1. **સાબિત કરો :** $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \dfrac{\pi}{4} + \dfrac{1}{2}\cos^{-1}(x^2)$

➠ ડા.બા. $= \tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

અહીં $x^2 = \cos\theta$ લેતાં, $\theta = \cos^{-1}(x^2)$

$0 < x^2 < 1$

$\therefore \cos\dfrac{\pi}{2} < \cos\theta < \cos 0$

$\therefore 0 < \theta < \dfrac{\pi}{2}$

$\therefore \tan^{-1}\left[\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$

$= \tan^{-1}\left[\dfrac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right]$

$= \tan^{-1}\left[\dfrac{\sqrt{2\cos^2\left(\frac{\theta}{2}\right)}+\sqrt{2\sin^2\left(\frac{\theta}{2}\right)}}{\sqrt{2\cos^2\left(\frac{\theta}{2}\right)}-\sqrt{2\sin^2\left(\frac{\theta}{2}\right)}}\right]$

$\left[\because 0 < \theta < \dfrac{\pi}{2} \quad \therefore 0 < \dfrac{\theta}{2} < \dfrac{\pi}{4}\right]$

$= \tan^{-1}\left[\dfrac{\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)}\right]$

$= \tan^{-1}\left[\dfrac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right]$

[∵ અંશ તથા છેદના દરેક પદને $\cos\left(\dfrac{\theta}{2}\right)$ વડે ભાગતાં]

➠ ડા.બા. $= \tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

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$= \tan^{-1}\left[\frac{\sqrt{2\cos^2\left(\frac{\theta}{2}\right)}+\sqrt{2\sin^2\left(\frac{\theta}{2}\right)}}{\sqrt{2\cos^2\left(\frac{\theta}{2}\right)}-\sqrt{2\sin^2\left(\frac{\theta}{2}\right)}}\right]$

$\left[\begin{array}{l}\because 0 < \theta < \frac{\pi}{2} \\ \therefore 0 < \frac{\theta}{2} < \frac{\pi}{4}\end{array}\right]$

$= \tan^{-1}\left[\frac{\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)}\right]$

$= \tan^{-1}\left[\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right]$

[$\because$ અંશ તથા છેદના દરેક પદને $\cos\left(\frac{\theta}{2}\right)$ વડે ભાગતાં]

2. સાદું રૂપ આપો : $\cos^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right)$, $x \in \left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$

➡ અહીં, $\frac{3}{5} = \cos\alpha$ મૂકો.

$\therefore \sin\alpha = \sqrt{1-\cos^2\alpha}$

$= \sqrt{1-\frac{9}{25}}$

$= \sqrt{\frac{16}{25}}$

$\therefore \sin\alpha = \frac{4}{5}$

$\therefore \cos^{-1}\left[\frac{3}{5}\cos x + \frac{4}{5}\sin x\right]$

$= \cos^{-1}[(\cos\alpha\ \cos x + \sin\alpha\ \sin x)]$

$= \cos^{-1}[\cos(\alpha - x)]$

$= \alpha - x$ ....(i)

હવે $\sin\alpha = \frac{4}{5}$ અને $\cos\alpha = \frac{3}{5}$ છે.

$\therefore \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{4}{3}$

$\therefore \alpha = \tan^{-1}\left(\frac{4}{3}\right)$

$$\therefore \cos^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right) = \alpha - x$$

$$= \tan^{-1}\left(\frac{4}{3}\right) - x$$

3. સાબિત કરો : $\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{77}{85}\right)$

➡ ડા.બા. $= \sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right)$

$$= \tan^{-1}\left(\frac{\frac{8}{17}}{\sqrt{1 - \frac{64}{289}}}\right) + \tan^{-1}\left(\frac{\frac{3}{5}}{\sqrt{1 - \frac{9}{25}}}\right)$$ (∵ આંતર સંબંધી સૂત્ર)

$$= \tan^{-1}\left(\frac{8}{15}\right) + \tan^{-1}\left(\frac{3}{4}\right)$$

$$= \tan^{-1}\left(\frac{\frac{8}{15} + \frac{3}{4}}{1 - \frac{24}{60}}\right)$$

$$\left(\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)\right)$$

$$= \tan^{-1}\left(\frac{32 + 45}{60 - 24}\right)$$

$$= \tan^{-1}\left(\frac{77}{36}\right)$$

$$= \sin^{-1}\left(\frac{\frac{77}{36}}{\sqrt{1 + \left(\frac{77}{36}\right)^2}}\right)$$

$$= \sin^{-1}\left(\frac{\frac{77}{36}}{\sqrt{\frac{1296 + 5929}{(36)^2}}}\right)$$

$$= \sin^{-1}\left(\frac{77}{\sqrt{7225}}\right)$$

$$= \sin^{-1}\left(\frac{77}{85}\right)$$

= જ.બા.

➡ ડા.બા. $= \sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right)$

$$= \tan^{-1}\left(\frac{\frac{8}{17}}{\sqrt{1 - \frac{64}{289}}}\right) + \tan^{-1}\left(\frac{\frac{3}{5}}{\sqrt{1 - \frac{9}{25}}}\right)$$ (∵ આંતર સંબંધી સૂત્ર)

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$$= \sin^{-1}\left(\frac{77}{\sqrt{7225}}\right)$$

$$= \sin^{-1}\left(\frac{77}{85}\right)$$

= જ.બા.

4. $\sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{63}{16}\right)$ **સાબિત કરો.**

➡ ડા.બા. = $\sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$

$$= \tan^{-1}\left(\frac{\frac{5}{13}}{\sqrt{1 - \frac{25}{169}}}\right) + \tan^{-1}\left(\frac{\sqrt{1 - \frac{9}{25}}}{\frac{3}{5}}\right)$$

($\because$ આંતર સંબંધી સૂત્ર પરથી)

$$= \tan^{-1}\left(\frac{\frac{5}{13}}{\frac{12}{13}}\right) + \tan^{-1}\left(\frac{\frac{4}{5}}{\frac{3}{5}}\right)$$

$$= \tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\left(\frac{4}{3}\right)$$

$$= \tan^{-1}\left(\frac{\frac{5}{12} + \frac{4}{3}}{1 - \frac{5}{12} \times \frac{4}{3}}\right)$$

$$= \tan^{-1}\left(\frac{15 + 48}{36 - 20}\right)$$

$$= \tan^{-1}\left(\frac{63}{16}\right)$$

= જ.બા.

5. **સાબિત કરો :** $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$

➡ ડા.બા. = $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{4} + \frac{2}{9}}{1 - \frac{1}{4}\left(\frac{2}{9}\right)}\right)$$

($\because$ $\tan^{-1}$ ના સરવાળા સૂત્ર મુજબ)

$$= \tan^{-1}\left(\frac{9 + 8}{36 - 2}\right)$$

$= \tan^{-1}\left(\frac{17}{34}\right)$

$= \tan^{-1}\left(\frac{1}{2}\right)$

$= \sin^{-1}\left(\frac{\frac{1}{2}}{\sqrt{1+\frac{1}{4}}}\right)$

$\left(\because \tan^{-1}(x) = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) \text{ છે.}\right)$

$= \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$

= જ.બા.

6. સાબિત કરો કે $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$ તથા અન્ય કિંમત $\frac{4+\sqrt{7}}{3}$ શા માટે શક્ય નથી ?

➡ અહીં, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)$

ધારો કે $\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right) = \theta$

$\therefore \sin^{-1}\left(\frac{3}{4}\right) = 2\theta$

$\therefore \sin(2\theta) = \frac{3}{4}$

$\therefore \frac{2\tan\theta}{1+\tan^2\theta} = \frac{3}{4}$

$\therefore 8\tan\theta = 3 + 3\tan^2\theta$

$\therefore 3\tan^2\theta - 8\tan\theta + 3 = 0$

જે $\tan\theta$ માં દ્વિઘાત સમીકરણ છે.

$\therefore \Delta = b^2 - 4ac$

$= 64 - 4(3)(3)$

$= 64 - 36$

$= 28$

$\therefore \sqrt{\Delta} = \sqrt{28}$

$= 2\sqrt{7}$

$\tan\theta = \frac{-b \pm \sqrt{\Delta}}{2a}$

$= \frac{8 \pm 2\sqrt{7}}{2 \times 3} = \frac{4 \pm \sqrt{7}}{3}$

પરંતુ $\frac{4+\sqrt{7}}{3} > 1$ થશે જે શક્ય નથી.

કારણ કે $\tan\theta$ નું મહત્તમ મૂલ્ય 1 હોય.

$\therefore \tan\theta = \frac{4-\sqrt{7}}{3}$ હોઈ શકે.

આમ, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$ સાબિત થાય છે.

7. જો $a_1, a_2, a_3, \ldots\ldots, a_n$ પદની સમાંતર શ્રેણીનો સામાન્ય તફાવત d હોય તો $\tan\left\{\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\right.$

$\tan^{-1}\left(\dfrac{d}{1+a_2a_3}\right) + ... + \tan^{-1}\left(\dfrac{d}{1+a_{n-1}\cdot a_n}\right)\bigg\}$ **નું મૂલ્ય મેળવો.**

➡ અહીં, a_1, a_2, a_3, ..., a_n સમાંતર શ્રેણીના n પદ છે.

$\therefore$ d (સામાન્ય તફાવત) $= a_2 - a_1 = a_3 - a_2 = a_4 - a_3$

$= .... = a_n - a_{n-1}$.

હવે $\tan\left\{\tan^{-1}\left(\dfrac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\dfrac{d}{1+a_2a_3}\right) + \tan^{-1}\left(\dfrac{d}{1+a_3a_4}\right) + ... + \tan^{-1}\left(\dfrac{d}{1+a_{n-1}\cdot a_n}\right)\right\}$

$$= \tan\left\{\tan^{-1}\left(\frac{a_2 - a_1}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{a_3 - a_2}{1+a_2a_3}\right) + ... + \tan^{-1}\left(\frac{a_n - a_{n-1}}{1+a_{n-1}\cdot a_n}\right)\right\}$$

$$= \tan\{\tan^{-1}(a_2) - \tan^{-1}(a_1) + \tan^{-1}(a_3) - \tan^{-1}(a_2) + \tan^{-1}(a_4) - \tan^{-1}(a_3) + ...$$
$$+ \tan^{-1}(a_n) - \tan^{-1}(a_{n-1})\}$$

$$= \tan\left(\tan^{-1}(a_n) - \tan^{-1} a_1\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{a_n - a_1}{1+a_n\cdot a_1}\right)\right)$$

$$= \frac{a_n - a_1}{1+a_n\cdot a_1}$$

જે જરૂરી મૂલ્ય છે.

8. કિંમત મેળવો : $4\tan^{-1}\left(\dfrac{1}{5}\right) - \tan^{-1}\left(\dfrac{1}{239}\right)$

➡ અહીં, $4\tan^{-1}\left(\dfrac{1}{5}\right) - \tan^{-1}\left(\dfrac{1}{239}\right)$

$$= 2\left\{2\tan^{-1}\left(\frac{1}{5}\right)\right\} - \tan^{-1}\left(\frac{1}{239}\right)$$

$$= 2\left\{\tan^{-1}\left(\frac{\frac{2}{5}}{1-\frac{1}{25}}\right)\right\} - \tan^{-1}\left(\frac{1}{239}\right)$$

$\left(\because 2\tan^{-1}x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)\right.$ હોય.$\Big)$

$$= 2\tan^{-1}\left(\frac{\frac{2}{5}}{\frac{24}{25}}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$= 2\tan^{-1}\left(\frac{5}{12}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$= \tan^{-1}\left(\frac{2\left(\frac{5}{12}\right)}{1-\frac{25}{144}}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$= \tan^{-1}\left(\frac{\frac{10}{12}}{\frac{144-25}{144}}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$= \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$= \tan^{-1}\left(\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119}\left(\frac{1}{239}\right)}\right)$$

$$\left(\because \tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}\left(\frac{x - y}{1 + xy}\right) \text{ છે.}\right)$$

$$= \tan^{-1}\left(\frac{28680 - 119}{28441 + 120}\right)$$

➡ અહીં, $4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$

$$= 2\left\{2\tan^{-1}\left(\frac{1}{5}\right)\right\} - \tan^{-1}\left(\frac{1}{239}\right)$$

$$= 2\left\{\tan^{-1}\left(\frac{\frac{2}{5}}{1 - \frac{1}{25}}\right)\right\} - \tan^{-1}\left(\frac{1}{239}\right)$$

$$\left(\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right) \text{ હોય.}\right)$$

$$= 2\tan^{-1}\left(\frac{\frac{2}{5}}{\frac{24}{25}}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$= 2\tan^{-1}\left(\frac{5}{12}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$= \tan^{-1}\left(\frac{2\left(\frac{5}{12}\right)}{1 - \frac{25}{144}}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$= \tan^{-1}\left(\frac{\frac{10}{12}}{\frac{144 - 25}{144}}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$= \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$= \tan^{-1}\left(\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119}\left(\frac{1}{239}\right)}\right)$$

$$\left(\because \tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}\left(\frac{x - y}{1 + xy}\right) \text{ છે.}\right)$$

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