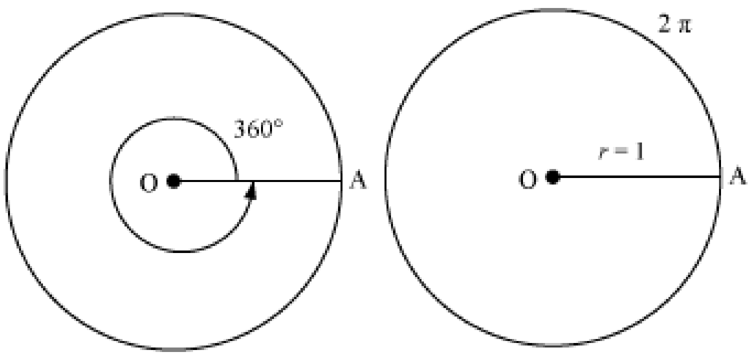
Trigonometric Functions

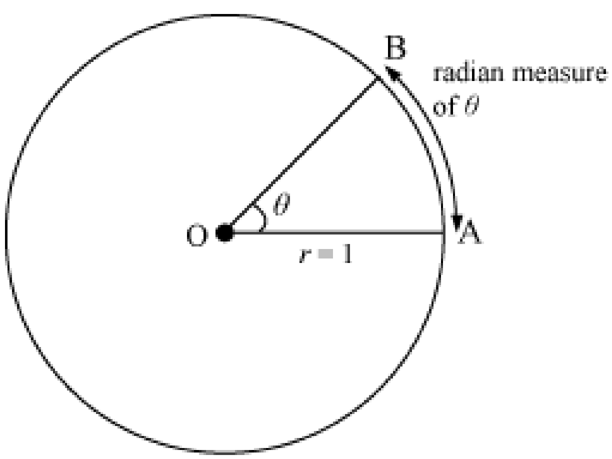
Degree and Radian Measures

- Degree and radian measure are the two ways of measuring an angle, depending on the two ways to define a circle: the angle at the centre of a circle is 360° and the circumference of a unit circle (circle whose radius is 1) is 2π.



According to the figure, 360° corresponds to 2π.

- If θ is any angle in degrees, then its radian measure represents the length of arc of unit circle corresponding to the angle θ.



- If θ isthe measure of an angle in degrees, then it is written as $\theta°$ and if α is the radian measure of an angle, then it is simply written as α.

- If an angle is $\left(\frac{1}{360}\right)^{th}$ of a complete revolution, then the angle is said to have a measure of **one degree,** written as 1°.

- A degree is divided into 60 minutes and a minute is divided into 60 seconds. One sixtieth of a degree is called a **minute,** written as 1′ and one sixtieth of a minute is called a **second**, written as 1″.
  $1° = 60'$
  $1' = 60''$

- Angle subtended at the centre by an arc of length 1 unit in a unit circle is said to have a measure of **one radian**.

- Length of the arc(l) = Angle subtended at the centre by the arc(θ) × Radius(r), where θ is in radians.

Conversion of Degree to Radian and Radian to Degree

- A circle subtends an angle of 2π radians or 360° at the centre. So,
  2π radians = 360°
  π radian = 180°
  $1° = \frac{\pi}{180}$ radians
  Or, 1 radian = $\left(\frac{180}{\pi}\right)^{\circ}$

  - To convert an angle from **degree measure to radian measure**, multiply the angle with $\frac{\pi}{180}$.
    - For e.g., $60° = 60 \times \frac{\pi}{180}$ radian $= \frac{\pi}{3}$ radian
  - To convert an angle from **radian measure to degree measure**, multiply the angle with $\frac{180}{\pi}$.
  - For e.g., $\frac{\pi}{6}$ radian $= \left(\frac{\pi}{6} \times \frac{180}{\pi}\right)^{\circ} = 30°$

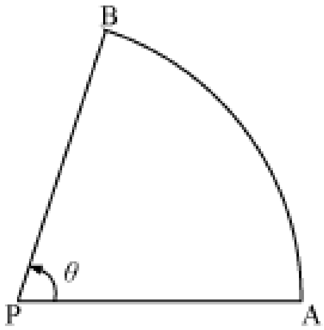
Solved Examples

Example 1:

A horse is tied to a pole with the help of a rope measuring 22 m. The horse moves in a circular path keeping the rope tight and covers a distance of 66 m. Find the angle (in degrees) traced out by the horse at the pole while moving.

Solution:

Let the pole be at point O. The horse starts from position A and covers 66 m and stops at B.



From the figure, it is clear that the horse moves along the arc of the circle with radius equal to the length of the rope.

Hence, we have

θ = $\frac{l}{r}$

l = 66 m

r = 22 m

θ = $\frac{66}{22} = 3$ radians

However, we have to express the answer in degrees.

π radian = 180°

3 radians = $\frac{540^\circ}{\pi} = \left(\frac{540 \times 7}{22}\right)^\circ$ = 171° 49’ 5.4’’

Thus, the horse moved by 171° 49’ 5.4’’.

Example 2:

Find the angle between the hour hand and the minute hand at half past four.

Solution:

Angle traced by the hour hand in 12 hours = 360°

Angle traced by the hour hand in 4 hours 30 min ($\frac{9}{2}$ hrs) = $\left(\frac{360}{12}\times\frac{9}{2}\right)^{\circ}=135^{\circ}$

Angle traced by the minute hand in 60 min = 360°

Angle traced by the minute hand in 30 min = $\left(\frac{360}{60}\times 30\right)^{\circ}=180^{\circ}$

Thus, the angle between two hands = 180° − 135° = 45°

= $\left(45\times\frac{\pi}{180}\right)=\frac{\pi}{4}$ radians

Sign of Trigonometric Functions

- The sign of any trigonometric function depends on the quadrant in which the angle of that trigonometric function lies. For e.g., the sign of trigonometric function sin *θ* depends on the quadrant in which angle *θ* lies.
- The signs of the six trigonometric functions in different quadrants are given in the table below.

	I	**II**	**III**	**IV**
sin *x*	+	+	−	−
cos *x*	+	−	−	+
tan *x*	+	−	+	−
cosec *x*	+	+	−	−

sec *x*	+	−	−	+
cot *x*	+	−	+	−

- If we are given the value of a trigonometric function along with the quadrant, then we can calculate the value as well as the sign of any other trigonometric function.
- **Example:** If $\sin x = -\frac{5}{13}$ and *x* lies in quadrant **III**, then we can calculate the other trigonometric functions as

$$\sin x = \frac{-5}{13}$$
$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{-13}{5}$$
$$\sin^2 x + \cos^2 x = 1$$
$$\cos^2 x = 1 - \sin^2 x$$
$$\cos^2 x = 1 - \left(\frac{-5}{13}\right)^2$$
$$\cos x = \pm\frac{12}{13}$$

Since *x* lies in the third quadrant, cos *x* is negative.

$$\therefore \cos x = \frac{-12}{13}$$
$$\therefore \sec x = \frac{1}{\cos x} = \frac{-13}{12}$$
$$\tan x = \frac{\sin x}{\cos x} = \frac{5}{12}$$
$$\text{and } \cot x = \frac{1}{\tan x} = \frac{12}{5}$$

- An example based on the above concept is explained in the given video.

Solved Examples

Example 1:

If $\cos\theta = \frac{1}{\sqrt{2}}$ and $\frac{3\pi}{2} < \theta < 2\pi$, then find the value of $\frac{2+\operatorname{cosec}\theta}{2-\cot\theta}$.

Solution:

$$\cos\theta = \frac{1}{\sqrt{2}}$$
$$\sin\theta = \pm\sqrt{1-\cos^2\theta}$$
$$\sin\theta = \pm\frac{1}{\sqrt{2}}$$

However, θ lies in the fourth quadrant, where sin θ is negative.

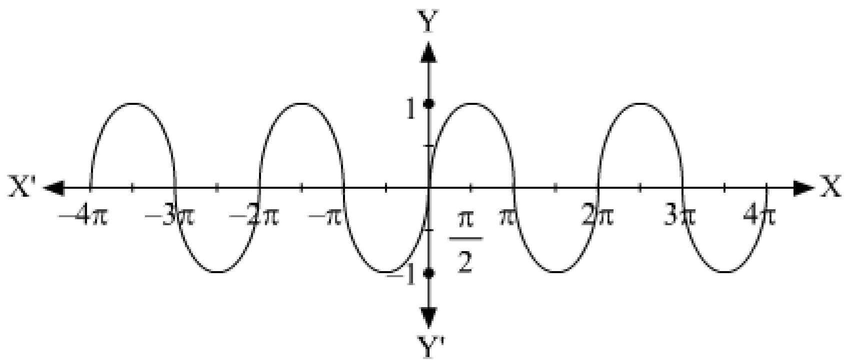
$$\therefore \sin\theta = -\frac{1}{\sqrt{2}} \Rightarrow \operatorname{cosec}\theta = -\sqrt{2}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta} = -1$$

$$\therefore \frac{2+\operatorname{cosec}\theta}{2-\cot\theta} = \frac{2+\left(-\sqrt{2}\right)}{2-(-1)} = \frac{2-\sqrt{2}}{3}$$

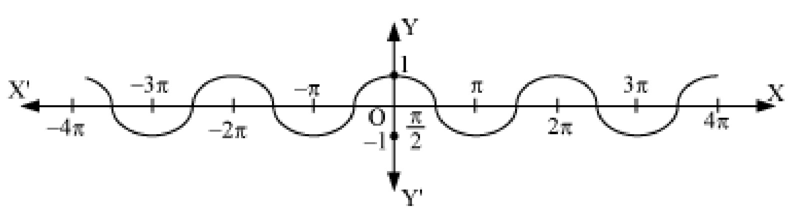
Trigonometric Functions

- Like any function, trigonometric functions also have domain and range and can be represented on a graph. The explanations of each of these graphs for all trigonometric functions is as follows:

sin *x*	**Domain** → *x*: *x* ∈ R
	Range → [−1, 1]

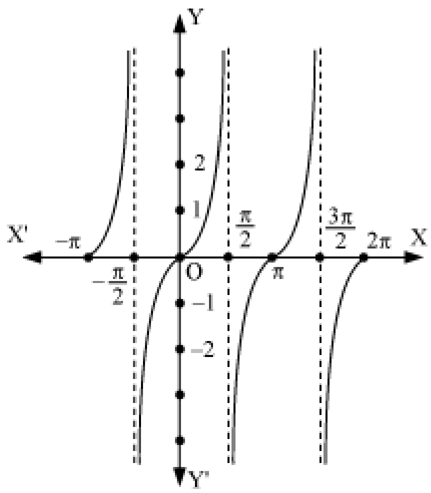
<table>
<tr><td></td><td>Periodicity → 2π</td></tr>
<tr><td colspan="2">

$y = \sin x$</td></tr>
</table>

<table>
<tr><td rowspan="3">cos x</td><td>Domain → x: x ∈ R</td></tr>
<tr><td>Range → [−1, 1]</td></tr>
<tr><td>Periodicity → 2π</td></tr>
<tr><td colspan="2">

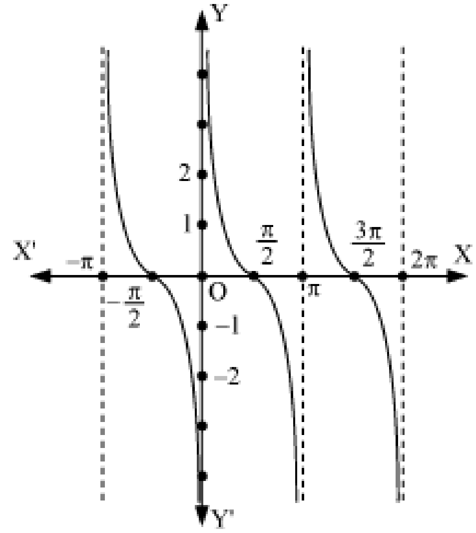
$y = \cos x$</td></tr>
</table>

tan *x*	**Domain** → *x*: *x* ∈ R and $x \neq (2n+1)\frac{\pi}{2}, n \in Z$
	Range → (−∞, ∞)
	Periodicity → π

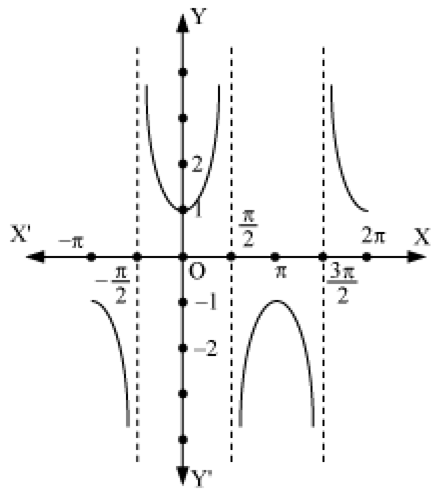


$y = \tan\ x$

cot *x*	**Domain** → *x*: *x* ∈ R and *x* ≠ *n*π, *n* ∈ Z
	Range → (−∞, ∞)
	Periodicity → π

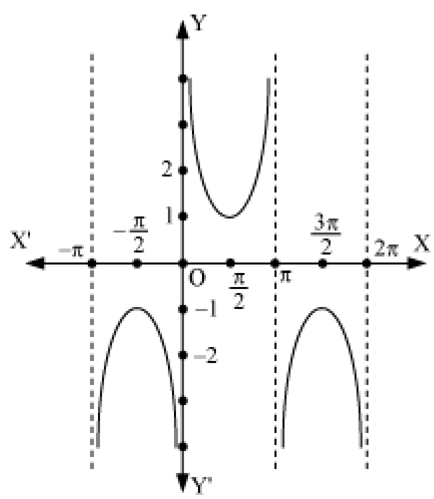
<table>
<tr><td colspan="2">

$y = \cot\ x$</td></tr>
<tr><td rowspan="3">sec x</td><td>Domain → x: x ∈ R and $x \neq (2n+1)\frac{\pi}{2}, n \in Z$</td></tr>
<tr><td>Range → y : y ∈ R, y ≤ − 1 or y ≥ 1</td></tr>
<tr><td>Periodicity → 2π</td></tr>
</table>



$y = \sec x$

cosec *x*	**Domain** → x: $x \in$ R and $x \neq n\pi$, $n \in$ Z
	Range → y : $y \in$ R, $y \geq 1$ or $y \leq -1$
	Periodicity → 2π



$y = \operatorname{cosec} x$

- Let us go through the following video to understand that how to find the value of sin840°.

Solved Examples

Example 1:

Find the value of tan (1485°).

Solution:

We know that the values of tan x repeat after an interval of π or 180°.

∴ tan (1485°) = tan (45° + 8 × 180°) = tan 45°
= 1

Trigonometric Identities

The two very important results that are extremely helpful in solving many questions and proving identities are

- $\sin(-x) = -\sin x$
- $\cos(-x) = \cos x$

We have the following trigonometric identities related to sum and difference of two angles:

- $\cos(x + y) = \cos x \cos y - \sin x \sin y$

To know the proof of this identity, let us go through the following video.

- $\cos(x - y) = \cos x \cos y + \sin x \sin y$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\sin(x - y) = \sin x \cos y - \cos x \sin y$
- $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

- $\cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot y + \cot x}$

- $\cot(x-y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$

With the help of the first identity, we can easily deduce the rest of the identities. Hence, let us go through the following video to know how these can be deduced.

- The following identities are the relations between trigonometric functions of an angle and the trigonometric function of twice (or thrice) that angle.

- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

- $\sin 2x = 2\sin x \cos x = \dfrac{2\tan x}{1+\tan^2 x}$

- $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

- $\sin 3x = 3\sin x - 4\sin^3 x$

- $\cos 3x = 4\cos^3 x - 3\cos x$

- $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

- The following are some identities involving addition and subtraction of trigonometric functions:

- $\cos x + \cos y = 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

- $\cos x - \cos y = -2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$

- $\sin x + \sin y = 2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

- $\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$
- Using the identities listed in the previous point, we can derive the following identities:
- $2\cos x\cos y = \cos(x+y) + \cos(x-y)$
- $-2\sin x\sin y = \cos(x+y) - \cos(x-y)$
- $2\sin x\cos y = \sin(x+y) + \sin(x-y)$
- $2\cos x\sin y = \sin(x+y) - \sin(x-y)$

Solved Examples

Example 1:

Prove that $\tan\theta(\sin 5\theta + \sin 3\theta) = \tan 4\theta(\sin 5\theta - \sin 3\theta)$.

Solution:

L.H.S. = $\tan\theta(\sin 5\theta + \sin 3\theta)$

$$= \tan\theta \times 2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)$$

$$= \frac{\sin\theta}{\cos\theta} \times 2\sin 4\theta\cos\theta = 2\sin\theta\sin 4\theta$$

R.H.S. = $\tan 4\theta(\sin 5\theta - \sin 3\theta)$

$$= \tan 4\theta \times 2\sin\left(\frac{5\theta-3\theta}{2}\right)\cos\left(\frac{5\theta+3\theta}{2}\right)$$

$$= \frac{\sin 4\theta}{\cos 4\theta} \times 2\sin\theta \times \cos 4\theta = 2\sin\theta \times \sin 4\theta$$

∴ L.H.S. = R.H.S.

Example 2:

Prove $\sqrt{(\cos x + \cos y)^2 + (\sin x + \sin y)^2} = 2\cos\left(\frac{x-y}{2}\right)$

Solution:

Solving the expression in square root,

$$(\cos x+\cos y)^2+(\sin x+\sin y)^2$$
$$=\left\{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right\}^2+\left\{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right\}^2$$
$$=4\cos^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right)+4\sin^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right)$$
$$=4\cos^2\left(\frac{x-y}{2}\right)\left\{\cos^2\left(\frac{x+y}{2}\right)+\sin^2\left(\frac{x+y}{2}\right)\right\}$$
$$=4\cos^2\left(\frac{x-y}{2}\right)$$
$$\therefore \sqrt{(\cos x+\cos y)^2+(\sin x+\sin y)^2}=\sqrt{4\cos^2\left(\frac{x-y}{2}\right)}$$
$$=2\cos\left(\frac{x-y}{2}\right)$$

Trigonometric Equations

Principal Solutions of Trigonometric Equations

- Equations that involve trigonometric functions of a variable are called Trigonometric Equations.
- Trigonometric Equations have two types of solutions: Principal Solutions and General Solutions.
- The principal solutions of a trigonometric equation with variable *x* are those solutions for which $0 \le x < 2\pi$.
- To find the principal solutions of a trigonometric equation, find the first few values of the variable for which the equation holds true. Among these values, select those values that lie in the range $0 \le x < 2\pi$, for they are the principal solutions. Ensure that you don't miss a value that lies in this range.
- For example, consider the trigonometric equation $\sin x = \frac{1}{\sqrt{2}}$.

Now, we know that $\sin\frac{\pi}{4}=\sin\frac{3\pi}{4}=\sin\frac{9\pi}{4}=\sin\frac{11\pi}{4}=\frac{1}{\sqrt{2}}$

However, among these values, only $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ lie in the range $0 \le x < 2\pi$.
Thus, the principal values of the trigonometric equation are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

General Solutions of Trigonometric Equations

- The general solution of a trigonometric equation is an expression involving integer '*n*', which gives all possible solutions of the equation.

  There are some important theorems for finding the general solution of a trigonometric equation:

  If *x* and *y* are not odd multiples of x/2, then tan *x* = tan *y* implies $x = n\pi + y$, where $n \in \mathbf{Z}$.

- To find the general solution of a trigonometric equation, first reduce the equation, if required, to any of the following three forms:

- sin *x* = sin *y*

- cos *x* = cos *y*

- tan *x* = tan *y*

  Next, apply the corresponding theorem to write down the general solution of the equation.

  For example, consider the trigonometric equation $2\sin x\cos x = -\frac{1}{2}$.
  Now, $2\sin x\cos x = \sin 2x$

$$\therefore \sin 2x = -\frac{1}{2}$$

$$\Rightarrow \sin 2x = \sin\frac{7\pi}{6} \qquad \left(\sin\frac{7\pi}{6} = -\frac{1}{2}\right)$$

We now need to apply the theorem "For real numbers *x* and *y*, if sin *x* = sin *y*, then $x = n\pi + (-1)^n y$, where $n \in \mathbf{Z}$."

$$2x = n\pi + (-1)^n \frac{7\pi}{6}$$

$$\therefore x = \frac{n\pi}{2} + (-1)^n \frac{7\pi}{12}$$

This is the general solution of the trigonometric equation.

Solved Examples

Example 1:

Solve the equation $\left(\frac{\tan\theta + \tan 2\theta}{\sqrt{3}}\right)$ + $\tan\theta \tan 2\theta = 1$.

Solution:

We have $\left(\frac{\tan\theta + \tan 2\theta}{\sqrt{3}}\right)$ + $\tan\theta \tan 2\theta = 1$

$\Rightarrow \left(\frac{\tan\theta + \tan 2\theta}{\sqrt{3}}\right) = 1 - \tan\theta \tan 2\theta$

$$\Rightarrow \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\ \tan 2\theta} = \sqrt{3}$$

$$\Rightarrow \tan(2\theta + \theta) = \sqrt{3}$$

$$\Rightarrow \tan 3\theta = \sqrt{3}$$

$$\Rightarrow \tan 3\theta = \tan\frac{\pi}{3}$$

$$3\theta = n\pi + \frac{\pi}{3}, n \in Z$$

$$\theta = \frac{n\pi}{3} + \frac{\pi}{9},\ n \in Z$$

Example 2:

Solve the equation $3\cos^2\theta + \sin^2\theta = 2$.

Solution:

$3\cos^2\theta + \sin^2\theta = 2$

$3(1 - \sin^2\theta) + \sin^2\theta = 2$ [$\cos^2\theta = 1 - \sin^2\theta$]

$3 - 3\sin^2\theta + \sin^2\theta = 2$

$2\sin^2\theta = 1$

$\sin^2\theta = \frac{1}{2}$

$$\frac{1-\cos 2\theta}{2} = \frac{1}{2}$$

$$\cos 2\theta = 0$$

$$\cos 2\theta = \cos\frac{\pi}{2}$$

$$\Rightarrow \theta = n\pi + \frac{\pi}{4}, n \in Z$$