Exercise 10.4

Q1E

Given $r = f(\theta) = e^{-\theta/4}, \frac{\pi}{2} \le \theta \le \pi$

Area $A = \int_a^b \frac{1}{2} r^2 d\theta$

$$\Rightarrow A = \frac{1}{2} \int_{\pi/2}^{\pi} e^{-2\theta/4} d\theta$$

$$= \frac{1}{2} \int_{\pi/2}^{\pi} e^{-\theta/2} d\theta$$

$$= \frac{1}{2} \left[\frac{e^{-\theta/2}}{-\frac{1}{2}} \right]_{\pi/2}^{\pi}$$

$$= e^{-\pi/4} - e^{-\pi/2}$$

$$=0.24805854$$

Q2E

Given $r = f(\theta) = \cos\theta, \ 0 \le \theta \le \frac{\pi}{6}$

Area $A = \int_a^b \frac{1}{2} r^2 d\theta$

$$A = \int_0^{\pi/6} \frac{1}{2} \cos^2 \theta d\theta$$

$$= \frac{1}{2} \int_0^{\pi/6} \frac{1+\cos 2\theta}{2} d\theta$$

$$= \frac{1}{4} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/6}$$

$$= \frac{1}{4} \left[\frac{\pi}{6} + \frac{1}{2} \sin \frac{\pi}{3} \right]$$

$$= \boxed{\frac{\pi}{24} + \frac{\sqrt{3}}{16}}$$

Q3E

Given $r^2 = 9\sin 2\theta,\ 0 \le \theta \le \frac{\pi}{2}$

Area $A = \int_a^b \frac{1}{2} r^2 d\theta$

$$A = \int_0^{\pi/2} \frac{1}{2} 9\sin 2\theta d\theta$$

$$= \int_0^{\pi/2} \frac{9\sin 2\theta}{2} d\theta$$

$$= \frac{9}{2}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{-9}{4}\left[\cos\pi - \cos 0\right]$$

$$= \boxed{\frac{9}{2}}$$

Q4E

Given $r = \tan\theta,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{3}$

Area $A = \int_a^b \frac{1}{2} r^2 d\theta$

$$A = \int_{\pi/6}^{\pi/3} \frac{1}{2}\tan^2\theta d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1}{2}(\sec^2\theta - 1) d\theta$$

$$= \frac{1}{2}\left[\tan\theta - \theta\right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left[\frac{2}{\sqrt{3}} - \frac{\pi}{6}\right]$$

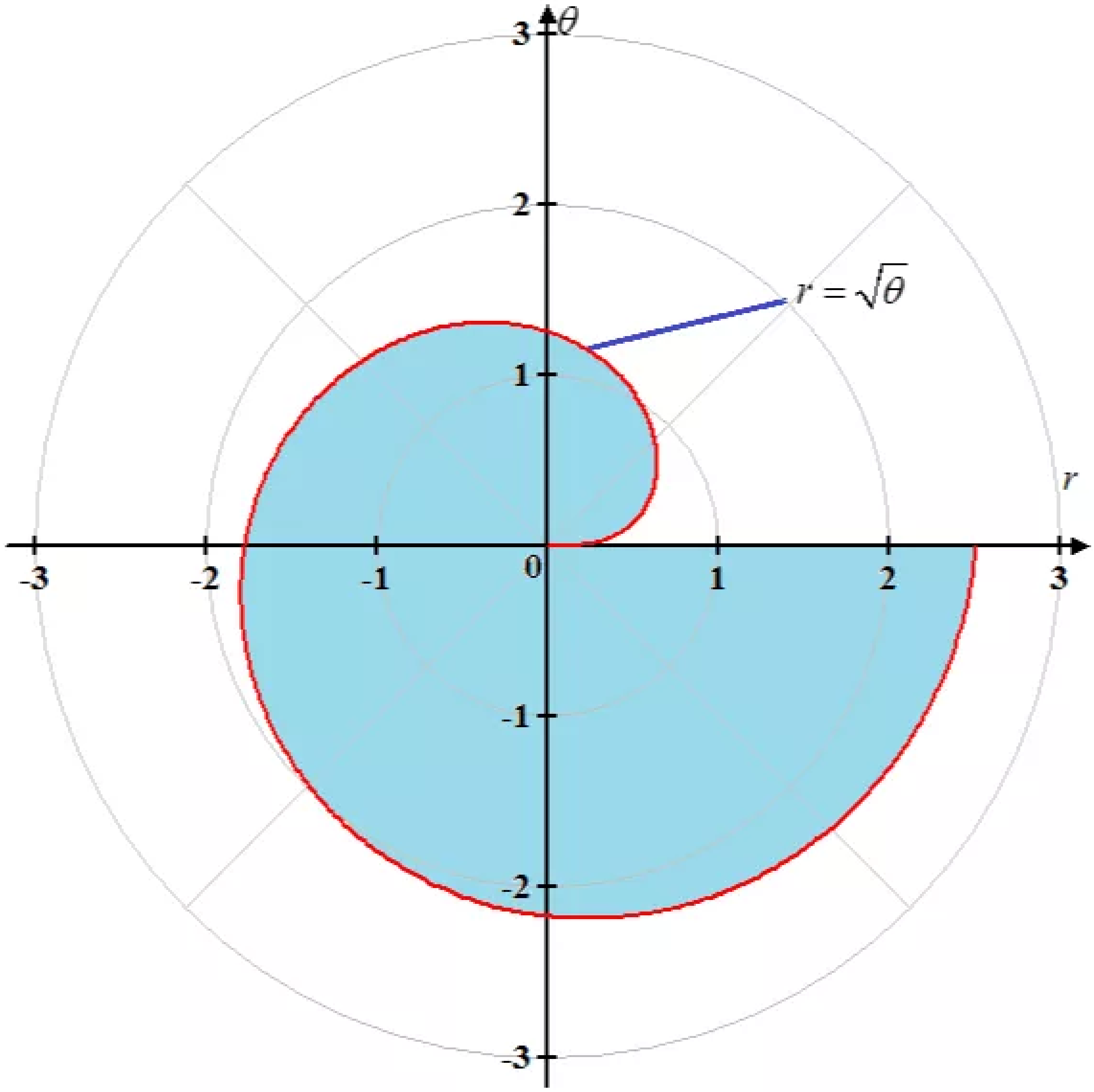
$$= \frac{1}{\sqrt{3}} - \frac{\pi}{12}$$

Q5E

Consider the polar curve

$r = \sqrt{\theta}$

First sketch a graph of the curve enclosed by the region need to be determined.



Find the area of the shaded region of the cardioid $r = 1 + \cos\theta$:

From the graph, observe that the region enclosed by shaded loop is swept out by a ray that rotates from $\theta = 0$ to $\theta = \pi$.

Recall that,

The area of the region bounded by the curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$ is

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

So, the area of the region bounded by the curve $r = 1 + \cos\theta$ and by the rays $\theta = 0$ and $\theta = \pi$ is

$$A = \int_a^b \frac{1}{2}[r]^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left[\sqrt{\theta}\right]^2 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \theta d\theta$$

$$= \frac{1}{2}\left[\frac{1}{2}\theta^2\right]_0^{2\pi}$$ Use power rule

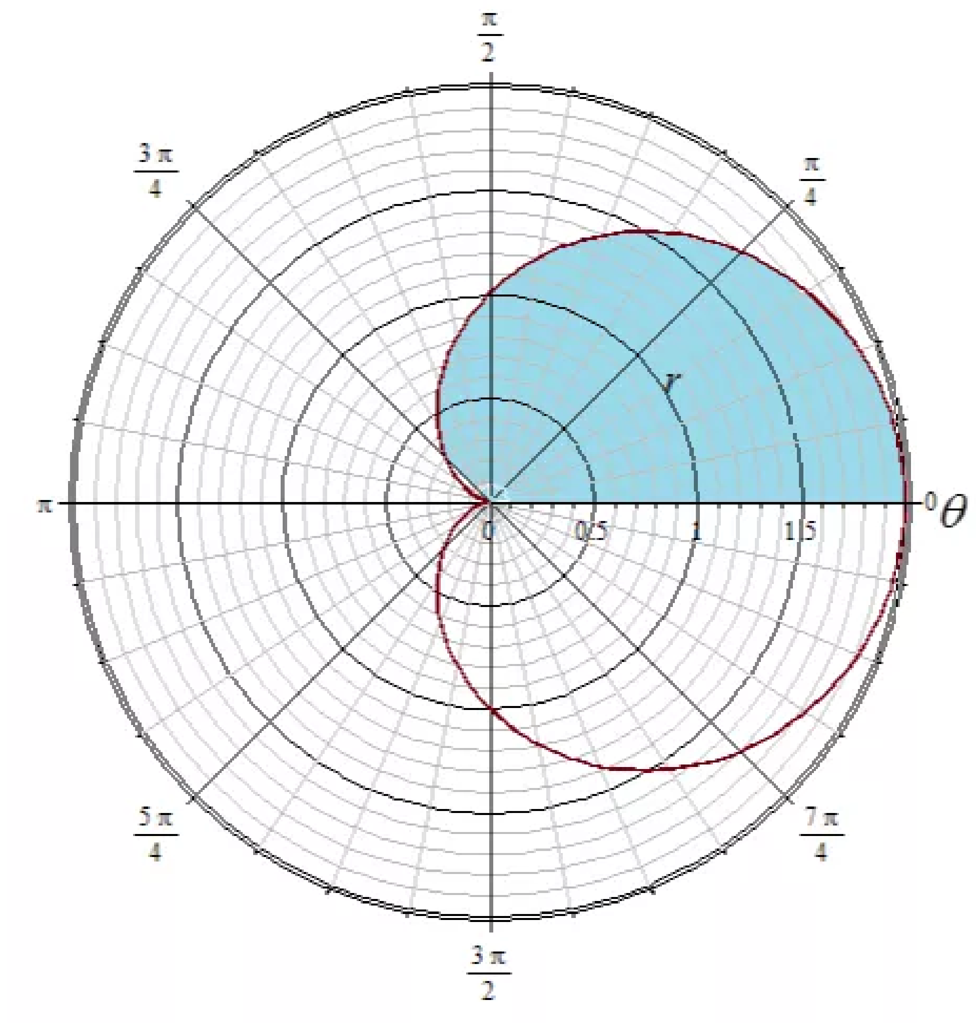
$$= \frac{1}{2}\left[\frac{1}{2}(2\pi)^2 - \frac{1}{2}(0)^2\right]$$ Apply the limits

$$= \pi^2$$

Thus, the area of the shaded region is $\boxed{\pi^2}$.

Q6E

Consider graph of the polar curve $r = 1 + \cos\theta$



Find the area of the shaded region of the cardioid $r=1+\cos\theta$:

From the graph, observe that the region enclosed by shaded loop is swept out by a ray that rotates from $\theta=0$ to $\theta=\pi$.

Recall that,

The area of the region bounded by the curve $r=f(\theta)$ and by the rays $\theta=a$ and $\theta=b$ is

$$A=\int_a^b \frac{1}{2}r^2 d\theta$$

So, the area of the region bounded by the curve $r=1+\cos\theta$ and by the rays $\theta=0$ and $\theta=\pi$ is

$$A=\int_0^\pi \frac{1}{2}(1+\cos\theta)^2\, d\theta$$

$=\frac{1}{2}\left(\int_0^\pi 1d\theta+2\int_0^\pi \cos\theta d\theta+\int_0^\pi \cos^2\theta d\theta\right)$ Use $\int kf(x)dx=k\int f(x)dx$

$=\frac{1}{2}\left(\int_0^\pi 1d\theta+2\int_0^\pi \cos\theta d\theta+\int_0^\pi \left(\frac{1+\cos 2\theta}{2}\right)d\theta\right)$ Since: $\cos^2\theta=\frac{1+\cos 2\theta}{2}$

$=\frac{1}{2}\left(\int_0^\pi 1d\theta+2\int_0^\pi \cos\theta d\theta+\int_0^\pi \frac{1}{2}d\theta+\int_0^\pi \left(\frac{\cos 2\theta}{2}\right)d\theta\right)$

Use $\int(f+g)(x)dx=\int f(x)dx+\int g(x)dx$

$=\frac{1}{2}\left(\int_0^\pi \frac{3}{2}d\theta+2\int_0^\pi \cos\theta d\theta+\int_0^\pi \left(\frac{\cos 2\theta}{2}\right)d\theta\right)$

$=\frac{1}{8}\left[6\theta+8\sin\theta+\sin 2\theta\right]_0^\pi$

$=\frac{1}{8}\left[6(\pi)+8\sin\pi+\sin 2\pi-6(0)+8\sin(0)+\sin 2(0)\right]$

$=\boxed{\frac{3\pi}{4}}$.

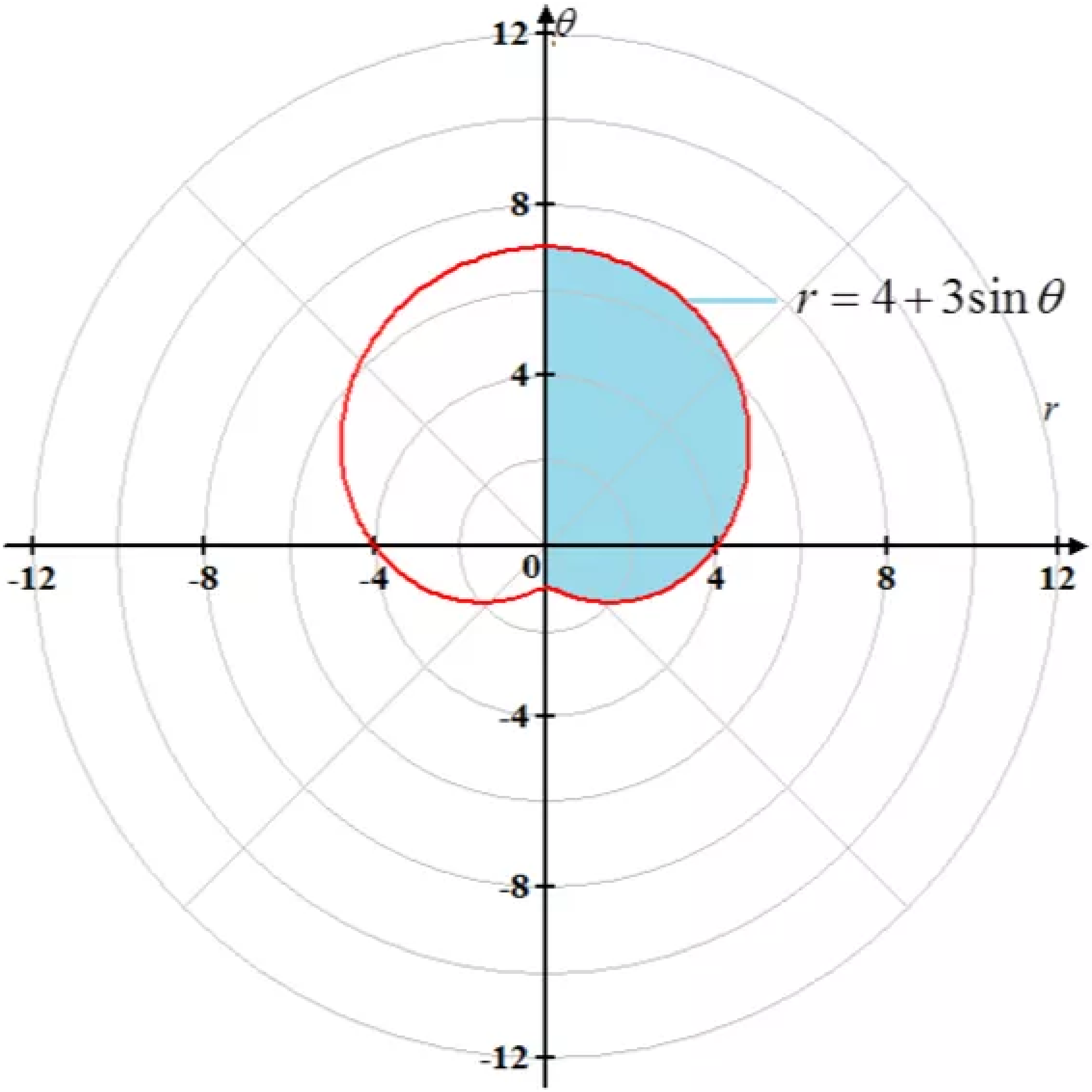
Therefore, the area of the shaded region is $\boxed{\frac{3\pi}{4}}$

Q7E

Consider the polar curve

$r = 4 + 3\sin\theta$

First sketch a graph of the curve enclosed by the region need to be determined.



It is required find the area of the shaded region shown in the above figure.

By observing the graph notice that shaded region is the region enclosed by the curve $r = 4 + 3\sin\theta$ is swept out by a ray that rotates from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.

Therefore, the area of the shaded region is

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}(4 + 3\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(16 + 24\sin\theta + 9\sin^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[16 + 24\sin\theta + 9\left(\frac{1-\cos 2\theta}{2}\right)\right] d\theta$$

Use the double angle identity $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[16 + 24\sin\theta + \frac{9}{2} - \frac{9\cos 2\theta}{2}\right] d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[\frac{41}{2} + 24\sin\theta - \frac{9}{2}\cos 2\theta\right] d\theta$$ Simplify

Continue the above steps.

$$A = \frac{1}{2}\left[\int_{-\pi/2}^{\pi/2} \frac{41}{2} d\theta + \int_{-\pi/2}^{\pi/2} 24\sin\theta d\theta - \int_{-\pi/2}^{\pi/2} \frac{9}{2}\cos 2\theta d\theta\right]$$

Use $\int (f+g)(x)dx = \int f(x)dx + \int g(x)dx$

$$= \frac{1}{2}\left[\frac{41}{2}\int_{-\pi/2}^{\pi/2} d\theta + 24\int_{-\pi/2}^{\pi/2} \sin\theta d\theta - \frac{9}{2}\int_{-\pi/2}^{\pi/2} \cos 2\theta d\theta\right]$$

Use $\int (kf)(x)dx = k\int f(x)dx$

$$= \frac{1}{2}\left[\frac{41}{2}[\theta]_{-\pi/2}^{\pi/2} + 24[-\cos\theta]_{-\pi/2}^{\pi/2} - \frac{9}{2}\left[\frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}\right]$$

$$= \frac{1}{2}\left[\frac{41}{2}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] + 24\left[-\cos\frac{\pi}{2} + \cos\left(-\frac{\pi}{2}\right)\right] - \frac{9}{2}\left[\frac{\sin 2\left(\frac{\pi}{2}\right)}{2} - \frac{\sin 2\left(-\frac{\pi}{2}\right)}{2}\right]\right]$$

Apply the limits

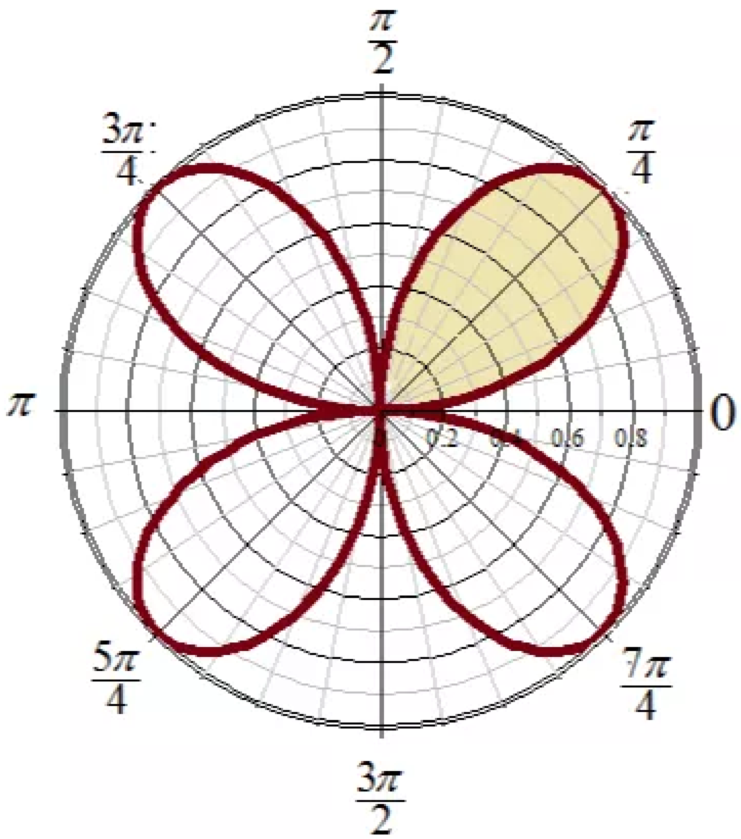
$$= \frac{1}{2}\left[\frac{41}{2}[\pi] + 24[-0+0] - \frac{9}{2}\left[\frac{0}{2} + \frac{0}{2}\right]\right]$$

$$= \frac{1}{2}\left[\frac{41}{2}[\pi] + 0 - 0\right]$$

$$= \boxed{\frac{41\pi}{4}}$$

Q8E

Consider graph of the polar curve $r = \sin 2\theta$



Find the area of the shaded region of the four-leaved rose $r = \sin 2\theta$:

From the graph, observe that the region enclosed by shaded loop is swept out by a ray that rotates from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

Recall that,

The area of the region bounded by the curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$ is

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

So, the area of the region bounded by the curve $r = \sin 2\theta$ and by the rays $\theta = 0$ and $\theta = \frac{\pi}{2}$ is

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2}(\sin 2\theta)^2 \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\sin^2 2\theta\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\frac{1-\cos 4\theta}{2}\right) d\theta$$ Since: $\sin^2 2\theta = \frac{1-\cos 4\theta}{2}$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} (1-\cos 4\theta) d\theta$$ Since: $\int kf(x)dx = k\int f(x)dx$

$$= \frac{1}{4}\left(\theta - \frac{\sin 4\theta}{4}\right)_0^{\frac{\pi}{2}}$$ Apply integration

$$= \frac{1}{4}\left[\left(\frac{\pi}{2} - \frac{\sin 4\left(\frac{\pi}{2}\right)}{4}\right) - \left(0 - \frac{\sin 0}{4}\right)\right]$$ Apply the limits

$$= \frac{1}{4}\left[\left(\frac{\pi}{2} - \frac{\sin 2\pi}{4}\right) - 0\right]$$ Since: $\sin 0 = 0$

$$= \frac{1}{4}\left[\left(\frac{\pi}{2} - 0\right) - 0\right]$$ Since: $\sin 2\pi = 0$

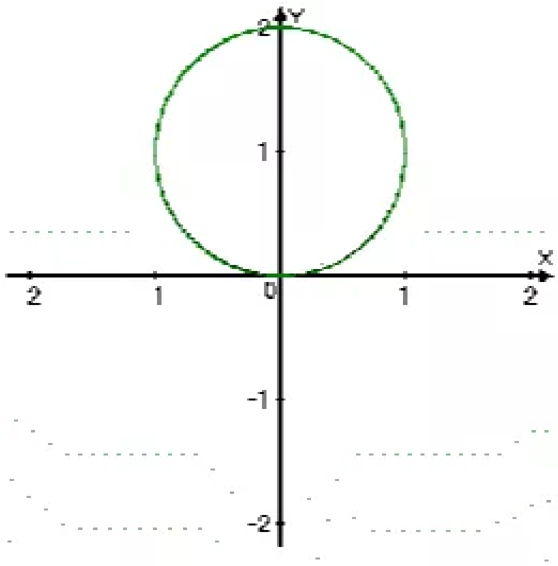
$$= \frac{\pi}{8}$$ Multiply

Therefore, the area of the shaded region is $\boxed{\frac{\pi}{8}}$

Q9E

Given $r = 2\sin\theta$

The sketch of this polar equation is



Observe that this is a circle formed by $0 \le \theta \le \pi$

Area $A = \int_a^b \frac{1}{2} r^2 \, d\theta$

$$A = \int_0^{\pi} \frac{1}{2} r^2 \, d\theta$$

$$= 2 \cdot \frac{1}{2} \int_0^{\pi/2} 4\sin^2\theta \, d\theta$$

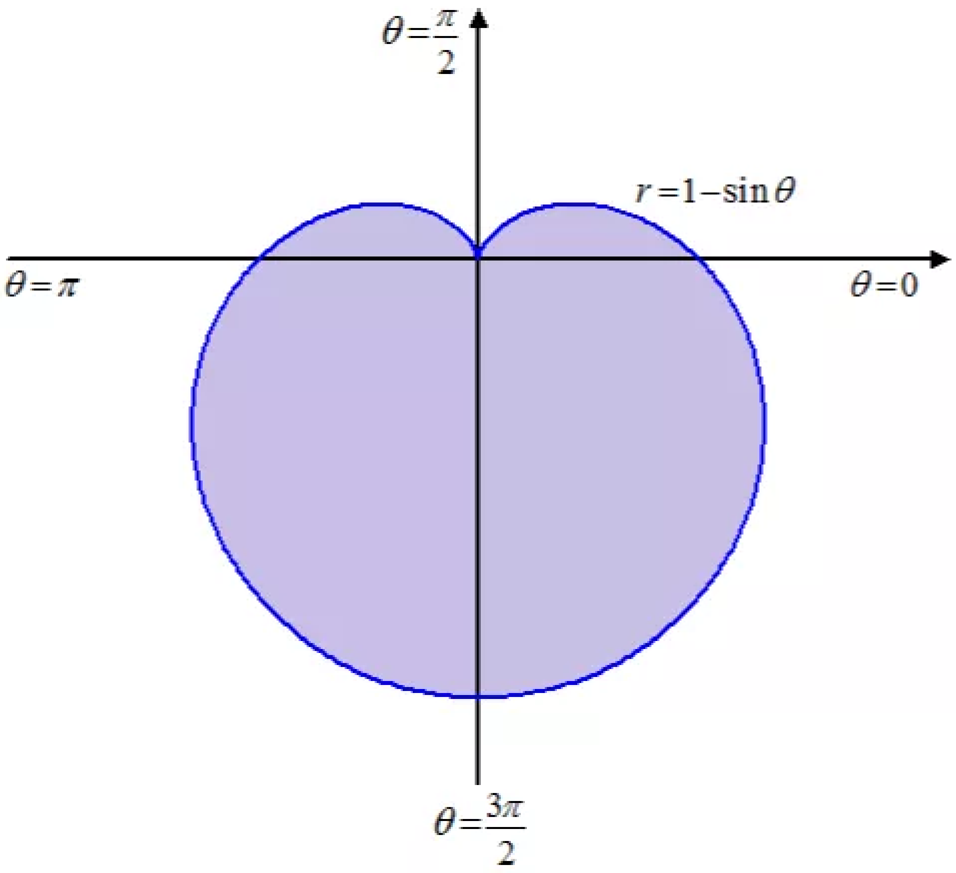
$$= 4 \cdot \frac{1}{2} \cdot \frac{\pi}{2} \qquad \left(\because \int_0^{\pi/2} \sin^{2k}\theta \, d\theta = \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdot \cdots \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right)$$

$$= \boxed{\pi}$$

Q10E

Consider the curve $r = 1 - \sin\theta$

Sketch the graph of the curve is shown below:



Observe that the rotation of the entire curve takes place when $0 \le \theta \le 2\pi$

The area of the polar curve $r = f(\theta)$ in the interval $[a,b]$ is

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

Therefore, the required area of the given curve is

$$A=\int_a^b \frac{1}{2}r^2 d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}(1-\sin\theta)^2\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\left(1-2\sin\theta+\sin^2\theta\right)d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\left(1-2\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$=\frac{1}{4}\int_0^{2\pi}(2-4\sin\theta+1-\cos 2\theta)\,d\theta$$

$$=\frac{1}{4}\int_0^{2\pi}(3-4\sin\theta-\cos 2\theta)\,d\theta$$

Therefore,

$$A=\frac{1}{4}\left(3\theta+4\cos\theta-\frac{\sin 2\theta}{2}\right)_0^{2\pi}$$

$$=\frac{1}{4}\left[(6\pi+4-0)-(0+4-0)\right]$$

$$=\frac{1}{4}[6\pi]$$

$$=\boxed{\frac{3\pi}{2}}$$

Q11E

Area $A=\int_a^b \frac{1}{2}r^2 d\theta$

$$=\int_0^{2\pi}\frac{1}{2}(3+2\cos\theta)^2\,d\theta$$

$$=\frac{1}{2}\left\{\int_0^{2\pi}9\,d\theta+\int_0^{2\pi}4\cos^2\theta\,d\theta+\int_0^{2\pi}12\cos\theta\,d\theta\right\}$$

$$=\frac{1}{2}\left\{18\pi+4\cdot 4\int_0^{\pi/2}\cos^2\theta\,d\theta+12\left(\sin(2\pi)-\sin(0)\right)\right\}$$

$$\left(\because \cos^2\theta \text{ is an even function}\right)$$

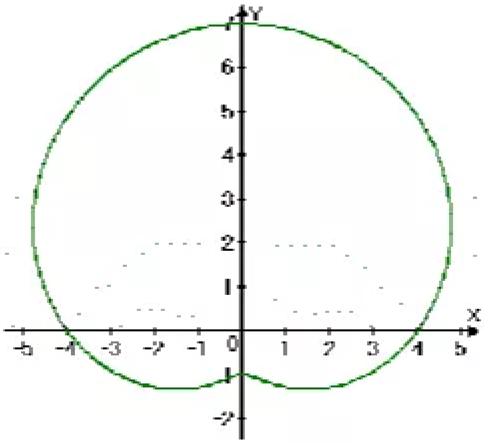
$$=\frac{1}{2}\left\{18\pi+16\cdot\frac{1}{2}\cdot\frac{\pi}{2}+12(1-1)\right\}\left(\because \int_0^{\pi/2}\sin^{2k}\theta\,d\theta=\frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}\cdot\ldots\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$=11\pi$$

Q12E

Given polar equation is $r = 4+3\sin\theta$

Its sketch is



Area $A = \int_a^b \frac{1}{2} r^2 d\theta$

$\int_0^{2\pi} \frac{1}{2}(4+3\sin\theta)^2 \, d\theta$

$= \frac{1}{2}\int_0^{2\pi} \left(16+9\sin^2\theta+24\sin\theta\right) d\theta$

$= \frac{1}{2}\int_0^{2\pi} 16\, d\theta + \frac{1}{2}\int_0^{2\pi} 9\sin^2\theta\, d\theta + \frac{1}{2}\int_0^{2\pi} 24\sin\theta\, d\theta$

$= 8(2\pi - 0) + \frac{9}{2}\cdot 4 \cdot \int_0^{\pi/2} \sin^2\theta\, d\theta - 12\left(\cos(2\pi) - \cos(0)\right)$

$\because \sin^2\theta$ is an even function, we changed the limits.

$= 16\pi + 18\cdot\frac{1}{2}\cdot\frac{\pi}{2} - 12(1-1)\left(\because \int_0^{\pi/2} \sin^{2k}\theta\, d\theta = \frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}\cdot\ldots\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$

$= \frac{41\pi}{2}$

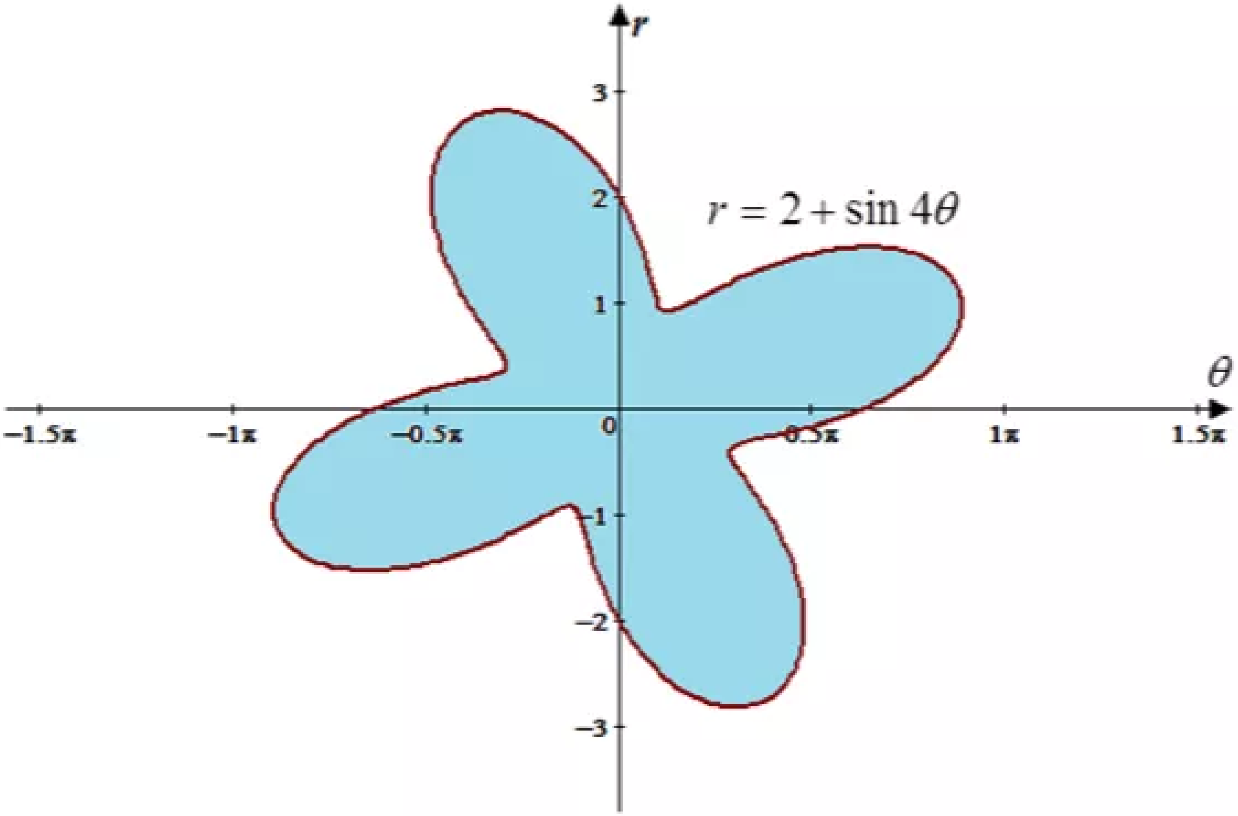
Q13E

Consider the following polar equation:

$r = 2+\sin 4\theta$

The objective is to sketch the curve and find the area that it encloses.

The sketch of the curve is shown below:



The rotation required is $0 \le \theta \le 2\pi.$

The formula for the area *A* of the polar curve is,

$$A = \int_a^b \frac{1}{2} r^2 \, d\theta$$

Use the above formula to find the area of the given curve.

Therefore, the area of the polar curve is,

$$\text{Area} = \int_0^{2\pi} \frac{1}{2}(2+\sin 4\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(4 + 4\sin 4\theta + \sin^2 4\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(4 + 4\sin 4\theta + \frac{1-\cos 8\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(\frac{9}{2} + 4\sin 4\theta - \frac{\cos 8\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\left[\frac{9}{2}\theta - \cos 4\theta - \frac{\sin 8\theta}{16}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[\left(\frac{9}{2}(2\pi) - \cos 8\pi - \frac{\sin 16\pi}{16}\right) - \left(0 - \cos 0 - \frac{\sin 0}{16}\right)\right]$$

$$= \frac{1}{2}(9\pi - 1 - 0 + 1 + 0)$$

$$= \frac{9\pi}{2}$$

Thus, the area of the given polar curve is $\boxed{A = \frac{9\pi}{2}}$.

Q14E

Area in polar coordinate is calculated by area of a sector of a circle and is given by;

$$A = \frac{r^2\theta}{2}$$

And the curve of equation

$$r = a \pm b\cos(\theta)$$

$$r = a \pm b\sin(\theta)$$

Is known as cardioids.

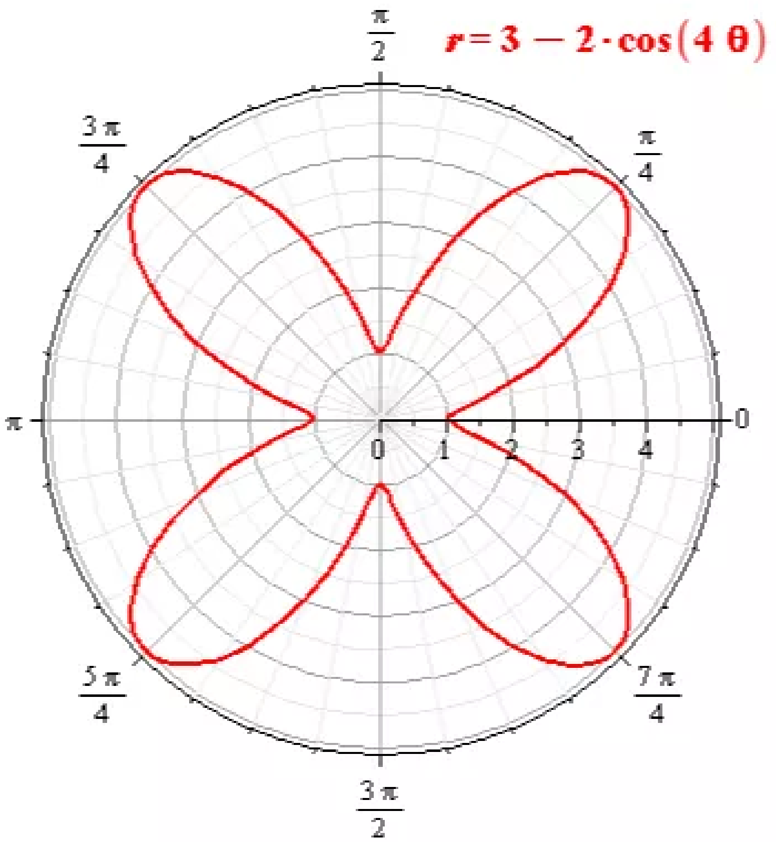
Given polar curve is

$$r = 3 - 2\cos 4\theta$$

The rotation require is;

$$0 \le \theta \le 2\pi$$

Its sketch is given as;



Area of the curve is;

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(3 - 2\cos 4\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(9 + 4\cos^2 4\theta - 12\cos 4\theta\right) d\theta$$

$$= \frac{1}{2}\left\{\int_0^{2\pi} 9 \, d\theta + 4\int_0^{2\pi} \cos^2 4\theta \, d\theta - 12\int_0^{2\pi} \cos 4\theta \, d\theta\right\} \ \ldots(1)$$

Consider

$$\int_0^{2\pi} \cos^2 4\theta \, d\theta$$

Suppose

$$4\theta = t$$

$$d\theta = \frac{dt}{4}$$

At

$$\theta = 0$$

$$t = 0$$

$$\theta = 2\pi$$

$$t = 8\pi$$

Therefore

$$\int_0^{2\pi} \cos^2 4\theta \, d\theta = \int_0^{8\pi} \cos^2 t \, \frac{dt}{4}$$

$$= \frac{1}{4} \cdot 16 \cdot \int_0^{\pi/2} \cos^2 t \, dt$$

$$= 4 \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

Use this result in (1), we get

$$A = \frac{1}{2}\left\{18\pi + 4\pi - 12\left(\frac{\sin 8\pi - \sin 0}{4}\right)\right\}$$

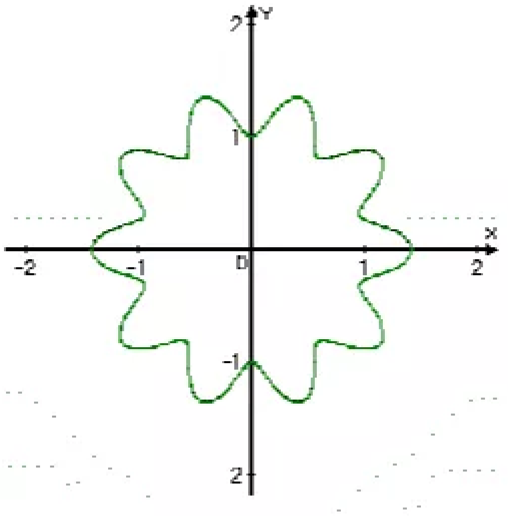
$$= 11\pi$$

The required solution is $\boxed{A = 11\pi}$.

Q15E

Given polar equation is $r=\sqrt{1+\cos^2(5\theta)}$

Its sketch is



Area of the curve is $A=\int_a^b \frac{1}{2}r^2\,d\theta$

$$\int_0^{2\pi}\frac{1}{2}\left(\sqrt{1+\cos^2(5\theta)}\right)^2 d\theta$$
$$=\frac{1}{2}\int_0^{2\pi}\left(1+\cos^2(5\theta)\right)d\theta$$
$$=\frac{1}{2}\left\{\int_0^{2\pi}1\,d\theta+\int_0^{2\pi}\cos^2(5\theta)\,d\theta\right\}---(*)$$

Consider $\int_0^{2\pi}\cos^2(5\theta)\,d\theta$:

Suppose $5\theta=t, d\theta=\frac{dt}{5}$

$$\theta=0\Rightarrow t=0$$
$$\theta=2\pi\Rightarrow t=10\pi$$

So, $\int_0^{2\pi}\cos^2(5\theta)\,d\theta=\int_0^{10\pi}\cos^2 t\,\frac{dt}{5}$

$=\frac{1}{5}\cdot 20\cdot\int_0^{\pi/2}\cos^2 t\,dt$ since $\cos^2 t$ is even function.

$$=4\cdot\frac{1}{2}\cdot\frac{\pi}{2}\left(\because\int_0^{\pi/2}\cos^{2k}\theta\,d\theta=\frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}\cdot\ldots\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

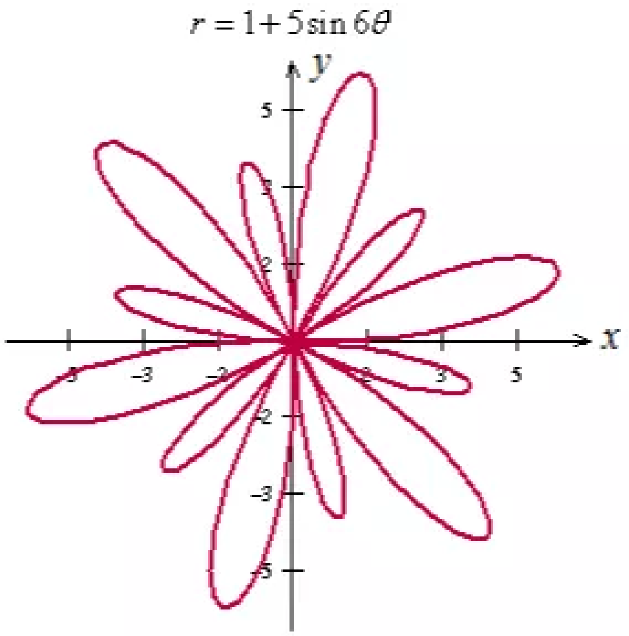
Using this result in (*), we get $\frac{1}{2}\{2\pi + \pi\}$

So, the required area is $\frac{3\pi}{2}$

Q16E

Consider the region $r = 1 + 5\sin 6\theta$

Graph of the region is shown below:



Now to find the area of the region.

The area of the polar curve $r = f(\theta)$ in the interval $[a,b]$ is

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

From figure, it is bounded between the lines $\theta=0, \theta=2\pi$

Here $r=f(\theta)=1+5\sin 6\theta$

Therefore, the required area of the given curve is

$$A=\int_a^b \frac{1}{2} r^2 d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}(1+5\sin 6\theta)^2 d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}(1+25\sin^2 6\theta+10\sin 6\theta)d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\left(1+25\left(\frac{1-\cos 12\theta}{2}\right)+10\sin 6\theta\right)d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\left(\frac{27}{2}+\frac{25\cos 12\theta}{2}+10\sin 6\theta\right)d\theta$$

$$=\frac{1}{2}\left[\frac{27}{2}\theta-\frac{25\sin 12\theta}{2\times 12}-\frac{10\cos 6\theta}{6}\right]_0^{2\pi}$$

$$=\frac{1}{2}\left[\left(\frac{27}{2}(2\pi)-\frac{25\sin 12(2\pi)}{24}-\frac{10\cos 6(2\pi)}{6}\right)-\frac{10}{6}\right]$$

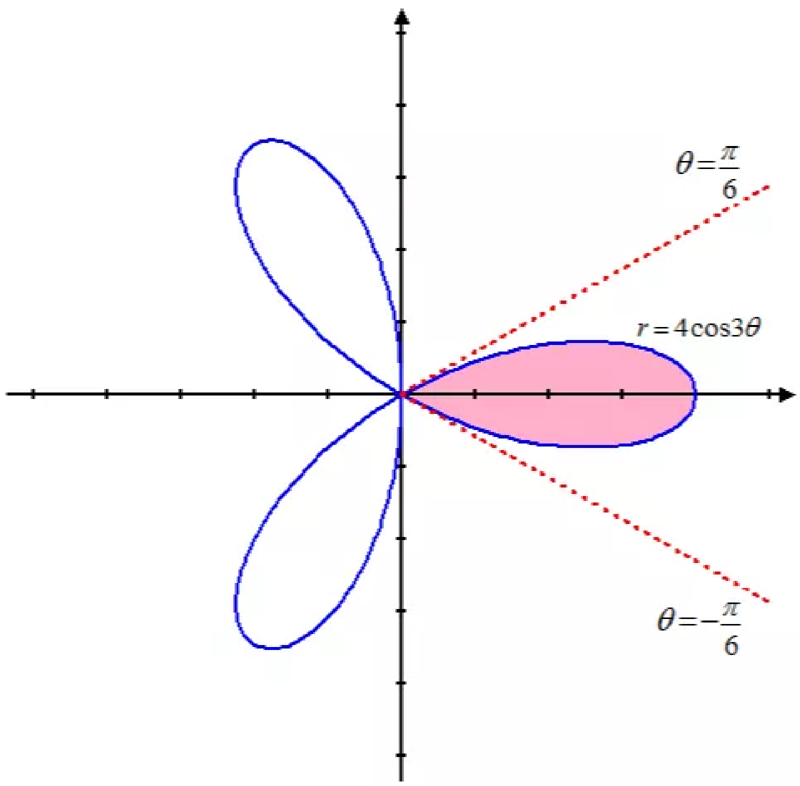
$$=\frac{27\pi}{2}$$

Hence the required area is $\boxed{\frac{27\pi}{2}}$.

Q17E

Consider the curve $r=4\cos 3\theta$

Sketch the graph of the curve is shown below:



Observe that the region enclosed by the right loop is swept out by a ray that rotates from

$\theta = -\frac{\pi}{6}$ to $\theta = \frac{\pi}{6}$.

The area of the polar curve $r = f(\theta)$ in the interval $[a,b]$ is

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

Therefore, the required area of the loop is

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

$$= \int_{-\pi/6}^{\pi/6} \frac{1}{2}\left(16\cos^2 3\theta\right) d\theta$$

$$= 8\int_{-\pi/6}^{\pi/6} \left(\frac{1+\cos 6\theta}{2}\right) d\theta$$

$$= 4\left(\theta + \frac{\sin 6\theta}{6}\right)_{-\pi/6}^{\pi/6}$$

$$= 4\left[\left(\frac{\pi}{6}+0\right) - \left(-\frac{\pi}{6}-0\right)\right]$$

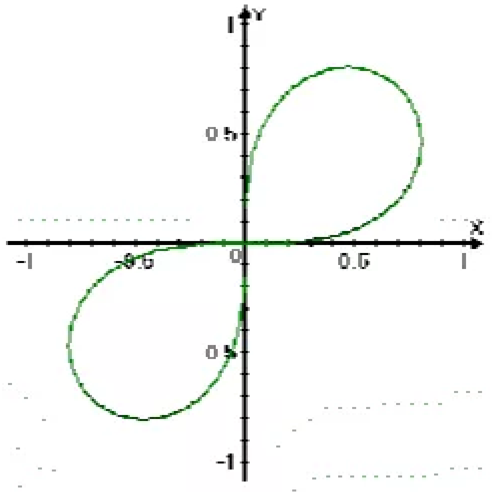
$$= 4\left(\frac{2\pi}{6}\right)$$

$$= \boxed{\frac{4\pi}{3}}.$$

Q18E

Given polar curve is $r^2 = \sin 2\theta$

Its sketch is



Observe that for the upper loop requires 0 to $\pi / 2$ rotation and the lower loop is from π to $3\pi / 2$

Area of one loop is $A = \int_a^b \frac{1}{2} r^2 d\theta$

$$\frac{1}{2}\int_0^{\pi/2} \sin 2\theta \, d\theta$$

suppose $2\theta = t, d\theta = \frac{dt}{2}$

$\theta = 0 \Rightarrow t = 0, \quad \theta = \frac{\pi}{2} \Rightarrow t = \pi$

$$\therefore \frac{1}{2}\int_0^{\pi/2} \sin 2\theta d\theta = \frac{1}{4}\int_0^{\pi} \sin t dt$$

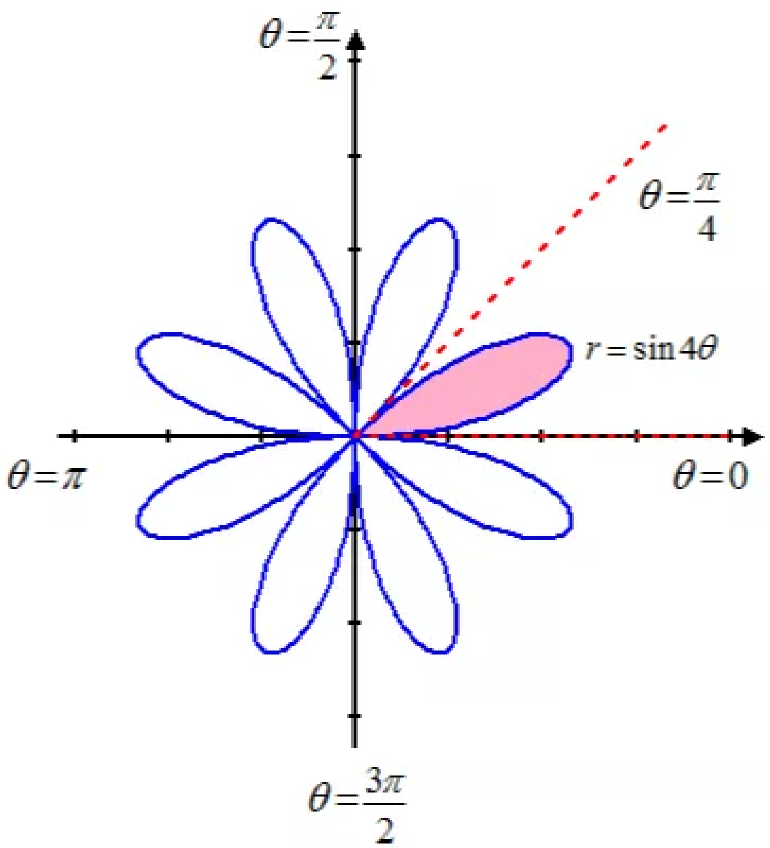
$$= \frac{1}{4}\left(\cos(0) - \cos(\pi)\right)$$

$$= \frac{1}{4}(1+1) = \frac{1}{2}$$

Q19E

Consider the curve $r = \sin 4\theta$

Sketch the graph of the curve is shown below:



Now need to find the area of the shaded portion.

Observe that the region enclosed by the right loop is swept out by a ray that rotates from

$\theta = 0$ to $\theta = \frac{\pi}{4}$.

The area of the polar curve $r = f(\theta)$ in the interval $[a,b]$ is

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

Therefore, the required area of the loop is

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \sin^2 4\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(\frac{1-\cos 8\theta}{2}\right) d\theta \quad \text{Since } \cos 2A = 1 - 2\sin^2 A$$

$$= \frac{1}{4}\left(\theta - \frac{\sin 8\theta}{8}\right)_0^{\pi/4}$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{4} - 0\right) - (0 - 0)\right]$$

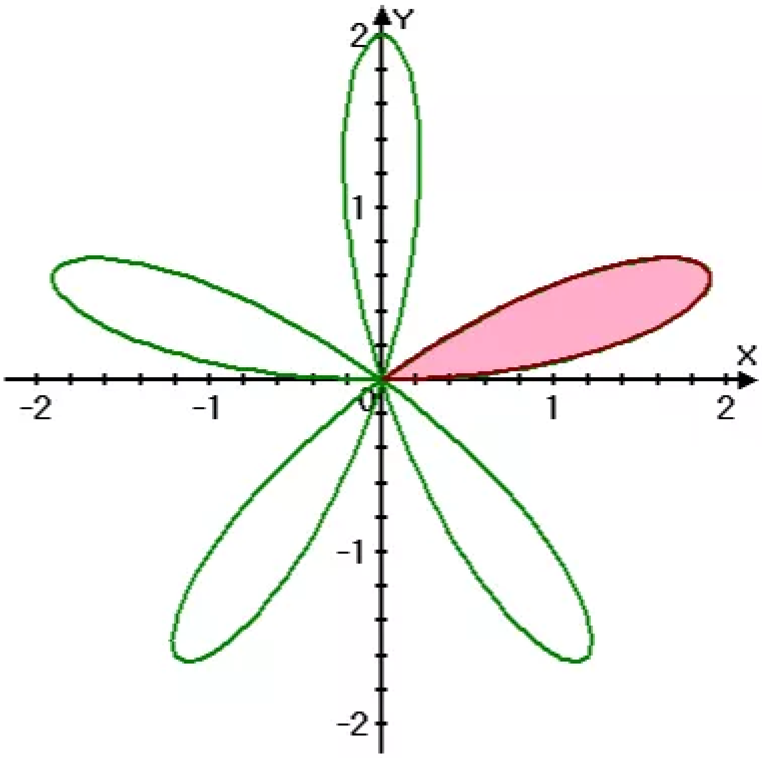
$$= \boxed{\frac{\pi}{16}}$$

Q20E

2657-10.4-20E SA: 9514 SR: 5589

Given polar equation is $r = 2\sin 5\theta$

The sketch of the curve and one loop of it is



For the rotation of entire curve it requires 0 to π and for one loop, it requires $\frac{\pi}{5}$

Area of the loop is $A = \int_a^b \frac{1}{2} r^2 d\theta$

$$= \frac{1}{2} \int_0^{\pi/5} 4\sin^2 5\theta \, d\theta$$

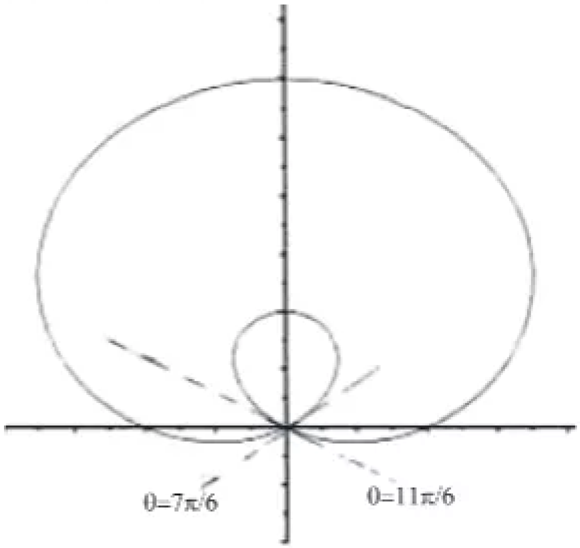
$$= \frac{4}{2} \int_0^{2\pi/5} \frac{1-\cos(10\theta)}{2} d\theta \qquad \because \cos(2t) = 1 - 2\sin^2 t$$

$$= \left[\theta - \frac{1}{10}\sin 10\theta \right]_0^{2\pi/5}$$

$$= \boxed{\frac{2\pi}{5}}$$

Q21E

First we sketch the curve



Now we have to find the interval in which the inner loop exists.
For this we take $r=0$

$$\Rightarrow \quad 1+2\sin\theta=0$$

$$\Rightarrow \quad \sin\theta=-\frac{1}{2}$$

$$\Rightarrow \quad \theta=\frac{7\pi}{6},\frac{11\pi}{6}$$

So inner loop lies between the lines $\theta=7\pi/6$ and $\theta=11\pi/6$

Since curve is symmetric about the vertical polar axis. So we can find the area of the loop by making double the area between $\theta=7\pi/6$ and $\theta=3\pi/2$

Then Area of the loop is $A=2\int_{7\pi/6}^{3\pi/2}\frac{1}{2}r^2\,d\theta$

$$\text{Therefore } A=\int_{7\pi/6}^{3\pi/2}(1+2\sin\theta)^2\,d\theta$$

$$=\int_{7\pi/6}^{3\pi/2}\left(1+2(1-\cos 2\theta)+4\sin\theta\right)d\theta \qquad \left[1-\cos 2\theta=2\sin^2\theta\right]$$

$$=\int_{7\pi/6}^{3\pi/2}(3-2\cos 2\theta+4\sin\theta)\,d\theta$$

$$=\left[3\theta-\sin 2\theta+4(-\cos\theta)\right]_{7\pi/6}^{3\pi/2}$$

$$=\left[3\theta-\sin 2\theta-4\cos\theta\right]_{7\pi/6}^{3\pi/2}$$

$$=\left[3\frac{3\pi}{2}-0-0-3.\frac{7\pi}{6}+\frac{\sqrt{3}}{2}-\frac{4\sqrt{3}}{2}\right]$$

$$=\left[\frac{9\pi}{2}-\frac{7\pi}{2}-\frac{3}{2}\sqrt{3}\right]$$

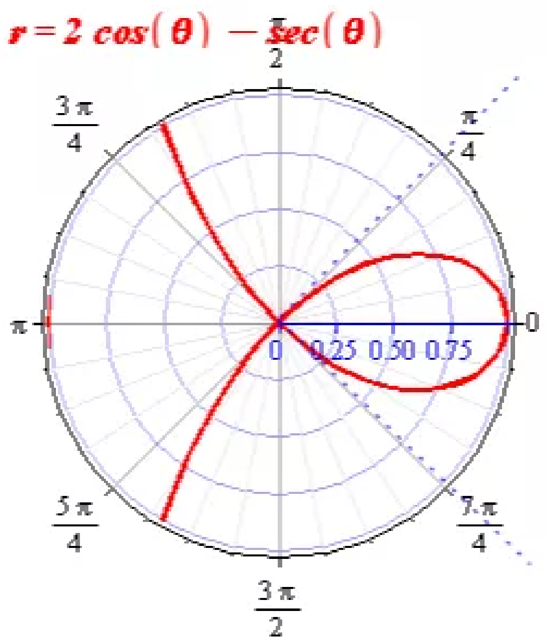
$$\boxed{A=\pi-\left(3\sqrt{3}/2\right)}$$

Q22E

Given the curve $r = 2\cos\theta - \sec\theta$;

The objective is to find the area enclosed by the loop of the strophoid $r = 2\cos\theta - \sec\theta$

First sketch the curve $r = 2\cos\theta - \sec\theta$



First, find the interval in which loop exists

For this, take r = 0

$$2\cos\theta - \sec\theta = 0$$

$$2\cos\theta - \frac{1}{\cos\theta} = 0$$

$$2\cos^2\theta - 1 = 0$$

$$\cos 2\theta = 0 \ \left(\text{since } \cos 2\theta = 2\cos^2\theta - 1\right)$$

$2\theta = \pi/2, -\pi/2$ (Since, loop is in first and fourth quadrant only)

$$\theta = \frac{\pi}{4}, \frac{-\pi}{4}$$

So, loop is in the interval $-\pi/4 \le \theta \le \pi/4$

The area of the region enclosed by the loop is;

$$A=\int_a^b \frac{1}{2}r^2 d\theta$$

$$A=\int_{-\pi/4}^{\pi/4} \frac{1}{2}r^2\, d\theta$$

$$=\frac{1}{2}\int_{-\pi/4}^{\pi/4}(2\cos\theta-\sec\theta)^2\, d\theta$$

$$=\frac{1}{2}\int_{-\pi/4}^{\pi/4}\left(4\cos^2\theta+\sec^2\theta-4\cos\theta.\sec\theta\right)d\theta$$

Substitute $\cos^2\theta=\frac{1+\cos 2\theta}{2}$

$$A=\frac{1}{2}\int_{-\pi/4}^{\pi/4}\left(4\left\{\frac{1+\cos 2\theta}{2}\right\}+\sec^2\theta-4\right)d\theta$$

$$=\frac{1}{2}\int_{-\pi/4}^{\pi/4}\left(2+2\cos 2\theta+\sec^2\theta-4\right)d\theta$$

$$=\frac{1}{2}\int_{-\pi/4}^{\pi/4}\left(2\cos 2\theta+\sec^2\theta-2\right)d\theta$$

$$=\frac{1}{2}\left[\sin 2\theta+\tan\theta-2\theta\right]_{-\pi/4}^{\pi/4}$$

$$A=\frac{1}{2}\left[\sin\frac{\pi}{2}+\tan\frac{\pi}{4}-\frac{2\pi}{4}+\sin\frac{\pi}{2}+\tan\frac{\pi}{4}-\frac{2\pi}{4}\right]$$

$$=\frac{1}{2}\left[1+1+-\frac{\pi}{2}+1+1-\frac{\pi}{2}\right]$$

$$=\frac{1}{2}[4-\pi]$$

$$=2-\pi/2$$

Hence, area of the region enclosed by the strophoid is $\boxed{2-\frac{\pi}{2}}$.

Q23E

Consider the curves $r=2\cos\theta,\ \ r=1$

The objective is to find the area inside the first curve and outside the second curve.

To find the value of θ at the intersections of the two curves, set the equations equal to each other.

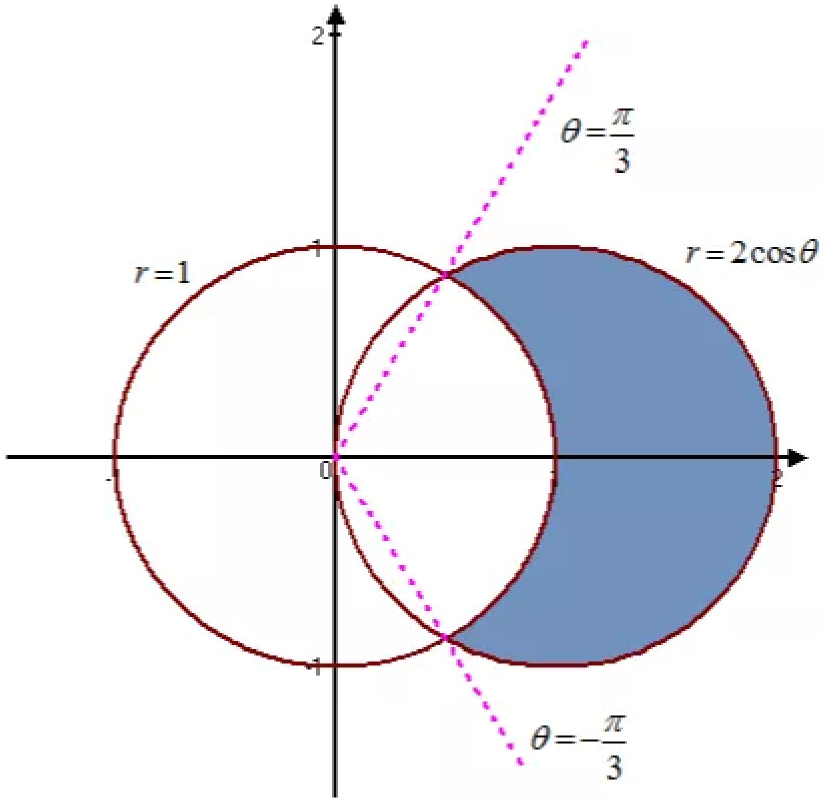
$$2\cos\theta=1$$

$$\cos\theta=\frac{1}{2}$$

$$\theta=-\frac{\pi}{3},\frac{\pi}{3}$$

Therefore θ lies between $-\frac{\pi}{3}\le\theta\le\frac{\pi}{3}$.

The graph of the region is shown below:



Therefore, the required area of the region is

$$A=\int_a^b \frac{1}{2}\left(r_1^2-r_2^2\right)d\theta$$

$$=\int_{-\pi/3}^{\pi/3}\frac{1}{2}\left((2\cos\theta)^2-1^2\right)d\theta$$

$$=2\int_0^{\pi/3}\frac{1}{2}\left((2\cos\theta)^2-1^2\right)d\theta \quad \begin{cases}\text{Since the function is symmetric} \\ \text{about the } x \text{ axis}\end{cases}$$

$$=\int_0^{\pi/3}\left(4\cos^2\theta-1\right)d\theta$$

$$=\int_0^{\pi/3}\left(4\left(\frac{1+\cos 2\theta}{2}\right)-1\right)d\theta \quad \text{Use the double angle formula}$$

$$=\int_0^{\pi/3}(1+2\cos 2\theta)d\theta$$

$$=\left[\theta+\sin 2\theta\right]_0^{\pi/3}$$

$$=\boxed{\frac{\pi}{3}+\frac{\sqrt{3}}{2}}$$

Q24E

First curve, $r = 1\text{-}\sin\ \theta$

Second curve, $r = 1$

Then for the points of intersection we must have

$$1 - \sin\theta = 1$$
$$\Rightarrow \sin\theta = 0$$
$$\Rightarrow \theta = \pi, 2\pi$$

Area, $A = \frac{1}{2}\int_{\pi}^{2\pi}\left[(1-\sin\theta)^2 - 1^2\right]d\theta$

$$= \frac{1}{2}\int_{\pi}^{2\pi}\left[1 - 2\sin\theta + \sin^2\theta - 1\right]d\theta$$

$$= \frac{1}{2}\int_{\pi}^{2\pi}\left[-2\sin\theta + \frac{1}{2}(1-\cos 2\theta)\right]d\theta$$

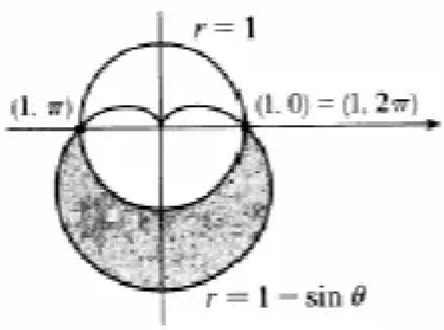
$$= \frac{1}{2}\left[2\cos\theta + \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{2}\left[2(\cos 2\pi - \cos\pi) + \frac{1}{2}(2\pi - \pi) - \frac{1}{2}(\sin 4\pi - \sin 2\pi)\right]$$

$$= \frac{1}{2}\left[2(1-(-1)) + \frac{\pi}{2}\right]$$

$$= \frac{1}{2}\left[2(2) + \frac{\pi}{2}\right]$$

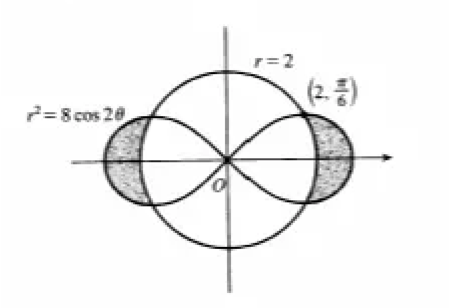
$$= \boxed{2 + \frac{\pi}{4}}$$



Q25E

$r^2 = 8\cos 2\theta$ has two loops on x-axis

And $r = 2$ is a circle at centre



Then for the points of intersection we must have

$$8\cos 2\theta = 4$$

$$\cos 2\theta = \frac{4}{8}$$

$$= \frac{1}{2}$$

$$\Rightarrow 2\theta = \frac{\pi}{3}, \frac{-\pi}{3}$$

$$\Rightarrow \theta = \frac{\pi}{6}, \frac{-\pi}{6}$$

$$\text{Area} = 2\left[\frac{1}{2}\int_{-\pi/6}^{\pi/6}[8\cos 2\theta - 4]d\theta\right]$$

$$= 2\left[\frac{1}{2}2\int_{0}^{\pi/6}4(2\cos 2\theta - 1)d\theta\right]$$

$$= 2\left[4\int_{0}^{\pi/6}(2\cos 2\theta - 1)d\theta\right]$$

$$= 8[\sin 2\theta - \theta]_0^{\pi/6}$$

$$= 8\left[\left(\sin\frac{\pi}{3} - \sin 0\right) - \left(\frac{\pi}{6} - 0\right)\right]$$

$$= 8\left[\frac{\sqrt{3}}{2} - 0 - \frac{\pi}{6}\right]$$

$$= \boxed{4\sqrt{3} - 4\pi/3}$$

Q26E

First we sketch the curves on the same set of polar axis

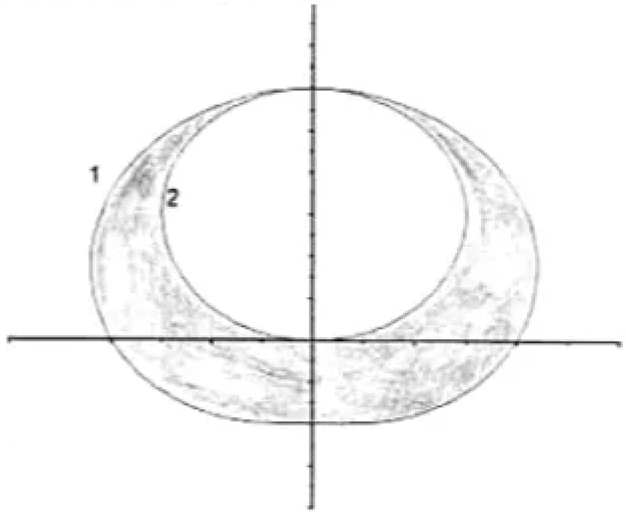


Fig. 1

We see that both curves are symmetric about vertical polar axis, so the region that lies inside the first curve and out side the second curve, is symmetric about vertical axis.

So area of enclosed region is

$$A = 2\int_{-\pi/2}^{\pi/2} \frac{1}{2}(2+\sin\theta)^2\, d\theta - 2\int_{0}^{\pi/2} \frac{1}{2}(3\sin\theta)^2\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[4+\sin^2\theta + 4\sin\theta\right] d\theta - \int_{0}^{\pi/2} 9\sin^2\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[4+\frac{1-\cos 2\theta}{2} + 4\sin\theta\right] d\theta - 9\int_{0}^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(4+\frac{1}{2}-\frac{1}{2}\cos 2\theta + 4\sin\theta\right) d\theta - \frac{9}{2}\int_{0}^{\pi/2} (1-\cos 2\theta)\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{9}{2}-\frac{1}{2}\cos 2\theta + 4\sin\theta\right) d\theta - \frac{9}{2}\int_{0}^{\pi/2} (1-\cos 2\theta)\, d\theta$$

Therefore area $A = \left[\frac{9}{2}\theta - \frac{1}{4}\sin 2\theta - 4\cos\theta\right]_{-\pi/2}^{\pi/2} - \frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$

$$= \left[\frac{9\pi}{4} - 0 - 0 + \frac{9\pi}{4} + 0 + 0\right] - \frac{9}{2}\left[\frac{\pi}{2} - 0 - 0\right]$$

$$= \frac{9\pi}{4} + \frac{9\pi}{4} - \frac{9\pi}{4}$$

Or $\boxed{A = 9\pi/4}$

Q27E

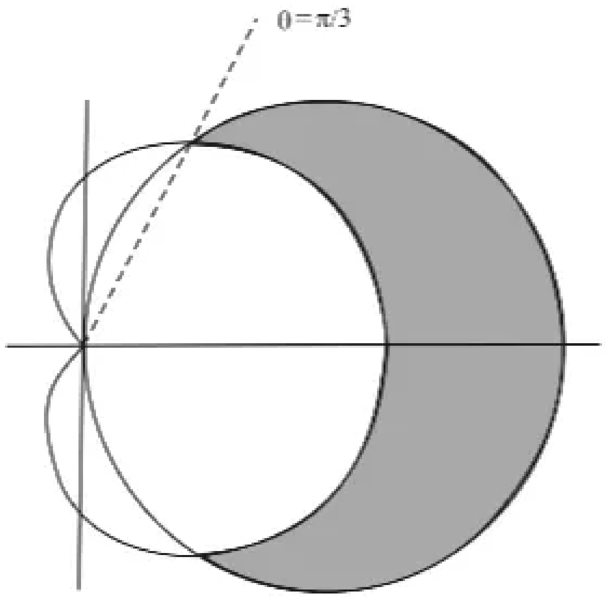
First we find the points of intersection of the curves $r = 3\cos\theta$, $r = 1+\cos\theta$

We must have

$$3\cos\theta = 1+\cos\theta$$

$$\Rightarrow \cos\theta = \tfrac{1}{2}$$

$$\Rightarrow \theta = \tfrac{\pi}{3}, -\tfrac{\pi}{3}$$



So area of bounded region is

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left[(3\cos\theta)^2 - (1+\cos\theta)^2\right]d\theta$$

$$= \int_{0}^{\pi/3}\left[9\cos^2\theta - 1 - \cos^2\theta - 2\cos\theta\right]d\theta$$

$$= \int_{0}^{\pi/3}\left[8\cos^2\theta - 2\cos\theta - 1\right]d\theta$$

$$= \int_{0}^{\pi/3}\left[8\left(\frac{1+\cos 2\theta}{2}\right) - 2\cos\theta - 1\right]d\theta \qquad \{\cos 2\theta = 2\cos^2\theta - 1$$

$$= \int_{0}^{\pi/3}\left[4 + 4\cos 2\theta - 2\cos\theta - 1\right]d\theta$$

$$= \int_{0}^{\pi/3}\left[3 + 4\cos 2\theta - 2\cos\theta\right]d\theta$$

So area of bounded region is

$$A=\frac{1}{2}\int_{-\pi/3}^{\pi/3}\left[(3\cos\theta)^2-(1+\cos\theta)^2\right]d\theta$$

$$=\int_{0}^{\pi/3}\left[9\cos^2\theta-1-\cos^2\theta-2\cos\theta\right]d\theta$$

$$=\int_{0}^{\pi/3}\left[8\cos^2\theta-2\cos\theta-1\right]d\theta$$

$$=\int_{0}^{\pi/3}\left[8\left(\frac{1+\cos 2\theta}{2}\right)-2\cos\theta-1\right]d\theta \qquad \{\cos 2\theta=2\cos^2\theta-1$$

$$=\int_{0}^{\pi/3}\left[4+4\cos 2\theta-2\cos\theta-1\right]d\theta$$

$$=\int_{0}^{\pi/3}\left[3+4\cos 2\theta-2\cos\theta\right]d\theta$$

Therefore the area $A=\left[3\theta+\frac{4\sin 2\theta}{2}-2\sin\theta\right]_0^{\pi/3}$

$$=\left[\pi+\frac{4\sin(2\pi/3)}{2}-2\sin(\pi/3)\right]$$

$$=\left[\pi+2\left(\sqrt{3}/2\right)-2\left(\sqrt{3}/2\right)\right]=\boxed{\pi}$$

Q29E

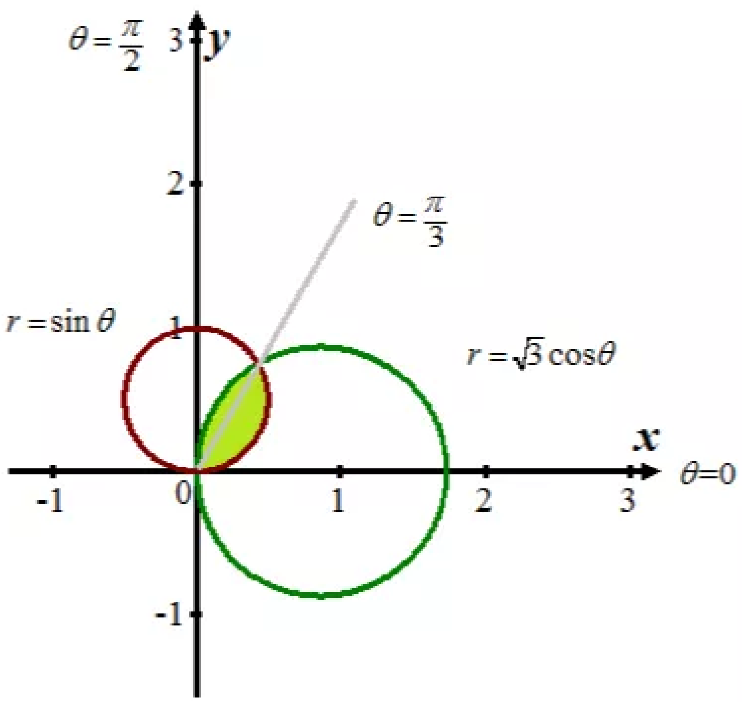
The equations of two curves are,

$r=\sqrt{3}\cos\theta$ and $r=\sin\theta$

The objective is to find the area of the region that lies inside the both curves.

Sketch the curves $r=\sqrt{3}\cos\theta$ and $r=\sin\theta$ and locate the bounded region.

The area that lies inside the both curves is shown below:



Set the both equations equal and then find the limits:

$$\sqrt{3}\cos\theta = \sin\theta$$
$$\tan\theta = \sqrt{3}$$
$$\theta = \frac{\pi}{3}$$

Compute the area by using the area formula:

$$A = \int_a^b \frac{1}{2}\left(r^2\right)d\theta$$

Substitute the limits in the integrals:

$$A = \int_0^{\frac{\pi}{3}} \frac{1}{2}(\sin\theta)^2\, d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}\left(\sqrt{3}\cos\theta\right)^2 d\theta$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2\theta d\theta + \frac{3}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2\theta d\theta$$

Use the following formulas to obtain the area:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$
$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

Area between the curves:

$$A = \int_0^{\frac{\pi}{3}} \frac{1}{2}\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{3}{2}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{3}} (1-\cos(2\theta))d\theta + \frac{3}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1+\cos(2\theta))d\theta$$

$$A = \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Bigg]_0^{\frac{\pi}{3}} + \frac{3}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Bigg]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$A = \frac{1}{4}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) + \frac{3}{4}\left(\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$
$$= \frac{\pi}{12} - \frac{\sqrt{3}}{16} + \frac{3\pi}{8} - \frac{3\pi}{12} - \frac{3\sqrt{3}}{16}$$
$$= \frac{3\pi}{8} - \frac{\pi}{6} - \frac{\sqrt{3}}{4}$$
$$A = \boxed{\frac{5\pi}{24} - \frac{\sqrt{3}}{4}}.$$

Therefore, the area of the region, $A = \boxed{\frac{5\pi}{24} - \frac{\sqrt{3}}{4}}$.

Q30E

Consider the curves $r = 1 + \cos\theta,\ \ r = 1 - \cos\theta$

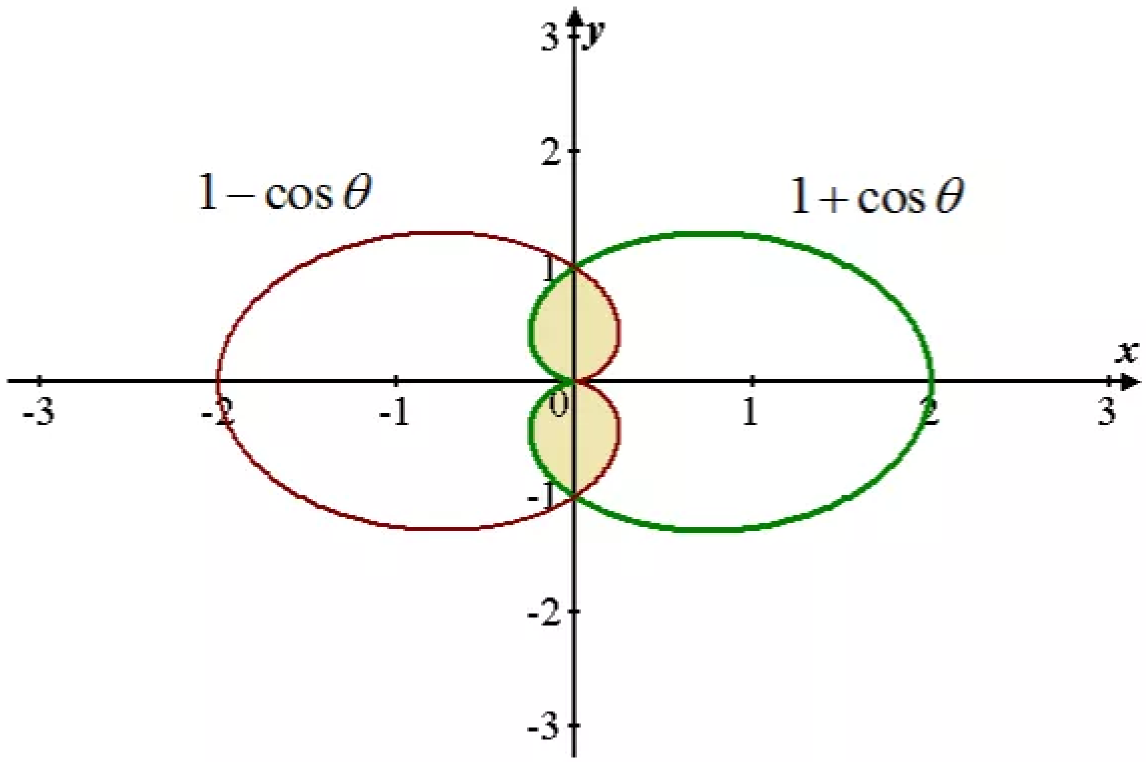
The objective is to find the area that lies inside both the curves.

To find the value of θ at the intersections of the two curves, set the equations equal to each other.

$$1 + \cos\theta = 1 - \cos\theta$$
$$2\cos\theta = 0$$
$$\cos\theta = 0$$
$$\Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

Therefore θ lies between $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$.

The graph of the region is shown below:



Now, compute the area by using the area formula.

$$A = \int_a^b \frac{1}{2}\left(r^2\right) d\theta$$

Since, the curve is symmetric about the y axis.

So, from the graph using symmetry, we can set up the integral on positive y axis as, 0 to $\frac{\pi}{2}$ for $r = 1 - \cos\theta$ (red curve) and $\frac{\pi}{2}$ to π for $r = 1 + \cos\theta$ (green) curve.

Substitute the limits in the integrals:

$$A = 2\left[\int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos\theta)^2\, d\theta + \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}(\cos\theta+1)^2\, d\theta\right]$$

$$= \left[\int_0^{\frac{\pi}{2}} (1-\cos\theta)^2\, d\theta + \int_{\frac{\pi}{2}}^{\pi} (\cos\theta+1)^2\, d\theta\right]$$

$$= \left[\int_0^{\frac{\pi}{2}} \left(1-2\cos\theta+\cos^2\theta\right) d\theta + \int_{\frac{\pi}{2}}^{\pi} \left(1+2\cos\theta+\cos^2\theta\right) d\theta\right]$$

$$= \left[\int_0^{\frac{\pi}{2}} \left(1-2\cos\theta+\frac{\cos 2\theta}{2}+\frac{1}{2}\right) d\theta + \int_{\frac{\pi}{2}}^{\pi} \left(1+2\cos\theta+\frac{\cos 2\theta}{2}+\frac{1}{2}\right) d\theta\right]$$

$$= \left[\left(-2\sin\theta+\frac{\sin 2\theta}{4}+\frac{3}{2}\theta\right)\Big|_0^{\frac{\pi}{2}} + \left(\frac{3}{2}\theta+2\sin\theta+\frac{\sin 2\theta}{4}\right)\Big|_{\frac{\pi}{2}}^{\pi}\right]$$

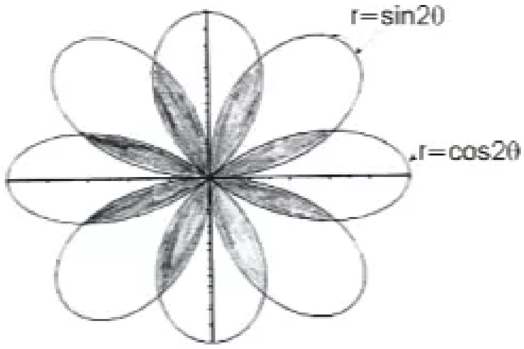
$$= \left[\left[-2\sin\left(\frac{\pi}{2}\right)+\frac{3\pi}{4}\right]+\left[\frac{3\pi}{2}-\left(\frac{3\pi}{4}+2\sin\left(\frac{\pi}{2}\right)\right)\right]\right]$$

$$= \left[-4+\frac{3\pi}{2}\right]$$

Hence, the required area is $\boxed{-4+\frac{3\pi}{2}}$.

Q31E

First we sketch the curve



We see that there are 8 symmetrical leaves enclosed by the curves $\cos 2\theta$ *and* $\sin 2\theta$. For finding the point of intersection we must have

$$\sin 2\theta = \cos 2\theta$$

$$\Rightarrow 2\theta = \pi/4 \quad \Rightarrow \theta = \pi/8$$

Total area of the enclosed region is 16 times the area of the region $0 \le \theta \le \pi/8$ bounded by the curve $r = \sin 2\theta$

Area for the region $A=16\int_0^{\pi/8}\frac{1}{2}(\sin 2\theta)^2\,d\theta$

$$=\frac{16}{2}\int_0^{\pi/8}\left(\sin^2 2\theta\right)d\theta$$

$$=8\int_0^{\pi/8}\left(\frac{1-\cos 4\theta}{2}\right)d\theta \qquad \left[\cos 2\theta=1-2\sin^2\theta\right]$$

$$=4\int_0^{\pi/8}(1-\cos 4\theta)\,d\theta$$

Therefore the area $\Rightarrow A=4\left[\theta-\frac{\sin 4\theta}{4}\right]_0^{\pi/8}$

$$\Rightarrow A=4\left[\frac{\pi}{8}-\frac{\sin \pi/2}{4}\right]$$

$$\Rightarrow A=4\left[\frac{\pi}{8}-\frac{1}{4}\right]$$

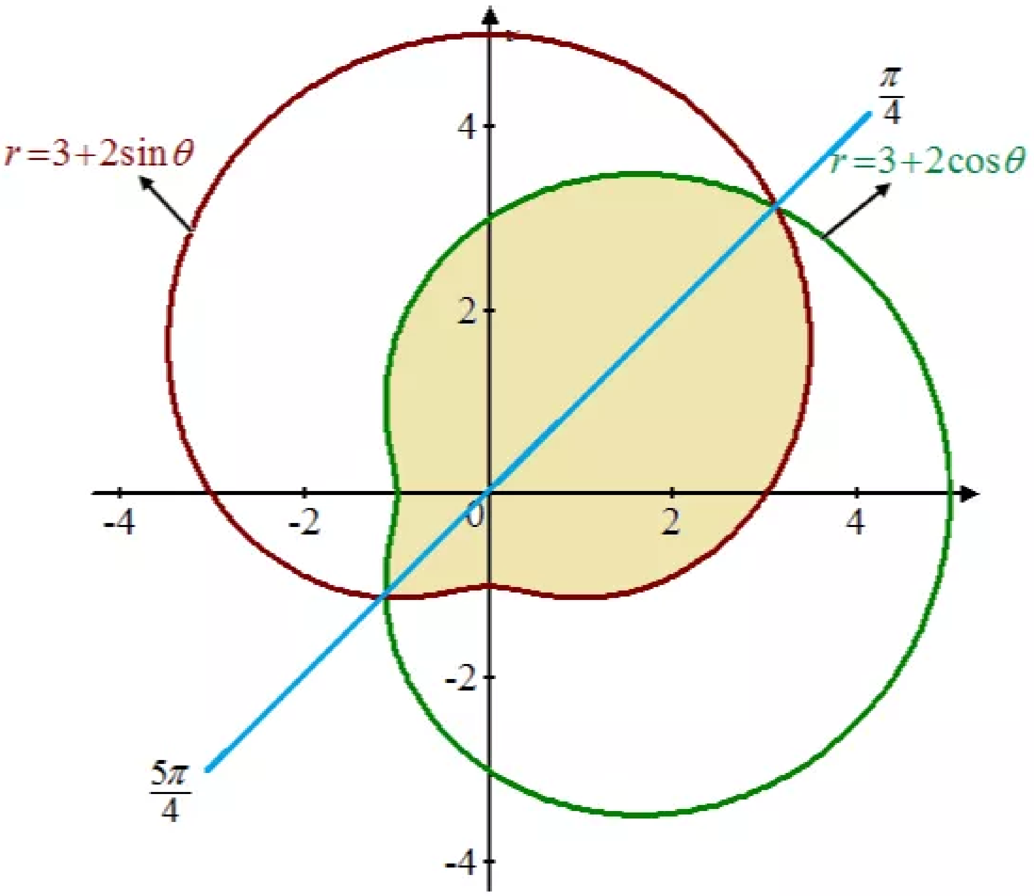
$$\Rightarrow \boxed{A=\frac{\pi}{2}-1}$$

Q32E

Consider the two curves

$r=3+2\cos\theta$ and $r=3+2\sin\theta$

The area that lies inside the both curves as shown in below figure:



To find the limits of integration,

$$3+2\cos\theta = 3+2\sin\theta$$
$$\cos\theta = \sin\theta$$
$$\tan\theta = 1$$

This occurs at $\theta = \frac{\pi}{4}, \frac{5\pi}{4}\left(\text{or} -\frac{3\pi}{4}\right)$

To compute the area, use the formula

$$A = \int_a^b \frac{1}{2} r^2 \, d\theta$$

To find the area that lies inside both the curves, find the area inside one whole limacon and subtract away the area outside the other curve.

Thus,

$$A = \frac{1}{2}\int_0^{2\pi} (3+2\cos\theta)^2 \, d\theta - \frac{1}{2}\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} (3+2\cos\theta)^2 - (3+2\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(9+12\cos\theta+4\cos^2\theta\right) d\theta -$$

$$\frac{1}{2}\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \left[9+12\cos\theta+4\cos^2\theta-9-12\sin\theta-4\sin^2\theta\right] d\theta$$

Since: $(a+b)^2 = a^2+2ab+b^2$

$$= \frac{1}{2}\int_0^{2\pi} \left(9+12\cos\theta+4\left(\frac{1+\cos 2\theta}{2}\right)\right) d\theta -$$

$$\frac{1}{2}\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \left[12\cos\theta+4\left(\frac{1+\cos 2\theta}{2}\right)-12\sin\theta-4\left(\frac{1-\cos 2\theta}{2}\right)\right] d\theta$$

Since: $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ and $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

Continuation to the above steps,

$$A=\frac{1}{2}\int_{0}^{2\pi}\left(9+12\cos\theta+2\left(1+\cos2\theta\right)\right)d\theta-$$

$$\frac{1}{2}\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}\left[12\cos\theta+2\left(1+\cos2\theta\right)-12\sin\theta-2\left(1-\cos2\theta\right)\right]d\theta$$

Simplify

$$=\frac{1}{2}\int_{0}^{2\pi}\left(9+12\cos\theta+2+2\cos2\theta\right)d\theta-$$

$$\frac{1}{2}\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}\left[12\cos\theta-12\sin\theta+2+2\cos2\theta-2+2\cos2\theta\right]d\theta$$

Use distributive property

$$=\frac{1}{2}\int_{0}^{2\pi}\left(11+12\cos\theta+2\cos2\theta\right)d\theta-\frac{1}{2}\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}\left[12\cos\theta-12\sin\theta+4\cos2\theta\right]d\theta$$

Simplify

$$A=\frac{1}{2}\left[11\theta+12\sin\theta+\sin2\theta\right]_{0}^{2\pi}-\frac{1}{2}\left[12\sin\theta+12\cos\theta+2\sin2\theta\right]_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}$$

Apply integration

Apply the limits,

$$A=\frac{1}{2}\left[11(2\pi)-0\right]-\frac{1}{2}\left[\begin{array}{l}\left(12\cdot\sin\left(\frac{\pi}{4}\right)+12\cos\left(\frac{\pi}{4}\right)+2\sin2\left(\frac{\pi}{4}\right)\right)-\\\left(12\cdot\sin\left(-\frac{3\pi}{4}\right)+12\cos\left(-\frac{3\pi}{4}\right)+2\sin2\left(-\frac{3\pi}{4}\right)\right)\end{array}\right]$$

$$=\frac{1}{2}\left[22\pi\right]-\frac{1}{2}\left[\begin{array}{l}\left(12\cdot\frac{\sqrt{2}}{2}+12\cdot\frac{\sqrt{2}}{2}+2\cdot1\right)-\\\left(12\cdot\left(-\frac{\sqrt{2}}{2}\right)+12\left(-\frac{\sqrt{2}}{2}\right)+2\cdot1\right)\end{array}\right]$$

$$=11\pi-\frac{1}{2}\left[6\sqrt{2}+6\sqrt{2}+2+6\sqrt{2}+6\sqrt{2}-2\right]$$

$$=11\pi-12\sqrt{2}$$

Therefore, the area that lies inside the curves $r=3+2\cos\theta$ and $r=3+2\sin\theta$ is

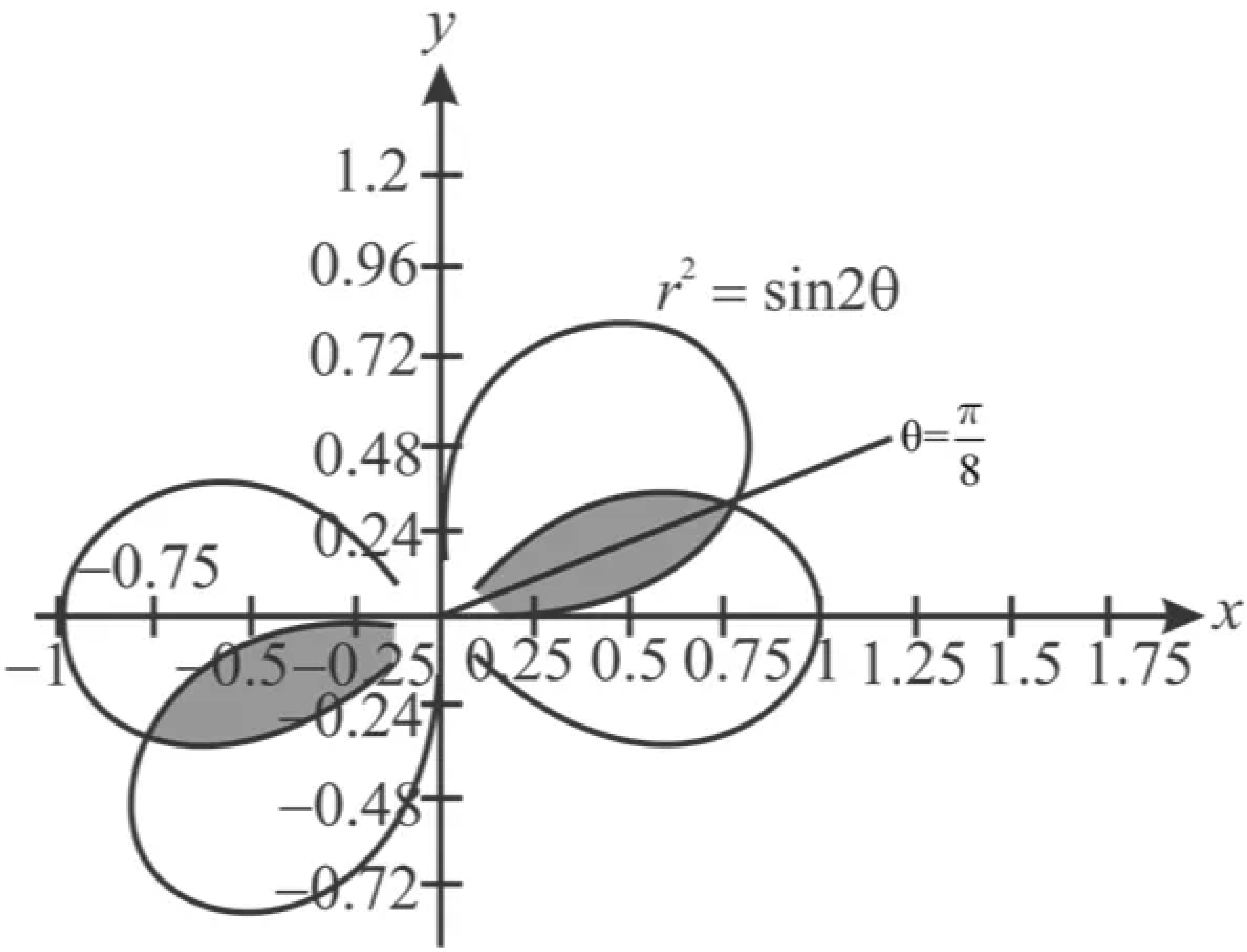
$$\boxed{11\pi-12\sqrt{2}}$$

Q33E

The objective is to find the area of the region that lies inside both curves $r^2 = \sin 2\theta$, and $r^2 = \cos 2\theta$

The two curves intersect at $\theta = \frac{\pi}{8}, \frac{5\pi}{8}$

Figure showing the area of region:



$\text{Total area}=2(\text{area that lies in first quadrant})\ (\text{By symmetry})$ ...... (1)

From the figure observe that the area in both the curves in the first quadrant is twice the area covered for $0 \le \theta \le \frac{\pi}{8}$ in the curve $r^2 = \sin 2\theta$

The formula for the area of a region in polar coordinates is given by $A = \frac{1}{2}\int_a^b r^2 d\theta$ for $r = f(\theta)$ where the angle θ goes from *a* to *b*

So area in the first quadrant is given by;

$\left(\text{since } r^2 = \sin 2\theta\right)$

$$= 2\int_0^{\frac{\pi}{8}} \frac{1}{2}(\sin 2\theta)d\theta$$

$$= \int_0^{\frac{\pi}{8}} (\sin 2\theta)d\theta$$

$$= -\frac{1}{2}[\cos 2\theta]_0^{\frac{\pi}{8}}$$

$$= -\frac{1}{2}\left(\cos\frac{\pi}{4} - \cos 0\right)$$

$$= -\frac{1}{2}\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{\left(\sqrt{2} - 1\right)}{2\sqrt{2}}$$

Now from (1) the total area that lies in both the curves is;

$$= 2\left(\frac{\sqrt{2} - 1}{2\sqrt{2}}\right)$$

$$= \left(\frac{\sqrt{2} - 1}{\sqrt{2}}\right)$$

$$= \left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= \left(1 - \frac{1}{2}\sqrt{2}\right)$$

Thus, area that lies in both the curve is given by $\boxed{\left(1-\frac{1}{2}\sqrt{2}\right)}$

$$A=2\left[\int_{\frac{\pi}{8}}^{\frac{\pi}{2}}\frac{1}{2}(\cos 2\theta)\,d\theta+\int_{0}^{\frac{\pi}{8}}\frac{1}{2}(\sin 2\theta)\,d\theta\right]$$

$$A=\left[\int_{\frac{\pi}{8}}^{\frac{\pi}{2}}(\cos 2\theta)\,d\theta+\int_{0}^{\frac{\pi}{8}}(\sin 2\theta)\,d\theta\right]$$

$$A=\left[\frac{1}{2}\sin 2\theta\Big|_{\frac{\pi}{8}}^{\frac{\pi}{2}}+\left(-\frac{1}{2}\cos 2\theta\right)\Big|_{0}^{\frac{\pi}{8}}\right]$$

$$A=\frac{1}{2}\left[\sin\left(2\frac{\pi}{2}\right)-\sin\left(2\frac{\pi}{8}\right)-\cos\left(2\frac{\pi}{8}\right)+\cos(0)\right]$$

$$A=\frac{1}{2}\left[-\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{\sqrt{2}}{2}\right)+1\right]$$

$$A=\frac{1}{2}\left[1-\sqrt{2}\right]$$

$$A=\frac{1}{2}-\frac{\sqrt{2}}{2}$$

Thus, $\boxed{A=\frac{1}{2}-\frac{\sqrt{2}}{2}}$

Q34E

We sketch the curves $r=a\sin\theta$ and $r=b\cos\theta$

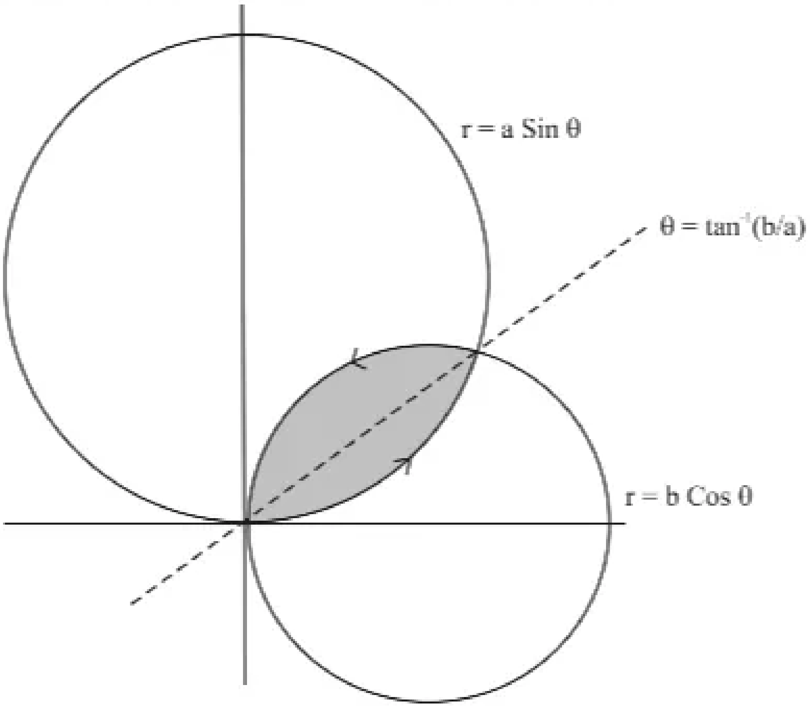


Fig. 1

For finding the points of intersection we must have

$$a\sin\theta = b\cos\theta$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{b}{a}$$

$$\Rightarrow \tan\theta = {}^{b}\!/_{a} \qquad \Rightarrow \theta = \tan^{-1}\left({}^{b}\!/_{a}\right)$$

Area of shaded region

$$A = \int_0^{\tan^{-1}(b/a)} \frac{1}{2}(a\sin\theta)^2 d\theta + \int_{\tan^{-1}(b/a)}^{\pi/2} \frac{1}{2}(b\cos\theta)^2 d\theta$$

$$= \frac{a^2}{2}\int_0^{\tan^{-1}(b/a)} \sin^2\theta d\theta + \frac{b^2}{2}\int_{\tan^{-1}(b/a)}^{\pi/2} \cos^2\theta d\theta$$

$$= \frac{a^2}{2}\int_0^{\tan^{-1}(b/a)} \frac{1-\cos 2\theta}{2} d\theta + \frac{b^2}{2}\int_{\tan^{-1}(b/a)}^{\pi/2} \frac{1+\cos 2\theta}{2} d\theta$$

$$\text{Thus } A = \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\tan^{-1}(b/a)} + \frac{b^2}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\tan^{-1}\frac{b}{a}}^{\pi/2}$$

$$= \frac{a^2}{4}\left[\tan^{-1}\frac{b}{a} - \frac{1}{2}\sin\left(2\tan^{-1}\frac{b}{a}\right)\right] + \frac{b^2}{4}\left[\frac{\pi}{2} + 0 - \tan^{-1}\frac{b}{a} - \frac{1}{2}\sin\left(2\tan^{-1}\frac{b}{a}\right)\right]$$

$$= \frac{a^2}{4}\tan^{-1}\frac{b}{a} - \frac{b^2}{4}\tan^{-1}\frac{b}{a} - \frac{a^2}{8}\sin\left(2\tan^{-1}\frac{b}{a}\right) - \frac{b^2}{8}\sin\left(2\tan^{-1}\frac{b}{a}\right) + \frac{b^2\pi}{8}$$

$$= \frac{(a^2-b^2)}{4}\tan^{-1}\frac{b}{a} - \frac{(a^2+b^2)}{8}\sin\left(2\tan^{-1}\frac{b}{a}\right) + \frac{b^2\pi}{8}$$

$$\text{But } \sin\left(2\tan^{-1}\frac{b}{a}\right) = 2\sin\left(\tan^{-1}\frac{b}{a}\right)\cos\left(\tan^{-1}\frac{b}{a}\right)$$

$$= 2\sin\theta\ \cos\theta$$

$$= 2\frac{b}{\sqrt{a^2+b^2}}\frac{a}{\sqrt{a^2+b^2}}$$

$$= \frac{2ab}{a^2+b^2}$$

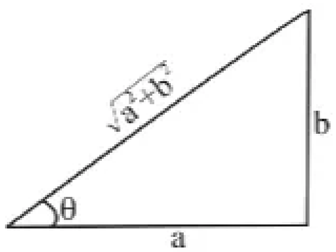


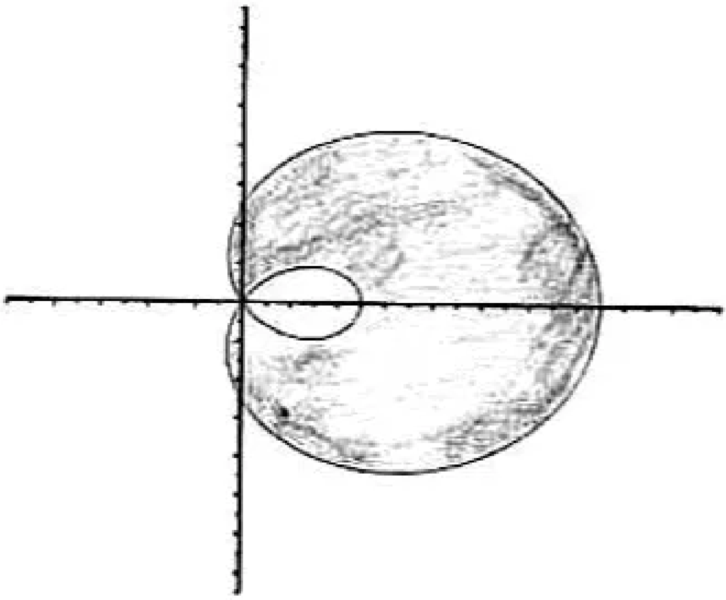
Fig. 2

Therefore $$A=\frac{\left(a^2-b^2\right)}{4}\tan^{-1}\frac{b}{a}-\frac{a^2+b^2}{8}\frac{2ab}{a^2+b^2}+\frac{b^2\pi}{8}$$

$$\boxed{A=\frac{a^2-b^2}{4}\tan^{-1}\frac{b}{a}-\frac{ab}{4}+\frac{b^2\pi}{8}},\quad a>0,b>0$$

Q35E

First we sketch the curve $r=\frac{1}{2}+\cos\theta$



Taking $\qquad r=0$

$$\Rightarrow\frac{1}{2}+\cos\theta=0$$

$$\Rightarrow\cos\theta=-\frac{1}{2}$$

$$\Rightarrow\theta=\frac{2\pi}{3}$$

Area of shaded region

$$A=2\left[\int_0^{2\pi/3}\frac{1}{2}\left(\frac{1}{2}+\cos\theta\right)^2 d\theta-\int_{2\pi/3}^{\pi}\frac{1}{2}\left(\frac{1}{2}+\cos\theta\right)^2 d\theta\right]$$

$$=\int_0^{2\pi/3}\left(\frac{1}{4}+\cos^2\theta+\cos\theta\right)d\theta-\int_{2\pi/3}^{\pi}\left(\frac{1}{4}+\cos^2\theta+\cos\theta\right)d\theta$$

$$=\int_0^{2\pi/3}\left(\frac{1}{4}+\frac{1+\cos 2\theta}{2}+\cos\theta\right)d\theta-\int_{2\pi/3}^{\pi}\left(\frac{1}{4}+\frac{1+\cos 2\theta}{2}+\cos\theta\right)d\theta$$

Here we used $\cos 2\theta=2\cos^2\theta-1$

Therefore $A = \int_0^{2\pi/3}\left(\frac{3}{4}+\frac{1}{2}\cos 2\theta+\cos\theta\right)d\theta - \int_{2\pi/3}^{\pi}\left(\frac{3}{4}+\frac{1}{2}\cos 2\theta+\cos\theta\right)d\theta$

$$= \left[\frac{3}{4}\theta+\frac{1}{4}\sin 2\theta+\sin\theta\right]_0^{2\pi/3} - \left[\frac{3}{4}\theta+\frac{1}{4}\sin 2\theta+\sin\theta\right]_{2\pi/3}^{\pi}$$

$$= \left[\frac{3}{4}\times\frac{2\pi}{3}+\frac{1}{4}\sin\frac{4\pi}{3}+\sin\frac{2\pi}{3}\right]-\left[\frac{3}{4}\pi+0+0-\left(\frac{3}{4}\times\frac{2\pi}{3}+\frac{1}{4}\sin\frac{4\pi}{3}+\sin\frac{2\pi}{3}\right)\right]$$

$$= \left[\frac{\pi}{2}+\frac{1}{4}\left(-\sqrt{3}/2\right)+\left(\sqrt{3}/2\right)\right]-\left[\frac{3\pi}{4}-\frac{\pi}{2}-\frac{1}{4}\left(-\sqrt{3}/2\right)-\left(\sqrt{3}/2\right)\right]$$

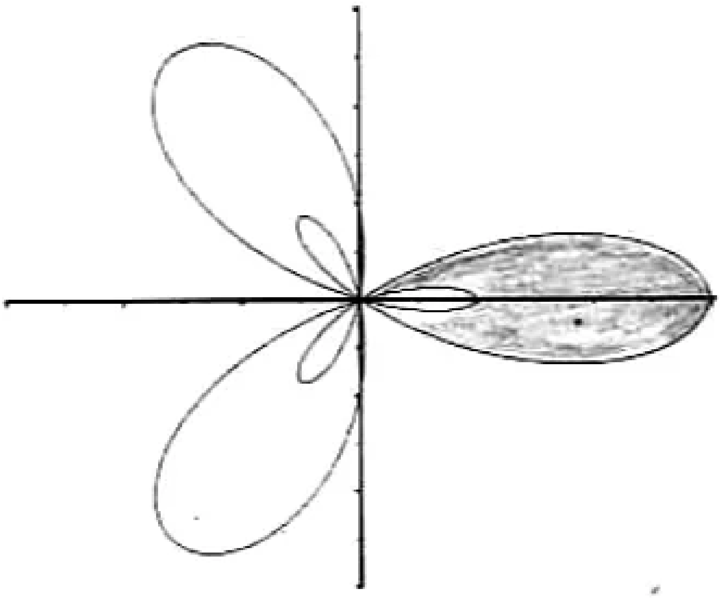
$$= \left[\frac{\pi}{2}-\frac{\sqrt{3}}{8}+\frac{\sqrt{3}}{2}-\frac{\pi}{4}-\frac{\sqrt{3}}{8}+\frac{\sqrt{3}}{2}\right]$$

$$= \left[\frac{\pi}{4}+\frac{3\sqrt{3}}{4}\right]$$

$$\Rightarrow \boxed{A = \left(\pi+3\sqrt{3}\right)/4}$$

Q36E

First we sketch the curve



Putting $r = 0$

$$\Rightarrow 1+2\cos 3\theta = 0$$

$$\Rightarrow 2\cos 3\theta = -1$$

$$\Rightarrow \cos 3\theta = -1/2$$

$$3\theta = \frac{2\pi}{3},\frac{4\pi}{3},\frac{8\pi}{3},\frac{10\pi}{3} \quad \ldots\ldots\ldots$$

$$\theta = \frac{2\pi}{9},\frac{4\pi}{9},\frac{8\pi}{9},\frac{10\pi}{9},\ldots\ldots$$

So area of the region between a large loop and the small loop is

$$A = 2\int_0^{2\pi/9} \frac{1}{2}(1+2\cos 3\theta)^2\, d\theta - \int_{2\pi/9}^{4\pi/9} \frac{1}{2}(1+2\cos 3\theta)^2\, d\theta$$

$$= \int_0^{2\pi/9} \left(1+4\cos^2 3\theta+4\cos 3\theta\right) d\theta - \frac{1}{2}\int_{2\pi/9}^{4\pi/9} \left(1+4\cos^2 3\theta+4\cos 3\theta\right) d\theta$$

$$= \int_0^{2\pi/9} \left(1+2(1+\cos 6\theta)+4\cos 3\theta\right) d\theta - \frac{1}{2}\int_{2\pi/9}^{4\pi/9} \left(1+2(1+\cos 6\theta)+4\cos 3\theta\right) d\theta$$

Here we used $\boxed{\cos 2\theta = 2\cos^2\theta - 1}$

$$= \int_0^{2\pi/9} \left(3+2\cos 6\theta+4\cos 3\theta\right) d\theta - \frac{1}{2}\int_{2\pi/9}^{4\pi/9} \left(3+2\cos 6\theta+4\cos 3\theta\right) d\theta$$

Thus $A = \left[3\theta + \frac{1}{3}\sin 6\theta + \frac{4}{3}\sin 3\theta\right]_0^{2\pi/9} - \frac{1}{2}\left[3\theta + \frac{1}{3}\sin 6\theta + \frac{4}{3}\sin 3\theta\right]_{2\pi/9}^{4\pi/9}$

$$= \left[3\times\frac{2\pi}{9} + \frac{1}{3}\sin\frac{6\times 2\pi}{9} + \frac{4}{3}\sin\frac{2\pi}{3}\right] - \frac{1}{2}\left[3\times\frac{4\pi}{9} + \frac{1}{3}\sin\frac{6\times 4\pi}{9} + \frac{4}{3}\sin\frac{3\times 4\pi}{9} - \left(3\times\frac{2\pi}{9} + \frac{1}{3}\sin\frac{6\times 2\pi}{9} + \frac{4}{3}\sin\frac{2\pi}{3}\right)\right]$$

$$= \frac{3}{2}\left[\frac{2\pi}{3} + \frac{1}{3}\sin\frac{4\pi}{3} + \frac{4}{3}\sin\frac{2\pi}{3}\right] - \frac{1}{2}\left[\frac{4\pi}{3} + \frac{1}{3}\sin\frac{8\pi}{3} + \frac{4}{3}\sin\frac{4\pi}{3}\right]$$

$$= \frac{3}{2}\left[\frac{2\pi}{3} + \frac{1}{3}\left(\frac{-\sqrt{3}}{2}\right) + \frac{4}{3}\left(\frac{\sqrt{3}}{2}\right)\right] - \frac{1}{2}\left[\frac{4\pi}{3} + \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right) + \frac{4}{3}\left(\frac{-\sqrt{3}}{2}\right)\right]$$

$$= \frac{3}{2}\left[\frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right] - \frac{1}{2}\left[\frac{4\pi}{3} - \frac{\sqrt{3}}{2}\right]$$

$$= \pi + \frac{3\sqrt{3}}{4} - \frac{2\pi}{3} + \frac{\sqrt{3}}{4}$$

$$= \frac{\pi}{3} + \sqrt{3}$$

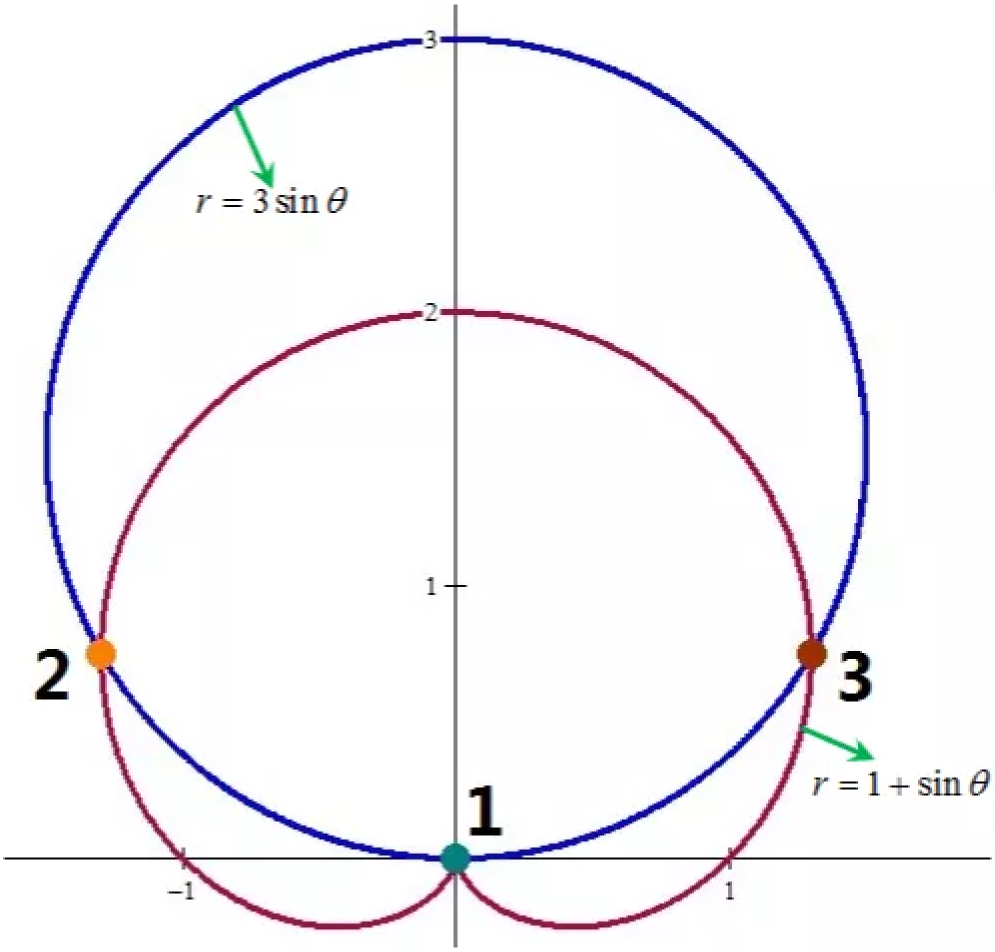
$$\Rightarrow \boxed{A = \left(\pi + 3\sqrt{3}\right)/3}$$

Q37E

Consider the two curves

$r = 1 + \sin\theta$ and $r = 3\sin\theta$

The graph of the curves $r = 1 + \sin\theta$ and $r - 3\sin\theta$ as shown below:



From the graph, observe that there are three intersection points.

Also, one of the indicated by **1** on the graph is the origin.

To find the other two intersections, let

$1 + \sin\theta = 3\sin\theta$

$-2\sin\theta + 1 = 0$ Subtract $2\sin\theta$ from both sides

$-2\sin\theta = -1$ Subtract 1 from both sides

$2\sin\theta = 1$ Multiply both sides by -1

$\sin\theta = \frac{1}{2}$ Divide both sides by 2.

On the interval $0 \le \theta \le \frac{\pi}{2}$, $\theta = \frac{\pi}{6}$

On the interval $\frac{\pi}{2} \le \theta \le \pi$, $\theta = \frac{5\pi}{6}$

To find *r*, substitute θ values in either of the equations.

$$r = 3\sin\theta$$
$$= 3\cdot\frac{1}{2}$$
$$= \frac{3}{2}$$

Thus,

$$r = \frac{3}{2}$$

Thus, the other two points are $\left(\frac{3}{2}, \frac{\pi}{6}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$

Since the value of θ that satisfy both equations between $0 \le \theta \le \frac{\pi}{2}$ is $\theta = \frac{\pi}{6}$, so the point indicated by **3** on the graph is $\left(\frac{3}{2}, \frac{\pi}{6}\right)$.

Since the value of θ that satisfy both equations between $\frac{\pi}{2} \le \theta \le \pi$ is $\theta = \frac{5\pi}{6}$, so the point indicated by **2** on the graph is $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$

Therefore, all points of intersection of two polar curves $r = 1 + \sin\theta$ and $r - 3\sin\theta$ are

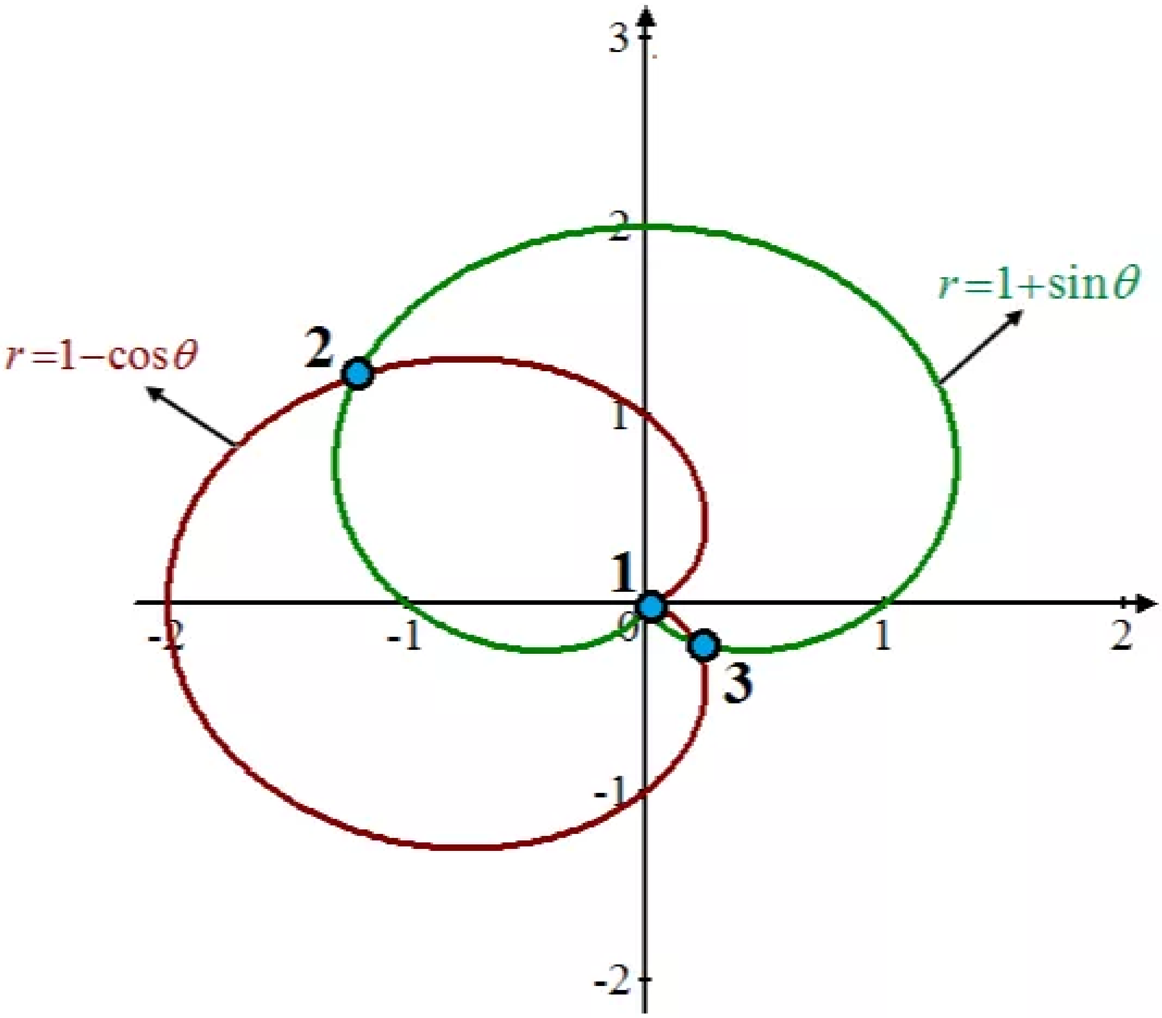
$$\boxed{(0,0), \left(\frac{3}{2}, \frac{\pi}{6}\right) \text{ and } \left(\frac{3}{2}, \frac{5\pi}{6}\right)}.$$

Q38E

Consider the two curves

$r = 1 - \cos\theta$ and $r = 1 + \sin\theta$

The graph of the curves $r = 1 - \cos\theta$ and $r = 1 + \sin\theta$ as shown below:



From the graph, observe that there are three intersection points.

Also, one of the indicated by **1** on the graph is the origin.

To find the other two intersections, let

$$1 - \cos\theta = 1 + \sin\theta$$
$$\sin\theta = -\cos\theta$$
$$\tan\theta = -1$$

On the interval $\frac{\pi}{2} \le \theta \le \pi$, $\theta = \frac{3\pi}{4}$

On the interval $\frac{3\pi}{2} \le \theta \le 2\pi$, $\theta = \frac{7\pi}{4}$

To find *r*, substitute θ values in either of the equations.

$$
\begin{aligned}
r &= 1+\sin\theta \\
&= 1+\sin\left(\frac{3\pi}{4}\right) \\
&= 1+\frac{\sqrt{2}}{2} \\
&= \frac{2+\sqrt{2}}{2}
\end{aligned}
$$

And

$$
\begin{aligned}
r &= 1+\sin\theta \\
&= 1+\sin\left(\frac{7\pi}{4}\right) \\
&= 1-\frac{\sqrt{2}}{2} \\
&= \frac{2-\sqrt{2}}{2}
\end{aligned}
$$

Thus, the other two points are $\left(\frac{2+\sqrt{2}}{2},\frac{3\pi}{4}\right)$ and $\left(\frac{2-\sqrt{2}}{2},\frac{7\pi}{4}\right)$

Since the value of θ that satisfy both equations between $\frac{\pi}{2}\le\theta\le\pi$ is $\theta=\frac{3\pi}{4}$, so the point indicated by **2** on the graph is $\left(\frac{2+\sqrt{2}}{2},\frac{3\pi}{4}\right)$.

Since the value of θ that satisfy both equations between $\frac{3\pi}{2}\le\theta\le 2\pi$ is $\theta=\frac{7\pi}{4}$, so the point indicated by **3** on the graph is $\left(\frac{2-\sqrt{2}}{2},\frac{7\pi}{4}\right)$.

Therefore, all points of intersection of two polar curves $r=1+\sin\theta$ and $r-3\sin\theta$ are

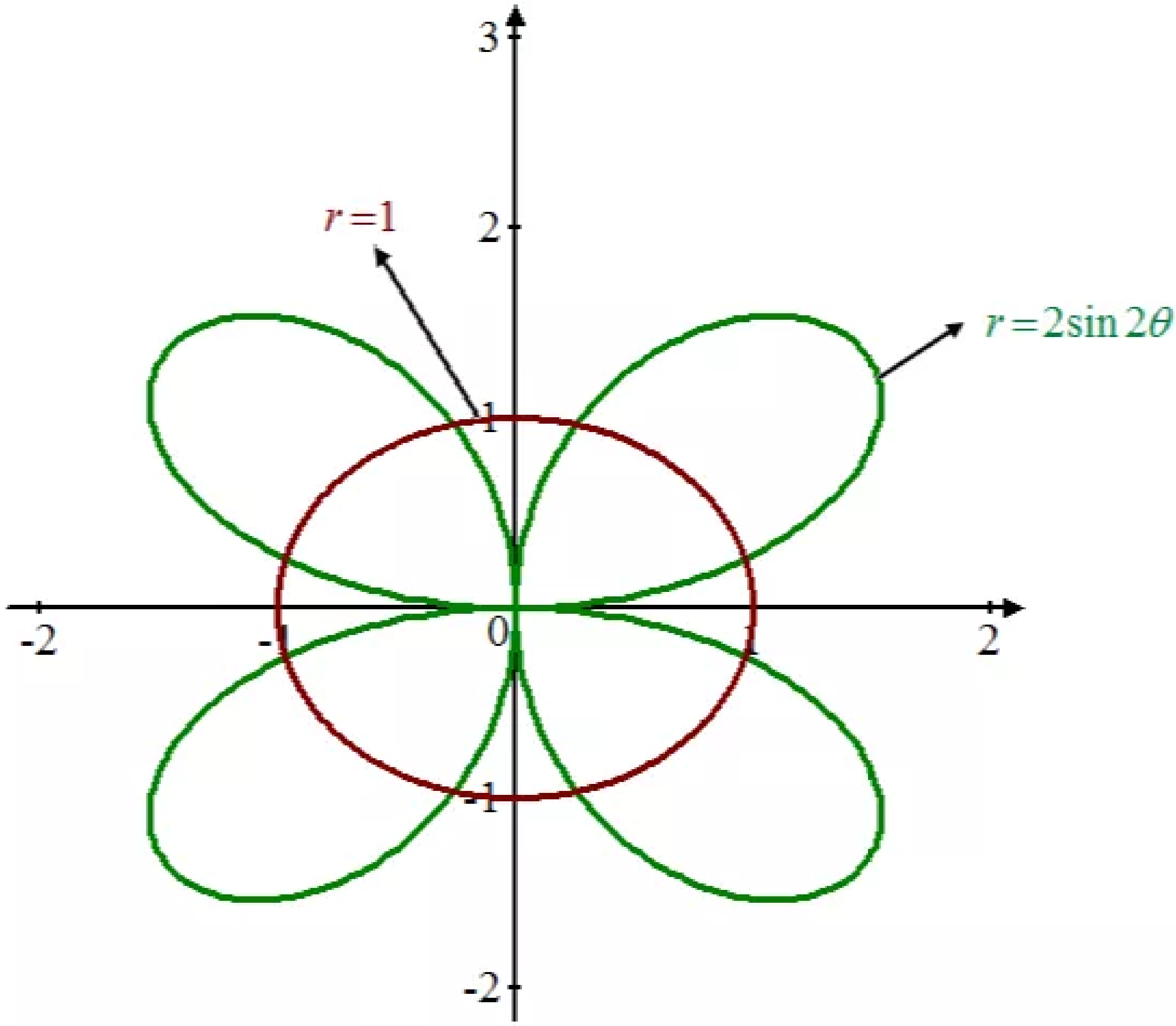
$$
\boxed{(0,0),\left(\frac{2+\sqrt{2}}{2},\frac{3\pi}{4}\right)\text{ and }\left(\frac{2-\sqrt{2}}{2},\frac{7\pi}{4}\right)}.
$$

Q39E

Consider the two curves

$r = 2\sin 2\theta$ and $r = 1$

The graph of the curves $r = 2\sin 2\theta$ and $r = 1$ as shown below:



To find the intersection points, let

$$2\sin 2\theta = 1$$

$$\sin 2\theta = \frac{1}{2}$$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

Thus, the values of θ between 0 and 2π that satisfy both equations are

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

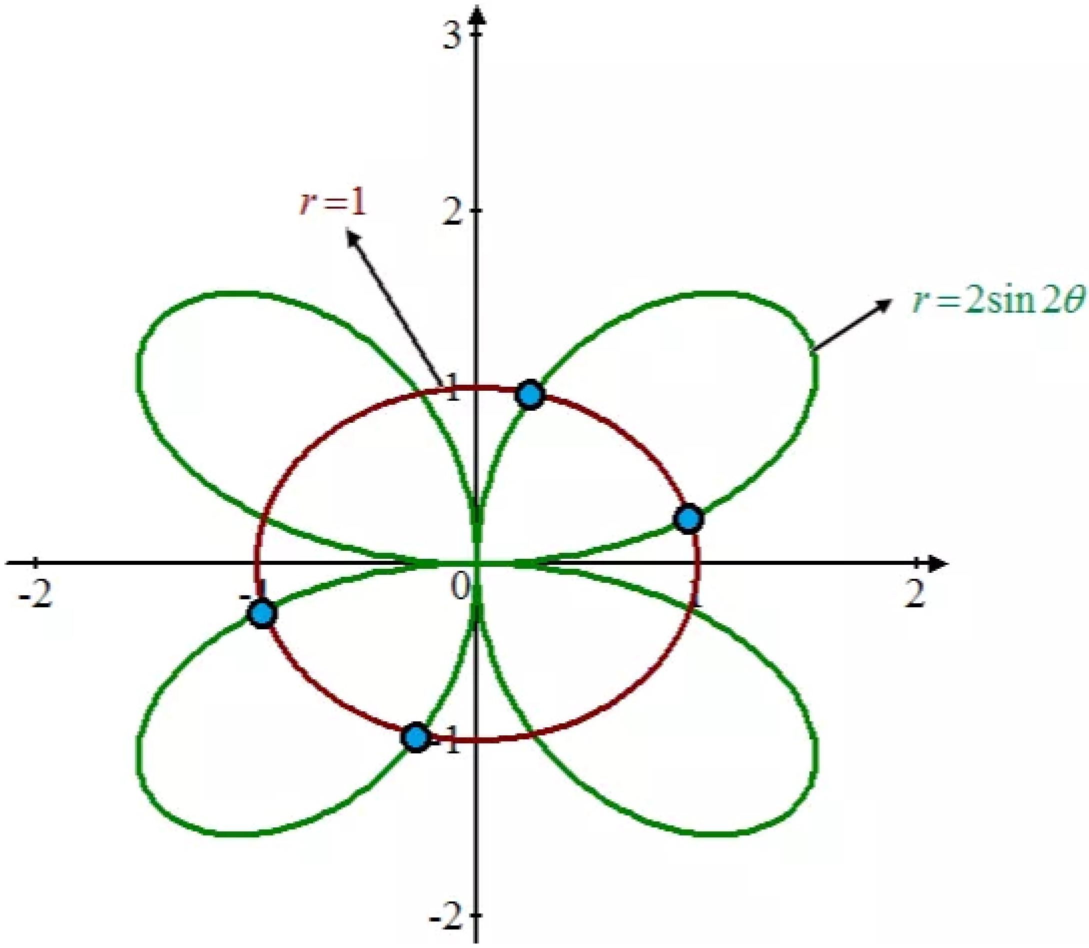
To find *r*, substitute θ values in either of the equations.

$$r = 2\sin 2\theta$$
$$= 2\cdot\frac{1}{2}$$
$$= 1$$

Thus, the curves have four points intersection-namely

$$\boxed{\left(1,\frac{\pi}{12}\right),\left(1,\frac{5\pi}{12}\right),\left(1,\frac{13\pi}{12}\right),\text{ and }\left(1,\frac{17\pi}{12}\right)}$$

These can be found on the graph using symmetry.



Q40E

We sketch the curves $r = \cos 3\theta$, $\quad r = \sin 3\theta$

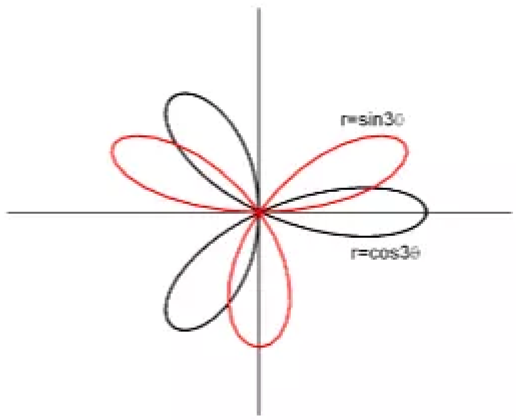


Fig.1

For the points of intersection we must have

$$\cos 3\theta = \sin 3\theta$$
$$\Rightarrow \tan 3\theta = 1$$
$$3\theta = \pi/4,\ 5\pi/4,\ 9\pi/4,\ 13\pi/4,\ 17\pi/4,\ \dots\dots\dots$$

Then $\theta = \pi/12,\ 5\pi/12,\ 9\pi/12,\ 13\pi/12,\ 17\pi/12, \dots\dots\dots$

Corresponding points are

$$\left(1/\sqrt{2},\ \pi/12\right), \left(-1/\sqrt{2},\ 5\pi/12\right), \left(1/\sqrt{2},\ 9\pi/12\right), \left(-1/\sqrt{2},\ 13\pi/12\right), \left(\frac{1}{\sqrt{2}},\ 17\pi/12\right), \dots$$

But from figure we see that there are only four points of intersection

$$\boxed{\left(1/\sqrt{2},\ \pi/12\right), \left(-1/\sqrt{2},\ 5\pi/12\right), \left(1/\sqrt{2},\ 9\pi/12\right), \text{ and the pole}}$$

Q41E

We sketch the curves $r = \sin\theta, \qquad r = \sin 2\theta$

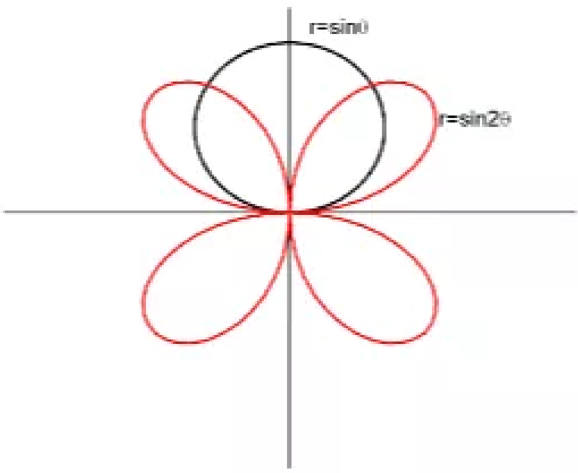


Fig.1

For points of intersection we must have

$$\sin\theta = \sin 2\theta$$
$$\Rightarrow \sin\theta = 2\sin\theta\cos\theta \qquad (\sin 2\theta = 2\sin\theta\cos\theta)$$
$$\Rightarrow \sin\theta - 2\sin\theta\cos\theta = 0$$
$$\Rightarrow \sin\theta(1-2\cos\theta) = 0$$
$$\Rightarrow \sin\theta = 0 \quad or \quad (1-2\cos\theta) = 0$$
$$\Rightarrow \theta = 0, \quad or \quad \cos\theta = 1/2$$
$$\Rightarrow \theta = \pi/3,\ 5\pi/3,\ 7\pi/3, \ldots$$

So points of intersection are $(0,0), \left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$

But from figure we see that there in one another point of intersection $\left(\sqrt{3}/2, 2\pi/3\right)$

So there are three points of intersection $\boxed{\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right), \left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right) \text{ and pole}}$

Q42E

For points of intersection we must have

$$\sin 2\theta = \cos 2\theta$$
$$\Rightarrow \tan 2\theta = 1$$
$$\Rightarrow 2\theta = \pi/4,\ 5\pi/4,\ 9\pi/4,\ 13\pi/4,\ 17\pi/4, \ldots\ldots\ldots$$
$$\Rightarrow \theta = \pi/8,\ 5\pi/8,\ 9\pi/8,\ 13\pi/8,\ 17\pi/8, \ldots\ldots\ldots$$
$$\Rightarrow \theta = \pi/8, 9\pi/8 \quad \left(\begin{array}{l}\text{since points of intersection}\\ \text{lies in first and third quadrants}\end{array}\right)$$

Corresponding points are $\left(1/\sqrt[4]{2},\ \pi/8\right), \left(1/\sqrt[4]{2},\ 9\pi/8\right)$

From the figure we see that both the curves are passing through pole.
But from figure we see that there are only three points of intersection.

$$\boxed{\left(1/\sqrt[4]{2},\ \pi/8\right), \left(1/\sqrt[4]{2},\ 9\pi/8\right) \text{ and pole}}$$

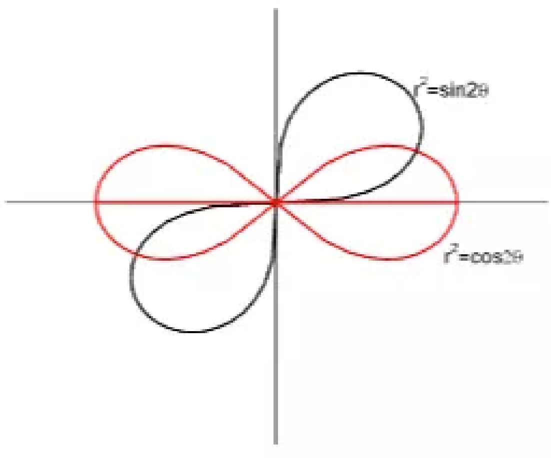


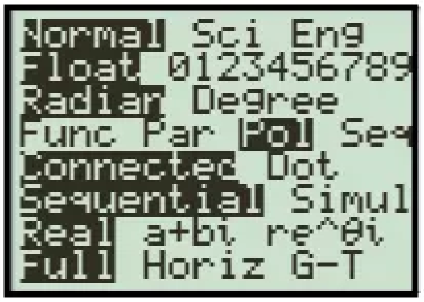
Fig.1

Q43E

Consider the cardioid $r = 1 + \sin\theta$ and the spiral loop $r = 2\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Find the points of intersection using graphing utility:

Step1: Press [MODE] key then set calculator as **Pol** mode

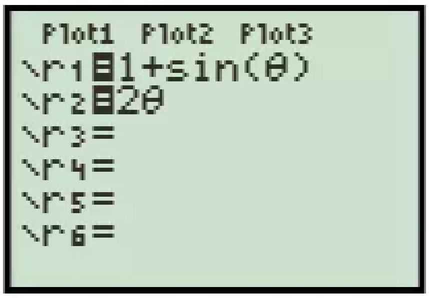


Step2: Press [Y=] key then enter the function and polynomial as

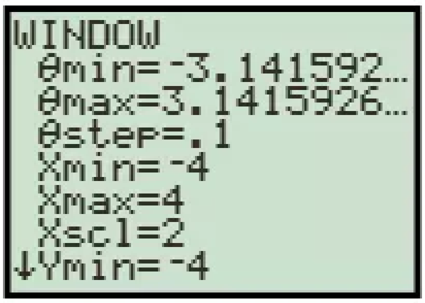
$r_1 =$ [1] [+] [sin] [X,T,θ,n] [)]

$r_2 =$ [2] [X,T,θ,n]

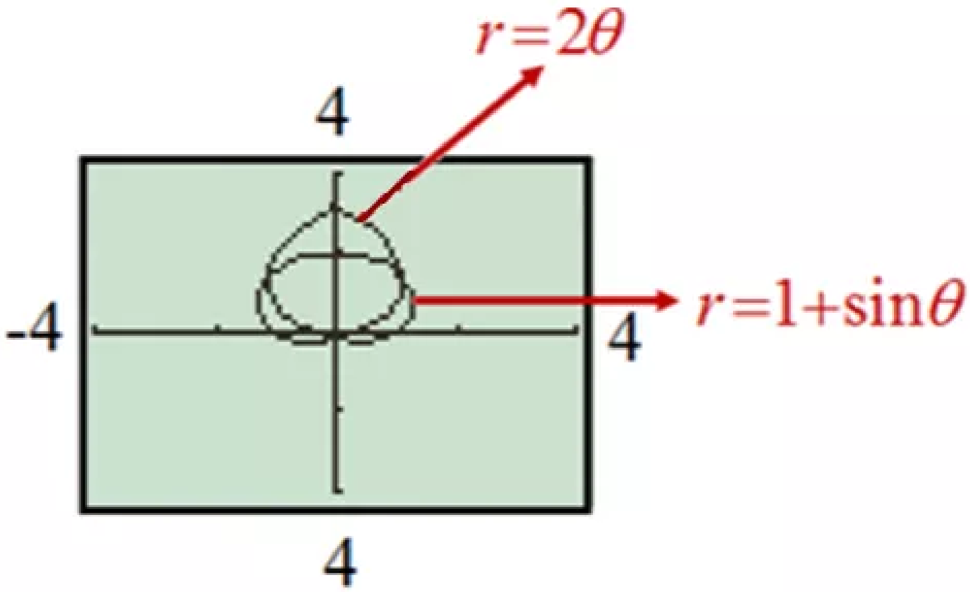
The display as shown below:



Step3: Press [WINDOW] key then set window settings as follows:



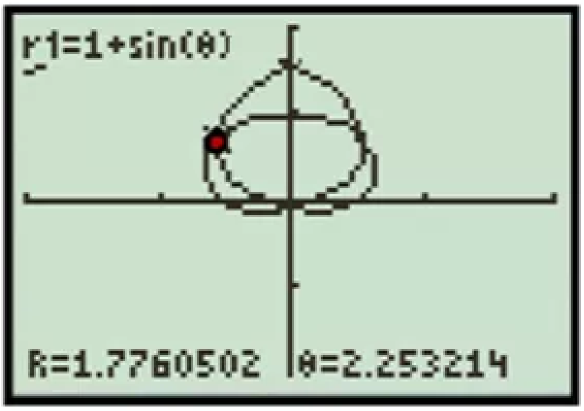
Step4: Press GRAPH key. The display as shown below

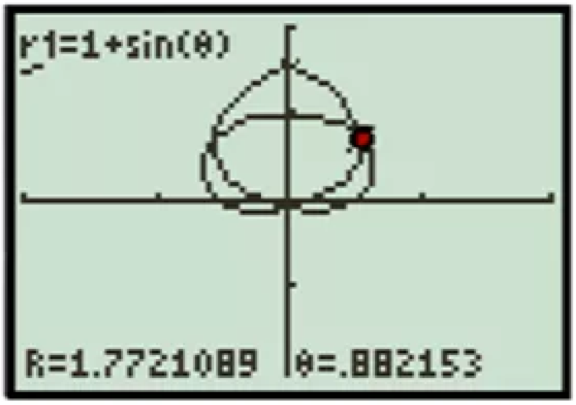


Step5: Press TRACE key. Use the right and left arrows to travel along the curve.

Find two intersection points by placing the cursor to near the points of intersection.

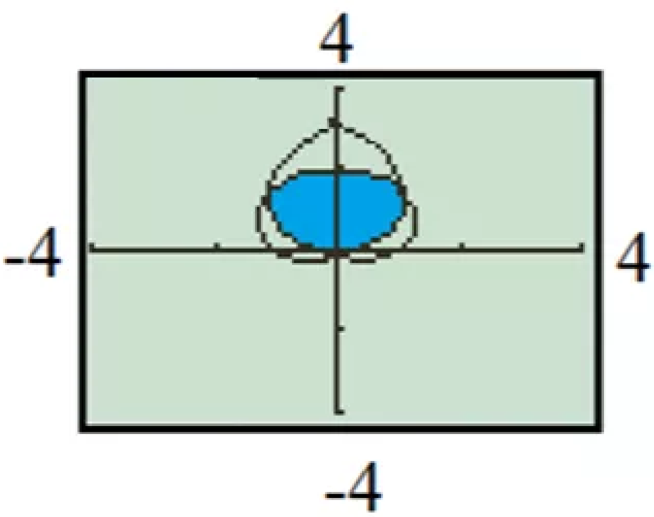
The display as shown below:





Thus, the values of θ are $\boxed{\theta \approx 0.89 \text{ and } \theta \approx 2.25}$

The area that lies inside the both curves as shown in below figure:



To compute the area, use the formula

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

From the graph, the area lies inside the both curves $r = 1 + \sin\theta$ and $r = 2\theta$ from 0 to $\frac{\pi}{2}$ is,

$$A = \frac{1}{2}\int_0^{0.89} (2\theta)^2 + \frac{1}{2}\int_{0.89}^{\frac{\pi}{2}} (1+\sin\theta)^2$$

Since the region is symmetric about the vertical axis $\theta = \frac{\pi}{2}$, so

$$A = 2\left(\frac{1}{2}\int_0^{0.89} (2\theta)^2 d\theta + \frac{1}{2}\int_{0.89}^{\frac{\pi}{2}} (1+\sin\theta)^2 d\theta\right)$$

$$= 2\cdot\frac{1}{2}\left(\int_0^{0.89} (2\theta)^2 d\theta + \int_{0.89}^{\frac{\pi}{2}} (1+\sin\theta)^2 d\theta\right)$$

$$= \int_0^{0.89} (2\theta)^2 d\theta + \int_{0.89}^{\frac{\pi}{2}} (1+\sin\theta)^2 d\theta$$

To solve this integral, use CAS.

The maple input command:

```
Int((2*theta)^2,theta=0..0.89)+int((1+sin(theta))^2,theta=0.89..pi/2);
```

Output:

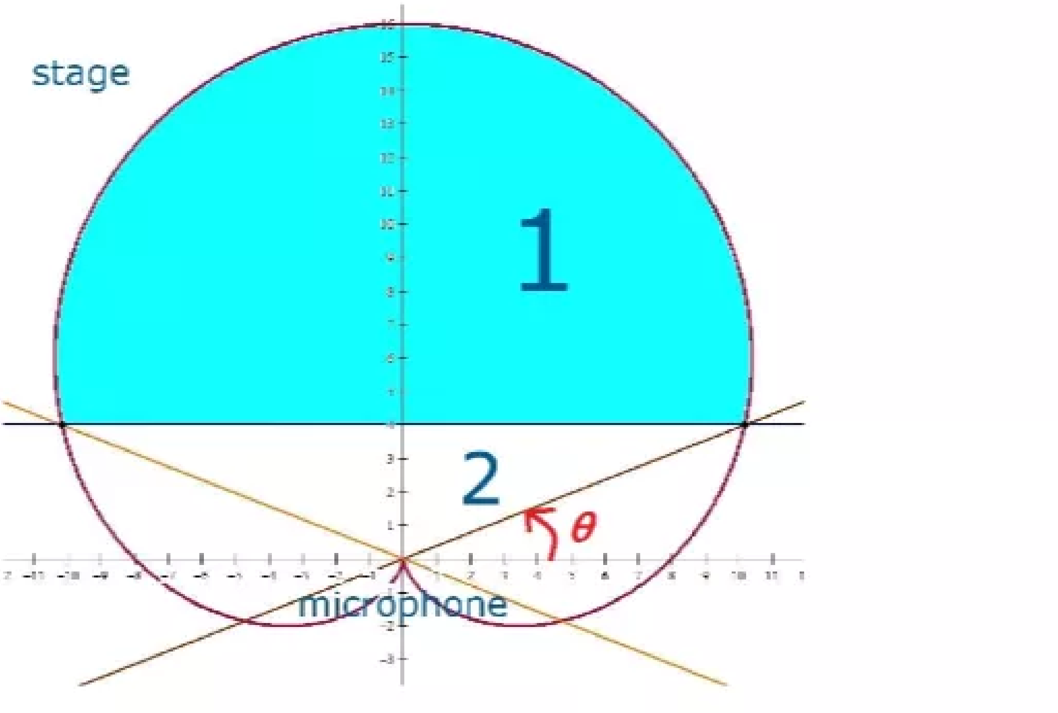
$$> int\left((2\cdot\theta)^2, \theta=0..0.89\right) + int\left((1+\sin(\theta))^2, \theta=0.89..\frac{\text{Pi}}{2}\right);$$

3.464526362

Therefore, the area that lies inside both curves $r = 1 + \sin\theta$ and $r = 2\theta$ is approximately $\boxed{3.46}$

Q44E

In a live performance, we set up the microphone at the pole, 4 meters away from the stage, and given that the pickup region is $r = 8 + 8\sin\theta$. And we need to find the area the pickup region overlay with the stage.
First we sketch a graph to see which area we need to calculate, and the light blue region is the area we need to calculate:



First we convert the function $y = 4$ to the polar coordinates, we get:

$$r\sin\theta = 4$$

$$r = \frac{4}{\sin\theta}$$

To find the region of θ, we need to find the value of θ at the intersections of the two curves, we let:

$$\Rightarrow \frac{4}{\sin\theta} = 8 + 8\sin\theta$$

$$\Rightarrow 8\sin^2\theta + 8\sin\theta - 4 = 0$$

$$\Rightarrow \sin\theta = \frac{\sqrt{3}-1}{2}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

The light blue area from the graph above, we can find the interval of θ of the sum of the area 1 and area 2, which is $\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) \le \theta \le \frac{\pi}{2}$.

To find the area bounded by two polar curves, we let $\Re$ be a region, that is bounded by curves with polar equations $r=f(\theta)$, $r=g(\theta)$, $\theta=a$, and $\theta=b$, where

$f(\theta)\geq g(\theta)\geq 0$, and $0\leq b-a\leq 2\pi$.

To find the area, we need to use areas formula in polar coordinates:
The formula for the area A of the polar region $\Re$ is

$$A=\int_a^b \frac{1}{2}\left[f(\theta)\right]^2 d\theta$$

In this question, the goal is to find the area 1, then multiple by 2. To calculate the area 1, we can find the sum of the area 1 and area 2, and then subtract area 2 to get the answer.

The sum of the area 1 and area 2 is the area bounded by the curve $r=8+8\sin\theta$, where $\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)\leq\theta\leq\frac{\pi}{2}$.

So we can calculate it by using the formula and then get:

$$A_1+A_2=\frac{1}{2}\int_{\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\pi}{2}}(8+8\sin\theta)^2 d\theta$$

$$=32\int_{\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\pi}{2}}\left(1+2\sin\theta+\sin^2\theta\right)d\theta$$

$$=32\int_{\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\pi}{2}}\left(1+2\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$=16\int_{\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\pi}{2}}\left(3+4\sin\theta-\cos 2\theta\right)d\theta$$

Continuous to the above

$$=16\left[3\theta-4\cos\theta-\frac{\sin 2\theta}{2}\right]_{\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\pi}{2}}$$

$$=16\left[\begin{array}{l}3\left(\frac{\pi}{2}\right)-4\cos\left(\frac{\pi}{2}\right)-\frac{\sin 2\left(\frac{\pi}{2}\right)}{2}-3\left(\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)\right)\\+4\cos\left(\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)\right)+\frac{\sin 2\left(\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)\right)}{2}\end{array}\right]$$

$$\approx 244.83936$$

From the graph we can see that area 2 is a triangle, so we can calculate the area of it using the formula:

$$A_2 = \frac{1}{2}bh$$
$$= \frac{1}{2}\cdot\frac{4}{\tan\theta}\cdot 4$$
$$= 2\cdot\frac{4}{\tan\left(\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)\right)}$$
$$\approx 20.339678$$

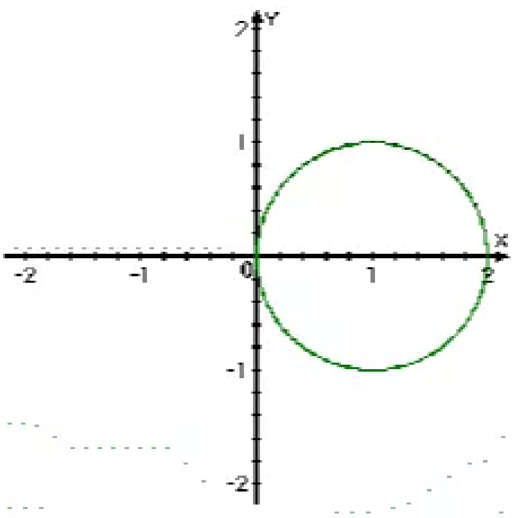
Hence, we can calculate the light blue area that we need:

$$A = 2(A_1 + A_2 - A_2)$$
$$= 2(244.83936 - 20.339678)$$
$$= 448.9994$$

So the area on the stage also within the pickup range will be approximated $\boxed{448.9994\ m^2}$.

Q45E

Given $r = 2\cos\theta,\ 0 \le \theta \le \pi$



For the entire rotation, $0 \le \theta \le \pi$ is enough.

The arc length of the given curve is $L=\int_a^b \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

Here $a=0$ and $b=\pi$

$r=2\cos\theta, \frac{dr}{d\theta}=-2\sin\theta$

$$\therefore L=\int_0^\pi \sqrt{4\cos^2\theta+(-2\sin\theta)^2}\,d\theta$$
$$=\int_0^\pi \sqrt{4}\,d\theta$$
$$=\int_0^\pi 2\,d\theta$$
$$=[2\theta]_0^\pi$$
$$=\boxed{2\pi}$$

Q46E

Consider the function:

$y=f(x), \quad a\le x\le b$

The derivative of the above function f' that continuous over the closed interval $[a,b]$, so the formula used to determine the arc length of the curve is:

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Consider the function:

$r=5^\theta,\ 0\le\theta\le 2\pi$

Substitute the values in the formula of arc length as shown below:

$a=0$
$b=2\pi$

Consider the function shown below:

$$r=5^\theta$$
$$\ln r=\theta\ln 5$$
$$\frac{1}{r}\times\frac{dr}{d\theta}=\ln 5$$
$$\frac{dr}{d\theta}=r\ln 5$$
$$=5^\theta\ln 5$$

So, the value of the derivative is:

$$\frac{dr}{d\theta}=5^\theta\ln 5$$

Determine the arc length of the above curve:

$$L=\int_a^b \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$=\int_0^{2\pi} \sqrt{5^{2\theta}+\left(5^{\theta}\ln 5\right)^2}\,d\theta$$

$$=\sqrt{1+(\ln 5)^2}\int_0^{2\pi} 5^{\theta}\,d\theta$$

$$=\sqrt{1+(\ln 5)^2}\left[\frac{5^{\theta}}{\ln 5}\right]_0^{2\pi}$$

$$=\sqrt{1+(\ln 5)^2}\left[\frac{5^{2\pi}-1}{\ln 5}\right]$$

Evaluate the value of L is:

$$L=29.016$$

Hence, the length of the polar curve is $\boxed{29.016}$.

Q47E

We have the polar equation as $r=\theta^2$ then $\frac{dr}{d\theta}=2\theta$

The length of the polar curve is

$$L=\int_0^{2\pi} \sqrt{\left(\theta^2\right)^2+\left(2\theta\right)^2}\,d\theta$$

$$=\int_0^{2\pi} \sqrt{\theta^4+4\theta^2}\,d\theta$$

$$=\int_0^{2\pi} \theta\sqrt{\theta^2+4}\,d\theta$$

Substitute $\theta^2+4=u \quad \Rightarrow 2\theta d\theta=du$

When $\begin{cases}\theta=0 & \Rightarrow u=4\\ \theta=2\pi & \Rightarrow u=4\pi^2+4\end{cases}$

Then $L = \int_4^{4\pi^2+4} \sqrt{u}\,\frac{du}{2}$

$$= \frac{1}{2}\left[\frac{2u^{3/2}}{3}\right]_4^{4\pi^2+4}$$

$$= \frac{1}{3}\left[\left(4\pi^2+4\right)^{3/2} - 4^{3/2}\right]$$

$$= \frac{1}{3}\left[\left(4\pi^2+4\right)^{3/2} - 8\right]$$

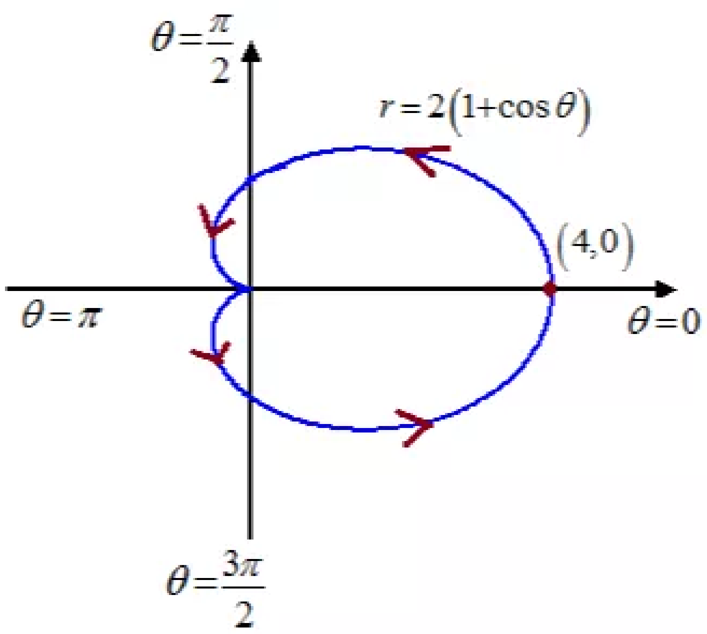
$$= \frac{1}{3}\left[4^{3/2}\left(\pi^2+1\right)^{3/2} - 8\right]$$

$$= \boxed{\frac{8}{3}\left\{\left(\pi^2+1\right)^{3/2} - 1\right\}}$$

Q48E

Consider the curve $r = 2(1+\cos\theta)$

The graph of the curve is shown below:



Observe that the red dot on the curve is the initial point and final point. So, $o \le \theta \le 2\pi$.

Always, the length of the curve with polar equation $r = f(\theta)$, $a \le \theta \le b$, is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Since $r = 2(1+\cos\theta)$

Then $\frac{dr}{d\theta} = -2\sin\theta$

Therefore, the required length of the curve is

$$L = \int_0^{2\pi} \sqrt{4(1+\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4(1+2\cos\theta+\cos^2\theta)+4\sin^2\theta d\theta}$$

$$= \int_0^{2\pi} \sqrt{4(2+2\cos\theta)} d\theta$$

$$= 2\sqrt{2}\int_0^{2\pi} \sqrt{1+\cos\theta}\, d\theta$$

$= 2\sqrt{2}\cdot 2\int_0^{\pi} \sqrt{1+\cos\theta}\, d\theta$ Since $\sqrt{1+\cos\theta}$ is an even function.

Since $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

So, $1+\cos\theta = 2\cos^2\frac{\theta}{2}$

Therefore,

$$L = 4\sqrt{2}\int_0^{\pi} \sqrt{2\cos^2\frac{\theta}{2}}\, d\theta$$

$$= 8\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta$$

$$= 8\left(2\sin\frac{\theta}{2}\right)_0^{\pi}$$

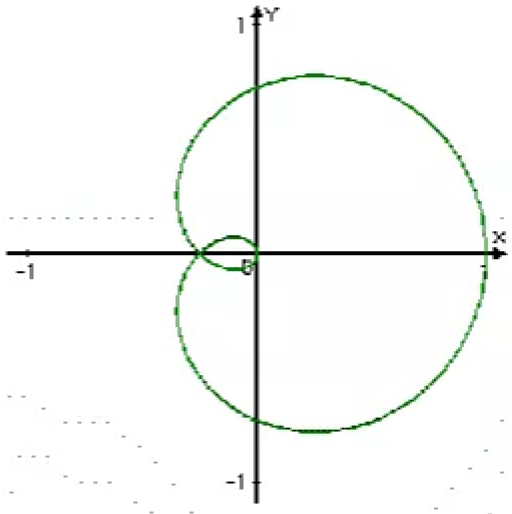
$$= 16(1-0)$$

$$= \boxed{16}$$

Q49E

Given polar curve is $r=\cos^4\left(\frac{\theta}{4}\right)$

Its sketch is



The entire rotation requires 4π units

Observe that the curve is symmetric about X axis.

So, the length of the arc above X axis is equal to the length of the arc below X axis.

For the rotation about X axis, the angle required is 2π units.

So,

$$L=\int_a^b \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\int_0^{4\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$
$$=2\int_0^{2\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Observe that the curve is symmetric about X axis.

So, the length of the arc above X axis is equal to the length of the arc below X axis.

For the rotation about X axis, the angle required is 2π units.

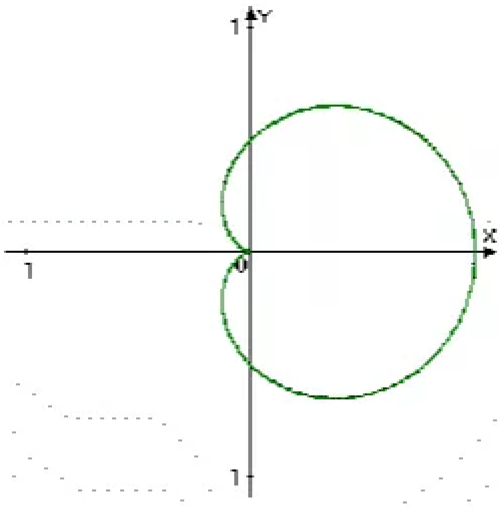
So,

$$L=\int_a^b \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\int_0^{4\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$
$$=2\int_0^{2\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= 2\int_0^{2\pi} \sqrt{\cos^8\left(\frac{\theta}{4}\right)+\cos^6\left(\frac{\theta}{4}\right)\sin^2\left(\frac{\theta}{4}\right)}\, d\theta$$
$$= 2\int_0^{2\pi} \sqrt{\cos^6\left(\frac{\theta}{4}\right)(\cos^2\left(\frac{\theta}{4}\right)+\sin^2\left(\frac{\theta}{4}\right))}\, d\theta$$
$$= 2\int_0^{2\pi} \sqrt{\cos^6\left(\frac{\theta}{4}\right)}d\theta$$
$$= \boxed{\frac{16}{3}}$$

Q50E

Given polar curve is $r = \cos^2\left(\frac{\theta}{2}\right)$



The curve requires 2π units of rotation and is symmetric about X axis.

So, the length of the arc is twice the length of the arc above X axis.

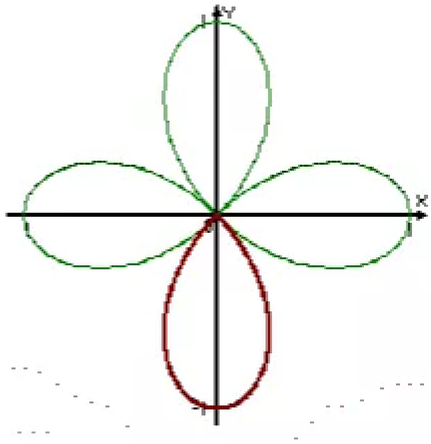
The arc above X axis requires π units of rotation.

So, Length $L = 2\int_a^b \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta = 2\int_0^{\pi} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$$= 2\int_0^{\pi} \sqrt{\cos^4\left(\frac{\theta}{2}\right)+\cos^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$$
$$= 2\int_0^{\pi} \sqrt{\cos^2\left(\frac{\theta}{2}\right)\left\{\cos^2\left(\frac{\theta}{2}\right)+\sin^2\left(\frac{\theta}{2}\right)\right\}}\, d\theta$$
$$= 2\int_0^{\pi} \sqrt{\cos^2\left(\frac{\theta}{2}\right)}d\theta$$
$$= 2\cdot 2$$
$$= 4$$

Q51E

Given polar curve is $r = \cos 2\theta$



Its graph is

The entire rotation requires 2π angle. For the red color part of the curve requires $\pi/2$ rotation and it is easy to see that 4 such curves are required for the entire rotation.

Length

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

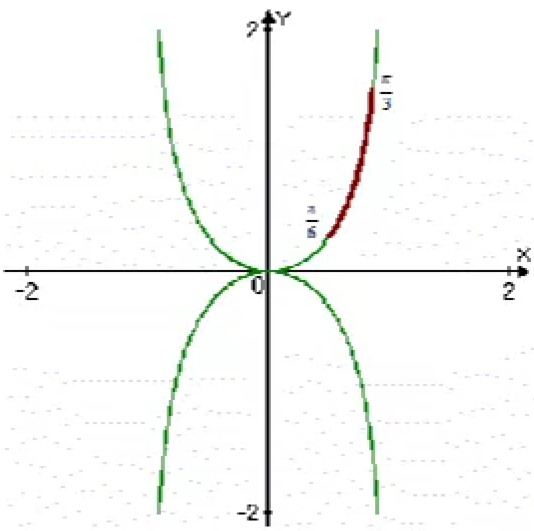
$$= \int_0^{\pi/2} \sqrt{(\cos 2\theta)^2 + (-2\sin 2\theta)^2}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{1 + 3\sin^2(2\theta)}\, d\theta$$

As we have been directed to use the calculator for this integration, the answer is 2.4221

Q52E

Given polar curve is $r = \tan\theta, \frac{\pi}{6} \le \theta \le \frac{\pi}{3}$



the entire curve requires 2π rotation and the piece of red curve is the required one between $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$

Length of the curve is $L=\int_a^b \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

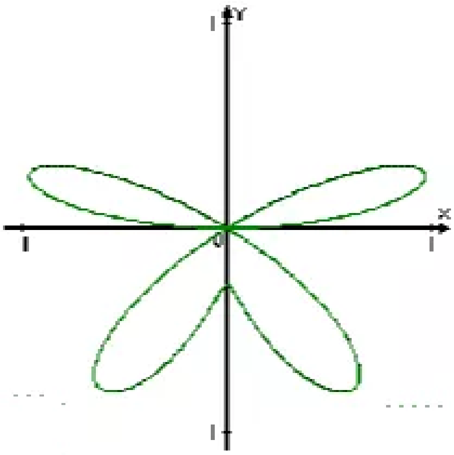
$$=\int_{\pi/6}^{\pi/3} \sqrt{\tan^2\theta+\sec^4\theta}\,d\theta$$

$$=\boxed{1.5302997}$$

Q53E

Given polar curve is $r=\sin(6\sin\theta)$

The graph of the curve is



For the complete rotation, the angle required is π

Length of the curve is $L=\int_a^b \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

$$=\int_0^\pi \sqrt{\sin^2(6\sin\theta)+\left(6\cos\theta\cdot\cos(6\sin\theta)\right)^2}\,d\theta$$

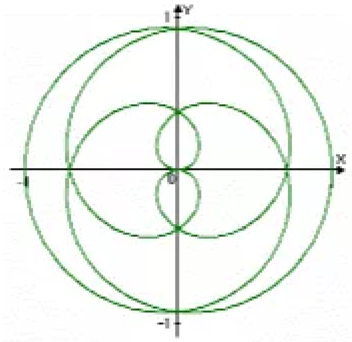
$$=\int_0^\pi \sqrt{\sin^2(6\sin\theta)+36\cos^2\theta\cdot\cos^2(6\sin\theta)}\,d\theta$$

$$=\int_0^\pi \sqrt{1+35\cos^2\theta\cdot\cos^2(6\sin\theta)}\,d\theta$$

$$=\boxed{8.0091}$$

Q54E

Given polar curve is $r = \sin\left(\frac{\theta}{4}\right)$. Its graph is



For the entire rotation, the angle of rotation required is 8π

We can easily see that the entire curve is symmetric about the origin.

So, we calculate the length of the curve in one quadrant and multiply with 4 to get the total length of the curve.

Length of the curve in one quadrant is $L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$$= \int_0^{2\pi} \sqrt{\sin^2\left(\frac{\theta}{4}\right) + \left(\frac{1}{4}\cos\left(\frac{\theta}{4}\right)\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{1 - \frac{15}{16}\cos^2\left(\frac{\theta}{4}\right)}\, d\theta$$

Using calculator, the result is 4.28921

So, the length of the entire curve is 17.15684

Q55E

(A) Parametric equations of the curve $r = f(\theta)$ are

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

By product rule, differentiating with respect to θ

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta, \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

Then $ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

$$= \sqrt{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2}\, d\theta$$

$$= \sqrt{\left(\frac{dr}{d\theta}\right)^2 \cos^2\theta + r^2\sin^2\theta - 2r\sin\theta\cos\theta\frac{dr}{d\theta} + \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + r^2\cos^2\theta + 2r\sin\theta\cos\theta\frac{dr}{d\theta}} \quad d\theta$$

$$= \sqrt{\left(\frac{dr}{d\theta}\right)^2 \left(\cos^2\theta + \sin^2\right) + r^2\left(\cos^2\theta + \sin^2\theta\right)} \;\; d\theta$$

$$= \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} \qquad\qquad \left(\sin^2\theta + \cos^2\theta = 1\right)$$

Given that f' is continuous and $0 \le a < b \le \pi$

Then area of the surface generated by rotating the polar curve $r = f(\theta)$ about the polar axis is given by $S = \int_a^b 2\pi y \; ds$

$$\boxed{S = \int_a^b 2\pi r\sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \; d\theta}$$

(B) Curve is $r^2 = \cos 2\theta$

We sketch the curve

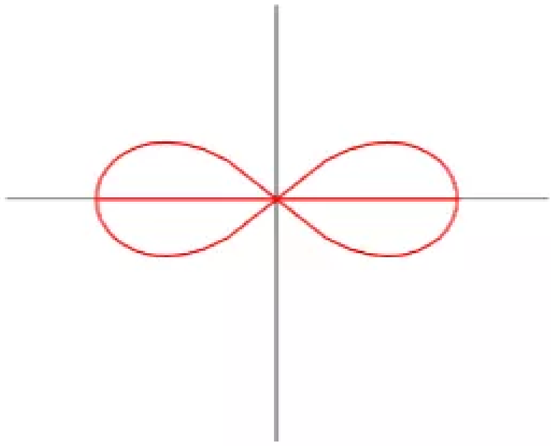


Fig.1

We take $r = 0 \Leftrightarrow \cos 2\theta = 0$

$\Rightarrow 2\theta = \pi/2,\ 3\pi/2$

$\Rightarrow \theta = \pi/4,\ 3\pi/4$

Since the curve is symmetric about the polar axis so if we double the surface area obtained by rotating the curve from $\theta = 0$ to $\pi/4$ then we can find the total surface area obtained by rotating the complete curve about the line $\theta = \pi/2$

Surface area $s = 2\int_0^{\pi/4} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$

Since $r^2 = \cos 2\theta$

$\Rightarrow 2r\frac{dr}{d\theta} = -2\sin 2\theta$

$\Rightarrow \frac{dr}{d\theta} = -\frac{\sin 2\theta}{r}$

$\Rightarrow \left(\frac{dr}{d\theta}\right)^2 = \frac{\sin^2 2\theta}{r^2}$

$= \frac{\sin^2 2\theta}{\cos 2\theta}$

Then $r^2 + \left(\frac{dr}{d\theta}\right)^2 = \cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}$

$= \frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}$

$= \frac{1}{\cos 2\theta}$

$= \sec 2\theta$

Then $S = 2\int_0^{\pi/4} 2\pi\sqrt{\cos 2\theta}\ \sin\theta\sqrt{\frac{1}{\cos 2\theta}}\ d\theta$

$= 4\pi\int_0^{\pi/4} \sin\theta\ d\theta$

$= 4\pi\left[-\cos\theta\right]_0^{\pi/4}$

$= 4\pi\left[-\cos\frac{\pi}{4} + \cos 0\right]$

$= 4\pi\left[-\frac{1}{\sqrt{2}} + 1\right]$

$= 4\pi\left[-\sqrt{2}/2 + 1\right]$

$= \boxed{2\pi\left(2 - \sqrt{2}\right)}$

Q56E

Consider the polar curve, $r = f(\theta), a \le \theta \le b.$

Recall that,

"If the curve given by the parametric equations $x = f(t), y = g(t), \alpha \le t \le \beta,$ is rotating about the y-axis, where f', g' are continuous and $g(t) \ge 0,$ then the area of the surface generated by the curve rotating about the y-axis is,

$\int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$" ...... (1)

The parametric equations of the polar curve $r = f(\theta)$ as,

$$x = r\cos\theta$$
$$= f(\theta)\cos\theta$$

and

$$y = r\sin\theta$$
$$= f(\theta)\sin\theta$$

Use θ as parameter instead of *t*.

Differentiate *x* with respect to θ,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(r\cos\theta)$$
$$= \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

Differentiate *y* with respect to θ,

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(r\sin\theta)$$
$$= \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

Consider,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$
$$= \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2$$
$$= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta + \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\cos^2\theta$$
$$= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta + r^2\sin^2\theta + \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + r^2\cos^2\theta$$
$$= \left(\frac{dr}{d\theta}\right)^2 \left(\cos^2\theta + \sin^2\theta\right) + r^2\left(\cos^2\theta + \sin^2\theta\right)$$
$$= \left(\frac{dr}{d\theta}\right)^2 + r^2 \text{ Since: } \cos^2\theta + \sin^2\theta = 1$$

By using (1) and the above information, the area of the surface generated by the curve $r = f(\theta), a \le \theta \le b$ rotating about the line $\theta = \frac{\pi}{2}$ is,

$$S = \boxed{\int_a^b 2\pi(r\cos\theta)\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta}$$

(b)

Consider the curve,

$$r^2 = \cos 2\theta$$

Rewrite it as,

$$r = \sqrt{\cos 2\theta}$$

The derivative of *r* with respect to θ is,

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{d}{d\theta}\left(\sqrt{\cos 2\theta}\right) \\ &= \frac{1}{2\sqrt{\cos 2\theta}}(-2\sin 2\theta) \\ &= \frac{\sin 2\theta}{\sqrt{\cos 2\theta}} \end{aligned}$$

The graph of the above curve is shown below:

(b)

Consider the curve,

$$r^2 = \cos 2\theta$$

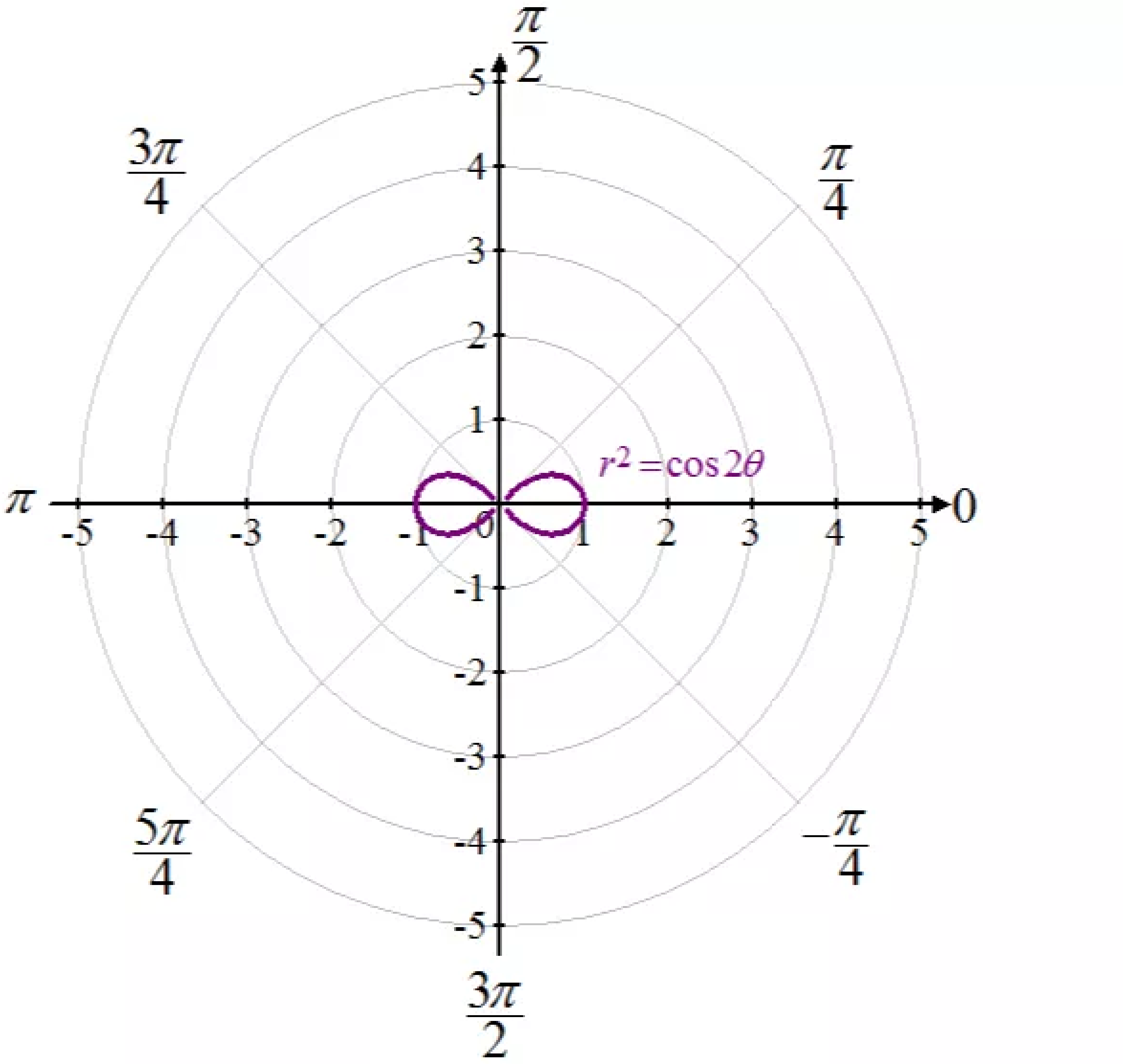
Rewrite it as,

$$r = \sqrt{\cos 2\theta}$$

The derivative of *r* with respect to θ is,

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{d}{d\theta}\left(\sqrt{\cos 2\theta}\right) \\ &= \frac{1}{2\sqrt{\cos 2\theta}}(-2\sin 2\theta) \\ &= \frac{\sin 2\theta}{\sqrt{\cos 2\theta}} \end{aligned}$$

The graph of the above curve is shown below:



The curve $r^2 = \cos 2\theta$ is defined on the interval $-\frac{\pi}{4}$ to $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ to $\frac{5\pi}{4}$.

By symmetry, use any one of the limits.

To find the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the line $\theta = \frac{\pi}{2}$

, use the result in part (a) with the above information.

$$S = \int_a^b 2\pi (r\cos\theta)\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\pi\left(\sqrt{\cos 2\theta}\cos\theta\right)\sqrt{\left(\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}\right)^2 + \cos 2\theta d\theta}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\pi\left(\sqrt{\cos 2\theta}\cos\theta\right)\sqrt{\frac{\sin^2 2\theta}{\cos 2\theta} + \cos 2\theta d\theta}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\pi\left(\sqrt{\cos 2\theta}\cos\theta\right)\sqrt{\frac{\left(\sin^2 2\theta + \cos^2 2\theta\right)}{\cos 2\theta}} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\pi \cos\theta \left(\sin^2 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\pi \cos\theta d\theta$$

$$= 2\pi \left[\sin\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 2\pi \left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right]$$

$$= 2\pi \left[\frac{2\sqrt{2}}{2}\right]$$

$$= 2\sqrt{2}\pi$$

Therefore, the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the line $\theta = \frac{\pi}{2}$ is $\boxed{2\sqrt{2}\pi}$