smaller radius at points M and N respectively.

$\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ are tangents to the circle with smaller radius,

So, OM = ON = radius of a smaller circle.

Hence, the chords $\overline{AB}$ and $\overline{CD}$ are at equidistance from the centre of the circle with larger radius.

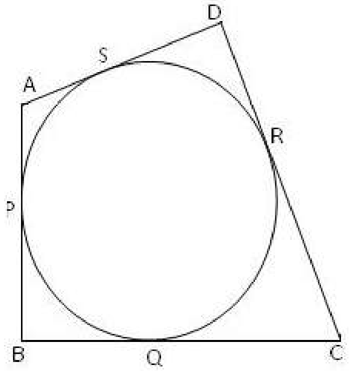
∴ PQ = RS

∴ $\overline{PQ} = \overline{RS}$

Question 7:

A circle touches all the sides of □ ABCD. If AB = 5, BC = 8, CD = 6. Find AD.

Solution :



If circle touches all the sides of a quadrilateral, then

AB + CD = BC + DA

∴ 5 + 6 = 8 + DA

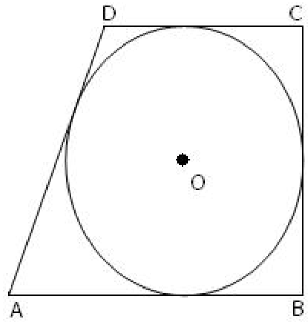
∴ 11 = 8 + DA

∴ AD = 3

Question 8:

A circle touches all the sides of $\square$ ABCD. If $\overline{AB}$ is the largest side then prove that $\overline{CD}$ is the smallest side.

Solution :



Given : A circle touches all the sides of ▫ABCD. $\overline{AB}$ is the largest side.

To prove: $\overline{CD}$ is the smallest side of ▫ABCD.

Proof: The circle touches all the sides of ▫ABCD.

$\therefore AB + CD = BC + DA \quad \ldots \quad \ldots (1)$

Given is $\overline{AB}$ is the largest side of ▫ABCD.

$\therefore AB > BC$

$\therefore AB = BC + m, \quad \text{where } m \in R^+$

Then, from (1),

$BC + m + CD = BC + DA$

$\therefore CD + m = DA$

$\therefore CD < DA \ \left(\because m \in R^+\right)$

Hence, $\overline{CD}$ is smaller than $\overline{DA} \quad \ldots \ \ldots (2)$

But, $\overline{AB}$ is the largest side.

$\therefore AB > DA$

$\therefore AB = DA + n, \quad \text{where } n \in R^+$

Then, from (1)

$DA + n + CD = BC + DA$

$\therefore CD + n = BC$

$\therefore CD < BC \ \left(\because n \in R^+\right)$

$\therefore \overline{CD}$ is smaller than $\overline{BC}. \quad \ldots \quad \ldots (3)$

$\overline{AB}$ is the largest side of ▫ABCD.

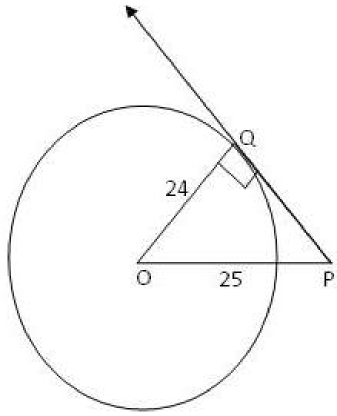
Hence, $\overline{CD}$ is smaller than $\overline{AB}. \quad \ldots \quad \ldots (4)$

Thus, from (2), (3) and (4), $\overline{CD}$ is the smallest side of ▫ABCD.

Question 9:

P is a point in the exterior of a circle having centre O and radius 24. OP = 25. A tangent from P touches the circle at Q. Find PQ.

Solution :



P lies in the exterior of a circle having centre O and $\overleftrightarrow{PQ}$ is a tangent.

$\therefore \overline{OQ} \perp \overline{PQ}$

In ΔOQP, $m\angle OQP = 90°$,

Here $OP = 25$ and $OQ = 24$

Now, $OP^2 = OQ^2 + PQ^2$

$\therefore PQ^2 = OP^2 - OQ^2$

$= (25)^2 - (24)^2$

$= 625 - 576$

$= 49$

$\therefore PQ = 7$

Question 10:

Select a proper option (a), (b), (c) or (d) from given options :

Question 10(1):

P is in exterior of ⊙(0, 15). A tangent from P touches the circle at T. If PT = 8, then OP=……

Solution :

a. 17

Here, $m\angle OTP = 90°$

$\therefore OP^2 = OT^2 + TP^2$

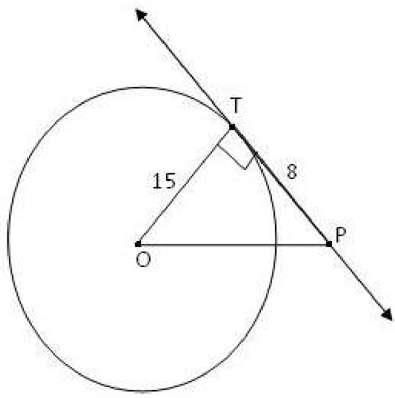
$\therefore OP^2 = (15)^2 + (8)^2$

$= 255 + 64$

$= 289$

$= (17)^2$

$\therefore OP = 17$



Question 10(2):

$\overleftrightarrow{PA}, \overleftrightarrow{PB}$ touch ⊙(O, r) at A and B. If m∠AOB = 80, then m∠OPB = ……

Solution :

b. 50°

ΔPOA and ΔPOB are congruent right angled triangle.

So $m\angle BOP = \frac{1}{2} m\angle AOB$

$= \frac{1}{2} \times 80$

$= 40$

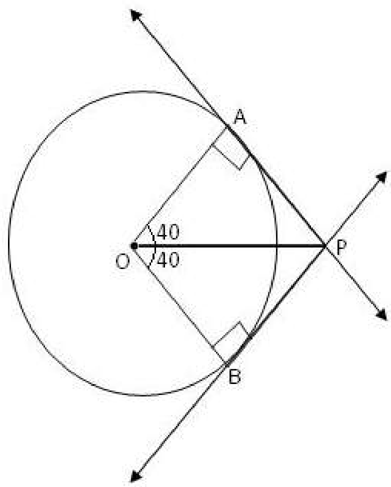
Now, in right angled ΔOBP,

$m\angle BOP + m\angle B + m\angle OPB = 180°$

$\therefore 40 + 90 + m\angle OPB = 180°$

$\therefore m\angle OPB + 130 = 180°$

$\therefore m\angle OPB = 180 - 130 = 50°$



Question 10(3):

A tangent from P, a point in the exterior of a circle, touches the circle at Q. If OP = 13, PQ = 5, then the diameter of the circle is ……

Solution :

d. 24

In right angled ΔOQP.

OP = 13 and PQ = 5.

Now $OP^2 = OQ^2 + PQ^2$

$\therefore (13)^2 = r^2 + (5)^2$

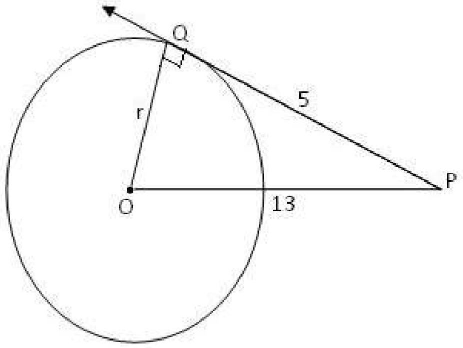
$\therefore r^2 = 169 - 25 = 144 = 12^2$

$\therefore r = 12$

Diameter of a circle

= 2× radius

= 2r = 2×12 = 24



Question 10(4):

In ΔABC, AB = 3, BC = 4, AC = 5, then the radius of the circle touching all the three sides is …..

Solution :

b. 1

Here, AB = 3, BC = 4 and AC = 5,

$AB^2 + BC^2 = (3)^2 + (4)^2$

$= 9 + 16 = 25$

$= 5^2 = AC^2$

$\therefore AB^2 + BC^2 = AC^2$

$\therefore$ ΔABC is a right angled triangle and ∠B is a right angle.

We know that, the radius of the circle touching all the three sides is $\frac{AB+BC-AC}{2}$.

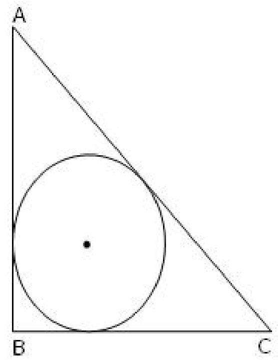
$\therefore$ The required radius of a circle

$= \frac{AB + BC - AC}{2}$

$= \frac{3 + 4 - 5}{2}$

$= \frac{2}{2}$

$= 1$



Question 10(5):

$\overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$ touch the circle with centre O at A and B respectively. If m∠OPB = 30 and OP = 10, then radius of the circle = ……..

Solution :

a. 5

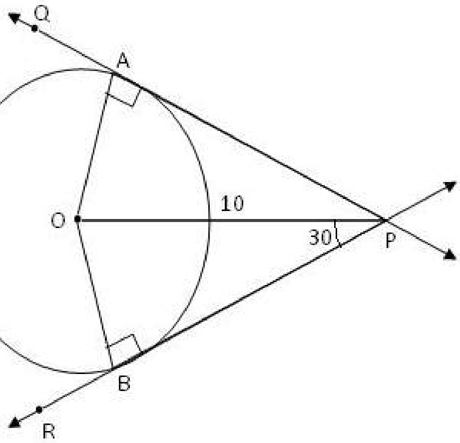
In right angled ΔOBP, $m\angle OPB = 30°$ and $OP = 10$.

Now, $\sin 30 = \frac{OB}{OP}$

$\therefore \frac{1}{2} = \frac{OB}{10}$

$\therefore OB = \frac{10}{2} = 5$

$\therefore$ Radius of a circle = OB = 5



Question 10(6):

The points of contact of the tangents from an exterior point P to the circle with centre O are A and B. If m∠OPB = 30, then m∠AOB = ………

Solution :

d. 120°

In right angled ΔOBP, m∠OPB = 30°.

Further,

m∠BOP + m∠OPB, m∠B = 180° (angles in linear pair)

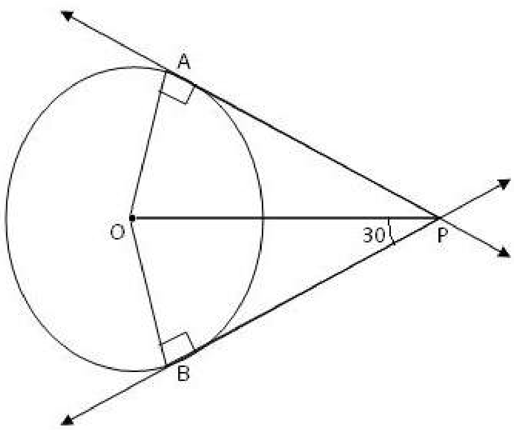
∴ m∠BOP + 30° + 90° = 180°

∴ m∠BOP = 180° – 120°

∴ m∠BOP = 60°

Now, m∠AOB = 2m∠BOP

= 2×60 = 120°



Question 10(7):

A chord of ⊙(O, 5) touches ⊙(O, 3). Therefore the length of the chord = ……….

Solution :

a. 8

Here, radius of a smaller circle OM = 3 and the radius of a bigger circle OB = 5.

In right angled ΔOMB,

$OB^2 = OM^2 + MB^2$

$\therefore (5)^2 = (3)^2 + MB^2$

$\therefore MB^2 = 25 - 9 = 16$

$\therefore MB = 4$

The length of a chord

$AB = 2 \times MB = 2 \times 4 = 8$