Motion In A Straight Line

Concepts of Differentiation and Integration

Calculus

Calculus is basically a way of calculating rate of changes (similar to slopes, but called derivatives in calculus), areas, volumes, and surface areas (for starters).

It's easy to calculate these kinds of things with algebra and geometry if the shapes you're interested in are simple. For example, if you have a straight line you can calculate the slope easily. But if you want to know the slope at an arbitrary point (any random point) on the graph of some function like x-squared or some other polynomial, then you would need to use calculus. In this case, calculus gives you a way of "zooming in" on the point you're interested in to find the slope exactly at the point. This called a derivative.

If you have a cube or a sphere, you can calculate the volume and surface area easily. If you have an odd shape, you need to use calculus. You use calculus to make a infinite number or really small slices of the object you're interested in, determine to sizes of the slices, and then add all those sizes up. This process is called integration. It turns out that integration is the reverse of derivation (finding a derivative).

In summary, calculus is a tool that lets you do calculation with complicated curves shapes, etc. that you would normally not be able to do with just algebra and geometry.

Differentiation and Integration

Differentiation is the process of obtaining the derived function $f'(x)$ from the function $f(x)$, where $f'(x)$ is the derivative of f at x.

The derivatives of certain common functions are given in the Table of derivatives,

Table of derivatives :

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\text{cosec}^2 x$

sec *x*	sec *x* tan *x*
cosec *x*	-(cosec *x*)(cot *x*)
ln *x*	1/x
e^x	e^x

Many other functions can be differentiated using the following rules of differentiation:

(i) If $h(x) = k\,f(x)$ for all x, where k is a constant, then $h'(x) = k\,f'(x)$.

(ii) If $h(x) = f(x) + g(x)$ for all x, then $h'(x) = f'(x) + g'(x)$.

(iii) The product rule: If $h(x) = f(x)g(x)$ for all x, then $h'(x) = f(x)g'(x) + f'(x)+g'(x)$.

(iv) The reciprocal rule: If $h(x) = 1/f(x)$ and $f(x) \neq 0$ for all x, then

$$h'\left(x\right) = -\frac{f'(x)}{(f(x))^2}$$

(v) The quotient rule: If $h(x) = f(x)/g(x)$ and $g(x) \neq 0$ for all x, then

$$h'\left(x\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

(vi) The chain rule: If $h(x) = (f \circ g)(x) = f(g(x))$ for all x, then $h'(x) = f'(g(x))g'(x)$.

Example:

$$f\left(x\right) = x^n$$
$$f'\left(x\right) = \frac{d}{dx} f\left(x\right) = \frac{d}{dx}\left(x^n\right)$$
$$\Rightarrow f'\left(x\right) = nx^{n-1}$$

Integration is the process of finding an anti-derivative of a given function *f*. 'Integrate *f*' means 'find an anti-derivative of *f*'. Such an anti-derivative may be called an indefinite integral of *f* and be denoted by $\int f(x)dx$

The term 'integration' is also used for any method of evaluating a definite integral.

$$\int_a^b f\left(x\right)dx$$

The definite integral can be evaluated if an anti-derivative Φ of *f* can be found, because then its value is Φ(*b*) − Φ(*a*). (This is provided that *a* and *b* both belong to an interval in which *f* is continuous.)

However, for many functions *f*, there is no anti-derivative expressible in terms of elementary functions, and other methods for evaluating the definite integral have to be sought, one such being so-called numerical integration.

Example:

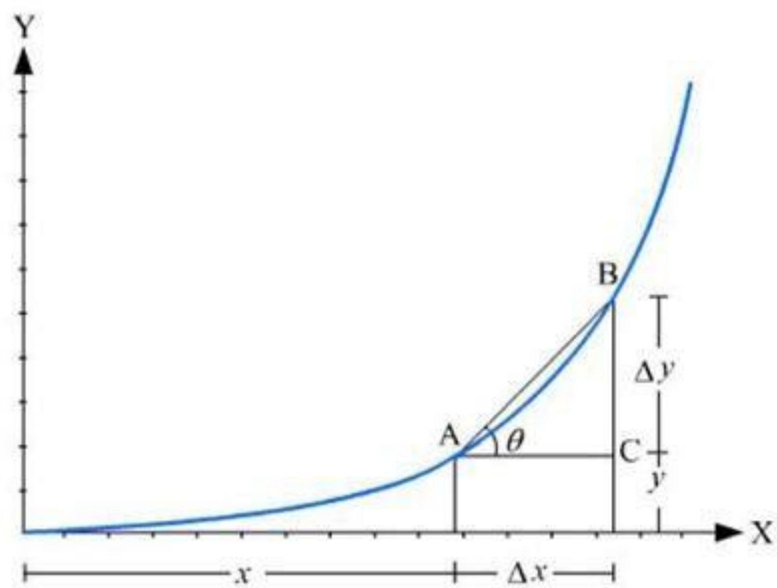
$$f\left(x\right) = x^n$$

$$\Rightarrow \int f\left(x\right)dx = \int x^n dx$$

$$\Rightarrow \int f\left(x\right)dx = \frac{x^{n+1}}{n+1} + \text{constant}$$

Differential and Integral Calculus

Differential calculus



Let *x* and *y* be two quantities interrelated in such a way that for each value of *x* there is one and only one value of *y*.

The graph represents the *y* versus *x* curve. Any point in the graph gives an unique values of *x* and *y*. Let us consider the point A on the graph. We shall increase *x* by a small amount Δ*x*, and the corresponding change in *y* be Δ*y*.

Thus, when x change by Δx, y change by Δy and the rate of change of y with respect to x is equal to $\frac{\Delta y}{\Delta x}$

In the triangle ABC, coordinate of A is (x, y); coordinate of B is $(x + \Delta x, y + \Delta y)$

The rate $\frac{\Delta y}{\Delta x}$ can be written as,

$$\frac{\Delta y}{\Delta x}=\frac{BC}{AC}=\tan\theta=\text{slope of the line AB}$$

But this cannot be the precise definition of the rate because the rate also varies between the point A and B. So, we must take very small change in x. That is Δx is nearly equal to zero. As we make Δx smaller and smaller the slope $\tan\theta$ of the line AB approaches the slope of the tangent at A. This slope of the tangent at A gives the rate of change of y with respect to x at A.

This rate is denoted by $\frac{dy}{dx}$

and,

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

Question:

Find the slope of the curve $y = 1 + x^2$ at $x = 5$.

Solution:

$y = 1 + x^2$

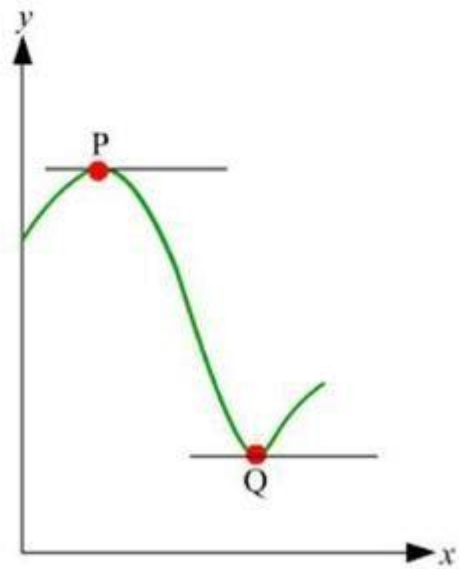
We know slope is given by, $\frac{dy}{dx}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(1+x^2)=0+2x=2x$$

So, at $x = 5$ the slope is $= 2 \times 5 = 10$

Maxima and Minima

Let x and y be two quantities interrelated in manner as shown in the graph below:



At the points P and Q the tangents to the curve is parallel to the x-axis.

Hence, its slope $\tan\theta = 0$.

But we know that the slope of the curve at any point equals the rate of change $\frac{dy}{dx}$ at the point.

Thus, at maximum (at P) or at minimum (at Q),

$$\frac{dy}{dx} = 0$$

Just before the maximum the slope is positive, at the maximum it is zero and just after the maximum it is negative. This implies, $\frac{dy}{dx}$ decrease at a maximum.

$$\therefore \frac{d}{dx}\left(\frac{dy}{dx}\right) < 0 \text{ at maximum}$$

$$\text{Or, } \frac{d^2y}{dx^2} < 0 \text{ at maximum}$$

Similarly, at the minimum:

$$\frac{d^2y}{dx^2} < 0 \text{ at maximum}$$

Question: A particle is thrown vertically upwards. At what time does the particle reach the maximum height? Find it using differential calculus.

Solution:

The equation of motion of the particle is given by.

$$y = ut = \frac{1}{2}gt^2$$

Where, *y* is the vertical displacement, *u* is the initial velocity, g is the acceleration due to gravity, *t* is the time.

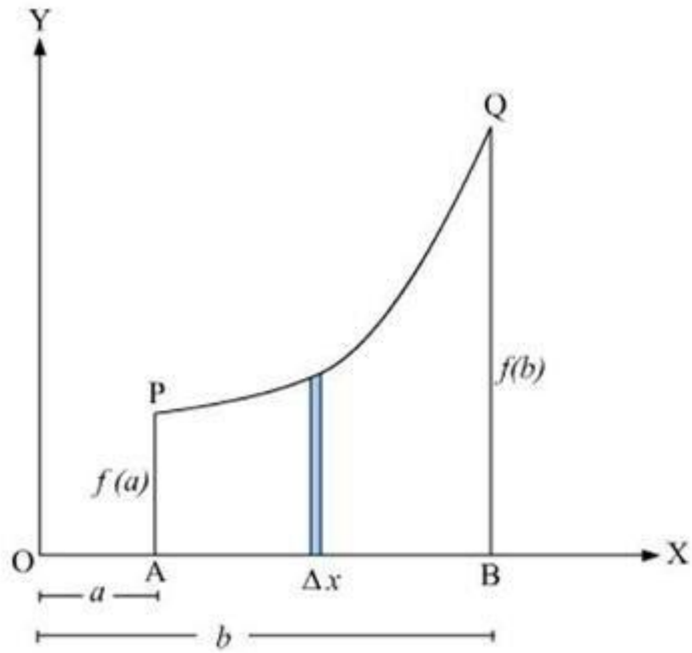
The vertical displacement is a function of time. Therefore, it will be maximum when $\frac{dy}{dt} = 0.$

$$y = ut - \frac{1}{2}gt^2$$
$$\Rightarrow \frac{dy}{dt} = \frac{d}{dt}\left(ut - \frac{1}{2}gt^2\right)$$
$$\Rightarrow \frac{dy}{dt} = u - gt$$
$$\because \frac{dy}{dt} = 0$$
$$\Rightarrow u - gt = 0$$
$$\Rightarrow t = \frac{u}{g}$$

Thus, the particle reaches maximum height at the time, $t = \frac{u}{g}$

Integral calculus

In the graph we have a curve PQ representing the relation between *x* and *y*. The equation of the curve is, $y = f(x)$



We shall find out the area under this curve. That is we need the area of APQB. To find this we shall first consider a very thin rectangle touching the curve and standing on the x-axis. The width of the rectangle is so that both the edges of the side near the curve actually touch the curve almost at the same point which means that Δx is so small that it tends to zero.

Area of this thin rectangle is = $f(x)\ \Delta x$

We shall take n such rectangles and fill the area. The area of APQB in the sum of the area of the rectangles. This may be written as

$$S = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x$$

This quantity is also denoted as,

$$S = \int_a^b f(x)\, dx$$

Question: The equation of motion of a particle is given by, $x = 1 + t^3$. Calculate the distance travelled by the particle from time $t = 0$ s to $t = 5$ s.

Solution:

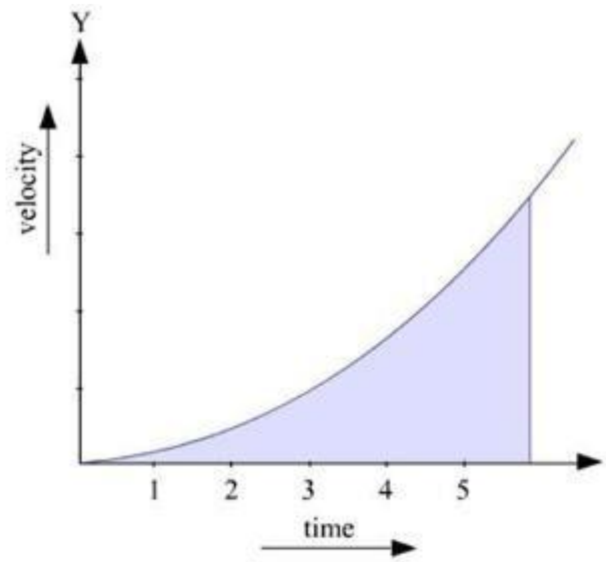
The motion equation is given by,

$x = 1 + t^3$

The velocity at any instance is,

$$v=\frac{dx}{dt}=\frac{d}{dx}(1+t^3)=3t^2$$

The velocity time graph is:



The are under this graph is the distance covered.

The area can be found by integrating the curve from t = 0s to t = 5s

$$S=\int_0^5 (3t^2)\,dt$$

$$\Rightarrow S=3\left[\frac{t^3}{3}\right]_0^5=125$$

Therefore the distance travelled by the particle in the given time interval is 125 m.

Motion in a Straight Line

Rest and Motion

We use the word ‘rest’ very often. For example, when someone is doing no work or lying on the bed, we often say that the person is resting. This means that the person is not moving. Scientifically as well, the word ‘rest’ has a similar meaning.

Scientifically, we say an object is at **rest** when the position of the object **does not change** with **time**, with respect to its **surroundings.**

Similarly, **motion** is defined as the change of position of an object with **time**, with respect to its **surroundings**.

What do we mean by **with respect to the surroundings**?

We know that a moving train is in motion because its position changes with time. Now, consider a person sitting in the train. For someone standing on the platform, the person sitting in the train is in motion. But for the co-passengers, the person is at rest as the position of the person does not change with time. Therefore, we need to consider the surroundings or the point of observation while describing the state of motion of an object. The surroundings is called reference frame.

What the above discussion shows us is that **rest and motion are relative**. They can be different for different observers. If someone views the Earth from the universe, then all the things on the Earth (such as houses, trees, a moving train, etc.) are in motion for that person. But for a person on the Earth's surface, things such as houses, trees, etc., are at rest. So, when we say that an object is at rest, what we really mean is that the object is at rest with respect to its surroundings.

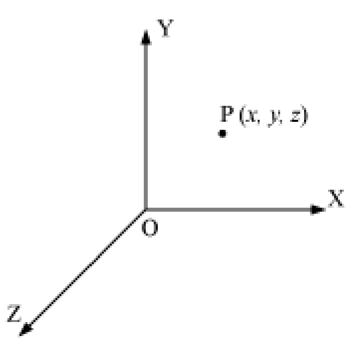
Point Mass Object

If the distance covered by an object is much greater than its size during its motion, then the object is considered as point mass object.

Rectilinear Motion – What it means?

Motion of a body that moves along a straight line such as the motion of a car moving on a straight road

Position – How to Describe it?



- Locating an object requires finding its position relative to a reference point.
- Reference point is often taken as the origin of a coordinate system.

- The coordinates (x, y, z) of the object describe the position of the object with respect to the coordinate axes.

- Coordinate system along with time constitutes a frame of reference.

Path Length

- Length of the actual path traversed by a body in a given time

- It is a **scalar quantity**. Therefore, only magnitude is important, not the direction of movement. (Implies that path length can never be negative)

Displacement

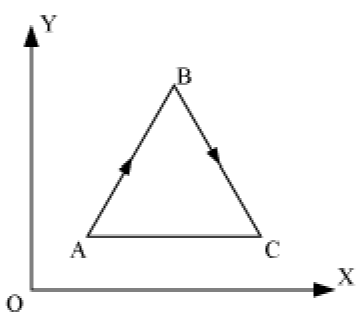
- A change of position ΔR from coordinate $R_1(x_1, y_1, z_1)$ to coordinate $R_2(x_2, y_2, z_2)$

- This is the shortest distance between the initial and final position of a body.

- It is a **vector quantity**. Therefore, both magnitude and direction are important to describe displacement. (Implies that displacement can be negative depending on the initial and final positions of a body in a coordinate system)

How Path Length and Displacement are Different?

Consider this example.



If an object goes from A to B and then B to C in time 't', then

- **path length = AB + BC** (arithmetic sum of the distances)

- displacement, $\Delta x = \overrightarrow{AC}$ (shortest distance between points A and C)

Problems based on Path Distance and Displacement

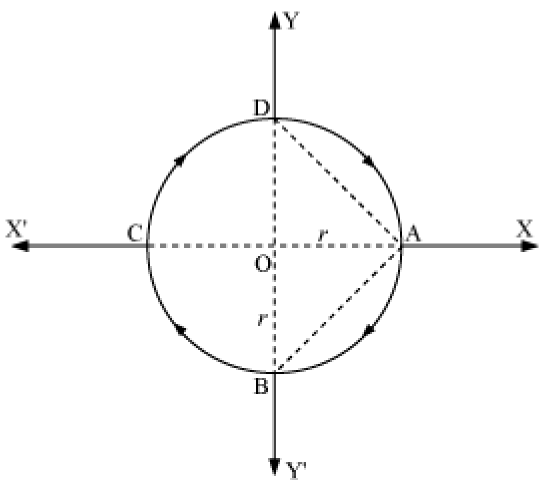
Example – A particle moves along a circle of radius 'r'. It starts from A and moves clockwise. Calculate the distance travelled by the particle and its displacement in each case. Take centre of the circle as the origin.

1. From A to B

2. From A to C

3. From A to D

4. In one complete revolution



Solution

(i) Distance travelled by the particle from A to B $= \frac{2\pi r}{4} = \frac{\pi r}{2}$

Displacement = $\left|\overrightarrow{AB}\right| = \sqrt{OA^2 + OB^2}$

$= \sqrt{r^2 + r^2} = \sqrt{2}r$ direction is along negative X and negative Y axis.

(ii) Distance travelled by particle from A to $C = \frac{2\pi r}{2} = \pi r$

Displacement $= |\overrightarrow{AC}| = 2r$ direction is along negative X axis.

(iii) Distance travelled from A to D $= \frac{2\pi r}{4} \times 3 = \frac{3}{2}\pi r$

Displacement$= \left|\overrightarrow{AD}\right| = \sqrt{r^2 + r^2} = \sqrt{2}r$ Direction is along Positive X and Y axis.

(iv) For one complete revolution i.e., motion from A to A,

Total distance travelled = $2\pi r$

Displacement = 0 [∵ the final position coincides with the initial position]

Speed, Velocity and Acceleration

Speed

- Speed = $\frac{\text{Distance travelled}}{\text{Time taken}} = \frac{\text{Path length}}{\text{Travelling time}}$
- It is a scalar quantity, which means that it requires no direction (it implies that speed cannot be negative).
- Instantaneous speed is the speed at a particular instant (when the interval of time is infinitely small).

i.e., instantaneous speed $= \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$

Average speed:

- Average speed of a particle is defined as the total distance travelled by the particle divided by the total time taken during which the motion took place.

- Suppose that a car is covering a distance of 160 km from A to B and covers successive 40 km distances in time 1.2 h, 1.4 h, 1.6 h, and 0.9 h respectively. The speed of car is different at different at every successive interval. In such cases, we need to find the average speed.

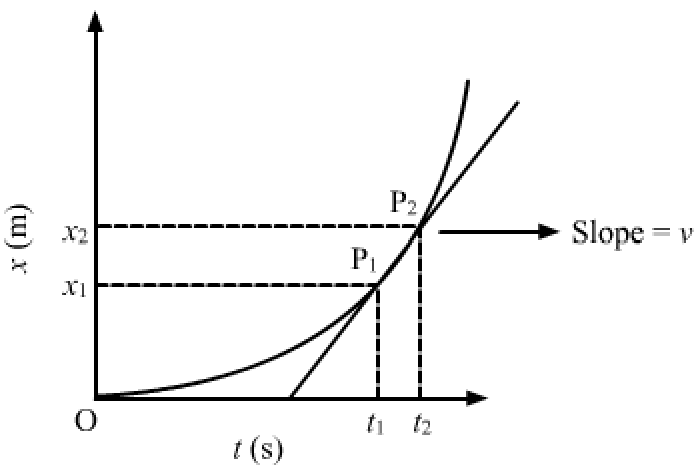
$$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time taken}}$$

Velocity

- Velocity = $\dfrac{\text{Displacement}}{\text{Time interval}} = \dfrac{\text{Final position} - \text{Initial position}}{\text{Time interval}}$

- It is a vector quantity. Therefore, the direction of movement is taken into consideration (it implies that velocity contains algebraic sign).

- In a **position-time graph**, the slope of the curve indicates the velocity and the angle of the slope with the *x*-axis indicates the direction.



- Average velocity is the ratio of the change in displacement Δx to the time interval Δt in which the change in displacement occurred.
- If x_1 and x_2 are the positions of a particle at time t_1 and t_2, respectively, then the magnitude of displacement of

the particle in time interval $\Delta t = (t_2 - t_1)$ is $\Delta \vec{x} = \vec{x_2} - \vec{x_1}$.

Average velocity, $\vec{v}_{avg} = \frac{\text{Displacement}}{\text{Time taken}} = \frac{\vec{x_2} - \vec{x_1}}{t_2 - t_1} = \frac{\Delta \vec{x}}{\Delta t}$

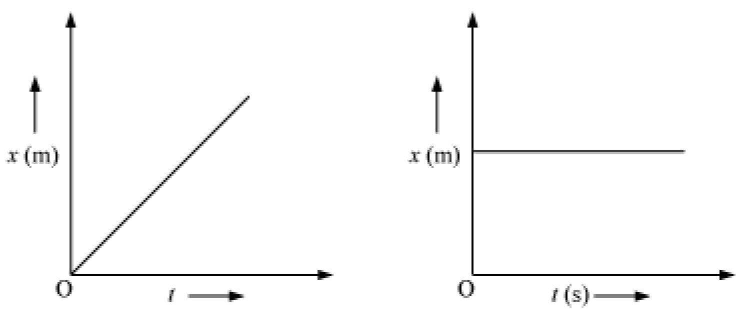
- Instantaneous velocity is the velocity at a particular instant (slope at a particular point on the *x*-*t* curve).

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

- When a motion is not uniform, sometimes instantaneous velocity has more importance than average velocity.

Uniform Motion: What it Means

In uniform motion, a body undergoes equal displacements in equal intervals of time.

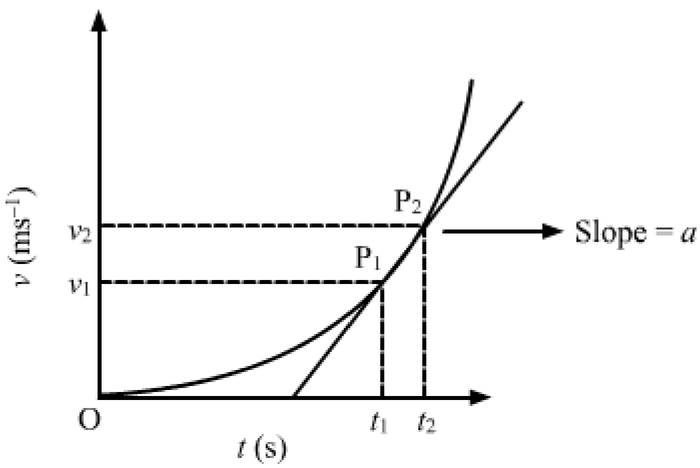


Position – time graph of an object in uniform motion

Position – time graph of a stationary object

Acceleration

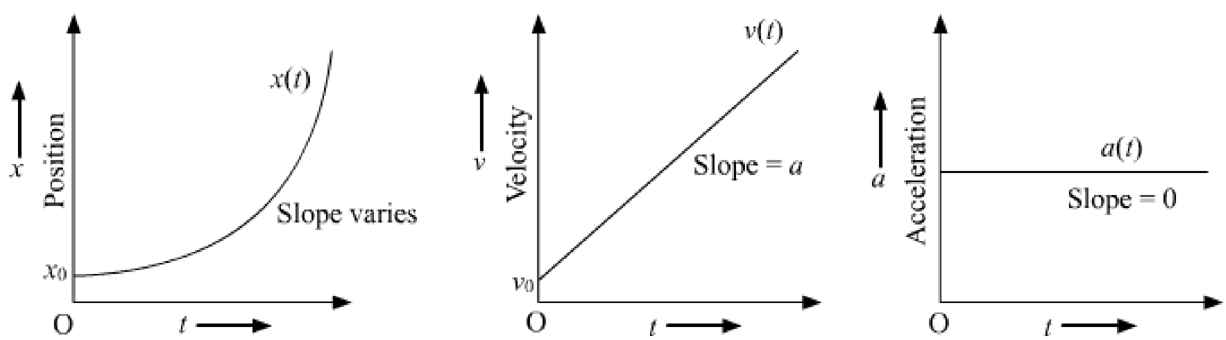
- Acceleration is the rate of change of the velocity of an object.
- Average acceleration, $\vec{a} = \frac{\text{Change in velocity}}{\text{Time taken}} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta \vec{v}}{\Delta t}$
- In a velocity-time graph, the slope of the curve indicates the average acceleration and the angle of the slope indicates the direction of change of the velocity.



- Instantaneous acceleration is the acceleration at a particular instant (slope at a particular point on the *v-t* curve).

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2 x}{dt^2}$$

In this case, the rate of change of velocity with time remains constant. Graphically, such motion can be represented as

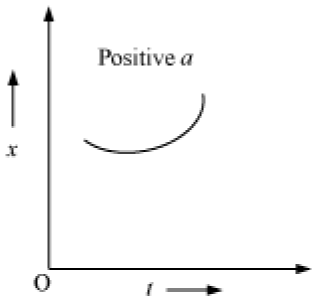


Position $x(t)$ of a particle moving with constant acceleration

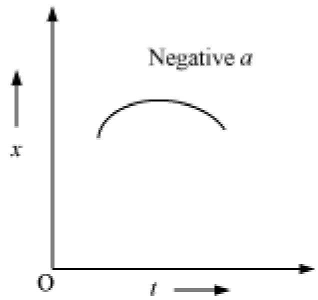
Its velocity $v(t)$, given at each point by the slope of the curve in (a).

Its (constant) acceleration, equal to the (constant) slope of the curve of $v(t)$.

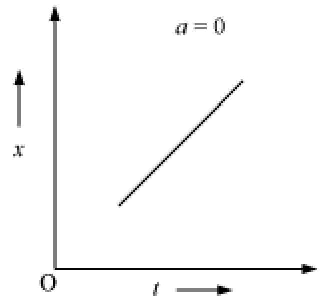
Graphical Representation of Accelerating Bodies



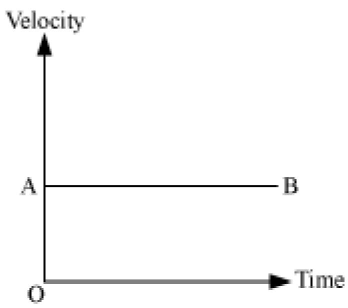
Position – time graph for motion with positive acceleration



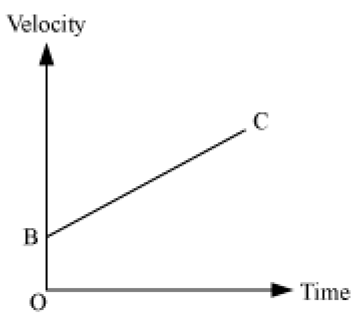
Position – time graph for motion with negative acceleration



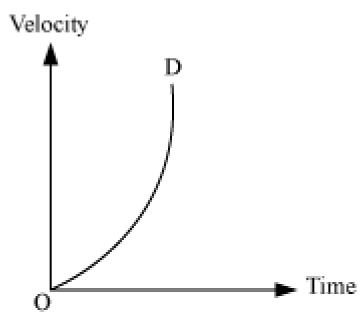
Position – time graph for motion with zero acceleration



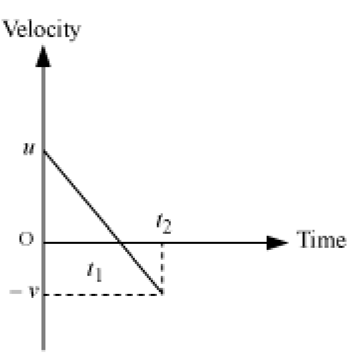
Object moving with constant velocity



Object moving with positive constant acceleration having some initial velocity



Object moving with increasing acceleration, having zero initial velocity



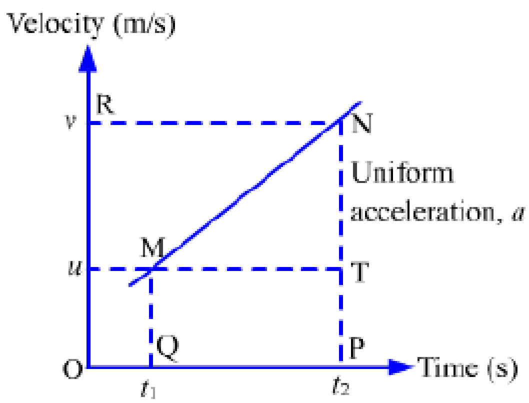
Object moving with uniform negative acceleration having positive initial velocity

First Equation of Motion

Velocity–Time Relation

Suppose a body is moving under a **uniform acceleration** in a given time interval. We can relate the change in the **velocity** of the moving body with the acceleration and time taken by using the one-dimensional velocity-time equation.

The velocity-time equation can be used for obtaining the final velocity, after time *t*, of a uniformly accelerating body.



Velocity–Time relation through the graphical method

Suppose a body is moving in a straight line, with an initial velocity u and under a uniform acceleration a. Its velocity becomes v after time t. The motion of this body is represented by the given velocity-time graph.

We can obtain the velocity-time equation if the velocities of a body (u and v) at times t_1 and t_2 are given, as shown in the velocity-time graph.

Initial velocity, u = MQ

Final velocity, v = NP

Time taken, t = QP = $(t_2 - t_1)$

Acceleration, a = Slope of line MN = NT/ MT = (NP - TP)/ (OP - OQ)

It is clear from the graph that TP = MQ

So, $a = (v - u)/ t_2 - t_1$

or, a $(t_2 - t_1) = v - u$ or, $v = u + a\,(t_2 - t_1)$

For initial time $t_1 = 0$, the equation reduces to: $v = u + a\,t_2$ or $v = u + at$ (as $t_2 = t$)

This is the **first equation of kinematics** and it is independent of the distance travelled. It is also known as the first equation of motion.

Solved Examples

Easy

Example 1: **On spotting a prey, a cheetah runs directly towards it with constant acceleration. The time taken by the cheetah is 50 s and its velocity, as it catches its prey, is 25 m/s. If we assume that the cheetah was initially at rest, then what is its acceleration?**

Solution:

It is given that:

Initial velocity (u) of the cheetah = 0

Its final velocity, v = 25 m/s

Time taken (t) by it to catch its prey= 50 s

We can determine the acceleration (a) of the cheetah using the relation:

$$a = \frac{v-u}{t}$$
$$= \frac{25-0}{50} = \frac{25}{50} = 0.5\ \text{m/s}^2$$

Medium

Example 2: **A motorcyclist is travelling at a constant velocity of 10 m/s. In order to overtake a car, he accelerates at the rate of 0.2 m/s^2. If he overtakes the car in 60 seconds, then what is his velocity while overtaking?**

Solution:

It is given that:

Initial velocity (u) of the motorcyclist = 10 m/s

His acceleration, a = 0.2 m/s^2

Time taken (t) by him to overtake the car = 60 s

Using the first equation of motion, we can compute the velocity (v) of the motorcyclist while overtaking the car.

$v = u + at$

$\Rightarrow v = 10 + 0.2 \times 60$

$\Rightarrow v = 10 + 12$

$\Rightarrow \therefore v = 22$ m/s

Hard

Example 3: **A train is moving under a constant acceleration of 150 km/h^2. It attains a velocity of**
125 km/h in half-hour. What is the initial velocity of the train in SI unit?

Solution:

It is given that:

Final velocity (v) of the train= 125 km/h

Time taken (t) by it to attain the above velocity= 0.5 h

Its acceleration (a) = 150 km/h^2

Using the first equation of motion, we can compute the initial velocity (u) of the train.

$v = u + at$

$\Rightarrow u = v - at$

$\Rightarrow u = 125 - 150 \times 0.5$

$\Rightarrow u = 125 - 75$

$\Rightarrow \therefore u = 50$ km/h

$$\text{Since } 1\,\text{km/h} = \left(\frac{5}{18}\right)\text{m/s}$$

$$50\,\text{km/h} = 50\left(\frac{5}{18}\right)\text{m/s}$$

$$= 13.89\ \text{m/s}$$

Therefore, the initial velocity of the train is 13.89 m/s.

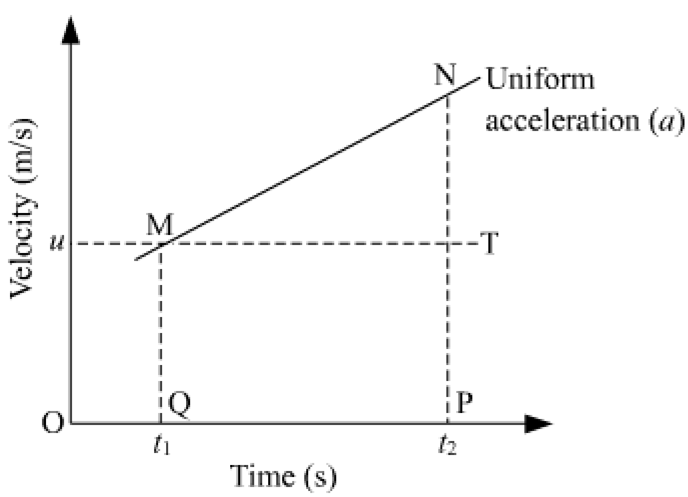
Second Equation of Motion

Position−Time Relation

Suppose a body moving under a **uniform acceleration**

covers a certain **distance**
in a given time interval. We can relate the change in the **velocity**
of the moving body with the acceleration, distance covered and time taken by using the one-dimensional position−time equation.The position−time equation is used to obtain the distance travelled by a uniformly accelerating body in a given interval of time.

Suppose a body is moving in a straight line, with an initial velocity u and under a uniform acceleration a. The distance covered by the moving body from time t_1 to time t_2 is represented in the given velocity−time graph.



It is clear from the graph that:

Initial velocity, u = MQ = TP

Time, t = PQ = $(t_2 - t_1)$ = MT

Change in velocity, NT = $a\ (t_2 - t_1)$

Distance, s = Area of trapezium QMNP

= Area of rectangle QMTP + Area of triangle MTN

= (MQ × QP) + ($\frac{1}{2}$ × NT × MT)

$= [u \times (t_2 - t_1)] + \frac{1}{2} \times a\ (t_2 - t_1) \times (t_2 - t_1)$

$\therefore s = u\ (t_2 - t_1) + \frac{1}{2}\ a\ (t_2 - t_1)^2$

For $t_1 = 0$ and $t_2 = t$, the equation reduces to:

$$\boxed{s = ut + \frac{1}{2}\ at^2}$$

This is the **second equation of kinematics** or the second equation of motion.

Average Velocity for Uniformly Accelerated Motion

We can obtain the relation for average velocity using the velocity−time and position−time equations.

The velocity−time equation is given as:

$v = u + at$

The position−time equation is given as:

$s = ut + \frac{1}{2}\ at^2$

$\Rightarrow \frac{s}{t} = u + \frac{1}{2}\ at$

$$\Rightarrow \frac{s}{t} = \frac{2u + at}{2}$$

$$\Rightarrow \frac{s}{t} = \frac{u + (u + at)}{2}$$

$$\Rightarrow \frac{s}{t} = \frac{u + v}{2}$$ (Using $v = u + at$)

$\therefore$Average velocity (v_{av}) = $\frac{s}{t} = \frac{u + v}{2}$

Did You Know?

If the acceleration is zero, then the second equation of motion denotes the distance travelled as the product of the initial velocity and time.

$$s = ut + \frac{1}{2}at^2$$
$$\Rightarrow s = ut + \frac{1}{2} \times 0 \times t^2$$
$$\Rightarrow s = ut$$

Solved Examples

Easy

Example 1: **A motorcyclist is travelling at a constant velocity of 10 m/s. He overtakes a car by accelerating at the rate of 0.2 m/s^2. If he overtakes the car in 60 s, then how much distance does he cover before overtaking the car?**

Solution:

It is given that:

Initial velocity (u) of the motorcyclist = 10 m/s

His acceleration, a = 0.2 m/s^2

Time taken (t) by him to overtake the car = 60 s

Using the second equation of motion, we can compute the distance covered (s) by the motorcyclist before overtaking the car.

$$s = ut + \frac{1}{2}at^2$$
$$\Rightarrow s = 10 \times 60 + \frac{1}{2} \times 0.2 \times 60^2$$
$$\Rightarrow s = 600 + \frac{1}{2} \times 0.2 \times 3600$$
$$\Rightarrow s = 600 + \frac{1}{2} \times 720$$
$$\Rightarrow s = 600 + 360$$
$$\Rightarrow \therefore s = 960 \text{ m}$$

Medium

Example 2: **A train moving at a speed of 180 km/h comes to a stop at a constant acceleration in 15 min after covering a distance of 25 km. What is its acceleration?**

Solution:

It is given that:

Initial velocity (u) of the train = 180 km/h

Distance covered (s) by it = 25 km

Time taken (t) by it to cover the above distance = 15 min = 0.25 h

Using the second equation of motion, we can compute the acceleration (a) of the train.

$$s = ut + \frac{1}{2}at^2$$
$$\Rightarrow 25 = 180 \times 0.25 + \frac{1}{2} \times a \times 0.25^2$$
$$\Rightarrow 25 = 45 + \frac{1}{2} \times a \times 0.0625$$
$$\Rightarrow 25 - 45 = 0.03125a$$
$$\Rightarrow a = \frac{-20}{0.03125}$$
$$\Rightarrow \therefore a = -640 \text{ km/h}^2$$

Hence, the train is retarding at a rate of 640 km/h^2. Note that since the speed of the train is decreasing, the acceleration comes out to be negative.

Hard

Example 3: **Brakes are applied on a car moving at a velocity of 72 km/h. It decelerates uniformly at the rate of 4 m/s^2 until it stops after 5 s. How far does the car go before it stops?**

Solution:

It is given that:

Initial velocity (u) of the car = 72 km/h

$= 72 \times \left(\frac{5}{18}\right)$ m/s

= 20 m/s

Its acceleration, $a = -4$ m/s^2 (since the car decelerates)

Time taken (t) by it to stop = 5 s

Using the second equation of motion, we can compute the distance covered (s) by the car before stopping.

$s = ut + \frac{1}{2}at^2$

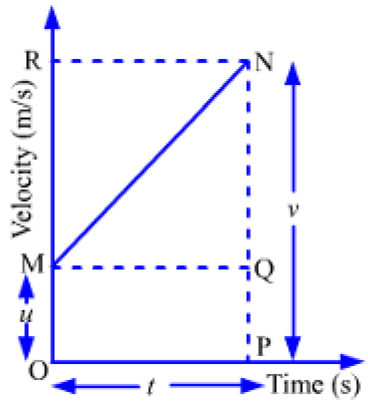
$\Rightarrow s = 20 \times 5 + \frac{1}{2} \times (-4) \times (5)^2$

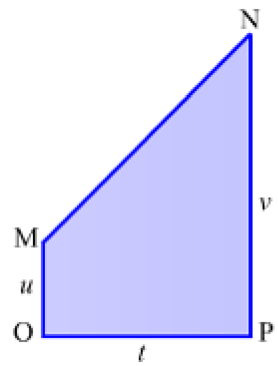
$\Rightarrow s = 100 - 50$

$\therefore s = 50$ m

Third Equation of Motion

Position–Velocity Relation





Suppose a body is moving in a straight line, with an initial velocity u and under a uniform acceleration a. Its velocity becomes v after time t and it covers a distance s in the given time interval. The motion of this body is represented in the given velocity–time graph.

It is clear from the graph that:

Initial velocity, u = MO = QP

Final velocity, v = OR = NP

The straight line MN represents the velocity–time curve.

Distance (s) covered by the body = Area of trapezium OMNP

$= \frac{1}{2} \times (\text{OM} + \text{PN}) \times \text{OP} \quad = \frac{1}{2} \times (u + v) \times t$

$\therefore s = \frac{1}{2}(u + v)\, t$ … (i)

Now, let us eliminate time t from this equation.

The velocity-time equation is given as:

$v = u + at$

$\therefore t = \frac{v - u}{a}$ … (ii)

On substituting the value of t from equation (ii) in equation (i), we obtain:

$$s = \frac{1}{2} \times (u + v) \times \left(\frac{v-u}{a}\right)$$

$$\Rightarrow s = \frac{(u+v)(v-u)}{2a}$$

$$\Rightarrow s = \frac{v^2 - u^2}{2a}$$

$$\therefore \boxed{v^2 - u^2 = 2as}$$

This is the **third equation of kinematics**. It is independent of time. It is also known as the third equation of motion.

Deriving the Second Equation of Motion

The third equation of motion is given as:

$v^2 = u^2 + 2as$ … (i)

The first equation of motion is given as:

$v = u + at$ … (ii)

On eliminating velocity v from equation (i) with the help of equation (ii), we obtain:

$(u + at)^2 = u^2 + 2as$

$\Rightarrow u^2 + 2uat + a^2t^2 = u^2 + 2as$

$\Rightarrow 2uat + a^2t^2 = 2as$

$\Rightarrow s = \frac{1}{2a} (2uat + a^2t^2)$

$\Rightarrow s = ut + \frac{1}{2} at^2$ … (iii)

This is the second equation of motion.

Solved Examples

Easy

Example 1: **On applying the brakes, a cyclist travelling initially at 2 m/s comes to a halt at a constant retardation of 2 m/s^2. How much distance does the cyclist cover before coming to rest?**

Solution:

It is given that:

Initial velocity (u) of the cyclist = 2 m/s

His final velocity, v = 0

His acceleration, a = −2 m/s^2 (since he is decelerating)

Using the third equation of motion, we can compute the distance covered (s) by the cyclist before stopping.

$$v^2 - u^2 = 2as$$
$$\Rightarrow v^2 = u^2 + 2as$$
$$\Rightarrow 0^2 = 2^2 - 2 \times 2 \times s$$
$$\Rightarrow 4 = 4s$$
$$\Rightarrow s = \frac{4}{4}$$
$$\Rightarrow \therefore s = 1 \text{ m}$$

Medium

Example 2: **A car covers 40 m in 8.5 s while applying brakes to a final speed of 2.8 m/s.**

(i) What is the initial speed of the car?

(ii) What is its acceleration?

Solution:

It is given that:
Final velocity (v) of the car = 2.8 m/s
Distance covered, $s = 40$ m
Time taken (t) to cover the above distance = 8.5 s

Let us take:
Initial velocity of the car = u
Acceleration of the car = a

We have the relation:

$$s = ut + \frac{1}{2}at^2$$

$$\Rightarrow 40 = 8.5u + \frac{1}{2}a \times 8.5^2 \qquad ...(1)$$

We know that:

$$v = u + at$$

$$\Rightarrow u = v - at$$

$$\Rightarrow u = 2.8 - a \times 8.5 \qquad ...(2)$$

From (1) and (2) we get:

$$40 = 8.5(2.8 - a \times 8.5) + \frac{1}{2}a \times 8.5^2$$

$$\Rightarrow \therefore a = -0.45 \text{ m/s}^2$$

On substituting the value of a in (2), we get:

$$u = 2.8 - (-0.45) \times 8.5$$

$$\Rightarrow \therefore u = 6.63 \text{ m/s}$$

Thus,
(i) The initial velocity of the car is 6.63 m/s.
(ii) The acceleration of the car is -0.45 m/s^2.

Hard

Example 3: **When the brakes are applied, a racing car stops within 0.0229 of a mile from a speed of**
60 mi/h and within 0.0399 of a mile from a speed of 80 mi/h.

(i) What is the braking acceleration of the car for 60 mi/h to rest?

(ii) What is its braking acceleration for 80 mi/h to rest?

(iii) What is its braking acceleration for 80 mi/h to 60 mi/h?

Solution:

(i) In the first case:
Inital velocity (u) of the car = 60 mi/h = 26.82 m/s
Its final velocity, $v = 0$
Distance covered, $s = 0.0229$ mile = 36.88 m
Let the acceleration of the car be a.

We know that:
$v^2 - u^2 = 2as$
$\Rightarrow v^2 = u^2 + 2as$
$\Rightarrow 0 = 26.82^2 + 2 \times a \times 36.88$
$\Rightarrow \therefore a = -9.75 \text{ m/s}^2$
Thus, the braking acceleration of the car for 60 mi/h to rest is -9.75 m/s^2.

(ii) In the second case:
Initial velocity (u) of the car = 80 mi/h = 35.76 m/s
Its final velocity, $v = 0$
Distance covered, $s = 0.0399$ mile = 64.31 m
Again, let the acceleration of the car be a.

We know that:
$v^2 - u^2 = 2as$
$\Rightarrow v^2 = u^2 + 2as$
$\Rightarrow 0 = 35.76^2 + 2 \times a \times 64.31$
$\Rightarrow \therefore a = -9.94 \text{ m/s}^2$
Thus, the braking acceleration of the car for 80 mi/h to rest is -9.94 m/s^2.

(iii) In the third case:

Initial velocity (u) of the car = 80 mi/h = 35.76 m/s

Its final velocity, v = 60 mi/h = 26.82 m/s

Distance covered, s = Distance covered in the second case – Distance covered in the first case

$= 64.31 - 36.88 = 27.43$ m

Again, the acceleration is taken as a.

We know that:

$v^2 - u^2 = 2as$

$\Rightarrow v^2 = u^2 + 2as$

$\Rightarrow 26.82^2 = 35.76^2 + 2 \times a \times 27.43$

$\Rightarrow \therefore a = -10.19 \text{ m/s}^2$

Thus, the braking acceleration of the car for 80 mi/h to 60 mi/h is -10.19 m/s^2.

Kinematics Equations for Uniformly Accelerated Motion

Terminology

$u \rightarrow$ Initial velocity

$v \rightarrow$ Final velocity

$a \rightarrow$ Acceleration

$t \rightarrow$ Time

$x_0 \rightarrow$ Initial position

$x \rightarrow$ Position after time t

Equations of Motion by Calculus Method

- **Velocity–Time Relation**

Acceleration, $a = \frac{dv}{dt}$ or $dv = adt$

Integrating the above,

$$\int_u^v dv = a\int_0^t dt$$

$\Rightarrow v - u = at$

$\Rightarrow v = u + at$

- **Displacement–Time Relation**

  Instantaneous velocity, $v = \dfrac{dx}{dt}$ or $dx = vdt$

  $\Rightarrow dx = (u + at)\, dt$ [From velocity–time relation]

  Integrating the above relation,

  $$\int_{x_0}^{x} dx = \int_0^t (u+at)dt = u\int_0^t dt + a\int_0^t tdt$$

  $\Rightarrow (x)_{x_0}^{x} = u(t)_0^t + a\left(\dfrac{t^2}{2}\right)_0^t$

  $\Rightarrow (x - x_0) = ut + \dfrac{1}{2}at^2$

  $\Rightarrow x = x_0 + ut + \dfrac{1}{2}at^2$

- **Velocity–Displacement Relation**

  $$a = \frac{dv}{dt} = \frac{dv}{dx}\times\frac{dx}{dt} = \frac{dv}{dx}v$$

  $\Rightarrow adx = vdv$

  Integrating the above expression,

  $\int_{x_0}^{x} adx = \int_u^v vdv$ or $a(x)_{x_0}^{x} = \left(\dfrac{v^2}{2}\right)_u^v$

$\Rightarrow a(x - x_0) = \frac{v^2}{2} - \frac{u^2}{2}$

$\Rightarrow v^2 - u^2 = 2a\ (x - x_0)$

Putting $(x - x_0) = S$, we obtain [S is the displacement]

$\boxed{v^2 - u^2 = 2aS}$

As $v^2 = u^2 + 2aS$,

$\Rightarrow 0 = (30)^2 + 2\ (-10)\ S$

$\Rightarrow S = 45$ m

As $v = u + at$,

$0 = 30 + (-10)\ t_1$

$\Rightarrow t_1 = 3$ s

(ii) For vertical downward motion,

$u = 0\ \text{ms}^{-1}$

$a = 10\ \text{ms}^{-2}$

$S = (45 + 20)$ m = 65 m

$t = t_2$ (say)

As $S = ut + \frac{1}{2}at^2$,

$65 = 0 + \frac{1}{2}(10)t_2^2$

$\Rightarrow 5t_2^2 = 65$

$\Rightarrow t_2^2 = \frac{65}{5}$ = 13 s

$\Rightarrow t_2 = \sqrt{13}\ \text{s} = 3.6\ \text{s}$

∴ Total time = $t_1 + t_2$ = (3 + 3.6) s = 6.6 s

Relative Velocity

Relative Velocity

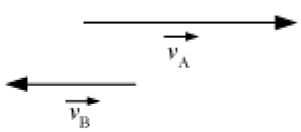
The relative velocity of a body **A** with respect to another body **B** (v_{AB}) is the time rate at which **A** changes its position with respect to **B**.

Case 1: Both bodies move in the same direction

If **A** and **B** are moving in the same direction, then the resultant relative velocity is

$v_{AB} = v_A - v_B$

Case 2: **The bodies move in opposite directions**



If **A** and **B** are moving in the opposite directions, then the resultant relative velocity is $v_{AB} = v_A + v_B$

Example 1 – Two cars **X** and **Y** are moving with speeds of 60 km h^{-1} and 80 km h^{-1} respectively along parallel straight paths. Both the cars started from the same position. What is the position of car **X** with respect to **Y** after 15 minutes?

Solution

Speed of car **X**, v_x = 60 km h^{-1}

Speed of car **Y**, v_y = 80 km h^{-1}

Both the cars are moving in the same direction.

Thus, relative velocity of car **X** with respect to car **Y**, $v_{xy} = v_x - v_y$

$= (60 - 80)\ \text{km h}^{-1}$

$= -20\ \text{km h}^{-1}$

∴ Separation of car **X** with respect to car **Y** after 15 minutes

is $\frac{-20 \times 15}{60} = -5$ km [∵ Distance = Speed × Time]

Example 2 – Two trains, 110 m and 90 m in length, are running in opposite directions with velocities 75 km h^{-1} and 64 km h^{-1}. In what time will they completely cross each other?

Solution

Here, $v_A = 75\ \text{km h}^{-1}$

$v_B = -64\ \text{km h}^{-1}$ [Trains are moving in opposite directions]

Length of train **A**, $l_A = 110$ m

Length of train **B**, $l_B = 90$ m

Relative velocity of the two trains is

$v_{AB} = v_A - v_B = 75 - (-64)$

$= (75 + 64)\ \text{km h}^{-1}$

$= 139\ \text{km h}^{-1}$

$= 139\ \text{km h}^{-1}$

$= 38.6\ \text{m s}^{-1}$

Total distance to be travelled by each train for completely crossing the other train

$= (110 + 90) = 200$ m

∴ Time taken by each train to cross the other train $= \frac{200\ \text{m}}{38.6\ \text{m s}^{-1}}$

$= 5.2$ s