CHAPTER 4

Methods of Differentiation

- Geometrical Meaning of a Derivative
- Standard Derivatives
- Differentiation of Inverse Trigonometric Functions
- Theorems on Derivatives
- Differentiation of Composite Functions (Chain Rule)
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GEOMETRICAL MEANING OF A DERIVATIVE

The essence of calculus is the **derivative**. The derivative is the instantaneous rate of change of a function with respect to one of its variables. This is equivalent to finding the slope of the **tangent line** to the function at a point. Let us use the view of derivatives as tangents to motivate a geometric definition of the derivative (Fig. 4.1).

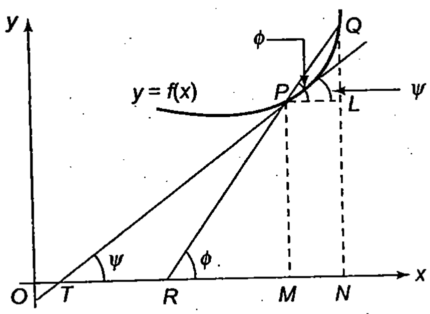

Fig. 4.1

Let $P(x_0, f(x_0))$ and $Q(x_0+h, f(x_0+h))$ be two points very close to each other on the curve $y = f(x)$. Draw PM and QN perpendiculars from P and Q on x-axis, and draw PL as perpendicular from P on QN. Let the chord PQ produced meet the x-axis at R and $\angle QPL = \angle QRN = \phi$.
Now in right-angled triangle QLP,

$$\tan\phi = \frac{QL}{PL} = \frac{NQ-NL}{MN} = \frac{NQ-MP}{ON-OM}$$

$$= \frac{f(x_0+h)-f(x_0)}{(x_0+h)-x_0}$$

$$= \frac{f(x_0+h)-f(x_0)}{h} \quad (1)$$

when $h \to 0$, the point Q moving along the curve tends to P, i.e., $Q \to P$. The chord PQ approaches the tangent line PT at the point P and then $\phi \to \psi$. Now, applying $\lim\limits_{h\to 0}$ in equation (1), we get

$$\lim_{h\to 0}\tan\phi = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

$$\Rightarrow \quad \tan\psi = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

$$\text{or} \quad f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

This definition of derivative is also called the **first principle of derivative.** Clearly, the domain of definition of $f'(x)$ is wherever the above limit exists.

Example 4.1 Find the derivative of $e^{\sqrt{x}}$ w.r.t. x using the first principle.

Sol. Let $f(x) = e^{\sqrt{x}}$, then $f(x+h) = e^{\sqrt{x+h}}$

$$\therefore \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{e^{\sqrt{x+h}} - e^{\sqrt{x}}}{h}$$

$$= e^{\sqrt{x}}\lim_{h\to 0}\left(\frac{e^{\sqrt{x+h}-\sqrt{x}}-1}{h}\right)$$

$$= e^{\sqrt{x}}\lim_{h\to 0}\left(\frac{e^{\sqrt{x+h}-\sqrt{x}}-1}{\sqrt{x+h}-\sqrt{x}}\right)\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)$$

$$= e^{\sqrt{x}}\lim_{h\to 0}\left(\frac{e^{\sqrt{x+h}-\sqrt{x}}-1}{\sqrt{x+h}-\sqrt{x}}\right)\times \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= e^{\sqrt{x}}\lim_{h\to 0}\left(\frac{e^{y}-1}{y}\right)\lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})},$$

where $y = \sqrt{x+h} - \sqrt{x}$ ($\because$ when $h \to 0$, $y \to 0$)

$$= e^{\sqrt{x}}\times 1\times\left(\frac{1}{\sqrt{x}+\sqrt{x}}\right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

Example 4.2 If $f(x) = x\tan^{-1}x$, find $f'(\sqrt{3})$ using the first principle.

Sol. We have $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$\therefore f'(\sqrt{3}) = \lim_{h\to 0}\frac{f(\sqrt{3}+h)-f(\sqrt{3})}{h}$$

$$= \lim_{h\to 0}\frac{(\sqrt{3}+h)\tan^{-1}(\sqrt{3}+h)-\sqrt{3}\tan^{-1}\sqrt{3}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{3}\left[\tan^{-1}(\sqrt{3}+h)-\tan^{-1}\sqrt{3}\right]+h\tan^{-1}(\sqrt{3}+h)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{3}}{h}\tan^{-1}\left(\frac{\sqrt{3}+h-\sqrt{3}}{1+\sqrt{3}(\sqrt{3}+h)}\right)+\lim_{h\to 0}\tan^{-1}(\sqrt{3}+h)$$

$$= \sqrt{3}\lim_{h\to 0}\left\{\frac{\tan^{-1}\left(\frac{h}{4+\sqrt{3}h}\right)}{\frac{h}{4+\sqrt{3}h}}\right\}\frac{1}{4+\sqrt{3}h}+\lim_{h\to 0}\tan^{-1}(\sqrt{3}+h)$$

$$\Rightarrow f'(\sqrt{3}) = \sqrt{3}\times 1\times\frac{1}{4}+\tan^{-1}\sqrt{3}$$

$$= \frac{\sqrt{3}}{4}+\tan^{-1}\sqrt{3}$$

Example 4.3 Find the derivative of $\sqrt{4-x}$ w.r.t. x using the first principle.

Sol. Let $f(x)=\sqrt{4-x}$, then $f(x+h)=\sqrt{4-(x+h)}$

$$\therefore \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{4-(x+h)}-\sqrt{4-x}}{h}$$

$$= \lim_{h\to 0}\frac{\left\{\sqrt{4-(x+h)}-\sqrt{4-x}\right\}\left\{\sqrt{4-(x+h)}+\sqrt{4-x}\right\}}{h\left\{\sqrt{4-(x+h)}+\sqrt{4-x}\right\}}$$

$$= \lim_{h\to 0}\frac{4-(x+h)-(4-x)}{h\left\{\sqrt{4-(x+h)}+\sqrt{4-x}\right\}}$$

$$= \lim_{h\to 0}\frac{-h}{h\left\{\sqrt{4-x-h}+\sqrt{4-x}\right\}} = \frac{-1}{2\sqrt{4-x}}$$

Example 4.4 Using the first principle, prove that

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right)=\frac{-f'(x)}{[f(x)]^2}.$$

Sol. Let $\phi=\frac{1}{f(x)}$, then $\phi(x+h)=\frac{1}{f(x+h)}$

$$\therefore \frac{d}{dx}(\phi(x)) = \lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{f(x+h)}-\frac{1}{f(x)}}{h}$$

$$= \lim_{h\to 0}\frac{f(x)-f(x+h)}{hf(x)f(x+h)}$$

$$= \lim_{h\to 0}\frac{f(x)-f(x+h)}{h}\ \lim_{h\to 0}\frac{1}{f(x)f(x+h)}$$

$$= -f'(x)\frac{1}{f(x)f(x)}$$

[$f(x)$ is differentiable $\Rightarrow f(x)$ is continuous $\Rightarrow \lim_{h\to 0} f(x+h)=f(x)$]

$$= \frac{-f'(x)}{\{f(x)\}^2}$$

Concept Application Exercise 4.1

1. Differentiate the following functions with respect to x using the first principle:
   a. $\sqrt{\sin x}$ **b.** $\cos^2 x$
   c. $\tan^{-1} x$ **d.** $\log_e x$
2. Using the first principle, prove that
$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x))+g(x)\frac{d}{dx}(f(x)).$$

STANDARD DERIVATIVES

a. $\frac{d}{dx}x^n = nx^{n-1},\ x\in R,\ n\in R, x>0$

b. $\frac{d}{dx}(e^x)=e^x$

c. $\frac{d}{dx}(a^x)=a^x \ln a$

d. $\frac{d}{dx}(\ln|x|)=\frac{1}{x}$

e. $\frac{d}{dx}(\log_a x)=\frac{1}{x}\log_a e$

f. $\frac{d}{dx}(\sin x)=\cos x$

g. $\frac{d}{dx}(\cos x)=-\sin x$

h. $\frac{d}{dx}(\tan x)=\sec^2 x$

i. $\frac{d}{dx}(\cot x)=-\text{cosec}^2 x$

j. $\frac{d}{dx}(\sec x)=\sec x\tan x$

k. $\frac{d}{dx}(\text{cosec}\, x)=-\text{cosec}\, x\cot x$

l. $\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{(1-x^2)}}$

m. $\frac{d}{dx}(\cos^{-1} x)=\frac{-1}{\sqrt{(1-x^2)}}$

n. $\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$

o. $\frac{d}{dx}(\cot^{-1} x)=\frac{-1}{1+x^2}$

p. $\frac{d}{dx}(\sec^{-1} x)=\frac{1}{x\sqrt{(x^2-1)}}$

q. $\frac{d}{dx}(\text{cosec}^{-1} x)=\frac{-1}{x\sqrt{(x^2-1)}}$

Some Standard Substitutions

Expression	Substitution
$\sqrt{a^2-x^2}$	$x=a\sin\theta$ or $a\cos\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$ or $a\cot\theta$

$\sqrt{x^2 - a^2}$ $\quad x = a\sec\theta$ or a cosecθ

$\sqrt{\dfrac{a+x}{a-x}}$ or $\sqrt{\dfrac{a-x}{a+x}}$ $\quad x = a\cos\theta$ or $a\cos 2\theta$

Example 4.5 If $y = (1 + x^{1/4})(1 + x^{1/2})(1 - x^{1/4})$, then find $\dfrac{dy}{dx}$.

Sol. $y = (1 + x^{1/4})(1 + x^{1/2})(1 - x^{1/4})$

$= (1 + x^{1/4})(1 - x^{1/4})(1 + x^{1/2})$

$= (1 - x^{1/2})(1 + x^{1/2})$

$= 1 - x$

$\Rightarrow \dfrac{dy}{dx} = -1$

Example 4.6 If $f(x) = x|x|$, then prove that $f'(x) = 2|x|$.

Sol. $f(x) = \begin{cases} -x^2, & x < 0 \\ x^2, & x \ge 0 \end{cases}$

$\Rightarrow f'(x) = \begin{cases} -2x, & x < 0 \\ 2x, & x > 0 \end{cases}$

$\therefore f'(x) = 2|x|$

Example 4.7 If $y = \sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$, $x \in \left(0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right)$, then find $\dfrac{dy}{dx}$.

Sol. We have

$$y = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \sqrt{\frac{2\sin^2 x}{2\cos^2 x}} = \sqrt{\tan^2 x}$$

$$\Rightarrow y = |\tan x|, \text{ where } x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$$

$$\Rightarrow y = \begin{cases} \tan x, & x \in \left(0, \frac{\pi}{2}\right) \\ -\tan x, & x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} \sec^2 x, & \text{if } x \in \left(0, \frac{\pi}{2}\right) \\ -\sec^2 x, & \text{if } x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

Example 4.8 If $y = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!}$, then show that $\dfrac{dy}{dx} - y + \dfrac{x^n}{n!} = 0$.

Sol. $$\frac{dy}{dx} = 0 + \frac{1}{1!} + \frac{1}{2!}(2x) + \frac{1}{3!}(3x^2) + \cdots + \frac{1}{n!}(nx^{n-1})$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!}$$

$$= \left\{1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}\right\} - \frac{x^n}{n!}$$

$$= y - \frac{x^n}{n!}$$

$$\Rightarrow \frac{dy}{dx} - y + \frac{x^n}{n!} = 0$$

DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

Example 4.9 Find $\dfrac{dy}{dx}$ for $y = \sin^{-1}(\cos x)$, where $x \in (0, 2\pi)$.

Sol. We have

$$\sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x)$$

$$= \begin{cases} \frac{\pi}{2} - x, & \text{if } 0 < x \le \pi \\ \frac{\pi}{2} - (2\pi - x), & \text{if } \pi < x < 2\pi \end{cases}$$

$$= \begin{cases} \frac{\pi}{2} - x, & \text{if } 0 < x \le \pi \\ x - \frac{3\pi}{2}, & \text{if } \pi < x < 2\pi \end{cases}$$

Clearly, it is not differentiable at $x = \pi$. Therefore,

$$\frac{d}{dx}\{\sin^{-1}(\cos x)\} = \begin{cases} -1, & \text{if } 0 < x < \pi \\ 1, & \text{if } \pi < x < 2\pi \end{cases}$$

Example 4.10 Differentiate $\sin^{-1}(2x\sqrt{1-x^2})$ with respect to x, if

a. $-\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$ $\qquad$ b. $\dfrac{1}{\sqrt{2}} < x < 1$

c. $-1 < x < -\dfrac{1}{\sqrt{2}}$.

Sol. Let $y = \sin^{-1}(2x\sqrt{1-x^2})$

Substituting $x = \sin\theta$, where $\theta = \sin^{-1}x$, and $\theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$,

we get $y = \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta)$

a. $-\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}} \Rightarrow -\dfrac{\pi}{4} < \theta < \dfrac{\pi}{4} \Rightarrow -\dfrac{\pi}{2} < 2\theta < \dfrac{\pi}{2}$

$\Rightarrow y = \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1}x$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2}{\sqrt{1-x^2}}$

b. $\dfrac{1}{\sqrt{2}} < x < 1 \Rightarrow \dfrac{\pi}{4} < \theta < \dfrac{\pi}{2} \Rightarrow \dfrac{\pi}{2} < 2\theta < \pi$

$\Rightarrow y = \sin^{-1}(\sin 2\theta) = \sin^{-1}(\sin(\pi - 2\theta)) = \pi - 2\theta$

$\Rightarrow y = \pi - 2\sin^{-1}x$

$\Rightarrow \dfrac{dy}{dx} = 0 - \dfrac{2}{\sqrt{1-x^2}} = -\dfrac{2}{\sqrt{1-x^2}}$

c. $-1 < x < -\dfrac{1}{\sqrt{2}} \Rightarrow -\dfrac{\pi}{2} < \theta < -\dfrac{\pi}{4} \Rightarrow -\pi < 2\theta < -\dfrac{\pi}{2}$

$$\Rightarrow y = \sin^{-1}(\sin 2\theta) = \sin^{-1}(-\sin(\pi + 2\theta)) = \sin^{-1}(\sin(-\pi - 2\theta)) = -\pi - 2\theta$$

$$\Rightarrow y = -\pi - 2\sin^{-1}x$$

$$\Rightarrow \frac{dy}{dx} = -0 - \frac{2}{\sqrt{1-x^2}} = -\frac{2}{\sqrt{1-x^2}}$$

Example 4.11 Find $\dfrac{dy}{dx}$ for $y = \tan^{-1}\left\{\dfrac{1-\cos x}{\sin x}\right\}$, $-\pi < x < \pi$.

Sol. $y = \tan^{-1}\left\{\dfrac{1-\cos x}{\sin x}\right\} = \tan^{-1}\left\{\dfrac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right\}$

$$= \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2} \quad \left(\because -\pi < x < \pi \Rightarrow -\frac{\pi}{2} < \frac{x}{2} < \frac{\pi}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}$$

Example 4.12 If $y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$, and $0 < x < 1$, then find $\dfrac{dy}{dx}$.

Sol. $y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$, where $0 < x < 1$

$$= \sin^{-1}[x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}]$$

$$= \sin^{-1}x - \sin^{-1}\sqrt{x}$$

$[\because \sin^{-1}x - \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})]$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$$

Example 4.13 Find $\dfrac{dy}{dx}$ for $y = \tan^{-1}\sqrt{\dfrac{a-x}{a+x}}$, $-a < x < a$.

Sol. $y = \tan^{-1}\left\{\sqrt{\dfrac{a-x}{a+x}}\right\}$, where $-a < x < a$

Substituting $x = a\cos\theta$, we get

$$y = \tan^{-1}\left\{\sqrt{\frac{a - a\cos\theta}{a + a\cos\theta}}\right\}$$

$$= \tan^{-1}\left\{\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\}$$

$$= \tan^{-1}\left\{\sqrt{\tan^2\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\left|\tan\frac{\theta}{2}\right|$$

Also for $-a < x < a$, $-1 < \cos\theta < 1$

$$\Rightarrow \theta \in (0, \pi) \Rightarrow \frac{\theta}{2} \in \left(0, \frac{\pi}{2}\right)$$

$$\therefore y = \tan^{-1}\left|\tan\frac{\theta}{2}\right| = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2} \times \frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\frac{d}{dx}\left(\frac{x}{a}\right) = -\frac{1}{2\sqrt{a^2-x^2}}$$

Concept Application Exercise 4.2

Find $\dfrac{dy}{dx}$ for the following functions:

1. $y = \log\left\{e^x\left(\dfrac{x-2}{x+2}\right)^{3/4}\right\}$
2. $y = \sec^{-1}\left(\dfrac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\dfrac{\sqrt{x}-1}{\sqrt{x}+1}\right)$
3. $y = \tan^{-1}\dfrac{4x}{1+5x^2} + \tan^{-1}\dfrac{2+3x}{3-2x}$
4. $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$, where $x \neq 0$
5. $y = \tan^{-1}\left(\dfrac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$, where $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ and $\dfrac{a}{b}\tan x > -1$
6. $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$, where $-1 < x < 1, x \neq 0$
7. $y = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) + \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$, where $0 < x < \infty$
8. $y = \tan^{-1}\dfrac{3a^2x - x^3}{a(a^2-3x^2)}$
9. $y = \sin^{-1}\left(\dfrac{5x+12\sqrt{1-x^2}}{13}\right)$
10. $y = \tan^{-1}\left(\dfrac{x}{1+\sqrt{1-x^2}}\right)$

THEOREMS ON DERIVATIVES

a. $\dfrac{d}{dx}\{f_1(x) \pm f_2(x)\} = \dfrac{d}{dx}f_1(x) \pm \dfrac{d}{dx}f_2(x)$.

b. $\dfrac{d}{dx}(kf(x)) = k\dfrac{d}{dx}f(x)$, where k is any constant.

c. $\dfrac{d}{dx}\{f_1(x)\,f_2(x)\} = f_1(x)\dfrac{d}{dx}f_2(x) + f_2(x)\dfrac{d}{dx}f_1(x)$.

In general,

$$\frac{d}{dx}\{f_1(x).f_2(x).f_3(x)...\}=\left(\frac{d}{dx}f_1(x)\right)(f_2(x)f_3(x)...)+\left(\frac{d}{dx}f_2(x)\right)(f_1(x)f_3(x)...)+\left(\frac{d}{dx}f_3(x)\right)(f_1(x))f_2(x)...)+\cdots$$

d. $$\frac{d}{dx}\left\{\frac{f_1(x)}{f_2(x)}\right\}=\frac{f_2(x)\frac{d}{dx}f_1(x)-f_1(x)\frac{d}{dx}f_2(x)}{[f_2(x)]^2}.$$

Example 4.14 Find $\frac{dy}{dx}$ for $y=x\sin x\log x$.

Sol. We have

$$\frac{d}{dx}(x\sin x\log x)=\left\{\frac{d}{dx}(x)\right\}\sin x\log x+x\frac{d}{dx}(\sin x)\ \log x+x\sin x\frac{d}{dx}(\log x)$$

$$=1\times\sin x\times\log x+x\times\cos x\times\log x+x\times\sin x\times\frac{1}{x}$$

$$=\sin x\log x+x\cos x\log x+\sin x$$

Example 4.15 If $y=\sqrt{\frac{1-x}{1+x}}$, prove that $(1-x^2)\frac{dy}{dx}+y=0$.

Sol. We have

$$y=\sqrt{\frac{1-x}{1+x}}$$

Differentiating w. r. t. x, we get

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{(1/2)-1}\frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

$$=\frac{1}{2}\sqrt{\frac{1+x}{1-x}}\frac{(1+x)\frac{d}{dx}(1-x)-(1-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$=\frac{1}{2}\sqrt{\frac{1+x}{1-x}}\frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}$$

$$=-\sqrt{\frac{1+x}{1-x}}\frac{1}{(1+x)^2}$$

$$\Rightarrow(1-x^2)\frac{dy}{dx}=-\sqrt{\frac{1+x}{1-x}}\frac{1}{(1+x)^2}(1-x^2)$$

$$\Rightarrow(1-x^2)\frac{dy}{dx}=-\sqrt{\frac{1-x}{1+x}}$$

$$\Rightarrow(1-x^2)\frac{dy}{dx}=-y$$

$$\Rightarrow(1-x^2)\frac{dy}{dx}+y=0$$

Example 4.16 Find the sum of the series $1+2x+3x^2+(n-1)x^{n-2}$ using differentiation.

Sol. We know that $1+x+x^2+\cdots+x^{n-1}=\frac{1-x^n}{1-x}$

Differentiating both sides w.r.t x, we get

$$0+1+2x+3x^2+\cdots+(n-1)x^{n-2}=\frac{(1-x)\frac{d}{dx}(1-x^n)-(1-x^n)\frac{d}{dx}(1-x)}{(1-x)^2}$$

$$\Rightarrow 1+2x+3x^2+\cdots+(n-1)x^{n-2}=\frac{-(1-x)nx^{n-1}+(1-x^n)}{(1-x)^2}$$

$$\Rightarrow 1+2x+3x^2+\cdots+(n-1)x^{n-2}=\frac{-nx^{n-1}+(n-1)x^n+1}{(1-x)^2}$$

Example 4.17 If $\sin y=x\cos(a+y)$, show that $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\cos a}$, and find the value of dy/dx at $x=0$.

Sol. We have

$x=\sin y/\cos(a+y)$ (1)

Differentiating w.r.t. y, we get

$$\Rightarrow\frac{dx}{dy}=\frac{\cos y\cos(a+y)+\sin y\sin(a+y)}{\cos^2(a+y)}$$

$$\Rightarrow\frac{dx}{dy}=\frac{\cos(a+y-y)}{\cos^2(a+y)}$$

$$\Rightarrow\frac{dy}{dx}=\frac{\cos^2(a+y)}{\cos a}$$

Putting $x=0$ in equation (1), $\sin y=0\Rightarrow y=n\pi,\ \forall\ n\in I$

$$\therefore\left[\frac{dy}{dx}\right]_{x=0}=\frac{\cos^2(a+n\pi)}{\cos a}=\frac{\cos^2 a}{\cos a}=\cos a$$

DIFFERENTIATION OF COMPOSITE FUNCTIONS (CHAIN RULE)

If $f(x)$ and $g(x)$ are two differentiable functions, then fog is also differentiable, and $(fog)'(x)=f'(g(x)).g'(x)$

$$\text{or, }\frac{d}{dx}\{(fog)(x)\}=\frac{d}{d\,g(x)}\{(fog)(x)\}\frac{d}{dx}(g(x)).$$

or

If y is a function of t and t is a function of x, then

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$$

Thus, if $y=f(t)$ and $t=\phi(x)$, then $\frac{dy}{dx}=f'(t)$, and

$$\frac{dt}{dx}=\phi'(x)$$

$\therefore\ \frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=f'(t)\phi'(x)$. This rule is called **Chain Rule.**

This chain rule can be extended as follows:
Let $y=f(t), t=\phi(z), z=\psi(x)$,

then $\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dz}\times\frac{dz}{dx}=f'(t)\,\phi'(z)\,\psi'(x)$

Let $y=\log\sin x^3=\log t$

Putting $t=\sin x^3=\sin z$, and putting $z=x^3$

$$\Rightarrow \frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dz}\times\frac{dz}{dx}=(1/t)\cos z\, 3x^2$$
$$=(1/\sin x^3)(\cos x^3)\times 3x^2=3x^2\cot x^3$$

Example 4.18 Find $\frac{dy}{dx}$ for $y=\sin(x^2+1)$.

Sol. Let $y=\sin(x^2+1)$.
Putting $u=x^2+1$, we get $y=\sin u$

$$\therefore \frac{dy}{du}=\cos u \text{ and } \frac{du}{dx}=2x$$

Now, $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$$\Rightarrow \frac{dy}{dx}=\cos u\, 2x=2x\cos(x^2+1)$$

Example 4.19 If $y=\sqrt{\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}$, then find $\frac{dy}{dx}$.

Sol. $y=\sqrt{\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}$

Putting $\frac{x^2}{3}-1=v$, we get $\sin\left(\frac{x^2}{3}-1\right)=\sin v=u$, and

$\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}=\log u=z$,

we get $y=\sqrt{z}$, $z=\log u$, $u=\sin v$ and $v=\frac{x^2}{3}-1$

$$\therefore \frac{dy}{dz}=\frac{1}{2\sqrt{z}}, \frac{dz}{du}=\frac{1}{u}, \frac{du}{dv}=\cos v \text{ and } \frac{dv}{dx}=\frac{2x}{3}$$

Now, $\frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{du}\times\frac{du}{dv}\times\frac{dv}{dx}$

$$\Rightarrow \frac{dy}{dx}=\left(\frac{1}{2\sqrt{z}}\right)\left(\frac{1}{u}\right)(\cos v)\left(\frac{2x}{3}\right)=\frac{x}{3}\cdot\frac{\cos v}{u\sqrt{\log u}}$$

$$=\frac{x\cot\left(\frac{x^2}{3}-1\right)}{3\sqrt{\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}}$$

Example 4.20 Find $\frac{dy}{dx}$ for $y=\log(x+\sqrt{a^2+x^2})$.

Sol. $y=\log(x+\sqrt{a^2+x^2})$

Then $\frac{dy}{dx}=\frac{d}{dx}\{\log(x+\sqrt{a^2+x^2})\}$

$$=\frac{1}{x+\sqrt{a^2+x^2}}\frac{d}{dx}(x+\sqrt{a^2+x^2})$$

$$=\frac{1}{x+\sqrt{a^2+x^2}}\left\{1+\frac{1}{2}(a^2+x^2)^{-1/2}\times\frac{d}{dx}(a^2+x^2)\right\}$$

$$=\frac{1}{x+\sqrt{a^2+x^2}}\left\{1+\frac{1}{2\sqrt{a^2+x^2}}\times 2x\right\}$$

$$=\frac{1}{x+\sqrt{a^2+x^2}}\times\frac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}}$$

$$=\frac{1}{\sqrt{a^2+x^2}}$$

Concept Application Exercise 4.3

Find $\frac{dy}{dx}$ for the following functions:

1. $y=\sin^{-1}\sqrt{(1-x)}+\cos^{-1}\sqrt{x}$
2. $y=\sqrt{\sin\sqrt{x}}$
3. $y=e^{\sin x^2}$
4. $y=\log\sqrt{\sin\sqrt{e^x}}$
5. $y=a^{(\sin^{-1}x)^2}$
6. $y=\log_e\sqrt{\frac{1+\sin x}{1-\sin x}}$, where $x=\pi/3$
7. If $y=(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})$, then find $\frac{dy}{dx}$ at $x=0$.
8. If $x^y=e^{x-y}$, prove that $\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$.
9. If $x\sqrt{1+y}+y\sqrt{1+x}=0$, prove that $\frac{dy}{dx}=-\frac{1}{(x+1)^2}$.

DIFFERENTIATION OF IMPLICIT FUNCTIONS

If variables x and y are connected by a relation of the form $f(x,y) = 0$ and it is not possible or convenient to express y as a function x, i.e., in the form $y = \phi(x)$, then y is said to be an implicit function of x. To find $\frac{dy}{dx}$ in such a case, we differentiate both sides of the given relation with respect to x, keeping in mind that the derivative of $\phi(y)$ with respect to x is $\frac{d\phi}{dy} \times \frac{dy}{dx}$.

For example,

$$\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}, \quad \frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx}$$

It should be noted that $\frac{d}{dy}(\sin y) = \cos y$

but $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$

Similarly, we have $\frac{d}{dy}\left(y^3\right) = 3y^2$,

whereas $\frac{d}{dx}\left(y^3\right) = 3y^2\frac{dy}{dx}$

A direct formula for implicit functions

Let $f(x, y) = 0$. Take all the terms towards left side and put the left side equal to $f(x, y)$.

$$\text{Then } \frac{dy}{dx} = -\frac{\text{differentiation of } f \text{ w.r.t. } x \text{ keeping } y \text{ as constant}}{\text{differentiation of } f \text{ w.r.t. } y \text{ keeping } x \text{ as constant}}$$

Example 4.21 If $x^2+2xy+y^3=4$, find $\frac{dy}{dx}$.

Sol. We have

$x^2+2xy+y^3=4$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\left(x^2\right)+2\frac{d}{dx}(xy)+\frac{d}{dx}\left(y^3\right) = \frac{d}{dx}(4)$$

$$\Rightarrow 2x+2\left(x\frac{dy}{dx}+y.1\right)+3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2(x+y)}{\left(2x+3y^2\right)}$$

Alternative method

$$\frac{dy}{dx} = -\frac{\text{differentiation of } f \text{ w.r.t. } x \text{ keeping } y \text{ as constant}}{\text{differentiation of } f \text{ w.r.t. } y \text{ keeping } x \text{ as constant}}$$

$$= -\frac{2x+2y}{2x+3y^2}$$

Example 4.22 If $y = x + \cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\cdots}}}$

prove that $\frac{dy}{dx} = \frac{y}{2y-x}$.

Sol. We have

$$y = x + \cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\cdots}}}$$

$$\Rightarrow y = x+\frac{1}{y}$$

$$\Rightarrow y^2 = xy+1$$

$$\Rightarrow 2y\frac{dy}{dx} = y+x\frac{dy}{dx}+0 \quad \text{[Differentiating both sides w.r.t. } x\text{]}$$

$$\Rightarrow \frac{dy}{dx}(2y-x) = y$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2y-x}$$

Example 4.23 If $\sqrt{x}+\sqrt{y} = 4$, then find $\frac{dx}{dy}$ at $y = 1$.

Sol. Differentiating both sides of the given equation w.r.t. y, we get

$$\frac{1}{2\sqrt{x}}\frac{dx}{dy}+\frac{1}{2\sqrt{y}} = 0$$

$$\Rightarrow \frac{dx}{dy} = -\frac{\sqrt{x}}{\sqrt{y}} = \frac{\sqrt{y}-4}{\sqrt{y}}$$

$$\Rightarrow \left[\frac{dx}{dy}\right]_{y=1} = \frac{1-4}{1} = -3$$

Example 4.24 If $y = \sqrt{x \log_e x}$, then find $\frac{dy}{dx}$ at $x = e$.

Sol. $$\frac{dy}{dx} = \frac{1}{2\sqrt{x\log_e x}}\frac{d}{dx}[x\log_e x]$$

$$= \frac{1}{2\sqrt{x\log_e x}}\left[x \times \frac{1}{x}+1\times\log_e x\right]$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=e} = \frac{1}{2\sqrt{e\times 1}}(1+1) = \frac{1}{\sqrt{e}} \quad (\because \log_e e = 1)$$

Concept Application Exercise 4.4

1. If $x^3+y^3=3axy$, find $\frac{dy}{dx}$.
2. If $\log(x^2+y^2)=2\tan^{-1}\left(\frac{y}{x}\right)$, show that $\frac{dy}{dx}=\frac{x+y}{x-y}$.
3. If $y=\sqrt{\sin x+y}$, then find $\frac{dy}{dx}$.
4. If $x=y\sqrt{1-y^2}$, then find $\frac{dy}{dx}$ in terms of y.
5. If $y=b\tan^{-1}\left(\frac{x}{a}+\tan^{-1}\frac{y}{x}\right)$, find $\frac{dy}{dx}$.
6. If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\cdots \text{to } \infty}}}$, prove that $\frac{dy}{dx}=\frac{\cos x}{2y-1}$.

DIFFERENTIATION OF FUNCTIONS IN PARAMETRIC FORM

Sometimes, x and y are given as functions of a single variable, i.e., $x=\phi(t)$, $y=\psi(t)$ are two functions and t is a variable. In such a case x and y are called parametric functions or parametric equations and t is called the parameter. To find $\frac{dy}{dx}$ in case of parametric functions, we first obtain the relationship between x and y by eliminating the parameter t and then we differentiate it with respect to x. But every time it is not convenient to eliminate the parameter. Therefore, $\frac{dy}{dx}$ can also be obtained by the following formula

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

Example 4.25 Find $\frac{dy}{dx}$ if $x=a(\theta-\sin\theta)$ and $y=a(1-\cos\theta)$.

Sol. We have, $x=a(\theta-\sin\theta)$ and $y=a(1-\cos\theta)$

$$\Rightarrow \frac{dx}{d\theta}=a(1-\cos\theta) \text{ and } \frac{dy}{d\theta}=a\sin\theta$$

$$\Rightarrow \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}$$

$$=\frac{a\sin\theta}{a(1-\cos\theta)}=\frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}=\cot\frac{\theta}{2}$$

Example 4.26 If $x=a\sec^3\theta$ and $y=a\tan^3\theta$, find $\frac{dy}{dx}$ at $\theta=\frac{\pi}{3}$.

Sol. We have $x=a\sec^3\theta$ and $y=a\tan^3\theta$

$$\frac{dx}{d\theta}=3a\sec^2\theta\frac{d}{d\theta}(\sec\theta)=3a\sec^3\theta\tan\theta$$

$$\text{and } \frac{dy}{d\theta}=3a\tan^2\theta\frac{d}{d\theta}(\tan\theta)=3a\tan^2\theta\sec^2\theta$$

$$\Rightarrow \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta}=\frac{\tan\theta}{\sec\theta}=\sin\theta$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\pi/3}=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$$

Example 4.27 Let $y=x^3-8x+7$ and $x=f(t)$. If $\frac{dy}{dt}=2$ and $x=3$ at $t=0$, then find the value of $\frac{dx}{dt}$ at $t=0$.

Sol. We have $y=x^3-8x+7$

$$\Rightarrow \frac{dy}{dx}=3x^2-8$$

It is given that when $t=0$, $x=3$.

$\therefore$ when $t=0$, $\frac{dy}{dx}=3.3^2-8=19$.

Also, $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$ (1)

Since, when $t=0$, $\frac{dy}{dx}=19$ and $\frac{dy}{dt}=2$,

$\therefore$ from (1), $19=\frac{2}{dx/dt}$

$$\Rightarrow \frac{dx}{dt}=\frac{2}{19}$$

Concept Application Exercise 4.5

1. If $x=\frac{2t}{1+t^2}$, $y=\frac{1-t^2}{1+t^2}$ then find $\frac{dy}{dx}$ at $t=2$.
2. If $x=a\cos^3\theta$, $y=b\sin^3\theta$, find $\frac{d^3y}{dx^3}$ at $\theta=0$.
3. If $x=\sqrt{a^{\sin^{-1}t}}$, $y=\sqrt{a^{\cos^{-1}t}}$, $a>0$ and $-1<t<1$, show that $\frac{dy}{dx}=-\frac{y}{x}$.
4. Find $\frac{dy}{dx}$ at $x=\pi/4$ for $x=a\left[\cos t+\frac{1}{2}\log\tan^2\frac{t}{2}\right]$ and $y=a\sin t$.

DIFFERENTIATION USING LOGARITHM

If $y = [f_1(x)]^{f_2(x)}$ or $y = f_1(x)\, f_2(x)\, f_3(x) \ldots$

or $y = \dfrac{f_1(x) f_2(x) f_3(x)\ldots}{g_1(x) g_2(x) g_3(x)\ldots}$

then it is convenient to take the logarithm of the function first and then differentiate.

Note: *Write $y = [f(x)]^{g(x)} = e^{g(x)\ln(f(x))}$ and differentiate easily or if $y = [f(x)]^{g(x)}$, then $\dfrac{dy}{dx}$ = Differential of y treating f(x) as constant + Differential of y treating g(x) as constant.*

For example, if $y = (\sin x)^{\log \cos x}$, find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = \text{(diff. of } y \text{ keeping base } \sin x \text{ as constant)}$$

$$+ \text{(diff. of } y \text{ keeping power } \log \cos x \text{ as constant)}$$

$$= (\sin x)^{\log\cos x} \log \sin x \frac{1}{\cos x}(-\sin x)$$

$$+ \log(\cos x).\ (\sin x)^{(\log\cos x - 1)} \cos x.$$

Example 4.28 If $x^m y^n = (x+y)^{m+n}$, prove that $\dfrac{dy}{dx} = \dfrac{y}{x}$.

Sol. We have $x^m y^n = (x+y)^{m+n}$.

Taking log on both sides, we get

$m \log x + n \log y = (m+n)\log(x+y)$

Differentiating both sides w.r.t. x, we get

$$m\frac{1}{x} + n\frac{1}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\frac{d}{dx}(x+y)$$

$$\Rightarrow \left(\frac{m}{x} + \frac{n}{y}\right)\frac{dy}{dx} = \frac{m+n}{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \left\{\frac{n}{y} - \frac{m+n}{x+y}\right\}\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \left\{\frac{nx+ny-my-ny}{y(x+y)}\right\}\frac{dy}{dx} = \left\{\frac{mx+nx-mx-my}{(x+y)x}\right\}$$

$$\Rightarrow \frac{nx-my}{y(x+y)}\frac{dy}{dx} = \frac{nx-my}{(x+y)x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

Example 4.29 Find $\dfrac{dy}{dx}$ for $y = (\sin x)^{\log x}$.

Sol. Let $y = (\sin x)^{\log x}$.

Then, $y = e^{\log x \log \sin x}$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{\log x \log \sin x}\frac{d}{dx}\{\log x \log \sin x\}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x}$$

$$\times \left\{\log \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\log \sin x)\right\}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x}\left\{\frac{\log \sin x}{x} + \log x \frac{1}{\sin x}\cos x\right\}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x}\left\{\frac{\log \sin x}{x} + \cot x \log x\right\}$$

Example 4.30 If $y = x^{x^{x^{\cdots\infty}}}$, find $\dfrac{dy}{dx}$.

Sol. Since by deleting a single term from an infinite series, it remains same.

Therefore, the given function may be written as

$y = x^y$

$\Rightarrow \log y = y \log x$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx} \times \log x + y\frac{1}{x} \qquad \text{[Diff. both sides w.r.t. } x\text{]}$$

$$\Rightarrow \frac{dy}{dx}\frac{\{1 - y\log x\}}{y} = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)}$$

Example 4.31 Find the derivative of $\dfrac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}$ w.r.t. x.

Sol. Let $y = \dfrac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}$

Taking log of both sides, we get

$$\log y = \frac{1}{2}\log x + \frac{3}{2}\log(x+4) - \frac{4}{3}\log(4x-3)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2x} + \frac{3}{2}\frac{1}{x+4} - \frac{4}{3} \times \frac{1}{4x-3} \times 4$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{1}{2x} + \frac{3}{2(x+4)} - \frac{16}{3(4x-3)}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}\left\{\frac{1}{2x} + \frac{3}{2(x+4)} - \frac{16}{3(4x-3)}\right\}$$

Use of logarithm helps in finding the sum of special series given in the following examples.

Example 4.32 If $x < 1$, prove that

$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \cdots\infty = \frac{1}{1-x}.$$

Sol. The given series is in the form

$$\frac{f_1'(x)}{f_1(x)}+\frac{f_2'(x)}{f_2(x)}+\frac{f_3'(x)}{f_3(x)}+\ldots\infty$$

Then consider the product $f_1(x)\times f_2(x)\times f_3(x)\ldots f_n(x)$

Now $(1-x)(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^{n-1}})$ (1)

$=(1-x^2)(1+x^2)(1+x^4)\ldots(1+x^{2^{n-1}})$

$=(1-x^4)(1+x^4)\ldots(1+x^{2^{n-1}})$

$\vdots$

$=(1-x^{2^{n-1}})(1+x^{2^{n-1}})$

$=1-x^{2^n}$

Now when $n\to\infty$, $x^{2^{n-1}}\to 0$ ($\because x<1$)

$\therefore$ taking $n\to\infty$, in (1), we get

$(1-x)(1+x)(1+x^2)(1+x^4)\ldots=1$

Taking logarithm, we get

$\log(1-x)+\log(1+x)+\log(1+x^2)+\log(1+x^4)+\ldots=0$

Differentiating w.r.t. 'x', we get

$$-\frac{1}{1-x}+\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\ldots=0$$

$$\text{or}\quad\frac{1}{x+1}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\ldots\infty=\frac{1}{1-x}$$

Concept Application Exercise 4.6

1. Find $\frac{dy}{dx}$ for $y=x^x$.
2. If $y^x=x^y$, then find $\frac{dy}{dx}$.
3. If $x=e^{y+e^{y+\ldots\text{to}\infty}}$, where $x>0$, then find $\frac{dy}{dx}$.
4. If $y=(\tan x)^{(\tan x)^{\tan x}}$, then find $\frac{dy}{dx}$ at $x=\pi/4$.
5. If $y=\frac{\sqrt{1-x^2}(2x+3)^{1/2}}{(x^2+2)^{2/3}}$, find $\frac{dy}{dx}$ at $x=0$.

DIFFERENTIATION OF ONE FUNCTION W.R.T. OTHER FUNCTION

Let $u=f(x)$ and $v=g(x)$ be the two functions of x. Then to find the derivative of $f(x)$ w.r.t. $g(x)$, i.e., to find $\frac{du}{dv}$, we use the formula: $\frac{du}{dv}=\frac{du/dx}{dv/dx}$.

Thus, to find the derivative of $f(x)$ w.r.t. $g(x)$, we first differentiate both w.r.t. x. and then divide the derivative of $f(x)$ w.r.t. x by the derivative of $g(x)$ w.r.t. x.

Example 4.33 Differentiate $\log\sin x$ w.r.t. $\sqrt{\cos x}$.

Sol. Let $u=\log\sin x$ and $v=\sqrt{\cos x}$

Then, $\frac{du}{dx}=\cot x$ and $\frac{dv}{dx}=-\frac{\sin x}{2\sqrt{\cos x}}$

$$\Rightarrow\frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{\cot x}{-\frac{\sin x}{2\sqrt{\cos x}}}$$

$$=-2\sqrt{\cos x}\cot x\,\text{cosec}\,x$$

Example 4.34 Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\tan^{-1}x$, where $x\neq 0$.

Sol. Let $u=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v=\tan^{-1}x$.

Putting $x=\tan\theta$,

we get $u=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

$$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{1}{2}\theta=\frac{1}{2}\tan^{-1}x$$

Thus, we have $u=\frac{1}{2}\tan^{-1}x$ and $v=\tan^{-1}x$

$$\Rightarrow\frac{du}{dx}=\frac{1}{2}\times\frac{1}{1+x^2}\text{ and }\frac{dv}{dx}=\frac{1}{1+x^2}$$

$$\Rightarrow\frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{1}{2(1+x^2)}(1+x^2)=\frac{1}{2}$$

Example 4.35 Find the derivative of $f(\tan x)$ w.r.t. $g(\sec x)$ at $x=\frac{\pi}{4}$, where $f'(1)=2$ and $g'(\sqrt{2})=4$.

Sol. Let $u=f(\tan x)$ and $v=g(\sec x)$

$$\Rightarrow\frac{du}{dx}=f'(\tan x)\sec^2 x$$

$$\text{and }\frac{dv}{dx}=g'(\sec x)\sec x\tan x$$

$$\Rightarrow\frac{du}{dv}=\frac{du}{dx}\Big/\frac{dv}{dx}=\frac{f'(\tan x)\sec^2 x}{g'(\sec x)\sec x\tan x}$$

$$\Rightarrow\left[\frac{du}{dv}\right]_{x=\frac{\pi}{4}}=\frac{f'\left(\tan\frac{\pi}{4}\right)}{g'\left(\sec\frac{\pi}{4}\right)\sin\frac{\pi}{4}}=\frac{f'(1)\sqrt{2}}{g'(\sqrt{2})}$$

$$=\frac{2\sqrt{2}}{4}=\frac{1}{\sqrt{2}}$$

Concept Application Exercise 4.7

1. Find the derivative of $\tan^{-1}\dfrac{2x}{1-x^2}$ w.r.t. $\sin^{-1}\dfrac{2x}{1+x^2}$.

2. Find the derivative of $\sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$ w.r.t. $\sqrt{1-x^2}$ at $x=\dfrac{1}{2}$.

3. If $y=f(x^3)$, $z=g(x^5)$, $f'(x)=\tan x$ and $g'(x)=\sec x$, then find the value of $\lim\limits_{x\to 0}\dfrac{(dy/dz)}{x}$.

DIFFERENTIATION OF DETERMINANTS

To differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged.

For example, if

$$\Delta(x)=\begin{vmatrix} f(x) & g(x) \\ u(x) & v(x)\end{vmatrix}, \text{ then}$$

$$\frac{d}{dx}\{\Delta(x)\}=\begin{vmatrix} f'(x) & g'(x) \\ u(x) & v(x)\end{vmatrix}+\begin{vmatrix} f(x) & g(x) \\ u'(x) & v'(x)\end{vmatrix}$$

Also,

$$\frac{d}{dx}\{\Delta(x)\}=\begin{vmatrix} f'(x) & g(x) \\ u'(x) & v(x)\end{vmatrix}+\begin{vmatrix} f(x) & g'(x) \\ u(x) & v'(x)\end{vmatrix}$$

Similar results hold for the differentiation of determinants of higher order. Following examples will illustrate the same.

Example 4.36 If $f(x)=\begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2\end{vmatrix}$, then prove that $f'(x)=3x^2+2x(a^2+b^2+c^2)$.

Sol. We have

$$f(x)=\begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2\end{vmatrix}$$

$$\Rightarrow f'(x)=\begin{vmatrix} 1 & 0 & 0 \\ ab & x+b^2 & bc \\ ac & bc & x+c^2\end{vmatrix}+\begin{vmatrix} x+a^2 & ab & ac \\ 0 & 1 & 0 \\ ac & bc & x+c^2\end{vmatrix}+\begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ 0 & 0 & 1\end{vmatrix}$$

$$\Rightarrow f'(x)=\begin{vmatrix} x+b^2 & bc \\ bc & x+c^2\end{vmatrix}+\begin{vmatrix} x+a^2 & ac \\ ac & x+c^2\end{vmatrix}+\begin{vmatrix} x+a^2 & ab \\ ab & x+b^2\end{vmatrix}$$

$$\Rightarrow f'(x)=[(x+b^2)(x+c^2)-b^2c^2]+[(x+a^2)(x+c^2)-a^2c^2]+[(x+a^2)(x+b^2)-a^2b^2]$$

$$\Rightarrow f'(x)=3x^2+2x(a^2+b^2+c^2)$$

Example 4.37 If $f(x)$, $g(x)$ and $h(x)$ are three polynomials of degree 2, then prove that

$$\phi(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x)\end{vmatrix}$$ is a constant polynomial.

Sol. Let $f(x)=a_1x^2+a_2x+a_3$, $g(x)=b_1x^2+b_2x+b_3$ and $h(x)=c_1x^2+c_2x+c_3$. Then,

$f'(x)=2a_1x+a_2$, $g'(x)=2b_1x+b_2$ and $h'(x)=2c_1x+c_2$

$f''(x)=2a_1$, $g''(x)=2b_1$, $h''(x)=2c_1$ and

$f'''(x)=g'''(x)=h'''(x)=0$

In order to prove that $\phi(x)$ is a constant polynomial, it is sufficient to show that $\phi'(x)=0$ for all values of x.

where,

$$\phi(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x)\end{vmatrix}$$

$$\Rightarrow \phi'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x)\end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ f''(x) & g''(x) & h''(x) \\ f''(x) & g''(x) & h''(x)\end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f'''(x) & g'''(x) & h'''(x)\end{vmatrix}$$

$$\Rightarrow \phi'(x)=0+0+\begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ 0 & 0 & 0\end{vmatrix}$$

$\Rightarrow \phi'(x)=0+0+0=0$ for all values of x

$\Rightarrow \phi(x)=$ constant for all

Hence, $\phi(x)$ is a constant polynomial.

Concept Application Exercise 4.8

1. If $y=\begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1\end{vmatrix}$, find $\dfrac{dy}{dx}$.

2. If $f(x)=\begin{vmatrix} x^n & n! & 2 \\ \cos x & \cos\dfrac{n\pi}{2} & 4 \\ \sin x & \sin\dfrac{n\pi}{2} & 8\end{vmatrix}$, then find the value of $\dfrac{d^n}{dx^n}[f(x)]_{x=0}$

HIGHER ORDER DERIVATIVES

If $y = y(x)$, then $\frac{dy}{dx}$, the derivative of y with respect to x, is itself, in general, a function of x and can be differentiated again. We call $\frac{dy}{dx}$ as the first-order derivative of y with respect to x and the derivates of $\frac{dy}{dx}$ w.r.t. x as the second-order derivative of y w.r.t. x, and it is denoted by $\frac{d^2y}{dx^2}$. Similarly, the derivative of $\frac{d^2y}{dx^2}$ w.r.t. x is termed as the third-order derivative of y w.r.t. x and is denoted by $\frac{d^3y}{dx^3}$ and so on. The nth order derivative of y w.r.t. x is denoted by $\frac{d^n y}{dx^n}$.

If $y = f(x)$, then the other alternative notations for

$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^2y}{dx^3}, \ldots, \frac{d^n y}{dx^n}$ are

$y_1, y_2, y_3, \ldots, y_n$

$y', y'', y''', \ldots, y^{(n)}$

$f'(x), f''(x), f'''(x), \ldots, f^n(x)$

The values of nth derivatives at $x = a$ are denoted by

$$y_n(a), y^n(a), D^n y(a), f^n(a) \text{ or } \left(\frac{d^n y}{dx^n}\right)_{x=a}.$$

Example 4.38 If $y = e^{\tan^{-1}x}$, then prove that

$$(1+x^2)\frac{d^2y}{dx^2} = (1-2x)\frac{dy}{dx}.$$

Sol. $y = e^{\tan^{-1}x} \Rightarrow \frac{dy}{dx} = \frac{e^{\tan^{-1}x}}{1+x^2}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(1+x^2)\frac{e^{\tan^{-1}x}}{(1+x^2)} - e^{\tan^{-1}x}(2x)}{(1+x^2)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(1-2x)e^{\tan^{-1}x}}{(1+x^2)^2} \Rightarrow \frac{d^2y}{dx^2}(1+x^2) = (1-2x)\frac{dy}{dx}$$

Example 4.39 If $y = (x^2 - 1)^m$, then the $(2m)$th differential coefficient of y is

a. m **b.** $(2m)!$

c. $2m$ **d.** $m!$

Sol. Expanding binomially, we get

$y = (x^2-1)^m = {}^mC_0 x^{2m} + {}^mC_1 x^{2m-2}(-1) + \cdots$

So on differentiating, all the terms, except first, reduce to zero, therefore

$$\frac{d^{2m}}{dx^{2m}}(x^2-1)^m = {}^mC_0 2m(2m-1)(2m-2)\ldots 1$$

$$= (2m)!$$

Example 4.40 If $y = x\log\{x/(a+bx)\}$, then show that

$$x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2.$$

Sol. Given $y/x = [\log x - \log(a+bx)]$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x} - \frac{b}{a+bx}$$

$$\Rightarrow x\frac{dy}{dx} - y = \frac{ax}{a+bx} \quad (1)$$

Differentiating again w.r.t. x

$$\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) - \frac{dy}{dx} = \frac{a^2}{(a+bx)^2}$$

$$\therefore x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} = \left(x\frac{dy}{dx} - y\right)^2 \text{ by (1)}$$

Example 4.41 If $y = (ax + b)/(x^2 + c)$, then show that $(2xy' + y)y''' = 3(xy'' + y')y''$, where a, b, c are constants and dashes denote differentiation w.r.t. x.

Sol. Given that $y(x^2 + c) = ax + b$

Differentiating w.r.t. x

$y'(x^2 + c) + 2xy = a$ (1)

Differentiating again w.r.t. x

$y''(x^2 + c) + y'2x + 2xy' + 2y = 0$

$\Rightarrow y''(x^2 + c) + 2(2xy' + y) = 0$ (2)

Differentiating again w.r.t. x

$y'''(x^2 + c) + 2xy'' + 2(2xy'' + 3y') = 0$

$\Rightarrow y'''(x^2 + c) + 6(xy'' + y') = 0$ (3)

Multiplying (2) by y''' and (3) by y'' and then subtracting, we get

$2(2xy' + y)y''' - 6(xy'' + y')y'' = 0$

$\Rightarrow (2xy' + y)y''' = 3(xy'' + y')y''$

Concept Application Exercise 4.9

1. Prove that $\frac{d^n}{dx^n}(e^{2x} + e^{-2x}) = 2^n[e^{2x} + (-1)^n e^{-2x}]$.
2. If $y = \sin(\sin x)$, and $\frac{d^2y}{dx^2} + \frac{dy}{dx}\tan x + f(x) = 0$, then find $f(x)$.
3. If $y = \log(1 + \sin x)$, prove that $y_4 + y_3y_1 + y_2^2 = 0$.
4. If $f(x) = (1 + x)^n$, then find the value of $f(0) + f'(0) + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \cdots + \frac{f^n(0)}{n!}$.

PROBLEMS BASED ON FIRST DEFINITION OF DERIVATIVE

Example 4.42 A function $f: R \to R$ satisfies the equation $f(x+y)=f(x)f(y)$ for all $x, y \in R$ and $f(x) \neq 0$ for all $x \in R$. If $f(x)$ is differentiable at $x=0$ and $f'(0)=2$, then prove that $f'(x)=2f(x)$.

Sol. We have $f(x+y)=f(x)f(y)$ for all $x, y \in R$

$\therefore f(0)=f(0)f(0) \Rightarrow f(0)\{f(0)-1\}=0 \Rightarrow f(0)=1$ $[\because f(0)\neq 0]$

Now, $f'(0)=2$

$$\Rightarrow \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=2$$

$$\Rightarrow \lim_{h\to 0}\frac{f(h)-1}{h}=2 \quad (\because f(0)=1) \qquad (1)$$

$$\text{Now, } f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h} \quad (\because f(x+y)=f(x)f(y))$$

$$=f(x)\left(\lim_{h\to 0}\frac{f(h)-1}{h}\right)=2f(x) \qquad \text{[Using (1)]}$$

Example 4.43 Let $f: R \to R$ satisfying $|f(x)| \le x^2$, $\forall\ x \in R$, differentiable at $x=0$ then find $f'(0)$.

Sol. Since, $|f(x)| \le x^2$, $\forall\ x \in R$ (1)

$\therefore$ At $x=0$, $|f(0)| \le 0 \Rightarrow f(0)=0$ (2)

$$\therefore f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)}{h} \qquad (3)$$

$$\text{Now, } \left|\frac{f(h)}{h}\right| \le |h| \qquad \text{(from (1))}$$

$$\Rightarrow -|h| \le \frac{f(h)}{h} \le |h|$$

$$\Rightarrow \lim_{h\to 0}\frac{f(h)}{h} \to 0 \qquad \text{(using Sandwich Theorem)} \quad (4)$$

$\therefore$ from (3) and (4), we get $f'(0)=0$.

Example 4.44 Suppose $p(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$. If $|p(x)| \le |e^{x-1}-1|$ for all $x \ge 0$, prove that $|a_1+2a_2+\cdots+na_n| \le 1$.

Sol. Given $p(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$

$\therefore p'(x)=0+a_1+2a_2x+\cdots+na_nx^{n-1}$

$\Rightarrow p'(1)=a_1+2a_2+\cdots+na_n$ (1)

Now, $|p(1)| \le 0$, $(\because |e^{1-1}-1|=|e^0-1|=|1-1|=0)$

$\Rightarrow |p(1)| \le 0 \Rightarrow p(1)=0$ $(\therefore |p(1)| \ge 0)$

As $|p(x)| \le |e^{x-1}-1|$, we get

$|p(1+h)| \le |e^h-1|\ \forall\ h>-1, h\neq 0$

$\Rightarrow |p(1+h)-p(1)| \le |e^h-1|$ $(\because p(1)=0)$

$$\Rightarrow \left|\frac{p(1+h)-p(1)}{h}\right| \le \left|\frac{e^h-1}{h}\right|$$

Taking limit as $h \to 0$, then

$$\Rightarrow \lim_{h\to 0}\left|\frac{p(1+h)-p(1)}{h}\right| \le \lim_{h\to 0}\left|\frac{e^h-1}{h}\right|$$

$\Rightarrow |p'(1)| \le 1$

$\Rightarrow |a_1+2a_2+\cdots+na_n| \le 1$ (from (1))

Example 4.45 Let $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ for all real x and y. If $f'(0)$ exists and equals -1 and $f(0)=1$, then find $f(2)$.

Sol. Since $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$

Replacing x by $2x$ and y by 0, then $f(x)=\frac{f(2x)+f(0)}{2}$

$\Rightarrow f(2x)+f(0)=2f(x) \Rightarrow f(2x)-2f(x)=-f(0)$ (1)

$$\text{Now, } f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f\left(\frac{2x+2h}{2}\right)-f(x)}{h}$$

$$=\lim_{h\to 0}\left\{\frac{\frac{f(2x)+f(2h)}{2}-f(x)}{h}\right\}$$

$$=\lim_{h\to 0}\left\{\frac{f(2x)+f(2h)-2f(x)}{2h}\right\}$$

$$=\lim_{h\to 0}\left\{\frac{f(2h)-f(0)}{2h}\right\} \qquad \text{(from (1))}$$

$=f'(0)$

$=-1\ \forall\ x \in R$ (given)

Integrating, we get $f(x)=-x+c$

Putting $x=0$, then $f(0)=0+c=1$ (given)

$\therefore c=1$

then $f(x)=1-x$

$\therefore f(2)=1-2=-1$

Alternative Method 1

$$\because f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$$

Differentiating both sides w.r.t. x treating y as constant.

$$\therefore f'\left(\frac{x+y}{2}\right).\frac{1}{2}=\frac{f'(x)+0}{2} \Rightarrow f'\left(\frac{x+y}{2}\right)=f'(x).$$

Replacing x by 0 and y by $2x$,

then $f'(x)=f'(0)=-1$ (given)

Integrating, we have $f(x)=-x+c$.

Putting $x=0$, $f(0)=0+c=1$ (given)

$\therefore c=1$

Hence, $f(x)=-x+1$

then $f(2)=-2+1=-1$.

Alternative Method 2 (Graphical Method)

Suppose $A(x, f(x))$ and $B(y, f(y))$ be any two points on the curve $y=f(x)$.

If M is the mid-point of AB, then co-ordinates of M are $\left(\frac{x+y}{2}, \frac{f(x)+f(y)}{2}\right)$

According to the graph, co-ordinates of P are $\left(\frac{x+y}{2}, f\left(\frac{x+y}{2}\right)\right)$

and $PL > ML \Rightarrow f\left(\frac{x+y}{2}\right) > \frac{f(x)+f(y)}{2}$

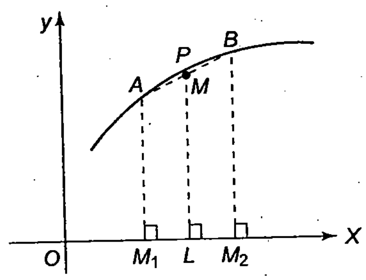

Fig. 4.2

But given $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ which is possible when $P \to M$,

i.e., P lies on AB. Hence, $y=f(x)$ must be a linear function.

Let $f(x)=ax+b \Rightarrow f(0)=0+b=1$ (given)

and $f'(x)=a \Rightarrow f'(0)=a=-1$ (given)

$\therefore f(x)=-x+1$

$\therefore f(2)=-2+1=-1$

Also in the given relation $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ satisfies the section formula for abscissa and ordinate on L.H.S. and R.H.S., respectively, which occurs only in the case of straight line.

Hence, $f(x)=ax+b$, from $f'(0)=-1$, $a=-1$, and from $f(0)=1$, $b=1 \Rightarrow f(x)=-x+1$.

Example 4.46 If $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$ for all $x, y \in R$ $(xy \neq 1)$ and $\lim_{x\to 0}\frac{f(x)}{x}=2$. Find $f\left(\frac{1}{\sqrt{3}}\right)$ and $f'(1)$.

Sol. $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$ (1)

Putting $x=y=0$, we get $f(0)=0$

Putting $y=-x$, we get $f(+x)+f(-x)=f(0)$

$\Rightarrow f(-x)=-f(x)$ (2)

also, $\lim_{x\to 0}\frac{f(x)}{x}=2$

Now, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ (3)

$=\lim_{h\to 0}\frac{f(x+h)+f(-x)}{h}$ (using (2) $-f(x)=f(-x)$)

$f'(x)=\lim_{h\to 0}\frac{f\left(\frac{x+h-x}{1-(x+h)(-x)}\right)}{h}$ (using (1))

$\Rightarrow f'(x)=\lim_{h\to 0}\left[\frac{f\left(\frac{h}{1+x(x+h)}\right)}{h}\right]$

$\Rightarrow f'(x)=\lim_{h\to 0}\frac{f\left(\frac{h}{1+xh+x^2}\right)}{\left(\frac{h}{1+xh+x^2}\right)}\times\left(\frac{1}{1+xh+x^2}\right)$

$\Rightarrow f'(x)=\lim_{h\to 0}\frac{f\left(\frac{h}{1+xh+x^2}\right)}{\left(\frac{h}{1+xh+x^2}\right)}\times\lim_{h\to 0}\frac{1}{1+xh+x^2}$

$\left(\text{using } \lim_{x\to 0}\frac{f(x)}{x}=2\right)$

$\Rightarrow f'(x)=2\times\frac{1}{1+x^2}$ $\Rightarrow f'(x)=\frac{2}{1+x^2}$

Integrating both sides, we get

$f(x)=2\tan^{-1}(x)+c$, where $f(0)=0 \Rightarrow c=0$

Thus, $f(x)=2\tan^{-1}x$.

Hence, $f\left(\frac{1}{\sqrt{3}}\right)=2\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=2\frac{\pi}{6}=\frac{\pi}{3}$, and

$f'(1)=\frac{2}{1+1^2}=\frac{2}{2}=1$

Concept Application Exercise 4.10

1. Let $f(x+y)=f(x).f(y)$ for all x and y. Suppose $f(5)=2$ and $f'(0)=3$, find $f'(5)$.
2. Let $f(xy)=f(x)f(y)$ $\forall$ $x, y \in R$ and f is differentiable at $x=1$ such that $f'(1)=1$ also $f(1)\neq 0$, $f(2)=3$, then find $f'(2)$.
3. If $f\left(\frac{x+2y}{3}\right)=\frac{f(x)+2f(y)}{3}$ $\forall$ $x, y \in R$ and $f'(0)=1$, $f(0)=2$, then find $f(x)$.
4. Prove that $\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=f''(x)$ (without using L'Hopital's rule)

MISCELLANEOUS SOLVED PROBLEMS

1. Derivative of $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ with respect to $\cos^{-1}x^2$ is

   a. $-1/2$ **b.** $1/2$

   c. -1 **d.** 1

Sol. **a.** Let $u = \tan^{-1}\left\{\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$

$$= \tan^{-1}\left\{\frac{1-\sqrt{\dfrac{1-x^2}{1+x^2}}}{1+\sqrt{\dfrac{1-x^2}{1+x^2}}}\right\}$$

Let $x^2 = \cos 2\theta$

$$\Rightarrow u = \tan^{-1}\left\{\frac{1-\tan\theta}{1+\tan\theta}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\theta\right)\right\} = \frac{\pi}{4}-\theta = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x^2$$

Also $v = \cos^{-1}x^2$

$$\Rightarrow \frac{du}{dv} = -\frac{1}{2}$$

2. Let f and g be differentiable functions satisfying $g'(a) = 2$, $g(a) = b$ and $fog = I$ (identity function). Then $f'(b)$ is equal to

a. $\dfrac{1}{2}$ **b.** 2

c. $\dfrac{2}{3}$ **d.** None of these

Sol. **a.** Given $fog = I$

$\Rightarrow fog(x) = x$ for all x

$\Rightarrow f'(g(x))g'(x) = 1$ for all x

$$\Rightarrow f'(g(a)) = \frac{1}{g'(a)} = \frac{1}{2}$$

$$\Rightarrow \quad f'(b) = \frac{1}{2} \qquad (\because g(a) = b)$$

3. If $f(x) = \cot^{-1}\left(\dfrac{x^x - x^{-x}}{2}\right)$, then $f'(1)$ is

a. -1 **b.** 1

c. $\ln 2$ **d.** $-\ln 2$

Sol. **a.** $\dfrac{d}{dx}\left(x^x\right) = x^x(1+\log x)$, $\dfrac{d}{dx}\left(x^{-x}\right) = -x^{-x}(1+\log x)$

$$\Rightarrow f'(x) = \frac{d}{dx}(f(x))$$

$$= -\frac{1}{1+\left(\dfrac{x^x-x^{-x}}{2}\right)^2}\frac{d}{dx}\left(\frac{x^x-x^{-x}}{2}\right)$$

$$= -\frac{4}{4+\left(x^x-x^{-x}\right)^2}\frac{1}{2}(x^x(1+\log x)+x^{-x}(1+\log x))$$

$$= -2\frac{(x^x+x^{-x})(1+\log x)}{\left(x^x + x^{-x}\right)^2}$$

$$\Rightarrow f'(1) = -1$$

4. If $y = \tan^{-1}\dfrac{1}{1+x+x^2} + \tan^{-1}\dfrac{1}{x^2+3x+3} + \tan^{-1}\dfrac{1}{x^2+5x+7} + \cdots +$ upto n terms, then $y'(0)$ is

a. $-\dfrac{1}{1+n^2}$ **b.** $-\dfrac{n^2}{1+n^2}$

c. $\dfrac{n}{1+n^2}$ **d.** None of these

Sol. **b.** $y = \tan^{-1}\dfrac{1}{1+x+x^2} + \tan^{-1}\dfrac{1}{x^2+3x+3} + \cdots + n$ terms

$$= \tan^{-1}\frac{(x+1)-x}{1+x(1+x)} + \tan^{-1}\frac{(x+2)-(x+1)}{1+(x+1)(x+2)} + \cdots + n \text{ terms}$$

$$= \tan^{-1}(x+1) - \tan^{-1}x + \tan^{-1}(x+2) - \tan^{-1}(x+1) + \cdots + \tan^{-1}(x+n) - \tan^{-1}(x+(n-1))$$

$$= \tan^{-1}(x+n) - \tan^{-1}x$$

$$y'(x) = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$$

$$\Rightarrow y'(0) = \frac{1}{1+n^2} - 1 = \frac{-n^2}{1+n^2}$$

5. If $\sqrt{(1-x^6)} + \sqrt{(1-y^6)} = a(x^3-y^3)$, and $\dfrac{dy}{dx} = f(x, y)\sqrt{\left(\dfrac{1-y^6}{1-x^6}\right)}$, then

a. $f(x, y) = y/x$ **b.** $f(x, y) = y^2/x^2$

c. $f(x, y) = 2y^2/x^2$ **d.** $f(x, y) = x^2/y^2$

Sol. **d.** Let $x^3 = \cos p$ and $y^3 = \cos q$

Given $\sqrt{(1-x^6)} + \sqrt{(1-y^6)} = a(x^3-y^3)$

$$\Rightarrow \sqrt{(1-\cos^2 p)} + \sqrt{(1-\cos^2 q)} = a(\cos p - \cos q)$$

$$\Rightarrow \sin p + \sin q = a(\cos p - \cos q)$$

$$\Rightarrow 2\sin\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right) = -2a\sin\left(\frac{p-q}{2}\right)\sin\left(\frac{p+q}{2}\right)$$

$$\Rightarrow \tan\left(\frac{p-q}{2}\right) = -\frac{1}{a}$$

$$\Rightarrow p - q = \tan^{-1}\left(-\frac{1}{a}\right)$$

$$\Rightarrow \cos^{-1}x^3 - \cos^{-1}y^3 = \tan^{-1}\left(-\frac{1}{a}\right)$$

Differentiate w.r.t. x, we have $-\dfrac{3x^2}{\sqrt{1-x^6}} + \dfrac{3y^2}{\sqrt{1-y^6}}\dfrac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}.$$

Hence, $f(x, y) = x^2/y^2$

6. If $f(x)=\cos x.\cos 2x.\cos 4x.\cos 8x.\cos 16x$, then $f'\left(\frac{\pi}{4}\right)$ is

a. $\sqrt{2}$ b. $\frac{1}{\sqrt{2}}$

c. 1 d. None of these

Sol. a. $f(x)=\frac{2\sin x\cos x\cos 2x\cos 4x\cos 8x\cos 16x}{2\sin x}$

$=\frac{\sin 32x}{2^5\sin x}$

$\Rightarrow f'(x)=\frac{1}{32}\times\frac{32\cos 32x\sin x-\cos x\sin 32x}{\sin^2 x}$

$\Rightarrow f'\left(\frac{\pi}{4}\right)=\frac{32\times\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\times 0}{32\left(\frac{1}{\sqrt{2}}\right)^2}=\sqrt{2}$

7. If $f(x)=|x|^{|\sin x|}$, then $f'\left(-\frac{\pi}{4}\right)$ equals

a. $\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\ln\frac{4}{\pi}-\frac{2\sqrt{2}}{\pi}\right)$

b. $\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\ln\frac{4}{\pi}+\frac{2\sqrt{2}}{\pi}\right)$

c. $\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\ln\frac{\pi}{4}-\frac{2\sqrt{2}}{\pi}\right)$

d. $\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\ln\frac{\pi}{4}+\frac{2\sqrt{2}}{\pi}\right)$

Sol. a. In the neighbourhood of $-\pi/4$, we have
$f(x)=(-x)^{-\sin x}=e^{-\sin x\log(-x)}$

$\Rightarrow f'(x)=e^{-\sin x\log(-x)}\left(-\cos x\log(-x)-\frac{\sin x}{x}\right)$

$\Rightarrow f'(x)=(-x)^{-\sin x}\left(-\cos x\log(-x)-\frac{\sin x}{x}\right)$

$\Rightarrow f'(-\pi/4)=\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{-1}{\sqrt{2}}\log\frac{\pi}{4}+\frac{4}{\pi}\times\frac{-1}{\sqrt{2}}\right)$

$=\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\log\frac{4}{\pi}-\frac{2\sqrt{2}}{\pi}\right)$

or use the concept that it is even function

8. If $y=\sqrt{(a-x)(x-b)}-(a-b)\tan^{-1}\sqrt{\frac{a-x}{x-b}}$, then $\frac{dy}{dx}$ is equal to

a. 1 b. $\sqrt{\frac{a-x}{x-b}}$

c. $\sqrt{(a-x)(x-b)}$ d. $\frac{1}{\sqrt{(a-x)(b-x)}}$

Sol. b. Let $x=a\cos^2\theta+b\sin^2\theta$
$\therefore a-x=a-a\cos^2\theta-b\sin^2\theta=(a-b)\sin^2\theta$, and

$x-b=a\cos^2\theta+b\sin^2\theta-b=(a-b)\cos^2\theta$

$\therefore\ y=(a-b)\sin\theta\cos\theta-(a-b)\tan^{-1}\tan\theta$

$=\frac{a-b}{2}\sin 2\theta-(a-b)\theta$

$\therefore\ \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{(a-b)\cos 2\theta-(a-b)}{(b-a)\sin 2\theta}=\frac{1-\cos 2\theta}{\sin 2\theta}$

$=\tan\theta=\sqrt{\frac{a-x}{x-b}}$

9. If $x=a\cos\theta, y=b\sin\theta$, then $\frac{d^3y}{dx^3}$ is

a. $-\frac{3b}{a^3}\operatorname{cosec}^4\theta\cot^4\theta$ b. $\frac{3b}{a^3}\operatorname{cosec}^4\theta\cot\theta$

c. $-\frac{3b}{a^3}\operatorname{cosec}^4\theta\cot\theta$ d. None of these

Sol. c. We have $y=b\sin\theta, x=a\cos\theta$.

Therefore, $\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=-\frac{b}{a}\cot\theta$

$\Rightarrow\ \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{d\theta}\left(-\frac{b}{a}\cot\theta\right)\cdot\frac{d\theta}{dx}$

$=\frac{b}{a}\operatorname{cosec}^2\theta\frac{d\theta}{dx}=-\frac{b}{a^2}\operatorname{cosec}^3\theta$

$\Rightarrow\ \frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d}{d\theta}\left(-\frac{b}{a^2}\operatorname{cosec}^3\theta\right)\frac{d\theta}{dx}$

$=-\frac{b}{a^2}3\operatorname{cosec}^2\theta(-\operatorname{cosec}\theta\cot\theta)\frac{d\theta}{dx}$

$=\frac{3b}{a^2}\operatorname{cosec}^3\theta\cot\theta\times\frac{-1}{a\sin\theta}=-\frac{3b}{a^3}\operatorname{cosec}^4\theta\cot\theta$

10. If $f(x)=\begin{vmatrix}(1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3}\\(1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3}\\(1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3}\end{vmatrix}$, then the coefficient of x in the expansion of $f(x)$ is

a. 1 b. 0

c. −1 d. 2

Sol. b. We have $f(x)=\begin{vmatrix}(1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3}\\(1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3}\\(1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3}\end{vmatrix}$

$=a_0+a_1x+a_2x^2+\cdots$

$\Rightarrow a_1=f'(0)=\begin{vmatrix}a_1b_1 & a_1b_2 & a_1b_3\\1 & 1 & 1\\1 & 1 & 1\end{vmatrix}+\begin{vmatrix}1 & 1 & 1\\a_2b_1 & a_2b_2 & a_2b_3\\1 & 1 & 1\end{vmatrix}$

$+\begin{vmatrix}1 & 1 & 1\\1 & 1 & 1\\a_3b_1 & a_3b_2 & a_3b_3\end{vmatrix}=0$

EXERCISES

Subjective Type

Solutions on page 4.29

1. Let $f(x)=x+\cfrac{1}{2x+\cfrac{1}{2x+\cfrac{1}{2x+\cdots\infty}}}$.

   Compute the value of $f(50).f'(50)$.

2. If $x^2+y^2=R^2$ (where $R>0$) and $k=\dfrac{y''}{\sqrt{(1+y'^2)^3}}$, then find k in terms of R alone.

3. If $y=\dfrac{x^2}{2}+\dfrac{1}{2}x\sqrt{x^2+1}+\log_e\sqrt{x+\sqrt{x^2+1}}$ prove that $2y=xy'+\log_e y'$ where y' denotes the derivative w.r.t. x.

4. If $y=\dfrac{2}{\sqrt{a^2-b^2}}\left(\tan^{-1}\left(\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{x}{2}\right)\right)$, then show that $\dfrac{d^2y}{dx^2}=\dfrac{b\sin x}{(a+b\cos x)^2}$.

5. Differentiate $\tan^{-1}\dfrac{x}{1+\sqrt{(1-x^2)}}+\sin\left\{2\tan^{-1}\sqrt{\left(\dfrac{1-x}{1+x}\right)}\right\}$ w.r.t. x.

6. If $y=(1/2^{n-1})\cos(n\cos^{-1}x)$, then prove that y satisfies the differential equation $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+n^2y=0$.

7. If $x\in\left(0,\dfrac{\pi}{2}\right)$, then show that

   $$\frac{d}{dx}\cos^{-1}\left\{\frac{7}{2}(1+\cos 2x)+\sqrt{(\sin^2 x-48\cos^2 x)}\sin x\right\}=1+\frac{7\sin x}{\sqrt{\sin^2 x-48\cos^2 x}}.$$

8. If $f(x)=\cos^{-1}\dfrac{1}{\sqrt{13}}(2\cos x-3\sin x)+\sin^{-1}\dfrac{1}{\sqrt{13}}\times(2\cos x+3\sin x)$ w.r.t. $\sqrt{1+x^2}$, then find $df(x)/dx$ at $x=3/4$.

9. If $|a_1\sin x+a_2\sin 2x+\cdots+a_n\sin nx|\le|\sin x|$ for $x\in R$, then prove that $|a_1+2a_1+3a_3+\cdots+na_n|\le 1$.

10. Given that $\cos\dfrac{x}{2}.\cos\dfrac{x}{4}.\cos\dfrac{x}{8}\cdots=\dfrac{\sin x}{x}$, then find the sum $\dfrac{1}{2^2}\sec^2\dfrac{x}{2}+\dfrac{1}{2^4}\sec^2\dfrac{x}{4}+\cdots$.

11. If $0<x<1$, then prove that

    $$\frac{1-2x}{1-x+x^2}+\frac{2x-4x^3}{1-x^2+x^4}+\frac{4x^3-8x^7}{1-x^4+x^8}+\cdots\infty=\frac{1+2x}{1+x+x^2}.$$

12. If $\dfrac{d}{dx}[(x^m-A_1x^{m-1}+A_2x^{m-2}-\cdots+(-1)^mA_m)e^x]=x^me^x$, find the value of A_r, where $0<r\le m$.

13. Let $f(x)$ and $g(x)$ be two functions having finite non-zero 3rd order derivatives $f''(x)$ and $g''(x)$ for all $x\in R$. If $f(x)g(x)=1$ for all $x\in R$, then prove that $\dfrac{f'''}{f'}-\dfrac{g'''}{g'}=3\left(\dfrac{f''}{f}-\dfrac{g''}{g}\right)$.

14. If $g(x)=\dfrac{f(x)}{(x-a)(x-b)(x-c)}$, where $f(x)$ is a polynomial of degree <3, then prove that

    $$\frac{dg(x)}{dx}=\begin{vmatrix}1 & a & f(a)(x-a)^{-2}\\ 1 & b & f(b)(x-b)^{-2}\\ 1 & c & f(c)(x-c)^{-2}\end{vmatrix}\div\begin{vmatrix}a^2 & a & 1\\ b^2 & b & 1\\ c^2 & c & 1\end{vmatrix}.$$

15. If $f(x)=e^{-1/x}$, where $x>0$. Let for each positive integer n, P_n be the polynomial such that $\dfrac{d^nf(x)}{dx^n}=P_n\left(\dfrac{1}{x}\right)e^{-1/x}$ for all $x>0$. Show that $P_{n+1}(x)=x^2\left[P_n(x)-\dfrac{d}{dx}P_n(x)\right]$.

16. Let $f: R\to R$ is a function satisfies condition $f(x+y^3)=f(x)+[f(y)]^3$ for all $x,y\in R$. If $f'(0)\ge 0$. Find $f(10)$.

17. Let $f(x+y)=f(x)+f(y)+2xy-1$ for all real x and y and $f(x)$ be differentiable function. If $f'(0)=\cos\alpha$, then prove that $f(x)>0\ \forall\ x\in R$.

18. If $f\left(\dfrac{x+y}{3}\right)=\dfrac{2+f(x)+f(y)}{3}$ for all real x and y and $f'(2)=2$, then determine $y=f(x)$.

19. If f, g and h are differentiable functions of x and $\Delta(x)=\begin{vmatrix}f & g & h\\ (xf)' & (xg)' & (xh)'\\ (x^2f)' & (x^2g)' & (x^2h)'\end{vmatrix}$, then prove that

    $$\Delta'(x)=\begin{vmatrix}f & g & h\\ f' & g' & h'\\ (x^3f'')' & (x^3g'')' & (x^3h'')'\end{vmatrix}.$$

20. If $y=f(a^x)$ and $f'(\sin x)=\log_e x$, then find $\dfrac{dy}{dx}$, if it exists, where $\left(\dfrac{\pi}{2}<x<\pi\right)$.

Objective Type

Solutions on page 4.33

Each question has four choices a, b, c, and d, out of which *only* *one* is correct.

1. $\frac{dy}{dx}$ for $y=\tan^{-1}\left\{\sqrt{\frac{1+\cos x}{1-\cos x}}\right\}$, where $0<x<\pi$, is
   a. –1/2 b. 0
   c. 1 d. –1
2. If $f(x)=|x^2-5x+6|$, then $f'(x)$ equals
   a. $2x-5$ for $2<x<3$ b. $5-2x$ for $2<x<3$
   c. $2x-5$ for $x>2$ d. $5-2x$ for $x<3$
3. If $y=\tan^{-1}\left(\frac{\log(e/x^2)}{\log(ex^2)}\right)+\tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$, then $\frac{d^2y}{dx^2}$ is
   a. 2 b. 1
   c. 0 d. –1
4. If $f(0)=0, f'(0)=2$, then the derivative of $y=f(f(f(f(x))))$ at $x=0$ is,
   a. 2 b. 8
   c. 16 d. 4
5. If $y=ax^{n+1}+bx^{-n}$, then $x^2\frac{d^2y}{dx^2}$ is equal to
   a. $n(n-1)y$ b. $n(n+1)y$
   c. ny d. n^2y
6. If $y=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}$, then $\frac{dy}{dx}$ is equal to
   a. y b. $y+\frac{x^n}{n!}$
   c. $y-\frac{x^n}{n!}$ d. $y-1-\frac{x^n}{n!}$
7. If $y=a\sin x+b\cos x$, then $y^2+\left(\frac{dy}{dx}\right)^2$ is a
   a. function of x b. function of y
   c. function of x and y d. constant
8. $\frac{d}{dx}\sqrt{\frac{1-\sin 2x}{1+\sin 2x}}$ is equal to $(0<x<\pi/2)$
   a. $\sec^2 x$ b. $-\sec^2\left(\frac{\pi}{4}-x\right)$
   c. $\sec^2\left(\frac{\pi}{4}+x\right)$ d. $\sec^2\left(\frac{\pi}{4}-x\right)$
9. If $y=\left(x+\sqrt{x^2+a^2}\right)^n$, then $\frac{dy}{dx}$ is
   a. $\frac{ny}{\sqrt{x^2+a^2}}$ b. $-\frac{ny}{\sqrt{x^2+a^2}}$
   c. $\frac{nx}{\sqrt{x^2+a^2}}$ d. $-\frac{nx}{\sqrt{x^2+a^2}}$
10. If $f(x)=\sqrt{1+\cos^2(x^2)}$, then $f'\left(\frac{\sqrt{\pi}}{2}\right)$ is
    a. $\sqrt{\pi}/6$ b. $-\sqrt{(\pi/6)}$
    c. $1/\sqrt{6}$ d. $\pi/\sqrt{6}$
11. $\frac{d}{dx}\cos^{-1}\sqrt{\cos x}$ is equal to
    a. $\frac{1}{2}\sqrt{1+\sec x}$ b. $\sqrt{1+\sec x}$
    c. $-\frac{1}{2}\sqrt{1+\sec x}$ d. $-\sqrt{1+\sec x}$
12. If $y=\log_{\sin x}(\tan x)$, then $\left(\frac{dy}{dx}\right)_{\pi/4}$ is equal to
    a. $\frac{4}{\log 2}$ b. $-4\log 2$
    c. $\frac{-4}{\log 2}$ d. None of these
13. If $y=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$, then $(1-x^2)\frac{dy}{dx}$ is equal to
    a. $x+y$ b. $1+xy$
    c. $1-xy$ d. $xy-2$
14. If $y=\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$ $(0<x<\pi/2)$, then $\frac{dy}{dx}=$
    a. $\frac{1}{2}$ b. $\frac{2}{3}$ c. 3 d. 1
15. If $y=x^{(x^x)}$, then $\frac{dy}{dx}$ is
    a. $y[x^x(\log ex)\log x+x^x]$
    b. $y[x^x(\log ex)\log x+x]$
    c. $y[x^x(\log ex)\log x+x^{x-1}]$
    d. $y[x^x(\log_e x)\log x+x^{x-1}]$
16. $\frac{d}{dx}\left[\sin^2\cot^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}\right]$ is equal to
    a. –1 b. $\frac{1}{2}$ c. $-\frac{1}{2}$ d. 1
17. If $y=ae^{mx}+be^{-mx}$, then $\frac{d^2y}{dx^2}-m^2y$ is equal to
    a. $m^2(ae^{mx}-be^{-mx})$ b. 1
    c. 0 d. None of these

18. $\frac{d^n}{dx^n}(\log x) =$

a. $\frac{(n-1)!}{x^n}$ b. $\frac{n!}{x^n}$

c. $\frac{(n-2)!}{x^n}$ d. $(-1)^{n-1}\frac{(n-1)!}{x^n}$

19. If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \cdots\infty}}}$, then $\frac{dy}{dx}$ is

a. $\frac{x}{2y-1}$ b. $\frac{x}{2y+1}$

c. $\frac{1}{x(2y-1)}$ d. $\frac{1}{x(1-2y)}$

20. The differential coefficient of $f(\log_e x)$ with respect to x, where $f(x) = \log_e x$, is

a. $\frac{x}{\log_e x}$ b. $\frac{1}{x}\log_e x$

c. $\frac{1}{x\log_e x}$ d. None of these

21. If $y = \sec(\tan^{-1} x)$, then $\frac{dy}{dx}$ at $x = 1$ is

a. $\cos\frac{\pi}{4}$ b. $\sin\frac{\pi}{2}$

c. $\sin\frac{\pi}{6}$ d. $\cos\frac{\pi}{3}$

22. If $f'(x) = \sqrt{2x^2-1}$ and $y = f(x^2)$, then $\frac{dy}{dx}$ at $x = 1$ is

a. 2 b. 1

c. – 2 d. None of these

23. If $u = f(x^3)$, $v = g(x^2)$, $f'(x) = \cos x$ and $g'(x) = \sin x$, then $\frac{du}{dv}$ is

a. $\frac{3}{2}x\cos x^3 \operatorname{cosec} x^2$ b. $\frac{2}{3}\sin x^3 \sec x^2$

c. $\tan x$ d. None of these

24. $x = t\cos t$, $y = t + \sin t$, then $\frac{d^2x}{dy^2}$ at $t = \frac{\pi}{2}$ is

a. $\frac{\pi+4}{2}$ b. $-\frac{\pi+4}{2}$

c. – 2 d. None of these

25. If $f(x) = \sqrt{1-\sin 2x}$, then $f'(x)$ is equal to

a. $-(\cos x + \sin x)$, for $x \in (\pi/4, \pi/2)$

b. $\cos x + \sin x$, for $x \in (0, \pi/4)$

c. $-(\cos x + \sin x)$, for $x \in (0, \pi/4)$

d. $\cos x - \sin x$, for $x \in (\pi/4, \pi/2)$

26. If $y = x - x^2$, then the derivative of y^2 with respect to x^2 is

a. $1-2x$ b. $2-4x$

c. $3x-2x^2$ d. $1-3x+2x^2$

27. The first derivative of the function

$\left[\cos^{-1}\left(\sin\sqrt{\frac{1+x}{2}}\right)+x^x\right]$ with respect to x at $x = 1$ is

a. 3/4 b. 0

c. 1/2 d. – 1/2

28. If $y = \sin px$ and y_n is the nth derivative of y, then $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$ is

a. 1 b. 0

c. – 1 d. None of these

29. A function f, defined for all positive real numbers, satisfies the equation $f(x^2) = x^3$ for every $x > 0$. Then the value of $f'(4) =$

a. 12 b. 3

c. 3/2 d. Cannot be determined

30. Suppose $f(x) = e^{ax} + e^{bx}$, where $a \neq b$, and that $f''(x) - 2f'(x) - 15f(x) = 0$ for all x. Then the product ab is

a. 25 b. 9

c. – 15 d. – 9

31. If $y = \frac{(a-x)\sqrt{a-x}-(b-x)\sqrt{x-b}}{\sqrt{a-x}+\sqrt{x-b}}$, then $\frac{dy}{dx}$ wherever it is defined is

a. $\frac{x+(a+b)}{\sqrt{(a-x)(x-b)}}$ b. $\frac{2x-a-b}{2\sqrt{a-x}\sqrt{x-b}}$

c. $-\frac{(a+b)}{2\sqrt{(a-x)(x-b)}}$ d. $\frac{2x+(a+b)}{2\sqrt{(a-x)(x-b)}}$

32. The function $f(x) = e^x + x$, being differentiable and one to one, has a differentiable inverse $f^{-1}(x)$. The value of $\frac{d}{dx}(f^{-1})$ at the point $f(\log 2)$ is

a. $\frac{1}{\ln 2}$ b. $\frac{1}{3}$

c. $\frac{1}{4}$ d. None of these

33. Let $h(x)$ be differentiable for all x and let $f(x) = (kx + e^x)h(x)$, where k is some constant. If $h(0) = 5$, $h'(0) = -2$ and $f'(0) = 18$, then the value of k is

a. 5 b. 4

c. 3 d. 2.2

34. If $y = \tan^{-1}\left(\frac{2^x}{1+2^{2x+1}}\right)$, then $\frac{dy}{dx}$ at $x = 0$ is

a. 1 b. 2

c. ln 2 d. None of these

35. The nth derivative of the function $f(x) = \dfrac{1}{1-x^2}$ (where $x \in (-1, 1)$) at the point $x = 0$ where n is even is

a. 0 b. $n!$
c. n^nC_2 d. 2^nC_2

36. $\dfrac{d^{20}y}{dx^{20}}(2\cos x \cos 3x)$ is equal to

a. $2^{20}(\cos 2x - 2^{20}\cos 3x)$ b. $2^{20}(\cos 2x + 2^{20}\cos 4x)$
c. $2^{20}(\sin 2x + 2^{20}\sin 4x)$ d. $2^{20}(\sin 2x - 2^{20}\sin 4x)$

37. If $y = \sqrt{\dfrac{1-x}{1+x}}$, then $(1-x^2)\dfrac{dy}{dx}$ is equal to

a. y^2 b. $1/y$
c. $-y$ d. $-y/x$

38. The derivative of $y = (1-x)(2-x)\ldots(n-x)$ at $x = 1$ is

a. 0 b. $(-1)(n-1)!$
c. $n! - 1$ d. $(-1)^{n-1}(n-1)!$

39. If $y = \cos^{-1}\left(\dfrac{5\cos x - 12\sin x}{13}\right)$, where $x \in \left(0, \dfrac{\pi}{2}\right)$, then $\dfrac{dy}{dx}$ is

a. 1 b. -1
c. 0 d. None of these

40. If $y = \tan^{-1}\sqrt{\dfrac{x+1}{x-1}}$, then $\dfrac{dy}{dx}$ is

a. $\dfrac{-1}{2|x|\sqrt{x^2-1}}$ b. $\dfrac{-1}{2x\sqrt{x^2-1}}$
c. $\dfrac{1}{2x\sqrt{x^2-1}}$ d. None of these

41. If $\sin^{-1}\left(\dfrac{x^2-y^2}{x^2+y^2}\right) = \log a$, then $\dfrac{dy}{dx}$ is equal to

a. $\dfrac{x}{y}$ b. $\dfrac{y}{x^2}$
c. $\dfrac{x^2-y^2}{x^2+y^2}$ d. $\dfrac{y}{x}$

42. If $y = \cos^{-1}(\cos x)$, then $\dfrac{dy}{dx}$ at $x = \dfrac{5\pi}{4}$ is

a. 1 b. -1
c. $\dfrac{1}{\sqrt{2}}$ d. None of these

43. If $e^x = \dfrac{\sqrt{1+t}-\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}}$ and $\tan\dfrac{y}{2} = \sqrt{\dfrac{1-t}{1+t}}$, then $\dfrac{dy}{dx}$ at $t = \dfrac{1}{2}$ is

a. $-\dfrac{1}{2}$ b. $\dfrac{1}{2}$
c. 0 d. None of these

44. If $x^2 + y^2 = t - \dfrac{1}{t}$ and $x^4 + y^4 = t^2 + \dfrac{1}{t^2}$, then $x^3 y\dfrac{dy}{dx} =$

a. 0 b. 1
c. -1 d. None of these

45. If $y^{1/m} = (x + \sqrt{1+x^2})$, then $(1 + x^2)y_2 + xy_1$ is (where y_r represents rth derivative of y w.r.t. x)

a. m^2y b. my^2
c. m^2y^2 d. None of these

46. Suppose the function $f(x) - f(2x)$ has the derivative 5 at $x = 1$ and derivative 7 at $x = 2$. The derivative of the function $f(x) - f(4x)$ at $x = 1$ has the value equal to

a. 19 b. 9
c. 17 d. 14

47. If $f(x) = \sin^{-1}\cos x$, then the value of $f(10) + f'(10)$ is

a. $11 - \dfrac{7\pi}{2}$ b. $\dfrac{7\pi}{2} - 11$
c. $\dfrac{5\pi}{2} - 11$ d. None of these

48. If $(\sin x)(\cos y) = 1/2$, then d^2y/dx^2 at $(\pi/4, \pi/4)$ is

a. -4 b. -2
c. -6 d. 0

49. A function f satisfies the condition, $f(x) = f'(x) + f''(x) + f'''(x) + \cdots$ where $f(x)$ is a differentiable function indefinitely and dash denotes the order of derivative. If $f(0) = 1$, then $f(x)$ is

a. $e^{x/2}$ b. e^x
c. e^{2x} d. e^{4x}

50. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then $\dfrac{dy}{dx}$ is equal to

a. $\sqrt{\dfrac{1-x^2}{1-y^2}}$ b. $\sqrt{\dfrac{1-y^2}{1-x^2}}$
c. $\sqrt{\dfrac{x^2-1}{1-y^2}}$ d. $\sqrt{\dfrac{y^2-1}{1-x^2}}$

51. If $y = x^2 + \cfrac{1}{x^2 + \cfrac{1}{x^2 + \cfrac{1}{x^2 + \cdots\infty}}}$, then $\dfrac{dy}{dx}$ is

a. $\dfrac{2xy}{2y - x^2}$ b. $\dfrac{xy}{y + x^2}$
c. $\dfrac{xy}{y - x^2}$ d. $\dfrac{2xy}{2 + \dfrac{x^2}{y}}$

52. $\frac{d}{dx}\left[\tan^{-1}\left(\frac{\sqrt{x}(3-x)}{1-3x}\right)\right]=$

a. $\frac{1}{2(1+x)\sqrt{x}}$ b. $\frac{3}{(1+x)\sqrt{x}}$

c. $\frac{2}{(1+x)\sqrt{x}}$ d. $\frac{3}{2(1+x)\sqrt{x}}$

53. Let $g(x)$ be the inverse of an invertible function $f(x)$ which is differentiable at $x = c$, then $g'(f(c))$ equals

a. $f'(c)$ b. $\frac{1}{f'(c)}$

c. $f(c)$ d. None of these

54. If $f(x) = x + \tan x$ and f is inverse of g, then $g'(x)$ equals

a. $\frac{1}{1+[g(x)-x]^2}$ b. $\frac{1}{2-[g(x)-x]^2}$

c. $\frac{1}{2+[g(x)-x]^2}$ d. None of these

55. If $y\sqrt{x^2+1}=\log(\sqrt{x^2+1}-x)$, then

$(x^2+1)\frac{dy}{dx}+xy+1=$

a. 0 b. 1

c. 2 d. None of these

56. If $y=\frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$, then $\frac{dy}{dx}$ is equal to

a. $\frac{ay}{x\sqrt{a^2-x^2}}$ b. $\frac{ay}{\sqrt{a^2-x^2}}$

c. $\frac{ay}{x\sqrt{x^2-a^2}}$ d. None of these

57. If $f(x) = x^4\tan(x^3) - x\ln(1+x^2)$, then the value of $\frac{d^4(f(x))}{dx^4}$ at $x = 0$ is

a. 0 b. 6

c. 12 d. 24

58. Let $g(x)$ be the inverse of an invertible function $f(x)$, which is differentiable for all real x, then $g''(f(x))$ equals

a. $-\frac{f''(x)}{(f'(x))^3}$ b. $\frac{f'(x)f''(x)-(f'(x))^3}{f'(x)}$

c. $\frac{f'(x)f''(x)-(f'(x))^2}{(f'(x))^2}$ d. None of these

59. If $f(x) = |\log_e|x||$, then $f'(x)$ equals

a. $\frac{1}{|x|}$, where $x \neq 0$

b. $\frac{1}{x}$ for $|x| > 1$ and $-\frac{1}{x}$ for $|x| < 1$

c. $-\frac{1}{x}$ for $|x| > 1$ and $\frac{1}{x}$ for $|x| < 1$

d. $\frac{1}{x}$ for $x > 0$ and $-\frac{1}{x}$ for $x < 0$

60. If $y = |\cos x| + |\sin x|$, then $\frac{dy}{dx}$ at $x = \frac{2\pi}{3}$ is

a. $\frac{1-\sqrt{3}}{2}$ b. 0

c. $\frac{1}{2}(\sqrt{3}-1)$ d. None of these

61. If g is the inverse function of f and $f'(x) = \sin x$, then $g'(x)$ is

a. $\operatorname{cosec}\{g(x)\}$ b. $\sin\{g(x)\}$

c. $-\frac{1}{\sin\{g(x)\}}$ d. None of these

62. If $x = \phi(t)$, $y = \psi(t)$, then $\frac{d^2y}{dx^2}$ is

a. $\frac{\phi'\psi''-\psi'\phi''}{(\phi')^2}$ b. $\frac{\phi'\psi''-\psi'\phi''}{(\phi')^3}$

c. $\frac{\phi''}{\psi''}$ d. $\frac{\psi''}{\phi''}$

63. $f(x) = e^x - e^{-x} - 2\sin x - \frac{2}{3}x^3$, then the least value of n for which $\left.\frac{d^n}{dx^n}f(x)\right|_{x=0}$ is non-zero is

a. 5 b. 6

c. 7 d. 8

64. If $f(x)$ satisfies the relation

$f\left(\frac{5x-3y}{2}\right)=\frac{5f(x)-3f(y)}{2}$ $\forall x, y \in R$, and $f(0) = 3$

and $f'(0) = 2$, then the period of $\sin(f(x))$ is

a. 2π b. π

c. 3π d. 4π

65. Instead of the usual definition of derivative $Df(x)$, if we define a new kind of derivative, $D^*F(x)$ by the formula $D^*(x) = \lim_{h\to 0}\frac{f^2(x+h)-f^2(x)}{h}$, where $f^2(x)$ means $[f(x)]^2$. If $f(x) = x\log x$, then $D^*f(x)|_{x=e}$ has the value

a. e b. $2e$

c. $4e$ d. None of these

66. If $f(x) = 2\sin^{-1}\sqrt{1-x} + \sin^{-1}\left(2\sqrt{x(1-x)}\right)$, where $x \in \left(0, \frac{1}{2}\right)$, then $f'(x)$ is

a. $\frac{2}{\sqrt{x(1-x)}}$ b. zero

c. $-\frac{2}{\sqrt{x(1-x)}}$ d. π

67. If $f''(x) = -f(x)$ and $g(x) = f'(x)$ and

$F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$ and given that $F(5) = 5$, then $F(10)$ is

a. 5 **b.** 10
c. 0 **d.** 15

68. The derivative of $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x = 0$ is

a. 1/8 **b.** 1/4
c. 1/2 **d.** 1

69. The nth derivative of xe^x vanishes when

a. $x = 0$ **b.** $x = -1$
c. $x = -n$ **d.** $x = n$

70. If $y^2 = ax^2 + bx + c$, then $y^3 \frac{d^2y}{dx^2}$ is

a. a constant **b.** a function of x only
c. a function of y only **d.** a function of x and y

71. If $y = \sin x + e^x$, then $\frac{d^2x}{dy^2} =$

a. $(-\sin x + e^x)^{-1}$ **b.** $\frac{\sin x - e^x}{(\cos x + e^x)^2}$

c. $\frac{\sin x - e^x}{(\cos x + e^x)^3}$ **d.** $\frac{\sin x + e^x}{(\cos x + e^x)^3}$

72. If $u = x^2 + y^2$ and $x = s + 3t$, $y = 2s - t$, then $\frac{d^2u}{ds^2}$ equals to

a. 12 **b.** 32
c. 36 **d.** 10

73. Let $y = t^{10} + 1$ and $x = t^8 + 1$, then $\frac{d^2y}{dx^2}$ is

a. $\frac{5}{2}t$ **b.** $20t^8$

c. $\frac{5}{16t^6}$ **d.** None of these

74. If $y = x\log\left(\frac{x}{a+bx}\right)$, then $x^3\frac{d^2y}{dx^2}$ equals to

a. $x\frac{dy}{dx} - y$ **b.** $\left(x\frac{dy}{dx} - y\right)^2$

c. $y\frac{dy}{dx} - x$ **d.** $\left(y\frac{dy}{dx} - x\right)^2$

75. Let $u(x)$ and $v(x)$ be differentiable functions such that $\frac{u(x)}{v(x)} = 7$. If $\frac{u'(x)}{v'(x)} = p$ and $\left(\frac{u(x)}{v(x)}\right)' = q$, then $\frac{p+q}{p-q}$ has the value equal to

a. 1 **b.** 0
c. 7 **d.** −7

76. If $ax^2 + 2hxy + by^2 = 1$, then $\frac{d^2y}{dx^2}$ is

a. $\frac{h^2 - ab}{(hx + by)^2}$ **b.** $\frac{ab - h^2}{(hx + by)^2}$

c. $\frac{h^2 + ab}{(hx + by)^2}$ **d.** None of these

77. If $x = t^2$, $y = t^3$, then $\frac{d^2y}{dx^2} =$

a. $\frac{3}{2}$ **b.** $\frac{3}{(4t)}$ **c.** $\frac{3}{2(t)}$ **d.** $\frac{3t}{2}$

78. If $y = x + e^x$, then $\frac{d^2x}{dy^2}$ is

a. e^x **b.** $-\frac{e^x}{(1+e^x)^3}$

c. $-\frac{e^x}{(1+e^x)^2}$ **d.** $\frac{-1}{(1+e^x)^3}$

79. If $f(x) = |\sin x - |\cos x||$, then the value $f'(x)$ at $x = 7\pi/6$ is

a. positive **b.** $\frac{1-\sqrt{3}}{2}$
c. 0 **d.** none of these

80. If graph of $y = f(x)$ is symmetrical about y-axis and that of $y = g(x)$ is symmetrical about the origin. If $h(x) = f(x) \cdot g(x)$, then $\frac{d^3h(x)}{dx^3}$ at $x = 0$ is

a. can not be determined **b.** $f(0) \cdot g(0)$
c. 0 **d.** none of these

81. If $x = \log p$ and $y = \frac{1}{p}$, then

a. $\frac{d^2y}{dx^2} - 2p = 0$ **b.** $\frac{d^2y}{dx^2} + y = 0$

c. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ **d.** $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$

82. Let $y = \ln(1 + \cos x)^2$, then the value of $\frac{d^2y}{dx^2} + \frac{2}{e^{y/2}}$ equals

a. 0 **b.** $\frac{2}{1 + \cos x}$

c. $\frac{4}{(1 + \cos x)}$ **d.** $\frac{-4}{(1 + \cos x)^2}$

83. Let $f(x) = \lim_{h\to 0} \frac{(\sin(x+h))^{\ln(x+h)} - (\sin x)^{\ln x}}{h}$, then $f\left(\frac{\pi}{2}\right)$ is

a. equal to 0 **b.** equal to 1

c. $\ln\frac{\pi}{2}$ **d.** non-existent

Multiple Correct Answers Type

Solutions on page 4.42

Each question has four choices a, b, c, and d, out of which *one* or *more* answers are correct.

1. If $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$, then $\frac{dy}{dx}$ is equal to
   a. $\frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{2\sqrt{x}}$ b. $\frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{2x}$
   c. $\frac{1}{2\sqrt{x}}\sqrt{y^2-4}$ d. $\frac{1}{2\sqrt{x}}\sqrt{y^2+4}$

2. Let $y = \sqrt{x + \sqrt{x + \sqrt{x + \cdots \infty}}}$, then $\frac{dy}{dx}$ is equals to
   a. $\frac{1}{2y-1}$ b. $\frac{x}{x+2y}$
   c. $\frac{1}{\sqrt{1+4x}}$ d. $\frac{y}{2x+y}$

3. If 1 is a twice repeated root of the equation $ax^3 + bx^2 + bx + d = 0$, then
   a. $a = b = d$ b. $a + b = 0$
   c. $b + d = 0$ d. $a = d$

4. If $x^3 - 2x^2y^2 + 5x + y - 5 = 0$ and $y(1) = 1$, then
   a. $y'(1) = 4/3$ b. $y''(1) = -4/3$
   c. $y''(1) = -8\frac{22}{27}$ d. $y'(1) = 2/3$

5. $f(x) = |x^2 - 3|x| + 2|$, then which of the following is/are true
   a. $f'(x) = 2x - 3$ for $x \in (0, 1) \cup (2, \infty)$
   b. $f'(x) = 2x + 3$ for $x \in (-\infty, -2) \cup (-1, 0)$
   c. $f'(x) = -2x - 3$ for $x \in (-2, -1)$
   d. None of these

6. If $y = \frac{x^4 - x^2 + 1}{x^2 + \sqrt{3}x + 1}$ and $\frac{dy}{dx} = ax + b$, then the value of $a - b$ is
   a. $\cot\frac{\pi}{8}$ b. $\cot\frac{\pi}{12}$
   c. $\tan\frac{5\pi}{12}$ d. $\tan\frac{5\pi}{8}$

7. Let $f(x) = \frac{\sqrt{x - 2\sqrt{x-1}}}{\sqrt{x-1} - 1}x$, then
   a. $f'(10) = 1$
   b. $f'(3/2) = -1$
   c. Domain of $f(x)$ is $x \geq 1$
   d. Range of $f(x)$ is $(-2, -1] \cup (2, \infty)$

8. If $y = x^{(\log x)^{\log(\log_e x)}}$, then $\frac{dy}{dx}$ is
   a. $\frac{y}{x}((\ln x^{\infty x-1}) + 2\ln x \ln(\ln x))$
   b. $\frac{y}{x}(\log x)^{\log(\log x)}(2\log(\log x) + 1)$
   c. $\frac{y}{x \ln x}[(\ln x)^2 + 2\ln(\ln x)]$
   d. $\frac{y}{x}\frac{\log y}{\log x}[2\log(\log x) + 1]$

9. Which of the following is/are true?
   a. $\frac{dy}{dx}$ for $y = \sin^{-1}(\cos x)$, where $x \in (0, \pi)$, is -1
   b. $\frac{dy}{dx}$ for $y = \sin^{-1}(\cos x)$, where $x \in (\pi, 2\pi)$, is 1
   c. $\frac{dy}{dx}$ for $y = \cos^{-1}(\sin x)$, where $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, is -1
   d. $\frac{dy}{dx}$ for $y = \cos^{-1}(\sin x)$, where $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, is -1

10. If $f(x-y), f(x)\ f(y)$ and $f(x+y)$ are in A.P. for all x, y, and $f(0) \neq 0$, then
    a. $f(4) = f(-4)$ b. $f(2) + f(-2) = 0$
    c. $f'(4) + f'(-4) = 0$ d. $f'(2) = f'(-2)$

11. If $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$, then $\frac{dy}{dx}$ is
    a. $\frac{-2}{1+x^2}$ for all x b. $\frac{-2}{1+x^2}$ for all $|x| < 1$
    c. $\frac{2}{1+x^2}$ for $|x| > 1$ d. None of these

12. $f: R^+ \to R$ be a continuous function satisfying $f\left(\frac{x}{y}\right) = f(x) - f(y)\ \forall\ x, y \in R^+$. If $f'(1) = 1$, then
    a. 'f' is unbounded b. $\lim_{x\to 0} f\left(\frac{1}{x}\right) = 0$
    c. $\lim_{x\to 0}\frac{f(1+x)}{x} = 1$ d. $\lim_{x\to 0} x \cdot f(x) = 0$

13. If $f_n(x) = e^{f_{n-1}(x)}$ for all $n \in N$ and $f_0(x) = x$, then $\frac{d}{dx}\{f_n(x)\}$ is
    a. $f_n(x)\frac{d}{dx}\{f_{n-1}(x)\}$
    b. $f_n(x)\, f_{n-1}(x)$
    c. $f_n(x)\, f_{n-1}(x) \ldots f_2(x) \cdot f_1(x)$
    d. None of these

Reasoning Type

Solutions on page 4.44

Each question has four choices a, b, c, and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. if both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1
b. if both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1

c. if STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.
d. if STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** Let $f(x) = x[x]$ and [.] denotes greatest integral function, when x is not an integral, then rule for $f'(x)$ is given by $[x]$.
   Statement 2: $f'(x)$ does not exist for any $x \in$ integer.
2. **Statement 1:** If $f(x)$ is an odd function, then $f'(x)$ is an even function.
   Statement 2: If $f'(x)$ is an even function, then $f(x)$ is an odd function.
3. **Statement 1:** Let $f: R \to R$ is a real-valued function $\forall\ x, y \in R$ such that $|f(x)-f(y)| \le |x-y|^3$, then $f(x)$ is a constant function.
   Statement 2: If derivative of the function w.r.t. x is zero, then function is constant.
4. **Statement 1:** For $f(x) = \sin x, f'(\pi) = f'(3\pi)$.
   Statement 2: For $f(x) = \sin x, f(\pi) = f(3\pi)$.
5. **Statement 1:** If differentiable function $f(x)$ satisfies the relation $f(x) + f(x-2) = 0\ \forall\ x \in R$, and if $\left(\frac{d}{dx}f(x)\right)_{x=a} = b$, then $\left(\frac{d}{dx}f(x)\right)_{a+4000} = b$.
   Statement 2: $f(x)$ is a periodic function with period 4.
6. If for some differentiable function $f(\alpha) = 0$ and $f'(\alpha) = 0$,
   Statement 1: Then sign of $f(x)$ does not change in the neighborhood of $x = \alpha$.
   Statement 2: α is repeated root of $f(x) = 0$.
7. Consider function $f(x)$ satisfies the relation, $f(x+y^3) = f(x) + f(y^3),\ \forall\ x, y \in R$ and differentiable for all x.
   Statement 1: If $f'(2) = a$, then $f'(-2) = a$.
   Statement 2: $f(x)$ is an odd function.

Linked Comprehension Type

Solutions on page 4.44

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c, and d, out of which *only one* is correct.

For Problems 1 – 3

$f(x)$ is a polynomial function $f: R \to R$ such that $f(2x) = f'(x) f''(x)$.

1. The value of $f(3)$ is
   a. 4 b. 12
   c. 15 d. None of these
2. $f(x)$ is
   a. one-one and onto
   b. one-one and into
   c. many-one and onto
   d. many-one and into
3. Equation $f(x) = x$ has
   a. three real and positive roots
   b. three real and negative roots
   c. one real roots
   d. three real roots such that sum of roots is zero

For Problems 4 – 6

$f: R \to R, f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$ for all $x \in R$.

4. The value of $f(1)$ is
   a. 2 b. 3 c. –1 d. 4
5. $f(x)$ is
   a. one-one and onto b. one-one and into
   c. many-one and onto d. many-one and into.
6. The value of $f'(1) + f''(2) + f'''(3)$ is
   a. 0 b. –1 c. 2 d. 3

For Problems 7 – 9

Repeated roots : If equation $f(x) = 0$, where $f(x)$ is a polynomial function, and if it has roots $\alpha, \alpha, \beta, \ldots$ or α root is repeated root, then $f(x) = 0$ is equivalent to $(x-\alpha)^2(x-\beta)\ldots = 0$, from which we can conclude that $f'(x) = 0$ or $2(x-\alpha)[(x-\beta)\ldots] + (x-\alpha)^2[(x-\beta)\ldots]' = 0$ or $(x-\alpha)[2\{(x-\beta)\ldots\} + (x-\alpha)\{(x-\beta)\ldots\}'] = 0$ has root α. Thus, if α root occurs twice in equation, then it is common in equations $f(x) = 0$ and $f'(x) = 0$.
Similarly, if α root occurs thrice in equation, then it is common in the equations $f(x) = 0, f'(x) = 0$ and $f''(x) = 0$.

7. If $x - c$ is a factor of order m of the polynomial $f(x)$ of degree $n(1 < m < n)$, then $x = c$ is a root of the polynomial (where $f^r(x)$ represent rth derivative of $f(x)$ w.r.t. x)
   a. $f^m(x)$ b. $f^{m-1}(x)$
   c. $f^n(x)$ d. None of these
8. If $a_1x^3 + b_1x^2 + c_1x + d_1 = 0$ and $a_2x^3 + b_2x^2 + c_2x + d_2 = 0$ have a pair of repeated roots common, prove that
$$\begin{vmatrix} 3a_1 & 2b_1 & c_1 \\ 3a_2 & 2b_2 & c_2 \\ a_2b_1 - a_1b_2 & c_1a_2 - c_2a_1 & d_1a_2 - d_2a_1 \end{vmatrix} = 0.$$
9. If α root occurs p times and β root occurs q times in polynomial equation $f(x) = 0$ or n degree$(1 < p, q < n)$, then which of the following is not true. (where $f^r(x)$ represents rth derivative of $f(x)$ w.r.t. x)
   a. if $p < q < n$, then α and β are two of the roots of the equation $f^{p-1}(x) = 0$
   b. if $q < p < n$, then α and β are two of the roots of the equation $f^{q-1}(x) = 0$
   c. If $p < q < n$, then equations $f(x) = 0$ and $f^p(x) = 0$ have exactly one root common
   d. If $q < p < n$, then equations $f^q(x) = 0$ and $f^p(x) = 0$ have exactly two roots common

For Problems 10 – 12

Equation $x^n - 1 = 0, n > 1, n \in N$, has roots $1, a_1, a_2, \ldots, a_{n-1}$

10. The value of $(1-a_1)(1-a_2)\ldots(1-a_{n-1})$ is
   a. $n^2/2$ b. n
   c. $(-1)^n n$ d. None of these
11. The value of $\sum_{r=1}^{n-1} \frac{1}{2-a_r}$ is
   a. $\frac{2^{n-1}(n-2)+1}{2^n-1}$ b. $\frac{2^n(n-2)+1}{2^n-1}$
   c. $\frac{2^{n-1}(n-1)-1}{2^n-1}$ d. None of these

12. The value of $\sum_{r=1}^{n-1} \frac{1}{1-a_r}$ is

a. $\frac{n}{4}$ b. $\frac{n(n-1)}{2}$

c. $\frac{n-1}{2}$ d. None of these

For Problems 13 – 15

$f(x)=x^2+xg'(1)+g''(2)$ and $g(x)=f(1)x^2+xf'(x)+f''(x)$

13. The value of $f(3)$ is

a. 1 b. 0

c. -1 d. -2

14. The value of $g(0)$ is

a. 0 b. -3

c. 2 d. None of these

15. The domain of the function $\sqrt{\frac{f(x)}{g(x)}}$ is

a. $(-\infty, 1]\cup(2, 3]$ b. $(-2, 0]\cup(1, \infty)$

c. $(-\infty, 0]\cup(2/3, 3]$ d. None of these

For Problems 16 – 18

$g(x+y)=g(x)+g(y)+3xy(x+y)\ \forall\ x, y\in R$ and $g'(0)=-4$.

16. Number of real roots of the equation $g(x)=0$ is

a. 2 b. 0 c. 1 d. 3

17. For which of the following values of x, $\sqrt{g(x)}$ is not defined?

a. $[-2, 0]$ b. $[2, \infty)$

c. $[-1, 1]$ d. None of these

18. The value of $g'(1)$ is

a. 0 b. 1

c. -1 d. None of these

For Problems 19 – 21

A curve is represented parametrically by the equations $x=f(t)=a^{\ln(b^t)}$ and $y=g(t)=b^{-\ln(a^t)}$ $a, b>0$ and $a\neq 1$, $b\neq 1$ where $t\in R$.

19. Which of the following is not a correct expression for $\frac{dy}{dx}$?

a. $\frac{-1}{f(t)^2}$ b. $-(g(t))^2$ c. $\frac{-g(t)}{f(t)}$ d. $\frac{-f(t)}{g(t)}$

20. The value of $\frac{d^2y}{dx^2}$ at the point where $f(t)=g(t)$ is

a. 0 b. $\frac{1}{2}$ c. 1 d. 2

21. The value of $\frac{f(t)}{f'(t)}\cdot\frac{f''(-t)}{f'(-t)}+\frac{f(-t)}{f'(-t)}\cdot\frac{f''(t)}{f'(t)}\ \forall\ t\in R$, is equal to

a. -2 b. 2 c. -4 d. 4

Matrix-Match Type

Solutions on page 4.46

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d, in column I have to be matched with statements p, q, r, s in column II. If the correct match are a-p, a-s, b-r, c-p, c-q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows

	p	q	r	s
a	(p) shaded	(q)	(r)	(s) shaded
b	(p)	(q) shaded	(r) shaded	(s)
c	(p) shaded	(q) shaded	(r)	(s)
d	(p)	(q)	(r)	(s) shaded

1.

Column I	Column II
a. Differentiable function $f(x)$ satisfies the relation $f(1-x)=f(1+x)$ for all $x\in R$	p. Graph of $f'(x)$ is symmetrical about point (1, 0)
b. Differentiable function $f(x)$ satisfies the relation $f(2-x)+f(x)=0$ for all $x\in R$	q. Graph of $f'(x)$ is symmetrical about line $x=1$
c. Differentiable function $f(x)$ satisfies the relation $f(x+2)+f(x)=0$ for all $x\in R$	r. $f'(-1)=f'(3)$
d. Differentiable function $f(x)$ satisfies the relation $f(x)+f(y)+f(x).f(y)=1$ for all x, y and $f(x)>0$	s. $f'(x)$ has period 4

2.

Column I	Column II
a. $y=f(x)$ be given by $x=t^5-5t^3-20t+7$ and $y=4t^3-3t^2-18t+3$, then $-5\times\frac{dy}{dx}$ at $t=1$	p. 0
b. $P(x)$ be a polynomial of degree 4, with $P(2)=-1, P'(2)=0, P''(2)=2, P'''(2)=-12$ and $P^{iv}(2)=24$, then $P''(3)$	q. -2
c. $y=\frac{1}{x}$, then $\frac{\frac{dy}{\sqrt{1+y^4}}}{\frac{dx}{\sqrt{1+x^4}}}$	r. 2
d. $f\left(\frac{2x+3y}{5}\right)=\frac{2f(x)+3f(y)}{5}$ and $f'(0)=p$ and $f(0)=q$, then $f''(0)$	s. -1

3.

Column I	Column II
a. $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then $\frac{dy}{dx}=-\frac{2}{1+x^2}$	p. for $x<0$
b. $y=\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$, then $\frac{dy}{dx}=-\frac{1}{1+x^2}$	q. for $x>1$
c. $y=\lvert e^{\lvert x\rvert}-e\rvert$, then $\frac{dy}{dx}>0$	r. for $x<-1$
d. $u=\log\lvert 2x\rvert$, $v=\lvert\tan^{-1}x\rvert$, then $du/dv>2$	s. for $-1<x<0$

4. Match the value of x in column II where derivative of the function in column I is negative.

Column I	Column II
a. $y=\lvert x^2-2\lvert x\rvert\rvert$	p. (1,2)
b. $y=\lvert\log_e\lvert x\rvert\rvert$	q. (–3, –2)
c. $y=x[x/2]$, where [·] represents greatest integer function	r. (–1, 0)
d. $y=\lvert\sin x\rvert$	s. (0, 1)

Integer Type

Solutions on page 4.48

1. If $f'(x)=\phi(x)$ and $\phi'(x)=f(x)$ for all x. Also $f(3)=5$ and $f'(3)=4$. Then the value of $[f(10)]^2-[\phi(10)]^2$ is
2. If $y=f(x)$ is an odd differentiable function defined on $(-\infty, \infty)$ such that $f'(3)=-2$, then $|f'(-3)|$ equals
3. If x^3+3x^2-9x+c is of the form $(x-\alpha)^2(x-\beta)$, then positive value of c is
4. If graph of $y=f(x)$ is symmetrical about the point (5, 0) and $f'(7)=3$, then the value of $f'(3)$ is
5. Let $g(x)=f(x)\sin x$, where $f(x)$ is a twice differentiable function on $(-\infty, \infty)$ such that $f'(-\pi)=1$. The value of $|g''(-\pi)|$ equals
6. Let $f(x)=(x-1)(x-2)(x-3)\cdots(x-n)$, $n\in N$ and $f'(n)=5040$, then the value of n is
7. $y=f(x)$, where f satisfies the relation $f(x+y)=2f(x)+xf(y)+y\sqrt{f(x)}$ $\forall x, y\in R$ and $f'(0)=0$, then $f(6)$ is equal to
8. If function f satisfies the relation $f(x)\times f'(-x)=f(-x)\times f'(x)$ for all x, and $f(0)=3$, now if $f(3)=3$, then the value of $f(-3)$ is
9. If $y=\dfrac{a+bx^{3/2}}{x^{5/4}}$ and $y'=0$ at $x=5$, then the value of a^2/b^2 is
10. Let $y=\dfrac{2^{\log_{2^{1/4}}x}-3^{\log_{27}(x^2+1)^3}-2x}{7^{4\log_{49}x}-x-1}$ and $\dfrac{dy}{dx}=ax+b$, then the value of $a+b$ is
11. $\displaystyle\lim_{h\to 0}\frac{(e+h)^{\ln(e+h)}-e}{h}$ is
12. If the function $f(x)=-4e^{\frac{1-x}{2}}+1+x+\dfrac{x^2}{2}+\dfrac{x^3}{3}$ and $g(x)=f^{-1}(x)$, then the reciprocal of $g'\left(\dfrac{-7}{6}\right)$ is
13. Suppose that $f(0)=0$ and $f'(0)=2$, and let $g(x)=f(-x+f(f(x)))$. The value of $g'(0)$ is equal to
14. Suppose $f(x)=e^{ax}+e^{bx}$, where $a\neq b$, and that $f''(x)-2f'(x)-15f(x)=0$ for all x. Then the value of $|ab|/3$ is
15. A non-zero polynomial with real coefficients has the property that $f(x)=f'(x)\cdot f''(x)$. If a is the leading coefficient of $f(x)$, then the value of $1/(2a)$ is
16. A function is represented parametrically by the equations $x=\dfrac{1+t}{t^3}$; $y=\dfrac{3}{2t^2}+\dfrac{2}{t}$, then the value of $\left|\dfrac{dy}{dx}-x\left(\dfrac{dy}{dx}\right)^3\right|$ is
17. Let $z=(\cos x)^5$ and $y=\sin x$. Then the value of $2\dfrac{d^2z}{dy^2}$ at $x=\dfrac{2\pi}{9}$ is
18. Let $g(x)=\begin{cases}\dfrac{x^2+x\tan x-x\tan 2x}{ax+\tan x-\tan 3x}; & x\neq 0\\ 0; & x=0\end{cases}$. If $g'(0)$ exists and is equal to non-zero value b, then $52\dfrac{b}{a}$ is equal to

Archives

Solutions on page 4.50

Subjective

1. Find the derivative of $\sin(x^2+1)$ with respect to x from first principle. **(IIT-JEE, 1978)**
2. Find the derivative of **(IIT-JEE, 1979)**

$$f(x)=\begin{cases}\dfrac{x-1}{2x^2-7x+5} & \text{when } x\neq 1\\ -\dfrac{1}{3} & \text{when } x=1\end{cases}\quad \text{at } x=1$$

3. Given $y=\dfrac{5x}{\sqrt[3]{(1-x)^2}}+\cos^2(2x+1)$, find $\dfrac{dy}{dx}$. **(IIT-JEE, 1980)**
4. Let $y=e^{x\sin x^3}+(\tan x)^x$, find $\dfrac{dy}{dx}$. **(IIT-JEE, 1981)**
5. Let f be a twice differentiable function such that $f''(x)=-f(x)$, and $f'(x)=g(x)$, $h(x)=[f(x)]^2+[g(x)]^2$. Find $h(10)$ if $h(5)=11$. **(IIT-JEE, 1982)**
6. If α be a repeated root of a quadratic equation $f(x)=0$ and $A(x)$, $B(x)$ and $C(x)$ be polynomials of degree 3, 4, and 5, respectively, then show that $\begin{vmatrix}A(x) & B(x) & C(x)\\ A(\alpha) & B(\alpha) & C(\alpha)\\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}$ is divisible by $f(x)$, where prime denotes the derivates. **(IIT-JEE, 1984)**
7. Find the derivates with respect to x of the function

$$(\log_{\cos x}\sin x)(\log_{\sin x}\cos x)^{-1}+\sin^{-1}\left(\frac{2x}{1+x^2}\right) \text{ at } x=\frac{\pi}{4}.$$

(IIT-JEE, 1984)

8. If $x = \operatorname{cosec}\theta - \sin\theta$ and $y = \operatorname{cosec}^n\theta - \sin^n\theta$, then show that $\left(x^2+4\right)\left(\frac{dy}{dx}\right)^2 = n^2\left(y^2+4\right)$. (IIT-JEE, 1989)

9. Find $\frac{dy}{dx}$ at $x=-1$, when $(\sin y)^{\sin\left(\frac{\pi}{2}x\right)} + \frac{\sqrt{3}}{2}\sec^{-1}(2x) + 2^x \tan(\log(x+2)) = 0$. (IIT-JEE, 1991)

10. If $y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$, prove that $\frac{y'}{y} = \frac{1}{x}\left(\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right)$. (IIT-JEE, 1998)

Objective

Fill in the blanks

1. If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin x^2$, then $\frac{dy}{dx} =$ ________. (IIT-JEE, 1982)

2. If $f_r(x), g_r(x), h_r(x), r = 1, 2, 3$ are polynomials such that $f_r(a) = g_r(a) = h_r(a), r = 1, 2, 3$ and

$$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix},$$ then $F'(x)$ at $x = a$ is ________. (IIT-JEE, 1985)

3. If $f(x) = \log_x(\log x)$, then $f'(x)$ at $x = e$ is ________. (IIT-JEE, 1985)

4. The derivative of $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ with respect to $\sqrt{1-x^2}$ at $x = \frac{1}{2}$ is ________. (IIT-JEE, 1986)

5. If $f(9) = 9, f'(9) = 4$, then $\lim_{x\to 9} \frac{\sqrt{f(x)}-3}{\sqrt{x}-3} =$ ________. (IIT-JEE, 1988)

6. If $f(x) = |x-2|$ and $g(x) = f[f(x)]$, then $g'(x) =$ ________, for $x > 20$ (IIT-JEE, 1990)

7. If $xe^{xy} = y + \sin^2 x$, then at $x = 0$, $\frac{dy}{dx} =$ ________. (IIT-JEE, 1996)

8. Let $F(x) = f(x)\,g(x)\,h(x)$ for all real x, where $f(x), g(x)$ and $h(x)$ are differentiable functions. At some point x_0, $F'(x_0) = 21F(x_0), f'(x_0) = 4f(x_0), g'(x_0) = -7g(x_0)$ and $h'(x_0) = kh(x_0)$. Then $k =$ ________. (IIT-JEE, 1997)

9. If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is ________. (IIT-JEE, 2009)

True or false

1. The derivative of an even function is always an odd function. (IIT-JEE, 1983)

Multiple choice questions with one correct answer

1. If $f(a) = 2, f'(a) = 1, g(a) = -1, g'(a) = 2$, then the value of $\lim \frac{g(x)f(a) - g(a)f(x)}{}$ is

a. -5 b. $\frac{1}{5}$
c. 5 d. None of these
(IIT-JEE, 1983)

2. If $y^2 = P(x)$, a polynomial of degree 3, then $2\frac{d}{dx}\left(y^2\frac{d^2y}{dx^2}\right) =$ (IIT-JEE, 1988)

a. $P'''(x) + P'(x)$ b. $P''(x)P'''(x)$
c. $P(x)P'''(x)$ d. a constant

3. Let $f(x)$ be a quadratic expression which is positive for all the real values of x. If $g(x) = f(x) + f'(x) + f''(x)$, then for any real x,

a. $g(x) < 0$ b. $g(x) > 0$
c. $g(x) = 0$ d. $g(x) \geq 0$ (IIT-JEE, 1990)

4. If $y = (\sin x)^{\tan x}$, then $\frac{dy}{dx} =$ (IIT-JEE, 1994)

a. $(\sin x)^{\tan x}(1 + \sec^2 x \log \sin x)$
b. $\tan x(\sin x)^{\tan x - 1}.\cos x$
c. $(\sin x)^{\tan x}\sec^2 x \log \sin x$
d. $\tan x\,(\sin x)^{\tan x - 1}$

5. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where p is a constant.

Then $\frac{d^3}{dx^3}(f(x))$ at $x = 0$ is

a. p b. $p - p^3$
c. $p + p^3$ d. independent of p
(IIT-JEE, 1997)

6. Let $f: R \to R$ be such that $f(1) = 3$ and $f'(1) = 6$. Then $\lim_{x\to 0}\left(\frac{f(1+x)}{f(1)}\right)^{1/x} =$

a. 1 b. $e^{1/2}$ c. e^2 d. e^3
(IIT-JEE, 2002)

7. $\lim_{h\to 0} \frac{f(2h+2+h^2) - f(2)}{f(h-h^2+1) - f(1)}$, given that $f'(2) = 6$ and $f'(1) = 4$

a. does not exist b. is equal to $-3/2$
c. is equal to $3/2$ d. is equal to 3
(IIT-JEE, 2004)

8. If $f(x)$ is differentiable and strictly increasing function, then the value of $\lim_{x\to 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$ is

a. 1 b. 0 c. -1 d. 2
(IIT-JEE, 2004)

9. If y is a function of x and $\log(x+y) - 2xy = 0$, then the value of $y'(0)$ is

a. 1 b. -1 c. 2 d. 0
(IIT-JEE, 2004)

10. If $x^2 + y^2 = 1$, then

a. $yy'' - 2(y')^2 + 1 = 0$ b. $yy'' + (y')^2 + 1 = 0$
c. $yy'' + (y')^{-2} - 1 = 0$ d. $yy'' + 2(y')^2 + 1 = 0$
(IIT-JEE, 2000)

11. $\frac{d^2x}{dy^2}$ is equal to

a. $\left(\frac{d^2y}{dx^2}\right)^{-1}$ b. $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

c. $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ d. $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

(IIT-JEE, 2007)

Reasoning Type

1. Let $f(x) = 2 + \cos x$ for all real x
   Statement 1: For each real t, there exists a point c in $[t, t-\pi]$ such that $f'(c) = 0$ because
   Statement 2: $f(t) = f(t + 2\pi)$ for each real t.
   a. Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1.
   b. Statement 1 is true, Statement 2 is true; Statement 2 is a not correct explanation for Statement 1.
   c. Statement 1 is true, Statement 2 is false.
   d. Statement 1 is false, Statement 2 is true.

(IIT-JEE, 2007)

Integer Type

1. Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.
   Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is (IITJEE 2011)

ANSWERS AND SOLUTIONS

Subjective Type

1. $f(x) = x + \cfrac{1}{x + x + \cfrac{1}{2x + \cfrac{1}{2x + \cdots\infty}}} = x + \frac{1}{x + f(x)}$

$\Rightarrow f(x) - x = \frac{1}{x + f(x)}$

$\Rightarrow f^2(x) - x^2 = 1$

differentiating w.r.t. x

$\Rightarrow 2f(x).\, f'(x) - 2x = 0$

$\Rightarrow f(x).f'(x) = x$

$\Rightarrow f(50).f'(50) = 50$

2. $x^2 + y^2 = R^2$

Differentiating w.r.t. x, $\Rightarrow 2x + 2yy' = 0$ (1)

$\Rightarrow y' = -\frac{x}{y}$ (2)

Differentiating (1) w.r.t. x, $\Rightarrow 1 + yy'' + (y')^2 = 0$

$\Rightarrow y'' = -\frac{1 + (y')^2}{y}$

Given $k = \frac{y''}{\left(1 + (y')^2\right)^{3/2}} = -\frac{1 + (y')^2}{y\left(1 + (y')^2\right)^{3/2}}$

$= -\frac{1}{y\sqrt{1 + (y')^2}} = -\frac{1}{y\sqrt{1 + \frac{x^2}{y^2}}}$

$= -\frac{1}{\sqrt{y^2 + x^2}} = -\frac{1}{R}$

3. $y = \frac{x^2}{2} + \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\log_e\left(x + \sqrt{x^2+1}\right)$

$\Rightarrow y' = x + \frac{1}{2}\left[\frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1}\right] + \frac{1}{2\sqrt{x^2+1}}$

$= x + \frac{1}{2}\left[\frac{2x^2+1}{\sqrt{x^2+1}}\right] + \frac{1}{2\sqrt{x^2+1}} = x + \sqrt{x^2+1}$

Also $2y = x^2 + x\sqrt{x^2+1} + \log_e(x + \sqrt{x^2+1})$

$= x\left(x + \sqrt{x^2+1}\right) + \log_e\left(x + \sqrt{x^2+1}\right)$

$= xy' + \log_e y'$, hence proved.

4. $y = A\tan^{-1}\left(B\tan\frac{x}{2}\right)$, where

$A = \frac{2}{\sqrt{a^2 - b^2}}$, $B = \sqrt{\frac{a-b}{a+b}}$

$AB = \frac{2}{\sqrt{(a-b)(a+b)}}\sqrt{\frac{a-b}{a+b}} \Rightarrow AB = \frac{2}{a+b}$.

$\frac{dy}{dx} = \frac{AB\sec^2\frac{x}{2} \times \frac{1}{2}}{1 + B^2\tan^2\frac{x}{2}}$

$= \frac{1}{a+b}.\frac{(a+b)}{(a+b)\cos^2\frac{x}{2} + (a-b)\sin^2\frac{x}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{a + b\cos x}$ (1)

$\Rightarrow \frac{d^2y}{dx^2} = \frac{b\sin x}{(a + b\cos x)^2}$

5. Putting $x = \cos\theta$

$y = \tan^{-1}\left(\frac{\cos\theta}{1 + \sin\theta}\right) + \sin\left\{2\tan^{-1}\sqrt{\left(\frac{1 - \cos\theta}{1 + \cos\theta}\right)}\right\}$

$= \tan^{-1}\frac{\sin\left(\frac{\pi}{2} - \theta\right)}{1 + \cos\left(\frac{\pi}{2} - \theta\right)} + \sin\left(2\tan^{-1}\tan\left(\frac{1}{2}\theta\right)\right)$

$$= \tan^{-1} \frac{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)} + \sin\left(2.\frac{1}{2}\theta\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) + \sin\theta$$

$$= \frac{\pi}{4}-\frac{\theta}{2} + \sqrt{1-\cos^2\theta}$$

$$= \frac{\pi}{4}-\frac{\cos^{-1}x}{2} + \sqrt{1-x^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{(1-x^2)}} + \frac{-2x}{2\sqrt{(1-x^2)}}$$

$$= \frac{1-2x}{2\sqrt{(1-x^2)}}$$

6. $y = (1/2^{n-1})\cos(n\cos^{-1}x)$

$$\therefore \frac{dy}{dx} = -\frac{1}{2^{n-1}}\sin\left(n\cos^{-1}x\right)\left[\frac{-n}{\sqrt{(1-x^2)}}\right]$$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = \frac{n^2}{2^{2n-2}}\sin^2(n\cos^{-1}x)$$

$$= \frac{n^2}{2^{2n-2}}[1-\cos^2(n\cos^{-1}x)]$$

$$= n^2\left[\frac{1}{2^{2n-2}}-y^2\right]$$

Differentiating both sides w.r.t. x

$$\left(1-x^2\right)2\frac{dy}{dx}\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = -2n^2y\frac{dy}{dx}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0$$

7. $y = \cos^{-1}\left\{\frac{7}{2}(1+\cos 2x) + \sqrt{(\sin^2 x - 48\cos^2 x)}\sin x\right\}$

$$= \cos^{-1}\{(7\cos x)(\cos x) + \sqrt{1-49\cos^2 x}\sqrt{1-\cos^2 x}\}$$

$$= \cos^{-1}(\cos x) - \cos^{-1}(7\cos x) \qquad (\because \cos x < 7\cos x)$$

$$= x - \cos^{-1}(7\cos x)$$

Now differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 1 + \frac{7\sin x}{\sqrt{1-49\cos^2 x}} = 1 + \frac{7\sin x}{\sqrt{\sin^2 x - 48\cos^2 x}}$$

8. $f(x) = \cos^{-1}\frac{1}{\sqrt{13}}(2\cos x - 3\sin x)$

$$+ \sin^{-1}\frac{1}{\sqrt{13}}(2\cos x + 3\sin x)$$

$$= \cos^{-1}\left[\frac{1}{\sqrt{13}}\sqrt{13}\cos\left(x+\tan^{-1}\frac{3}{2}\right)\right]$$

$$+ \sin^{-1}\left[\frac{1}{\sqrt{13}}\sqrt{13}\sin\left(x+\tan^{-1}\frac{2}{3}\right)\right]$$

$$= \cos^{-1}\left[\cos\left(x+\tan^{-1}\frac{3}{2}\right)\right] + \sin^{-1}\left[\sin\left(x+\tan^{-1}\frac{2}{3}\right)\right]$$

$$= 2x + \tan^{-1}\frac{3}{2} + \tan^{-1}\frac{2}{3}$$

$$= 2x + \frac{\pi}{2}$$

$\Rightarrow f'(3/4) = 2$

Now let $g(x) = \sqrt{1+x^2} \Rightarrow g'(x) = \dfrac{x}{\sqrt{1+x^2}}$

$\Rightarrow g'(3/4) = 3/5 \Rightarrow f'(3/4)/g'(3/4) = 10/3.$

9. Clearly, we can get $a_1 + 2a_2 + 3a_3 + \cdots + na_n$, by differentiating $a_1\sin x + a_2\sin 2x + \cdots + a_n\sin nx$ and putting $x = 0$.

Thus, we have to prove that $|f'(0)| \le 1$

Let $f(x) = a_1\sin x + a_2\sin 2x + \cdots + a_n\sin nx$

$\Rightarrow f'(x) = a_1\cos x + 2a_2\cos 2x + \cdots + na_n\cos nx$

$\Rightarrow f'(0) = a_1 + 2a_2 + \cdots + na_n$

Also given $|f(x)| \le |\sin x|$ for $x \in R$ (1)

Put $x = 0 \Rightarrow |f(0)| \le 0 \Rightarrow f(0) = 0$

Now, $f'(0) = \lim_{h\to 0}\dfrac{f(h)-f(0)}{h}$

$\Rightarrow f'(0) = \lim_{h\to 0}\dfrac{f(h)}{h}$ (as $f(0) = 0$)

$\Rightarrow |f'(0)| = \lim_{h\to 0}\left|\dfrac{f(h)}{h}\right| \le \lim_{h\to 0}\left|\dfrac{\sin h}{h}\right| = 1$ (as $|f(x)| \le \sin x|$)

Hence, $|f'(0)| \le 1$.

10. We have $\cos\frac{x}{2}.\cos\frac{x}{4}.\cos\frac{x}{8}\cdots = \dfrac{\sin x}{x}$

Taking log on both sides, we get

$$\log\cos\frac{x}{2} + \log\cos\frac{x}{4} + \log\cos\frac{x}{8}\cdots = \log\sin x - \log x$$

Differentiating both sides with respect to x, we get

$$-\frac{1}{2}\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} - \frac{1}{4}\frac{\sin\frac{x}{4}}{\cos\frac{x}{4}} - \frac{1}{8}\frac{\sin\frac{x}{8}}{\cos\frac{x}{8}} = \frac{\cos x}{\sin x} - \frac{1}{x}$$

$$\Rightarrow -\frac{1}{2}\tan\frac{x}{2}-\frac{1}{4}\tan\frac{x}{4}-\frac{1}{8}\tan\frac{x}{8}\cdots = \cot x-\frac{1}{x}$$

Differentiating both sides with respect to x, we get

$$-\frac{1}{2^2}\sec^2\frac{x}{2}-\frac{1}{4^2}\sec^2\frac{x}{4}-\frac{1}{8^2}\sec^2\frac{x}{8}\cdots = -\text{cosec}^2 x+\frac{1}{x^2}$$

$$\Rightarrow \frac{1}{2^2}\sec^2\frac{x}{2}+\frac{1}{4^2}\sec^2\frac{x}{4}+\frac{1}{8^2}\sec^2\frac{x}{8}\cdots = \text{cosec}^2 x-\frac{1}{x}.$$

11. The given series is in the form

$$\frac{f_1'(x)}{f_1(x)}+\frac{f_2'(x)}{f_2(x)}+\frac{f_3'(x)}{f_3(x)}+\cdots\infty$$

Then consider the product $f_1(x)\cdot f_2(x)\cdot f_3(x)\ldots f_n(x)$

Also,

$$(1+x+x^2)(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)\cdots(1-x^{2^{n-1}}+x^{2^n})$$

$$=(1+x^2+x^4)(1-x^2+x^4)(1-x^4+x^8)\cdots(1-x^{2^{n-1}}+x^{2^n})$$

$$=(1+x^4+x^8)(1-x^4+x^8)\ldots(1-x^{2^{n-1}}+x^{2^n})$$

$$\vdots$$

$$=\left(1+x^{2^n}+x^{2^{n+1}}\right)$$

When $n\to\infty$, $x^{2^n}, x^{2^{n+1}}\to 0$, as $x<1$

$\Rightarrow (1+x+x^2)(1-x+x^2)(1-x^2+x^4)\ldots\infty=1$

Taking logarithm of both sides, we get

$$\log\left(1+x+x^2\right)+\log\left(1-x+x^2\right)+\log(1-x^2+x^4)+\log(1-x^4+x^8)+\cdots=0$$

Differentiating both sides w. r. t. x, we get

$$\frac{1+2x}{1+x+x^2}+\frac{-1+2x}{1-x+x^2}+\frac{-2x+4x^3}{1-x^2+x^4}+\frac{-4x^3+8x^7}{1-x^4+x^8}+\cdots=0$$

$$\text{or}\quad \frac{1-2x}{1-x+x^2}+\frac{2x-4x^3}{1-x^2+x^4}+\frac{4x^3-8x^7}{1-x^4+x^8}+\cdots=\frac{1+2x}{1+x+x^2}$$

12. $x^m e^x=\dfrac{d}{dx}[(x^m-A_1x^{m-1}+A_2x^{m-2}-\cdots+(-1)^m A_m)e^x]$

$= [x^m - A_1x^{m-1} + A_2x^{m-2}+\cdots+(-1)^{m-1}A_{m-1}x + (-1)^m A_m]e^x+[mx^{m-1}-A_1(m-1)x^{m-2}+A_2(m-2)x^{m-3}+\cdots+(-1)^{m-1}A_{m-1}]e^x$

$=x^m e^x+(-A_1+m)x^{m-1}e^x+\{A_2-A_1(m-1)\}x^{m-2}e^x+\cdots+(-1)^{m-1}(-A_m+A_{m-1})e^x$

$\Rightarrow -A_1+m=0, A_2-A_1(m-1)=0, \ldots, -A_m+A_{m-1}=0$

$\Rightarrow A_1=m, A_2=A_1(m-1)=m(m-1)=m!/(m-2)!$

$A_3=A_2(m-2)=m(m-1)(m-2)=m!/(m-3)!\ldots$

$A_m=A_{m-1}=m!$

$\Rightarrow A_r=m!/(m-r)!$

13. We have $f(x)\,g(x)=1$. Differentiating with respect to x, we get

$$f'g+fg'=0 \qquad (1)$$

Differentiating (1) w.r.t. x, we get

$$f''g+2f'g'+fg''=0 \qquad (2)$$

Differentiating (2) w.r.t. x, we get

$$f'''g+g'''f+3f''g'+3g''f'=0$$

$$\Rightarrow \frac{f'''}{f'}(fg')+\frac{g'''}{g'}(f'g)+\frac{3f''}{f}(f'g)+\frac{3g''}{g}(fg')=0$$

$$\Rightarrow \left(\frac{f'''}{f'}+\frac{3g''}{g}\right)(f'g)=-\left(\frac{g'''}{g'}+\frac{3f''}{f}\right)(f'g)$$

$$\Rightarrow -\left(\frac{f'''}{f'}+\frac{3g''}{g}\right)(fg')=\left(\frac{g'''}{g'}+\frac{3f''}{g}\right)fg' \qquad \text{[Using (1)]}$$

$$\Rightarrow \frac{f'''}{f'}+\frac{3g''}{g}=\frac{g'''}{g'}+\frac{3f''}{f} \Rightarrow \frac{f'''}{f'}-\frac{g'''}{g'}=3\left(\frac{f''}{f}-\frac{g''}{g}\right)$$

14. By partial fractions, we have $g(x)=\dfrac{f(a)}{(x-a)(a-b)(a-c)}+\dfrac{f(b)}{(b-a)(x-b)(b-c)}+\dfrac{f(c)}{(c-a)(c-b)(x-c)}$

$$\Rightarrow g(x)=\frac{1}{(a-b)(b-c)(c-a)}\times\left[\frac{f(a)(c-b)}{(x-a)}+\frac{f(b)(a-c)}{(x-b)}+\frac{f(c)(b-a)}{(x-c)}\right]$$

$$\Rightarrow g(x)=\begin{vmatrix}1 & a & f(a)/(x-a)\\ 1 & b & f(b)/(x-b)\\ 1 & c & f(c)/(x-c)\end{vmatrix}\div\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$$

$$\Rightarrow \frac{dg(x)}{dx}=\begin{vmatrix}1 & a & -f(a)(x-a)^{-2}\\ 1 & b & -f(b)(x-b)^{-2}\\ 1 & c & -f(c)(x-c)^{-2}\end{vmatrix}\div\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$$

$$=\begin{vmatrix}1 & a & f(a)(x-a)^{-2}\\ 1 & b & f(b)(x-b)^{-2}\\ 1 & c & f(c)(x-c)^{-2}\end{vmatrix}\div\begin{vmatrix}a^2 & a & 1\\ b^2 & b & 1\\ c^2 & c & 1\end{vmatrix}$$

15. From the given condition of the problem

$$\frac{d^{n+1}f(x)}{dx^{n+1}}=P_{n+1}\left(\frac{1}{x}\right)e^{-1/x}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{d^n f(x)}{dx^n}\right)=P_{n+1}\left(\frac{1}{x}\right)e^{-1/x}$$

$$\Rightarrow \frac{d}{dx}\left(P_n\left(\frac{1}{x}\right)e^{-1/x}\right)=P_{n+1}\left(\frac{1}{x}\right)e^{-1/x}$$

$$\Rightarrow e^{-1/x}\frac{dP_n\left(\frac{1}{x}\right)}{dx}+P_n\left(\frac{1}{x}\right)\frac{de^{-1/x}}{dx}=P_{n+1}\left(\frac{1}{x}\right)e^{-1/x}$$

$$\Rightarrow e^{-1/x}\frac{dP_n\left(\frac{1}{x}\right)}{dx\left(\frac{1}{x}\right)}\times\frac{d\left(\frac{1}{x}\right)}{dx}+$$

$$P_n\left(\frac{1}{x}\right)\frac{de^{-1/x}}{d\left(\frac{-1}{x}\right)}\times\frac{d\left(\frac{-1}{x}\right)}{dx}=P_{n+1}\left(\frac{1}{x}\right)e^{-1/x}$$

$$\Rightarrow -\frac{1}{x^2}\frac{dP_n\left(\frac{1}{x}\right)}{d\left(\frac{1}{x}\right)}+\frac{1}{x^2}P_n\left(\frac{1}{x}\right)=P_{n+1}\left(\frac{1}{x}\right)\quad\left(\text{Put}\frac{1}{x}=y\right)$$

$$\Rightarrow P_{n+1}(y)=y^2\left[P_n(y)-\frac{dP_n(y)}{dy}\right]$$

$$\Rightarrow P_{n+1}(x)=x^2\left[P_n(x)-\frac{dP_n(x)}{dx}\right]$$

16. Given $f(x+y^3)=f(x)+[f(y)]^3$ (1)

and $f'(0)\ge 0$ (2)

Replacing x, y by 0

$f(0)=f(0)+f(0)^3 \Rightarrow f(0)=0$ (3)

Also $f'(0)=\lim\limits_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\frac{f(h)}{h}$ (4)

Let $I=f'(0)=\lim\limits_{h\to 0}\frac{f(0+(h^{1/3})^3)-f(0)}{(h^{1/3})^3}$

$$=\lim_{h\to 0}\frac{f((h^{1/3}))^3}{(h^{1/3})^3}=\lim_{h\to 0}\left(\frac{f(h^{1/3})}{(h^{1/3})}\right)^3=I^3$$

$\Rightarrow I=I^3$

or $I=0, 1, -1$ as $f'(0)\ge 0$ $\quad(\therefore f'(0)=0, 1)$ (5)

Thus, $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{f(x+(h^{1/3})^3)-f(x)}{(h^{1/3})^3}$$

$$f'(x)=\lim_{h\to 0}\frac{f(x)+(f(h^{1/3}))^3-f(x)}{(h^{1/3})^3}\quad\text{[using (1)]}$$

$$\Rightarrow f'(x)=\lim_{h\to 0}\left(\frac{f(h^{1/3})}{(h^{1/3})}\right)^3=(f'(0))^3$$

$\Rightarrow f'(x)=0, 1$ $\quad$ [as $f'(0)=0, 1$ using (5)]

Integrating both sides, we get

$f(x)=c$ or $x+c$ $\quad$ as $f(0)=0$

$\Rightarrow\quad f(x)=0$ $\quad$ or x

Thus, $\quad f(10)=0$ $\quad$ or 10

17. We have

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)+f(h)+2hx-1-f(x)}{h}$$

(using the given definition)

$$=\lim_{h\to 0}\left(2x+\frac{f(h)-1}{h}\right)$$

Now substituting $x=y=0$ in the given functional relation, we get

$f(0)=f(0)+f(0)+0-1\Rightarrow f(0)=1$

$\therefore f'(x)=2x+\lim\limits_{h\to 0}\frac{f(h)-f(0)}{h}=2x+f'(0)$

$\Rightarrow f'(x)=2x+\cos\alpha$

Integrating, we get $f(x)=x^2+x\cos\alpha+C$

Here, $x=0$ and $f(0)=1$

$\therefore 1=C$

$\Rightarrow f(x)=x^2+x\cos\alpha+1.$

It is a quadratic in x with discriminant.

$D=\cos^2\alpha-4<0$

and coefficient of $x^2=1>0.$

$\therefore f(x)>0\ \forall\ x\in R$

Alternative Method

$f(x+y)=f(x)+f(y)+2xy-1$ (1)

Differentiate w.r.t. x keeping y as constant

$f'(x+y)=f'(x)+2y$

Put $x=0$ and $y=x$

We get $f'(x)=f'(0)+2x$

$\Rightarrow\ f'(x)=\cos\alpha+2x$

$\Rightarrow\ f(x)=x\cos\alpha+x^2+c$ (2)

Put $x=y=0$ in (1), we get $f(0)=f(0)+f(0)-1\Rightarrow f(0)=1$

Then from (2), we get $f(x)=x^2+(\cos\alpha)x+1$

18. $\because f\left(\frac{x+y}{3}\right)=\frac{2+f(x)+f(y)}{3}$ (1)

Replacing x by $3x$ and y by 0, then $f(x)=\frac{2+f(3x)+f(0)}{3}$

$\Rightarrow\ f(3x)-3f(x)+2=-f(0)$ (2)

In (1) putting $x=0$ and $y=0$, then

we get, $f(0)=2$ (3)

$$\text{Now, } f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f\left(\frac{3x+3h}{3}\right)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{2+f(3x)+f(3h)}{3}-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(3x)-3f(x)+f(3h)+2}{3h}$$

$$=\lim_{h\to 0}\frac{f(3h)-f(0)}{3h}\quad\text{[from (2)]}$$

$= f'(0) = c$ (say)

$\therefore f'(x) = c$

At $x = 2, f'(2) = c = 2$ (given)

$\therefore f'(x) = 2$

Integrating both sides, we get

$f(x) = 2x + d$ [from (2)]

$\therefore \quad d = 2$

then $f(x) = 2x + 2$. Hence, $y = 2x + 2$.

Alternative Method

$$f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3} \quad (1)$$

Differentiating w.r.t. x keeping y as constant,

we get $f'\left(\frac{x+y}{3}\right)\frac{1}{3} = \frac{f'(x)}{3}$

Put $x = 2$ and $y = 3x - 2$

we get $f'(x) = 2$

Integrating, we get $f(x) = 2x + c$

Now, put $x = y = 0$ in (1), we get $3f(0) = 2 + 2f(0)$

$\Rightarrow f(0) = 2$

Hence, $f(x) = 2x + 2$

19. $(xf)' = xf' + f$

and $(x^2 f)'' = (2xf + x^2 f')' = 2 + 2xf' + x^2 f''$

$$\Rightarrow \Delta = \begin{vmatrix} f & g & h \\ xf'+f & xg'+g & xh'+h \\ 2f+4xf'+x^2f'' & 2g+4xg'+x^2g'' & 2h+4xh'+x^2f'' \end{vmatrix}$$

$R_2 \to R_2 - R_1$ and then $R_3 \to R_3 - 4R_2 - 2R_1$

$$\Rightarrow \Delta = \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2f'' & x^2g'' & x^2h'' \end{vmatrix}$$

Taking x common from R_2 and multiplying with R_3

$$\Rightarrow \Delta = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix}$$

$$\Rightarrow \frac{d\Delta}{dx} = \begin{vmatrix} f' & g' & h' \\ f' & g' & h' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

$$= 0 + 0 + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

20. Given that $f'(\sin x) = \dfrac{df(\sin x)}{d(\sin x)} = \log_e x$

$= \log_e(\pi - \sin^{-1}(\sin x))$

$(\because \sin^{-1}\sin x = \pi - x \text{ for } x \in (\pi/2, \pi))$

$$\Rightarrow \frac{df(t)}{dt} = \log_e(\pi - \sin^{-1} t)$$

$$\Rightarrow f'(t) = \log_e(\pi - \sin^{-1} t) \quad (1)$$

and $y = f(a^x)$

$$\frac{dy}{dx} = f'(a^x)a^x \log_e a$$

$$= a^x \log_e a \log_e(\pi - \sin^{-1} a^x)$$

(using (1))

Objective Type

1. a. $y = \tan^{-1}\left\{\sqrt{\dfrac{1+\cos x}{1-\cos x}}\right\}$

$$= \tan^{-1}\left\{\sqrt{\frac{2\cos^2 x/2}{2\sin^2 x/2}}\right\}$$

$$= \tan^{-1}\left|\cot\frac{x}{2}\right| = \tan^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow y = \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \frac{x}{2}\right)\right\} = \frac{\pi}{2} - \frac{x}{2}$$

$$\therefore \frac{dy}{dx} = 0 - \frac{1}{2} = -\frac{1}{2}$$

2. b. $f(x) = |x^2 - 5x + 6| = \begin{cases} x^2 - 5x + 6 & \text{if } x \ge 3 \text{ or } x \le 2 \\ -(x^2 - 5x + 6), & \text{if } 2 < x < 3 \end{cases}$

$$\Rightarrow \quad f'(x) = \begin{cases} (2x-5), & \text{if } x > 3 \text{ or } x < 2 \\ -(2x-5), & \text{if } 2 < x < 3 \end{cases}$$

3. c. We have $y = \tan^{-1}\left(\dfrac{\log e - \log x^2}{\log e + \log x^2}\right) + \tan^{-1}\left(\dfrac{3 + 2\log x}{1 - 6\log x}\right)$

$$= \tan^{-1}\left(\frac{1 - 2\log x}{1 + 2\log x}\right) + \tan^{-1}\left(\frac{3 + 2\log x}{1 - 6\log x}\right)$$

$= \tan^{-1} 1 - \tan^{-1}(2\log x) + \tan^{-1} 3 + \tan^{-1}(2\log x)$

$= \tan^{-1} 1 + \tan^{-1} 3$

$$\Rightarrow \frac{dy}{dx} = 0 \Rightarrow \frac{d^2y}{dx^2} = 0$$

4.c. $y'(x) = f'(f(f(f(x))))f'(f(f(x)))f'(f(x))f'(x)$

$\Rightarrow y'(0) = f'(f(f(f(0))))f'(f(f(0)))f'(f(0))f'(0)$

$= f'(f(f(0)))f'(f(0))f'(0)f'(0)$

$= f'(f(0))f'(0)f'(0)f'(0)$

$= f'(0)f'(0)f'(0)f'(0)$

$= (f'(0))^4 = 2^4 = 16$

5.b. $y = ax^{n+1} + bx^{-n}$

$\Rightarrow \frac{dy}{dx} = (n+1)ax^n - nbx^{-n-1}$

$\Rightarrow \frac{d^2y}{dx^2} = n(n+1)ax^{n-1} + n(n+1)bx^{-n-2}$

$\Rightarrow x^2\frac{d^2y}{dx^2} = n(n+1)y$

6.c. $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

$\Rightarrow \frac{dy}{dx} = 0 + 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!}$

$\Rightarrow \frac{dy}{dx} + \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$

$\Rightarrow \frac{dy}{dx} = y - \frac{x^n}{n!}$

7.d. $y = a\sin x + b\cos x$

Differentiating with respect to x, we get

$\frac{dy}{dx} = a\cos x - b\sin x$

Now, $\left(\frac{dy}{dx}\right)^2 = (a\cos x - b\sin x)^2$

$= a^2\cos^2 x + b^2\sin^2 x - 2ab\sin x\cos x$, and

$y^2 = (a\sin x + b\cos x)^2$

$= a^2\sin^2 x + b^2\cos^2 x + 2ab\sin x\cos x$

So, $\left(\frac{dy}{dx}\right)^2 + y^2 = a^2(\sin^2 x + \cos^2 x) + b^2(\sin^2 x + \cos^2 x)$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 + y^2 = (a^2 + b^2) =$ constant.

8.b. $y = \sqrt{\frac{1-\sin 2x}{1+\sin 2x}} = \frac{\cos x - \sin x}{\cos x + \sin x} = \frac{1-\tan x}{1+\tan x} = \tan\left(\frac{\pi}{4} - x\right)$

$\Rightarrow \frac{dy}{dx} = -\sec^2\left(\frac{\pi}{4} - x\right)$

9.a. $\frac{dy}{dx} = \frac{d}{dx}\left[\left(x + \sqrt{x^2+a^2}\right)^n\right]$

$= n\left(x + \sqrt{x^2+a^2}\right)^{n-1} \cdot \frac{d}{dx}\left(x + \sqrt{x^2+a^2}\right)$

$= n(x + \sqrt{x^2+a^2})^{n-1}\left(\frac{\sqrt{x^2+a^2} + x}{\sqrt{x^2+a^2}}\right)$

$= \frac{n(x + \sqrt{x^2+a^2})^n}{\sqrt{x^2+a^2}}$

$= \frac{ny}{\sqrt{x^2+a^2}}$

10.b. $f(x) = \sqrt{1+\cos^2(x^2)}$

$\Rightarrow f'(x) = \frac{1}{2\sqrt{1+\cos^2(x)^2}}(2\cos x^2)(-\sin x^2)(2x)$

$\Rightarrow f'(x) = \frac{-x\sin 2x^2}{\sqrt{1+\cos^2(x^2)}}$

$\Rightarrow f'\left(\frac{\sqrt{\pi}}{2}\right) = \frac{-\frac{\sqrt{\pi}}{2}\sin\frac{2\pi}{4}}{\sqrt{1+\cos^2\frac{\pi}{4}}} = \frac{-\frac{\sqrt{\pi}}{2}1}{\sqrt{\frac{3}{2}}}$

$\therefore f'\left(\frac{\sqrt{\pi}}{2}\right) = -\sqrt{\frac{\pi}{6}}$

11.a. $\frac{d}{dx}\cos^{-1}\sqrt{\cos x} = \frac{\sin x}{2\sqrt{\cos x}\sqrt{1-\cos x}}$

$= \frac{\sqrt{1-\cos^2 x}}{2\sqrt{\cos x}\sqrt{1-\cos x}} = \frac{1}{2}\sqrt{\frac{1+\cos x}{\cos x}}$

12.c. $y = \frac{\log\tan x}{\log\sin x}$

$\Rightarrow \frac{dy}{dx} = \frac{(\log\sin x)\left(\frac{\sec^2 x}{\tan x}\right) - (\log\tan x)(\cot x)}{(\log\sin x)^2}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{\pi/4} = \frac{-4}{\log 2}$ (On simplification)

13.b. $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$

$\frac{dy}{dx} = \frac{\sqrt{1-x^2}\frac{1}{\sqrt{1-x^2}} - \left(\sin^{-1} x\right)\frac{1}{2}\frac{(-2x)}{\sqrt{1-x^2}}}{1-x^2}$

$\Rightarrow (1-x^2)\frac{dy}{dx} = 1 + x\left(\frac{\sin^{-1} x}{\sqrt{1-x^2}}\right) = 1 + xy$

14.a. $y = \cot^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$

$= \cot^{-1}\left[\frac{2+2\cos x}{2\sin x}\right] = \cot^{-1}\left[\frac{1+\cos x}{\sin x}\right]$

$= \cot^{-1}\left[\cot\frac{x}{2}\right] = \frac{x}{2}$

$\therefore \frac{dy}{dx} = \frac{1}{2}$

15.c. $y = x^{(x^x)}$

$\Rightarrow \log y = x^x \log x$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{dz}{dx}\log x + \frac{1}{x}z$ (where $x^x = z$)

$\Rightarrow \frac{dy}{dx} = x^{(x^x)}\left[x^x(\log ex)\log x + x^{x-1}\right]$

$\left(\because \frac{dz}{dx} = x^x \log ex\right)$

16.b. Let $y = \sin^2 \cot^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

$\Rightarrow y = \sin^2 \cot^{-1}\left\{\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\} = \sin^2 \cot^{-1}\left(\tan\frac{\theta}{2}\right)$

$\Rightarrow y = \sin^2\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$

$= \cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2} = \frac{1+x}{2}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}$

17.c. $y = ae^{mx} + be^{-mx}$

$\Rightarrow \frac{dy}{dx} = ame^{mx} - mbe^{-mx}$

Again $\frac{d^2y}{dx^2} = am^2 e^{mx} + m^2 be^{-mx}$

$\Rightarrow \frac{d^2y}{dx^2} = m^2\left(ae^{mx} + be^{-mx}\right) \Rightarrow \frac{d^2y}{dx^2} = m^2 y$

$\Rightarrow \frac{d^2y}{dx^2} - m^2 y = 0$

18.d. Let $y = \log x$

$\Rightarrow y_1 = \frac{1}{x},\ y_2 = \frac{-1}{x^2},\ y_3 = \frac{2}{x^3},\ \ldots,\ y_n = \frac{(-1)^{n-1}(n-1)!}{x^n}.$

19.c. $y = \sqrt{\log x + y}$

$\Rightarrow y^2 = \log x + y$

$\Rightarrow 2y\frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{x(2y-1)}$

20.c. $f(\log_e x) = \log_e(\log_e x)$

$\therefore \frac{df(\log_e x)}{dx} = \frac{1}{\log_e x} \times \frac{1}{x}$

21.a. $y = \sec(\tan^{-1} x) = \sec\left(\sec^{-1}\sqrt{1+x^2}\right) = \sqrt{1+x^2}$

Differentiating w.r.t. x, we have $\frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = \frac{1}{\sqrt{2}}$

22.a. $y = f(x^2) \Rightarrow \frac{dy}{dx} = f'(x^2)2x = 2x\sqrt{2(x^2)^2 - 1}$

At $x = 1$, $\frac{dy}{dx} = 2 \times 1 \times \sqrt{2-1} = 2$

23.a. $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{f'(x^3)3x^2}{g'(x^2)2x} = \frac{\cos x^3 3x^2}{\sin x^2 2x}$

$= \frac{3}{2}x\cos x^3 \operatorname{cosec} x^2$

24.b. $\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{\cos t - t\sin t}{1+\cos t}$

$\therefore \frac{d^2x}{dy^2} = \frac{\frac{d}{dt}\left(\frac{dx}{dy}\right)}{\frac{dy}{dt}}$

$= \frac{\frac{(-2\sin t - t\cos t)(1+\cos t) - (\cos t - t\sin t)(-\sin t)}{(1+\cos t)^2}}{1+\cos t}$

Now, put $t = \pi/2$

25.c. $f(x) = \sqrt{1-\sin 2x} = \sqrt{(\cos x - \sin x)^2}$

$= |\cos x - \sin x|$

$= \begin{cases} \cos x - \sin x, & \text{for } 0 \le x \le \pi/4 \\ -(\cos x - \sin x), & \text{for } \pi/4 < x \le \pi/2 \end{cases}$

$\therefore f'(x) = \begin{cases} -(\cos x + \sin x), & \text{for } 0 < x < \pi/4 \\ (\cos x + \sin x), & \text{for } \pi/4 < x < \pi/2. \end{cases}$

26.d. Let $u = y^2$ and $v = x^2$

$\therefore \frac{du}{dx} = \frac{d}{dx}y^2 = \left(\frac{d}{dy}y^2\right)\left(\frac{dy}{dx}\right)$

$= 2y(1-2x) = 2(x-x^2)(1-2x) = 2x(1-x)(1-2x)$ (1)

and $\frac{dv}{dx} = 2x$ (2)

Hence, $\frac{du}{dv} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)} = \frac{2x(1-x)(1-2x)}{2x}$ (from (1) and (2))

$= (1-x)(1-2x) = 1 - 3x + 2x^2$

27.a. $f(x)=\cos^{-1}\left[\cos\left(\frac{\pi}{2}-\sqrt{\frac{1+x}{2}}\right)\right]+x^x=\frac{\pi}{2}-\sqrt{\frac{1+x}{2}}+x^x$

$\Rightarrow f'(x)=-\frac{1}{\sqrt{2}}\times\frac{1}{2\sqrt{1+x}}+x^x(1+\log x)$

$\Rightarrow f'(1)=-\frac{1}{4}+1=\frac{3}{4}$

28.b. $D=\begin{vmatrix}\sin px & p\cos px & -p^2\sin px\\ -p^3\cos px & p^4\sin px & p^5\cos px\\ -p^6\sin px & -p^7\cos px & p^8\sin px\end{vmatrix}$

$=p^9\begin{vmatrix}\sin px & p\cos px & -p^2\sin px\\ -\cos px & p\sin px & p^2\cos px\\ -\sin px & -p\cos px & p^2\sin px\end{vmatrix}$

$=-p^9\begin{vmatrix}\sin px & p\cos px & -p^2\sin px\\ \cos px & p\sin px & p^2\cos px\\ \sin px & p\cos px & -p^2\sin px\end{vmatrix}=0$

29.b. $2xf'(x^2)=3x^2 \Rightarrow 4f'(2)=12\Rightarrow f'(4)=3.$

30.c. $(a^2-2a-15)e^{ax}+(b^2-2b-15)e^{bx}=0$

$\Rightarrow (a^2-2a-15)=0$ and $b^2-2b-15=0$

$\Rightarrow (a-5)(a+3)=0$ and $(b-5)(b+3)=0$

$\Rightarrow a=5$ or -3 and $b=5$ or -3

$\therefore a\neq b$ hence $a=5$ and $b=-3$

or $a=-3$ and $b=5$

$\Rightarrow ab=-15$

31.b. $y=\frac{(a-x)^{3/2}+(x-b)^{3/2}}{\sqrt{a-x}+\sqrt{x-b}}$

$=\frac{\left(\sqrt{a-x}+\sqrt{x-b}\right)\left(a-x-\sqrt{a-x}\sqrt{x-b}+x-b\right)}{\sqrt{a-x}+\sqrt{x-b}}$

$=a-b-\sqrt{a-x}\sqrt{x-b}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2\sqrt{a-x}}\sqrt{x-b}-\frac{1}{2\sqrt{x-b}}\sqrt{a-x}$

$=\frac{2x-a-b}{2\sqrt{a-x}\sqrt{x-b}}$

32.b. $f(g(x))=x$

$\Rightarrow f'(g(x))g'(x)=1$

$\Rightarrow (e^{g(x)}+1)g'(x)=1$

$\Rightarrow (e^{g(f(\log 2))}+1)g'(f(\log 2))=1$

$\Rightarrow (e^{\log 2}+1)g'(f(\log 2))=1$

$\Rightarrow g'(f(\log 2))=1/3$

33.c. $f'(x)=(kx+e^x)h'(x)+h(x)(k+e^x)$

$f'(0)=h'(0)+h(0)(k+1)$

$\Rightarrow 18=-2+5(k+1)\Rightarrow k=3$

34. d. $y=\tan^{-1}\left(\frac{2^{x+1}-2^x}{1+2^x.2^{x+1}}\right)=\tan^{-1}2^{(x+1)}-\tan^{-1}2^x$

$\Rightarrow y'=\frac{2^{x+1}\ln 2}{1+\left(2^{x+1}\right)^2}-\frac{2^x\ln 2}{1+\left(2^x\right)^2}$

$\Rightarrow y'(0)=-\frac{1}{10}\ln 2$

35.b. $f(x)=1+x^2+x^4+x^6+\cdots\infty$, where $|x|\le 1$

$\Rightarrow f^n(0)=n!$, where n is even.

36.b. $y=2\cos x\cos 3x=\cos 4x+\cos 2x$

$\Rightarrow \frac{d^{20}y}{dx^{20}}=4^{20}\cos 4x+2^{20}\cos 2x$

37.c. We have $y=\sqrt{\frac{1-x}{1+x}}$

Differentiating w.r.t. x, we get

$\frac{dy}{dx}=\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{1/2-1}\frac{d}{dx}\left(\frac{1-x}{1+x}\right)$

$=\frac{1}{2}\sqrt{\frac{1+x}{1-x}}\times\frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}$

$=-\sqrt{\frac{1+x}{1-x}}\frac{1}{(1+x)^2}$

$\Rightarrow (1-x^2)\frac{dy}{dx}=-\sqrt{\frac{1+x}{1-x}}\frac{1}{(1+x)^2}(1-x^2)$

$\Rightarrow (1-x)^2\frac{dy}{dx}=-\sqrt{\frac{1-x}{1+x}}$

$\Rightarrow (1-x^2)\frac{dy}{dx}=-y$

$\Rightarrow \left(1-x^2\right)\frac{dy}{dx}+y=0$

38.b. $\frac{dy}{dx}=-[(2-x)(3-x)\cdots(n-x)+(1-x)(3-x)\cdots(n-x)+\cdots(1-x)(2-x)\cdots(n-1-x)]$

At $x=1$

$\frac{dy}{dx}=-[(n-1)!+0+\cdots+0]=(-1)(n-1)!$

39.a. Let $\cos\alpha=\frac{5}{13}$, then $\sin\alpha=\frac{12}{13}$.

So, $y=\cos^{-1}(\cos\alpha.\cos x-\sin\alpha.\sin x)$

$\Rightarrow y=\cos^{-1}\{\cos(x+\alpha)\}=x+\alpha$ ($\because x+\alpha$ is in the first or the second quadrant)

$$\Rightarrow \frac{dy}{dx}=1$$

40.a. Let $x=\sec\theta$

Then $y=\tan^{-1}\sqrt{\frac{\sec\theta+1}{\sec\theta-1}}$

$$=\tan^{-1}\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}=\tan^{-1}\left(\cot\frac{\theta}{2}\right)$$

$$\Rightarrow y=\tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}=\frac{\pi}{2}-\frac{1}{2}\sec^{-1}x$$

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{2}\times\frac{1}{|x|\sqrt{x^2-1}}$$

41.d. We have $\sin^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right)=\log a$

$$\Rightarrow \frac{x^2-y^2}{x^2+y^2}=\sin(\log a)$$

$\Rightarrow \frac{1-\tan^2\theta}{1+\tan^2\theta}=\sin(\log a)$, on putting $y=x\tan\theta$

$\Rightarrow \cos 2\theta=\sin(\log a)$

$\Rightarrow 2\theta=\cos^{-1}(\sin(\log a))$

$$\Rightarrow \theta=\frac{1}{2}\cos^{-1}(\sin(\log a))$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{2}\cos^{-1}(\sin(\log a))$$

$$\Rightarrow \frac{y}{x}=\tan\left(\frac{1}{2}\cos^{-1}(\sin(\log a))\right)$$

Differentiating w.r.t. x

$$\Rightarrow \frac{x\frac{dy}{dx}-y}{x^2}=0$$

$$\Rightarrow x\frac{dy}{dx}-y=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x}$$

42.b. $y=\cos^{-1}(\cos x)=\cos^{-1}\{\cos[2\pi-(2\pi-x)]\}$.

$=\cos^{-1}[\cos(2\pi-x)]$

$=2\pi-x$

$$\therefore \frac{dy}{dx}=-1 \quad \text{at } x=\frac{5\pi}{4}$$

43.a. Let $t=\cos 2\theta$

Then $e^x=\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}$

$$=\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\frac{1-\tan\theta}{1+\tan\theta}=\tan\left(\frac{\pi}{4}-\theta\right)$$

$$\tan\frac{y}{2}=\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}=\tan\theta$$

At $t=\frac{1}{2}$, $\cos 2\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{6}$

Then $x=\log\tan\frac{\pi}{12}$, $y=\frac{\pi}{3}$

Differentiating w.r.t. θ, $e^x\frac{dx}{d\theta}=-\sec^2\left(\frac{\pi}{4}-\theta\right)$ and

$$\frac{1}{2}\sec^2\frac{y}{2}\frac{dy}{d\theta}=\sec^2\theta$$

$$\therefore \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{2\sec^2\theta\cos^2\frac{y}{2}}{-e^{-x}\sec^2\left(\frac{\pi}{4}-\theta\right)}$$

At $t=\frac{1}{2}$, i.e., $\theta=\frac{\pi}{6}$, $\frac{dy}{dx}=\frac{2\sec^2\frac{\pi}{6}\cos^2\frac{\pi}{6}}{-e^{-\log\tan\pi/12}\sec^2\frac{\pi}{12}}$

$$\therefore \frac{dy}{dx}=\frac{2}{-\cot\frac{\pi}{12}\sec^2\frac{\pi}{12}}$$

$$=-2\tan\frac{\pi}{12}\cos^2\frac{\pi}{12}=-\sin\frac{\pi}{6}=-\frac{1}{2}$$

44.b. We have $x^2+y^2=t-\frac{1}{t}$ and $x^4+y^4=t^2+\frac{1}{t^2}$

$$\Rightarrow (x^2+y^2)^2=t^2+\frac{1}{t^2}-2$$

$\Rightarrow (x^2+y^2)^2=x^4+y^4-2$

$\Rightarrow 2x^2y^2=-2$

$\Rightarrow x^2y^2=-1$

$$\Rightarrow y^2=-\frac{1}{x^2}$$

$$\Rightarrow 2y\frac{dy}{dx}=\frac{2}{x^3}$$

$$\Rightarrow x^3y\frac{dy}{dx}=1$$

45.a. We have

$$y^{1/m} = \left(x+\sqrt{1+x^2}\right)$$

$$\Rightarrow y = \left(x+\sqrt{1+x^2}\right)^m$$

$$\Rightarrow \frac{dy}{dx} = m\left(x+\sqrt{1+x^2}\right)^{m-1}\left(1+\frac{x}{\sqrt{x^2+1}}\right)$$

$$= m\frac{\left(x+\sqrt{1+x^2}\right)^m}{\sqrt{1+x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{my}{\sqrt{1+x^2}}$$

$$\Rightarrow y_1^2(1+x^2) = m^2y^2$$

$$\Rightarrow 2y_1y_2(1+x^2)+2xy_1^2 = 2m^2yy_1$$

$$\Rightarrow y_2(1+x^2)+xy_1 = m^2y$$

46.a. $y=f(x)-f(2x) \Rightarrow y'=f'(x)-2f'(2x)$

$\Rightarrow y'(1)=f'(1)-2f'(2)=5$, and (1)

$y'(2)=f'(2)-2f'(4)=7$ (2)

Now let $y=f(x)-f(4x)$

$\Rightarrow y'=f'(x)-4f'(4x)$

$\Rightarrow y'(1)=f'(1)-4f'(4)$ (3)

Substituting the value of $f'(2)=7+2f'(4)$ in (1), we get

$f'(1)-2(7+2f'(4))=5$

$f'(1)-4f'(4)=19$

47.a. $f(10)=\sin^{-1}\cos 10=\sin^{-1}\sin\left(\frac{\pi}{2}-10\right)$

$$= -\sin^{-1}\sin\left(10-\frac{\pi}{2}\right)$$

$$= -\sin^{-1}\sin\left(3\pi-10+\frac{\pi}{2}\right) = -\left(3\pi+\frac{\pi}{2}-10\right) = 10-\frac{7\pi}{2}$$

$$f'(x) = \frac{-\sin x}{\sqrt{1-\cos^2 x}} = \frac{-\sin x}{|\sin x|} \Rightarrow f'(10) = \frac{-\sin 10}{|\sin 10|} = 1.$$

So, $f(10)+f'(10)=11-\frac{7\pi}{2}$

48.a. $(\sin x)(\cos y)=\frac{1}{2}$

$$\Rightarrow (\cos x)(\cos y) - \sin y \sin x\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = (\cot x)(\cot y)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\text{cosec}^2 x.\cot y - \text{cosec}^2 y \cot x.\frac{dy}{dx}$$

Now $\left(\frac{dy}{dx}\right)_{(\pi/4,\,\pi/4)} = 1$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{(\pi/4,\,\pi/4)} = -(2)(1)-(2)(1)(1) = -4$$

49.a. Given $f=f'+f''+f'''+\cdots\infty$

$\Rightarrow f'=f''+f'''+f''''+\cdots\infty$

$\Rightarrow f-f'=f'$

$\Rightarrow f=2f'$

Hence, $\frac{f'}{f}=1/2 \Rightarrow \int\frac{f'}{f}dx = \int\frac{1}{2}dx$

$\Rightarrow \log f(x) = x/2+c$

$\Rightarrow f(x)=e^{x/2+c}$

Also, $f(0)=1 \Rightarrow c=0 \Rightarrow f(x)=e^{x/2}$

50.b. Putting $x=\sin\theta$ and $y=\sin\phi$

$\cos\theta+\cos\phi = a(\sin\theta-\sin\phi)$

$$\Rightarrow 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} = a\left(2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}\right)$$

$$\Rightarrow \frac{\theta-\phi}{2} = \cot^{-1}a$$

$$\Rightarrow \theta-\phi = 2\cot^{-1}a$$

$$\Rightarrow \sin^{-1}x-\sin^{-1}y = 2\cot^{-1}a$$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

51.a. $y=x^2+\frac{1}{y}$

$$\Rightarrow y^2 = x^2y+1$$

$$\Rightarrow 2y\frac{dy}{dx} = y.2x+x^2\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{2y-x^2}.$$

52.d. $\frac{d}{dx}\left(\tan^{-1}\frac{\left(\sqrt{x}(3-x)\right)}{1-3x}\right)$

Put $\sqrt{x}=\tan\theta \Rightarrow \theta=\tan^{-1}\sqrt{x}$

$$\frac{d}{dx}\left(\tan^{-1}\frac{\left(\tan\theta(3-\tan^2\theta)\right)}{1-3\tan^2\theta}\right)$$

$$\frac{d}{dx}\left(\tan^{-1}\frac{\left(3\tan\theta-\tan^3\theta\right)}{1-3\tan^2\theta}\right)$$

$$\frac{d}{dx}\left[\tan^{-1}(\tan 3\theta)\right] = \frac{d}{dx}(3\theta)$$

$$= \frac{d}{dx}\left[3\tan^{-1}\sqrt{x}\right] = \frac{3}{2\sqrt{x}\,(1+x)}$$

53.b. Since $g(x)$ is the inverse of function $f(x)$, therefore $gof(x)=I(x)$ for all x

Now $gof(x)=I(x), \forall x$

$\Rightarrow (gof)'(x) = 1, \forall x$

$\Rightarrow g'(f(x))f'(x) = 1, \forall x$

$\Rightarrow g'(f(x)) = \dfrac{1}{f'(x)}, \forall x$

$\Rightarrow g'(f(c)) = \dfrac{1}{f'(c)}$ (putting $x = c$)

54.c. $f(x) = x + \tan x$

$f(f^{-1}(y)) = f^{-1}(y) + \tan f^{-1}(y)$

$y = g(y) + \tan g(y)$

$x = g(x) + \tan g(x)$

Differentiating, we get $1 = g'(x) + \sec^2 g(x)\, g'(x)$

$$\Rightarrow g'(x) = \frac{1}{1+\sec^2 g(x)} = \frac{1}{2+[g(x)-x]^2}$$

55.a. $y\sqrt{x^2+1} = \log\left\{\sqrt{x^2+1} - x\right\}$

Differentiating both sides w.r.t. x., we get

$$\frac{dy}{dx}\sqrt{x^2+1} + y\frac{1}{2\sqrt{x^2+1}}2x = \frac{1}{\sqrt{x^2+1}-x} \times \left\{\frac{1}{2}\frac{2x}{\sqrt{x^2+1}} - 1\right\}$$

$$\Rightarrow (x^2+1)\frac{dy}{dx} + xy = \sqrt{x^2+1}\frac{-1}{\sqrt{x^2+1}}$$

$$\Rightarrow (x^2+1)\frac{dy}{dx} + xy + 1 = 0$$

56.a. $y = \dfrac{\sqrt{a+x} - \sqrt{a-x}}{\sqrt{a+x} + \sqrt{a-x}}$

$$\Rightarrow y = \frac{\left(\sqrt{a+x} - \sqrt{a-x}\right)^2}{(a+x)-(a-x)}$$

$$\Rightarrow y = \frac{(a+x)+(a-x)-2\left(\sqrt{a^2-x^2}\right)}{2x}$$

$$= \frac{2a - 2\sqrt{a^2-x^2}}{2x} = \frac{a-\sqrt{a^2-x^2}}{x} \quad (1)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{x\left[-\frac{1}{2\sqrt{a^2-x^2}}(-2x)\right] - \left(a - \sqrt{a^2-x^2}\right)}{x^2}$$

$$= \frac{x^2 - a\sqrt{a^2-x^2} + a^2 - x^2}{x^2\sqrt{a^2-x^2}} = \frac{a\left(a-\sqrt{a^2-x^2}\right)}{x^2\sqrt{a^2-x^2}}$$

$$= \frac{a}{x\sqrt{a^2-x^2}}\left[\frac{a-\sqrt{a^2-x^2}}{x}\right] = \frac{ay}{x\sqrt{a^2-x^2}} \quad \text{[by (1)]}$$

57.a. As $f(x) = x^4\tan(x^3) - x\ln(1+x^2)$ is odd, $\Rightarrow \dfrac{d^3 f(x)}{dx^3}$ is even

$$\Rightarrow \frac{d^4 f(x)}{dx^4} = 0 \text{ at } x = 0.$$

58.a. Given that $g^{-1}(x) = f(x) \Rightarrow x = g(f(x))$ or $g'(f(x))f'(x) = 1$

$$\Rightarrow g'(f(x)) = \frac{1}{f'(x)}$$

$$\Rightarrow g''(f(x))f'(x) = \frac{-f''(x)}{[f'(x)]^2} \Rightarrow g''(f(x)) = \frac{-f''(x)}{[f'(x)]^3}.$$

59.b. For $x > 1$, we have $f(x) = |\log|x|| = \log x$

$\Rightarrow f'(x) = \dfrac{1}{x}$

For $x < -1$, we have $f(x) = |\log|x|| = \log(-x)$

$\Rightarrow f'(x) = \dfrac{1}{x}$

For $0 < x < 1$, we have $f(x) = |\log|x|| = -\log x$

$\Rightarrow f'(x) = \dfrac{-1}{x}$

For $-1 < x < 0$, we have $f(x) = -\log(-x)$

$\Rightarrow f'(x) = \dfrac{-1}{x}$.

$$\text{Hence, } f'(x) = \begin{cases} \dfrac{1}{x}, & |x| > 1 \\ -\dfrac{1}{x}, & |x| < 1 \end{cases}$$

60.c. In neighbourhood of $x = \dfrac{2\pi}{3}$, $|\cos x| = -\cos x$ and $|\sin x| = \sin x$

$\Rightarrow y = -\cos x + \sin x$

$\Rightarrow \dfrac{dy}{dx} = \sin x + \cos x$

$\Rightarrow$ At $x = \dfrac{2\pi}{3}$, $\dfrac{dy}{dx} = \sin\dfrac{2\pi}{3} + \cos\dfrac{2\pi}{3} = \dfrac{\sqrt{3}-1}{2}$.

61.a. Since g is the inverse function of f, we have $f\{g(x)\} = x$

$\Rightarrow \dfrac{d}{dx}(f\{g(x)\}) = 1$

$\Rightarrow f'\{g(x)\}.g'(x) = 1$

$\Rightarrow \sin\{g(x)\}\, g'(x) = 1$

$\Rightarrow g'(x) = \dfrac{1}{\sin\{g(x)\}}$

62.b. We have $x = \phi(t)$, $y = \psi(t)$. Therefore,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\psi'}{\phi'}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\psi'}{\phi'}\right) = \frac{d}{dt}\left(\frac{\psi'}{\phi'}\right)\frac{dt}{dx}$$

$$= \frac{\phi'\psi'' - \psi'\phi''}{\phi'^2}\frac{1}{\phi'} = \frac{\phi'\psi'' - \psi'\phi''}{\phi'^3}$$

63.c. $f(x)=e^x-e^{-x}-2\sin x-\frac{2}{3}x^3$

$f^{I}(x)=e^x+e^{-x}-2\cos x-2x^2$
$f^{II}(x)=e^x-e^{-x}+2\sin x-4x$
$f^{III}(x)=e^x+e^{-x}+2\cos x-4$
$f^{IV}(x)=e^x-e^{-x}-2\sin x$
$f^{V}(x)=e^x+e^{-x}-2\cos x$
$f^{VI}(x)=e^x-e^{-x}+2\sin x$
$f^{VII}(x)=e^x+e^{-x}+2\cos x$
Clearly, $f^{VII}(0)$ is non-zero.

64.b. Given $f\left(\frac{5x-3y}{2}\right)=\frac{5f(x)-3f(y)}{2}$

$\Rightarrow f\left(\frac{5x-3y}{5-3}\right)=\frac{5f(x)-3f(y)}{5-3}$, which satisfies section formula for abscissa on L.H.S. and ordinate on R.H.S.
Hence, $f(x)$ must be the linear function (as only straight line satisfies such section formula).
Hence, $f(x)=ax+b$
But $f(0)=3\Rightarrow b=3, f'(0)=2\Rightarrow a=2.$
Thus, $f(x)=2x+3\Rightarrow$ Period of $\sin(f(x))=\sin(2x+3)$ is π.

65.c. $D^*(x)=\lim_{h\to 0}\frac{f^2(x+h)-f^2(x)}{h}$

$=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}(f(x+h)+f(x))$

$=2f(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=2f(x)\times f'(x)$
$\Rightarrow D^*(x\log x)=2x\log x\,(1+\log x)$
$\Rightarrow D^*f(x)|_{x=e}=4e$

66.b. $\sqrt{x}=\cos\theta$

$x\in\left(0,\frac{1}{2}\right)\Rightarrow\sqrt{x}=\cos\theta\in\left(0,\frac{1}{\sqrt{2}}\right)$

$\Rightarrow\theta\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

$\Rightarrow 2\theta\in\left(\frac{\pi}{2},\pi\right)$

$\Rightarrow f(x)=2\sin^{-1}\sqrt{1-\cos^2\theta}+\sin^{-1}(2\sqrt{\cos^2\theta\sin^2\theta})$
$=2\sin^{-1}(\sin\theta)+\sin^{-1}(2\sin\theta\cos\theta)$
$=2\theta+\sin^{-1}(\sin 2\theta)$
$=2\theta+\pi-2\theta$
$=\pi$
$\Rightarrow f'(x)=0$

67.a. $F'(x)=\left[f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right)+g\left(\frac{x}{2}\right)g'\left(\frac{x}{2}\right)\right]$

Here $g(x)=f'(x)$
and $g'(x)=f''(x)=-f(x)$
so $F'(x)=f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right)-f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right)=0$
$\Rightarrow F(x)$ is a constant function.
$\Rightarrow F(10)=5$

68.b. Let $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, and $z=\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$

Putting $x=\tan\theta$ in y, we get

$y=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{1}{2}\tan^{-1}x$

$\Rightarrow\frac{dy}{dx}=\frac{1}{2(1+x^2)}$

Putting $x=\sin\theta$ in z, we get

$z=\tan^{-1}\left(\frac{2\sin\theta\cos\theta}{\cos 2\theta}\right)=\tan^{-1}(\tan 2\theta)=2\theta=2\sin^{-1}x$

$\Rightarrow\frac{dz}{dx}=\frac{2}{\sqrt{1-x^2}}$

Thus, $\frac{dy}{dz}=\frac{\frac{dy}{dx}}{\frac{dz}{dx}}=\frac{1}{4(1+x^2)}\sqrt{1-x^2}\Rightarrow\left(\frac{dy}{dz}\right)_{x=0}=\frac{1}{4}$

69.c. $f(x)=xe^x$
$f'(x)=e^x+xe^x$
$f''(x)=e^x+e^x+xe^x$
$f'''(x)=2e^x+e^x+xe^x=3e^x+xe^x$
...
...
$f^n(x)=ne^x+xe^x.$
Now, $f^n(x)=0$
$\Rightarrow ne^x+xe^x=0\Rightarrow x=-n$

70.a. $y^2=ax^2+bx+c$

$\Rightarrow 2y\frac{dy}{dx}=2ax+b$

$\Rightarrow 2\left(\frac{dy}{dx}\right)^2+2y\frac{d^2y}{dx^2}=2a$

$\Rightarrow y\frac{d^2y}{dx^2}=a-\left(\frac{dy}{dx}\right)^2$

$\Rightarrow y\frac{d^2y}{dx^2}=a-\left(\frac{2ax+b}{2y}\right)^2$

$\Rightarrow y\frac{d^2y}{dx^2}=\frac{4ay^2-(2ax+b)^2}{4y^2}$

$\Rightarrow 4y^3\frac{d^2y}{dx^2}=4a\left(ax^2+bx+c\right)-\left(4a^2x^2+4abx+b^2\right)$

$\Rightarrow 4y^3\frac{d^2y}{dx^2}=4ac-b^2=$ constant.

71.c. $y = \sin x + e^x \Rightarrow \dfrac{dy}{dx} = \cos x + e^x$

$$\Rightarrow \frac{dx}{dy} = (\cos x + e^x)^{-1} \quad (1)$$

Again, $\dfrac{d^2x}{dy^2} = -(\cos x + e^x)^{-2}(-\sin x + e^x)\dfrac{dx}{dy}$

Substituting the value of $\dfrac{dy}{dx}$ from (1)

$$\frac{d^2x}{dy^2} = \frac{(\sin x - e^x)}{(\cos x + e^x)^2}(\cos x + e^x)^{-1} = \frac{\sin x - e^x}{(\cos x + e^x)^3}$$

72.d. $u = x^2 + y^2, x = s + 3t, y = 2s - t$

Now, $\dfrac{dx}{ds} = 1, \dfrac{dy}{ds} = 2$ (1)

$\dfrac{d^2x}{ds^2} = 0, \dfrac{d^2y}{ds^2} = 0$ (2)

Now $u = x^2 + y^2, \dfrac{du}{ds} = 2x\dfrac{dx}{ds} + 2y\dfrac{dy}{ds}$

$$\frac{d^2u}{ds^2} = 2\left(\frac{dx}{ds}\right)^2 + 2x\frac{d^2x}{ds^2} + 2\left(\frac{dy}{ds}\right)^2 + 2y\left(\frac{d^2y}{ds^2}\right)$$

From (1) and (2), $\dfrac{d^2u}{ds^2} = 2\times1 + 0 + 2\times4 + 0 = 10.$

73.c. Here, $y = t^{10} + 1$ and $x = t^8 + 1$

$\therefore t^8 = x - 1 \Rightarrow t^2 = (x-1)^{1/4}$

So, $y = (x-1)^{5/4} + 1$

Differentiate both sides w.r.t. x, we get $\dfrac{dy}{dx} = \dfrac{5}{4}(x-1)^{1/4}$

Again, differentiate both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{5}{16}(x-1)^{-3/4}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{5}{16(x-1)^{3/4}} = \frac{5}{16(t^2)^3} = \frac{5}{16t^6}$$

74.b. From the given relation $\dfrac{y}{x} = \log x - \log(a + bx)$

Differentiating w.r.t. x, we get $\dfrac{\left(x\dfrac{dy}{dx}\right) - y}{x^2} =$

$$\frac{1}{x} - \frac{b}{a+bx} = \frac{a}{x(a+bx)}$$

$$\therefore x\frac{dy}{dx} - y = \frac{ax}{a+bx} \quad (1)$$

Differentiating again w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \frac{(a+bx)a - ax.b}{(a+bx)^2}$$

$$\Rightarrow x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2}$$

$$\Rightarrow x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} = \left(x\frac{dy}{dx} - y\right)^2 \text{ [by (1)]}$$

75.a. $u(x) = 7v(x) \Rightarrow u'(x) = 7v'(x) \Rightarrow p = 7$ (given)

Again $\dfrac{u(x)}{v(x)} = 7 \Rightarrow \left(\dfrac{u(x)}{v(x)}\right)' = 0 \Rightarrow q = 0$

Now $\dfrac{p+q}{p-q} = \dfrac{7+0}{7-0} = 1$

76.a. $ax^2 + 2hxy + by^2 = 1$

Differentiating both sides w.r.t. x, we get

$$2ax + 2hx\frac{dy}{dx} + 2hy + 2by\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{ax + hy}{hx + by}$$

Again differentiating w.r.t. x, we get

$$\Rightarrow \frac{d^2y}{dx^2}$$

$$= -\left[\frac{(hx+by)\left(a + h\dfrac{dy}{dx}\right) - (ax+hy)\left(h + b\dfrac{dy}{dx}\right)}{(hx+by)^2}\right]$$

$$= -\frac{\left[y(ab - h^2) + \dfrac{dy}{dx}(h^2x - abx)\right]}{(hx+by)^2}$$

$$= \frac{(h^2 - ab)\left(y - x\dfrac{dy}{dx}\right)}{(hx+by)^2}$$

$$= \frac{(h^2 - ab)}{(hx+by)^2}\left[y + x\frac{ax+hy}{hx+by}\right]$$

$$= \frac{h^2 - ab}{(hx+by)^2}$$

77. b. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2}{2t} = \dfrac{3}{2}\sqrt{x} \Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{3}{4\sqrt{x}} = \dfrac{3}{4t}$

78. b. $y = x + e^x \Rightarrow \dfrac{dy}{dx} = 1 + e^x \Rightarrow \dfrac{dx}{dy} = \dfrac{1}{1+e^x}$

$$\Rightarrow \frac{d}{dy}\left(\frac{dx}{dy}\right)=\frac{d}{dy}\left(\frac{1}{1+e^x}\right) \Rightarrow \frac{d^2x}{dy^2}=\frac{d}{dx}\left(\frac{1}{1+e^x}\right)\frac{dx}{dy}$$

$$\Rightarrow \frac{d^2x}{dy^2}=\frac{-e^x}{(1+e^x)^2}\frac{1}{(1+e^x)}=-\frac{e^x}{(1+e^x)^3}$$

79. a. In the neighbourhood of $x = 7\pi/6$, we have $f(x) = |\sin x + \cos x| = -\sin x - \cos x$

$\Rightarrow f'(x) = -\cos x + \sin x \Rightarrow f'(7\pi/6) = -\cos(7\pi/6) + \sin(7\pi/6)$

$= \frac{\sqrt{3}-1}{2}$

80. a. $y = f(x)$ is an even function and $y = g(x)$ is an odd function.

$\Rightarrow h(x) = f(x)\,g(x)$ is an odd function.

$\Rightarrow h(x) = -h(-x)$

$\Rightarrow h'(x) = h'(-x)$

$\Rightarrow h''(x) = -h''(-x)$

$\Rightarrow h'''(x) = h'''(-x)$

Now, we cannot determine the value of $h'''(0)$.

81. c. $\frac{dy}{dx}=\frac{-\frac{1}{p^2}}{\frac{1}{p}}=-\frac{1}{p}=-y \Rightarrow \frac{d^2y}{dx^2}=-\frac{dy}{dx}$

$\Rightarrow \frac{d^2y}{dx^2}+\frac{dy}{dx}=0$

82. a. $y = 2\ln(1+\cos x)$

$\frac{dy}{dx}=\frac{-2\sin x}{1+\cos x}$

$\frac{d^2y}{dx^2}=-2\left[\frac{(1+\cos x)\cos x-\sin x(-\sin x)}{(1+\cos x)^2}\right]$

$=-2\left[\frac{\cos x+1}{(1+\cos x)^2}\right]=\frac{-2}{(1+\cos x)}$

Now $2e^{-y/2}=2\cdot e^{-\frac{\ln(1+\cos x)^2}{2}}=\frac{2}{(1+\cos x)}$

$\therefore \frac{d^2y}{dx^2}+\frac{2}{e^{y/2}}=0$

83. a. Let $g(x)=(\sin x)^{\ln x}=e^{\ln x\cdot\ln(\sin x)}$

$f(x)=g'(x)=(\sin x)^{\ln x}\left[\cot x(\ln x)+\frac{\ln(\sin x)}{x}\right]$

Hence, $f\left(\frac{\pi}{2}\right)=g'\left(\frac{\pi}{2}\right)=1(0+0)=0.$

Multiple Correct Answers Type

1. a, c.

$$\frac{dy}{dx}=\frac{e^{\sqrt{x}}}{2\sqrt{x}}-\frac{e^{-\sqrt{x}}}{2\sqrt{x}}=\frac{e^{\sqrt{x}}-e^{-\sqrt{x}}}{2\sqrt{x}}$$

$$=\frac{\sqrt{\left(e^{\sqrt{x}}+e^{-\sqrt{x}}\right)^2-4}}{2\sqrt{x}}=\frac{\sqrt{y^2-4}}{2\sqrt{x}}$$

2. a, c, d.

$y^2=x+y \Rightarrow \frac{dy}{dx}=\frac{1}{2y-1}$

Also $y=\frac{x}{y}+1 \Rightarrow \frac{dy}{dx}=\frac{y}{2x+y}$

Also $y^2-y-x=0 \Rightarrow y=\frac{1\pm\sqrt{1+4x}}{2}$

$\Rightarrow y=\frac{1+\sqrt{1+4x}}{2}$. (as $y>0$)

$\Rightarrow y'=\frac{1}{4}\frac{4}{\sqrt{1+4x}}=\frac{1}{\sqrt{1+4x}}$

3. b, c, d.

1 is a root of $f(x)=0, f'(x)=0$ and $f''(x)=0$, or

1 is a root of $ax^3+bx^2+bx+d=0$ (1)

$3ax^2+2bx+b=0$ (2)

$\Rightarrow a+2b+d=0$

$a+b=0$

$\Rightarrow b+d=0$ and $a=d$.

4. a, c.

$x^3-2x^2y^2+5x+y-5=0$

Differentiating w.r.t. x, we get

$\Rightarrow 3x^2-4xy^2-4x^2y\frac{dy}{dx}+5+\frac{dy}{dx}=0$

$\Rightarrow y'=\frac{dy}{dx}=\frac{3x^2-4xy^2+5}{4x^2y-1}$

$y'(1)=\frac{3-4+5}{4-1}=\frac{4}{3}$

Also, y''

$$=\frac{(6x-4y^2-8xyy')(4x^2y-1)-(8xy+4x^2y')(3x^2-4xy^2+5)}{(4x^2y-1)^2}$$

$$\Rightarrow y''(1)=\frac{(6-4-8.\frac{4}{3})(4-1)-(8+4.\frac{4}{3})(3-4+5)}{(4-1)^2}$$

$=-8\frac{22}{27}$

5. a, b, c.

$f(x)=|x^2-3|x|+2|$

$$=\begin{cases}|x^2-3x+2|, & x\ge 0\\ |x^2+3x+2|, & x<0\end{cases}$$

$$=\begin{cases}x^2-3x+2, & x^2-3x+2\ge 0, & x\ge 0\\ -x^2+3x-2, & x^2-3x+2<0, & x\ge 0\\ x^2+3x+2, & x^2+3x+2\ge 0, & x<0\\ -x^2-3x-2, & x^2+3x+2<0, & x<0\end{cases}$$

$$=\begin{cases}x^2-3x+2, & x\in[0,1]\cup[2,\infty)\\ -x^2+3x-2, & x\in(1,2)\\ x^2+3x+2, & x\in(-\infty,-2]\cup[-1,0)\\ -x^2-3x+2, & x\in(-2,-1)\end{cases}$$

$$\Rightarrow f'(x)=\begin{cases}2x-3, & x\in(0,1)\cup(2,\infty)\\ -2x+3, & x\in(1,2)\\ 2x+3, & x\in(-\infty,-2)\cup(-1,0)\\ -2x-3, & x\in(-2,-1)\end{cases}$$

6. b, c.

$$y=\frac{(x^2+1)^2-3x^2}{x^2+\sqrt{3}x+1}=\frac{(x^2+1+\sqrt{3}x)(x^2+1-\sqrt{3}x)}{x^2+1+\sqrt{3}x}$$

$$\frac{dy}{dx}=2x-\sqrt{3}\Rightarrow a=2 \text{ and } b=-\sqrt{3}$$

$$a-b=2+\sqrt{3}=\tan\frac{5\pi}{12}=\cot\frac{\pi}{12}$$

7. a, b, d.

$$f(x)=\frac{\sqrt{(\sqrt{x-1})^2+1-2\sqrt{x-1}}}{\sqrt{x-1}-1}x$$

$$=\frac{|\sqrt{x-1}-1|}{\sqrt{x-1}-1}x$$

$$=\begin{cases}-x \text{ if } x\in[1,2)\\ x \quad \text{if } x\in(2,\infty)\end{cases}$$

8. b, d.

$$y=x^{(\log x)^{\log(\log x)}}$$

$\Rightarrow \log y=(\log x)(\log x)^{\log(\log x)}$ (1)

Taking log of both sides, we get

$\Rightarrow \log(\log y)=\log(\log x)+\log(\log x)\log(\log x)$

Differentiating w.r.t. x, we get

$$\frac{1}{\log y}\cdot\frac{1}{y}\frac{dy}{dx}=\frac{1}{x\log x}+\frac{2\log(\log x)}{\log x}\frac{1}{x}$$

$$=\frac{2\log(\log x)+1}{x\log x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x}\cdot\frac{\log y}{\log x}(2\log(\log x)+1)$$

Substituting the value of y from (1), we get

$$\frac{dy}{dx}=\frac{y}{x}(\log x)^{\log(\log x)}(2\log(\log x)+1)$$

9. a, b, c.

We have $\sin^{-1}(\cos x)=\frac{\pi}{2}-\cos^{-1}(\cos x)$

$$=\begin{cases}\frac{\pi}{2}-x, & \text{if } 0<x\le\pi\\ \frac{\pi}{2}-(2\pi-x), & \text{if } \pi<x<2\pi\end{cases}$$

$$=\begin{cases}\frac{\pi}{2}-x, & \text{if } 0<x\le\pi\\ x-\frac{3\pi}{2}, & \text{if } \pi<x<2\pi\end{cases}$$

$$\therefore \frac{d}{dx}\{\sin^{-1}(\cos x)\}=\begin{cases}-1, & \text{if } 0<x<\pi\\ 1, & \text{if } \pi<x<2\pi\end{cases}$$

We have $\cos^{-1}(\sin x)=\frac{\pi}{2}-\sin^{-1}(\sin x)$

$$=\begin{cases}\frac{\pi}{2}-x, & \text{if } -\frac{\pi}{2}<x\le\frac{\pi}{2}\\ \frac{\pi}{2}-(\pi-x), & \text{if } \frac{\pi}{2}<x<\frac{3\pi}{2}\end{cases}$$

$$=\begin{cases}\frac{\pi}{2}-x, & \text{if } -\frac{\pi}{2}<x\le\frac{\pi}{2}\\ x-\frac{\pi}{2}, & \text{if } \frac{\pi}{2}<x<\frac{3\pi}{2}\end{cases}$$

$$\therefore \frac{d}{dx}(\cos^{-1}(\sin x))=\begin{cases}-1, & \text{if } -\frac{\pi}{2}<x<\frac{\pi}{2}\\ 1, & \text{if } \frac{\pi}{2}<x<\frac{3\pi}{2}\end{cases}$$

10. a, c.

$f(x-y), f(x)f(y)$ and $f(x+y)$ are in A.P.

$\Rightarrow f(x+y)+f(x-y)=2f(x)f(y)$ for all x, y

Putting $x=0, y=0$ in (1),

we get $f(0)+f(0)=2f(0)f(0)$

$\Rightarrow f(0)=1 \quad (\because f(0)\neq 0)$

Putting $x=0, y=x$,

we get $f(x)+f(-x)=2f(0)f(x)$

$\Rightarrow f(x)=f(-x)$ (1)

$\Rightarrow f(4)=f(-4), f(3)=f(-3)$

Differentiating (1) w.r.t. x, $f'(x)+f'(-x)=0$

$\Rightarrow f'(4)+f'(-4)=0$

11. b, c.

$$y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-1}{\sqrt{1-\frac{4x^2}{(1+x^2)^2}}}\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{1+x^2}{\sqrt{(1-x^2)^2}}\frac{2(1+x^2)-4x^2}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx}=-2\frac{(1+x^2)}{|1-x^2|}\frac{1-x^2}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = -2\left(\frac{1-x^2}{|1-x^2|}\right)\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2}, & \text{if } |x|>1 \\ \frac{-2}{1+x^2}, & \text{if } |x|<1 \end{cases}$$

12. a, c, d.

$$f'(x) = \lim_{x\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}x} = \frac{f'(1)}{x} = \frac{1}{x}$$

$\Rightarrow f(x) = \ln x$ as $f(1)=0$

13. a, c.

$$\frac{d}{dx}\{f_n(x)\} = \frac{d}{dx}\{e^{f_{n-1}(x)}\}$$

$$= e^{f_{n-1}(x)}\frac{d}{dx}\{f_{n-1}(x)\} = f_n(x)\frac{d}{dx}\{f_{n-1}(x)\}$$

$$= f_n(x)\cdot\frac{d}{dx}\{e^{f_{n-2}(x)}\} = f_n(x).e^{f_{n-2}(x)}\frac{d}{dx}\{f_{n-2}(x)\}$$

$$= f_n(x)\, f_{n-1}(x)\frac{d}{dx}\{f_{n-2}(x)\}$$

...

$$= f_n(x) f_{n-1}(x) \cdots f_2(x)\frac{d}{dx}\{f_1(x)\}$$

$$= f_n(x)\cdot f_{n-1}(x)\cdots f_2(x)\frac{d}{dx}\{e^{f_0(x)}\}$$

$$= f_n(x)\cdot f_{n-1}(x)\cdots f_2(x)\, e^{f_0(x)}\frac{d}{dx}\{f_0(x)\}$$

Use $e^{f_0(x)} = f_1(x)$ and $f_0(x) = x$

Reasoning Type

1. a. $f(x) = x[x] = \begin{cases} -x, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ x, & 1 \le x < 2 \\ 2x, & 2 \le x < 3 \\ \cdots & \cdots \end{cases}$

$\Rightarrow f'(x) = \begin{cases} -1, & -1 < x < 0 \\ 0, & 0 < x < 1 \\ 1, & 1 < x < 2 \\ 2, & 2 < x < 3 \\ \cdots & \cdots \end{cases} \Rightarrow f'(x) = [x]$

2. c. Statement 1 is always true, but Statement 2 is not always true, as if $f'(x) = \cos x$, then $f(x)$ can be $\sin x$ which is odd function, but if $f(x) = -\sin x + 2$, then $f(x)$ is neither odd nor even.

3. a. Since $|f(x)-f(y)| \le |x-y|^3$, where $x \ne y$

$$\therefore \left|\frac{f(x)-f(y)}{x-y}\right| \le |x-y|^2$$

Taking lim as $y \to x$, we get

$$\lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right| \le \lim_{y\to x}|x-y|^2$$

$$\Rightarrow \left|\lim_{y\to x}\frac{f(x)-f(y)}{x-y}\right| \le \left|\lim_{y\to x}(x-y)^2\right|$$

$\Rightarrow |f'(x)| \le 0$

$\Rightarrow |f'(x)| = 0$ $\quad (\because |f'(x)| \ge 0)$

$\therefore f'(x) = 0$

$\Rightarrow f(x) = c$ (constant)

4. b. Both the statement are true, but Statement 2 is not correct explanation of Statement 1.

Statement 1 is true as period of $\sin x$ is 2π.

Or, in general if for $y=f(x), f(a)=f(b)$, we cannot say $f'(a) = f'(b)$.

5. a. $f(x)+f(x-2)=0$ (1)

Replace x by $x-2 \Rightarrow f(x-2)+f(x-4)=0$ (2)

From (1) and (2), $f(x)-f(x-4)=0$

Replace x by $x+4 \Rightarrow f(x+4)=f(x)$.

$\Rightarrow f(x)=f(x+4)=f(x+8)=\cdots=f(x+4000)$

$\Rightarrow f'(x)=f'(x+4000)$.

Hence, both the statements are true and Statement 2 is correct explanation of Statement 1.

Hence, $f(x)$ is periodic with period 4.

6. d. Statement 2 is true as $f(\alpha)=0$ and $f'(\alpha)=0$, then definitely α is repeated root of $f(x)=0$.

But from data, we are not sure how many times a root repeats.

Also $f(x) = (x-\alpha)^n \times g(x)$, which changes sign at $x = \alpha$, when n is odd and does not if n is even. Hence, Statement 1 is false.

7. a. Given $f(x+y^3)=f(x)+f(y^3)\ \forall\ x, y \in R$

Put $x=y=0$, we get $f(0+0)=f(0)+f(0) \Rightarrow f(0)=0$.

Now, put $y=-x^{1/3}$, we get $f(0)=f(x)+f(-x)$

$\Rightarrow f(x)+f(-x)=0$

$\Rightarrow f(x)$ is an odd function

$\Rightarrow f'(x)$ is an even function

$\Rightarrow f(-2)=a$

Linked Comprehension Type

For Problems 1 – 3

1. b, 2. a, 3. d.

Sol. Suppose degree of $f(x) = n$, then degree of $f' = n-1$ and deg $f'' = n-2$, so $n = n-1+n-2$

Hence, $n=3$.

So put $f(x) = ax^3 + bx^2 + cx + d$. (where $a \neq 0$)

From $f(2x) = f'(x).f''(x)$,

we have $8ax^3 + 4bx^2 + 2cx + d$

$= (3ax^2 + 2bx + c)(6ax + 2b)$

$= 18a^2x^3 + 18abx^2 + (6ac + 4b^2)x + 2bc$.

Comparing coefficients of terms, we have

$18a^2 = 8a \Rightarrow a = 4/9$

$18ab = 4b \Rightarrow b = 0$

$2c = 6ac + 4b^2 \Rightarrow c = 0$

$d = 2bc \Rightarrow d = 0$

$\Rightarrow f(x) = \frac{4x^3}{9}$, which is clearly one-one and onto.

$\Rightarrow f(3) = 12$.

Also, $\frac{4x^3}{9} = x \Rightarrow x = 0, x = \pm 3/2$.

Hence sum of roots of equation is zero.

For Problems 4 – 6

4. d, 5. c, 6. d.

Sol. Here, $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$

Put $f'(1) = a, f''(2) = b, f'''(3) = c$ (1)

$\therefore \quad f(x) = x^3 + ax^2 + bx + c$

$\Rightarrow \quad f'(x) = 3x^2 + 2ax + b$, or

$f'(1) = 3 + 2a + b$ (2)

$\Rightarrow \quad f''(x) = 6x + 2a$, or

$f''(2) = 12 + 2a$ (3)

$\Rightarrow \quad f'''(x) = 6$, or

$f'''(3) = 6$ (4)

From (1) and (4), $c = 6$

From (1), (2) and (3), we have $a = -5, b = 2$

$\therefore f(x) = x^3 - 5x^2 + 2x + 6$

$f'(x) = 3x^2 - 10x + 2$.

For Problems 7 – 9

Sol. 7. b, 9. d.

7.b. From the given information, we have $f(x) = (x-c)^m g(x)$, where $g(x)$ is polynomial of degree $n-m$,

Then $x = c$ is common root for the equations $f(x) = 0, f^1(x) = 0, f^2(x) = 0, \cdots, f^{m-1}(x) = 0$,

where $f^r(x)$ represent rth derivative of $f(x)$ w.r.t. x.

8. Let $f(x) = a_1x^3 + b_1x^2 + c_1x + d_1 = 0$ has roots α, α, β, then $g(x) = a_2x^3 + b_2x^2 + c_2x + d_2 = 0$ must have roots α, α, γ

$\Rightarrow a_1\alpha^3 + b_1\alpha^2 + c_1\alpha + d_1 = 0$, and (1)

$a_2\alpha^3 + b_2\alpha^2 + c_2\alpha + d_2 = 0$ (2)

α is also a root of equations $f'(x) = 3a_1x^2 + 2b_1x + c_1 = 0$ and $g'(x) = 3a_2x^2 + 2b_2x + c_2 = 0$

$\Rightarrow 3a_1\alpha^2 + 2b_1\alpha + c_1 = 0$, and (3)

$3a_2\alpha^2 + 2b_2\alpha + c_2 = 0$ (4)

Also from $a_2(1) - a_1(2)$, we have

$(a_2b_1 - a_1b_2)\alpha^2 + (c_1a_2 - c_2a_1)\alpha + d_1a_2 - d_2a_1 = 0$ (5)

Eliminating α from (3), (4) and (5) we have

$$\begin{vmatrix} 3a_1 & 2b_1 & c_1 \\ 3a_2 & 2b_2 & c_2 \\ a_2b_1 - a_1b_2 & c_1a_2 - c_2a_1 & d_1a_2 - d_2a_1 \end{vmatrix} = 0.$$

9. d.

For Problems 10 – 12

10. b, 11. a, 12. c.

Sol.

10.b. Since $1, a_1, a_2, \cdots, a_{n-1}$ are roots of $x^n - 1 = 0$, then

$x^n - 1 = (x-1)(x-a_1)(x-a_2)\cdots(x-a_{n-1})$ (1)

$\Rightarrow \frac{x^n - 1}{x-1} = (x-a_1)(x-a_2)\cdots(x-a_{n-1})$

$\Rightarrow \lim_{x\to 1}\frac{x^n - 1}{x-1} = \lim_{x\to 1}[(x-a_1)(x-a_2)\cdots(x-a_{n-1})]$

$\Rightarrow (1-a_1)(1-a_2)\cdots(1-a_{n-1}) = n$.

11.a. From (1), $\log(x^n - 1) = \log(x-1) + \log(x-a_1) + \cdots + \log(x-a_{n-1})$

Differentiating w.r.t. x, we get

$\frac{nx^{n-1}}{x^n - 1} = \frac{1}{x-1} + \frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_{n-1}}$ (2)

Putting $x = 2$ in (2), we get

$\frac{n2^{n-1}}{2^n - 1} = 1 + \frac{1}{2-a_1} + \frac{1}{2-a_2} + \cdots + \frac{1}{2-a_{n-1}}$

$\Rightarrow \frac{1}{2-a_1} + \frac{1}{2-a_2} + \cdots + \frac{1}{2-a_{n-1}} = \frac{n2^{n-1}}{2^n - 1} - 1$

$= \frac{n2^{n-1} - 2^n + 1}{2^n - 1}$

$= \frac{2^{n-1}(n-2) + 1}{2^n - 1}$

12. c. From (2), $\frac{nx^{n-1}}{x^n - 1} - \frac{1}{x-1} = \frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_{n-1}}$

$\Rightarrow \frac{nx^{n-1} - 1(1 + x + x^2 + \cdots + x^{n-1})}{x^n - 1}$

$= \frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_{n-1}}$

$\Rightarrow \lim_{x\to 1}\frac{nx^{n-1} - 1(1 + x + x^2 + \cdots + x^{n-1})}{x^n - 1}$

$= \lim_{x\to 1}\left(\frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_{n-1}}\right)$

$\Rightarrow \lim_{x\to 1}\frac{n(n-1)x^{n-2} - (1 + 2x + \cdots + (n-1)x^{n-2})}{nx^{n-1}}$

$= \frac{1}{1-a_1} + \frac{1}{1-a_2} + \cdots + \frac{1}{1-a_{n-1}}$ (applying L Hopital's Rule on L.H.S.)

$$\Rightarrow \frac{n(n-1)-(1+2+\cdots+(n-1))}{n}=\frac{1}{1-a_1}+\frac{1}{1-a_2}+\cdots+\frac{1}{1-a_{n-1}}$$

$$\Rightarrow \frac{1}{1-a_1}+\frac{1}{1-a_2}+\cdots+\frac{1}{1-a_{n-1}}=\frac{n-1}{2}$$

For Problems 13 – 15

13. b, 14. c, 15. c.

Sol.

Here put $g'(1)=a, g''(2)=b$ (1)

Then $f(x)=x^2+ax+b, f(1)=1+a+b \Rightarrow f'(x)=2x+a,$ $f''(x)=2.$

$\therefore g(x)=(1+a+b)x^2+(2x+a)x+2=x^2(3+a+b)+ax+2.$

$\Rightarrow g'(x)=2x(3+a+b)+a$ and $g''(x)=2(3+a+b).$

Hence, $g'(1)=2(3+a+b)+a$ (2)

$g''(2)=2(3+a+b)$ (3)

From (1), (2) and (3), we have $a=2(3+a+b)+a$

and $b=2(3+a+b)$

$\Rightarrow 3+a+b=0$ and $b+2a+6=0$

Hence, $b=0$ and $a=-3$. So, $f(x)=x^2-3x$ and $g(x)=-3x+2.$

$$\sqrt{\frac{f(x)}{g(x)}}=\sqrt{\frac{x^2-3x}{-3x+2}} \text{ is defined if } \frac{x^2-3x}{-3x+2}\ge 0$$

$$\Rightarrow \frac{x(x-3)}{(x-2/3)}\le 0 \Rightarrow x\in(-\infty,0]\cup(2/3,3]$$

For Problems 16 – 18

Sol.

16. d, 17. c, 18. c

$g(x+y)=g(x)+g(y)+3x^2y+3xy^2$ (1)

$\Rightarrow g'(x+y)=g'(x)+6yx+3y^2$ (differentiating w.r.t. x keeping y as constant)

Put $x=0$

$\Rightarrow g'(y)=g'(0)+3y^2$

$\Rightarrow g'(y)=-4+3y^2$

$\Rightarrow g'(x)=-4+3x^2$

$\Rightarrow g(x)=-4x+x^3+c$

Now put $x=y=0$ in (1), we get $g(0)=g(0)+g(0)+0$

$\Rightarrow g(0)=0$

$\Rightarrow g(x)=x^3-4x$

$g(x)=0 \Rightarrow x^3-4x=0 \Rightarrow x=0,2,-2.$ Hence, three roots.

$\sqrt{g(x)}=\sqrt{x^3-4x}$ is defined if $x^3-4x\ge 0$ or $x\in[-2,0]\cup[2,\infty).$

Also, $g'(x)=3x^2-4 \Rightarrow g'(1)=-1$

For Problems 19 – 21

Sol.

19. d, 20. d, 21. b.

$$x=f(t)=a^{\ln(b^t)}=a^{t\ln b} \quad (1)$$

$$y=g(t)=b^{-\ln(a^t)}=(b^{\ln a})^{-t}=(a^{\ln b})^{-t}=a^{-t\ln b}$$

$$\therefore y=g(t)=a^{\ln(b^{-t})}=f(-t) \quad (2)$$

From equations (1) and (2)

$xy=1$

19. d. $\because y=\dfrac{1}{x}$

$$\therefore \frac{dy}{dx}=-\frac{1}{x^2}=-\frac{1}{f^2(t)}$$

Also, $xy=1 \Rightarrow -\dfrac{1}{f^2(t)}=-g^2(t)$

$$\therefore \frac{dy}{dx}=-\frac{1}{x^2}=-\frac{y^2}{1}$$

Also, $xy=1 \Rightarrow \dfrac{dy}{dx}=-\dfrac{y}{x}=-\dfrac{g(t)}{f(t)}$

20. d $f(t)=g(t) \Rightarrow f(t)=f(-t) \Rightarrow t=0$

$\{\because f(t)$ is one-one function$\}$

At $t=0, x=y=1$

$$\because xy=1 \quad \therefore \frac{dy}{dx}=\frac{-1}{x^2} \text{ and } \frac{d^2y}{dx^2}=\frac{2}{x^3}$$

At $x=1$, $\dfrac{d^2y}{dx^2}=2$

21. b. $\because xy=1$

$\therefore fg=1$

$\therefore fg'+gf'=0$

$\therefore fg''+g'f'+g'f'+gf''=0$

$\Rightarrow fg''+gf''+2g'f'=0$

$$\Rightarrow \frac{f}{f'}\frac{g''}{g'}+\frac{gf''}{g'f'}=-2 \quad (3)$$

From equation (2), $g(t)=f(-t)$

$\therefore g'(t)=-f'(-t)$

and $g''(t)=f''(-t)$

substituting in equation (3)

$$\frac{f(t)}{f'(t)}\cdot\frac{f''(-t)}{-f'(-t)}+\frac{f(-t)}{f'(-t)}\cdot\frac{f''(t)}{f'(t)}=-2$$

$$\frac{f(t)}{f'(t)}\cdot\frac{f''(-t)}{f'(-t)}+\frac{f(-t)}{-f'(-t)}\cdot\frac{f''(t)}{f'(t)}=2$$

Matrix-Match Type

1. a. → p; b. → q, r; c. → s, r; d. → q, r.

Sol.

a. $f(1-x)=f(1+x)$

$\Rightarrow -f'(1-x)=f'(1+x).$

Hence, graph of $f(x)$ is symmetrical about point $(1,0)$ (as if $f(x)=-f(-x)$, then $f(x)$ is odd and its graph is symmetrical about $(0,0)$. Now shift the graph at $(1,0)$).

b. $f(2-x)+f(x)=0$

Replace x by $1+x$, then $f(2-(1+x))+f(1+x)=0$

$\Rightarrow f(1-x)+f(1+x)=0$

$\Rightarrow -f'(1-x)+f'(1+x)=0$

$\Rightarrow f'(1-x)=f'(1+x)$ (1)

$\Rightarrow$ Graph of $f'(x)$ is symmetrical about line $x=1$.

Also, put $x=2$ in (1), we get $f'(-1)=f'(3)$.

c. $f(x+2)+f(x)=0$ (1)

Replace x by $x+2$, we get $f(x+4)+f(x+2)=0$ (2)

From (1) and (2), we have $f(x)=f(x+4)$

Hence, $f(x)$ is periodic with period 4.

Also, $f'(x)=f'(x+4)$. Hence $f'(x)$ is periodic with period 4.

Put $x=-1$ in $f'(x)=f'(x+4)$, we get $f'(-1)=f'(3)$.

d. Putting $x=0, y=0$, we get $2f(0)+\{f(0)\}^2=1$

$\Rightarrow f(0)=\sqrt{2}-1$ $\{\because f(0)>0\}$

Putting $y=x$, $2f(x)+\{f(x)\}^2=1$

Diff. w.r.t. x, we get

$2f'(x)+2f(x).f'(x)=0$ or $f'(x)\{1+f(x)\}=0$

$\Rightarrow f'(x)=0$, because $f(x)>0$.

2. **a.** $\rightarrow$ **q; b.** $\rightarrow$ **r; c.** $\rightarrow$ **s; d.** $\rightarrow$ **p.**

Sol.

a. $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{12t^2-6t-18}{5t^4-15t^2-20}$

$\Rightarrow \left.\dfrac{dy}{dx}\right|_{t=1}=\dfrac{12-6-18}{5-15-20}=\dfrac{2}{5}$

$\Rightarrow 5\left.\dfrac{dy}{dx}\right|_{t=1}=2$ at $t=1$.

b. Let us take $P(x)=a(x-2)^4+b(x-2)^3+c(x-2)^2+d(x-2)+e$

$-1=P(2)=e$

$0=P'(2)=d$

$2=P''(2)=2c\Rightarrow c=1$

$-12=P'''(2)=6b\Rightarrow b=-2$

$24=p''''(2)=24a \Rightarrow a=1$

Thus, $P''(x)=12(x-2)^2-12(x-2)+2$

$\Rightarrow P''(3)=12-12(1)+2=2$

c. Here $\sqrt{(1+y^4)}=\sqrt{\left(1+\dfrac{1}{x^4}\right)}=\dfrac{\sqrt{1+x^4}}{x^2}$ $\left(\because y=\dfrac{1}{x}\right)$

$\Rightarrow \dfrac{\sqrt{1+y^4}}{\sqrt{1+x^4}}=\dfrac{1}{x^2}$ (1)

But $y=\dfrac{1}{x}$

$\therefore \dfrac{dy}{dx}=-\dfrac{1}{x^2}$ (2)

From (1) and (2), $\dfrac{\sqrt{1+y^4}}{\sqrt{1+x^4}}=-\dfrac{dy}{dx}$

$\Rightarrow \dfrac{\dfrac{dy}{\sqrt{1+y^4}}}{\dfrac{dx}{\sqrt{1+x^2}}}=-1$

d. Obviously, $f(x)$ is a linear function.

Also from $f'(0)=p$ and $f(0)=q$, $f(x)=px+q$.

$\Rightarrow f''(0)=0$

3. **a.** $\rightarrow$ **q, r; b.** $\rightarrow$ **p, r, s; c.** $\rightarrow$ **q, s; d.** $\rightarrow$ **q, r.**

Sol.

a. We know that

$$2\tan^{-1}x=\begin{cases}\sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } -1\le x\le 1\\ \pi-\sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } x>1\\ -\pi-\sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } x<-1\end{cases}$$

$\Rightarrow \dfrac{dy}{dx}=-\dfrac{2}{1+x^2}$ if $x<-1$ or $x>1$

b. $\cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)=\begin{cases}\tan^{-1}x, & x\ge 0\\ -\tan^{-1}x, & x<0\end{cases}$

$\Rightarrow \dfrac{dy}{dx}=-\dfrac{1}{1+x^2}$ if $x<0$

c. $y=|e^{|x|}-e|=\begin{cases}|e^x-e|, & x\ge 0\\ |e^{-x}-e|, & x<0\end{cases}=\begin{cases}e^x-e, & x\ge 1\\ e-e^x, & 0\le x<1\\ e-e^{-x}, & -1\le x<0\\ e^{-x}-e, & x<-1\end{cases}$

$\Rightarrow \dfrac{dy}{dx}>0$ if $x>1$ of $-1<x<0$.

d. $u=\log|2x|$, $v=|\tan^{-1}x|$

$\Rightarrow \dfrac{du}{dx}=\dfrac{1}{x}$, and $\dfrac{dv}{dx}=\begin{cases}\dfrac{1}{1+x^2}, & x>0\\ -\dfrac{1}{1+x^2}, & x<0\end{cases}$

$\Rightarrow \dfrac{du}{dv}=\begin{cases}\dfrac{1+x^2}{x}, & x>0\\ -\dfrac{1+x^2}{x}, & x<0\end{cases}$

Now we know that $\dfrac{1+x^2}{x}=x+\dfrac{1}{x}>2$ if $x>1$ and <-2 if $x<-1$

$\Rightarrow \dfrac{du}{dv}>2$ if $x<-1$ or $x>1$

4. a. $\to$ **p, q, r; b.** $\to$ **q, s; c.** $\to$ **q, r; d.** $\to$ **r.**

Sol.

a. p, q, r

The graph of $y = |x^2 - 2|x||$

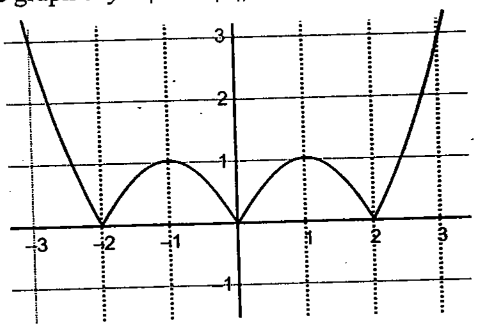

Fig. 4.3

From the graph dy/dx is negative for p, q, r

b. q, s

The graph of $y = |\log |x||$

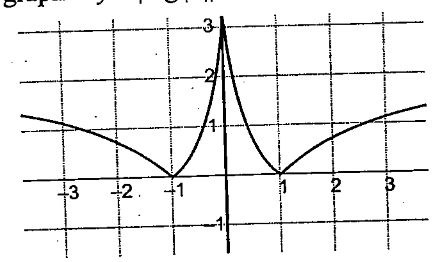

Fig. 4.4

From the graph dy/dx is negative for q, s

c. q, r

$$y = x[x/2] = \begin{cases} -x, & -4 \le x < -2 \\ -x, & -2 \le x < 0 \\ 0, & 0 \le x < 2 \\ x, & 2 \le x < 4 \end{cases}$$

Hence dy/dx is negative for q, r

d. q

The graph of $y = |\sin x|$

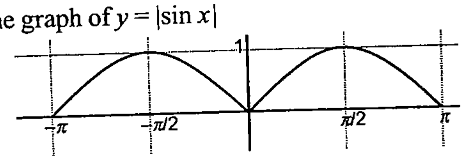

Fig. 4.5

From the graph dy/dx is negative for q

Integer Type

1. (9) $\dfrac{d}{dx}\{[(f(x)]^2 - [\phi(x)]^2\}$

$= 2[f(x)\cdot f'(x) - \phi(x)\cdot \phi'(x)]$

$= 2[f(x)\cdot \phi(x) - \phi(x)\cdot f(x)]$ $[\because f'(x) = \phi(x) \text{ and } \phi'(x) = f(x)]$

$= 0$

$\Rightarrow [f(x)]^2 - [\phi(x)]^2 = \text{constant}$

$\therefore [f(10)]^2 - [\phi(10)]^2 = [f(3)]^2 - [\phi(3)]^2 = [f(3)]^2 - [f'(3)]^2 = 25 - 16 = 9.$

2. (2) Since $f(x)$ is odd. Therefore $f(-x) = -f(x) \Rightarrow f'(-x)(-1) = -f'(x)$

$\Rightarrow f'(-x) = f'(x)$ $\therefore f'(-3) = f'(3) = -2.$

3. (5) Here $x = \alpha$ is a repeated root of the equation $f(x) = 0$ hence $x = \alpha$ is also a root of the equation $f'(x) = 0$ i.e., $3x^2 + 6x - 9 = 0$

or $x^2 + 2x - 3 = 0$ or $(x + 3)(x - 1) = 0$

has the root α once which can be either -3, or 1.

If $\alpha = 1$, then $f(x) = 0$ gives $c - 5 = 0$ or $c = 5$

If $\alpha = -3$, then $f(x) = 0$ gives $-27 + 27 + 27 + c = 0$ $\therefore c = -27$

4. (3) We have $f(5 - x) = -f(5 + x) \Rightarrow -f'(5 - x) = -f'(5 + x)$

$\Rightarrow f'(5-2) = f'(5+2) \Rightarrow f'(3) = f'(7) = 3$

5. (2) We have $g(x) = f(x)\sin x$ (1)

On differentiating equation (1) w.r.t. x, we get

$g'(x) = f(x)\cos x + f'(x)\sin x$ (2)

Again differentiating equation (2) w.r.t. x, we get

$g''(x) = f(x)(-\sin x) + f'(x)\cos x + f'(x)\cos x + f''(x)\sin x$ (3)

$\Rightarrow g''(-\pi) = 2f'(-\pi)\cos(-\pi) = 2\times 1\times(-1) = -2$

Hence $g''(-\pi) = -2$

6. (8) $ln(f(x)) = ln(x-1) + ln(x-2) + \cdots + ln(x-n)$

$$\Rightarrow f'(x) = f(x)\left[\frac{1}{x-1} + \frac{1}{x-2} + \cdots \frac{1}{x-n}\right]$$

$\Rightarrow f'(x) = (x-2)(x-3)(x-n) + (x-1)(x-3)\cdots(x-n) + \cdots + (x-1)(x-2)\cdots(x-(n-1))$

$\Rightarrow f'(n) = (n-1)(n-2)(n-3)\cdot 3.2\cdot 1$ (all other factors except the last vanishes when $x = n$)

$\Rightarrow 5040 = (n-1)!$

$\Rightarrow n = 8$

7. (9) $$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x+0)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{2f(x) + xf(h) + h\sqrt{f(x)} - 2f(x) - xf(0) - 0\sqrt{f(x)}}{h}$$

as $f(0) = 0$

$$\Rightarrow \lim_{h\to 0} x\left(\frac{f(h) - f(0)}{h - 0}\right) + \sqrt{f(x)} = f'(0) + \sqrt{f(x)}$$

$\Rightarrow f'(x) = \sqrt{f(x)}$ $\quad (\because f'(0) = 0)$

$$\Rightarrow \int \frac{f'(x)}{\sqrt{f(x)}}dx = \int dx$$

$\Rightarrow 2\sqrt{f(x)} = x + c$

$\Rightarrow f(x) = \dfrac{x^2}{4}$ $\quad (\because f(0) = 0)$

8. (3) $f(x)\times f'(-x) = f(-x)\times f'(x)$

$\Rightarrow f'(x)\times f(-x) - f(x)\times f'(-x) = 0$

$\Rightarrow \dfrac{d}{dx}[f(x)f(-x)] = 0$

$\Rightarrow f(x)f(-x) = k$

Given $(f(0))^2 = k = 9 \Rightarrow k = 9$

Then $f(3)f(-3) = 9 \Rightarrow f(-3) = 3$

9. (5) $y=\dfrac{a+bx^{3/2}}{x^{5/4}} \Rightarrow y'=\dfrac{\frac{3}{2}bx^{1/2}x^{5/4}-\frac{5}{4}x^{1/4}(a+bx^{3/2})}{x^{5/2}}$

According to the question,

$$0=\frac{\frac{3}{2}b5^{1/2}5^{5/4}-\frac{5}{4}5^{1/4}(a+b5^{3/2})}{5^{5/2}}$$

$\Rightarrow \dfrac{3b}{2}5^{7/4}-a\dfrac{5^{5/4}}{4}-5b\dfrac{5^{7/4}}{4}=0$

$\Rightarrow b5^{7/4}=a5^{5/4}$

$\Rightarrow b\sqrt{5}=a$

$\Rightarrow a:b=\sqrt{5}:1$

10. (3) $y=\dfrac{x^4-(x^2+2x+1)}{x^2-x-1}=x^2+x+1$

$\therefore \dfrac{dy}{dx}=2x+1=ax+b$

hence $a=2$ and $b=1$

11. (2) Limit is $f'(e)$ where $f(x)=x^{\ln x}=e^{\ln^2 x}$

$\Rightarrow g'(f(x))f'(x)=e^{\ln^2 x}\cdot\dfrac{2\ln x}{x}$

$\Rightarrow f'(e)=e\cdot\dfrac{2}{e}=2$

12. (5) We have $(gof)(x)=x$

$\Rightarrow g'(f(x))f'(x)=1$

when $f(x)=-\dfrac{7}{6} \Rightarrow x=1$

$\Rightarrow g'(f(x))\,g'\left(-\dfrac{7}{6}\right)f'(1)=1$

Hence $g'\left(-\dfrac{7}{6}\right)=\dfrac{1}{f'(1)}=\dfrac{1}{5}$

13. (6) $g(x)=f(-x+f(f(x)));\quad f(0)=0;\quad f'(0)=2$

$g'(x)=f'(-x+f(f(x)))\cdot[-1+f'(f(x))\cdot f'(x)]$

$g'(0)=f'(f(0))\cdot[-1+f'(0)\cdot f'(0)]$

$=f'(0)[-1+(2)(2)]$

$=(2)(3)=6$

14. (5) According to question $(a^2-2a-15)e^{ax}+(b^2-2b-15)e^{bx}=0$

$\Rightarrow (a^2-2a-15)=0$ and $b^2-2b-15=0$

$\Rightarrow (a-5)(a+3)=0$ and $(b-5)(b+3)=0$

$\Rightarrow a=5$ or -3 and $b=5$ or -3

$\therefore a\neq b$ hence $a=5$ and $b=-3$

or $a=-3$ and $b=5$

$\Rightarrow ab=-15$

15. (9) Let degree of $f(x)$ is n; degree of $f'(x)=n-1$

degree of $f''(x)$ is $(n-2)$

Hence $n=(n-1)+(n-2)=2n-3$

$\therefore n=3$

Hence $f(x)=ax^3+bx^2+cx+d \quad (a\neq 0)$

$f'(x)=3ax^2+2bx+c$

$f''(x)=6ax+2b$

$\therefore ax^3+bx^2+cx+d=(3ax^2+2bx+c)(6ax+2b)$

$\therefore 18a^2=a \Rightarrow a=\dfrac{1}{18}$

16. (1) $\dfrac{dx}{dt}=-\dfrac{3}{t^4}-\dfrac{2}{t^3}=-\left(\dfrac{3+2t}{t^4}\right)$

$\dfrac{dy}{dt}=-\left(\dfrac{3}{t^3}+\dfrac{2}{t^2}\right)=-\left(\dfrac{3+2t}{t^3}\right)$

$\Rightarrow \dfrac{dy}{dx}=t$

$\Rightarrow \dfrac{dy}{dx}-x\left(\dfrac{dy}{dx}\right)^3=t-\left(\dfrac{1+t}{t^3}\right)\cdot t^3=-1$

17. (5) $z=(\cos x)^5;\ y=\sin x$

$\dfrac{dz}{dx}=-5\cos^4 x\cdot\sin x;\ \dfrac{dy}{dx}=\cos x$

$\therefore \dfrac{dz}{dy}=-5\cos^3 x\cdot\sin x$

Now $\dfrac{d^2z}{dy^2}=\dfrac{d}{dx}\left(\dfrac{dz}{dy}\right)\cdot\dfrac{dx}{dy}$

$=-5\dfrac{d}{dx}[\cos^3 x\cdot\sin x]\dfrac{1}{\cos x}$

$=-5[\cos^4 x-3\sin^2 x\cdot\cos^2 x]\dfrac{1}{\cos x}$

$=-5(\cos^3 x-3\sin^2 x\cdot\cos x)$

$=-5(\cos^3 x-3\cos x(1-\cos^2 x))$

$=-5(4\cos^3 x-3\cos x)$

$=-5\cos 3x$

$\therefore \left.\dfrac{d^2z}{dy^2}\right|_{x=\frac{2\pi}{9}}=-5\cos 120°=\dfrac{5}{2}$

18. (7) $g'(0)=b=\lim\limits_{x\to 0}\dfrac{x^2+x\tan x-x\tan 2x}{x(ax+\tan x-\tan 3x)}$

$=\lim\limits_{x\to 0}\dfrac{x+\tan x-\tan 2x}{ax+\tan x-\tan 3x}$

$$=\lim_{x\to 0}\frac{x+\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots\infty\right)-\left(2x+\frac{8x^3}{3}+\frac{2}{15}\cdot 32x^5+\cdots\infty\right)}{ax+\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots\infty\right)-\left(3x+\frac{27x^3}{3}+\frac{2}{15}243x^5+\cdots\infty\right)}$$

$$= \lim_{x \to 0} \frac{x^3\left(-\frac{7}{3}+\frac{-62}{15}x^2+\cdots\infty\right)}{(a+1-3)\,x+\left(\frac{1}{3}-9\right)x^3+\frac{2}{15}(-242)x^5+\cdots\infty}$$

b can be finite if $a+1-3=0$

$$\therefore\ a=2 \text{ and } b=\frac{-\frac{7}{3}}{\frac{1}{3}-9}=\left(\frac{-7}{3}\right)\left(\frac{3}{-26}\right)=\frac{7}{26}\Rightarrow 52\frac{b}{a}=7$$

Archives

Subjective

1. Let $f(x)=\sin(x^2+1)$, then, $f(x+h)=\sin[(x+h)^2+1]$

$$\therefore \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sin[(x+h)^2+1]-\sin[x^2+1]}{h}$$

$$\Rightarrow f'(x)=\lim_{h\to 0} 2\cos\left(\frac{2x^2+h^2+2xh+2}{2}\right)\times\frac{\sin\left(\frac{h^2+2xh}{2}\right)}{h}$$

$$=2\cos\left(x^2+1\right)\lim_{h\to 0}\frac{\sin\left[\frac{h^2+2xh}{2}\right]}{h\left[\frac{h+2x}{2}\right]}\left(\frac{h+2x}{2}\right)$$

$$=2x\cos(x^2+1)$$

2. $$f(x)=\begin{cases}\frac{x-1}{2x^2-7x+5}, & x\neq 1\\ -\frac{1}{3}, & x=1\end{cases}=\begin{cases}\frac{x-1}{(x-1)(2x-5)}, & x\neq 1\\ -\frac{1}{3}, & x=1\end{cases}$$

$$=\begin{cases}\frac{1}{2x-5}, & x\neq 1\\ -\frac{1}{3}, & x=1\end{cases}$$

$$\therefore f'(x)|_{x=1}=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{2(h+1)-5}+\frac{1}{3}}{h}$$

$$=\lim_{h\to 0}\frac{2h}{3h(2h-3)}$$

$$=\lim_{h\to 0}\frac{2}{(2h-3).3}$$

$$=-2/9.$$

3. Given $y=5\left[\frac{x}{(1-x)^{2/3}}\right]+\cos^2(2x+1)$

Differentiating both sides w.r.t x, we get

$$\frac{dy}{dx}=\frac{5\left[(1-x)^{2/3}-\frac{2}{3}(1-x)^{-1/3}(-1)\right]}{(1-x)^{4/3}}+2\cos(2x+1)(-\sin(2x+1))2$$

$$=\frac{5\left[(1-x)^{2/3}+\frac{2}{3(1-x)^{1/3}}\right]}{(1-x)^{4/3}}-2\sin(4x+2)$$

$$=\frac{5(3-3x+2)}{3(1-x)^{5/3}}-2\sin(4x+2)$$

$$=\frac{5(5-3x)}{3(1-x)^{5/3}}-2\sin(4x+2)$$

4. We are given that $y=e^{x\sin x^3}+(\tan x)^x$

$=u+v$

$u=e^{x\sin x^3}$ and $v=(\tan x)^x$

now $\frac{du}{dx}=e^{x\sin x^3}\frac{d}{dx}\left(x\sin x^3\right)$

$=e^{x\sin x^3}[3x^3\times\cos x^3+\sin x^3]$

$v=(\tan x)^x$

$\Rightarrow \log v=x\log\tan x$

Differentiating w.r.t. x, we get $\frac{1}{v}\frac{dv}{dx}=x\frac{1}{\tan x}\sec^2 x+\log\tan x$

$$\therefore \frac{dv}{dx}=(\tan x)^x\left(\frac{2x}{\sin 2x}+\log\tan x\right)$$

Hence, $$\frac{dy}{dx}=e^{x\sin x^3}\left(\sin x^3+3x^3\cos x^3\right)+(\tan x)^x\left(\frac{2x}{\sin 2x}+\log\tan x\right)$$

5. Given that f is twice differentiable function such that $f''(x)=-f(x)$ and $f'(x)=g(x)$, $h(x)=[f(x)]^2+[g(x)]^2$

$\Rightarrow h'(x)=2f(x)f'(x)+2g(x)g'(x)$

$=2f(x)g(x)+2g(x)f''(x)$ $\quad [\because g(x)=f'(x) \Rightarrow g'(x)=f''(x)]$

$=2f(x)g(x)+2g(x)(-f(x))$

$=2f(x)g(x)-2f(x)g(x)$

$=0$

$\therefore h'(x)=0,\ \forall\ x$

$\Rightarrow h$ is a constant function.

$\therefore h(5)=11$
$\Rightarrow h(10)=11$

6. Let $F(x)=\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ (1)

$$F'(x)=\begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} \quad (2)$$

Given that α is a repeated root of quadratic equation $f(x)=0$. Therfore, we must have $f(x)=(x-\alpha)^2$.

Now $F(\alpha)=\begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}=0$

($\because R_1$ and R_2 are identical)

and $F'(\alpha)=\begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}=0$

($\because R_1$ and R_3 are identical)

Thus, $x=\alpha$ is a root of $F(x)=0$ and $F'(x)=0$.
$\Rightarrow (x-\alpha)$ is a factor of $F'(x)$ also, or we can say $(x-\alpha)^2$ is a factor of $F(x)$.
$\Rightarrow F(x)$ is divisible by $f(x)$.

7. Given that $f(x)=(\log_{\cos x}\sin x)(\log_{\sin x}\cos x)^{-1}$

$$+\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$=\frac{\log_{\cos x}\sin x}{\log_{\sin x}\cos x}+\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$=\left(\frac{\log\sin x}{\log\cos x}\right)^2+2\tan^{-1}x$$

$=u+v.$

So that $f'(x)=\dfrac{du}{dx}+\dfrac{dv}{dx}$ (1)

Now,

$$\frac{du}{dx}=2\left(\frac{\log\sin x}{\log\cos x}\right)\left[\frac{\cot x\log\cos x+\tan x\log\sin x}{(\log\cos x)^2}\right]$$

$$\Rightarrow \left.\frac{du}{dx}\right]_{x=\pi/4}=2\left(\frac{\log(1/\sqrt{2})}{\log(1/\sqrt{2})}\right)$$

$$\times\left[\frac{1\log(1/\sqrt{2})+1\log(1/\sqrt{2})}{(\log(1/\sqrt{2}))^2}\right]$$

$=-8\log_2 e.$ (2)

Also, $\dfrac{dv}{dx}=\dfrac{2}{1+x^2}$

$$\Rightarrow \left.\frac{dv}{dx}\right]_{x=\pi/4}=\frac{2}{1+\frac{\pi^2}{16}}=\frac{32}{16+\pi^2} \quad (3)$$

$\Rightarrow$ From (1), $f'(x)]_{x=\pi/4}=-8\log_2 e+\dfrac{32}{16+\pi^2}$

8. As $x=\operatorname{cosec}\theta-\sin\theta$
$\Rightarrow x^2+4=(\operatorname{cosec}\theta-\sin\theta)^2+4=(\operatorname{cosec}\theta+\sin\theta)^2$ (1)
and $y^2+4=(\operatorname{cosec}^n\theta-\sin^n\theta)^2+4=(\operatorname{cosec}^n\theta+\sin^n\theta)^2$ (2)
Now,

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{n(\operatorname{cosec}^{n-1}\theta)(-\operatorname{cosec}\theta\cot\theta)-n\sin^{n-1}\theta\cos\theta}{-\operatorname{cosec}\theta\cot\theta-\cos\theta}$$

$$=\frac{n(\operatorname{cosec}^n\theta\cot\theta+\sin^{n-1}\theta\cos\theta)}{(\operatorname{cosec}\theta\cot\theta+\cos\theta)}$$

$$=\frac{n\cot\theta(\operatorname{cosec}^n\theta+\sin^n\theta)}{\cot\theta(\operatorname{cosec}\theta+\sin\theta)}$$

$$=\frac{n(\operatorname{cosec}^n\theta+\sin^n\theta)}{(\operatorname{cosec}\theta+\sin\theta)}=\frac{n\sqrt{y^2+4}}{\sqrt{x^2+4}} \text{ [From (1) and (2)]}$$

Squaring both sides, we get $\left(\dfrac{dy}{dx}\right)^2=\dfrac{n^2(y^2+4)}{(x^2+4)}$, or

$$(x^2+4)\left(\frac{dy}{dx}\right)^2=n^2(y^2+4).$$

9. Given $(\sin y)^{\sin\left(\frac{\pi x}{2}\right)}+\dfrac{\sqrt{3}}{2}\sec^{-1}(2x)$
$+2^x\tan[\log(x+2)]=0$ (1)
For $x=-1$, we have

$$(\sin y)^{\sin\left(\frac{-\pi}{2}\right)}+\frac{\sqrt{3}}{2}\sec^{-1}(-2)+2^{-1}\tan[\log(-1+2)]=0$$

$$\Rightarrow (\sin y)^{-1}+\frac{\sqrt{3}}{2}\left(\frac{2\pi}{3}\right)+\frac{1}{2}\tan 0=0$$

$\Rightarrow \dfrac{1}{\sin y}=-\dfrac{\pi}{\sqrt{3}} \Rightarrow \sin y=-\dfrac{\sqrt{3}}{\pi}$, when $x=-1$ (2)

Now, let $u=(\sin y)^{\sin\left(\frac{\pi x}{2}\right)}$
Taking log on both sides, we get

$$\log u=\sin\left(\frac{\pi x}{2}\right)\log\sin y$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx}=\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\log\sin y+\cot y\frac{dy}{dx}\sin\left(\frac{\pi x}{2}\right)$$

$$\Rightarrow \frac{du}{dx}=(\sin y)^{\sin\left(\frac{\pi x}{2}\right)}\left[\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\log\sin y\right.$$

$$+\sin\left(\frac{\pi x}{2}\right)\cot y\,\frac{dy}{dx}\Big] \quad (3)$$

Now differentiating (1), we get

$$\Rightarrow (\sin y)^{\sin\left(\frac{\pi x}{2}\right)}\left[\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\log\sin y\right.$$

$$\left.+\sin\left(\frac{\pi x}{2}\right)\cot y\,\frac{dy}{dx}\right]+\frac{\sqrt{3}}{2x\sqrt{4x^2-1}}+2^x(\log 2)$$

$$\tan(\log(x+2))+\frac{2^x\sec^2[\log(x+2)]}{x+2}=0.$$

At $x=-1$ and $\sin y=-\frac{\sqrt{3}}{\pi}$, we get

$$\Rightarrow\left(-\frac{\sqrt{3}}{\pi}\right)^{-1}\left[0+(-1)\sqrt{\frac{\pi^2}{3}-1}\left(\frac{dy}{dx}\right)_{x=-1}\right]$$

$$+\frac{\sqrt{3}}{-2\sqrt{3}}+0+2^{-1}=0.$$

$$\Rightarrow\frac{\pi}{\sqrt{3}\sqrt{3}}\sqrt{\pi^2-3}\left(\frac{dy}{dx}\right)_{x=-1}-\frac{1}{2}+\frac{1}{2}=0$$

$$\Rightarrow\frac{dy}{dx}=0$$

10. $$y=\frac{ax^2}{(x-a)(x-b)(x-c)}+\frac{bx}{(x-b)(x-c)}+\frac{c}{x-c}+1$$

$$\Rightarrow\quad y=\frac{ax^2}{(x-a)(x-b)(x-c)}+\frac{bx}{(x-b)(x-c)}+\left(\frac{c+x-c}{x-c}\right)$$

$$\Rightarrow\quad y=\frac{ax^2}{(x-a)(x-b)(x-c)}+\frac{bx}{(x-b)(x-c)}+\frac{x}{x-c}$$

$$\Rightarrow\quad y=\frac{ax^2}{(x-a)(x-b)(x-c)}+\frac{bx+x(x-b)}{(x-b)(x-c)}$$

$$\Rightarrow\quad y=\frac{ax^2}{(x-a)(x-b)(x-c)}+\frac{x^2}{(x-b)(x-c)}$$

$$\Rightarrow\quad y=\frac{ax^2+x^2(x-a)}{(x-a)(x-b)(x-c)}$$

$$\Rightarrow\quad y=\frac{x^3}{(x-a)(x-b)(x-c)}$$

$$\Rightarrow\quad \log y=\log\left\{\frac{x^3}{(x-a)(x-b)(x-c)}\right\}$$

$$\Rightarrow\quad \log y=3\log x-\{\log(x-a)+\log(x-b)+\log(x-c)\}$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{3}{x}-\left\{\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}\right\}$$

$$\Rightarrow\quad \frac{dy}{dx}=y\left\{\left(\frac{1}{x}-\frac{1}{x-a}\right)+\left(\frac{1}{x}-\frac{1}{x-b}\right)+\left(\frac{1}{x}-\frac{1}{x-c}\right)\right\}$$

$$\Rightarrow\quad \frac{dy}{dx}=y\left\{-\frac{a}{x(x-a)}-\frac{b}{x(x-b)}-\frac{c}{x(x-c)}\right\}$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{y}{x}\left\{\frac{a}{a-x}+\frac{b}{b-x}+\frac{c}{x-c}\right\}.$$

Objective

Fill in the blanks

1. $y=f\left(\frac{2x-1}{x^2+1}\right)$; $f'(x)=\sin x^2$

$$\frac{dy}{dx}=f'\left(\frac{2x-1}{x^2+1}\right)\frac{d}{dx}\left(\frac{2x-1}{x^2+1}\right)$$

$$=\sin\left(\frac{2x-1}{x^2+1}\right)^2\frac{2\left(x^2+1\right)-2x(2x-1)}{\left(x^2+1\right)^2}$$

$$=\frac{2+2x-2x^2}{\left(x^2+1\right)^2}\sin\left(\frac{2x-1}{x^2+1}\right)^2$$

2. Given that $F(x)=\begin{vmatrix} f_1(x) & f_2(x) & f_3(x)\\ g_1(x) & g_2(x) & g_3(x)\\ h_1(x) & h_2(x) & h_3(x)\end{vmatrix}$, (1)

where $f_r(x), g_r(x), h_r(x)$, $r=1,2,3$ are polynomials in x and hence differentiable and $f_r(a)=g_r(a)=h_r(a)$, 1, 2, 3,... (2)

Differentiating (1) w.r.t. x,

$$\Rightarrow F'(x)=\begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x)\\ g_1(x) & g_2(x) & g_3(x)\\ h_1(x) & h_2(x) & h_3(x)\end{vmatrix}$$

$$+\begin{vmatrix} f_1(x) & f_2(x) & f_3(x)\\ g_1'(x) & g_2'(x) & g_3'(x)\\ h_1(x) & h_2(x) & h_3(x)\end{vmatrix}+\begin{vmatrix} f_1(x) & f_2(x) & f_3(x)\\ g_1(x) & g_2(x) & g_3(x)\\ h_1'(x) & h_2'(x) & h_3'(x)\end{vmatrix}$$

$$\therefore\ F'(a)=\begin{vmatrix} f_1'(a) & f_2'(a) & f_3'(a)\\ g_1(a) & g_2(a) & g_3(a)\\ h_1(a) & h_2(a) & h_3(a)\end{vmatrix}$$

$$+\begin{vmatrix} f_1(a) & f_2(a) & f_3(a)\\ g_1'(a) & g_2'(a) & g_3'(a)\\ h_1(a) & h_2(a) & h_3(a)\end{vmatrix}+\begin{vmatrix} f_1(a) & f_2(a) & f_3(a)\\ g_1(a) & g_2(a) & g_3(a)\\ h_1'(a) & h_2'(a) & h_3'(a)\end{vmatrix}$$

$F'(a)=D_1+D_2+D_3$

From (2), $D_1=D_2=D_3=0$ (By the property of determinants that $D=0$ if two rows are identical)

$\Rightarrow F'(a)=$ zero.

3. Given, $f(x)=\log_x(\log x)=\frac{\log_e(\log_e x)}{(\log_e x)}$

$$\Rightarrow f'(x)=\frac{\frac{1}{\log_e x}\frac{1}{x}\log_e x-\frac{1}{x}\log_e(\log_e x)}{(\log_e x)^2}$$

$$= \frac{\frac{1}{x}\left[1-\log_e(\log_e x)\right]}{(\log_e x)^2}$$

at $x = e$, we get

$$f'(e) = \frac{\frac{1}{e}\left[1-\log_e(\log_e e)\right]}{(\log_e e)^2} = \frac{\frac{1}{e}[1-\log_e 1]}{(1)^2}$$

$$= \frac{1}{e}(1-0) = \frac{1}{e}$$

4. Let $u = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$; $v = \sqrt{1-x^2}$

We have, $u = \cos^{-1}(2x^2-1) = 2\cos^{-1} x$

$$\therefore \frac{du}{dx} = \frac{-2}{\sqrt{1-x^2}} \text{ and } \frac{dv}{dx} = \frac{-x}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{-2}{\sqrt{1-x^2}}}{\frac{-x}{\sqrt{1-x^2}}} = \frac{2}{x} \Rightarrow \left.\frac{du}{dv}\right|_{x=1/2} = 4$$

5. Given that $f(9)=9, f'(9)=4$

Then, $\lim_{x\to 9} \frac{\sqrt{f(x)}-3}{\sqrt{x}-3}$

$$= \lim_{x\to 9} \frac{\left(\sqrt{f(x)}-3\right)\left(\sqrt{f(x)}+3\right)}{\left(\sqrt{x}-3\right)\left(\sqrt{x}+3\right)} \lim_{x\to 9} \frac{\sqrt{x}+3}{\sqrt{f(x)}+3}$$

$$= \lim_{x\to 9} \frac{f(x)-9}{x-9} \times \left[\frac{3+3}{3+3}\right]$$

$$= \lim_{x\to 9} \frac{f(x)-f(9)}{x-9} \times 1 = f'(9) = 4$$

6. $f(x) = |x-2|$

$\Rightarrow g(x) = f(f(x)) = |f(x)-2|$

$= ||x-2|-2| = |x-2-2|$ (as $x > 20$) $= |x-4|$

$= x-4$ (as $x > 20$)

$\therefore g'(x) = 1$

7. Given $xe^{xy} = y + \sin^2 x$

Differentiating w.r.t. x, we get

$$e^{xy}\, 1 + x\, e^{xy}\left(y + x\frac{dy}{dx}\right) = \frac{dy}{dx} + 2\sin x \cos x$$

Put $x = 0 \Rightarrow 1 + 0 = \frac{dy}{dx} + 0 \Rightarrow \frac{dy}{dx} = 1$

8. $F(x) = f(x)\,g(x)\,h(x),\ \forall\ x \in R$

$f(x), g(x), h(x)$ are differentiable functions.

$\Rightarrow F'(x) = f'(x)\,g(x)\,h(x) + f(x)\,g'(x)\,h(x) + f(x)\,g(x)\,h'(x)$

At $x = x_0$,

$F'(x_0) = f'(x_0)\,g(x_0)\,h(x_0) + f(x_0)\,g'(x_0)\,h(x_0) + f(x_0)g(x_0)\,h'(x_0)$.

Using the given values of $F'(x_0), f'(x_0), g'(x_0)$ and $h'(x_0)$ we get $21F(x_0) = 4f(x_0)\,g(x_0)\,h(x_0) - 7f(x_0)\,g(x_0)\,h(x_0) + kh(x_0)f(x_0)\,g(x_0)$

$\Rightarrow 21 = 4 - 7 + k$ $\quad (\because F(x_0) = f(x_0)\,g(x_0)\,h(x_0))$

$\Rightarrow k = 24$

9. $f(0) = 1, f'(x) = 3x^2 + \frac{1}{2}e^{x/2}$

$g(f(x)) = x$

$\Rightarrow g'(f(x))\cdot f'(x) = 1$

Put $x = 0 \Rightarrow g'(1) = \frac{1}{f'(0)} = 2.$

True or false

1. Even function satisfies the relation $f(x) = f(-x)$
Diff. w.r.t x, $\Rightarrow f'(x) = -f'(-x)$, which is relation satisfied by an odd function.

Multiple choice questions with one correct answer

1.c. $\lim_{x\to a} \frac{g(x)\,f(a) - g(a)\,f(x)}{x-a}$

$$= \lim_{h\to 0} \frac{g(a+h)\,f(a) - g(a)f(a) + g(a)f(a) - g(a)f(a+h)}{h}$$

$$= \lim_{h\to 0} f(a)\left[\frac{g(a+h)-g(a)}{h}\right] - \lim_{h\to 0} g(a)\left[\frac{f(a+h)-f(a)}{h}\right]$$

$= f(a)\,g'(a) - g(a)f'(a)$

$= 2\times 2 - (-1)\times 1 = 5$

2.c. We have $y^2 = P(x)$, where $P(x)$ is a polynomial of degree 3 and hence thrice differentiable,

then $y^2 = P(x)$ (1)

Differentiate (1) w.r.t. x, we get

$$2y\frac{dy}{dx} = P'(x) \quad (2)$$

Again differentiate w.r.t. x, we get

$$2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = P''(x)$$

$$\Rightarrow \frac{[P'(x)]^2}{2y^2} + 2y\frac{d^2y}{dx^2} = P''(x) \quad \text{[Using (2)]}$$

$$\Rightarrow 4y^3\frac{d^2y}{dx^2} = 2y^2\, P''(x) - [P'(x)]^2$$

$$\Rightarrow 4y^3\frac{d^2y}{dx^2} = 2P(x)\, P''(x) - [P'(x)]^2 \quad \text{[Using(1)]}$$

$$\Rightarrow 2y^3\frac{d^2y}{dx^2} = P(x)\, P''(x) - \frac{1}{2}[P'(x)]^2.$$

Again differentiating with respect to x, we get

$$2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right) = P'''(x)\, P(x) + P''(x)$$

$$P'(x) - P'(x)P''(x)$$

$= P'''(x)\,P(x)$

3.b. Let $f(x) = ax^2 + bx + c$

As given that $f(x) > 0,\ \forall\ x \in R$

$\therefore\ a > 0$ and $b^2 - 4ac < 0$ (1)

Now, $g(x) = f(x) + f'(x) + f''(x)$

$= ax^2 + bx + c + 2ax + b + 2a$

$= ax^2 + (2a + b)x + (2a + b + c)$

Here $D = (2a + b)^2 - 4a(2a + b + c)$

$= 4a^2 + b^2 + 4ab - 8a^2 - 4ab - 4ac$

$= -4a^2 + (b^2 - 4ac) < 0$

Also $a > 0$ from (1), $\Rightarrow g(x) > 0,\ \forall\ x \in R$.

4.a. $y = (\sin x)^{\tan x}$

$\Rightarrow \log y = \tan x \log \sin x$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \log \sin x + \tan x \frac{1}{\sin x}.\cos x$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\tan x}[1 + \sec^2 x \log \sin x].$$

5.d. Given $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$ where p is constant

$$\Rightarrow f'''(x) = \begin{vmatrix} 6 & -\cos x & \sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$\Rightarrow f'''(x)]_{x=0} = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} = 0 \qquad (\because R_1 \equiv R_2)$$

= Independent of p

6.c. Given limit has 1^∞ form

$$\Rightarrow L = \lim_{x\to 0}\left[\frac{f(1+x)}{f(1)}\right]^{1/x} = e^{\lim_{x\to 0}\left[\frac{f(1+x)}{f(1)}-1\right]\frac{1}{x}}$$

$$= e^{\lim_{x\to 0}\left[\frac{f(1+x)-f(1)}{xf(1)}\right]} = e^{\lim_{x\to 0}\left[\frac{f'(1+x)}{f(1)}\right]}$$

(applying L' Hopital's Rule)

$$= e^{\frac{f'(1)}{f(1)}} = e^2$$

7.d. $$\lim_{h\to 0}\frac{f(2h + 2 + h^2) - f(2)}{f(h - h^2 + 1) - f(1)} \quad \left[\frac{0}{0}\text{ form}\right]$$

$\therefore$ Applying L' Hopital's Rule, we get

$$= \lim_{h\to 0}\frac{f'(2h + 2 + h^2)(2 + 2h)}{f'(h - h^2 + 1)(1 - 2h)}$$

$$= \frac{f'(2)2}{f'(1)1} = \frac{6\times 2}{4\times 1} = 3$$

8.c. $$L = \lim_{x\to 0}\frac{f(x^2) - f(x)}{f(x) - f(0)} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{f'(x^2)2x - f'(x)}{f'(x)} \text{ (applying L' Hopital's Rule)}$$

$$= \lim_{x\to 0}\frac{f'(x^2).2x}{f'(x)} - 1 = 0 - 1 = -1$$

9.a. $\log(x + y) = 2xy$ when $x = 0 \Rightarrow y = 1$

Differentiating w.r.t. x, we get

$$\Rightarrow \frac{1}{x+y}\left[1 + \frac{dy}{dx}\right] = 2y + \frac{2xdy}{dx}$$

Put $x = 0$ and $y = 1$

$$\Rightarrow \frac{1}{0+1}\left[1 + \frac{dy}{dx}\right] = 2 + 0$$

$$\Rightarrow \frac{dy}{dx} = 1 \text{ or } y'(0) = 1$$

10.b. $x^2 + y^2 = 1$

$\Rightarrow 2x + 2yy' = 0$

$\Rightarrow x + yy' = 0$

$\Rightarrow 1 + yy'' + (y')^2 = 0$

11.d. $$\frac{d^2x}{d^2y} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dx}\left(\frac{dx}{dy}\right)\frac{dx}{dy}$$

$$= \left\{\frac{d}{dx}\left[\frac{1}{\left(\frac{dy}{dx}\right)}\right]\right\}\frac{1}{\left(\frac{dy}{dx}\right)} = -\frac{1}{\left(\frac{dy}{dx}\right)^2}\frac{d^2y}{dx^2}\frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= -\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2}$$

Reasoning Type

1.b. Given $f(x) = 2 + \cos x$ which is continuous and differentiable every where $f'(x) = -\sin x = 0$ at $x = 0, \pi$

$\therefore$ Statement 1 is true.

Also, $f(t) = f(t + 2\pi)$ is true.

But Statements 1 and 2 are not related.

Integer Type

1. (1) $$f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right), \text{ where } \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

$$= \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{2\cos^2\theta - 1}}\right)\right)$$

$$= \sin(\sin^{-1}(\tan\theta)) = \tan\theta$$

$$\Rightarrow \frac{d(\tan\theta)}{d(\tan\theta)} = 1.$$