Continuity and Differentiability
Short Answer Type Questions

1. **Find the value of the constant k so that the function f defined below is continuous at** $x=0, where\ f(x)=\begin{cases}\dfrac{1-\cos 4x}{8x^2}, & x\neq 0.\\ k, & x=0\end{cases}$

Sol. It is given that the function f is continuous at x = 0. Therefore, $\lim_{x\to 0} f(x)=f(0)$

$$\Rightarrow \lim_{x\to 0}\frac{1-\cos 4x}{8x^2}=k$$

$$\Rightarrow \lim_{x\to 0}\frac{2\sin^2 2x}{8x^2}=k$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2=k$$

$$\Rightarrow k=1$$

Thus, f is continuous at x = 0 if k = 1.

2. **Discuss the continuity of the function** $f(x)=\sin x.\cos x$ **.**

Sol. Since sin x and cos x are continuous functions and product of two continuous function is a continuous function, therefore $f(x)=\sin x.\cos x$ is a continuous function.

3. **If** $f(x)=\begin{cases}\dfrac{x^3+x^2-16x+20}{(x-2)^2}, & x\neq 2\\ k & ,x=2\end{cases}$ **is continuous at x = 2, find the value of k.**

Sol. Given $f(2)$ = k.

Now, $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=\lim_{x\to 2}\dfrac{x^3+x^2-16x+20}{(x-2)^2}$

$$=\lim_{x\to 2}\frac{(x+5)(x-2)^2}{(x-2)^2}=\lim_{x\to 2}(x+5)=7$$

As f is continuous at x= 2, we have

$$\lim_{x\to 2} f(x)=f(2)$$

$$\Rightarrow k=7.$$

4. **Show that the function f defined by** $f(x)=\begin{cases}x\sin\dfrac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$ **is continuous at x = 0.**

Sol. Left hand limit at x = 0 is given by

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} x\sin\frac{1}{x}=0 \text{ [since,} -1<\sin\frac{1}{x}<1]$$

Similarly, $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} x\sin\dfrac{1}{x}=0. Moreover\ f(0)=0.$

Thus, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0). Hence\ f\ is\ continuous\ at\ x=0$

5. **Given** $f(x)=\frac{1}{x-1}$**. Find the points of discontinuity of the composite function** $y = f[f(x)]$.

Sol. We know that $f(x)=\frac{1}{x-1}$ is discontinuous at x = 1

Now, for $x \neq 1$,

$$f(f(x))=f\left(\frac{1}{x-1}\right)=\frac{1}{\frac{1}{x-1}-1}=\frac{x-1}{2-x}$$

Which is discontinuous at x = 2.

Hence, the points of discontinuity are x = 1 and x = 2.

6. **Let** $f(x) = x|x|$, **for all** $x \in \mathbf{R}$. **Discuss the derivability of** $f(x)$ **at x = 0**

Sol. We may rewrite f as $f(x)=\begin{cases} x^2, if\ x\geq 0 \\ -x^2, if\ x<0 \end{cases}$

Now, $Lf'(0)=\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{-h^2-0}{h}=\lim_{h\to 0^-}-h=0$

Now, $Rf'(0)=\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{h^2-0}{h}=\lim_{h\to 0^-}h=0$

Since the left-hand derivative and right hand derivative both are equal, hence f is differentiable at x = 0.

7. **Differentiate** $\sqrt{\tan\sqrt{x}}$ **w.r.t. x**

Sol. Let $y=\sqrt{\tan\sqrt{x}}$. Using chain rule, we have

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\tan\sqrt{x}}}.\frac{d}{dx}(\tan\sqrt{x})$$

$$=\frac{1}{2\sqrt{\tan\sqrt{x}}}.\sec^2\sqrt{x}\frac{d}{dx}(\sqrt{x})$$

$$=\frac{1}{2\sqrt{\tan\sqrt{x}}}(\sec^2\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)$$

$$=\frac{(\sec^2\sqrt{x})}{4\sqrt{x}\sqrt{\tan\sqrt{x}}}.$$

8. **If** $y=\tan(x+y),\ find\ \frac{dy}{dx}$.

Sol. Given y = tan (x + y). differentiating both sides w.r.t. x, we have

$$\frac{dy}{dx}=\sec^2(x+y)\frac{d}{dx}(x+y)$$

$$=\sec^2(x+y)\left(1+\frac{dy}{dx}\right)$$

$$or\quad [1-\sec^2(x+y)\frac{dy}{dx}=\sec^2(x+y)$$

Therefore, $\dfrac{dy}{dx}=\dfrac{\sec^2(x+y)}{1-\sec^2(x+y)}=-\operatorname{cosec}^2(x+y).$

9. **If $e^x+e^y=e^{x+y}$, prove that $\dfrac{dy}{dx}=-e^{y-x}$**

Sol. Given that $e^x+e^y=e^{x+y}$. Differentiating both sides w.r.t. x, we have

$$e^x+e^y\frac{dy}{dx}=e^{x+y}\left(1+\frac{dy}{dx}\right)$$

$$or\,(e^y-e^x+^y)\frac{dy}{dx}=e^x+^y-e^x,$$

Which implies that $\dfrac{dy}{dx}=\dfrac{e^{x+y}-e^x}{e^y-e^{x+y}}=\dfrac{e^x+e^y-e^x}{e^y-e^x-e^y}=-e^{y-x}$

10. **Find** $\dfrac{dy}{dx}$, *if* $y=\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right),-\dfrac{1}{\sqrt{3}}<x<\dfrac{1}{\sqrt{3}}.$

Sol. Put x = tanθ, where $\dfrac{-\pi}{6}<\theta<\dfrac{\pi}{6}.$

Therefore, $y=\tan^{-1}\left(\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$

$=\tan^{-1}(\tan 3\theta)$

$=3\theta\ (because\dfrac{-\pi}{2}<3\theta<\dfrac{\pi}{2})$

$=3\tan^{-1}x$

Hence, $\dfrac{dy}{dx}=\dfrac{3}{1\ \ x^2}.$

11. **If** $y=\sin^{-1}\left\{x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right\}$ *and* $0<x<1$, *then find* $\dfrac{dy}{dx}.$

Sol. We have $y=\sin^{-1}\left\{x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right\}$, *where* $0<x<1.$

Put $x=\sin A\ and\ \sqrt{x}=\sin B$

Therefore, $y=\sin^{-1}\left\{\sin A\sqrt{1-\sin^2 B}-\sin B\sqrt{1-\sin^2 A}\right\}$

$=\sin^{-1}\{\sin A\cos B-\sin B\cos A\}$

$=\sin^{-1}\{\sin(A-B)\}=A-B$

Thus $y=\sin^{-1}x-\sin^{-1}\sqrt{x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-\sqrt{(x)}^2}}.\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=\frac{1}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{x}\sqrt{1-x}}.$$

12. **If** $x=a\ sec^3\theta$ **and** $y=a\ tan^3\theta$, **find** $\dfrac{dy}{dx}$ $at\,\theta=\dfrac{\pi}{3}.$

Sol. We have x = a $\sec^3\theta$ and y = a $\tan^3\theta$.

Differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = 3a\sec^2\theta\frac{d}{d\theta}(\sec\theta) = 3a\sec^3\theta\tan\theta$$

and $\frac{dy}{d\theta} = 3a\tan^2\theta\frac{d}{d\theta}(\tan\theta) = 3a\tan^2\theta\sec^2\theta$

Thus $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta.$

Hence $\left(\frac{dy}{dx}\right)_{at\,\theta=\frac{\pi}{3}} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$

13. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

Sol. We have $x^y = e^{x-y}$, Taking logarithm on both sides, we get

$y\log x = x - y$

$\Rightarrow y(1+\log x) = x$

i.e. $y = \frac{x}{1+\log x}$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x).1 - x\left(\frac{1}{x}\right)}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}.$$

14. If $y = \tan x + \sec x$, prove that $\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}$.

Sol. We have $y = \tan x + \sec x$. Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \sec^2 x + \sec x\tan x$$

$$= \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} = \frac{1+\sin x}{\cos^2 x} = \frac{1+\sin x}{(1+\sin x)(1-\sin x)}.$$

Thus $\frac{dy}{dx} = \frac{1}{1-\sin x}$.

Now, differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{-(-\cos x)}{(1-\sin x)^2} = \frac{\cos x}{(1-\sin x)^2}$$

15. If $f(x) = |\cos x|$, find $f'\left(\frac{3\pi}{4}\right)$.

Sol. When $\frac{\pi}{2} < x < \pi$, cos x < 0 so that $|\cos x|$ = - cos x, i.e., f (x) = - cos x $\Rightarrow$ f ' (x) = sin x.

Hence, $f'\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$

16. **If** $f(x) = |\cos x - \sin x/,$ **find** $f'\frac{\pi}{6}.$

Sol. When $0 < x < \frac{\pi}{4}$, cos x >sin x, so that cos x – sin x > 0, i.e.,

$f(x) = \cos x - \sin x$

$\Rightarrow f'(x) = -\sin x - \cos x$

Hence, $f'\frac{\pi}{6} = -\sin\frac{\pi}{6} - \cos\frac{\pi}{6} = -\frac{1}{2}(1+\sqrt{3}).$

17. **Verify Rolle's theorem for the function,** $f(x) = \sin 2x \, in \left[0, \frac{\pi}{2}\right].$

Sol. Consider $f(x) = \sin 2x \, in \left[0, \frac{\pi}{2}\right]$. Not that:

(i) The function f is continuous in $\left[0, \frac{\pi}{2}\right]$ as f is a sine function, which is always continuous.

(ii) $f'(x) = 2\cos 2x,$ exists in $\left(0, \frac{\pi}{2},\right)$, hence f is derivable in $\left(0, \frac{\pi}{2}\right).$

(iii) $f(0) = sin 0 = 0$ and $f\left(\frac{\pi}{2}\right)$ = sin π = 0 $\Rightarrow f(0) = f\left(\frac{\pi}{2}\right).$

Conditions of Rolle's theorem are satisfied. Hence there exists at least one $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$. Thus

$2\cos 2c = 0 \Rightarrow 2c = \frac{\pi}{2} \Rightarrow c = \frac{\pi}{4}.$

18. **Verify mean value theorem for the function** $f(x) = (x-3)(x-6)(x-9)$ **in [3, 5].**

Sol. (i) Function f is continuous in [3, 5] as product of polynomial functions is a polynomial, which is continuous.

(ii) $f'(x) = 3x^2 - 36x + 99$ exists in (3, 5) and hence derivable in (3, 5).

Thus conditions of mean value theorem are satisfied. Hence, there exists at least one $c \in (3, 5),$ such that

$f'(c) = \frac{f(5) - f(3)}{5-3}$

$\Rightarrow 3c^2 - 36c + 99 = \frac{8-0}{2} = 4$

$\Rightarrow c = 6 \pm \sqrt{\frac{13}{3}}.$

Hence $c = 6 - \sqrt{\frac{13}{3}}$ (since other value is not permissible).

Long Answer Type Quesitons

19. If $f(x)=\dfrac{\sqrt{2}\cos x-1}{\cot x-1}, x\neq\dfrac{\pi}{4}$ **find the value of** $f\left(\dfrac{\pi}{4}\right)$ **so that** $f(x)$ **becomes continuous at** $x=\dfrac{\pi}{4}$.

Sol. Given, $f(x)=\dfrac{\sqrt{2}\cos x-1}{\cot x-1}, x\neq\dfrac{\pi}{4}$

Therefore, $\lim\limits_{x\to\frac{\pi}{4}} f(x)=\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sqrt{2}\cos x-1}{\cot x-1}$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\left(\sqrt{2}\cos x-1\right)\sin x}{\cos x-\sin x}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\left(\sqrt{2}\cos x-1\right)}{\left(\sqrt{2}\cos x+1\right)}.\frac{\left(\sqrt{2}\cos x+1\right)}{\left(\cos x-\sin x\right)}.\frac{\left(\cos x+\sin x\right)}{\left(\cos x+\sin x\right)}.\sin x$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{2\cos^2 x-1}{\cos^2 x-\sin^2 x}.\frac{\cos x+\sin x}{\sqrt{2}\cos x+1}.(\sin x)$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\cos 2x}{\cos 2x}.\left(\frac{\cos x+\sin x}{\sqrt{2}\cos x+1}\right).(\sin x)$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\left(\cos x+\sin x\right)}{\sqrt{2}\cos x+1}\sin x$$

$$=\frac{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)}{\sqrt{2}.\frac{1}{\sqrt{2}}+1}=\frac{1}{2}$$

Thus, $\lim\limits_{x\to\frac{\pi}{4}} f(x)=\dfrac{1}{2}$

If we define $f\left(\dfrac{\pi}{4}\right)=\dfrac{1}{2}$, then $f(x)$ will become continuous at $x=\dfrac{\pi}{4}$. Hence for f to be continuous at $x=\dfrac{\pi}{4}, f\left(\dfrac{\pi}{4}\right)=\dfrac{1}{2}$.

20. Show that the function f **given by** $f(x)=\begin{cases}\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, & if\ \ x\neq 0\\ 0, & if\ x=0\end{cases}$ **is discontinuous at x = 0.**

Sol. The left hand limit of f at x = 0 is given by

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} = \frac{0-1}{0+1} = -1$$

Similarly, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$

$$= \lim_{x\to 0^+} \frac{1-\frac{1}{e^{\frac{1}{x}}}}{1+\frac{1}{e^{\frac{1}{x}}}} = \lim_{x\to 0^+} \frac{1-e^{\frac{-1}{x}}}{1+e^{\frac{-1}{x}}} = \frac{1-0}{1+0} = 1$$

Thus $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$ therefore, $\lim_{x\to 0} f(x)$ does not exist. Hence f is discontinuous at x = 0.

21. **Let** $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2}, & if\ x<0 \\ = a, & if\ x=0 \\ \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & if\ x>0 \end{cases}$

For what value of a, f is continuous at x = 0?

Sol. Here $f(0) = a$ Left-hand limit of f at 0 is

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1-\cos 4x}{x^2} = \lim_{x\to 0^-} \frac{2\sin^2 2x}{x^2}$$

$$= \lim_{2x\to 0^-} 8\left(\frac{\sin 2x}{2x}\right)^2 = 8(1)^2 = 8.$$

and right hand limit of f at 0 is

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x}(\sqrt{16+\sqrt{x}}+4)}{(\sqrt{16+\sqrt{x}}+4)(\sqrt{16+\sqrt{x}}-4)}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x}(\sqrt{16+\sqrt{x}}+4)}{16+\sqrt{x}-16} = \lim_{x\to 0^+} \left(\sqrt{16+\sqrt{x}}+4\right) = 8$$

Thus, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 8.$ Hence f is continuous at x = 0 only if a = 8.

22. **Examine the differentiability of the function f defined by**

$$f(x) = \begin{matrix} 2x+3, & if\ -3 \le x < -2 \\ x+1, & if\ -2 \le x < 0 \\ x+2, & if\ 0 \le x \le 1 \end{matrix}$$

Sol. The only doubtful points for differentiability of $f(x)$ are x = –2 and x = 0.

Differentiability at x = - 2.

Now $L\,f'(-2)=\lim\limits_{h\to 0^-}\dfrac{f(-2+h)-f(-2)}{h}$

$=\lim\limits_{h\to 0^-}\dfrac{2(-2+h)+3-(-2+1)}{h}=\lim\limits_{h\to 0^-}\dfrac{2h}{h}=\lim\limits_{h\to 0^-}2=2.$

and $R\,f'(-2)=\lim\limits_{h\to 0^+}\dfrac{f(-2+h)-f(-2)}{h}$

$=\lim\limits_{h\to 0^-}\dfrac{-2+h+1-(-2+1)}{h}$

$=\lim\limits_{h\to 0^-}\dfrac{h-1-(-1)}{h}=\lim\limits_{h\to 0^+}\dfrac{h}{h}=1$

Thus $R\,f'(-2)\neq L\,f'(-2)$. Therefore f is not differentiable at x = -2.

Similarly, for differentiability at x = 0, we have

$L\,(f'(0)=\lim\limits_{h\to 0^-}\dfrac{f(0+h)-f(0)}{h}$

$=\lim\limits_{h\to 0^-}\dfrac{0+h+1-(0+2)}{h}$

$=\lim\limits_{h\to 0}\dfrac{h-1}{h}=\lim\limits_{h\to 0^-}1-\dfrac{1}{h}$

which does not exist. Hence f is not differentiable at x = 0.

23. **Differentiate** $\tan^{-1}\dfrac{\sqrt{1-x^2}}{x}$ **with respect to** $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$**, where** $x\in\dfrac{1}{\sqrt{2}},1$.

Sol. Let $u=\tan^{-1}\dfrac{\sqrt{1-x^2}}{x}$ and $v=\cos^{-1}\left(2x\sqrt{1-x^2}\right)$.

We want to find $\dfrac{du}{dv}=\dfrac{\frac{du}{dx}}{\frac{dv}{dx}}$

Now $u=\tan^{-1}\dfrac{\sqrt{1-x^2}}{x}$. $Put\ x=\sin\theta.\left(\dfrac{\pi}{4}<\theta<\dfrac{\pi}{2}\right).$

Then $u=\tan^{-1}\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}=\tan^{-1}(\cot\theta)$

$=\tan^{-1}\left\{\tan\left(\dfrac{\pi}{2}-\theta\right)\right\}=\dfrac{\pi}{2}-\theta=\dfrac{\pi}{2}-\sin^{-1}x$

Hence $\dfrac{du}{dx}=\dfrac{-1}{\sqrt{1-x^2}}$.

Now $v=\cos^{-1}\left(2x\sqrt{1-x^2}\right)$

$=\dfrac{\pi}{2}-\sin^{-1}(2x\sqrt{1-x^2})$

$=\dfrac{\pi}{2}-\sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})=\dfrac{\pi}{2}-\sin^{-1}(\sin 2\theta)$

$$= \frac{\pi}{2} - \sin^{-1}\{\sin(\pi - 2\theta)\} \text{ [since } \frac{\pi}{2} < 2\theta < \pi]$$

$$= \frac{\pi}{2} - (\pi - 2\theta) = \frac{-\pi}{2} + 2\theta$$

$$\Rightarrow v = \frac{-\pi}{2} + 2\sin^{-1} x$$

$$\Rightarrow \frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}.$$

Hence $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{-1}{\sqrt{1-x^2}}}{\frac{2}{\sqrt{1-x^2}}} = \frac{-1}{2}.$

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 24 to 35.

24. The function $f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & if\ x \neq 0 \\ k & , if\ x = 0 \end{cases}$ **is continuous at x = 0, then the value of k is**

(A) 3
(B) 2
(C) 1
(D) 1.5

Sol. (B) is the Correct answer.

25. The function $f(x) = [x]$, **where [x] denotes the greatest integer function, is continuous at**

(A) 4
(B) - 2
(C) 1
(D) 1.5

Sol. (D) is the correct answer. The greatest integer function[x] is discontinuous at all integral values of x. Thus D is the correct answer.

26. The number of points at which the function $f(x) = \frac{1}{x - [x]}$ **is not continuous is**

(A) 1
(B) 2
(C) 3
(D) None of these

Sol. (D) is the correct answer. As x – [x] = 0, when x is an integer so $f(x)$ is discontinuous for all $x \in \mathbf{Z}$.

27. The function given by $f(x) = \tan x$ **is discontinuous on the set**

(A) $\{n\pi : n \in \mathbf{Z}\}$

(B) $\{2n\pi : n \in \mathbf{Z}\}$

(C) $\left\{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\right\}$

(D) $\left\{\frac{n\pi}{2} : n \in \mathbf{Z}\right\}$

Sol. C is the correct answer.

28. Let $f(x) = |\cos x|$. **Then**

(A) f **is everywhere differentiable.**

(B) f **is everywhere continuous but not differentiable at** $n = n\pi,\ n \in \mathbf{Z}$.

(C) f **is everywhere continuous but not differentiable at** $x = (2n+1)\frac{\pi}{2},\ n \in \mathbf{Z}$

(D) None of these.

Sol. C is the correct answer.

29. The function $f(x) = |x| + |x-1|$ **is**

(A) continuous at x = 0 as well as at x = 1.

(B) continuous at x = 1 but not at x = 0.

(C) discontinuous at x = 0 as well as at x = 1.

(D) continuous at x = 0 but not at x = 1.

Sol. Correct answer is A.

30. The value of k which makes the function defined by

$$f(x) = \begin{cases} \sin\frac{1}{x}, & if\ x \neq 0 \\ k, & if\ x = 0 \end{cases}, continuous\ at\ x = 0\ is$$

(A) 8

(B) 1

(C) -1

(D) None of these

Sol. (D) is the correct answer. Indeed $\lim_{x \to 0} \sin\frac{1}{x}$ does not exist.

31. The set of points where the functions f given by $f(x) = |x-3|\cos x$ **differentiable is**

(A) R

(B) $\mathbf{R} - \{3\}$

(C) $(0, \infty)$

(D) None of these

Sol. B is the correct answer.

32. Differential coefficient of sec ($\tan^{-1}$ x) w.r.t. x is

(A) $\frac{x}{\sqrt{1+x^2}}$

(B) $\frac{x}{1+x^2}$

(C) $x\sqrt{1+x^2}$

(D) $\frac{1}{\sqrt{1+x^2}}$

Sol. (A) is the correct answer.

33. **If $u=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, then $\frac{du}{dv}$ is**

(A) $\frac{1}{2}$

(B) x

(C) $\frac{1-x^2}{1+x^2}$

(D) 1

Sol. (D) is the correct answer.

34. **The value of c in Rolle's Theorem for the function $f(x)=e^x \sin x,\ x\in[0,\pi]$ is**

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{2}$

(D) $\frac{3\pi}{4}$

Sol. (D) is the correct answer.

35. **The value of c in Mean value theorem for the function $f(x)=x(x-2), x\in[1,2]$ is**

(A) $\frac{3}{2}$

(B) $\frac{2}{3}$

(C) $\frac{1}{2}$

(D) $\frac{3}{2}$

Sol. (A) is the correct answer.

36. **Match the following**

COLUMN – I	COLUMN – II
(A) If a function $f(x)=\begin{cases}\frac{\sin 3x}{x}, & if\ x\neq 0\\ \frac{k}{2}, & if\ x=0\end{cases}$ **is continuous at x = 0, then k is equal to**	**(A) |x|**
(B) Every continuous function is differentiable	**(B) True**
(C) An example of a function which is continuous everywhere but not differentiable at exactly one point	**(C) 6**
(D) The identify function i.e. $f(x)=x\forall x\in R$ **is a continuous function.**	**(D) False**

Sol. $A\to c, B\to d, C\to a, D\to b$

Fill in the blanks in each of the Examples 37 to 41.

37. **The number of points at which the function** $f(x)=\frac{1}{\log|x|}$ **is discontinuous is __________.**

Sol. The given function is discontinuous at x = 0, ± 1 and hence the number of points of discontinuity is 3.

38. **If** $f(x)=\begin{cases} ax+1 \text{ if } x\geq 1 \\ x+2 \text{ if } x<1 \end{cases}$ **is continuous, then a should be equal to __________.**

Sol. A = 2

39. **The derivative of** $log_{10}x$ **w.r.t. x is __________.**

Sol. $(\log_{10} e)\frac{1}{x}$.

40. **If** $y=\sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)+\sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$, **then** $\frac{dy}{dx}$ **is equal to _________.**

Sol. 0.

41. **The derivative of sin x w.r.t. cos x is _________.**

Sol. −cot x

Sate whether the statements are True of False in each of the Exercises 42 to 46.

42. **For continuity, at x = a, each of** $\lim_{x\to a^+} f(x)$ **and** $\lim_{x\to a^-} f(x)$ **is equal to** $f(a)$**.**

Sol. True.

43. **y = |x – 1| is a continuous function.**

Sol. True.

44. **A continuous function can have some points where limit does not exist.**

Sol. False.

45. **|sin x| is a differentiable function for every value of x.**

Sol. False.

46. **cos |x| is differentiable everywhere.**

Sol. True.

Continuity and Differentiability

Objective Type Questions

Choose the correct answers from the given four options in each of the Exercises 83 to 96.

83. **If** $f(x) = 2x$ **and** $f(x) = \frac{x^2}{2} + 1,$ **then which of the following can be a discontinuous function**

(A) $f(x) + g(x)$

(B) $f(x) - g(x)$

(C) $f(x).g(x)$

(D) $\frac{g(x)}{f(x)}$

Sol. (D) We know that, if f and g be continuous functions, then

(A) $f + g$ is continuous

(B) $f - g$ is continuous.

(C) fg is continuous

(D) $\frac{f}{g}$ is continuous at these points, where $g(x) \neq 0$.

Here, $\frac{g(x)}{f(x)} = \frac{\frac{x^2}{2}+1}{2x} = \frac{x^2+2}{4x}$

Which is discontinuous at x = 0.

84. **The function** $f(x) = \frac{4 - x^2}{4x - x^3}$

(A) discontinuous at only one point

(B) discontinuous at exactly two points

(C) discontinuous at exactly three points

(D) None of these

Sol. (C) We have, $f(x) = \frac{4-x^2}{4x-x^3} = \frac{(4-x^2)}{x(4-x^2)}$

$= \frac{(4-x^2)}{x(2^2-x^2)} = \frac{4-x^2}{x(2+x)(2-x)}$

Clearly, $f(x)$ is discontinuous at exactly three points $x = 0,\ x = -2\ and\ x = 2.$

85. **The set of points where the function** f **given by** $f(x) = |2x-1|\ sinx$ **is differentiable is**

(A) R

(B) $\mathbf{R} - \left\{\frac{1}{2}\right\}$

(C) $(0, \infty)$

(D) None of these

Sol. (B) We have, $f(x) = |2x-1|\ sinx$

At $x = \frac{1}{2}$, $f(x)$ is not differentiable

Hence, $f(x)$ is differentiable in $\mathbf{R}-\left\{\frac{1}{2}\right\}$

$$\because Rf'\left(\frac{1}{2}\right)=\lim_{h\to 0}\frac{f\left(\frac{1}{2}+h\right)-f\left(\frac{1}{2}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\left|2\left(\frac{1}{2}+h\right)-1\right|\sin\left(\frac{1}{2}+h\right)-0}{h}$$

$$=\lim_{h\to 0}\frac{|2h|.\sin\left(\frac{1+2h}{2}\right)}{h}=2.\sin\frac{1}{2}$$

And $Lf'\left(\frac{1}{2}\right)=\lim_{h\to 0}\frac{f\left(\frac{1}{2}-h\right)-f\left(\frac{1}{2}\right)}{-h}$

$$=\lim_{h\to 0}\frac{\left|2\left(\frac{1}{2}-h\right)^{-1}\right|\sin\left(\frac{1}{2}-h\right)-0}{-h}$$

$$=\lim_{h\to 0}\frac{|0-2h|-\sin\left(\frac{1}{2}-h\right)}{-h}=2\sin\left(\frac{1}{2}\right)$$

$$\because Rf'\left(\frac{1}{2}\right)\neq Lf'\left(\frac{1}{2}\right)$$

So, $f(x)$ is not differentiable at $x=\frac{1}{2}$.

86. The function $f(x)=cot\,x$ is discontinuous on the set

(A) $\{x=n\pi:n\in \mathbf{Z}\}$

(B) $\{x=2n\pi:n\in \mathbf{Z}\}$

(C) $\left\{x=(2n+1)\frac{\pi}{2};n\in \mathbf{Z}\right\}$

(D) $\left\{x=\frac{n\pi}{2};n\in \mathbf{Z}\right\}$

Sol. (a) We have, $f(x)=cot\,x$ is continuous in $R-\{n\pi:n\in Z\}$.

Since, $f(x)=\cot x=\frac{\cos x}{\sin x}$ [since, $\sin x=0$ *at* $n\pi, n\in Z$]

Hence, $f(x)=cot\,x$ is discontinuous on the set $\{x=n\pi:n\in Z\}$.

87. The function $f(x)=e^{|x|}$ is

(A) continuous everywhere but not differentiable at x = 0

(B) continuous and differentiable everywhere

(C) not continuous at x = 0
(D) None of these.

Sol. (A) Let $u(x) = |x|$ and $v(x) = e^x$

$\therefore f(x) = vou(x) = v[u(x)]$

$= v|x| = e^{|x|}$

Since, $u(x)$ and $v(x)$ are both continuous functions.

So, $f(x)$ is also continuous function but $u(x) = |x|$ is not differentiable at x = 0,

whereas $v(x) = e^x$ is differentiable at everywhere.

Hence, $f(x)$ is continuous everywhere but not differentiable at x = 0.

88. If $f(x) = x^2 \sin \frac{1}{x}$, where $x \neq 0$, then the value of the function f at x = 0, so that the function is continuous at x = 0, is
(A) 0
(B) –1
(C) 1
(D) None of these

Sol. (A) $\because f(x) = x^2 \sin\left(\frac{1}{x}\right), where\, x \neq 0$

Hence, value of the function f at $x = 0$, so that it is continuous at $x = 0$, is 0.

89. If $f(x) = \begin{cases} mx + 1, & if\ x \leq \frac{\pi}{2} \\ \sin x + n, & if\ x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$ then

(A) $m = 1,\ n = 0$

(B) $m = \frac{n\pi}{2} + 1$

(C) $n = \frac{m\pi}{2}$

(D) $m = n = \frac{\pi}{2}$

Sol. (C) We have, $f(x) = \begin{cases} mx + 1, & if\ x \leq \frac{\pi}{2} \\ \sin x + n, & if\ x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$

$$\therefore LHL = \lim_{x \to \frac{\pi^-}{2}} (mx + 1) = \lim_{h \to 0} \left[m\left(\frac{\pi}{2} - h\right) + 1 \right] = \frac{m\pi}{2} + 1$$

and $RHL = \lim_{x \to \frac{\pi^+}{2}} (\sin x + n) = \lim_{h \to 0} \left[\sin\left(\frac{\pi}{2} + h\right) + n \right]$

$= \lim_{h \to 0} \cos \text{h} + n = 1 + n$

$\therefore LHL = RHL \left[\text{to be continuous at } x = \frac{\pi}{2} \right]$

$\Rightarrow m.\frac{\pi}{2}+1=n+1$

$\therefore n=m.\frac{\pi}{2}$

90. **Let $f(x)=|sin\ x|$. Then**

(A) f **is everywhere differentiable**

(B) f **is everywhere continuous but not differentiable at** $x=n\pi, n\in \mathbf{Z}$.

(C) f **is everywhere continuous but not differentiable at** $x=(2n+1)\frac{\pi}{2},\ n\in \mathbf{Z}$.

(D) None of these

Sol. (B) We have, $f(x)=|sin\ x|$.

Let $f(x)=vou(x)=v[u(x)]$ [where, $u(x)=sin\ x\ and\ v(x)=|x|$]

$=v(sin\,x)=|sin\,x|$

Where, $u(x)\ and\ v(x)$ are both continuous.

Hence, $f(x)=vo\ u(x)$ is also a continuous function but $v(x)$ is not differentiable at x = 0

So, $f(x)$ is not differentiable where $\sin x=0 \Rightarrow x=n\pi,\ n\in Z$

Hence, $f(x)$ is continuous everywhere but not differentiable at $x=n\pi,\ n\in Z$.

91. **If $y=\log\left(\frac{1-x^2}{1+x^2}\right)$ then $\frac{dy}{dx}$ is equal to**

(A) $\frac{4x^3}{1-x^4}$

(B) $\frac{-4x}{1-x^4}$

(C) $\frac{1}{4-x^4}$

(D) $\frac{-4x^3}{1-x^4}$

Sol. (B) We have, $y=\log\left(\frac{1-x^2}{1+x^2}\right)$

$$\therefore \frac{dy}{dx}=\frac{1}{\frac{1-x^2}{1+x^2}}.\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$=\frac{(1+x^2)}{(1-x^2)}.\frac{(1+x^2).(-2x)-(1-x^2).2x}{(1+x^2)^2}$$

$$=\frac{-2x[1+x^2+1-x^2]}{(1-x^2).(1+x^2)}=\frac{-4x}{1-x^4}$$

92. **If $y=\sqrt{\sin x+y}$, then $\frac{dy}{dx}$ is equal to**

(A) $\frac{\cos x}{2y-1}$

(B) $\frac{\cos x}{1-2y}$

(C) $\frac{\sin x}{1-2y}$

(D) $\frac{\sin x}{2y-1}$

Sol. (a) $\because y=(\sin x+y)^{1/2}$

$\therefore \frac{dy}{dx}=\frac{1}{2}(\sin x+y)^{-1/2}.\frac{d}{dx}(\sin x+y)$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2}.\frac{1}{(\sin x+y)^{1/2}}.\left(\cos x+\frac{dy}{dx}\right)$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2y}\left(\cos x+\frac{dy}{dx}\right)$ $[\because (\sin x+y)^{1/2}=y]$

$\Rightarrow \frac{dy}{dx}\left(1-\frac{1}{2y}\right)=\frac{\cos x}{2y}$

$\therefore \frac{dy}{dx}=\frac{\cos x}{2y}.\frac{2y}{2y-1}=\frac{\cos x}{2y-1}$

93. The derivative of $cos^{-1}\left(2x^2-1\right)$ **w.r.t.** $cos^{-1}x$ **is**

(A) 2

(B) $\frac{-1}{2\sqrt{1-x^2}}$

(C) $\frac{2}{x}$

(D) $1-x^2$

Sol. (a) let u = $u=cos^{-1}\left(2x^2-1\right)$ and $v=cos^{-1}x$

$\therefore \frac{dv}{dx}=\frac{+-1}{\sqrt{1-(2x^2-1)^2}}.4x=\frac{-4x}{\sqrt{1-(4x^4+1-4x^2)}}$

$=\frac{-4x}{\sqrt{-4x^4+4x^2}}=\frac{-4x}{\sqrt{4x^2(1-x^2)}}$

$=\frac{-2}{\sqrt{1-x^2}}$

and $\frac{du}{dx}=\frac{-1}{\sqrt{1-x^2}}$

$\therefore \frac{dx}{dv}=\frac{du/dx}{dv/dx}=\frac{-2/\sqrt{1-x^2}}{-1\sqrt{1-x^2}}=2$

94. If $x=t^2,\ y=t^3$**, then** $\frac{d^2y}{dx^2}$ **is**

(A) $\frac{3}{2}$

(B) $\frac{3}{4t}$

(C) $\frac{3}{2t}$

(D) $\frac{3}{4}$

Sol. (B) We have, $x = t^2$, $y = t^3$,

$\therefore \frac{dx}{dt} = 2t \text{ and } \frac{dy}{dt} = 3t^2$

$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2}t$

On further differentiating w.r.t. x, we get

$\frac{d^2y}{dx^2} = \frac{3}{2}.\frac{d}{dt}t.\frac{dt}{dx}$

$= \frac{3}{2}.\frac{1}{2t} \left[\because \frac{dt}{dx} = \frac{1}{2t}\right]$

$= \frac{3}{4t}$

95. The value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}]$ is

(A) 1

(B) – 1

(C) $\frac{3}{2}$

(D) $\frac{1}{3}$

Sol. (A) $\because f'(c) = 0$ $[\because f'(x) = 3x^2 - 3]$

$\Rightarrow 3c^2 - 3 = 0$

$\Rightarrow c^2 = \frac{3}{3} = 1$

$\Rightarrow c = \pm 1, where 1 \in (0, \sqrt{3})$

$\therefore c = 1$

96. For the function $f(x) = x + \frac{1}{x}, x \in [1, 3]$, the value of c for mean value theorem is

(A) 1

(B) $\sqrt{3}$

(C) 2

(D) None of these

Sol. (b) $\because f'(c) = \frac{f(b) - f(a)}{b - a}$

$$\Rightarrow 1-\frac{1}{c^2}=\frac{\left[3+\frac{1}{3}\right]-\left[1+\frac{1}{1}\right]}{3-1}\quad \left[\begin{array}{l}\because f'(x)=1-\frac{1}{x^2}\\ and\, b=3, a=1\end{array}\right]$$

$$\Rightarrow \frac{c^2-1}{c^2}=\frac{\frac{10}{3}-2}{2}$$

$$\Rightarrow \frac{c^2-1}{c^2}=\frac{4}{3\times 2}=\frac{2}{3}$$

$$\Rightarrow 3(c^2-1)=2c^2$$

$$\Rightarrow 3c^2-2c^2=3$$

$$\Rightarrow c^2=3\Rightarrow c=\pm\sqrt{3}$$

$$\because c=\sqrt{3}\in(1,3)$$

Fill in the blanks in each of the Exercises 97 to 101:

97. An example of a function which is continuous everywhere but fails to be differentiable exactly at two points is ____________.

Sol. $|x|+|x-1|$ is continuous everywhere but fails to be differentiable exactly at points $x=0$ *and* $x=1$.

So, there can be more such examples of functions.

98. Derivative of x^2 w.r.t. x^3 is ____________.

Sol. Derivative of x^2 w.r.t. x^3, is $\frac{2}{3x}$

Let $u=x^2$ *and* $v=x^3$

$$\therefore \frac{du}{dx}=2x \text{ and } \frac{dv}{dx}=3x^2$$

$$\Rightarrow \frac{du}{dv}=\frac{2x}{3x^2}=\frac{2}{3x}$$

99. If $f(x)=|\cos x|$, then $f'\left(\frac{\pi}{4}\right)=$ __________.

Sol. If $f(x)=|\cos x|$, then $f'\left(\frac{\pi}{4}\right)$

$$\because 0<x<\frac{\pi}{2}, \cos x>0.$$

$$f(x)=+\cos x$$

$$\therefore f'(x)=(-\sin x)$$

$$\Rightarrow f'\left(\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}=\frac{-1}{\sqrt{2}}\quad\left[\because \sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}\right]$$

100. If $f(x)=|\cos x-\sin x|$, then $f'\left(\frac{\pi}{3}\right)=$ __________.

Sol. $\because f(x)=|\cos x-\sin x|$

$$\therefore f'\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}+1}{2}$$

We know that, $\frac{\pi}{4}<x<\frac{\pi}{2}, \sin x>\cos x$

$\therefore \cos x-\sin x \leq 0$ *i.e.*, $f(x)=-(\cos x-\sin x)$

$f'(x)=-[-\sin x-\cos x]$

$$\therefore f'\left(\frac{\pi}{3}\right)=-\left(\frac{-\sqrt{3}}{2}-\frac{1}{2}\right)=\left(\frac{\sqrt{3}+1}{2}\right)$$

101. For the curve $\sqrt{x}+\sqrt{y}=1, \frac{dy}{dx} at \left(\frac{1}{4},\frac{1}{4}\right)$ is ___________.

Sol. For the curve $\sqrt{x}+\sqrt{y}=1, \frac{dy}{dx} at \left(\frac{1}{4},\frac{1}{4}\right)$

We have, $\sqrt{x}+\sqrt{y}=1$

$$\Rightarrow \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\left(\frac{1}{4},\frac{1}{4}\right)}=\frac{-\frac{1}{2}}{\frac{1}{2}}=-1$$

State True of False for the statements in each of the Exercises 102 to 106.

102. Rolle's theorem is applicable for the function $f(x)=|x-1|$ *in* $[0,\ 2]$.

Sol. False

Hence, $f(x)=|x-1|$ *in* $[0,\ 2]$. is not differentiable at $x=1\in(0,2)$.

103. If f is continuous on its domain D, then $|f|$ is also continuous on D.

Sol. True

104. The composition of two continuous function is a continuous function.

Sol. True

105. Trigonometric and inverse-trigonometric functions are differentiable in their respective domain.

Sol. True

106. If $f.g$ is continuous at $x=a$, then f and g are separately continuous at $x=a$.

Sol. False

Let $f(x)=\sin x\ and\ g(x)=\cot x$

$$\therefore f(x).g(x)=\sin x.\frac{\cos x}{\sin x}=\cos x$$

which is continuous at $x=a$. but $cot\, x$ is not continuous at $x=a$.

Continuity and Differentiability

Short Answer Type Questions

1. Examine the continuity of the function $f(x) = x^3 + 2x^2 - 1 \; at \; x = 1$

Sol. We have, $f(x) = x^3 + 2x^2 - 1 \; at \; x = 1$

$\therefore \lim_{x \to 1^+} f(x) = \lim_{h \to 0} (1+h)^3 + 2(1+h)^2 - 1 = 2$

and $\lim_{x \to 1^-} f(x) = \lim_{h \to 0} (1-h)^3 + 2(1-h)^2 - 1 = 2$

$\therefore \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$

and $f(1) = 1 + 2 - 1 = 2$

So, $f(x)$ is continuous at x = 1.

Find which of the functions in Exercises 2 to 10 is continuous or discontinuous at the indicated points:

2. $f(x) = \begin{cases} 3x+5, \; if \; x \geq 2 \\ x^2, \; if \; x < 2 \end{cases} \; at \; x = 2.$

Sol. We have, $f(x) = \begin{cases} 3x+5, \; if \; x \geq 2 \\ x^2, \; if \; x < 2 \end{cases} \; at \; x = 2.$

At x = 2, $LHL = \lim_{x \to 2^-} (x)^2$

$= \lim_{h \to 0} (2-h)^2 = \lim_{h \to 0} (4 + h^2 - 4h) = 4$

And $RHL = \lim_{x \to 2^+} (3x+5)$

$= \lim_{h \to 0} [3(2+h)+5] = 11$

Since, $LHL \neq RHL \; at \; x = 2$

So, $f(x)$ is discontinuous at x = 2.

3. $f(x) = \begin{cases} \frac{1-\cos 2x}{x^2}, \; if \; x \neq 0 \\ 5, \quad if \; x = 0 \end{cases} \; at \; x = 0$

Sol. We have $f(x) = \begin{cases} \frac{1-\cos 2x}{x^2}, \; if \; x \neq 0 \\ 5, \quad if \; x = 0 \end{cases} \; at \; x = 0$

At x = 0 $LHL = \lim_{x \to 0^-} \frac{1 - \cos 2x}{x^2}$

$= \lim_{h \to 0} \frac{1 - \cos 2(0-h)}{(0-h)^2}$

$= \lim_{h \to 0} \frac{1 - \cos 2h}{h^2} \quad [\because \cos(-\theta) = \cos\theta]$

$$= \lim_{h\to 0} \frac{1-1+2\sin^2 h}{h^2} \quad [\because \cos 2\theta = 1-2\sin^2\theta]$$

$$= \lim_{h\to 0} \frac{2\ (\sin h)^2}{(h)^2} \quad \left[\because \lim_{h\to 0}\frac{\sin h}{h} = 1\right]$$

$$= 2$$

$$RHL = \lim_{x\to 0^+} \frac{1-\cos 2x}{x^2}$$

$$= \lim_{x\to 0} \frac{1-\cos 2(0+h)}{(0+h)^2}$$

$$= \lim_{x\to 0} \frac{2\sin^2 h}{h^2} = 2 \quad \left[\because \lim_{x\to 0}\frac{\sin h}{h} = 1\right]$$

And $f(0) = 5$

Since, $LHL = RHL \neq f(0)$

Hence, $f(x)$ is not continuous at x = 0

4. $f(x) = \begin{cases} \dfrac{2x^2-3x-2}{x-2}, & if\ x \neq 2 \\ 5, & if\ x = 2 \end{cases}$ $at\ x = 2.$

Sol. We have, $f(x) = \begin{cases} \dfrac{2x^2-3x-2}{x-2}, & if\ x \neq 2 \\ 5, & if\ x = 2 \end{cases}$ $at\ x = 2.$

At x = 2 $LHL = \lim_{x\to 2^-} \frac{2x^2-3x-2}{x-2}$

$$= \lim_{h\to 0} \frac{2(2-h)^2-3(2-h)-2}{(2-h)-2}$$

$$= \lim_{h\to 0} \frac{8+2h^2-8h-6+3h-2}{-h}$$

$$= \lim_{h\to 0} \frac{2h^2-5h}{-h} = \lim_{h\to 0} \frac{h(2h-5)}{-h} = 5$$

$$RHL = \lim_{x\to 2^+} \frac{2x^2-3x-2}{x-2}$$

$$= \lim_{h\to 0} \frac{2(2+h)^2-3(2+h)-2}{(2+h)-2}$$

$$= \lim_{h\to 0} \frac{8+2h^2+8h-6-3h-2}{h}$$

$$= \lim_{h\to 0} \frac{2h^2+5h}{h} = \lim_{h\to 0} \frac{h(2h+5)}{h} = 5$$

$and\ f(2)=5$

$\therefore LHL = RHL = f(2)$

$So, f(x)\ is\ continuous\ at\ x=2.$

5. $f(x)=\begin{cases} \dfrac{|x-4|}{2(x-4)}, & if\ x\neq 4 \\ 0, & if\ x=4 \end{cases}\ at\ x=4.$

Sol. We have, $f(x)=\begin{cases} \dfrac{|x-4|}{2(x-4)}, & if\ x\neq 4 \\ 0, & if\ x=4 \end{cases}\ at\ x=4.$

At x = 4, $LHL=\lim_{x\to 4^-}\dfrac{|x-4|}{2(x-4)}$

$=\lim_{h\to 0}\dfrac{|4-h-4|}{2[(4-h)-4]}=\lim_{h\to 0}\dfrac{|0-h|}{(8-2h-8)}$

$=\lim_{h\to 0}\dfrac{h}{-2h}=\dfrac{-1}{2}\ and\ f(4)=0\neq LHL$

So, $f(x)$ is discontinuous at x = 4.

6. $f(x)=\begin{cases} |x|\cos\dfrac{1}{x}, & if\ x\neq 0 \\ 0, & if\ x=0 \end{cases}\ at\ x=0.$

Sol. We have, $f(x)=\begin{cases} |x|\cos\dfrac{1}{x}, & if\ x\neq 0 \\ 0, & if\ x=0 \end{cases}\ at\ x=0.$

At x = 0 $LHL=\lim_{x\to 0^-}|x|\cos\dfrac{1}{x}=\lim_{h\to 0}|0-h|\cos\dfrac{1}{0-h}$

$=\lim_{h\to 0}h\cos\left(\dfrac{-1}{h}\right)$

$=0\times[\text{an oscillating number between -1 and 1}]=0$

$RHL=\lim_{x\to 0^+}|x|\cos\dfrac{1}{x}$

$=\lim_{h\to 0}|0+h|\cos\dfrac{1}{(0+h)}$

$=\lim_{h\to 0}h\cos\dfrac{1}{h}$

$=0\times[\text{an oscillating number between -1 and 1}]=0$

$and\ f(0)=0$

Since, $LHL=RHL=f(0)$

$So, f(x)\ is\ continuous\ at\ x=0.$

7. $f(x)=\begin{cases} |x-a|\sin\dfrac{1}{x-a}, & if\ x\neq 0 \\ 0, & if\ x=a \end{cases}$ $at\ x=a.$

Sol. We have, $f(x)=\begin{cases} |x-a|\sin\dfrac{1}{x-a}, & if\ x\neq 0 \\ 0, & if\ x=a \end{cases}$ $at\ x=a.$

At x = a, $LHL=\lim\limits_{x\to a^-}|x-a|\sin\dfrac{1}{x-a}$

$=\lim\limits_{h\to 0}|a-h-a|\sin\left(\dfrac{1}{a-h-a}\right)$

$=\lim\limits_{h\to 0}-h\sin\left(\dfrac{1}{h}\right)$ $[\because \sin(-\theta)=-\sin\theta]$

$=0\times[\text{an oscillating number between -1 and 1}]=0$

$RHL=\lim\limits_{x\to a^+}|x-a|\sin\left(\dfrac{1}{x-a}\right)$

$=\lim\limits_{h\to 0}|a+h-a|\sin\left(\dfrac{1}{a+h-a}\right)=\lim\limits_{h\to 0}h\sin\dfrac{1}{h}$

$=0\times[\text{an osclillating number between-1 and 1}]=0$

$and\ f(a)=0$

$\therefore\ LHL=RHL=f(a)$

$So,\ f(x)\ is\ continuous\ at\ x=a.$

8. $f(x)=\begin{cases} \dfrac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, & if\ x\neq 0 \\ 0, & if\ x=0 \end{cases}$ $at\ x=0.$

Sol. We have, $f(x)=\begin{cases} \dfrac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, & if\ x\neq 0 \\ 0, & if\ x=0 \end{cases}$ $at\ x=0.$

At x = 0, $LHL=\lim\limits_{x\to 0^-}\dfrac{e^{1/x}}{1+e^{1/x}}=\lim\limits_{h\to 0}\dfrac{e^{1/0-h}}{1+e^{1/0-h}}$

$=\lim\limits_{h\to 0}\dfrac{e^{-1/h}}{1+e^{-1/h}}=\lim\limits_{h\to 0}\dfrac{1}{e^{1/h}(1+e^{-1/h})}$

$=\lim\limits_{h\to 0}\dfrac{1}{e^{1/h}+1}=\dfrac{1}{e^{\infty}+1}=\dfrac{1}{\infty+1}$ $[\because e^{\infty}=\infty]$

$=\dfrac{1}{\dfrac{1}{0}}=0$

$$RHL = \lim_{x \to 0^+} \frac{e^{1/x}}{1+e^{1/x}}$$

$$= \lim_{h \to 0} \frac{e^{1/0+h}}{1+e^{1/0+h}} = \lim_{h \to 0} \frac{e^{1/h}}{1+e^{1/h}}$$

$$= \lim_{h \to 0} \frac{1}{e^{-1/h}+1} = \frac{1}{e^{-\infty}+1}$$

$$= \frac{1}{0+1} = 1 \; [\because e^{-\infty} = 0]$$

Hence, $LHL \neq RHL \; at \; x = 0$

So, $f(x)$ is discontinuous at x = 0.

9. $f(x) = \begin{cases} \frac{x^2}{2}, & if \; 0 \le x \le 1 \\ 2x^2 - 3x + \frac{3}{2}, & if \; 1 < x \le 2 \end{cases} \quad at \; x = 1.$

Sol. We have, $f(x) = \begin{cases} \frac{x^2}{2}, & if \; 0 \le x \le 1 \\ 2x^2 - 3x + \frac{3}{2}, & if \; 1 < x \le 2 \end{cases} \quad at \; x = 1.$

At x = 1, $HL = \lim_{x \to 1^-} \frac{x^2}{2} = \lim_{h \to 0} \frac{(1-h)^2}{2}$

$$= \lim_{h \to 0} \frac{1+h^2-2h}{2} = \frac{1}{2}$$

$$RHL = \lim_{x \to 1^+} \left(2x^2 - 3x + \frac{3}{2} \right)$$

$$= \lim_{h \to 0} \left[2(1+h)^2 - 3(1+h) + \frac{3}{2} \right]$$

$$= \lim_{h \to 0} \left(2 + 2h^2 + 4h - 3 - 3h + \frac{3}{2} \right) = -1 + \frac{3}{2} = \frac{1}{2}$$

And $f(1) = \frac{1^2}{2} = \frac{1}{2}$

$\therefore LHL = RHL = f(1)$

Hence, $f(x)$ is continuous at x = 1.

10. $f(x) = |x| + |x-1| \; at \; x = 1$

Sol. We have, $f(x) = |x| + |x-1| \; at \; x = 1$

At x = 1, $LHL = \lim_{x \to 1^-} \left[|x| + |x-1| \right]$

$$= \lim_{h \to 0} \left[|1-h| + |1-h-1| \right] = 1 + 0 = 1$$

And $RHL = \lim_{x \to 1^-} \left[|x| + |x-1| \right]$

$$= \lim_{h \to 0} \left[|1+h| + |1+h-1| \right] = 1 + 0 = 1$$

$and\ f(1)=|1|+|0|=1$

$\therefore LHL=RHL=f(1)$

Hence, $f(x)$ is continuous at x = 1.

Find the value of k in each Exercise 11 to 14 so that the function f is continuous at the indicated point:

11. $f(x)=\begin{cases}3x-8, & if\ x\le 5\\ 2k, & if\ x>5\end{cases}$ $at\ x=5.$

Sol. We have, $f(x)=\begin{cases}3x-8, & if\ x\le 5\\ 2k, & if\ x>5\end{cases}$ $at\ x=5.$

Since, $f(x)$ is continuous at x = 5.

$\therefore LHL=RHL=f(5)$

Now, $LHL=\lim_{x\to 5^-}(3x-8)=\lim_{h\to 0}[3(5-h)-8]$

$=\lim_{h\to 0}[15-3h-8]=7$

$RHL=\lim_{x\to 5^+}2k=\lim_{h\to 0}2k=2k=7\ [\because LHL=RHL]$

And f(5) = 3×5 – 8 = 7

$2k=7\Rightarrow k=\frac{7}{2}$

12. $f(x)=\begin{cases}\frac{2^{x+2}-16}{4^x-16}, & if\ x\ne 2\\ k, & if\ x=2\end{cases}$ $at\ x=2.$

Sol. We have, $f(x)=\begin{cases}\frac{2^{x+2}-16}{4^x-16}, & if\ x\ne 2\\ k, & if\ x=2\end{cases}$ $at\ x=2.$

Since, $f(x)$ is continuous at x = 2.

$\therefore LHL=RHL=f(2)$

At x = 2, $\lim_{x\to 2}\frac{2^x.2^2-2^4}{4^x-4^2}=\lim_{x\to 2}\frac{4.(2^x-4)}{(2^x)^2-(4)^2}$

$=\lim_{x\to 2}\frac{4.(2^x-4)}{(2^x-4)(2^x+4)}\ [\because a^2-b^2=(a+b)(a-b)]$

$=\lim_{x\to 2}\frac{4}{2^x+4}=\frac{4}{8}=\frac{1}{2}$

$But\ f(2)=k$

$\therefore k=\frac{1}{2}$

13. $f(x)=\begin{cases}\frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & if\ -1\le x<0\\ \frac{2x+1}{x-1}, & if\ 0\le x\le 1\end{cases}$ $at\ x=0.$

Sol. We have $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & if\ -1\le x<0\\ \dfrac{2x+1}{x-1}, & if\ 0\le x\le 1\end{cases}$ $at\ x=0.$

$\therefore\ LHL=\lim_{x\to 0^-}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}$

$=\lim_{x\to 0^-}\left(\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}\right).\left(\dfrac{\sqrt{1+kx}+\sqrt{1-kx}}{\sqrt{1+kx}+\sqrt{1-kx}}\right)$

$=\lim_{x\to 0^-}\dfrac{1+kx-1+kx}{x[\sqrt{1+kx}+\sqrt{1-kx}}$

$=\lim_{x\to 0^-}\dfrac{2kx}{x\sqrt{1+kx}+\sqrt{1-kx}}$

$=\lim_{h\to 0}\dfrac{2k}{\sqrt{1+k(0-h)}+\sqrt{1-k(0-h)}}$

$=\lim_{h\to 0}\dfrac{2k}{\sqrt{1-kh}+\sqrt{1+kh}}\quad=\dfrac{2k}{2}=k$

and $f(0)=\dfrac{2\times 0+1}{0-1}=-1$

$\Rightarrow k=-1\,[\because LHL=RHL=f(0)]$

14. $f(x)=\begin{cases}\dfrac{1-\cos kx}{x\sin x}, & if\ x\ne 0\\ \dfrac{1}{2}, & if\ x=0\end{cases}$ $at\ x=0.$

Sol. We have, $f(x)=\begin{cases}\dfrac{1-\cos kx}{x\sin x}, & if\ x\ne 0\\ \dfrac{1}{2}, & if\ x=0\end{cases}$ $at\ x=0.$

At x = 0, $LHL=\lim_{x\to 0^-}\dfrac{1-\cos kx}{x\sin x}=\lim_{h\to 0}\dfrac{1-\cos k(0-h)}{(0-h)\sin(0-h)}$

$=\lim_{h\to 0}\dfrac{1-\cos\ (-kh)}{-h\sin(-h)}$

$=\lim_{h\to 0}\dfrac{1-\cos kh}{h\sin h}$ $[\because \cos(-\theta)=\cos\ \theta, \sin(-\theta)=-\sin\theta]$

$=\lim_{h\to 0}\dfrac{1-1+2\sin^2\dfrac{kh}{2}}{h\sin h}$ $\left[\because \cos\theta=1-2\sin^2\dfrac{\theta}{2}\right]$

$=\lim_{h\to 0}\dfrac{2\sin^2\dfrac{kh}{2}}{h\sin h}$

$$= \lim_{h\to 0} \frac{2\sin\frac{kh}{2}}{\frac{kh}{2}} \cdot \frac{\sin\frac{kh}{2}}{\frac{kh}{2}} \cdot \frac{1}{\frac{\sin h}{h}} \cdot \frac{k^2 h/4}{h}$$

$$= \frac{2k^2}{4} = \frac{k^2}{2} \left[\because \lim_{h\to 0}\frac{\sin h}{h} = 1\right]$$

Also, $f(0) = \frac{1}{2} \Rightarrow \frac{k^2}{2} = \frac{1}{2} \Rightarrow k = \pm 1$

15. Prove that the function f defined by $f(x) = \begin{cases} \frac{x}{|x|+2x^2}, & if\ x \neq 0 \\ k, & if\ x = 0 \end{cases}$ remains discontinuous at x = 0, regardless the choice of k.

Sol. $f(x) = \begin{cases} \frac{x}{|x|+2x^2}, & if\ x \neq 0 \\ k, & if\ x = 0 \end{cases}$

At x = 0, $LHL = \lim_{x\to 0^-} \frac{x}{|x|+2x^2} = \lim_{h\to 0} \frac{(0-h)}{|0-h|+2(0-h)^2}$

$$= \lim_{h\to 0} \frac{-h}{h+2h^2} = \lim_{h\to 0} \frac{-h}{h(1+2h)} = -1$$

$$RHL = \lim_{x\to 0^+} \frac{x}{|x|+2x^2} = \lim_{h\to 0} \frac{0+h}{|0+h|+2(0+h)^2}$$

$$= \lim_{h\to 0} \frac{h}{h+2h^2} = \lim_{h\to 0} \frac{h}{h(1+2h)} = 1$$

And f(0) = k

Since, $LHL \neq RHL$ for any value of k.

Hence, $f(x)$ is discontinuous at x = 0 regardless the choice of k.

16. Find the values of a and b such that the function f defined by

$$f(x) = \begin{cases} \frac{x-4}{|x-4|} + a, & if\ x < 4 \\ a + b, & if\ x = 4 \\ \frac{x-4}{|x-4|} + b, & if\ x > 4 \end{cases}$$

is a continuous function at x = 4.

Sol. We have, $f(x) = \begin{cases} \frac{x-4}{|x-4|} + a, & if\ x < 4 \\ a + b, & if\ x = 4 \\ \frac{x-4}{|x-4|} + b, & if\ x > 4 \end{cases}$

At x = 4, $LHL = \lim_{x \to 4^-} \frac{x-4}{|x-4|} + a$

$= \lim_{h \to 0} \frac{4-h-4}{|4-h-4|} + a = \lim_{h \to 0} \frac{-h}{h} + a$

$= -1 + a$

$RHL = \lim_{x \to 4^+} \frac{x-4}{|x-4|} + b$

$= \lim_{h \to 0} \frac{4+h-4}{|4+h-4|} + b = \lim_{h \to 0} \frac{h}{h} + b = 1 + b$

$f(4) = a + b \Rightarrow -1 + a = 1 + b = a + b$

$\Rightarrow$ -1+a=a+b and 1+b=a+b

$\therefore\ b = -1 \text{ and } a = 1$

17. Given the function $f(x) = \frac{1}{x+2}$. Find the points of discontinuity of the composite function $y = f(f(x))$.

Sol. We have $f(x) = \frac{1}{x+2}$

$\therefore y = f\{f(x)\}$

$= f\left(\frac{1}{x+2}\right) = \frac{1}{\frac{1}{x+2} + 2}$

$= \frac{1}{1+2x+4}.(x+2) = \frac{(x+2)}{(2x+5)}$

So, the function y will not be continuous at those points, where it is not defined as it is a rational function.

Therefore, $y = \frac{(x+2)}{(2x+5)}$ is not defined, when 2x + 5 = 0

$\therefore\ x = \frac{-5}{2}$

Hence, y is discontinuous at $x = \frac{-5}{2}$

18. Find all points of discontinuity of the function $f(t) = \frac{1}{t^2 + t - 2}$, where $t = \frac{1}{x-1}$.

Sol. We have $f(t) = \frac{1}{t^2 + t - 2}$, and $t = \frac{1}{x-1}$

$\therefore f(t) = \frac{1}{\left(\frac{1}{x^2 + 1 - 2x}\right) + \left(\frac{1}{x-1}\right) - \frac{2}{1}}$

$= \frac{1}{\left(\frac{1 + x - 1 + [-2(x-1)^2]}{(x^2 + 1 - 2x)}\right)}$

$$= \frac{x^2+1-2x}{x-2x^2-2+4x}$$

$$= \frac{x^2+1-2x}{-2x^2+5x-2}$$

$$= \frac{(x-1)^2}{-(2x^2-5x+2)}$$

$$= \frac{(x-1)^2}{(2x-1)(2-x)}$$

So, $f(t)$ is discontinuous at $2x-1=0 \Rightarrow x=1/2$.

And $2-x=0 \Rightarrow x=2$.

19. **Show that the function $f(x)=|\sin x+\cos x|$ is continuous at $x=\pi$.**

Sol. We have $f(x)=|\sin x+\cos x| \; at\, x=\pi$.

Let $g(x)=sinx+cosx$

And $h(x)=|x|$

$\therefore hog(x)=h[g(x)]$

$= h(sin\,x+cos\,x)$

$=|sin\,x+cos\,x|$

Since, $g(x)=sin\ x+cos\ x$ is a continuous function as it is forming with addition of two continuous function sin x and cos x.

Also, $h(x)=|x|$ is also a continuous function. Since, we know that composite functions of two continuous functions is also a continuous function.

Hence, $f(x)=|sin\ x+cos\ x|$ is a continuous function everywhere.

So, $f(x)$ is continuous at $x=\pi$

20. **Examine the differentiability of f, where f is defined by**

$$f(x)=\begin{cases} x[x], & if\ 0\le x<2 \\ (x-1)x, & if\ 2\le x<3 \end{cases} at\ x=2.$$

Sol. We have $f(x)=\begin{cases} x[x], & if\ 0\le x<2 \\ (x-1)x, & if\ 2\le x<3 \end{cases} at\ x=2.$

At x = 2, $Lf'(2)=\lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$

$$=\lim_{h\to 0}\frac{(2-h)[2-h]-(2-1)2}{-h}$$

$\{\because [a-h]=[a-1]$, where a is any positive number$\}$

$$=\lim_{h\to 0}\frac{(2-h)(1)-2}{-h}$$

$$=\lim_{h\to 0}\frac{2-h-2}{-h}=\lim_{h\to 0}\frac{-h}{-h}=1$$

$$Rf'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$
$$=\lim_{h\to 0}\frac{(2+h-1)(2+h)-(2-1).2}{h}$$
$$=\lim_{h\to 0}\frac{(1+h)(2+h)-2}{h}$$
$$=\lim_{h\to 0}\frac{2+h+2h+h^2-2}{h}$$
$$=\lim_{h\to 0}\frac{h^2+3h}{h}=\lim_{h\to 0}\frac{h(h+3)}{h}=3$$
$$\therefore Lf'(2)\neq Rf'(2)$$
So, $f(x)$ nor differentiable at x = 2.

21. $f(x)=\begin{cases} x^2\sin\frac{1}{x}, & if\ \ x\neq 0 \\ 0, & if\ \ x=0 \end{cases}$ $at\ x=0.$

Sol. We have, $f(x)=\begin{cases} x^2\sin\frac{1}{x}, & if\ \ x\neq 0 \\ 0, & if\ \ x=0 \end{cases}$ $at\ x=0.$

For differentiability at x = 0,
$$Lf'(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{x^2\sin\frac{1}{x}-0}{x-0}$$
$$=\lim_{h\to 0}\frac{(0-h)^2\sin\left(\frac{1}{0-h}\right)}{0-h}=\lim_{h\to 0}\frac{h^2\sin\left(\frac{-1}{h}\right)}{-h}$$
$$=\lim_{h\to 0}+h\sin\left(\frac{1}{h}\right)\ \ [\because \sin(-\theta)=-\sin\theta]$$
$$=0\times[\text{an oscillating number between -1 and 1}]=0$$
$$Rf'(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{x^2\sin\frac{1}{x}-0}{x-0}$$
$$=\lim_{h\to 0}\frac{(0+h)^2\sin\left(\frac{1}{0+h}\right)}{0+h}=\lim_{h\to 0}\frac{h^2\sin(1/h)}{h}$$
$$=\lim_{h\to 0}h\sin(1/h)$$
= 0 $\times$ [an oscillating number between -1 and 1] = 0
$$\because Lf'(0)=Rf'(0)$$
So, $f(x)$ is differentiable at x = 0.

22. $f(x)\begin{cases} 1+x, & if\ x\leq 2 \\ 5-x, & if\ x>2 \end{cases}$ $at\ x=2.$

Sol. We have, $f(x)\begin{cases}1+x, & if\ x\le 2\\ 5-x, & if\ x>2\end{cases} at\ x=2.$

For differentiability at x = 2.

$Lf'(2)=\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^-}\frac{(1+x)-(1+2)}{x-2}$

$=\lim_{h\to 0}\frac{(1+2-h)-3}{2-h-2}=\lim_{h\to 0}\frac{-h}{-h}=1$

$Rf'(2)=\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^+}\frac{(5-x)-3}{x-2}$

$=\lim_{h\to 0}\frac{5-(2+h)-3}{2+h-2}$

$=\lim_{h\to 0}\frac{5-2-h-3}{h}=\lim_{h\to 0}\frac{-h}{+h}$

$=-1$

$\because Lf'(2)\ne Rf'(2)$

So, $f(x)$ is not differentiable at x = 2.

23. Show that $f(x)=|x-5|$ is continuous but not differentiable at x = 5.

Sol. We have $f(x)=|x-5|$

$\therefore f(x)=\begin{cases}-(x-5), if\ x<5\\ x-5, \quad if\ x\ge 5\end{cases}$

For continuity at x = 5,

$LHL=\lim_{x\to 5^-}(-x+5)$

$=\lim_{h\to 0}[-(5-h)+5]=\lim_{h\to 0}h=0$

$RHL=\lim_{x\to 5^+}(x-5)$

$=\lim_{h\to 0}(5+h-5)=\lim_{h\to 0}h=0$

$\therefore f(5)=5-5=0$

$\Rightarrow LHL=RHL=f(5)$

Hence, $f(x)$ is continuous at x = 5.

Now, $Lf'(5)=\lim_{x\to 5^-}\frac{f(x)-f(5)}{x-5}$

$=\lim_{x\to 5^-}\frac{-x+5-0}{x-5}=-1$

$Rf'(5)=\lim_{x\to 5^+}\frac{f(x)-f(5)}{x-5}$

$=\lim_{x\to 5^+}\frac{x-5-0}{x-5}=1$

$\therefore Lf'(5)\ne Rf'(5)$

So, $f(x)=|x-5|$ is not differentiable at x = 5.

24. A function $f: R \rightarrow R$ satisfies the equation $f(x+y) = f(x).f(y)$ for all $x,\ y \in R, f(x) \neq 0.$ Suppose that the function is differentiable at $x = 0$ *and* $f'(0) = 2,$ then prove that $f'(x) = 2f(x).$

Sol. Let $f: R \rightarrow R$ satisfies the equation $f(x+y) = f(x).f(y),\ \forall x,\ y \in R, f(x) \neq 0.$

Let $f(x)$ is differentiable at $x = 0$ *and* $f'(0) = 2.$

$$\Rightarrow f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

$$\Rightarrow 2 = \lim_{x\to 0} \frac{f(x) - f(0)}{x}$$

$$\Rightarrow 2 = \lim_{h\to 0} \frac{f(0+h) - f(0)}{0+h}$$

$$\Rightarrow 2 = \lim_{h\to 0} \frac{f(0).f(h) - f(0)}{h}$$

$$\Rightarrow 2 = \lim_{h\to 0} \frac{f(0)[f(h) - 1]}{h} \quad [\because f(0) = f(h)]...(i)]$$

Also, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x).f(h) - f(x)}{h} \quad [\because f(x+y) = f(x).f(y)]$$

$$= \lim_{h\to 0} \frac{f(x)[f(h) - 1]}{h} = 2f(x) \text{ [using } Eq.(i)]$$

$\therefore f'(x) = 2f(x)$

Differentiate each of the following w.r.t. x (Exercises 25 to 43):

25. $2^{\cos^2 x}$

Sol. Let y = $2^{\cos^2 x}$

$\therefore \log y = \log 2^{\cos^2 x} = \cos^2 x.\log 2$

On differentiating w.r.t. x, we get

$$\frac{d}{dy}\log y.\frac{dy}{dx} = \frac{d}{dx}\log 2.\cos^2 x$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \log 2\frac{d}{dx}(\cos x)^2$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \log 2.[2\cos x].\frac{d}{dx}\cos x$$

$$= \log 2.2\cos x.(-\sin x)$$

$$= \log 2.[-(\sin 2x)]$$

$$\therefore \frac{dy}{dx} = -y.\log 2(\sin 2x)$$

$$= -2^{\cos^2 x}.\log 2(\sin 2x)$$

26. $\dfrac{8^x}{x^8}$

Sol. Let $y=\dfrac{8^x}{x^8} \Rightarrow \log y=\log\dfrac{8^x}{x^8}$

$\Rightarrow \dfrac{d}{dy}\log y.\dfrac{dy}{dx}=\dfrac{d}{dx}[\log 8^x-\log x^8]$

$\Rightarrow \dfrac{1}{y}.\dfrac{dy}{dx}=[x.\log 8-8.\log x]$

On differentiating w.r.t. x, we get

$\dfrac{1}{y}.\dfrac{dy}{dx}=\log 8.1-8.\dfrac{1}{x}$

$\Rightarrow \dfrac{1}{y}.\dfrac{dy}{dx}=\log 8-\dfrac{8}{x}$

$\therefore \dfrac{dy}{dx}=y\left(\log 8-\dfrac{8}{x}\right)=\dfrac{8^x}{x^8}\left(\log 8-\dfrac{8}{x}\right)$

27. $\log\left(x+\sqrt{x^2+a}\right)$

Sol. Let y = $\log\left(x+\sqrt{x^2+a}\right)$

$\therefore \dfrac{dy}{dx}=\dfrac{d}{dx}\log(x+\sqrt{x^2+a})$

$=\dfrac{1}{(x+\sqrt{x^2+a})}.\dfrac{d}{dx}[x+\sqrt{x^2+a}]$

$=\dfrac{1}{(x+\sqrt{x^2+a})}\left[1+\dfrac{1}{2}(x^2+a)^{-1/2}.2x\right]$

$=\dfrac{1}{(x+\sqrt{x^2+a})}.\left(1+\dfrac{x}{\sqrt{x^2+a}}\right)$

$=\dfrac{(\sqrt{x^2+a}+x)}{(x+\sqrt{x^2+a})(\sqrt{x^2+a})}=\dfrac{1}{(\sqrt{x^2+a})}$

28. $\log\left[\log\left(\log x^5\right)\right]$

Sol. Let y = $y=\log\left[\log\left(\log x^5\right)\right]$

$\therefore \dfrac{dy}{dx}=\dfrac{d}{dx}[\log(\log\log x^5)]$

$=\dfrac{1}{\log\ \log x^5}.\dfrac{d}{dx}(\log.\log x^5)$

$=\dfrac{1}{\log\ \log x^5}.\left(\dfrac{1}{\log x^5}\right).\dfrac{d}{dx}\log x^5$

$$=\frac{1}{\log\log x^5}\cdot\frac{1}{\log x^5}\cdot\frac{d}{dx}(5\log x)=\frac{5}{x.\log(\log x^5).\log(x^5)}$$

29. $\sin\sqrt{x}+\cos^2\sqrt{x}$

Sol. Let $y=\sin\sqrt{x}+\left(\cos\sqrt{x}\right)^2$

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}\sin(x^{1/2})+\frac{d}{dx}[\cos(x^{1/2})]^2$$

$$=\cos x^{1/2}\frac{d}{dx}x^{1/2}+2\cos(x^{1/2})\frac{d}{dx}[\cos(x^{1/2})]$$

$$=\cos(x^{1/2})\frac{1}{2}x^{-1/2}+2.\cos(x^{1/2}).\left[-\sin(x^{1/2}).\frac{d}{dx}x^{1/2}\right]$$

$$=\cos\sqrt{x}.\frac{1}{2\sqrt{x}}[-2\cos(x^{1/2})].\sin x^{1/2}.\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}[\cos(\sqrt{x})-\sin(2\sqrt{x})]$$

30. $\sin^n(ax^2+bx+c)$

Sol. Let $y=\sin^n(ax^2+bx+c)$

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}[\sin(ax^2+bx+c)]^n$$

$$=n.[\sin(ax^2+bx+c)]^{n-1}.\frac{d}{dx}\sin(ax^2+bx+c)$$

$$=n.\sin^{n-1}(ax^2+bx+c).\cos(ax^2+bx+c).\frac{d}{dx}(ax^2+bx+c)$$

$$=n.\sin^{n-1}(ax^2+bx+c).\cos(ax^2+bx+c).(2ax+b)$$

$$=n.(2ax+b).\sin^{n-1}(ax^2+bx+c).\cos(ax^2+bx+c)$$

31. $\cos\left(\tan\sqrt{x+1}\right)$

Sol. Let $y=\cos\left(\tan\sqrt{x+1}\right)$

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}\cos\left(\tan\sqrt{x+1}\right)=-\sin(\tan\sqrt{x+1}).\frac{d}{dx}(\tan\sqrt{x+1})$$

$$=-\sin(\tan\sqrt{x+1}).\sec^2\sqrt{x+1}.\frac{d}{dx}(x+1)^{1/2}\quad\left[\because\frac{d}{dx}(\tan x)=\sec^2 x\right]$$

$$=-\sin(\tan\sqrt{x+1}).(\sec\sqrt{x+1})^2.\frac{1}{2}(x+1)^{-1/2}.\frac{d}{dx}(x+1)$$

$$=\frac{-1}{2\sqrt{x+1}}.\sin(\tan\sqrt{x+1}).\sec^2(\sqrt{x+1})$$

32. $\sin x^2+\sin^2 x+\sin^2(x^2)$

Sol. Let $y=\sin x^2+\sin^2 x+\sin^2(x^2)$

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}\sin(x^2)+\frac{d}{dx}(\sin x)^2+\frac{d}{dx}(\sin x^2)^2$$

$$= \cos(x^2)\frac{d}{dx}(x^2) + 2\sin x.\frac{d}{dx}\sin x + 2\sin x^2.\frac{d}{dx}\sin x^2$$

$$= \cos x^2 2x + 2.\sin x.\cos x + 2\sin x^2 \cos x^2.\frac{d}{dx}x^2$$

$$= 2x\cos(x)^2 + 2.\sin x.\cos x + 2\sin x^2.\cos x^2.2x$$

$$= 2x\cos(x)^2 + \sin 2x + \sin 2(x)^2.2x$$

$$= 2x\cos(x)^2 + 2x.\sin 2(x^2) + \sin 2x$$

33. $\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$

Sol. Let $y = \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$$

$$= \frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{x+1}}\right)^2}}.\frac{d}{dx}\frac{1}{(x+1)^{1/2}} \quad \left[\because \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}\right]$$

$$= \frac{1}{\sqrt{\frac{x+1-1}{x+1}}}.\frac{d}{dx}.(x+1)^{-1/2}$$

$$= \sqrt{\frac{x+1}{x}}.\frac{-1}{2}(x+1)^{-\frac{1}{2}-1}.\frac{d}{dx}(x+1)$$

$$= \frac{(x+1)^{1/2}}{x^{1/2}}.\left(-\frac{1}{2}\right)(x+1)^{-3/2} = \frac{-1}{2\sqrt{x}}.\left(\frac{1}{x+1}\right)$$

34. $(\sin x)^{\cos x}$

Sol. Let $y = (\sin x)^{\cos x}$

$$\Rightarrow \log y = \log(\sin x)^{\cos x} = \cos x \log \sin x$$

$$\therefore \frac{d}{dy}\log y.\frac{dy}{dx} = \frac{d}{dx}(\cos x.\log \sin x)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \cos x.\frac{d}{dx}\log \sin x + \log \sin x.\frac{d}{dx}\cos x$$

$$= \cos x.\frac{1}{\sin x}.\frac{d}{dx}\sin x + \log \sin x.(-\sin x)$$

$$= \cot x.\cos x - \log(\sin x).\sin x \quad \left[\because \cot x = \frac{\cos x}{\sin x}\right]$$

$$\therefore \frac{dy}{dx} = y\left[\frac{\cos^2 x}{\sin x} - \sin x.\log(\sin x)\right]$$

$$= \sin x^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x.\log(\sin x)\right]$$

35. $\sin^m x.\cos^n x$

Sol. Let $y = \sin^m x.\cos^n x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}[(\sin x)^m.(\cos x)^n]$$

$$= (\sin x)^m.\frac{d}{dx}(\cos x)^n + (\cos x)^n.\frac{d}{dx}(\sin x)^m$$

$$= (\sin x)^m.n(\cos x)^{n-1}.\frac{d}{dx}\cos x + (\cos x)^n m(\sin x)^{m-1}.\frac{d}{dx}\sin x$$

$$= (\sin x)^m.n(\cos x)^{n-1}(-\sin x) + (\cos x)^n.m(\sin x)^{m-1}\cos x$$

$$= -n\sin^m x.\cos^{n-1} x.(\sin x) + m\cos^n x.\sin^{m-1} x.\cos x$$

$$= -n.\sin^m x.\sin x.\cos^n x.\frac{1}{\cos x} + m.\sin^m x.\frac{1}{\sin x}.\cos^n x.\cos x$$

$$= -n.\sin^m x.\cos^n x.\tan x + m\sin^m x.\cos^n x.\cot x$$

$$= \sin^m x.\cos^n x[-n\tan x + m\cot x]$$

36. $(x+1)^2(x+2)^3(x+3)^4$

Sol. Let $y = (x+1)^2(x+2)^3(x+3)^4$

$$\therefore \log y = \log\left\{(x+1)^2.(x+2)^3(x+3)^4\right\}$$

$$= \log(x+1)^2 + \log(x+2)^3 + \log(x+3)^4$$

and $\dfrac{d}{dy}\log y.\dfrac{dy}{dx} = \dfrac{d}{dx}[2\log(x+1)] + \dfrac{d}{dx}[3\log(x+2)] + \dfrac{d}{dx}[4\log(x+3)]$

$$\frac{1}{y}.\frac{dy}{dx} = \frac{2}{(x+1)}.\frac{d}{dx}(x+1) + 3.\frac{1}{(x+2)}.\frac{d}{dx}(x+2)$$

$$+4.\frac{1}{(x+3)}.\frac{d}{dx}(x+3)\ \left[\because \frac{d}{dx}(\log x) = \frac{1}{x}\right]$$

$$= \left[\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3}\right]$$

$$\therefore \frac{dy}{dx} = y\left[\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3}\right]$$

$$= (x+1)^2.(x+2)^3.(x+3)^4\left[\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3}\right]$$

$$= (x+1)^2.(x+2)^3.(x+3)^4$$

$$\left[\frac{2(x+2)(x+3) + 3(x+1)(x+3) + 4(x+1)(x+2)}{(x+1)(x+2)(x+3)}\right]$$

$$=\frac{(x+1)^2(x+2)^3(x+3)^4}{(x+1)(x+2)(x+3)}$$

$$[2(x^2+5x+6)+3(x^2+4x+3)+4(x^2+3x+2)]$$

$$=(x+1)(x+2)^2(x+3)^3$$

$$[2x^2+10x+12+3x^2+12x+9+4x^2+12x+8]$$

$$=(x+1)(x+2)^2(x+3)^3[9x^2+34x+29]$$

37. $\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right), \frac{-\pi}{4}<x<\frac{\pi}{4}$

Sol. Let $y=\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$$

$$=\frac{-1}{\sqrt{1-\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)^2}}.\frac{d}{dx}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$$

$$\left[\because \frac{d}{dx}(\cos x)=-\frac{1}{\sqrt{1-x^2}}\right]$$

$$=\frac{-1}{\sqrt{4-\frac{(\sin^2 x+\cos^2 x+2\sin x.\cos x)}{2}}}.\frac{1}{\sqrt{2}}(\cos x-\sin x)$$

$$=\frac{-1.\sqrt{2}}{\sqrt{1-\sin 2x}}.\frac{1}{\sqrt{2}}(\cos x-\sin x)$$

$$[\because 1-\sin 2x=(\cos x-\sin x)^2=\cos^2 x+\sin^2 x-2\sin x\cos x]$$

$$=\frac{-1(\cos x-\sin x)}{(\cos x-\sin x)}=-1$$

38. $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), -\frac{\pi}{4}<x<\frac{\pi}{4}$

Sol. Let $y=\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$$

$$=\frac{1}{1+\sqrt{\left(\frac{1-\cos x}{1+\cos x}\right)^2}}.\frac{d}{dx}\left[\frac{1-\cos x}{1+\cos x}\right]^{1/2} \quad \left[\because \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}\right]$$

$$=\frac{1}{1+\frac{1-\cos x}{1+\cos x}}\cdot\frac{1}{2}\left[\frac{1-\cos x}{1+\cos x}\right]^{1/2}\cdot\frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$=\frac{1}{\frac{1+\cos x+1-\cos x}{1+\cos x}}\cdot\frac{1}{2}\left[\frac{(1-\cos x)}{(1+\cos x)}\cdot\frac{(1-\cos x)}{(1+\cos x)}\right]^{-1/2}$$

$$\cdot\frac{(1+\cos x).\sin x+(1-\cos x).\sin x}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)}{2}\cdot\frac{1}{2}\left[\frac{(1-\cos x)^2}{(1-\cos^2 x)}\right]^{-1/2}\left[\frac{\sin x(1+\cos x+1-\cos x)}{(1+\cos x)^2}\right]$$

$$=\frac{(1+\cos x)}{2}\cdot\frac{1}{2}\left[\frac{(1-\cos x)^2}{(1-\cos^2 x)}\right]^{-1/2}\left[\frac{\sin x(1+\cos x+1-\cos x)}{(1+\cos x)^2}\right]$$

$$=\frac{(1+\cos x)}{2}\cdot\frac{1}{2}\left[\frac{(1-\cos x)^2}{\sin x}\right]^{-1/2}\frac{2\sin x}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)}{2}\cdot\frac{1}{2}\cdot\frac{\sin x}{(1-\cos x)}\cdot\frac{2\sin x}{(1+\cos x)^2}$$

$$=\frac{2\sin^2 x}{4(1+\cos x)(1-\cos x)}=\frac{1}{2}\cdot\frac{\sin^2 x}{(1-\cos^2 x)}$$

$$=\frac{1}{2}\cdot\frac{\sin^2 x}{\sin^2 x}=\frac{1}{2}$$

Alternate Method

Let $y=\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

$$=\tan^{-1}\left(\sqrt{\frac{1-1+2\sin^2\frac{x}{2}}{1+2\cos^2\frac{x}{2}-1}}\right)\quad\left[\because \cos=1-2\sin^2\frac{x}{2}=2\cos^2\frac{x}{2}-1\right]$$

$$=\tan^{-1}\left(\tan\frac{x}{2}\right)=\frac{x}{2}$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{2}$$

39. $\tan^{-1}(\sec x+\tan x), -\frac{\pi}{2}<x<\frac{\pi}{2}$

Sol. Let $y=\tan^{-1}(\sec x+\tan x)$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\tan^{-1}(\sec x+\tan x)$$

$$=\frac{1}{1+(\sec x+\tan x)^2}.\frac{d}{dx}(\sec x+\tan x)\ \left[\because \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}\right]$$

$$=\frac{1}{1+\sec^2 x+\tan^2 x+2\sec x\ .\ \tan x}.[\sec x.\tan x+\sec^2 x]$$

$$=\frac{1}{(\sec^2 x+\sec^2 x+2\sec x.\tan x)}.\sec x.(\sec x+\tan x)$$

$$=\frac{1}{2\sec x(\tan x+\sec x)}.\sec x(\sec x+\tan x)=\frac{1}{2}$$

40. $\tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right), \frac{-\pi}{2}<x<\frac{\pi}{2}\ and\ \frac{a}{b}\tan x>-1$

Sol. Let y = $\tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right)$

$$=\tan^{-1}\left[\frac{\frac{a\cos x}{b\cos x}-\frac{b\sin x}{b\cos x}}{\frac{b\cos x}{b\cos x}+\frac{a\sin x}{b\cos x}}\right]=\tan^{-1}\left[\frac{\frac{a}{b}-\tan x}{1+\frac{a}{b}\tan x}\right]$$

$$=\tan^{-1}\frac{a}{b}-\tan^{-1}\tan x\ \left[\because \tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$=\tan^{-1}\frac{a}{b}-x$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\tan^{-1}\frac{a}{b}\right)-\frac{d}{dx}(x)$$

$$=0-1\ \left[\because \frac{d}{dx}\left(\frac{a}{b}\right)=0\right]$$

$$=-1$$

41. $\sec^{-1}\left(\frac{1}{4x^3-3x}\right), 0<x<\frac{1}{\sqrt{2}}$

Sol. Let $y=\sec^{-1}\left(\frac{1}{4x^3-3x}\right)$...$(i)$

On putting x = $\cos\theta$ in Eq. (i), we get

$$y=\sec^{-1}\frac{1}{4\cos^3\theta-3\cos\theta}$$

$$=\sec^{-1}\frac{1}{\cos 3\theta}$$

$$=\sec^{-1}(\sec 3\theta)=3\theta$$

$$=3\cos^{-1}x\ [\because \theta=\cos^{-1}x]$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(3\cos^{-1}x)$$

$$=3.\frac{-1}{\sqrt{1-x^2}}$$

42. $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right), \frac{-1}{\sqrt{3}}<\frac{x}{a}<\frac{1}{\sqrt{3}}$

Sol. Let y = $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$

Put $x=a\tan\theta \Rightarrow \theta=\tan^{-1}\frac{x}{a}$

$$\therefore\ y=\tan^{-1}\left[\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right]\left[\because \tan 3\theta=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right]$$

$$=\tan^{-1}(\tan 3\theta)=3\theta$$

$$=3\tan^{-1}\frac{x}{a}\left[\because \theta=\tan^{-1}\frac{x}{a}\right]$$

$$\therefore\ \frac{dy}{dx}=3.\frac{d}{dx}\tan^{-1}\frac{x}{a}=3.\left[\frac{1}{1+\frac{x^2}{a^2}}\right].\frac{d}{dx}.\left(\frac{x}{a}\right)$$

$$=3.\frac{a^2}{a^2+x^2}.=\frac{1}{a}=\frac{3a}{a^2+x^2}$$

43. $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), -1<x<1, x\neq 0$

Sol. Let $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

Put $x^2=\cos 2\theta$

$$\therefore\ y=\tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{1+2\cos^2\theta-1}+\sqrt{1-1+2\sin^2\theta}}{\sqrt{1+2\cos^2\theta-1}-\sqrt{1-1+2\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right)=\tan^{-1}\left[\frac{\sqrt{2}(\cos\theta+\sin\theta)}{\sqrt{2}(\cos\theta-\sin\theta)}\right]$$

$$=\tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)=\tan^{-1}\left(\frac{\frac{\cos\theta+\sin\theta}{\cos\theta}}{\frac{\cos\theta-\sin\theta}{\cos\theta}}\right)$$

$$=\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}+\theta\right) \left[\because \tan(a+b)=\frac{\tan a+\tan b}{1-\tan a.\tan b}\right]$$

$$=\frac{\pi}{4}+\theta=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2 \left[\because 2\theta=\cos^{-1}x^2 \Rightarrow \theta=\frac{1}{2}\cos^{-1}x^2\right]$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\frac{\pi}{4}\right)+\frac{d}{dx}\left(\frac{1}{2}\cos^{-1}x^2\right)$$

$$=0+\frac{1}{2}.\frac{-1}{\sqrt{1-x^4}}.\frac{d}{dx}x^2=\frac{1}{2}.\frac{-2x}{\sqrt{1-x^4}}=\frac{-x}{\sqrt{1-x^4}}$$

Find $\frac{dy}{dx}$ of each of the functions expressed in parametric form in Exercises from 44 to 48.

44. $x=t+\frac{1}{t},\ y=t-\frac{1}{t}$

Sol. $\because\ x=t+\frac{1}{t}$ and $y=t-\frac{1}{t}$

$$\therefore \frac{dx}{dt}=\frac{d}{dt}\left(t+\frac{1}{t}\right) and \frac{dy}{dt}=\frac{d}{dt}\left(t-\frac{1}{t}\right)$$

$$\Rightarrow \frac{dx}{dt}=1+(-1)t^{-2}\ and\ \frac{dy}{dt}=1-(-1)t^{-2}$$

$$\Rightarrow \frac{dx}{dt}=1-\frac{1}{t^2}\ and\ \frac{dy}{dt}=1+\frac{1}{t^2}$$

$$\Rightarrow \frac{dx}{dt}=\frac{t^2-1}{t^2} and \frac{dy}{dt}=\frac{t^2+1}{t^2}$$

$$\therefore \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{t^2+1/t^2}{t^2-1/t^2}=\frac{t^2+1}{t^2-1}$$

45. $x=e^{\theta}\left(\theta+\frac{1}{\theta}\right), y=e^{-\theta}\left(\theta-\frac{1}{\theta}\right)$

Sol. $\therefore x=e^{\theta}\left(\theta+\frac{1}{\theta}\right) and\ y=e^{-\theta}\left(\theta-\frac{1}{\theta}\right)$

$$\therefore\ \frac{dx}{d\theta}=\frac{d}{d\theta}\left[e^{\theta}.\left(\theta+\frac{1}{\theta}\right)\right]$$

$$=e^{\theta}.\frac{d}{d\theta}\left(\theta+\frac{1}{\theta}\right)+\left(\theta+\frac{1}{\theta}\right).\frac{d}{d\theta}e^{\theta}$$

$$=e^{\theta}\left(1-\frac{1}{\theta^2}\right)+\left(\theta+\frac{1}{\theta}\right)e^{\theta}$$

$$=e^{\theta}\left(1-\frac{1}{\theta^2}+\theta+\frac{1}{\theta}\right)$$

$$=e^{\theta}\left(\frac{\theta^2-1+\theta^3+\theta}{\theta^2}\right)\ ...(i)$$

and $\frac{dy}{d\theta}=\frac{d}{d\theta}\left[e^{-\theta}.\left(\theta-\frac{1}{\theta}\right)\right]$

$=e^{-\theta}.\frac{d}{d\theta}\left(\theta-\frac{1}{\theta}\right)+\frac{d}{d\theta}e^{-\theta}\left(\theta-\frac{1}{\theta}\right)$

$=e^{-\theta}\left(1+\frac{1}{\theta^2}\right)+\left(\theta-\frac{1}{\theta}\right)e^{-\theta}.\frac{d}{d\theta}(-\theta)$

$=e^{-\theta}\left[\frac{\theta^2+1}{\theta^2}-\frac{\theta^2-1}{\theta}\right]=e^{-\theta}\left[\frac{\theta^2+1-\theta^3+\theta}{\theta}\right]$ ...(*ii*)

$\therefore\ \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{e^{-\theta}\left(\frac{\theta^2+1-\theta^3+\theta}{\theta^2}\right)}{e^{\theta}\left(\frac{\theta^2-1+\theta^3+\theta}{\theta^2}\right)}$

$=e^{-2\theta}\left(\frac{-\theta^3+\theta^2+\theta+1}{\theta^3+\theta^2+\theta-1}\right)$

46. $x=3\cos\theta-2\cos^3\theta, y=3\sin\theta-2\sin^3\theta.$

Sol. $\because\ x=3\cos\theta-2\cos^3\theta$ and $y=3\sin\theta-2\sin^3\theta$

$\therefore\ \frac{dx}{d\theta}=\frac{d}{d\theta}(3\cos\theta)-\frac{d}{d\theta}(2\cos^3\theta)$

$=3.(-\sin\theta)-2.3\cos^2\theta.\frac{d}{d\theta}.\cos\theta$

$=-3\sin\theta+6\cos^2\theta\sin\theta$

and $\frac{dy}{d\theta}=3\cos A-2.3\sin^2\theta.\frac{d}{d\theta}.\sin\theta$

$=3\cos\theta-6\sin^2\theta.\cos\theta$

Now, $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{3\cos\theta-6\sin^2\theta\cos\theta}{-3\sin\theta+6\cos^2\theta\sin\theta}$

$=\frac{3\cos\theta(1-2\sin^2\theta)}{3\sin\theta(-1+2\cos^2\theta)}=\cot\theta.\frac{\cos 2\theta}{\cos 2\theta}=\cot\theta\ 0$

47. $\sin x=\frac{2t}{1+t^2}$, $\tan y=\frac{2t}{1-t^2}$.

Sol. $\because\ \sin x=\frac{2t}{1+t^2}$ ...(*i*)

And $\tan y=\frac{2t}{1-t^2}$ ...(*ii*)

$\therefore\ \frac{d}{dx}\sin x.\frac{dx}{dt}=\frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$

$\Rightarrow\ \cos x\frac{dx}{dt}=\frac{(1+t^2).\frac{d}{dt}(2t)-(2t).\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$

$$=\frac{2(1+t^2)-2t.2t}{(1+t^2)^2}=\frac{2+2t^2-4t^2}{(1+t^2)^2}$$

$$\Rightarrow \frac{dx}{dt}=\frac{2(1-t^2)}{(1+t^2)^2}.\frac{1}{\cos x}$$

$$\Rightarrow \frac{dx}{dt}=\frac{2(1-t^2)}{(1+t^2)^2}.\frac{1}{\sqrt{1-\sin^2 x}}=\frac{2(1-t^2)}{(1+t^2)^2}.\frac{1}{\sqrt{1-\left(\frac{2t}{1+t^2}\right)^2}}$$

$$\frac{dx}{dt}=\frac{2(1-t^2)}{(1+t^2)^2}.\frac{(1+t^2)}{(1-t^2)}=\frac{2}{1+t^2}\quad ...(iii)$$

Also, $\frac{d}{dy}\tan y.\frac{dy}{dt}=\frac{d}{dt}\left(\frac{2t}{1-t^2}\right)$

$$\sec^2 y\frac{dy}{dt}=\frac{(1-t^2)\frac{d}{dt}.(2t)-2t.\frac{d}{dt}(1-t^2)}{(1-t^2)^2}$$

$$\frac{dy}{dt}=\frac{2-2t^2+4t^2}{(1-t^2)^2}.\frac{1}{\sec^2 y}$$

$$=\frac{2(1+t^2)}{(1-t^2)^2}.\frac{1}{(1+\tan^2 y)}=\frac{2(1+t^2)}{(1-t^2)^2}.\frac{1}{1+\frac{4t^2}{(1-t^2)^2}}$$

$$=\frac{2(1+t^2)}{(1-t^2)^2}.\frac{(1-t^2)^2}{(1+t^2)^2}=\frac{2}{1+t^2}\quad ...(iv)$$

$$\therefore \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2/1+t^2}{2/1+t^2}=1\ [from\,Eqs.(iii)\,and\,(iv)]$$

48. $x=\frac{1+\log t}{t^2},\ y=\frac{3+2\log t}{t}.$

Sol. $\because x=\frac{1+\log t}{t^2}$ and $y=\frac{3+2\log t}{t}$

$$\therefore \frac{dx}{dt}=\frac{t^2.\frac{d}{dt}(1+\log t)-(1+\log t).\frac{d}{dt}t^2}{(t^2)^2}$$

$$=\frac{t^2.\frac{1}{t}-(1+\log t).2t}{t^4}=\frac{t-(1+\log t).2t}{t^4}$$

$$=\frac{t}{t^4}[1-2(1+\log t)=\frac{-1-2\log t}{t^3}\quad ...(i)$$

and $\frac{dy}{dt}=\frac{t.\frac{d}{dt}(3+2\log t)-(3+2\log t).\frac{d}{dt}t}{t^2}$

$$= \frac{t.2.\frac{1}{t} - (3 + 2\log t).1}{t^2}$$

$$= \frac{2 - 3 - 2\log t}{t^2} = \frac{-1 - 2\log t}{t^2} \quad ...(ii)$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1 - 2\log t / t^2}{-1 - 2\log t / t^3} = t$$

49. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, then prove that $\frac{dy}{dx} = -\frac{y \log x}{x \log y}$

Sol. $\because x = e^{\cos 2t}$ and $y = e^{\sin 2t}$

$$\therefore \frac{dx}{dt} = \frac{d}{dt} e^{\cos 2t} = e^{\cos 2t} . \frac{d}{dt} \cos 2t$$

$$= e^{\cos 2t} . (-\sin 2t) . \frac{d}{dt}(2t)$$

$$\frac{dx}{dt} = -2e^{\cos 2t} .\sin 2t \quad ...(i)$$

and $\frac{dy}{dt} = \frac{d}{dt} e^{\sin 2t} = e^{\sin 2t} . \frac{d}{dt} \sin 2t$

$$= e^{\sin 2t} \cos 2t . \frac{d}{dt} 2t$$

$$= 2e^{\sin 2t} . \cos 2t \quad ...(ii)$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{\sin 2t}.\cos 2t}{-2e^{\cos 2t}.\sin 2t}$$

$$= \frac{e^{\sin 2t}.\cos 2t}{e^{\cos 2t}.\sin 2t} \quad ...(iii)$$

We know that, $\log x = \cos 2t . \log e = \cos 2t \quad ...(iv)$

And $\log y = \sin 2t . \log e = \sin 2t \quad ...(v)$

$$\therefore \frac{dy}{dx} = \frac{-y \log x}{x \log y}$$

[using Eqs.(iv) and (v) in Eq.(iii) and $x = e^{\cos 2t}, y = e^{\sin 2t}$]

Hence proved.

50. If $x = a\,sin2t(1 + cos2t)$ and $y = b\ cos2t(1 - cos2t)$, show that $\left(\frac{dy}{dx}\right)_{at\, t = \frac{\pi}{4}} = \frac{b}{a}$.

Sol. $\because x = a\,sin2t(1 + cos2t)$ and $y = b\ cos2t(1 - cos2t)$

$$\therefore \frac{dx}{dt} = a\left[\sin 2t . \frac{d}{dt}(1 + \cos 2t) + (1 + \cos 2t) . \frac{d}{dt} \sin 2t\right]$$

$$= a\left[\sin 2t . (-\sin 2t) . \frac{d}{dt} 2t + (1 + \cos 2t) . \cos 2t . \frac{d}{dt} 2t\right]$$

$$= -2a \sin^2 2t + 2a \cos 2t (1 + \cos 2t)$$

$$\Rightarrow \frac{dx}{dt} = -2a[\sin^2 2t - \cos 2t(1+\cos 2t)] \ ...(i)$$

and $\frac{dy}{dt} = b\left[\cos 2t.\frac{d}{dt}(1-\cos 2t) + (1-\cos 2t).\frac{d}{dt}\cos 2t\right]$

$$= b\left[\cos 2t.(\sin 2t)\frac{d}{dt}2t + (1-\cos 2t)(-\sin 2t).\frac{d}{dt}2t\right]$$

$$= b[2\sin 2t.\cos 2t + 2(1-\cos 2t)(-\sin 2t)]$$

$$= 2b[\sin 2t.\cos 2t - (1-\cos 2t)\sin 2t] \ ...(ii)$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2b[-\sin 2t.\cos 2t + (1-\cos 2t)\sin 2t]}{-2a[\sin^2 2t - \cos 2t(1+\cos 2t)]}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{t=\pi/4} = \frac{b}{a}\frac{\left[-\sin\frac{\pi}{2}\cos\frac{\pi}{2} + \left(1-\cos\frac{\pi}{2}\right)\sin\frac{\pi}{2}\right]}{\left[\sin^2\frac{\pi}{2} - \cos\frac{\pi}{2}\left(1+\cos\frac{\pi}{2}\right)\right]}$$

$$= \frac{b}{a}.\frac{(0+1)}{(1-0)} \left[\because \sin\frac{\pi}{2} = 1 \, and \cos\frac{\pi}{2} = 0\right]$$

$$= \frac{b}{a} \ \textit{Hence proved}.$$

51. **If $x = 3\,sin\,t - sin\,3t,\ y = 3\,cos t - cos\ 3t$, find $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$.**

Sol. $x = 3\,sin\,t - sin\,3t,\ y = 3\,cos t - cos\ 3t,$

$$\therefore \frac{dx}{dt} = 3.\frac{d}{dt}\sin t - \frac{d}{dt}\sin 3t$$

$$= 3\cos t - \cos 3t.\frac{d}{dt}3t = 3\cos t - 3\cos 3t \ ...(i)$$

and $\frac{dy}{dt} = 3.\frac{d}{dt}\cos t - \frac{d}{dt}\cos 3t$

$$= 3\sin t + \sin 3t.\frac{d}{dt}3t$$

$$\frac{dy}{dt} = 3\sin 3t - 3t\sin t \ ...(ii)$$

$$\therefore \frac{dy}{dt} = \frac{dy/dt}{dx/dt} = \frac{3(\sin 3t - \sin t)}{3(\cos t - \cos 3t)}$$

Now, $\left(\frac{dy}{dx}\right)_{t=\pi/3} = \frac{\sin\frac{3\pi}{3} - \sin\frac{\pi}{3}}{\left(\cos\frac{\pi}{3} - \cos 3\frac{\pi}{3}\right)} = \frac{0 - \sqrt{3}/2}{\frac{1}{2} - (-1)}$

$$= \frac{-\sqrt{3}/2}{3/2} = \frac{-\sqrt{3}}{3} = \frac{-1}{\sqrt{3}}$$

52. Differentiate $\frac{x}{\sin x}$ w.r.t. sin x.

Sol. Let $u = \frac{x}{\sin x}$ *and* $v = \sin x$

$$\therefore \frac{du}{dx} = \frac{\sin x.\frac{d}{dx}x - x.\frac{d}{dx}\sin x}{(\sin x)^2}$$

$$= \frac{\sin x - x\cos x}{\sin^2 x} \quad ...(i)$$

and $\frac{dv}{dx} = \frac{d}{dx}\sin x = \cos x \quad ...(ii)$

$$\therefore \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\sin x - x\cos x/\sin^2 x}{\cos x}$$

$$= \frac{\sin x - x\cos x}{\sin^2 x\cos x} = \frac{\frac{\sin x - x\cos x}{\cos x}}{\frac{\sin^2 x\cos x}{\cos x}}$$

[dividing by cos x in both numerator and denominator]

$$= \frac{\tan x - x}{\sin^2 x}$$

53. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. tan-1 x when $x \neq 0$.

Sol. Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ *and* $v = \tan^{-1} x$

$\therefore x = \tan\theta$

$$\Rightarrow u = \tan^{-1}\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}$$

$$= \tan^{-1}\frac{(\sec\theta - 1)\cos\theta}{\sin\theta}$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left[\frac{1-1+2\sin^2\theta/2}{2\sin\theta/2.\cos\theta/2}\right] \quad [\because \cos\theta = 1-2\sin^2\theta]$$

$$= \tan^{-1}\left[\tan\frac{\theta}{2}\right]$$

$$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1} x$$

$$\therefore \frac{du}{dx} = \frac{1}{2}\frac{d}{dx}\tan^{-1} x = \frac{1}{2}.\frac{1}{1+x^2} \quad ...(i)$$

and $\frac{dv}{dx} = \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} \quad ...(ii)$

$$\therefore \frac{du}{dv} = \frac{du/dx}{dv/dx}$$

$$= \frac{1/2(1+x^2)}{1/(1+x^2)} = \frac{(1+x^2)}{2(1+x^2)} = \frac{1}{2}$$

Find $\frac{dy}{dx}$ when x and y are connected by the relation given in each of the Exercises 54 to 57.

54. $\sin(xy) + \frac{x}{y} = x^2 - y$

Sol. We have, $\sin(xy) + \frac{x}{y} = x^2 - y$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(\sin xy) + \frac{d}{dx}\left(\frac{x}{y}\right) = \frac{d}{dx}x^2 - \frac{d}{dx}y$$

$$\Rightarrow \cos xy.\frac{d}{dx}(xy) + \frac{y\frac{d}{dx}x - x.\frac{d}{dx}y}{y^2} = 2x - \frac{dy}{dx}$$

$$\Rightarrow \cos xy.\left[x.\frac{d}{dx}y + y.\frac{d}{dx}.x\right] + \frac{y - x\frac{dy}{dx}}{y^2} = 2x - \frac{dy}{dx}$$

$$\Rightarrow x\cos xy.\frac{dy}{dx} + y\cos xy + \frac{y}{y^2} - \frac{x}{y^2}\frac{dy}{dx} = 2x - \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left[x\cos xy - \frac{x}{y^2} + 1\right] = 2x - y\cos xy - \frac{y}{y^2}$$

$$\therefore \frac{dy}{dx} = \left[\frac{2x\,y - y^2\cos x\,y - 1}{y}\right]\left[\frac{y^2}{x\,y^2\cos x\,y - x + y^2}\right]$$

$$= \frac{(2x\,y - y^2\cos x\,y - 1)\,y}{(x\,y^2\cos x\,y - x + y^2)}$$

55. $\sec(x + y) = xy$

Sol. We have, $\sec(x + y) = xy$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\sec(x+y) = \frac{d}{dx}(xy)$$

$$\Rightarrow \sec(x+y).\tan(x+y).\frac{d}{dx}(x+y) = x.\frac{d}{dx}y + y.\frac{d}{dx}x$$

$$\Rightarrow \sec(x+y).\tan(x+y).\left(1+\frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow \sec(x+y)\tan(x+y) + \sec(x+y).\tan(x+y).\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\Rightarrow \frac{dy}{dx}[\sec(x+y).\tan(x+y) - x] = y - \sec(x+y).\tan(x+y)$$

$$\therefore \frac{dy}{dx} = \frac{y - \sec(x+y).\tan(x+y)}{\sec(x+y).\tan(x+y) - x}$$

56. $\tan^{-1}(x^2+y^2) = a$

Sol. We have, $\tan^{-1}(x^2+y^2) = a$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\tan^{-1}(x^2+y^2) = \frac{d}{dx}(a)$$

$$\Rightarrow \frac{1}{1+(x^2+y^2)^2}.\frac{d}{dx}(x^2+y^2) = 0$$

$$\Rightarrow 2x + \frac{d}{dy}y^2.\frac{dy}{dx} = 0$$

$$\Rightarrow 2y.\frac{dy}{dx} = -2x$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

57. $(x^2+y^2)^2 = xy$

Sol. We have, $(x^2+y^2)^2 = xy$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(x^2+y^2)^2 = \frac{d}{dx}(xy)$$

$$\Rightarrow 2(x^2+y^2).\frac{d}{dx}(x^2+y^2) = x.\frac{d}{dx}y + y.\frac{d}{dx}x$$

$$\Rightarrow 2(x^2+y^2).\left(2x + 2y\frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow 2x^2.2x + 2x^2.2y\frac{dy}{dx} + 2y^2.2x + 2y^2.\ 2y\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\Rightarrow \frac{dy}{dx}[4x^2y + 4y^3 - x] = y - 4x^3 - 4xy^2$$

$$\therefore \frac{dy}{dx} = \frac{(y - 4x^3 - 4xy^2)}{(4x^2y + 4y^3 - x)}$$

58. **If** $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$**, then show that** $\frac{dy}{dx}.\frac{dx}{dy} = 1$.

Sol. We have, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(*i*)

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(ax^2) + \frac{d}{dx}(2hxy) + \frac{d}{dx}(by^2) + \frac{d}{dx}(2gx) + \frac{d}{dx}(2fy) + \frac{d}{dx}(c) = 0$$

$$\Rightarrow 2ax + 2h\left(x.\frac{dy}{dx} + y.1\right) + b.2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} + 0 = 0$$

$$\Rightarrow \frac{dy}{dx}[2hx + 2by + 2f] = -2ax - 2hy - 2g$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2(ax+hy+g)}{2(hx+by+f)}$$

$$= \frac{-(ax+hy+g)}{(hx+by+f)} \qquad ...(ii)$$

Now, differentiating Eq. (i) w.r.t. y, we get

$$\frac{d}{dx}(ax^2)+\frac{d}{dy}(2hxy)+\frac{d}{dx}(by^2)+\frac{d}{dx}(2gx)+\frac{d}{dy}(2fy)+\frac{d}{dy}(c)=0$$

$$\Rightarrow a.2x.\frac{dx}{dy}+2h.\left(x.\frac{d}{dy}y+y.\frac{d}{dy}x\right)+b.2y+2g.\frac{dx}{dy}+2f+0=0$$

$$\Rightarrow \frac{dx}{dy}[2ax+2hy+2g]=-2hx-2by-2f$$

$$\Rightarrow \frac{dx}{dy}=\frac{-2(hx+by+f)}{2(ax+hy+g)}=\frac{-(hx+by+f)}{(ax+hy+g)} \quad ...(iii)$$

$$\therefore \frac{dy}{dx}.\frac{dx}{dy}=\frac{-(ax+hy+g)}{(hx+by+f)}.\frac{-(hx+by+f)}{(ax+hy+g)} \quad [\text{using } Eqs.(ii) \text{ and } (iii)]$$

$= 1 = RHS$ *Hence proved.*

59. **If** $x=e^{\frac{x}{y}}$**, prove that** $\frac{dy}{dx}=\frac{x-y}{x\log x}$.

Sol. We have, $x=e^{\frac{x}{y}}$

$$\therefore \frac{d}{dx}x=\frac{d}{dx}e^{x/y}$$

$$\Rightarrow 1 = e^{x/y}.\frac{d}{dx}(x/y)$$

$$\Rightarrow 1 = e^{x/y}.\left[\frac{y.1-x.dy/dx}{y^2}\right]$$

$$\Rightarrow y^2 = y.e^{x/y} - x.\frac{dy}{dx}.e^{x/y}$$

$$\Rightarrow x.\frac{dy}{dx}.e^{x/y} = ye^{x/y} - y^2$$

$$\therefore \frac{dy}{dx}=\frac{y(e^{x/y}-y)}{x.e^{x/y}}$$

$$=\frac{(e^{x/y}-y)}{e^{x/y}.\frac{x}{y}} \quad \left[\because x=e^{x/y} \Rightarrow \log x = \frac{x}{y}\right]$$

$=\frac{x-y}{x.\log x}$ *Hence proved.*

60. **If** $y^x = e^{y-x}$**, prove that** $\frac{dy}{dx}=\frac{(1+\log y)^2}{\log y}$.

Sol. We have, $y^x = e^{y-x}$,

$\Rightarrow \log y^x = \log^{y-x}$

$\Rightarrow x\log y = y - x.\log_e = (y-x)$ $[\because \log_e = 1]$

$\Rightarrow \log y = \frac{(y-x)}{x}$ ...(i)

Now, differentiating w.r.t. x, we get

$$\frac{d}{dy}\log y.\frac{dy}{dx} = \frac{d}{dx}\frac{(y-x)}{x}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{x.\frac{d}{dx}(y-x) - (y-x).\frac{d}{dx}.x}{x^2}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{x\left(\frac{dy}{dx} - 1\right) - (y-x)}{x^2}$$

$$\Rightarrow \frac{x^2}{y}.\frac{dy}{dx} = x\frac{dy}{dx} - x - y + x$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{x^2}{y} - x\right) = -y$$

$$\therefore \frac{dy}{dx} = \frac{-y^2}{x^2 - xy} = \frac{-y^2}{x(x-y)}$$

$$= \frac{y^2}{x(y-x)}.\frac{x}{x} = \frac{y^2}{x^2}.\frac{1}{\frac{(y-x)}{x}}$$

$$= \frac{(1+\log y)^2}{\log y} \quad \left[\because \log y = \frac{y-x}{x}\log y = \frac{y}{x} - 1 \Rightarrow 1 + \log y = \frac{y}{x}\right]$$

Hence proved.

61. $y = (\cos x)^{(\cos x)^{(\cos x)\ldots\ldots\infty}}$**, show that** $\frac{dy}{dx} = \frac{y^2 \tan x}{y \log\cos x - 1}$.

Sol. We have, $y = (\cos x)^{(\cos x)^{(\cos x)\ldots\ldots\infty}}$

$\Rightarrow y = (\cos x)^y$

$\therefore \log y = \log(\cos x)^y$

$\Rightarrow \log y = y\log\cos x$

On differentiating w.r.t. x, we get

$$\frac{1}{y}.\frac{dy}{dx} = y.\frac{d}{dx}\log\cos x + \log\cos x.\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{y}{\cos x}.\frac{d}{dx}\cos + \log\cos x.\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{1}{y}-\log\cos x\right]=\frac{-y\sin x}{\cos x}=-y\tan x$$

$$\therefore \frac{dy}{dx}=\frac{-y^2\tan x}{(1-y\log\cos x)}$$

$$=\frac{y^2\tan x}{y\log\cos x-1} \quad \textit{Hence proved.}$$

62. **If** $x\,sin\left(a+y\right)+sina\,cos\left(a+y\right)=0,$ **prove that** $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$.

Sol. We have,

$$x\sin(a+y)+\sin a.\cos(a+y)=0$$

$$\Rightarrow x\sin(a+y)=-\sin a.\cos(a+y)$$

$$\Rightarrow x=\frac{-\sin a.\cos(a+y)}{\sin(a+y)}$$

$$\Rightarrow x=-\sin a.\cot(a+y)$$

$$\therefore \frac{dx}{dy}=-\sin a.[-\cos ec^2(a+y)].\frac{d}{dy}(a+y)$$

$$=\sin a.\frac{1}{\sin^2(a+y)}.1$$

$$=\frac{\sin^2(a+y)}{\sin a} \quad \textit{Hence proved.}$$

63. **If** $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$ **prove that** $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$.

Sol. We have,

$$\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$$

On putting $x=\sin\alpha$ *and* $y=sin\beta,$ we get

$$\sqrt{1-\sin^2\alpha}+\sqrt{1-\sin^2\beta}=a(\sin\alpha-\sin\beta)$$

$$\Rightarrow \cos\alpha+\cos\beta=a(\sin\alpha-\sin\beta)$$

$$\Rightarrow 2\cos\frac{\alpha+\beta}{2}.\cos\frac{\alpha-\beta}{2}=a\left(2\cos\frac{\alpha+\beta}{2}.\sin\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \cos\frac{\alpha-\beta}{2}=a\sin\frac{\alpha-\beta}{2}$$

$$\Rightarrow \cot\frac{\alpha-\beta}{2}=a$$

$$\Rightarrow \frac{\alpha-\beta}{2}=\cot^{-1}a$$

$$\Rightarrow \alpha-\beta=2\cot^{-1}a$$

$$\Rightarrow \sin^{-1}x-\sin^{-1}y=2\cot^{-1}a \quad [\because x=\sin\alpha \,and\, y=\sin\beta]$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=0$$

$\therefore \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = \sqrt{\frac{1-y^2}{1-x^2}}$ *Hence proved.*

64. **If $y = tan^{-1}x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.**

Sol. We have, $y = \tan^{-1}x$ [on differentiating w.r.t. x]

$\frac{dy}{dx} = \frac{1}{1+x^2}$ [again differentiating w.r.t. x]

Now, $\frac{d^2y}{dx^2} = \frac{d}{dx}(1+x^2)^{-1}$

$= -1(1+x^2)^{-2}.\frac{d}{dx}(1+x^2)$

$= -\frac{1}{(1+x^2)^2}.2x$

$= \frac{-2\tan y}{(1+\tan^2 y)^2}$ [$\because y = \tan^{-1}x \Rightarrow \tan y = x$]

$= \frac{-2\tan y}{(\sec^2 y)^2}$

$= -2\frac{\sin y}{\cos y}.\cos^2 y.\cos^2 y$

$= -\sin 2y.\cos^2 y$ [$\because \sin 2x = 2\sin x\cos x$]

Verify the Rolle's theorem for each of the functions in Exercises 65 to 69.

65. $f(x) = x(x-1)^2$ in $[0, 1]$.

Sol. We have, $f(x) = x(x-1)^2$ in $[0, 1]$.

(i) Since, $f(x) = x(x-1)^2$ is a polynomial function.

So, it is continuous in [0, 1].

(ii) Now, $f'(x) = x.\frac{d}{dx}(x-1)^2 + (x-1)^2\frac{d}{dx}x$

$= x.2(x-1).1 + (x-1)^2$

$= 2x^2 - 2x + x^2 + 1 - 2x$

$= 3x^2 - 4x + 1$ which exists in $(0,1)$

So, $f(x)$ is differentiable in (0, 1)

(iii) Now, $f(0) = 0$ and $f(1) = 0 \Rightarrow f(0) = f(1)$

f satisfies the above conditions of Rolle's theorem.

Hence, by Rolle's theorem $\exists c \in (0,1)$ such that

$f'(c) = 0$

$\Rightarrow 3c^2 - 4c + 1 = 0$

$\Rightarrow 3c^2 - 3c - c + 1 = 0$

$\Rightarrow 3c(c-1) - 1(c-1) = 0$

$\Rightarrow (3c-1)(c-1) = 0$

$\Rightarrow c = \frac{1}{3}, 1 \Rightarrow \frac{1}{3} \in (0.1)$

Thus, we see that there exists a real number c in the open interval (0, 1).
Hence, Rolle's theorem has been verified.

66. $f(x) = \sin^4 x + \cos^4 x in \left[0, \frac{\pi}{2}\right].$

Sol. We have, $f(x) = \sin^4 x + \cos^4 x in \left[0, \frac{\pi}{2}\right]$ ...(*i*)

(i) $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$

[since, $sin^4 x$ *and* $cos^4 x$ are continuous functions and we know that, if g and h be continuous functions, then $(g+h)$ is a continuous function.]

(ii) $f'(x) = 4(\sin x)^3 . \cos x + 4(\cos x)^3 . (-\sin x)$

$= 4\sin^3 x . \cos x - 4\sin x . \cos^3 x$

$= 4\sin x \cos x(\sin^2 x - \cos^2 x)$ which exists in $\left(0, \frac{\pi}{2}\right)$ ...(*ii*)

Hence, $f(x)$ is differentiable in $\left(0, \frac{\pi}{2}\right)$.

(iii) Also, $f(0) = 0 + 1 = 1 and f'\left(\frac{\pi}{2}\right) = 1 + 0 = 1$

$\Rightarrow f(0) = f\left(\frac{\pi}{2}\right)$

Conditions of Rolle's theorem are satisfied.

Hence, there exists atleast one $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$

$\therefore 4\sin c \cos c(\sin^2 c - \cos^2 c) = 0$

$\Rightarrow 4\sin c \cos c(-\cos 2c) = 0$

$\Rightarrow -2\sin 2c . \cos 2c = 0$

$\Rightarrow -\sin 4c = 0$

$\Rightarrow \sin 4c = 0$

$\Rightarrow 4c = \pi$

$\Rightarrow c = \frac{\pi}{4}$

and $\frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

Hence, Rolle's theorem has been verified.

67. $f(x) = log(x^2 + 2) - log 3 \ in \ [-1, 1].$

Sol. We have, $f(x) = log(x^2 + 2) - log 3$

(i) Logarithmic functions are continuous in their domain.

Hence, $f(x) = log(x^2+2) - log3$ is continuous in $[-1, 1]$

(ii) $f'(x) = \frac{1}{x^2+2}.2x - 0$

$= \frac{2x}{x^2+2}$, which exists in $(-1, 1)$.

Hence, $f(x)$ is differentiable in $(-1, 1)$.

(iii) $f(-1) = log\left[(-1)^2+2\right] - log3 = log3 - log3 = 0$ and

$f(1) = log(1^2+2) - log3 = log3 - log3 = 0$

$\Rightarrow f(-1) = f(1)$

Conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that

$f'(c) = 0$

$\Rightarrow \frac{2c}{c^2+2} = 0$

$\Rightarrow c = 0 \in (-1,1)$

Hence, Rolle's theorem has been verified.

68. $f(x) = x(x+3)e^{-x/2}$ *in* $[-3, 0]$.

Sol. We have, $f(x) = x(x+3)e^{-x/2}$

(i) $f(x)$ is a continuous function. [since, it is a combination of polynomial functions $x(x+3)$ and an exponential function $e^{-x/2}$ which are continuous functions]

So, $f(x) = x(x+3)e^{-x/2}$ is continuous in $[-3, 0]$.

(ii) $\therefore f'(x) = (x^2+3x).\frac{d}{dx}e^{-x/2} + e^{-x/2}.\frac{d}{dx}(x^2+3x)$

$= (x^2+3x).e^{-x/2}.\left(-\frac{1}{2}\right) + e^{-x/2}.(2x+3)$

$= e^{-x/2}\left[2x+3-\frac{1}{2}.(x^2+3x)\right]$

$= e^{-x/2}\left[\frac{4x+6-x^2-3x}{2}\right]$

$= e^{-x/2}.\frac{1}{2}[-x^2+x+6]$

$= \frac{-1}{2}e^{-x/2}[x^2-x-6]$

$= \frac{-1}{2}e^{-x/2}[x^2-3x+2x-6]$

$= \frac{-1}{2}e^{-x/2}[(x+2)(x-3)]\, which\ exists\ in\, (-3,0)$

Hence, $f(x)$ is differentiable in (-3, 0).

(iii) $\therefore f(-3) = -3(-3+3)e^{-3/2} = 0$

and $f(0) = 0(0+3)e^{-0/2} = 0$

$\Rightarrow f(-3) = f(0)$

Since, conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that $f'(c) = 0$

$\Rightarrow -\frac{1}{2}e^{-c/2}(c+2)(c-3) = 0$

$\Rightarrow c = -2, 3, where -2 \in (-3, 0)$

Therefore, Rolle's theorem has been verified.

69. $f(x) = \sqrt{4-x^2}\ in[-2, 2].$

Sol. We have, $f(x) = \sqrt{4-x^2} = (4-x^2)^{1/2}$

(i) $f(x) = \sqrt{4-x^2}$ is continuous function.

[since every polynomial function is a continuous function]

Hence, $f(x)$ is continuous in [-2, 2].

(ii) $f'(x) = \frac{1}{2}(4-x^2)^{-1/2}.(-2x)$

$= -x.\frac{1}{\sqrt{4-x^2}}$, which exists everywhere except at $x = \pm 2$.

Hence, $f(x)$ is differentiable in (-2, 2).

(iii) $f(-2) = \sqrt{(4-4)} = 0\ and\ f(2) = \sqrt{(4-4)} = 0$

$\Rightarrow f(-2) = f(2)$

Conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that f ' (c) = 0

$\Rightarrow -c\frac{1}{\sqrt{4-c^2}} = 0$

$\Rightarrow c = 0 \in (-2, 2)$

Hence, Rolle's theorem has been verified.

70. Discuss the applicability of Rolle's theorem on the function given by

$f(x) = \begin{cases} x^2+1, & if\ \ 0 \le x \le 1 \\ 3-x, & if\ \ 1 \le x \le 2 \end{cases}.$

Sol. We have, $f(x) = \begin{cases} x^2+1, & if\ \ 0 \le x \le 1 \\ 3-x, & if\ \ 1 \le x \le 2 \end{cases}.$

We know that, polynomial function is everywhere continuous and differentiability.

So, $f(x)$ is continuous and differentiable at all points except possibly at $x = 1$.

Now, check the differentiability at $x = 1$.

At $x = 1$,

$LDH = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x-1}$

$= \lim_{x \to 1} \frac{(x^2+1)-(1+1)}{x-1}$ $[\because f(x) = x^2+1, \forall 0 \le x \le 1]$

$= \lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x-1}$

$= 2$

and $RDH = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1} \frac{(3-x) f(1+1)}{(x-1)}$

$= \lim_{x \to 1} \frac{3-x-2}{x-1} = \lim_{x \to 1} \frac{-(x-1)}{x-1} = -1$

$\therefore$ $LHD \neq RHD$

So, $f(x)$ is not differentiable at x = 1.

Hence, polle's theorem is not applicable on the interval [0, 2]

71. Find the points on the curve $y = (\cos x - 1)\, in [0, 2\pi]$, where the tangent is parallel to x-axis.

Sol. The equation of the curve is $y = cos\ x - 1$.

Now, we have to find a point on the curve in $[0, 2\pi]$.

where the tangent is parallel to X-axis i.e., the tangent to the curve at x = c has a slope o, where $c \in]0, 2\pi[$.

Let us apply Rolle's theorem to get the point.

(i) y = cos x – 1 is a continuous function in $[0, 2\pi]$.

[since it is a combination of cosine function and a constant function]

(ii) $y' = -sinx$, which exists in $(0, 2\pi)$

Hence, y is differentiable in $(0, 2\pi)$

(iii) $y(0) = cos 0 - 1 = 0$ and $y(2\pi) = \cos 2\pi - 1 = 0$

$\therefore$ $y(0) = y(2\pi)$

Since, conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that

$f'(c) = 0$

$\Rightarrow -\sin c = 0$

$\Rightarrow$ $c = \pi\ or\ 0, where\ \pi \in (0, 2\pi)$

$\Rightarrow x = \pi$

$\therefore$ $y = \cos \pi - 1 = -2$

Hence, the required point on the curve, where the tangent drawn is parallel to the X-axis is $(\pi, -2)$.

72. Using Rolle's theorem, find the point on the curve $y = x(x-4), x \in [0, 4]$. where the tangent is parallel to x-axis.

Sol. We have, $y = x(x-4), x \in [0, 4]$

(i) y is a continuous function since $x(x-4)$ is a polynomial function.

Hence, $y = x(x-4)$ is continuous in $[0, 4]$

(ii) $y' = (x-4).1 + x.1 - 2x - 4$ which exists in $(0, 4)$.

Hence, y is differentiable in $(0,\ 4)$.

(iii) $y(0)=0(0-4)=0$

and $y(4) = 4(4-4) = 0$

$\Rightarrow y(0) = y(4)$

Since, conditions of Rolle's theorem are satisfied.

Hence, there exists a point c such that

$f'(c) = 0 \text{ in } (0,4)\ [\because f'(x) = y']$

$\Rightarrow 2c-4=0$

$\Rightarrow c=2$

$\Rightarrow x = 2;\ y = 2(2-4) = -4$

Thus, (2, -4) is the point on the curve at which the tangent drawn is parallel to X-axis.

Verify mean value theorem for each of the functions given Exercises 73 to 76.

73. $f(x)=\dfrac{1}{4x-1} \text{ in } [1,\ 4]$

Sol. We have, $f(x)=\dfrac{1}{4x-1} \text{ in } [1,\ 4]$

(i) $f(x)$ is continuous in [1, 4].

Also, at $x=\dfrac{1}{4}$. $f(x)$ is discontinuous.

Hence, f (x) is continuous in [1, 4].

(ii) $f'(x)=-\dfrac{4}{(4x-1)^2}$, which exists in (1, 4).

Since, conditions of mean value theorem are satisfied.

Hence, there exists a real number $c\in[1,\ 4]$ such that

$f'(c)=\dfrac{f(4)-f(1)}{4-1}$

$$\Rightarrow \frac{-4}{(4c-1)^2}=\frac{\frac{1}{16-1}-\frac{1}{4-1}}{4-1}=\frac{\frac{1}{15}-\frac{1}{3}}{3}$$

$$\Rightarrow \frac{-4}{(4c-1)^2}=\frac{1-5}{45}=\frac{-4}{45}$$

$$\Rightarrow (4c-1)^2=45$$

$$\Rightarrow 4c-1=\pm 3\sqrt{5}$$

$$\Rightarrow c=\frac{3\sqrt{5}+1}{4}\in(1,4)\ [neglecting\ (-ve)\ value]$$

Hence, mean value theorem has been verified.

74. $f(x)=x^3-2x^2-x+3 \text{ in } [0,\ 1]$

Sol. We have, $f(x)=x^3-2x^2-x+3 \text{ in } [0,\ 1]$

(i) Since, $f(x)$ is a polynomial function.

Hence, $f(x)$ is continuous in [0, 1]

(ii) $f'(x)=3x^2-4x-1$, which exists in (0, 1).

Hence, $f(x)$ is differentiable in (0, 1).

Since, continuous of mean value theorem are satisfied.

Therefore, by mean value theorem $\exists c\in(0,1)$, such that

$$f'(c)=\frac{f(1)-f(0)}{1-0}$$

$$\Rightarrow 3c^2-4c-1=\frac{[1-2-1+3]-[0+3]}{1-0}$$

$$\Rightarrow 3c^2-4c-1=\frac{-2}{1}$$

$$\Rightarrow 3c^2-4c+1=0$$

$$\Rightarrow 3c^2-3c-c+1=0$$

$$\Rightarrow 3c(c-1)-1(c-1)=0$$

$$\Rightarrow (3c-1)(c-1)=0$$

$$\Rightarrow c=1/3,1, where\frac{1}{3}\in(0,1)$$

Hence, the mean value theorem has been verified.

75. $f(x)=\sin x-\sin 2x\, in[0,\ \pi]$.

Sol. We have, $f(x)=\sin x-\sin 2x\, in[0,\ \pi]$.

(i) Since, we know that sine functions are continuous functions hence $f(x)=sin\,x-sin2x$ is a continuous function in $[0,\pi]$.

(ii) $f'(x)=\cos x-\cos 2x.2=\cos x-2\cos 2x$, which exists in $(0,\ \pi)$

So, $f(x)$ is differentiable in $(0,\ \pi)$. Continuous of mean value theorem are satisfied.

Hence, $\exists c\in(0,\pi)$ such that, $f'(c)=\dfrac{f(\pi)-f(0)}{\pi-0}$

$$\Rightarrow \cos\ c-2\cos 2c=\frac{\sin\pi-\sin 2\pi-\sin 0+\sin 2.0}{\pi-0}$$

$$\Rightarrow 2\cos 2c-\cos c=\frac{0}{\pi}$$

$$\Rightarrow 2.(2\cos^2 c-1)-\cos c=0$$

$$\Rightarrow 4\cos^2 c-2-\cos c=0$$

$$\Rightarrow 4\cos^2 c-\cos c-2=0$$

$$\Rightarrow \cos c=\frac{1\pm\sqrt{1+32}}{8}=\frac{1\pm\sqrt{33}}{8}$$

$$\therefore c=\cos^{-1}\left(\frac{1\pm\sqrt{33}}{8}\right)$$

Also, $\cos^{-1}\left(\dfrac{1\pm\sqrt{33}}{8}\right)\in(0,\pi)$

Hence, mean value theorem has been verified.

76. $f(x)=\sqrt{25-x^2}\ \ in\ [1,\ 5]$

Sol. We have, $f(x)=\sqrt{25-x^2}$ *in* [1, 5]

(i) Since, $f(x)=(25-x^2)^{1/2}$, where $25-x^2 \geq 0$

$\Rightarrow x^2 \leq \pm 5 \Rightarrow -5 \leq x \leq 5$

Hence, $f(x)$ is continuous in [1, 5].

(ii) $f'(x)=\frac{1}{2}(25-x^2)^{1/2}.-2x=-\frac{-x}{\sqrt{25-x^2}}$, which exists in (1, 5).

Hence, $f'(x)$ is differentiable in (1, 5).

Since, conditions of mean value theorem are satisfied.

By mean value theorem $\exists c \in (1,5)$ such that

$$f'(c)=\frac{f(5)-f(1)}{5-1} \Rightarrow \frac{-c}{\sqrt{25-c^2}}=\frac{0-\sqrt{24}}{4}$$

$$\Rightarrow \frac{c^2}{25-c^2}=\frac{24}{16}$$

$$\Rightarrow 16c^2=600-24c^2$$

$$\Rightarrow c^2=\frac{600}{40}=15$$

$$\therefore c=\pm\sqrt{15}$$

Also, $c=\sqrt{15}\in(1,5)$

Hence, the mean value theorem has been verified.

77. Find a point on the curve $y=(x-3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

Sol. We have, $y=(x-3)^2$, which is continuous in $x_1=3$ *and* $x_2=4$ *i.e.*, $[3,\ 4]$.

Also, $y'=2(x-3).1=2(x-3)$ which exists in (3, 4)

Hence, by mean value theorem there exists a point on the curve at which tangent drawn is parallel to the chord joining the points (3, 0) and (4, 1).

Thus, $f'(c)=\frac{f(4)-f(3)}{4-3}$

$$\Rightarrow 2(c-3)=\frac{(4-3)^2-(3-3)^2}{4-3}$$

$$\Rightarrow 2c-6=\frac{1-0}{1} \Rightarrow c=\frac{7}{2}$$

For $x=\frac{7}{2}$, $y=\left(\frac{7}{2}-3\right)^2=\left(\frac{1}{2}\right)^2=\frac{1}{4}$

So, $\left(\frac{7}{2},\frac{1}{4}\right)$ is the point on the curve at which tangent drawn is parallel to the chord joining the points (3, 0) and (4, 1).

78. Using mean value theorem, prove that there is a point on the curve $y=2x^2-5x+3$ between the points $A(1,\ 0)$ *and* $B(2,\ 1)$ where tangent is parallel to the chord AB. Also, find that point.

Sol. We have, $y=2x^2-5x+3$ which is continuous in [1, 2] as it is a polynomial function.
Also, y' = 4x – 5, which exists in (1, 2).
By mean value theorem, $\exists c\in(1,2)$ at which drawn tangent is parallel to the chord AB, where A and B are $(1,\ 0)$ *and* $(2,\ 1)$, respectively.

$$\therefore\ f'(c)=\frac{f(2)-f(1)}{2-1}$$

$$\Rightarrow 4c-5=\frac{(8-10+3)-(2-5+3)}{1}$$

$$\Rightarrow 4c-5=1$$

$$\Rightarrow c=\frac{6}{4}=\frac{3}{2}\in(1,2)$$

For $x=\frac{3}{2},\quad y=2\left(\frac{3}{2}\right)^2-5\left(\frac{3}{2}\right)+3$

$$=2\times\frac{9}{4}-\frac{15}{2}+3=\frac{9-15+6}{2}=0$$

Hence, $\left(\frac{3}{2},0\right)$ is the point on the curve $y=2x^2-5x+3$ between the points $A(1,\ 0)$ *and* $B(2,\ 1)$ where tangent is parallel to the chord AB.

Continuity and Differentiability

Long Answer Type Questions

79. Find the values of p and q so that $f(x)=\begin{cases}x^2+3x+p, & if\ x\le 1\\ qx+2, & if\ x>1\end{cases}$ **is differentiable at x = 1.**

Sol. We have, $f(x)=\begin{cases}x^2+3x+p, & if\ x\le 1\\ qx+2, & if\ x>1\end{cases}$ is differentiable at x=1.

$\therefore\ Lf'(1)=\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}$

$=\lim_{x\to 1^-}\frac{(x^2+3x+p)-(1+3+p)}{x-1}$

$=\lim_{h\to 0}\frac{[(1-h)^2+3(1-h)+p]-[1+3+p]}{(1-h)-1}$

$=\lim_{h\to 0}\frac{[1+h^2-2h+3-3h+p]-[4+p]}{-h}$

$=\lim_{h\to 0}\frac{[h^2-5h+p+4-4-p]}{-h}=\lim_{h\to 0}\frac{h[h-5]}{-h}$

$=\lim_{h\to 0}-[h-5]=5$

$Rf'(1)=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{(qx+2)-(1+3+p)}{x-1}$

$=\lim_{h\to 0}\frac{[q(1+h)+2]-(4+p)}{1+h-1}$

$=\lim_{h\to 0}\frac{[q+qh+2-4-p]}{h}=\lim_{h\to 0}\frac{qh+(q-2-p)}{h}$

$\Rightarrow q-2-p=0\ \Rightarrow p-q=-2\ \ ...(i)$

$\Rightarrow \lim_{h\to 0}\frac{qh+0}{h}=q$ [for existing the limit]

$if\ Lf'(1)=Rf'(1), then\ 5=q$

$\Rightarrow\ p-5=-2\Rightarrow\ p=3$

$\therefore\ p=3\ and\ q=5$

80. If $x^m.y^n=(x+y)^{m+n}$**, prove that**

$(i)\ \frac{dy}{dx}=\frac{y}{x}\ and\ (ii)\ \frac{d^2y}{dx^2}=0.$

Sol. We have, $x^m.y^n=(x+y)^{m+n}\ \ ...(i)$

(i) Differentiating Eq. (i) w.r.t. x, we get

$\frac{d}{dx}(x^m.y^n)=\frac{d}{dx}(x+y)^{m+n}$

$\Rightarrow x^m.\frac{d}{dy}y^n.\frac{dy}{dx}+y^n.\frac{d}{dx}x^m=(m+n)(x+y)^{m+n-1}\frac{d}{dx}(x+y)$

$\Rightarrow x^m.ny^{n-1}\frac{dy}{dx}+y^n.mx^{m-1}=(m+n)(x+y)^{m+n-1}\left(1+\frac{dy}{dx}\right)$

Long Answer Type Questions

79. **Find the values of p and q so that** $f(x)=\begin{cases} x^2+3x+p, & if\ x\le 1 \\ qx+2, & if\ x>1 \end{cases}$ **is differentiable at x = 1.**

Sol. We have, $f(x)=\begin{cases} x^2+3x+p, & if\ x\le 1 \\ qx+2, & if\ x>1 \end{cases}$ is differentiable at x=1.

$\therefore\ Lf'(1)=\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}$

$=\lim_{x\to 1^-}\frac{(x^2+3x+p)-(1+3+p)}{x-1}$

$=\lim_{h\to 0}\frac{[(1-h)^2+3(1-h)+p]-[1+3+p]}{(1-h)-1}$

$=\lim_{h\to 0}\frac{[1+h^2-2h+3-3h+p]-[4+p]}{-h}$

$=\lim_{h\to 0}\frac{[h^2-5h+p+4-4-p]}{-h}=\lim_{h\to 0}\frac{h[h-5]}{-h}$

$=\lim_{h\to 0}-[h-5]=5$

$Rf'(1)=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{(qx+2)-(1+3+p)}{x-1}$

$=\lim_{h\to 0}\frac{[q(1+h)+2]-(4+p)}{1+h-1}$

$=\lim_{h\to 0}\frac{[q+qh+2-4-p]}{h}=\lim_{h\to 0}\frac{qh+(q-2-p)}{h}$

$\Rightarrow\ q-2-p=0\ \Rightarrow\ p-q=-2\ \ ...(i)$

$\Rightarrow\ \lim_{h\to 0}\frac{qh+0}{h}=q$ [for existing the limit]

$if\ Lf'(1)=Rf'(1), then\ 5=q$

$\Rightarrow\ p-5=-2\Rightarrow\ p=3$

$\therefore\ p=3\ and\ q=5$

80. **If** $x^m.y^n=(x+y)^{m+n}$**, prove that**

$(i)\ \frac{dy}{dx}=\frac{y}{x}\ and\ (ii)\ \frac{d^2y}{dx^2}=0.$

Sol. We have, $x^m.y^n=(x+y)^{m+n}\ \ ...(i)$

(i) Differentiating Eq. (i) w.r.t. x, we get

$\frac{d}{dx}(x^m.y^n)=\frac{d}{dx}(x+y)^{m+n}$

$\Rightarrow x^m.\frac{d}{dy}y^n.\frac{dy}{dx}+y^n.\frac{d}{dx}x^m=(m+n)(x+y)^{m+n-1}\frac{d}{dx}(x+y)$

$\Rightarrow x^m.ny^{n-1}\frac{dy}{dx}+y^n.mx^{m-1}=(m+n)(x+y)^{m+n-1}\left(1+\frac{dy}{dx}\right)$

$$\Rightarrow \frac{dy}{dx}[x^m.ny^{n-1}-(m+n).(x+y)^{m+n-1}]=(m+n)(x+y)^{m+n-1}-y^n mx^{m-1}$$

$$\Rightarrow \frac{dy}{dx}[nx^m y^{n-1}-(m+n)(x+y)^{m+n-1}]=(m+n)(x+y)^{m+n-1}-\frac{y^{n-1}.y.mx^m}{x}$$

$$\therefore \frac{dy}{dx}=\frac{\frac{(m+n)(x+y)^{m+n}}{(x-y)}-\frac{y^{n-1}.y.mx^m}{x}}{\frac{n\,x^m y^n}{y}-(m+n)(x+y)^{m+n}\frac{1}{(x+y)}}$$

$$=\frac{\frac{x(m+n)(x+y)^{m+n}-(x+y).\,y.^{n-1}\,y.mx^m}{(x+y).x}}{\frac{(x+y)n\,x^m y^n-y(m+n)(x+y)^{m+n}}{(x+y).y}}$$

$$=\frac{\frac{x(m+n).x^m.y^n-m(x+y)\,y^n x^m}{(x+y).x}}{\frac{(x+y)nx^m.y^n-y(m+n).x^m.y^n}{(x+y).y}}\quad [\because (x+y)^{m+n}=x^m.y^n]$$

$$=\frac{x^m y^n[mx+nx-mx-my].(x+y)\,y}{x^m y^n[nx+ny-my-ny].(x+y).x}$$

$$=\frac{y}{x}\quad ...(i)$$

Hence proved.

(ii) Further, differentiating Eq. (ii) i.e., $\frac{dy}{dx}=\frac{y}{x}$ on both the sides w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{x.\frac{dy}{dx}-y.1}{x^2}$$

$$=\frac{x.\frac{y}{x}-y}{x^2}\quad \left[\because \frac{dy}{dx}=\frac{y}{x}\right]$$

$=0$ *Hence proved.*

81. **If** $x=\sin t$ *and* $y=\sin pt$, **prove that** $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+p^2y=0$.

Sol. We have, $x=\sin t$ *and* $y=\sin pt$,

$$\therefore \frac{dx}{dt}=\cos t \text{ and } \frac{dy}{dt}=\cos pt.p$$

$$\Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{p.\cos pt}{\cos t}\quad ...(i)$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{\cos t.\frac{d}{dt}(p.\cos pt)\frac{dt}{dx}-p\cos pt.\frac{d}{dt}\cos t.\frac{dt}{dx}}{\cos^2 t}$$

$$= \frac{[\cos t.p.(-\sin pt).p - p\cos pt.(-\sin t)]\frac{dt}{dx}}{\cos^2 t}$$

$$= \frac{[-p^2 \sin pt.\cos t + p\sin t.os\ pt].\frac{1}{\cos t}}{\cos^2 t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-p^2 \sin pt.\cos t + p\cos pt.\sin t}{\cos^3 t} \quad ...(ii)$$

Since, we have to prove

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$$

$$\therefore LHS = (1-\sin^2 t)\frac{[p^2 \sin pt.\cos t + p\cos pt.\sin t]}{\cos^3 t}$$

$$-\sin.\frac{p\cos pt}{\cos t} + p^2 \sin pt$$

$$= \frac{1}{\cos^3 t}\begin{bmatrix}(1-\sin^2 t)(-p^2 \sin pt.\cos t + p\cos pt.\sin t) \\ -p\cos pt.\sin t.\cos^2 t + p^2 \sin pt.\cos^3 t\end{bmatrix}$$

$$= \frac{1}{\cos^3 t}\begin{bmatrix}-p^2 \sin pt.\cos^3 t + p\cos pt.\sin t.\cos^2 t \\ -p\cos pt.\sin t.\cos^2 t + p^2 \sin pt.\cos^3 t\end{bmatrix} [\because 1-\sin^2 t = \cos^2 t]$$

$$= \frac{1}{\sin^3 t}.0$$

$$= 0 \quad \textit{Hence proved}.$$

82. **Find** $\frac{dy}{dx}, \textit{ if } y = x^{\tan x} + \sqrt{\frac{x^2+1}{2}}.$

Sol. We have, $\frac{dy}{dx}, \textit{ if } y = x^{\tan x} + \sqrt{\frac{x^2+1}{2}}.$

Taking, $u = x^{\tan x} \textit{ and } v = \sqrt{\frac{x^2+1}{2}}$

$\log u = \tan x \log x \quad ...(ii)$

and $v^2 = \frac{x^2+1}{2} \quad ...(iii)$

On, differentiating Eq. (ii) w.r.t. x, we get

$$\frac{1}{u}.\frac{du}{dx} = \tan x.\frac{1}{x} + \log x \sec^2 x$$

$$\Rightarrow \frac{du}{dx} = u\left[\frac{\tan x}{x} + \log x.\sec^2 x\right]$$

$$= x^{\tan x}\left[\frac{\tan x}{x} + \log x.\sec^2 x\right] ...(iv)$$

also, differentiating Eq. (iii) w.r.t. x, we get

$$2v.\frac{dv}{dx}=\frac{1}{2}(2x)\Rightarrow\frac{dv}{dx}=\frac{1}{4v}.(2x)$$

$$\Rightarrow\frac{dv}{dx}=\frac{1}{4.\sqrt{\frac{x^2+1}{2}}}.2x=\frac{x.\sqrt{2}}{2\sqrt{x^2+1}}$$

$$\Rightarrow\frac{dv}{dx}=\frac{x}{\sqrt{2(x^2+1)}}\quad ...(v)$$

Now, $y=u+v$

$$\therefore\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

$$=x^{\tan x}\left[\frac{\tan x}{x}+\log x.\sec^2 x\right]+\frac{x}{\sqrt{2(x^2+1)}}$$

Continuity and Differentiability

Objective Type Questions

Choose the correct answers from the given four options in each of the Exercises 83 to 96.

83. **If** $f(x)=2x$ **and** $f(x)=\frac{x^2}{2}+1,$ **then which of the following can be a discontinuous function**

(A) $f(x)+g(x)$

(B) $f(x)-g(x)$

(C) $f(x).g(x)$

(D) $\frac{g(x)}{f(x)}$

Sol. (D) We know that, if f and g be continuous functions, then

(A) $f+g$ is continuous

(B) $f-g$ is continuous.

(C) fg is continuous

(D) $\frac{f}{g}$ is continuous at these points, where $g(x)\neq 0$.

Here, $\frac{g(x)}{f(x)}=\frac{\frac{x^2}{2}+1}{2x}=\frac{x^2+2}{4x}$

Which is discontinuous at x = 0.

84. **The function** $f(x)=\frac{4-x^2}{4x-x^3}$

(A) discontinuous at only one point

(B) discontinuous at exactly two points

(C) discontinuous at exactly three points

(D) None of these

Sol. (C) We have, $f(x)=\frac{4-x^2}{4x-x^3}=\frac{(4-x^2)}{x(4-x^2)}$

$=\frac{(4-x^2)}{x(2^2-x^2)}=\frac{4-x^2}{x(2+x)(2-x)}$

Clearly, $f(x)$ is discontinuous at exactly three points $x=0,\ x=-2\ and\ x=2.$

85. **The set of points where the function** f **given by** $f(x)=|2x-1|\ sinx$ **is differentiable is**

(A) R

(B) $\mathbf{R}-\left\{\frac{1}{2}\right\}$

(C) $(0,\infty)$

(D) None of these

Sol. (B) We have, $f(x)=|2x-1|\ sinx$

At $x=\frac{1}{2}$, $f(x)$ is not differentiable