4.6 Lagrange's Mean Value Theorem

4.6.1 Definition

If a function $f(x)$,

(1) Is continuous in the closed interval [a, b] and

(2) Is differentiable in the open interval (a, b)

Then there is atleast one value $c \in (a,b)$, such that; $f'(c) = \frac{f(b) - f(a)}{b - a}$

4.6.2 Analytical Interpretation

First form: Consider the function, $\phi(x) = f(x) - \frac{f(b) - f(a)}{b - a} x$

Since, $f(x)$ is continuous in [a, b]

$\therefore$ $\phi(x)$ is also continuous in [a,b]

since, $f'(x)$ exists in (a, b) hence $\phi'(x)$ also exists in (a,b) and $\phi'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$ .....(i)

Clearly, $\phi(x)$ satisfies all the condition of Rolle's theorem

$\therefore$ There is atleast one value of c of x between a and b such that $\phi'(c) = 0$ substituting $x = c$ in (i) we get,

$f'(c) = \frac{f(b) - f(a)}{b - a}$ which proves the theorem.

Second form: If we write $b = a + h$ then $a < c < b, c = a + \theta h$ where $0 < \theta < 1$

Thus, the mean value theorem can be stated as follows:

If (i) $f(x)$ is continuous in closed interval [a, a+h]

(ii) $f'(x)$ exists in the open interval $(a, a+h)$ then there exists at least one number $\theta (0 < \theta < 1)$

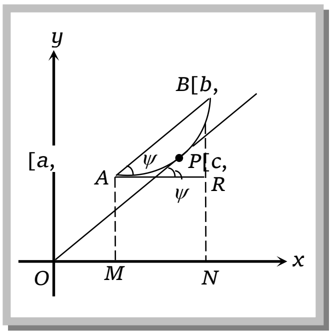
Such that $f(a+h) = f(a) + hf'(a + \theta h)$.

4.6.3 Geometrical Interpretation

Let $f(x)$ be a function defined on [a, b] and let APB be the curve represented by $y = f(x)$. Then co-ordinates of A and B are $(a, f(a))$ and $(b, f(b))$ respectively. Suppose the chord AB makes an angle ψ with the axis of x. Then from the triangle ARB, we have

$$\tan\psi = \frac{BR}{AR} \Rightarrow \tan\psi = \frac{f(b)-f(a)}{b-a}$$

By Lagrange's Mean value theorem, we have, $f'(c) = \frac{f(b)-f(a)}{b-a}$ $\therefore \tan\psi = f'(c)$



$\Rightarrow$ slope of the chord AB = slope of the tangent at $(c, f(c))$

Example: 1 In the mean-value theorem $\frac{f(b)-f(a)}{b-a} = f'(c)$, if $a = 0$, $b = \frac{1}{2}$ and $f(x) = x(x-1)(x-2)$, the value of c is **[MP PET 20**

(a) $1-\frac{\sqrt{15}}{6}$ (b) $1+\sqrt{15}$ (c) $1-\frac{\sqrt{21}}{6}$ (d) $1+\sqrt{21}$

Solution: (c) From mean value theorem $f'(c) = \frac{f(b)-f(a)}{b-a}$

$a = 0, f(a) = 0 \Rightarrow b = \frac{1}{2}, f(b) = \frac{3}{8}$

$f'(x) = (x-1)(x-2) + x(x-2) + x(x-1)$,

$f'(c) = (c-1)(c-2) + c(c-2) + c(c-1) = c^2 - 3c + 2 + c^2 - 2c + c^2 - c$, $f'(c) = 3c^2 - 6c + 2$

According to mean value theorem

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow 3c^2 - 6c + 2 = \frac{\left(\frac{3}{8}\right)-0}{\left(\frac{1}{2}\right)-0} = \frac{3}{4} \Rightarrow 3c^2 - 6c + \frac{5}{4} = 0$$

$$c = \frac{6 \pm \sqrt{36-15}}{2\times 3} = \frac{6\pm\sqrt{21}}{6} = 1 \pm \frac{\sqrt{21}}{6}.$$

Example: 2 From mean value theorem $f(b) - f(a) = (b-a)f'(x_1), a < x_1 < b$ if $f(x) = \frac{1}{x}$ then x_1

(a) $\sqrt{ab}$ (b) $\frac{2ab}{a+b}$ (c) $\frac{a+b}{2}$ (d) $\frac{b-a}{b+a}$

Solution: (a) $f'(x_1) = \frac{-1}{x_1^2}$, $\therefore \frac{-1}{x_1^2} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{ab} \Rightarrow x_1 = \sqrt{ab}$.

Example: 3 The abscissae of the points of the curve $y = x^3$ in the interval [–2, 2], where the slope of the tangent can be obtained by mean value theorem for the interval [– 2, 2] are

(a) $\pm\frac{2}{\sqrt{3}}$ (b) $\pm\frac{\sqrt{3}}{2}$ (c) $\pm\sqrt{3}$ (d) 0

Solution: (a) Given that equation of curve $y = x^3 = f(x)$

So $f(2)=8$ and $f(-2)=-8$

Now $f'(x)=3x^2 \Rightarrow f'(x)=\dfrac{f(2)-f(-2)}{2-(-2)} \Rightarrow \dfrac{8-(-8)}{4}=3x^2; \therefore x=\pm\dfrac{2}{\sqrt{3}}$.

Assignment

Lagrange's Mean Value Theorem

Basic Level

1. If from mean value theorem, $f'(x_1)=\frac{f(b)-f(a)}{b-a}$, then **[MP PET 1999]**

(a) $a<x_1\le b$ (b) $a\le x_1<b$ (c) $a<x_1<b$ (d) $a\le x_1\le b$

2. For the function $x+\frac{1}{x}, x\in[1,3]$, the value of c for the mean value theorem is **[MP PET 1997]**

(a) 1 (b) $\sqrt{3}$ (c) 2 (d) None of these

3. For the function $f(x)=e^x, a=0, b=1$, the value of c in mean value theorem will be **[DCE 2002]**

(a) log x (b) log $(e-1)$ (c) 0 (d) 1

Advance Level

4. If the function $f(x)=x^3-6ax^2+5x$ satisfies the conditions of Lagrange's mean value theorem for the interval [1, 2] and the tangent to the curve $y=f(x)$ at $x=\frac{7}{4}$ is parallel to the chord that joins the points of intersection of the curve with the ordinates $x=1$ and $x=2$. Then the value of a is

(a) $35/16$ (b) $35/48$ (c) $7/16$ (d) $5/16$

5. If $f(x)=\cos x, 0\le x\le\frac{\pi}{2}$, then the real number 'c' of the mean value theorem is **[MP PET 1994]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\sin^{-1}\left(\frac{2}{\pi}\right)$ (d) $\cos^{-1}\left(\frac{2}{\pi}\right)$

6. Let $f(x)$ satisfy all the conditions of mean value theorem in [0, 2]. If $f(0)=0$ and $|f'(x)|\le\frac{1}{2}$ for all x in [0, 2], then

(a) $f(x)\le 2$ (b) $|f(x)|\le 1$

(c) $f(x)=2x$ (d) $f(x)=3$ for at least one x in [0, 2]

7. The function $f(x)=(x-3)^2$ satisfies all the conditions of mean value theorem in [3, 4]. A point on $y=(x-3)^2$, where the tangent is parallel to the chord joining (3, 0) and (4, 1) is

(a) $\left(\frac{7}{2},\frac{1}{2}\right)$ (b) $\left(\frac{7}{2},\frac{1}{4}\right)$ (c) $(1,4)$ (d) (4, 1)

8. Let $f(x)$ and $g(x)$ are defined and differentiable for $x \geq x_0$ and $f(x_0)=g(x_0), f'(x)>g'(x)$ for $x > x_0$, then

(a) $f(x)< g(x)$ for some $x > x_0$

(b) $f(x)= g(x)$ for some $x > x_0$

(c) $f(x)> g(x)$ for all $x > x_0$

(d) None of these

9. Let *f* be differentiable for all *x*. If $f(1)=-2$ and $f'(x)\geq 2$ for all $x \in [1,6]$ then

(a) $f(6)<8$ (b) $f(6)\geq 8$ (c) $f(6)\geq 5$ (d) $f(6)\leq 5$

10. The value of *c* in Lagrange's theorem for the function $|x|$ in the interval [– 1, 1] is

(a) 0 (b) 1/2 (c) –1/2 (d) Non-existent in the interval

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Answer Sheet

Assignment (Basic and Advance Level)

1	2	3	4	5	6	7	8	9	10
c	b	b	b	c	b	b	c	b	d