Determinats

Short Answer Type Quesitons

1. **If** $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$ **then find x.**

Sol. We have $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$. This gives

$2x^2 - 40 = 18 - 40 \quad \Rightarrow x^2 = 9 \Rightarrow x = \pm 3.$

2. **If** $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}, \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$ **, then prove that** $\Delta + \Delta_1$ **= 0.**

Sol. We have $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$

Interchanging rows and columns, we get

$$\Delta_1 = \begin{vmatrix} 1 & yz & x \\ 1 & zx & y \\ 1 & xy & z \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} x & xyz & x^2 \\ y & xyz & y^2 \\ z & xyz & z^2 \end{vmatrix}$$

$= \frac{xyz}{xyz}\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix}$ Interchanging C_1 and C_2

$$= (-1)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = -\Delta \;\Rightarrow\; \Delta_1 + \Delta = 0$$

3. **Without expanding, show that**

$$\Delta = \begin{vmatrix} \text{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \text{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0$$

Sol. Applying $C_1 \to C_1 - C_2 - C_3$, we have

$$\Delta = \begin{vmatrix} \cos ec^2\theta - \cot^2\theta - 1 & \cot^2\theta & 1 \\ \cot^2\theta - \cos ec^2\theta + 1 & \cos ec^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = \begin{vmatrix} 0 & \cot^2\theta & 1 \\ 0 & \text{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = 0$$

4. **Show that** $\Delta = \begin{vmatrix} x & p & q \\ p & x & q \\ q & q & x \end{vmatrix} = (x-p)(x^2 + px - 2q^2)$

Sol. Applying $C_1 \to C_1 - C_2$, we have

$$\Delta = \begin{vmatrix} x-p & p & q \\ p-x & x & q \\ 0 & q & x \end{vmatrix} = (x-p)\begin{vmatrix} 1 & p & q \\ -1 & x & q \\ 0 & q & x \end{vmatrix}$$

$$= (x-p)\begin{vmatrix} 0 & p+x & 2q \\ -1 & x & q \\ 0 & q & x \end{vmatrix} \text{Applying } R_1 \to R_1 + R_2$$

Expanding along C_1. We have $\Delta = (x-p)\left(px + x^2 - 2q^2\right) = (x-p)\left(x^2 + px - 2q^2\right)$

5. **If** $\Delta = \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$**, then show that Δ is equal to zero.**

Sol. Interchanging rows and columns, we get $\Delta = \begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$

Taking '–1' common from R_1, R_2 and R_3, we get

$$\Delta = (-1)^3 \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix} = -\Delta \quad \Rightarrow \quad 2\Delta = 0 \text{ or } \Delta = 0$$

6. **Prove that** $\left(A^{-1}\right)' = \left(A'\right)^{-1}$**, where A is an invertible matrix.**

Sol. Since A is an invertible matrix, so it is non-singular.
We know that $|A| = |A'|$. But $|A| \neq 0$. So $|A'| \neq 0$ i.e. A' is invertible matrix.
Now, we know that $AA^{-1} = A^{-1}A = I$.
Taking transpose on both sides, we get $(A^{-1})' A' = A'(A^{-1})' = (I)' = I$
Hence $(A^{-1})'$ is inverse of A', i.e., $(A')^{-1} = (A^{-1})'$

<u>Long Answer Type Questions</u>

7. **If x= – 4 is a root of** $\Delta = \begin{vmatrix} x & 2 & 3 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix} = 0,$ **then find the other two roots.**

Sol. Applying $R_1 \to \left(R_1 + R_2 + R_3\right)$, we get

$$\begin{vmatrix} x+4 & x+4 & x+4 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$$

Taking (x+4) common from R_1, we get

$$\Delta = (x+4)\begin{vmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1,$ we get

$$\Delta = (x+4)\begin{vmatrix} 1 & 0 & 0 \\ 1 & x-1 & 0 \\ 3 & -1 & x-3 \end{vmatrix}$$

Expanding along R_1,
Δ = (x+ 4) [(x – 1) (x– 3) – 0]. Thus, Δ = 0 implies
$x = -4, 1, 3$

8. **In a triangle ABC, if** $\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0$ **then prove that ΔABC is an isoceles triangle.**

Sol. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ -\cos^2 A & -\cos^2 B & -\cos^2 C \end{vmatrix} R_3 \to R_3 - R_2$$

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B - \sin A & \sin C - \sin B \\ \cos^2 A & \cos^2 A - \cos^2 B & \cos^2 B - \cos^2 C \end{vmatrix} .(C_3 \to C_3 - C_2 \ and \ C_2 \to C_2 - C_1)$$

Expanding along R_1, we get

$$\Delta = (sinB - sinA)(sin^2 C - sin^2 B) - (sinC - sin\ B)(sin^2 B - sin^2 A)$$

= (sinB – sinA) (sinC – sinB) (sinC – sin A) = 0
⇒ either sinB – sinA = 0 or sinC – sinB or sinC – sinA = 0
⇒ A = B or B = C or C = A
i.e. triangle ABC is isosceles.

9. **Show that if the determinant** $\Delta = \begin{vmatrix} 3 & -2 & \sin 3\theta \\ -7 & 8 & \cos 2\theta \\ -11 & 14 & 2 \end{vmatrix} = 0,$ **then** $\sin\theta = 0 \ or \ \frac{1}{2}$

Sol. Applying $R_2 \to R_2 + 4R_1$ and $R_3 \to R_3 + 7R_1$, we get

$$\begin{vmatrix} 3 & -2 & \sin 3\theta \\ 5 & 0 & \cos 2\theta + 4\sin 3\theta \\ 10 & 0 & 2 + 7\sin 3\theta \end{vmatrix} = 0$$

or 2 [5 (2 + 7 sin3θ) – 10 (cos2θ+ 4sin3θ)] = 0
or 2 + 7 sin3θ– 2 cos2θ– 8 sin3θ= 0
or 2 – 2cos 2θ– sin 3θ= 0
sinθ(4sin^{2}θ+4sinθ– 3) = 0
or sinθ= 0 or (2sinθ– 1) = 0 or (2sinθ + 3) = 0
or sinθ= 0 or $sin\theta = \frac{1}{2}$.

Objective Type Questions

Choose the correct answer from the given four options in each of the Example 10 and 11.

10. **Let** $\Delta = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}$ *and* $\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}$, **then**

(A) $\Delta_1 = -\Delta$ **(B) $\Delta \neq \Delta_1$** **(C) $\Delta - \Delta_1 = 0$** **(D) None of these**

Sol. (C) is the correct answer since $\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix} = \begin{vmatrix} A & x & yz \\ B & y & zx \\ C & z & xy \end{vmatrix}$

$$= \frac{1}{xyz}\begin{vmatrix} Ax & x^2 & xyz \\ By & y^2 & xyz \\ Cz & z^2 & xyz \end{vmatrix} = \frac{xyz}{xyz}\begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix} = \Delta$$

11. **If x, y$\in$ R, then the determinant** $\Delta = \begin{vmatrix} \cos x & -\sin x & 1 \\ \sin x & \cos x & 1 \\ \cos(x+y) & -\sin(x+y) & 0 \end{vmatrix}$ **lies in the interval.**

(A) $\left[-\sqrt{2},, \sqrt{2}\right]$ **(B) [–1, 1]** **(C)** $\left[-\sqrt{2}, 1\right]$ **(D)** $\left[-1, -\sqrt{2}\right]$

Sol. The correct choice is A. Indeed applying $R_3 \to R_3 - \cos y R_1 + \sin y R_2$, we get

$$\Delta = \begin{vmatrix} \cos x & -\sin x & 1 \\ \sin x & \cos x & 1 \\ 0 & 0 & \sin y - \cos y \end{vmatrix}$$

Expanding along R_3, we have

$\Delta = (\sin y - \cos y)(\cos^2 x + \sin^2 x)$

$= (\sin y - \cos y) = \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin y - \frac{1}{\sqrt{2}}\cos y\right]$

$\sqrt{2}\left[\cos\frac{\pi}{4}\sin y - \sin\frac{\pi}{4}\cos y\right]$

$= \sqrt{2} sin \left(y - \frac{\pi}{4}\right)$

Hence $-2 \le \Delta \le 2$.

Fill in the blanks in each of the Examples 12 to 14.

12. **If A, B, C are the angles of a triangle, then**

$$\Delta = \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix} = \ldots\ldots\ldots\ldots$$

Sol. Answer is 0. Apply $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$.

13. **The determinant** $\Delta = \begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$ **is equal to ....**

Sol. Answer is 0. Taking $\sqrt{5}$ common from C_2 and C_3 and applying $C_1 \to C_3 - \sqrt{3}C_2$, we get the desired result.

14. **The value of the determinant**

$$\Delta = \begin{vmatrix} \sin^2 23^0 & \sin^2 67^0 & \cos 180^0 \\ -\sin^2 67^0 & -\sin^2 23^0 & \cos^2 180^0 \\ \cos 180^0 & \sin^2 23^0 & \sin^2 67^0 \end{vmatrix} = ...$$

Sol. Δ= 0. Apply $C_1 \to C_1 + C_2 + C_3$.

State whether the statements in the s 15 to 18 is True or False.

15. **The determinant**

$$\Delta = \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ -\cos x & \sin x & \cos y \end{vmatrix}$$ **is independent of x only.**

Sol. True. Apply $R_1 \to R_1 + sinyR_2 + cosyR_3$, and expand.

16. **The value of** $\begin{vmatrix} 1 & 1 & 1 \\ {}^nC_1 & {}^{n+2}C_1 & {}^{n+4}C_1 \\ {}^nC_2 & {}^{n+2}C_2 & {}^{n+4}C_2 \end{vmatrix}$ **is 8.**

Sol. True

17. **If** $A = \begin{bmatrix} x & 5 & 2 \\ 2 & y & 3 \\ 1 & 1 & z \end{bmatrix}, xyz = 80,\ 3x+2y+10z = 20,$ **then**

$$A\ adj.\ A = \begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}.$$

Sol. False.

18. **If** $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & x \\ 2 & 3 & 1 \end{bmatrix},\ A^{-1} = \begin{bmatrix} \frac{1}{2} & -4 & \frac{5}{2} \\ -\frac{1}{2} & 3 & -\frac{3}{2} \\ \frac{1}{2} & y & \frac{1}{2} \end{bmatrix}$ **then x= 1, y= – 1.**

Sol. True

Determinats

Objective Type Questions (M.C.Q.)

Choose the correct answer from given four options in each of the Exercises from 24 to 37.

24. **If** $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, **, then value of x is**

(A) 3
(B) ± 3
(C) ±6
(D) 6

Sol. (C) $\because \begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$

$\Rightarrow 2x^2 - 40 = 18 + 14$

$\Rightarrow 2x^2 = 32 + 40$

$\Rightarrow x^2 = \frac{72}{2} = 36$

$\therefore$ x=± 6

25. **The value of determinant** $\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$

(A) $a^3 + b^3 + c^3$
(B) 3 bc
(C) $a^3 + b^3 + c^3 - 3abc$
(D) None of these

Sol. We have,

$$\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix} = \begin{vmatrix} a+c & b+c+a & a \\ b+c & c+a+b & b \\ c+b & a+b+c & c \end{vmatrix} \left[\because C_1 \to C_1 + C_2 \; and \; C_2 \to C_2 + C_3\right]$$

$$= (a+b+c)\begin{vmatrix} a+c & 1 & a \\ b+c & 1 & b \\ c+b & 1 & c \end{vmatrix}$$ [taking (a + b + c) common from C_2]

$$= (a+b+c)\begin{vmatrix} a-b & 0 & a-c \\ 0 & 0 & b-c \\ c+b & 1 & c \end{vmatrix} \left[\because R_2 \to R_2 - R_3 \; and \; R_1 \to R_1 - R_3\right]$$

$= (a+b+c)\left[-(b-c).(a-b)\right]$ [expanding along R_2]

$= (a+b+c)(c-b)(a-b)$

26. **The area of a triangle with vertices (–3, 0), (3, 0) and (0, k) is 9 sq. units. Then, the value of k will be**

(A) 9
(B) 3
(C) –9

(D) 6

Sol. (B) We know that, area of a triangle with vertices $(x_1, y_1),(x_2 y_2)$ *and* (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\therefore\ \Delta = \frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix}$$

Expanding along $R_{1,}$

$$9 = \frac{1}{2}\left[-3(-k)-0+1(3k)\right]$$

$$\Rightarrow\ 18=3k+3k=6k$$

$$\therefore\ k=\frac{18}{6}=3$$

27. The determinant $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$ **equals**

(A) $abc(b-c)\ (c-\ a)\ (a\ -b)$

(B) $(b-c)\ (c-\ a)\ (a\ -b)$

(C) $(a\ +b\ +c)\ (b\ -c)\ (c-\ a)\ (a\ -b)$

(D) None of these

Sol. We have,

$$\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix} = \begin{vmatrix} b(b-a) & b-c & c(b-a) \\ a(b-a) & a-b & b(b-a) \\ c(b-a) & c-a & a(b-a) \end{vmatrix}$$

$$=(b-a)^2\begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$$

[on taking (b - a) common from C_1 and C_3 each]

$$=(b-a)^2\begin{vmatrix} b-c & b-c & c \\ a-b & a-b & b \\ c-a & c-a & a \end{vmatrix}\ [\because\ C_1 \to C_1 - C_3] = 0$$

[Since, two columns C_1 and C_2 are identical, so the value of determinant is zero]

28. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ **in the interval** $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ **is**

(A) 0
(B) 2
(C) 1
(D) 3

Sol. We have, $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$,

$$\begin{vmatrix} 2\cos x + \sin x & \cos x & \cos x \\ 2\cos x + \sin x & \sin x & \cos x \\ 2\cos x + \sin x & \cos x & \sin x \end{vmatrix} = 0$$

On taking $(2cosx + sinx)$ common from C₁, we get

$$\Rightarrow (2\cos x + \sin x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

$$\Rightarrow \ (2\cos x + \sin x)\begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & (\sin x - \cos x) \end{vmatrix} = 0$$

$[\because R_2 \to R_2 - R_1 \ and \ R_3 \to R_3 - R_1]$

Expanding along C_1,

$$(2\cos x + \sin x)\left[1.(\sin x - \cos x)^2\right] = 0$$

$$\Rightarrow (2\cos x + \sin x)(\sin x - \cos x)^2 = 0$$

Either $2\cos x = -\sin x$

$$\Rightarrow \ \cos x = -\frac{1}{2}\sin x$$

$$\Rightarrow \ \tan x = -2 \ \ ...(i)$$

But here for $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$, we get $-1 \le \tan x \le 1$ so, no solution possible.

and for $(\sin x - \cos x)^2 = 0, \sin x = \cos x$

$$\Rightarrow \tan x = 1 = \tan\frac{\pi}{4}$$

$$\therefore \ \ x = \frac{\pi}{4}$$

So, only one distinct real root exists.

29. If A, B and C are angles of a triangle, then the determinant

$\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$ **is equal to**

(A) 0

(B) – 1
(C) 1
(D) None of these

Sol. We have, $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$

Applying $C_1 \to aC_1 + bC_2 + cC_3$.

$$\begin{vmatrix} -a + b\cos C + c\,cosB & \cos C & \cos B \\ a\cos C - b + c\cos A & -1 & \cos A \\ a\cos B + b\cos A - c & \cos A & -1 \end{vmatrix}$$

Also, by projection rule in a triangle, we know that

$a = b\,cosC + c\,cosB$,

$b = c\,cosA + a\,cosC \; and$

$c = a\,cosB + b\,cosA$

Using above equation in column first, we get

$$\begin{vmatrix} -a + a & \cos C & \cos B \\ b - b & -1 & \cos A \\ c - c & \cos A & -1 \end{vmatrix} = \begin{vmatrix} 0 & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{vmatrix} = 0$$

[Since, determinant having all elements of any column or row gives value of determinant as zero]

30. **Let** $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$**, then** $\lim_{t \to 0} \frac{f(t)}{t^2}$ **is equal to**

(A) 0
(B) – 1
(C) 2
(D) 3

Sol. We have,

$$f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$$

Expanding along C_1,

$= \cos t\left(t^2 - 2t^2\right) - 2\sin t\left(t^2 - t\right) + \sin t\left(2t^2 - t\right)$

$= -t^2 \cos t - \left(t^2 - t\right) 2\sin t + \left(2t^2 - t\right)\sin t$

$= -t^2 \cos t - t^2.2\sin t + t.2\sin t + 2t^2 \sin t$

$= -t^2 \cos t + 2t\sin t$

$$\therefore \lim_{t \to 0} \frac{f(t)}{t^2} = \lim_{t \to 0} \frac{\left(-t^2 \cos t\right)}{t^2} + \lim_{t \to 0} \frac{2t\sin t}{t^2}$$

$= -\lim_{t\to 0} \cos t + 2.\lim_{t\to 0} \frac{\sin t}{t}$

$= -1 + 1 \left[\because \lim_{t\to 0} \frac{\sin t}{t} = 1 \; and \; \cos 0 = 1\right]$

= 0

31. **The maximum value of** $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1+\cos\theta & 1 & 1 \end{vmatrix}$ **is (θ is real number)**

(A) $\frac{1}{2}$

(B) $\frac{\sqrt{3}}{2}$

(C) $\sqrt{2}$

(D) $\frac{2\sqrt{3}}{4}$

Sol. Since,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1+\cos\theta & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 0 & \sin\theta & 1 \\ \cos\theta & 0 & 1 \end{vmatrix} \left[\because C_1 \to C_2 - C_3 \; and \; C_2 \to C_2 - C_3\right]$$

$= 1(\sin\theta.\cos\theta)$

$= \frac{1}{2}.2\sin\cos\theta = \frac{1}{2}\sin 2\theta$

Since, the maximum value of $sin2\theta$ is 1. So, for maximum value of θ should be 45^0

$\therefore \Delta - \frac{1}{2}\sin 2.45^0$

$= \frac{1}{2}\sin 90^0 = \frac{1}{2}.1 = \frac{1}{2}$

32. **If** $f(x)$ **=** $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$**, then**

(A) $f(a)$ **= 0**

(B) $f(b)$ **= 0**

(C) $f(0)$ **= 0**

(D) $f(1)$ **= 0**

Sol. We have,

$$f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$$

$$\Rightarrow f(a) = \begin{vmatrix} 0 & 0 & a-b \\ 2a & 0 & a-c \\ a+b & a+c & 0 \end{vmatrix}$$

$$= \left[(a-b)\{2a.(a+c)\}\right] \neq 0$$

$$\therefore f(b) = \begin{vmatrix} 0 & b-a & 0 \\ b+a & 0 & b-c \\ 2b & b+c & 0 \end{vmatrix}$$

$$= -(b-a)\left[2b(b-c)\right]$$

$$= -2b(b-a)(b-c) \neq 0$$

$$\therefore f(0) = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

$$= a(bc) - b(ac)$$

$$= abc - abc = 0$$

33. **If** $A = \begin{bmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix}, then\ A^{-1}$ **exists if**

(A) λ = 2
(B) λ ≠ 2
(C) λ ≠ – 2
(D) None of these

Sol. We have,

$$A = \begin{bmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix}$$

Expanding along R_1,

$$|A| = 2(6-5) - \lambda(-5) - 3(-2) = 2 + 5\lambda + 6$$

We know that, A^{-1} exists, if A is non-singular matrix i.e., $|A| \neq 0$.

$\therefore\ 2+5\lambda+6 \neq 0$

$\Rightarrow\ 5\lambda \neq -8$

$\therefore\ \lambda \neq \frac{-8}{5}$

So, A^{-1} exists if and only if $\lambda \neq \frac{-8}{5}$

34. **If A and B are invertible matrices, then which of the following is not correct?**

(A) $adj\ A = |A|.A^{-1}$

(B) $det(A)^{-1} = [det(A)]^{-1}$

(C) $(AB)^{-1} = B^{-1}A^{-1}$

(D) $(A+B)^{-1} = B^{-1} + A^{-1}$

Sol. (D) Since, A and B are invertible matrices, So, we can say that

$(AB)^{-1} = B^{-1}A^{-1}$ ......(i)

Also, $A^{-1} = \frac{1}{|A|}(adj\ A)$

$\Rightarrow adj\ \text{A} = |A|.A^{-1}$ ...(ii)

Also, $\det(A)^{-1} = [\det(A)]^{-1}$

$\Rightarrow \det(A)^{-1} = \frac{1}{[\det(A)]}$

$\Rightarrow \det(A).\det(A)^{-1} = 1$ ...(iii)

Which is true.

Again, $(A+B)^{-1} = \frac{1}{|(A+B)|} adj(A+B)$

$\Rightarrow (A+B)^{-1} \neq B^{-1} + A^{-1}$ ...(iv)

So, only option (d) is incorrect.

35. If x, y, z are all different from zero and $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$**, then value of** $x^{-1} + y^{-1} + z^{-1}$ **is**

(A) xyz

(B) $x^{-1}y^{-1}z^{-1}$

(C) –x–y–z

(D) –1

Sol. We have, $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$,

$\Rightarrow \begin{vmatrix} x & 0 & 1 \\ 0 & y & 1 \\ -z & -z & 1+z \end{vmatrix} = 0$

Expanding along R_1

$x[y(1+z)+z] - 0 + 1(yz) = 0$

$\Rightarrow x(y + yz + z) + yz = 0$

$\Rightarrow xy+xyz+xz+yz=0$

$\Rightarrow \frac{xy}{xyz}+\frac{xyz}{xyz}+\frac{xz}{xyz}+\frac{yz}{xyz}=0$ [on dividing (xyz) from both sides]

$\Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1=0$

$\Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=-1$

$\therefore\ x^{-1}+y^{-1}+z^{-1}=-1$

36. **The value of the determinant** $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ **is**

(A) $9x^2(x+y)$

(B) $9y^2(x+y)$

(C) $3y^2(x+y)$

(D) $7x^2(x+y)$

Sol. We have, $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

$=\begin{vmatrix} 3(x+y) & x+y & y \\ 3(x+y) & x & y \\ 3(x+y) & x+2y & -2y \end{vmatrix}$ $[\because\ C_1 \to C_1+C_2+C_3 \ and \ C_3 \to C_3-C_2]$

$=3(x+y)\begin{vmatrix} 1 & (x+y) & y \\ 1 & x & y \\ 1 & (x+2y) & -2y \end{vmatrix}$ [taking 3(x+y) common from first column]

$=3(x+y)\begin{vmatrix} 0 & y & 0 \\ 1 & x & y \\ 1 & (x+2y) & -2y \end{vmatrix}$ $[\because R_1 \to R_1-R_2]$

Expanding along R_1,

$=3(x+y)\left[-y(-2y)-y\right]$

$=3y^2.3(x+y)=9y^2(x+y)$

37. **There are two values of a which makes determinant,** $\Delta=\begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix}=86,$ **then**

sum of these number is

(A) 4

(B) 5

(C) – 4

(D) 9

Sol. We have,

$$\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86,$$

$\Rightarrow 1\left(2a^2+4\right)-2\left(-4a-20\right)+0=86$ [expanding along first column]

$\Rightarrow 2a^2+4+8a+40=86$

$\Rightarrow 2a^2+8a+44-86=0$

$\Rightarrow a^2+4a-21=0$

$\Rightarrow a^2+7a-3a-21=0$

$\Rightarrow (a+7)(a-3)=0$

a=-7 and 3

$\therefore$ Required sum=-7+3=-4

<u>Fill in the blanks</u>

38. If A is a matrix of order 3×3, then |3A| is equal to ______ .

Sol. If A is a matrix of order 3×3, then $|3A| = 3\times3\times3|A| = 27|A|$

39. If A is invertible matrix of order 3×3, then $|A^{-1}|$ is equal to ______ .

Sol. If A is invertible matrix of order 3×3, then $\left|A^{-1}\right| = \frac{1}{|A|}$. $\left[\text{since}, |A|.\left|A^{-1}\right| = 1\right]$

40. If $x,\ y,\ z\ \in R,$ then the value of determinant $\begin{vmatrix} \left(2^x+2^{-x}\right)^2 & \left(2^x-2^{-x}\right)^2 & 1 \\ \left(3^x+3^{-x}\right)^2 & \left(3^x-3^{-x}\right)^2 & 1 \\ \left(4^x+4^{-x}\right)^2 & \left(4^x-4^{-x}\right) & 1 \end{vmatrix}$ **is equal to**

_______.

Sol. We have, $\begin{vmatrix} \left(2^x+2^{-x}\right)^2 & \left(2^x-2^{-x}\right)^2 & 1 \\ \left(3^x+3^{-x}\right)^2 & \left(3^x-3^{-x}\right)^2 & 1 \\ \left(4^x+4^{-x}\right)^2 & \left(4^x-4^{-x}\right)^2 & 1 \end{vmatrix}$

$$= \begin{vmatrix} \left(2.2^x\right)\left(2.2^{-x}\right) & \left(2^x-2^{-x}\right)^2 & 1 \\ \left(2.3^x\right)\left(2.3^{-x}\right) & \left(3^x-3^{-x}\right)^2 & 1 \\ \left(2.4^x\right)\left(2.4^{-x}\right) & \left(4^x-4^{-x}\right)^2 & 1 \end{vmatrix} \left[\because (a+b)^2-(a-b)^2=4ab\right]$$

$[\because C_1 \to C_1 - C_2]$

$$= \begin{vmatrix} 4 & \left(2^x-2^{-x}\right)^2 & 1 \\ 4 & \left(3^x-3^{-x}\right)^2 & 1 \\ 4 & \left(4^x-4^{-x}\right)^2 & 1 \end{vmatrix} = 0$$ [Since, C_1 and C_3 are proportional to each other]

41. **If** $\cos 2\theta = 0$**, then** $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2$ **= ________.**

Sol. Since, $\cos 2\theta = 0$

$\Rightarrow \cos 2\theta = \cos\frac{\pi}{2} \Rightarrow 2\theta = \frac{\pi}{2}$

$\Rightarrow \theta = \frac{\pi}{4}$

$\therefore \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ *and* $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\therefore \begin{vmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{vmatrix}$$

Expanding along R_1,

$$= \left[-\frac{1}{\sqrt{2}}\left(\frac{1}{2}\right) + \frac{1}{\sqrt{2}}\left(-\frac{1}{2}\right)\right]^2 = \left[\frac{-2}{2\sqrt{2}}\right]^2 = \left(\frac{-1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

42. **If A is a matrix of order 3×3, then $(A^2)^{-1}$ = ________.**

Sol. If A is a matrix of order 3×3, then $\left(A^2\right)^{-1} = \left(A^{-1}\right)^2$.

43. **If A is a matrix of order 3×3, then number of minors in determinant of A are ________.**

Sol. If A is a matrix of order 3×3, then the number of minors in determinant of A are 9.
[Since, in a 3×3 matrix, these are 9 elements]

44. **The sum of the products of elements of any row with the co-factors of corresponding elements is equal to ________.**

Sol. The sum of the products of elements of any row with the co-factors of corresponding elements is equal to value of the determinant.

$$\text{Let } \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along R_1,

$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

= Sum of products of elements of R_1 with their corresponding cofactors.

45. **If x= – 9 is a root of** $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$**, then other two roots are ________.**

Sol. Since, $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$

Expanding along R_1,

$x\left(x^2-12\right)-3(2x-14)+7(12-7x)=0$

$\Rightarrow x^3-12x-6x+42+84-49x=0$

$\Rightarrow x^3-67x+126=0$ ...(i)

Here, $126\times1=9\times2\times7$

For x=2, $2^3-67\times2+126=134-134=0$

Hence, x = 2 is a root.

For x=7, $7^3-67\times7+126=469-469=0$

Hence, x = 7 is also a root.

46. $\begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix}$ **= ________.**

Sol. We have, $\begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix} = \begin{vmatrix} z-x & xyz & x-z \\ z-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix} \left[\because C_1 \to C_1 - C_3\right]$

$=(z-x)\begin{vmatrix} 1 & xyz & x-z \\ 1 & 0 & y-z \\ 1 & z-y & 0 \end{vmatrix}$

[taking (z - x) common from column 1]

Expanding along R_1,

$=(z-x)\left[1.\{-(y-z)(z-y)\}-xyz(z-y)+(x-z)(z-y)\right]$

$=(z-x)(z-y)(-y+z-xyz+x-z)$

$=(z-x)(z-y)(x-y-xyz)$

$=(z-x)(y-z)(y-x+xyz)$

47. **If** $f(x)=\begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix} = A+Bx+Cx^2+....then\ A=$ ______.

Sol. Since, $f(x) \ =(1+x)^{17}(1+x)^{23}(1+x)41\begin{vmatrix} 1 & (1+x)^2 & (1+x)^6 \\ 1 & (1+x)^6 & (1+x)^{11} \\ 1 & (1+x)^2 & (1+x)^6 \end{vmatrix}=0$

$\left[\text{since}, R_1\ and\ R_3\ are\ identical\right]$

$\therefore A=0$

State True or False for the statements of the following Exercises:

48. $\left(A^3\right)^{-1}=\left(A^{-1}\right)^3$ **where A is a square matrix and |A| ≠0.**

Sol. True

Since, $\left(A^n\right)^{-1}=\left(A^{-1}\right)^n$, where $n\in N$.

49. $(aA)^{-1} = \frac{1}{a}A^{-1}$**, where a is any real number and A is a square matrix.**

Sol. False

Since, we know that, if A is a non-singular square matrix, then for any scalar a (non-zero), aA is invertible such that

$$(aA) = \left(\frac{1}{a}A^{-1}\right) = \left(a.\frac{1}{a}\right)\left(A.A^{-1}\right)$$

i.e. (aA) is inverse of $\left(\frac{1}{a}A^{-1}\right)$ or $(aA)^{-1} = \frac{1}{a}A^{-1}$, where a is any non-zero scalar.

In the above statement a is any real number. So, we can conclude that above statement is false.

50. $\left|A^{-1}\right| \neq |A|^{-1}$**, where A is non-singular matrix.**

Sol. False

$|A^{-1}| = |A|^{-1}$, where A is a non-singular matrix.

51. If A and B are matrices of order 3 and |A| = 5, |B| = 3, then |3AB| = 27 × 5 × 3 = 405.

Sol. True

We know that, $|AB| = |A|.|B|$

$\therefore |3AB| = 27|AB|$

$= 27|A|.|B|$

$= 27 \times 5 \times 3 = 405$

52. If the value of a third order determinant is 12, then the value of the determinant formed by replacing each element by its co-factor will be 144.

Sol. True

Let A is the determinant

$\therefore\ |A| = 12$

Also, we know that, if A is a square matrix of order n, then $|adj\ A| = |A|^{n-1}$

For n= 3, $|adj\ A| = |A|^{3-1} = |A|^2$

$= (12)^2 = 144$

53. $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0,$ **where a, b, care in A.P.**

Sol. True

Since, a, b and c are in AP, then 2b = a + c

$$\therefore \begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0,$$

$$\Rightarrow \begin{vmatrix} 2x+4 & 2x+6 & 2x+a+c \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0, \quad [\because R_1 \to R_1 + R_3]$$

$$\Rightarrow \begin{vmatrix} 2(x+2) & 2(x+3) & 2(x+b) \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0, [\because\ 2b = a+c]$$

$\Rightarrow\ 0 = 0$ [since, R_1 *and* R_2 are in proportional to each other]

Hence, statement is true.

54. $|adj.\ A| = |A|^2$ **, where A is a square matrix of order two.**

Sol. False

If A is a square matrix of order n, then

$|adj.\ A| = |A|^{n-1}$

$\Rightarrow |adj.\ A| = |A|^{2-1} = |A| \quad [\because\ n = 2]$

55. **The determinant** $\begin{vmatrix} sinA & cosA & sinA + cosB \\ sinB & cosA & sinB + cosB \\ sinC & cosA & sinC + cosB \end{vmatrix}$ **is equal to zero.**

Sol. True

Since, $\begin{vmatrix} sinA & cosA & sinA + cosB \\ sinB & cosA & sinB + cosB \\ sinC & cosA & sinC + cosB \end{vmatrix}$

$= \begin{vmatrix} sinA & cosA & sinA \\ sinB & cosA & sinB \\ sinC & cosA & sinC \end{vmatrix} + \begin{vmatrix} sinA & cosA & cosB \\ sinB & cosA & cosB \\ sinC & cosA & cosB \end{vmatrix}$

$= 0 + \begin{vmatrix} sinA & cosA & cosB \\ sinB & cosA & cosB \\ sinC & cosA & cosB \end{vmatrix}$

[Since, in first determinant C_1 *and* C_3 are identicals]

$= \cos A . \cos B \begin{vmatrix} \text{sinA} & 1 & 1 \\ \text{sinB} & 1 & 1 \\ \text{sinC} & 1 & 1 \end{vmatrix}$

[taking cosA common from C_2 and cosB common from C_3]

=0 [since, C_2 and C_3 are identicals]

56. **If the determinant** $\begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$ **splits into exactly K determinants of order 3, each element of which contains only one term, then the value of K is 8.**

Sol. True

Since, $\begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$

$$=\begin{vmatrix} x & p & l \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix} = \begin{vmatrix} a & u & f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix} \text{ [splitting first row]}$$

$$=\begin{vmatrix} x & p & l \\ y & q & m \\ z+c & r+w & n+h \end{vmatrix} + \begin{vmatrix} x & p & l \\ b & v & g \\ z+c & r+w & n+h \end{vmatrix}$$

$$+\begin{vmatrix} a & u & f \\ y & q & m \\ z+c & r+w & n+h \end{vmatrix} + \begin{vmatrix} a & u & f \\ b & v & g \\ z+c & r+w & n+h \end{vmatrix} \text{ [splitting second row]}$$

Similarly, we can split these 4 determinants in 8 determinants by splitting each one in two determinants further. So, given statement n is true.

57. **Let** $\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 16$ **, then** $\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix} = 32$

Sol. True

We have $\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 16$

and we have to prove, $\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix} = 32$

$$\Delta_1 = \begin{vmatrix} 2p+2x+2a & a+x & a+p \\ 2q+2y+2b & b+y & b+q \\ 2r+2z+2c & c+z & c+r \end{vmatrix} [\because C_1 \to C_1 + C_2 + C_3]$$

$$= 2\begin{vmatrix} p & x-p & a+p \\ q & y-q & b+q \\ r & z-r & c+r \end{vmatrix}$$

[taking 2 common from C_1 and then $C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$]

$$= 2\begin{vmatrix} p & x & a+p \\ q & y & b+q \\ r & z & c+r \end{vmatrix} - \begin{vmatrix} p & p & a+p \\ q & q & b+q \\ r & r & c+r \end{vmatrix}$$

$$= 2\begin{vmatrix} p & x & a+p \\ q & y & b+q \\ r & z & c+r \end{vmatrix} - 0$$

[Since, two columns C_1 *and* C_2 are indenticals]

$$= 2\begin{vmatrix} p & x & a \\ q & y & b \\ r & z & c \end{vmatrix} + 2\begin{vmatrix} p & x & p \\ q & y & q \\ r & z & r \end{vmatrix}$$

$$= 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} + 0$$

[Since, C_1 *and* C_3 are identical in second determinant and in first determinant, $C_1 \leftrightarrow C_2$ and then $C_1 \leftrightarrow C_3$]

$= 2 \times 16$ $[\because \Delta = 16]$

= 32 Hence proved.

58. The maximum value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & (1+\sin\theta) & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$ *is* $\frac{1}{2}$.

Sol. True

since, $\begin{vmatrix} 1 & 1 & 1 \\ 1 & \sin\theta & 1 \\ 1 & 1 & \cos\theta \end{vmatrix}$ $[\because R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

On expanding along third row, we get the value of the determinant

$= \cos\theta.\sin\theta = \frac{1}{2}\sin 2\theta = \frac{1}{2}$

[when θ *is* 45^0 which gives maximum value]

Determinats

Short Answer Type Questions

Using the properties of determinants in Exercises 1 to 6, evaluate:

1. $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$

Sol. We have, $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix} = \begin{vmatrix} x^2-2x+2 & x-1 \\ 0 & x+1 \end{vmatrix}$ $[\because C_1 \to C_1 - C_2]$

$= (x^2-2x+2).(x+1)-(x-1).0$

$= x^3-2x^2+2x+x^2-2x+2$

$= x^3-x^2+2$

2. $\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & z+z \end{vmatrix}$

Sol. We have, $\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = \begin{vmatrix} a & -a & 0 \\ 0 & a & -a \\ x & y & a+z \end{vmatrix}$ $\begin{bmatrix} \because\ R_1 \to R_1 - R_2 \\ and\ R_2 \to R_2 - R_3 \end{bmatrix}$

$= \begin{vmatrix} a & 0 & 0 \\ 0 & a & -a \\ x & x+y & a+z \end{vmatrix}$ $[\because C_2 \to C_2 + C_1]$

$= a\left(a^2+az+ax=ay\right)$

$= a^2\left(a+z+x+y\right)$

3. $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$

Sol. We have, $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix} = x^2y^2z^2\begin{vmatrix} 0 & x & x \\ y & 0 & y \\ z & z & 0 \end{vmatrix}$

[taking x^2, y^2 *and* z^2 *common from* C_1, C_2 *and* C_3, *respectively*]

$= x^2y^2z^2\begin{vmatrix} 0 & 0 & x \\ y & -y & y \\ z & z & 0 \end{vmatrix}$ $[\because\ C_2 \to C_2 - C_3]$

$= x^2y^2z^2\left[x\left(yz+yz\right)\right]$

$= x^2y^2z^2.2xyz = 2x^3y^3z^3$

4. $$\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$$

Sol. We have, $\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

Applying, $C_1 \to C_1 + C_2 + C_{3,}$

$= \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$

= (x+y+z) $\begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$

[Taking x + y + z common from column C_1]

$= (x+y+z) \begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 2y+x & x-y \\ 0 & x-z & 2z+x \end{vmatrix}$

$[\because R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

Now, expanding along first column, we get

$(x+y+z).1[(2y+x)(2z+x)-(x-y)(x-z)]$

$= (x+y+z)(4yz+2yx+2xz+x^2-x^2+xz+yx-yz)$

$= (x+y+z)(3yz+3yx+3xz)$

$= 3(x+y+z)(yz+yx+xz)$

5. $$\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

Sol. We have, $\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} = \begin{vmatrix} 2x+4 & 2x+4 & 2x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$ $[\because R_1 \to R_1 + R_2]$

$= \begin{vmatrix} 2x & 2x & 2x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} + \begin{vmatrix} 4 & 4 & 0 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$

[Here, given determinant is expressed in sum of two determinants]

$= 2x \begin{vmatrix} 1 & 1 & 1 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} + 4 \begin{vmatrix} 1 & 1 & 0 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$

[taking 2x common from first row of first determinant and 4 from first row of second determinants]

Applying $C_1 \to C_1 - C_3 \; and \; C_2 \to C_2 - C_3$ in first and applying $C_1 \to C_1 - C_2$ in second, we get

$$= 2x\begin{vmatrix} 0 & 0 & 1 \\ 0 & 4 & x \\ -4 & -4 & x+4 \end{vmatrix} + 4\begin{vmatrix} 0 & 1 & 0 \\ -4 & x+4 & x \\ 0 & x & x+4 \end{vmatrix}$$

Expanding both the along first column, we get

$$2x\left[-4(-4)\right] + 4\left[4(x+4-0)\right]$$

$$= 2x \times 16 + 16(x+4)$$

$$= 32x + 16x + 64$$

=16(3x+4)

6. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Sol. We have, $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \left[\because \; R_1 \to R_1 + R_2 + R_3\right]$$

$$= (a+b+c) = \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$\left[taking \; (a+b+c) \; common \; from \; the \; first \; row\right]$

$$= (a+b+c) = \begin{vmatrix} 0 & 0 & 1 \\ 0 & -(a+b+c) & 2b \\ (a+b+c) & (a+b+c) & (c-a-b) \end{vmatrix}$$

$\left[\because C_1 \to C_1 - C_3 \; and \; C_2 \to C_2 - C_3\right]$

Expanding along R_1,

$$= (a+b+c)\left[1\left\{0 + \left(a+b+c^2\right)\right\}\right]$$

$$= (a+b+c)\left[(a+b+c)^2\right]$$

$$= (a+b+c)^3$$

Using the properties of determinants in Exercises 7 to 9, prove that:

7. $\begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix} = 0$

Sol. We have to prove,

$$\begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix} = 0$$

$$\therefore \quad LHS = \begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} xy^2z^2 & xyz & xy+xz \\ x^2yz^2 & xyz & yz+xy \\ x^2y^2z & xyz & xz+yz \end{vmatrix}$$

$\left[\because \; R_1 \to xR_1, R_2 \to yR_2, R_3 \to zR_3\right]$

$$= \frac{1}{xyz}(xyz)^2 \begin{vmatrix} yz & 1 & xy+xz \\ xz & 1 & yz+xy \\ xy & 1 & xz+yz \end{vmatrix}$$

[taking (xyz) common from C_1 and C_2]

$$= xyz\begin{vmatrix} yz & 1 & xy+yz+zx \\ xz & 1 & xy+yz+zx \\ xy & 1 & xy+yz+zx \end{vmatrix} [C_3 \to C_3 + C_1]$$

$$= xyz(xy+yz+zx)\begin{vmatrix} yz & 1 & 1 \\ xz & 1 & 1 \\ zy & 1 & 1 \end{vmatrix}$$

[taking (xy + yz+zx) common from C_3]

=0 [since, C_2 and C_3 are identicals]

=RHS Hence proved.

8. $$\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$$

Sol. We have to prove,

$$\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$$

$$\therefore LHS = \begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$$

$$= \begin{vmatrix} y+z+z+y & z & y \\ z+z+x+x & z+x & x \\ y+x+x+y & x & x+y \end{vmatrix} [\because C_1 \to C_1 + C_2 + C_3]$$

$$= 2\begin{vmatrix} (y+z) & z & y \\ (z+x) & z+x & x \\ (x+y) & x & x+y \end{vmatrix} [Taking\ 2\ common\ from\ C_1]$$

$$= 2\begin{vmatrix} y & z & y \\ 0 & z+x & x \\ y & x & x+y \end{vmatrix} [\because C_1 \to C_1 - C_2]$$

$$= 2\begin{vmatrix} 0 & z-x & -x \\ 0 & z+x & x \\ y & x & x+y \end{vmatrix} [\because R_1 \to R_1 - R_3]$$

$$= 2\left[y\left(xz - x^2 + xz + x^2\right)\right]$$

$= 4xyz = RHS$ Hence proved.

9. $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$

Sol. We have to prove,

$$= \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$$

$$\therefore\ LHS = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+2a-2a-1 & 2a+1-a-2 & 0 \\ 2a+1-3 & a+2-3 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$[\because R_1 \to R_1 - R_2\ and\ R_2 \to R_2 - R_3]$

$$= \begin{vmatrix} (a-1)(a+1) & (a-1) & 0 \\ 2(a-1) & (a-1) & 0 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^2 \begin{vmatrix} (a+1) & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

[taking (a-1) common from R_1 and R_2 each]

$$= (a-1)^2\left[1(a+1)-2\right] = (a-1)^3$$

=RHS Hence proved.

10. If A + B + C = 0, then prove that $\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$

Sol. We have, $\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$

$\therefore \ LHS = \begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$

$= 1\left(1-\cos^2 A\right) - \cos C\left(\cos C - \cos A.\cos B\right) + \cos B\left(\cos C.\cos A - \cos B\right)$

$= \sin^2 A - \cos^2 C + \cos A.\cos B.\cos C + \cos A.\cos B.\cos C - \cos^2 B$

$= \sin^2 A - \cos^2 B + 2\cos A.\cos B.\cos C - \cos^2 C$

$= -\cos(A+B).\cos(A-B) + 2\cos A.\cos B.\cos C - \cos^2 C$

$\left[\because \cos^2 B - \sin^2 A = \cos(A+B).\cos(A-B)\right]$

$= -\cos(-C).\cos(A-B) + \cos C\left(2\cos A.\cos B - \cos C\right)$ $\left[\because \cos(-\theta) = \cos\theta\right]$

$= -\cos C(\cos A.\cos B + \sin A.\sin B - 2\cos A.\cos B + \cos C)$

$= \cos C(\cos A.\cos B - \sin A.\sin B - \cos C)$

$= \cos C\left[\cos(A+B) - \cos C\right]$

$= \cos C\left(\cos C - \cos C\right) = 0 = RHS$ Hence proved.

11. If the co-ordinates of the vertices of an equilateral triangle with sides of length 'a' are $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3a^4}{4}$

Sol. Since, we know that area of a triangle with vertices $(x_1, y_1), (x_2, y_2)$ *and* (x_3, y_3) is given by $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$\Rightarrow \Delta^2 = \frac{1}{4}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 \quad .....(i)$

We know that, area of an equilateral triangle with side a,

$\Delta = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)a^2 = \frac{\sqrt{3}}{4}a^2$

$\Rightarrow \ \Delta^2 = \frac{3}{16}a^4 \quad ...(ii)$

from Eqs. (i) and (ii), $\frac{3}{16}a^4 = \frac{1}{4}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$

$$=\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3}{4}a^4$$ Hence proved.

12. Find the value of θ satisfying $\begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0.$

Sol. We have, $\begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0.$

$$\Rightarrow \begin{bmatrix} 0 & 1 & \sin 3\theta \\ -7 & 3 & \cos 2\theta \\ 14 & -7 & -2 \end{bmatrix} = 0. \quad [\because C_1 \to C_1 - C_2]$$

$$\Rightarrow 7\begin{bmatrix} 0 & 1 & \sin 3\theta \\ -1 & 3 & \cos 2\theta \\ 2 & -7 & -2 \end{bmatrix} = 0. \quad [\textit{taking 7 common from } C_1]$$

$\Rightarrow 7[0\text{-}1(2-2\cos 2\theta)+\sin 3\theta(7-6)]=0$ [expanding along R_1]

$\Rightarrow 7[-2(1-\cos 2\theta)+\sin 3\theta]=0$

$\Rightarrow -14+14\cos 2\theta+7\sin 3\theta=0$

$\Rightarrow 14\cos 2\theta+7\sin 3\theta=14$

$\Rightarrow 14(1-2\sin^2\theta)+7(3\sin\theta-4\sin^3\theta)=14$

$\Rightarrow -28\sin^2\theta+14+21\sin\theta-28\sin^3\theta=14$

$\Rightarrow -28\sin^2\theta-28\sin^3\theta+21\sin\theta=0$

$\Rightarrow 28\sin^3\theta+28\sin^2\theta-21\sin\theta=0$

$\Rightarrow 4\sin^3\theta+4\sin^2\theta-3\sin\theta=0$

$\Rightarrow \sin\theta(4\sin^2\theta+4\sin\theta-3)=0$

$\Rightarrow$ Either $\sin\theta=0$

$\Rightarrow \theta=n\pi \text{ or } 4\sin^2\theta+4\sin\theta-3=0$

$\therefore \sin\theta=\frac{-4\pm\sqrt{16+48}}{8}=\frac{-4\pm\sqrt{64}}{8}$

$=\frac{-4\pm 8}{8}=\frac{4}{8},\frac{-12}{8}$

$\sin\theta=\frac{1}{2},\frac{-3}{2}$

If $\sin\theta=\frac{1}{2}=\sin\frac{\pi}{6}$, *then*

$\theta=n\pi+(-1)^n\frac{\pi}{6}$

Hence, $\sin\theta = \frac{-3}{2}$ [not possible because $-1 \le \sin\theta \le 1$]

13. **If** $\begin{bmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{bmatrix} = 0$ **, then find values of x.**

Sol. Given, $\begin{bmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{bmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 12+x & 12+x & 12+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0 \ [\because R_1 \to R_1 + R_2 + R_3]$$

$$\Rightarrow (12+x)\begin{bmatrix} 1 & 1 & 1 \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{bmatrix} = 0$$ [taking (12+x) common from R_1]

$$\Rightarrow (12+x)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 8 & 4+x \\ 2x & 8 & 4-x \end{vmatrix} = 0 \ [\because C_1 \to C_1 - C_3 \ and \ C_2 \to C_2 + C_3]$$

$$\Rightarrow (12+x)[1.(-16x)] = 0$$

$$\Rightarrow (12+x)(-16x) = 0$$

$$\therefore x = -12, 0$$

14. **If** $a_1, a_2, a_3, ...a_n$ **are in G.P., then prove that the determinant** $\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$ **is independent of r.**

Sol. We Know that,

$a_{r+1} = AR^{(r+1)-1} = AR^r$

Where r=rth term of a GA, A=First term of a GP and R=Common ratio of GP

We have $\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$

$$= AR^r.AR^{r+6}.AR^{r+10}\begin{vmatrix} 1 & AR^4 & AR^8 \\ 1 & AR^4 & AR^8 \\ 1 & AR^6 & AR^{10} \end{vmatrix}$$

[taking $AR^r.AR^{r+6}.AR^{r+10}$ common from R_1, R_2 *and* R_3 respectively]

=0 [Since, R_1 *and* R_2 are identicals]

15. **Show that the points** $(a+5, a-4)$, $(a-2, a+3)$ *and* (a, a) **do not lie on a straight line for any value of a.**

Sol. Given, the point are $(a+5, a-4), (a-2, a+3)\ and\ (a, a)$

$$\therefore\ \Delta = \frac{1}{2}\begin{vmatrix} a+5 & a-4 & 1 \\ a-2 & a+3 & 1 \\ a & a & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 5 & -4 & 0 \\ -2 & 3 & 0 \\ a & a & 1 \end{vmatrix} \quad [\because R_1 \to R_1 - R_3\ and\ R_2 \to R_2 - R_3]$$

$$= \frac{1}{2}[1(15-8)]$$

$$= \frac{7}{2} \neq 0$$

Hence, given points form a triangle i.e. points do not lie in a straight line.

16. Show that the ΔABC is an isosceles triangle if the determinant

$$\Delta = \begin{bmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{bmatrix} = 0$$

Sol. We have, $\Delta = \begin{bmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{bmatrix} = 0$

$$\Delta = \begin{bmatrix} 0 & 0 & 1 \\ \cos A - \cos C & \cos B - \cos C & 1+\cos C \\ \cos^2 A + \cos A - \cos^2 C - \cos C & \cos^2 B + \cos B - \cos^2 C - \cos C & \cos^2 C + \cos C \end{bmatrix} = 0$$

$[\therefore C_1 \to C_1 - C_3\ and\ C_2 \to C_2 - C_3]$

$$\Rightarrow (\cos A - \cos C).(\cos B - \cos C)$$

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1+\cos C \\ \cos A + \cos C + 1 & \cos B + \cos C + 1 & \cos^2 C + \cos C \end{vmatrix} = 0$$

[taking (cosA - cosC) common from C_1 and (cosB - cosC) common from C_2]

$$\Rightarrow (\cos A - \cos C).(\cos B - \cos C)\left[(\cos B + \cos C + 1) - (\cos A + \cos C + 1)\right] = 0$$

$$\Rightarrow (\cos A - \cos C).(\cos B - \cos C)(\cos B + \cos C + 1 - \cos A - \cos C - 1) = 0$$

$$\Rightarrow (\cos A - \cos C).(\cos B - \cos C)(\cos B - \cos A) = 0$$

i.e., $\cos A = \cos C\ or\ \cos B = \cos C\ or\ \cos B = \cos A$

$\Rightarrow$ A=C or B=C or B=A

Hence, ABC is an isosceles triangle.

17. Find A^{-1} if $A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$ and show that $A^{-1} = \frac{A^2 - 3I}{2}$

Sol. We have, $A=\begin{vmatrix}0&1&1\\1&0&1\\1&1&0\end{vmatrix}$

$\therefore A_{11}=-1, A_{12}=1, A_{13}=1, A_{21}=1, A_{22}=-1, A_{23}=1, A_{31}=1, A_{32}=1, and\ A_{33}=-1$

$\therefore adj\ A=\begin{vmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{vmatrix}^T=\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}$

and $|A|=-1(-1)+1.1=2$

$\therefore A^{-1}=\frac{adjA}{|A|}=\frac{1}{2}\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}\quad ...(i)$

And $A^2=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}.\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}=\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\quad ...(ii)$

$\therefore \frac{A^2-3I}{2}=\frac{1}{2}\left\{\begin{vmatrix}2&1&1\\1&2&1\\1&1&2\end{vmatrix}-\begin{vmatrix}3&0&0\\0&3&0\\0&0&3\end{vmatrix}\right\}=\frac{1}{2}\begin{vmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{vmatrix}$

$=A^{-1}$ [Using Eq. (i)]

Hence proved.

Determinats

Long Answer Type Questions

18. **If** $A=\begin{bmatrix}1 & 2 & 0\\ -2 & -1 & -2\\ 0 & -1 & 1\end{bmatrix}$**, then find the value of A^{-1}.**

Using A^{-1}, solve the system of linear equations $x-2y=10,\quad 2x-y-z=8,$ $-2y+z=7.$

Sol. We have, $A=\begin{vmatrix}1 & 2 & 0\\ -2 & -1 & -2\\ 0 & -1 & 1\end{vmatrix}$ ...(i)

$\therefore\ |A|=1(-3)-2(-2)+0=1\neq 0$

$Now, A_{11}=-3, A_{12}=2, A_{13}=2, A_{21}=-2, A_{22}=1, A_{23}=1, A_{31}=-4, A_{32}=2\ and\ A_{33}=3$

$\therefore adj(A)=\begin{vmatrix}-3 & 2 & 2\\ -2 & 1 & 1\\ -4 & 2 & 3\end{vmatrix}^T=\begin{vmatrix}-3 & -2 & -4\\ 2 & 1 & 2\\ 2 & 1 & 3\end{vmatrix}$

$\therefore\ A^{-1}=\dfrac{adjA}{|A|}$

$=\dfrac{1}{1}\begin{vmatrix}-3 & -2 & -4\\ 2 & 1 & 2\\ 2 & 1 & 3\end{vmatrix}$

$\Rightarrow A^{-1}=\begin{vmatrix}-3 & -2 & -4\\ 2 & 1 & 2\\ 2 & 1 & 3\end{vmatrix}$ ...(i)

Also, we have the system of linear equations as

$x-2y=10$

$2x-y-z=8$

and $-2y+z=7$

In the form of CX=D,

$$\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & -1\\ 0 & -2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}10\\ 8\\ 7\end{bmatrix}$$

where, $C=\begin{bmatrix}1 & -2 & 0\\ 2 & -1 & -1\\ 0 & -2 & 1\end{bmatrix} X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} and\ D=\begin{bmatrix}10\\ 8\\ 7\end{bmatrix}$

We know that, $\left(A^T\right)^{-1}=\left(A^{-1}\right)^T$

$$\therefore C^T = \begin{vmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{vmatrix} = A \text{ [using Eq. (i)]}$$

$$\therefore X = C^{-1}D$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} -30+16+14 \\ -20+8+7 \\ -40+16+21 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}$$

$\therefore x = 0, y = -5 \text{ and } z = -3$

19. Using matrix method, solve the system of equations $3x + 2y - 2z = 3$, $x + 2y + 3z = 6$, **and** $2x - y + z = 2$.

Sol. Given system of equations is

$3x + 2y - 2z = 3$,

$x + 2y + 3z = 6$,

and $2x - y + z = 2$

In the form of AX=B

$$= \begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

For A^{-1}, $|A| = |3(5) - 2(1-6) + (-2)(-5)|$

$= |15 + 10 + 10| = |35| \neq 0$

$\therefore A_{11} = 5, A_{12} = 5, A_{13} = -5,$

$A_{21} = 0, A_{22} = 7, A_{23} = 7,$

$A_{31} = 10, A_{32} = -11, A_{33} = 4$

$$\therefore adj\ A = \begin{vmatrix} 5 & 5 & -5 \\ 0 & 7 & 7 \\ 10 & -11 & 4 \end{vmatrix}^T = \begin{vmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{vmatrix}$$

Now, $A^{-1} = \dfrac{adj\ A}{|A|} = \dfrac{1}{35} \begin{vmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{vmatrix}$

$For\ X = A^{-1}B,$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ 5 & 7 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

$$= \frac{1}{35}\begin{bmatrix} 15+20 \\ 15+42-22 \\ -15+42+8 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 35 \\ 35 \\ 35 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore x = 1,\ y = 1 \text{ and } z = 1$

20. Given $A = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$, find BA and use this to solve the system of equations $y + 2z = 7,\ x - y = 3,\ 2x + 3y + 4z = 17$.

Sol. We have, $A = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

$$\therefore BA = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}\begin{vmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{vmatrix} = \begin{vmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 6I$$

$$\therefore B^{-1} = \frac{A}{6} = \frac{1}{6}A = \frac{1}{6}\begin{vmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{vmatrix} \quad ...(i)$$

Also, $x - y = 3,\ 2x + 3y + 4z = 17 \text{ and } y + 2z = 7$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} \quad \text{[using Eq. (i)]}$$

$$= \frac{1}{6}\begin{bmatrix} 6+34-28 \\ -12+34-28 \\ 6-17+35 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$\therefore x = 2,\ y = -1 \text{ and } z = 4$

21. If a + b + c ≠ 0 and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, then prove that a = b = c.

Sol. We have, $A=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$=\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix}$ $[\because R_1 \to R_1+R_2+R_3]$

$=(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$

$=(a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b-a & c-a & a \\ c-b & a-b & b \end{vmatrix}$ $[\because C_1 \to C_1 - C_3 \; and \; C_2 \to C_2 - C_3]$

Expanding along R_1,

$=(a+b+c)\left[1(b-a)(a-b)-(c-a)(c-b)\right]$

$=(a+b+c)\left(ba-b^2-a^2+ab-c^2+cb+ac-ab\right)$

$=\frac{-1}{2}(a+b+c)\times(-2)\left(a^2-b^2-c^2+ab+bc+ca\right)$

$=\frac{-1}{2}(a+b+c)\left[a^2+b^2+c^2-2ab-2bc-2ca+a^2+b^2+c^2\right]$

$=-\frac{1}{2}(a+b+c)\left[a^2+b^2-2ab+b^2+c^2-2bc+c^2+a^2-2ac\right]$

$=\frac{-1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$

Also, A= 0

$=\frac{-1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]=0$

$(a-b)^2+(b-c)^2+(c-a)^2=0$ $[\because \text{ a+b+c} \neq 0\text{, given}]$

$\Rightarrow a-b=b-c=c-a=0$

$a=b=c$ *Hence proved.*

22. Prove that $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ **is divisible by (a + b + c) and find the quotient.**

Sol. Let $\Delta=\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$

$$= \begin{vmatrix} bc-a^2-ca+b^2 & ca-b^2-ab+c^2 & ab-c^2 \\ ca-b^2-ab+c^2 & ab-c^2-bc+a^2 & bc-a^2 \\ ab-c^2-bc+a^2 & bc-a^2-ca+b^2 & ca-b^2 \end{vmatrix}$$

$[\because C_1 \to C_1 - C_2 \text{ and } C_2 \to C_2 - C_3]$

$$= \begin{vmatrix} (b-a)(a+b+c) & (c-b)(a+b+c) & ab-c^2 \\ (c-b)(a+b+c) & (a-c)(a+b+c) & bc-a^2 \\ (a-c)(a+b+c) & (b-a)(a+b+c) & ca-b^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b-a & c-b & ab-c^2 \\ c-b & a-c & bc-a^2 \\ a-c & b-a & ca-b^2 \end{vmatrix}$$

[taking (a + b + c) common from C_1 and C_2 each]

$$= (a+b+c)^2 \begin{vmatrix} 0 & 0 & ab+bc+ca-(a^2+b^2+c^2) \\ c-b & a-c & bc-a^2 \\ a-c & b-a & ca-b^2 \end{vmatrix}$$

$[\because R_1 \to R_1 + R_2 + R_3]$

Now, expanding along R_1,

$$= (a+b+c)^2 \left[ab+bc+ca-(a^2+b^2+c^2)(c-b)(b-a)-(a-c)^2 \right]$$

$$= (a+b+c)^2 (ab+bc+ca-a^2-b^2-c^2)$$

$$(cb-ac-b^2+ab-a^2-c^2+2ac)$$

$$= (a+b+c)^2 (a^2+b^2+c^2-ab-bc-ca)$$

$$(a^2+b^2+c^2-ac-ab-bc)$$

$$= \frac{1}{2}(a+b+c)\left[(a+b+c)(a^2+b^2+c^2-ab-bc-ca)\right]$$

$$\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

$$= \frac{1}{2}(a+b+c)(a^3+b^3+c^3-3abc)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

Hence, given determinant is divisible by $(a+b+c)$ and quotient is

$$(a^3+b^3+c^3-3abc)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

23. If $x+y+z=0$, **prove that** $\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

Sol. Since, $x+y+z=0$, also we have to prove

$$\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$\therefore \text{LHS}=\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix}$$

$$= xa\left(za.ya - xb.xc\right) - yb(yc.ya - xb.zb) + zc\left(yc.xc - za.zb\right)$$

$$= xa\left(a^2yz - x^2bc\right) - yb(y^2ac - b^2xz) + zc\left(c^2xy - z^2ab\right)$$

$$= x\,yza^3 - x^3\,abc - y^3\,abc + b^3\,xyz + c^3\,xyz - z^3\,abc$$

$$= xyz(a^3+b^3+c^3) - abc\left(x^3+y^3+z^3\right)$$

$$= xyz(a^3+b^3+c^3) - abc\left(3xyz\right)$$

$$\left[\because\ x+y+z=0 \Rightarrow x^3+y^3+z^3-3xyz\right]$$

$$= xyz(a^3+b^3+c^3-3abc)\ \ldots(i)$$

Now, $RHS = xyz\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = xyz\begin{vmatrix} a+b+c & b & c \\ a+b+c & a & b \\ a+b+c & c & a \end{vmatrix}[\because C_1 \to C_1 + C_2 + C_3]$

$$= xyz\left(a+b+c\right)\begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix}$$

[taking (a + b + c) common from C_1]

$$= xyz\left(a+b+c\right)\begin{vmatrix} 0 & b-c & c-a \\ 0 & a-c & b-a \\ 1 & c & a \end{vmatrix}$$

$$\left[\because R_1 \to R_1 - R_3 \ and\ R_2 \to R_2 - R_3\right]$$

Expanding along C_1,

$$= xyz\left(a+b+c\right)\left[1\left(b-c\right)\left(b-a\right) - \left(a-c\right)\left(c-a\right)\right]$$

$$= xyz\left(a+b+c\right)\left(b^2 - ab - bc + ac + a^2 + c^2 - 2ac\right)$$

$$= xyz\left(a+b+c\right)\left(a^2+b^2+c^2-ab-bc-ca\right)$$

$$= xyz\left(a^3+b^3+c^3-3abc\right)\ \ldots(ii)$$

From Eqs. (i) and (ii),

LHS=RHS

$$\Rightarrow \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}\ Hence\ proved.$$