Chapter 17

MISCELLANEOUS PROPOSITIONS

On the four normals that can be drawn from any point in the plane of a central conic to the conic.

411. LET the equation to the conic be

$$Ax^2 + By^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

[If A and B be both positive, it is an ellipse; if one be positive and the other negative, it is a hyperbola.]

The equation to the normal at any point (x', y') of the curve is

$$\frac{x - x'}{Ax'} = \frac{y - y'}{By'}.$$

If this normal pass through the given point (h, k), we have

$$\frac{h - x'}{Ax'} = \frac{k - y'}{By'},$$

i.e. $$(A - B)\,x'y' + Bhy' - Akx' = 0 \quad \ldots\ldots\ldots (2).$$

This is an equation to determine the point (x', y') such that the normal at it goes through the point (h, k). It shews that the point (x', y') lies on the rectangular hyperbola

$$(A - B)\,xy + Bhy - Akx = 0 \quad \ldots\ldots\ldots\ldots (3).$$

The point (x', y') is therefore both on the curve (3) and on the curve (1). Also these two conics intersect in four points, real or imaginary. There are therefore four points, in general, lying on (1), such that the normals at them pass through the given point (h, k).

Also the hyperbola (3) passes through the origin and the point (h, k) and its asymptotes are parallel to the axes.

Hence *From a given point four normals can in general be drawn to a given central conic, and their feet all lie on a certain rectangular hyperbola, which passes through the given point and the centre of the conic, and has its asymptotes parallel to the axes of the given conic.*

412. *To find the conditions that the normals at the points where two given straight lines meet a central conic may meet in a point.*

Let the conic be

$$Ax^2 + By^2 = 1 \text{ .... } (1),$$

and let the normals to it at the points where it is met by the straight lines

$$l_1x + m_1y = 1 \text{ .... } (2),$$

and

$$l_2x + m_2y = 1 \text{ .... } (3)$$

meet in the point (h, k).

By Art. 384, the equation to any conic passing through the intersection of (1) with (2) and (3) is

$$Ax^2 + By^2 - 1 + \lambda (l_1x + m_1y - 1)(l_2x + m_2y - 1) = 0 \ldots (4).$$

Since these intersections are the feet of the four normals drawn from (h, k), then, by the last article, the conic

$$(A - B)xy + Bhy - Akx = 0 \text{ .... } (5)$$

passes through the same four points.

For some value of λ it therefore follows that (4) and (5) are the same.

Comparing these equations, we have, since the coefficients of x^2 and y^2 and the constant term in (5) are all zero,

$$A + \lambda l_1 l_2 = 0, \ B + \lambda m_1 m_2 = 0, \text{ and } -1 + \lambda = 0.$$

Therefore $\lambda = 1$, and hence

$$l_1 l_2 = -A, \text{ and } m_1 m_2 = -B \ldots\ldots\ldots\ldots (6).$$

The relations (6) are the required conditions.

Also, comparing the remaining coefficients in (4) and (5), we have

$$\frac{\lambda (l_1 m_2 + l_2 m_1)}{A - B} = \frac{-\lambda (l_1 + l_2)}{-Ak} = \frac{-\lambda (m_1 + m_2)}{Bh},$$

so that $$h = -\frac{A-B}{B}\frac{m_1 + m_2}{l_1 m_2 + l_2 m_1} \dots\dots\dots\dots\dots (7),$$

and $$k = \frac{A-B}{A}\frac{l_1 + l_2}{l_1 m_2 + l_2 m_1}\dots\dots\dots\dots\dots(8).$$

Cor. 1. If the given conic be an ellipse, we have

$$A = \frac{1}{a^2} \text{ and } B = \frac{1}{b^2}.$$

The relations (6) then give

$$a^2 l_1 l_2 = b^2 m_1 m_2 = -1\dots\dots\dots\dots\dots\dots(9),$$

and the coordinates of the point of concurrence are

$$h = \frac{a^2 - b^2}{a^2}\frac{m_1 + m_2}{l_1 m_2 + l_2 m_1} = l_1 (a^2 - b^2) \cdot \frac{1 - b^2 m_1^2}{a^2 l_1^2 + b^2 m_1^2},$$

and $$k = -\frac{a^2 - b^2}{b^2}\frac{l_1 + l_2}{l_1 m_2 + l_2 m_1} = -m_1 (a^2 - b^2) \cdot \frac{1 - a^2 l_1^2}{a^2 l_1^2 + b^2 m_1^2}.$$

Cor. 2. If the equations to the straight lines be given in the form $y = mx + c$ and $y = m'x + c'$, we have

$$m = -\frac{l_1}{m_1}, \quad c = \frac{1}{m_1}, \quad m' = -\frac{l_2}{m_2}, \text{ and } c' = \frac{1}{m_2}.$$

The relations (9) then give

$$mm' = \frac{b^2}{a^2} \text{ and } cc' = -b^2.$$

413. *If the normals at four points P, Q, R, and S of an ellipse meet in a point, the sum of their eccentric angles is equal to an odd multiple of two right angles.* [Cf. Art. 293.]

If α, β, γ, and δ be the eccentric angles of the four points, the equations to PQ and RS are

$$y = -x \cdot \frac{b}{a} \cot \frac{\alpha+\beta}{2} + \frac{b \cos \frac{\alpha-\beta}{2}}{\sin \frac{\alpha+\beta}{2}},$$

and
$$y = -x \cdot \frac{b}{a} \cot \frac{\gamma+\delta}{2} + \frac{b \cos \frac{\gamma-\delta}{2}}{\sin \frac{\gamma+\delta}{2}}. \quad \text{[Art. 259.]}$$

Since the normals at these points meet in a point, we have, by Art. 412, Cor. 2,

$$\frac{b^2}{a^2} = mm' = \frac{b^2}{a^2} \cot \frac{\alpha+\beta}{2} \cot \frac{\gamma+\delta}{2}.$$

$$\therefore \tan \frac{\alpha+\beta}{2} = \cot \frac{\gamma+\delta}{2} = \tan\left(\frac{\pi}{2} - \frac{\gamma+\delta}{2}\right).$$

$$\therefore \frac{\alpha+\beta}{2} = n\pi + \frac{\pi}{2} - \frac{\gamma+\delta}{2},$$

i.e.
$$\alpha+\beta+\gamma+\delta = (2n+1)\pi.$$

414. Ex. 1. *If the normals at the points A, B, C, and D of an ellipse meet in a point O, prove that $SA \cdot SB \cdot SC \cdot SD = \lambda^2 \cdot SO^2$, where S is one of the foci and λ is a constant.*

Let the equation to the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{..............................} \quad (1),$$

and let O be the point (h, k).

As in Art. 411, the feet of the normals drawn from O lie on the hyperbola

$$\left(\frac{1}{a^2} - \frac{1}{b^2}\right) xy + \frac{hy}{b^2} - \frac{kx}{a^2} = 0,$$

i.e.
$$a^2e^2xy = a^2hy - b^2kx \quad \text{........................} \quad (2).$$

The coordinates of the points A, B, C, and D are therefore found by solving (1) and (2).

From (2) we have
$$y = \frac{b^2kx}{a^2(h - e^2x)}.$$

Substituting in (1) and simplifying, we obtain

$$x^4a^2e^4 - 2ha^2e^2x^3 + x^2(a^2h^2 + b^2k^2 - a^4e^4) + 2he^2a^4x - a^4h^2 = 0 \ldots (3).$$

If x_1, x_2, x_3, and x_4 be the roots of this equation, we have (Art. 2),

$$\Sigma x_1 = \frac{2h}{e^2}, \quad \Sigma x_1 x_2 = \frac{a^2h^2 + b^2k^2 - a^4e^4}{a^2e^4},$$

$$\Sigma x_1 x_2 x_3 = -\frac{2ha^2}{e^2}, \text{ and } x_1 x_2 x_3 x_4 = -\frac{a^2h^2}{e^4}.$$

If S be the point $(-ae, 0)$ we have, by Art. 251,

$$SA = a + ex_1.$$

$$\therefore SA \,.\, SB \,.\, SC \,.\, SD = (a+ex_1)(a+ex_2)(a+ex_3)(a+ex_4)$$
$$= a^4 + a^3e\Sigma x_1 + a^2e^2\Sigma x_1x_2 + ae^3\Sigma x_1x_2x_3 + e^4x_1x_2x_3x_4$$
$$= \frac{b^2}{e^2}\{(h+ae)^2 + k^2\}, \text{ on substitution and simplification,}$$
$$= \frac{b^2}{e^2} \,.\, SO^2.$$

Aliter. If ρ stand for one of the quantities SA, SB, SC, or SD we have

$$\rho = a + ex,$$

i.e.

$$x = \frac{1}{e}(\rho - a).$$

Substituting this value in (3) we obtain an equation in the fourth degree, and easily have

$$\rho_1\rho_2\rho_3\rho_4 = \frac{b^2}{e^2}[(h+ae)^2 + k^2], \text{ as before.}$$

Ex. 2. *If the normals at four points P, Q, R, and S of a central conic meet in a point, and if PQ pass through a fixed point, find the locus of the middle point of RS.*

Let the equation to PQ be

$$y = m_1x + c_1 \quad \ldots\ldots\ldots\ldots(1),$$

and that to RS

$$y = m_2x + c_2 \quad \ldots\ldots\ldots\ldots(2).$$

If the equation to the given conic be $Ax^2 + By^2 = 1$, we then have (by Art. 412, Cor. 2)

$$m_1m_2 = \frac{A}{B} \quad \ldots\ldots\ldots\ldots(3),$$

and

$$c_1c_2 = -\frac{1}{B} \quad \ldots\ldots\ldots\ldots(4).$$

If (f, g) be the fixed point through which PQ passes, we have

$$g = m_1f + c_1 \quad \ldots\ldots\ldots\ldots(5).$$

Now the middle point of RS lies on the diameter conjugate to it, *i.e.* by Art. 376, on the diameter

$$y = -\frac{A}{Bm_2}x,$$

i.e., by (3),

$$y = -m_1x \quad \ldots\ldots\ldots\ldots(6).$$

Now, from (4) and (5),

$$c_2 = -\frac{1}{B(g - fm_1)},$$

so that, by (3), the equation to RS is

$$y = \frac{A}{Bm_1}x - \frac{1}{B(g - fm_1)} \quad \text{.......................(7)}.$$

Eliminating m_1 between (6) and (7), we easily have, as the equation to the required locus,

$$(Ax^2 + By^2)(gx + fy) + xy = 0.$$

Cor. From equation (6) it follows that the diameter conjugate to RS is equally inclined with PQ to the axis, and hence that the points P and Q and the ends of the diameter conjugate to RS are concyclic (Art. 400).

EXAMPLES XLVI

1. If the sum of the squares of the four normals drawn from a point O to an ellipse be constant, prove that the locus of O is a conic.

2. If the sum of the reciprocals of the distances from a focus of the feet of the four normals drawn from a point O to an ellipse be $\frac{4}{\text{lat. rect.}}$, prove that the locus of O is a parabola passing through that focus.

3. If four normals be drawn from a point O to an ellipse and if the sum of the squares of the reciprocals of perpendiculars from the centre upon the tangents drawn at their feet be constant, prove that the locus of O is a hyperbola.

4. The normals at four points of an ellipse are concurrent and they meet the major axis in G_1, G_2, G_3, and G_4; prove that

$$\frac{1}{CG_1} + \frac{1}{CG_2} + \frac{1}{CG_3} + \frac{1}{CG_4} = \frac{4}{CG_1 + CG_2 + CG_3 + CG_4}.$$

5. If the normals to a central conic at four points L, M, N, and P be concurrent, and if the circle through L, M, and N meet the curve again in P', prove that PP' is a diameter.

6. Shew that the locus of the foci of the rectangular hyperbolas which pass through the four points in which the normals drawn from any point on a given straight line meet an ellipse is a pair of conics.

7. If the normals at points of an ellipse, whose eccentric angles are α, β, and γ, meet in a point, prove that

$$\sin(\beta + \gamma) + \sin(\gamma + \alpha) + \sin(\alpha + \beta) = 0.$$

Hence, by page 235, Ex. 15, shew that if PQR be a maximum triangle inscribed in an ellipse, the normals at P, Q, and R are concurrent.

8. Prove that the normals at the points where the straight line $\frac{x}{a\cos\alpha}+\frac{y}{b\sin\alpha}=1$ meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet at the point

$$\left(-ae^2\cos^3\alpha,\quad \frac{a^2e^2}{b}\sin^3\alpha\right).$$

9. Prove that the loci of the point of intersection of normals at the ends of focal chords of an ellipse are the two ellipses

$$a^2y^2(1+e^2)^2+b^2(x\pm ae)(x\mp ae^3)=0.$$

10. Tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are drawn from any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=4$; prove that the normals at the points of contact meet on the ellipse $a^2x^2+b^2y^2=\frac{1}{4}(a^2-b^2)^2$.

11. Any tangent to the rectangular hyperbola $4xy=ab$ meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the points P and Q; prove that the normals at P and Q meet on a fixed diameter.

12. Chords of an ellipse meet the major axis in the point whose distance from the centre is $a\sqrt{\frac{a-b}{a+b}}$; prove that the normals at its ends meet on a circle.

13. From any point on the normal to the ellipse at the point whose eccentric angle is α two other normals are drawn to it; prove that the locus of the point of intersection of the corresponding tangents is the curve

$$xy+bx\sin\alpha+ay\cos\alpha=0.$$

14. Shew that the locus of the intersection of two perpendicular normals to an ellipse is the curve

$$(a^2+b^2)(x^2+y^2)(a^2y^2+b^2x^2)^2=(a^2-b^2)^2(a^2y^2-b^2x^2)^2.$$

15. ABC is a triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having each side parallel to the tangent at the opposite angular point; prove that the normals at A, B, and C meet at a point which lies on the ellipse

$$a^2x^2+b^2y^2=\frac{1}{4}(a^2-b^2)^2.$$

16. The normals at four points of an ellipse meet in a point (h, k). Find the equations of the axes of the two parabolas which pass through the four points. Prove that the angle between them is $2\tan^{-1}\frac{b}{a}$ and that they are parallel to one or other of the equi-conjugates of the ellipse.

17. Prove that the centre of mean position of the four points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the normals at which pass through the point (α, β), is the point

$$\left(\frac{1}{2}\frac{a^2\alpha}{a^2-b^2},\quad -\frac{1}{2}\frac{b^2\beta}{a^2-b^2}\right).$$

18. Prove that the product of the three normals drawn from any point to a parabola, divided by the product of the two tangents from the same point, is equal to one quarter of the latus rectum.

19. Prove that the conic $2aky=(2a-h)y^2+4ax^2$ intersects the parabola $y^2=4ax$ at the feet of the normals drawn to it from the point (h, k).

20. From a point (h, k) four normals are drawn to the rectangular hyperbola $xy=c^2$; prove that the centre of mean position of their feet is the point $\left(\frac{h}{4}, \frac{k}{4}\right)$, and that the four feet are such that each is the orthocentre of the triangle formed by the other three.

ANSWERS

7. Proceed as in Art. 413, and use, in addition, the second result of Art. 412, Cor. 2. From the two results, thus obtained, eliminate δ.

9. Take $l_1x+m_1y-1=0$ (Art. 412, Cor. 1) as a focal chord of the ellipse.

14. If the normals are perpendicular, so also are the tangents; the line $l_1x+m_1y-1=0$ is therefore the polar with respect to the ellipse of a point $(\sqrt{a^2+b^2}\cos\theta,\ \sqrt{a^2+b^2}\sin\theta)$ on the director circle.

15. The triangle ABC is a maximum triangle (Page 235, Ex. 15) inscribed in the ellipse.

20. Use the notation of Art. 333.

SOLUTIONS/HINTS

1. As in Art. 414, Ex. 1,

$$\Sigma x_1 = \frac{2a^2h}{a^2e^2}, \quad \Sigma x_1 x_2 = \frac{a^2h^2 + b^2k^2 - a^4e^4}{a^2e^4}.$$

$$\therefore \ \Sigma x_1{}^2 = \frac{2a^2}{a^4e^4}(a^2h^2 - b^2k^2 + a^4e^4).$$

Similarly,

$$\Sigma y_1 = -\frac{2b^2k}{a^2e^2}, \text{ and } \Sigma y_1{}^2 = \frac{2b^2}{a^4e^4}(b^2k^2 - a^2h^2 + a^4e^4).$$

If r_1, r_2 etc. be the lengths of the normals, then

$$\Sigma r_1{}^2 = \Sigma(h - x_1)^2 + \Sigma(k - y_1)^2$$

$$= 4(h^2 + k^2) - 2h\Sigma_1 x_1 - 2k\Sigma y_1 + \Sigma x_1{}^2 + \Sigma y_1{}^2$$

$$= 4(h^2 + k^2) - \frac{4a^2h^2}{a^2e^2} + \frac{4b^2k^2}{a^2e^2} + \frac{2a^2}{a^4e^4}(a^2h^2 - b^2k^2 + a^4e^4)$$

$$+ \frac{2b^2}{a^4e^4}(b^2k^2 - a^2h^2 + a^4e^4)$$

$$= \frac{2h^2(a^2 - 2b^2) + 2k^2(2a^2 - b^2)}{a^2 - b^2} + 2a^2 + 2b^2.$$

Hence the equation of the locus is

$$x^2(a^2 - 2b^2) + y^2(2a^2 - b^2) = \text{cons.},$$

which is the equation of a conic.

2. See Art. 414, Ex. 1. The equation whose roots are SA, SB, SC, SD is found by putting $x = \frac{1}{e}(\rho - a)$, *i.e.* is

$$a^2e^2(\rho - a)^4 - 2eha^2(\rho - a)^3 + (\rho - a)^2(a^2h^2 + b^2k^2 - a^4e^4)$$

$$+ 2e^3ha^4(\rho - a) - a^4e^2h^2 = 0.$$

Whence

$$\Sigma\frac{1}{SA} = \frac{4a^5e^2 + 6ha^4e + 2a(a^2h^2 + b^2k^2 - a^4e^4) - 2ha^4e^3}{a^6e^2 + 2ha^5e + a^2(a^2h^2 + b^2k^2 - a^4e^4) - 2ha^5e^3 - a^4e^2h^2}$$

$$= \frac{2a}{b^2} \text{ (given).}$$

After some reduction, this gives $aey^2 = b^2(x + ae)$ as the equation to the locus of (h, k), which is a parabola passing through S.

3. If p_1, p_2, etc., be the lengths of the perpendiculars, then, from the equation of the tangent at (x_1y_1), we have

$$\Sigma \frac{1}{p_1^2} = \Sigma \frac{x_1^2}{a^4} + \Sigma \frac{y_1^2}{b^4} = \frac{2}{a^2 . a^4e^4}(a^2h^2 - b^2k^2 + a^4e^4)$$
$$+ \frac{2}{b^2 . a^4e^4}(b^2k^2 - a^2h^2 + a^4e^4) \text{ [by Ex. 1]}$$
$$= \frac{2}{a^2e^2}\left\{\frac{k^2}{a^2} - \frac{h^2}{b^2}\right\} + \frac{2}{a^2} + \frac{2}{b^2}.$$

Hence the condition gives $\left\{\frac{y^2}{a^2} - \frac{x^2}{b^2}\right\} = \frac{1}{C^2}$ for the equation of the locus of (h, k), which is the equation to a hyperbola.

4. Since $CG_1 = e^2CN$, etc., from equation (3) of Art. 414, Ex. 1, we have

$$\Sigma \frac{1}{CG_1} = \frac{1}{e^2}\Sigma \frac{1}{x_1} = \frac{1}{e^2} . \frac{2he^2a^4}{a^4h^2} = \frac{2}{h},$$

and $\frac{4}{\Sigma CG_1} = \frac{4}{e^2\Sigma x_1} = \frac{4a^2e^4}{e^2 . 2ha^2e^2} = \frac{2}{h}$. $\therefore \Sigma \frac{1}{CG_1} = \frac{4}{\Sigma CG_1}$.

5. Let $\alpha, \beta, \gamma, \delta, \delta'$ be eccentric angles of L, M, N, P, P'.

Then $\alpha + \beta + \gamma + \delta' = 2n\pi$ (Ex. XXXII. 18),

and $\alpha + \beta + \gamma + \delta = (2m + 1)\pi$. [Art. 413.]

$\therefore \delta = \delta' + (2r + 1)\pi$. $\therefore \cos\delta = -\cos\delta'$, and $\sin\delta = -\sin\delta'$.

Hence PP' is a diameter.

6. If (h, k) lies on a given straight line, then

$$lh + mk = 1. \quad \text{.....................(1)}$$

By Art. 393, the equations to find the foci of conic (2) in Art. 414, Ex. 1, become

$$(a^2e^2y + b^2k)^2 - (a^2e^2x - a^2h)^2 = 0, \quad \text{.........(2)}$$

and $\frac{(a^2e^2y + b^2k)(a^2e^2x - a^2h)}{2a^2e^2} = a^2e^2xy - a^2hy + b^2kx.$ ...(3)

Now (2) gives

$$a^2e^2(x - y) = a^2h + b^2k, \quad \text{..................(4)}$$

or $a^2e^2(x + y) = a^2h - b^2k,$ ....(5)

and (3) gives $a^2e^4xy + kb^2e^2x - ha^2e^2y + b^2hk = 0.$ ....(6)

Eliminating h and k between (1), (4) and (6) we obtain a conic; also the elimination of h and k between (1), (5) and (6) gives another conic.

7. Comparing $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$,

and $$\frac{x}{a}\cos\frac{\gamma+\delta}{2}+\frac{y}{b}\sin\frac{\gamma+\delta}{2}=\cos\frac{\gamma-\delta}{2}$$

with $$l_1x+m_1y=1 \text{ and } l_2x+m_2y=1,$$

and using the condition $l_1l_2a^2=m_1m_2b^2=-1$ [Art. 412],

we obtain $\cos\frac{\alpha+\beta}{2}\cos\frac{\gamma+\delta}{2}+\cos\frac{\alpha-\beta}{2}\cos\frac{\gamma-\delta}{2}=0$;

whence

$$\cos\frac{\alpha+\beta+\gamma+\delta}{2}+\cos\frac{\alpha+\beta-\gamma-\delta}{2}$$
$$+\cos\frac{\alpha+\gamma-\beta-\delta}{2}+\cos\frac{\alpha+\delta-\beta-\gamma}{2}=0;$$

but $$\alpha+\beta+\gamma+\delta=(2n+1)\pi. \quad \text{[Art. 413.]}$$

$$\therefore \sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0.$$

The eccentric angles of P, Q, R, α are $\alpha+\frac{2\pi}{3}$, $\alpha+\frac{4\pi}{3}$. These satisfy the above condition; for

$$\sin 2\alpha+\sin\left(2\alpha+\frac{2\pi}{3}\right)+\sin\left(2\alpha+\frac{4\pi}{3}\right)$$
$$=\sin 2\alpha+\cos 2\alpha\frac{\sqrt{3}}{2}-\sin 2\alpha\frac{1}{2}-\sin 2\alpha\frac{1}{2}-\cos 2\alpha\frac{\sqrt{3}}{2}=0.$$

8. Let (h, k) be the point of intersection of the normals, and let $lx+my=1$ be the chord joining the feet of the other two normals which pass through (h, k). Then for some value of λ, the conics

$$a^2e^2xy+b^2kx-a^2hy=0, \quad \text{[Art. 414, Ex. 1 (2)]}$$

and

$$\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+\left(\frac{x}{a\cos\alpha}+\frac{y}{b\sin\alpha}-1\right)(lx+my-1)=0$$

[Art. 380]

must be identical.

Comparing coefficients we obtain

$$\lambda = 1,\ \frac{1}{a^2} + \frac{l}{a\cos\alpha} = 0, \text{ and } \frac{1}{b^2} + \frac{m}{b\sin\alpha} = 0.$$

$$\therefore\ la = -\cos\alpha, \text{ and } mb = -\sin\alpha \ \ldots\ldots\ldots(1)$$

and
$$\frac{\dfrac{l}{b\sin\alpha} + \dfrac{m}{a\cos\alpha}}{a^2e^2} = \frac{l + \dfrac{1}{a\cos\alpha}}{-b^2k} = \frac{m + \dfrac{1}{b\sin\alpha}}{a^2h}.$$

$$\therefore\ \frac{1}{a^2e^2 . ab\sin\alpha\cos\alpha} = \frac{\sin^2\alpha}{ab^2k\cos\alpha} = -\frac{\cos^2\alpha}{ba^2h\sin\alpha}, \text{ by } (1).$$

$$\therefore\ h = -ae^2\cos^3\alpha,\ k = \frac{a^2e^2}{b}\sin^3\alpha.$$

9. The chord $lx + my = 1$ passes through the focus if

$$lae = 1.$$

Substitute in the equations of Art. 412, Cor. 1. Then

$$\frac{h}{ae} = \frac{e^2 - e^2b^2m^2}{1 + e^2b^2m^2}, \text{ and } \frac{k}{ma^2e^2} = \frac{1 - e^2}{1 + e^2b^2m^2}. \quad \therefore\ k = \frac{e^2b^2 . m}{1 + e^2b^2m^2}.$$

$$\therefore\ \frac{h + ae}{ae} = \frac{1 + e^2}{1 + e^2b^2m^2} \text{ and } \frac{h - ae^3}{ae} = -\frac{e^2b^2m^2(1 + e^2)}{1 + e^2b^2m^2}.$$

$$\therefore\ \frac{(h + ae)(h - ae^3)}{a^2e^2} = -\frac{(1 + e^2)^2 e^2b^2m^2}{(1 + e^2b^2m^2)^2} = -\frac{(1 + e^2)^2 k^2}{e^2b^2}.$$

Therefore the locus of (h, k) is

$$a^2y^2(1 + e^2)^2 + b^2(x + ae)(x - ae^3) = 0.$$

Similarly, if the chord passes through the other focus.

10. The polar of $(2a\cos\phi,\ 2b\sin\phi)$, which is any point on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 4$, with respect to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is

$$\frac{2x\cos\phi}{a} + \frac{2y\sin\phi}{b} = 1. \quad \text{[Art. 274.]}$$

Put $l = \frac{2\cos\phi}{a}$, $m = \frac{2\sin\phi}{b}$ in the equations of Art. 414, Cor. 1.

$$\therefore\ h = \frac{2\cos\phi}{a}(a^2 - b^2)\frac{1 - 4\sin^2\phi}{4};$$

$$k = -\frac{2\sin\phi}{b}(a^2 - b^2)\frac{1 - 4\cos^2\phi}{4}.$$

$$\therefore\ 4a^2h^2 + 4b^2k^2 = (a^2 - b^2)^2\{\cos^2\phi\,(1 - 4\sin^2\phi)^2 + \sin^2\phi\,(1 - 4\cos^2\phi)^2\} = (a^2 - b^2)^2.$$

Hence the required locus is $a^2x^2 + b^2y^2 = \frac{1}{4}(a^2 - b^2)^2$.

11. The intersections of $lx + my = 1$ with $4xy = ab$ are given by $4lx^2 - 4x + abm = 0$.

Therefore this line will be a tangent if $lmab = 1$.

By Art. 412, Ex. 1, the intersection of the normals at the extremities are

$$\frac{x}{la^2e^2} = \frac{1 - b^2m^2}{a^2l^2 + b^2m^2} \text{ and } \frac{y}{-ma^2e^2} = \frac{1 - a^2l^2}{a^2l^2 + b^2m^2}.$$

Dividing,

$$-\frac{mx}{ly} = \frac{1 - b^2m^2}{1 - a^2l^2} = \frac{lmab - b^2m^2}{lmab - a^2l^2} = \frac{bm\,(al - bm)}{al\,(bm - al)} = -\frac{bm}{al}.$$

$\therefore\ ax = by$, which is a diameter.

12. If $lx + my = 1$ be the equation to the chord,

$$\frac{1}{l} = a\sqrt{\frac{a - b}{a + b}}.\quad \therefore\ a^2l^2 = \frac{a + b}{a - b}.$$

Also $l = \frac{1}{a}\sqrt{\frac{a + b}{a - b}} = \frac{1}{a}\cdot\frac{c}{a - b}$, where $c^2 = a^2 - b^2$.

Substitute in equations of Art. 412, Cor. 1.

$$\therefore\ x = \frac{c^3}{a\,(a - b)}\cdot\frac{1 - b^2m^2}{\frac{a + b}{a - b} + b^2m^2};$$

$$\therefore\ \frac{xa}{c} = \frac{c^2\,(1 - b^2m^2)}{(a + b) + (a - b)\,b^2m^2}. \quad \ldots\ldots\ldots(1)$$

and

$$y = \frac{m\,.\,c^2\,.\,2b}{(a + b) + (a - b)\,b^2m^2}; \quad \ldots\ldots\ldots\ldots(2)$$

(1) gives

$$b^2m^2 = \frac{c^3 - xa\,(a + b)}{c^3 + xa\,(a - b)}, \quad \ldots\ldots\ldots\ldots\ldots(3)$$

and then (2) gives $yac = bm\,[c^3 + xa\,(a - b)\}$. ....(4)

Eliminating m from (3) and (4), we have

$$y^2a^2c^2 = \{c^3 + xa(a-b)\}\{c^3 - xa(a+b)\} = c^6 - 2abc^3x - x^2a^2c^2.$$

This reduces to $\left(x + \frac{bc}{a}\right)^2 + y^2 = c^2$, which is a circle.

13. Let β and γ be the eccentric angles of the points of contact of the corresponding tangents.

Then $x = a\frac{\cos\frac{1}{2}(\beta+\gamma)}{\cos\frac{1}{2}(\beta-\gamma)}$, $y = b\frac{\sin\frac{1}{2}(\beta+\gamma)}{\cos\frac{1}{2}(\beta-\gamma)}$, [Art. 265]

and $\quad \sin(\beta+\gamma) + \sin(\alpha+\gamma) + \sin(\beta+\alpha) = 0.$ [Ex. 7.]

$$\therefore\ 2\sin\frac{\beta+\gamma}{2}\cos\frac{\beta+\gamma}{2} + 2\sin\left\{\alpha + \frac{\beta+\gamma}{2}\right\}\cos\frac{\beta-\gamma}{2} = 0.$$

$$\therefore\ \frac{\sin\frac{\beta+\gamma}{2}.\cos\frac{\beta+\gamma}{2}}{\cos^2\frac{\beta-\gamma}{2}} + \sin\alpha.\frac{\cos\frac{\beta+\gamma}{2}}{\cos\frac{\beta-\gamma}{2}} + \cos\alpha.\frac{\sin\frac{\beta+\gamma}{2}}{\cos\frac{\beta-\gamma}{2}} = 0.$$

$$\therefore\ \frac{xy}{ab} + \frac{x\sin\alpha}{a} + \frac{y\cos\alpha}{b} = 0,$$

i.e. $$xy + bx\sin\alpha + ay\cos\alpha = 0.$$

14. If the normals are perpendicular, so also are the corresponding tangents, which therefore intersect on the director circle.

If $A^2 = a^2 + b^2$, any point on the director circle is

$$(A\cos\theta,\ A\sin\theta),$$

the polar of which is $\frac{Ax\cos\theta}{a^2} + \frac{Ay\sin\theta}{b^2} = 1$ with regard to the ellipse. Hence the intersection of the perpendicular normals is given by the equations of Art. 412, Ex. 1, on putting $l = \frac{A\cos\theta}{a^2}$, $m = \frac{A\sin\theta}{b^2}$.

$$\therefore\ x = (a^2 - b^2)\frac{A\cos\theta}{a^2}\frac{1 - \frac{(a^2+b^2)\sin^2\theta}{b^2}}{\frac{(a^2+b^2)\cos^2\theta}{a^2} + \frac{(a^2+b^2)\sin^2\theta}{b^2}}$$

$$= \frac{(a^2-b^2)}{a^2+b^2}.A\cos\theta.\frac{b^2\cos^2\theta - a^2\sin^2\theta}{b^2\cos^2\theta + a^2\sin^2\theta},$$

and $$y = -(a^2 - b^2)\frac{A \sin\theta}{b^2} \frac{1 - \dfrac{(a^2+b^2)\cos^2\theta}{a^2}}{\dfrac{(a^2+b^2)\cos^2\theta}{a^2} + \dfrac{(a^2+b^2)\sin^2\theta}{b^2}}$$

$$= \frac{a^2-b^2}{a^2+b^2} . A \sin\theta \frac{b^2\cos^2\theta - a^2\sin^2\theta}{b^2\cos^2\theta + a^2\sin^2\theta}.$$

$$\therefore \frac{\sin\theta}{y} = \frac{\cos\theta}{x} = \frac{1}{\sqrt{x^2+y^2}}.$$

Substitute; $\therefore x = \dfrac{a^2-b^2}{a^2+b^2} . \dfrac{Ax}{\sqrt{x^2+y^2}} . \dfrac{b^2x^2 - a^2y^2}{b^2x^2+a^2y^2}$.

$$\therefore (a^2+b^2)(x^2+y^2)(b^2x^2+a^2y^2)^2 = (a^2-b^2)^2(b^2x^2-a^2y^2)^2.$$

15. Let α, β, γ be the eccentric angles of A, B, C $[\alpha < \beta < \gamma]$.

Then $\tan\alpha = \tan\dfrac{\beta+\gamma}{2}$, etc. Hence on comparing with a figure we have $\dfrac{\beta+\gamma}{2} = \alpha + \pi$, $\dfrac{\gamma+\alpha}{2} = 2\beta$, and $\dfrac{\alpha+\beta}{2} = \gamma - \pi$.

$$\therefore \beta = \alpha + \frac{2\pi}{3}, \text{ and } \gamma = \alpha + \frac{4\pi}{3}.$$

Hence, as in Ex. 7 (part ii), the normals at A, B, C are concurrent.

Comparing $lx + my = 1$ with

$$\frac{x}{a}\cos\frac{1}{2}(\alpha+\beta) + \frac{y}{b}\sin\frac{1}{2}(\alpha+\beta) = \cos\frac{1}{2}(\alpha-\beta),$$

we have

$al = 2\cos\frac{1}{2}(\alpha+\beta)$, $bm = 2\sin\frac{1}{2}(\alpha+\beta)$, since $\cos\frac{1}{2}(\alpha-\beta) = \frac{1}{2}$.

By Art. 412, Cor. 1,

$$x = \frac{2\cos\frac{1}{2}(\alpha+\beta)}{a}(a^2-b^2)\frac{1-4\sin^2\frac{1}{2}(\alpha+\beta)}{4},$$

and $$y = \frac{2\sin\frac{1}{2}(\alpha+\beta)}{b}(a^2-b^2)\frac{1-4\cos^2\frac{1}{2}(\alpha+\beta)}{4}.$$

$$\therefore 4a^2x^2 + 4b^2y^2 = (a^2-b^2)^2, \text{ as in Ex. 10.}$$

16. The equation of any conic passing through the intersections of the ellipse and the conic through the feet of the four normals is

$$b^2x^2 + a^2y^2 - a^2b^2 + 2\lambda\{(a^2 - b^2)xy + b^2kx - a^2hy\} = 0.$$

[Art. 414, Ex. 1 and Art. 380.]

This is a parabola if $\lambda^2(a^2 - b^2)^2 = a^2b^2$ [Art. 357], *i.e.* if

$$\lambda = \pm\frac{ab}{a^2 - b^2}.$$

The terms of the second degree in the equations of the two parabolas are therefore $\{bx - ay\}^2$ and $\{bx + ay\}^2$.

Therefore the axes are parallel to the lines

$$bx \pm ay = 0 \text{ [Art. 360]},$$

the angle between which is $2\tan^{-1}\frac{b}{a}$.

They are parallel to the equi-conjugate diameters by Art. 289.

17. By Art. 414, Ex. 1 $\frac{1}{4}(x_1 + x_2 + x_3 + x_4) = \frac{1}{2}\frac{ha^2}{a^2 - b^2}$;

similarly, $\frac{1}{4}(y_1 + y_2 + y_3 + y_4) = -\frac{1}{2}\frac{hb^2}{a^2 - b^2}$.

18. In the equation $y^2 = 4ax$, move the origin to the point (h, k) and change to polars, so that the equation becomes $(r\sin\theta + k)^2 = 4a(r\cos\theta + h)$, *i.e.*

$$r^2\sin^2\theta + 2r(k\sin\theta - 2a\cos\theta) + k^2 - 4ah = 0.$$

The condition for equal roots, viz.

$$h\tan^2\theta - k\tan\theta + a = 0$$

gives the direction of the two tangents through (h, k) and, if θ_1, θ_2 be the roots, the product of the lengths of the two tangents

$$= r_1r_2 = \frac{k^2 - 4ah}{\sin\theta_1\sin\theta_2}.$$

Now (as in Art. 235, Ex. 1)

$$\frac{k}{h} = \tan\theta_1 + \tan\theta_2 = \frac{\sin(\theta_1 + \theta_2)}{\cos\theta_1\cos\theta_2},$$

and $\frac{a}{h} = \tan\theta_1\tan\theta_2$. $\therefore\ \frac{h - a}{h} = \frac{\cos(\theta_1 + \theta_2)}{\cos\theta_1\cos\theta_2}$.

Square and add; $\therefore \frac{(h-a)^2+k^2}{h^2} = \frac{1}{\cos^2\theta_1 \cos^2\theta_2}$, whence

$\sin\theta_1 \sin\theta_2 = \tan\theta_1 \tan\theta_2 \times \cos\theta_1 \cos\theta_2 \frac{a}{\sqrt{(h-a)^2+k^2}}$.

$$\therefore\ al_1 l_2 = (k^2 - 4ah)\sqrt{(h-a)^2+k^2},$$

where l_1, l_2 are the lengths of the tangents.

The equation to the tangent at (x', y') to the parabola

$$(y+k)^2 = 4a(x+h)$$

is $\quad yy' - 2a(x+x') + k(y+y') + k^2 - 4ah = 0$ [Art. 372],

whence the equation to the normal is

$$2a(y-y') + (y'+k)(x-x') = 0.$$

If this passes through the origin, $2ay' + x'(y'+k) = 0$.

Since (x', y') lies on the curve,

$$\therefore\ (y'+k)^2 - 4a(x'+h) = 0.$$

Eliminating x', $y'^3 + Ay'^2 + By' + k(k^2-4ah) = 0$.

$$\therefore\ k(k^2-4ah) = y_1y_2y_3 = \rho_1\rho_2\rho_3 \sin\phi_1 \sin\phi_2 \sin\phi_3,$$

where ρ_1, ρ_2, ρ_3 are the lengths of the normals and ϕ_1, ϕ_2, ϕ_3 the angles they make with the axis of x.

Also ϕ_1, ϕ_2, ϕ_3 are the roots of the equation

$$k = h\tan\phi - 2a\tan\phi - a\tan^3\phi,$$

i.e. $\quad a\sin^3\phi + (2a-h)\sin\phi\cos^2\phi + k\cos^3\phi = 0,$

i.e. $\quad \sin^2\phi\,\{(2a-h) + (h-a)\sin^2\phi\}^2 = k^2(1-\sin^2\phi)^3,$

i.e. $\quad \sin^6\phi\,\{(h-a)^2+k^2\} + A'\sin^4\phi + B'\sin^2\phi - k^2 = 0.$

$$\therefore\ \sin\phi_1 \sin\phi_2 \sin\phi_3 = \frac{k}{\sqrt{(h-a)^2+k^2}}.$$

$$\therefore\ \rho_1\rho_2\rho_3 = (k^2-4ah)\sqrt{(h-a)^2+k^2} = al_1l_2.$$

19. The normal at any point (x', y') of the parabola is

$$(x-x')y' + 2a(y-y') = 0,$$

and hence goes through (h, k) if $(h-x')y' + 2a(k-y') = 0$.

The feet of the normals from (h, k) therefore lie on the curves

$$xy + 2ay = 2ak + hy \ \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and

$$y^2 = 4ax. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Multiplying (1) by y, and (2) by x, and subtracting, it follows that a curve through the intersections of (1) and (2) is $2ay^2 = 2aky + hy^2 - 4ax^2$. Hence, etc.

20. The normal at $\left(x', \frac{c^2}{x'}\right)$ passes through (h, k) if

$$x'(h - x') = \frac{c^2}{x'}\left(k - \frac{c^2}{x'}\right). \quad \text{[Art. 332.]}$$

Hence the abscissae of the feet of the four normals are given by $x^4 - hx^3 + c^2kx - c^4 = 0$.

$$\therefore \frac{1}{4}(x_1 + x_2 + x_3 + x_4) = \frac{h}{4}, \text{ and } x_1x_2x_3x_4 + c^4 = 0.$$

Similarly $$\frac{1}{4}(y_1 + y_2 + y_3 + y_4) = \frac{k}{4}.$$

Also the chord joining (x_1, y_1) and (x_2, y_2) is perpendicular to the chord joining (x_3, y_3) and (x_4, y_4) if

$$\frac{c^2}{x_1x_2} \cdot \frac{c^2}{x_3x_4} + 1 = 0 \quad \text{[Art. 329]},$$

i.e. if $x_1x_2x_3x_4 + c^4 = 0$, which is true.

Confocal Conics.

415. Def. Two conics are said to be confocal when they have both foci common.

To find the equation to conics which are confocal with the ellipse $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{.......................} \quad (1).$$

All conics having the same foci have the same centre and axes.

The equation to any conic having the same centre and axes as the given conic is $$\frac{x^2}{A} + \frac{y^2}{B} = 1 \quad \text{.......................} \quad (2).$$

The foci of (1) are at the points $(\pm\sqrt{a^2 - b^2}, 0)$.

The foci of (2) are at the points $(\pm\sqrt{A - B}, 0)$.

These foci are the same if $A - B = a^2 - b^2$,

i.e. if $A - a^2 = B - b^2 = \lambda$ (say) $\therefore A = a^2 + \lambda$, and $B = b^2 + \lambda$.

The equation (2) then becomes

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

which is therefore the required equation, the quantity λ determining the particular confocal.

416. *For different values of* λ *to trace the conic given by the equation*

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \;\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

First, let λ be very great; then $a^2+\lambda$ and $b^2+\lambda$ are both very great and, the greater that λ is, the more nearly do these quantities approach to equality. A circle of infinitely great radius is therefore a confocal of the system.

Let λ gradually decrease from infinity to zero; the semi-major axis $\sqrt{a^2+\lambda}$ gradually decreases from infinity to a, and the semi-minor axis from infinity to b. When λ is positive, the equation (1) therefore represents an ellipse gradually decreasing in size from an infinite circle to the standard ellipse

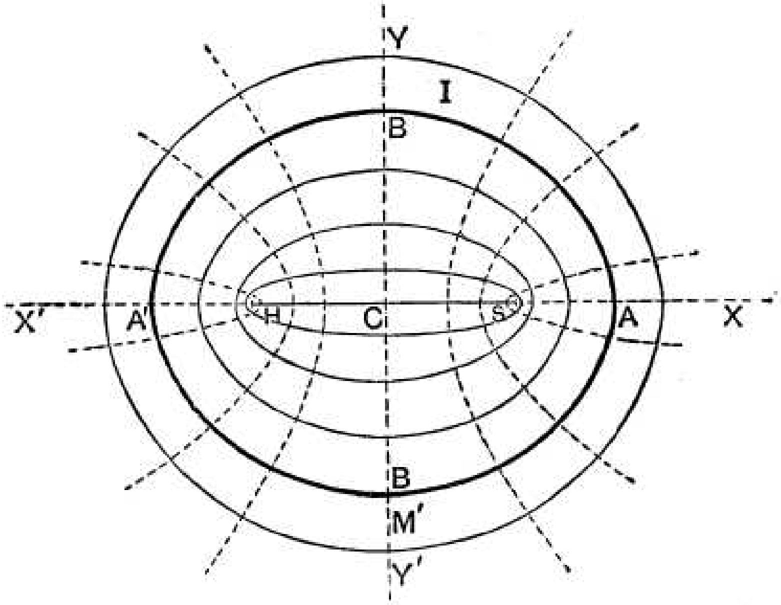
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

This latter ellipse is marked I in the figure.

Next, let λ gradually decrease from 0 to $-b^2$. The semi-major axis decreases from a to $\sqrt{a^2-b^2}$, and the semi-minor axis from b to 0.

For these values of λ the confocal is still an ellipse, which always lies within the ellipse I; it gradually decreases in size until, when λ is a quantity very slightly greater than $-b^2$, it is an extremely narrow ellipse very nearly coinciding with the line SH, which joins the two foci of all curves of the system.

Next, let λ be less than $-b^2$; the semi-minor axis $\sqrt{b^2+\lambda}$ now becomes imaginary and the curve is a hyperbola; when λ is very slightly less than $-b^2$ the curve is a hyperbola very nearly coinciding with the straight lines SX and HX'.



[As λ passes through the value $-b^2$ it will be no ed that the confocal instantaneously changes from the line-ellipse SH to the line-hyperbola SX and HX'.]

As λ gets less and less, the semi-transverse axis $\sqrt{a^2+\lambda}$ becomes less and less, so that the ends of the transverse axis of the hyperbola gradually approach to C, and the hyperbola widens out as in the figure.

When $\lambda = -a^2$, the transverse axis of the hyperbola vanishes, and the hyperbola degenerates into the infinite double line YOY'.

When λ is less than $-a^2$, both semi-axes of the conic become imaginary, and therefore the confocal becomes wholly imaginary.

417. *Through any point in the plane of a given conic there can be drawn two conics confocal with it; also one of these is an ellipse and the other a hyperbola.*

Let the equation to the given conic be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and let the given point be (f, g).

Any conic confocal with the given conic is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \text{ .... (1).}$$

If this go through the point (f, g), we have

$$\frac{f^2}{a^2+\lambda}+\frac{g^2}{b^2+\lambda}=1 \text{ .... (2).}$$

This is a quadratic equation to determine λ and therefore gives two values of λ.

Put $b^2+\lambda=\mu$, and hence

$$a^2+\lambda=\mu+a^2-b^2=\mu+a^2e^2.$$

The equation (2) then becomes

$$\frac{f^2}{\mu+a^2e^2}+\frac{g^2}{\mu}=1,$$

i.e.

$$\mu^2+\mu(a^2e^2-f^2-g^2)-g^2a^2e^2=0 \text{ .... (3).}$$

On applying the criterion of Art. 1 we at once see that the roots of this equation are both real.

Also, since its last term is negative, the product of these roots is negative, and therefore one value of μ is positive and the other is negative.

The two values of $b^2+\lambda$ are therefore one positive and the other negative. Similarly, the two values of $a^2+\lambda$ can be shewn to be both positive.

On substituting in (2) we thus obtain an ellipse and a hyperbola.

418. *Confocal conics cut at right angles.*

Let the confocals be

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1, \text{ and } \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1,$$

and let them meet at the point (x', y').

The equations to the tangents at this point are

$$\frac{xx'}{a^2+\lambda_1}+\frac{yy'}{b^2+\lambda_1}=1, \text{ and } \frac{xx'}{a^2+\lambda_2}+\frac{yy'}{b^2+\lambda_2}=1.$$

These cut at right angles if (Art 69)

$$\frac{x'^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+\frac{y'^2}{(b^2+\lambda_1)(b^2+\lambda_2)}=0 \;\ldots\ldots\; (1).$$

But, since (x', y') is a common point of the two confocals, we have

$$\frac{x'^2}{a^2+\lambda_1}+\frac{y'^2}{b^2+\lambda_1}=1, \quad\text{and}\quad \frac{x'^2}{a^2+\lambda_2}+\frac{y'^2}{b^2+\lambda_2}=1.$$

By subtraction, we have

$$x'^2\left(\frac{1}{a^2+\lambda_1}-\frac{1}{a^2+\lambda_2}\right)+y'^2\left(\frac{1}{b^2+\lambda_1}-\frac{1}{b^2+\lambda_2}\right)=0,$$

i.e.
$$\frac{x'^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+\frac{y'^2}{(b^2+\lambda_1)(b^2+\lambda_2)}=0 \ldots\ldots (2).$$

The condition (1) is therefore satisfied and hence the two confocals cut at right angles.

Cor. From equation (2) it is clear that the quantities $b^2+\lambda_1$ and $b^2+\lambda_2$ have opposite signs; for otherwise we should have the sum of two positive quantities equal to zero. Two confocals, therefore, which intersect, are one an ellipse and the other a hyperbola.

419. *One conic and only one conic, confocal with the conic* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *can be drawn to touch a given straight line.*

Let the equation to the given straight line be

$$x\cos\alpha+y\sin\alpha=p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Any confocal of the system is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The straight line (1) touches (2) if

$$p^2=(a^2+\lambda)\cos^2\alpha+(b^2+\lambda)\sin^2\alpha \text{ (Art. 264)},$$

i.e. if
$$\lambda=p^2-a^2\cos^2\alpha-b^2\sin^2\alpha.$$

This only gives one value for λ and therefore there is only one conic of the form (2) which touches the straight line (1).

Also $\lambda+a^2=p^2+(a^2-b^2)\sin^2\alpha=$ a real quantity. The conic is therefore real.

EXAMPLES XLVII

1. Prove that the difference of the squares of the perpendiculars drawn from the centre upon parallel tangents to two given confocal conics is constant.

2. Prove that the equation to the hyperbola drawn through the point of the ellipse, whose eccentric angle is α, and which is confocal with the ellipse, is

$$\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = a^2 - b^2.$$

3. Prove that the locus of the points lying on a system of confocal ellipses, which have the same eccentric angle α, is a confocal hyperbola whose asymptotes are inclined at an angle 2α.

4. Shew that the locus of the point of contact of tangents drawn from a given point to a system of confocal conics is a cubic curve, which passes through the given point and the foci.

If the given point be on the major axis, prove that the cubic reduces to a circle.

5. Prove that the locus of the feet of the normals drawn from a fixed point to a series of confocals is a cubic curve which passes through the given point and the foci of the confocals.

6. A point P is taken on the conic whose equation is

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1,$$

such that the normal at it passes through a fixed point (h, k); prove that P lies on the curve

$$\frac{x}{y-k} + \frac{y}{x-h} = \frac{a^2-b^2}{hy-kx}.$$

7. Two tangents at right angles to one another are drawn from a point P, one to each of two confocal ellipses; prove that P lies on a fixed circle. Shew also that the line joining the points of contact is bisected by the line joining P to the common centre.

8. From a given point a pair of tangents is drawn to each of a given system of confocals; prove that the normals at the points of contact meet on a straight line.

9. Tangents are drawn to the parabola $y^2 = 4x\sqrt{a^2-b^2}$, and on each is taken the point at which it touches one of the confocals

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1;$$

prove that the locus of such points is a straight line.

10. Normals are drawn from a given point to each of a system of confocal conics, and tangents at the feet of these normals; prove that the locus of the middle points of the portions of these tangents intercepted between the axes of the confocals is a straight line.

11. Prove that the locus of the pole of a given straight line with respect to a series of confocals is a straight line which is the normal to that confocal which the straight line touches.

12. A series of parallel tangents is drawn to a system of confocal conics; prove that the locus of the points of contact is a rectangular hyperbola.

Shew also that the locus of the vertices of these rectangular hyperbolas, for different directions of the tangents, is the curve $r^2 = c^2 \cos 2\theta$, where $2c$ is the distance between the foci of the confocals.

13. The locus of the pole of any tangent to a confocal with respect to any circle, whose centre is one of the foci, is obtained and found to be a circle; prove that, if the circle corresponding to each confocal be taken, they are all coaxal.

14. Prove that the two conics

$$ax^2 + 2hxy + by^2 = 1 \quad \text{and} \quad a'x^2 + 2h'xy + b'y^2 = 1$$

can be placed so as to be confocal, if

$$\frac{(a-b)^2 + 4h^2}{(ab-h^2)^2} = \frac{(a'-b')^2 + 4h'^2}{(a'b'-h'^2)^2}.$$

ANSWERS

11. The locus can be shewn to be a straight line which is perpendicular to the given straight line; also the given straight line touches one of the confocals and its pole with respect to that confocal is its point of contact; this point of contact therefore lies on the locus, which is therefore the normal.

14. As in Art. 366, use the Invariants of Art. 135.

SOLUTIONS/HINTS

1. The lines

$$x \cos \alpha + y \sin \alpha = p, \quad \text{and} \quad x \cos \alpha + y \sin \alpha = p'$$

will touch $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$, respectively,

if $$p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha,$$

and $p'^2 = (a^2 + \lambda) \cos^2 \alpha + (b^2 + \lambda) \sin^2 \alpha$, whence $p'^2 - p^2 = \lambda$.

2. The conic $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ will pass through the point $(a\cos\alpha,\ b\sin\alpha)$ if $\frac{a^2\cos^2\alpha}{a^2+\lambda}+\frac{b^2\sin^2\alpha}{b^2+\lambda}=1.$

Whence $\lambda=0$ or $-(a^2\sin^2\alpha+b^2\cos^2\alpha)$.

3. Eliminating λ from

$$x=\sqrt{a^2+\lambda}\cos\alpha,\quad y=\sqrt{b^2+\lambda}\sin\alpha,$$

we obtain
$$\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=a^2-b^2.$$

The asymptotes are $y\pm x\tan\alpha=0$.

4. The tangent at (h,k) to the conic $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ passes through the fixed point (f,g) if

$$\frac{fh}{a^2+\lambda}+\frac{gk}{b^2+\lambda}=1. \quad \dots\dots\dots\dots\dots\dots(1)$$

Also
$$\frac{h^2}{a^2+\lambda}+\frac{k^2}{b^2+\lambda}=1. \quad \dots\dots\dots\dots\dots\dots(2)$$

Therefore, on subtraction,

$$\frac{h(f-h)}{a^2+\lambda}+\frac{k(g-k)}{b^2+\lambda}=0,$$

giving
$$\lambda=-\frac{b^2h(f-h)+a^2k(g-k)}{h(f-h)+k(g-k)}.$$

Substitute for λ in (2), and we have

$$\frac{h}{f-h}-\frac{k}{g-k}=\frac{a^2-b^2}{h(f-h)+k(g-k)}.$$

The locus of (h,k) is thus the cubic curve

$$\frac{x}{f-x}-\frac{y}{g-y}=\frac{a^2-b^2}{fx+gy-x^2-y^2}.$$

If $g=0$, this locus becomes the circle

$$f(x^2+y^2)-x(a^2-b^2+f^2)+(a^2-b^2)f=0.$$

5. By Ex. XXXIII, 23 the straight line $lx + my = n$ is a normal to the conic

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \text{ if } \frac{a^2+\lambda}{l^2} + \frac{b^2+\lambda}{m^2} = \frac{(a^2-b^2)^2}{n^2}.$$

This gives only one value for λ.

6. The normal at the point (x_1, y_1) to the conic

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

passes through the fixed point (h, k) if

$$(a^2+\lambda)\frac{h-x_1}{x_1} = (b^2+\lambda)\frac{k-y_1}{y_1}. \quad \text{.........(1)}$$

Also $$\frac{x_1^2}{a^2+\lambda} + \frac{y_1^2}{b^2+\lambda} = 1. \quad \text{..................(2)}$$

(1) gives $$\frac{a^2+\lambda}{x_1(k-y_1)} = \frac{b^2+\lambda}{y_1(h-x_1)} = \frac{a^2-b^2}{x_1k-hy_1},$$

and then (2) gives $\frac{x_1}{k-y_1} + \frac{y_1}{h-x_1} = \frac{a^2-b^2}{kx_1-hy_1}$. Hence the required locus.

Now one of the confocals passes through the point (h, k) and the corresponding foot of the normal is clearly the point (h, k) itself.

Also one of the confocals is the line-ellipse joining the foci S and H, the normals to which are the lines joining (h, k) to S and H, so that two of the feet of normals are the points S and H.

7. Any tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$x\cos\alpha + y\sin\alpha = \sqrt{(a^2\cos^2\alpha + b^2\sin^2\alpha)},$$

and a perpendicular tangent to

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

is $$x\sin\alpha - y\cos\alpha = \sqrt{\{(a^2+\lambda)\sin^2\alpha + (b^2+\lambda)\cos^2\alpha\}}.$$

Square and add; $\therefore\ x^2 + y^2 = a^2 + b^2 + \lambda$.

The locus of P is therefore a circle.

Let $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$ ...(i), and $\dfrac{xx_2}{a^2+\lambda} + \dfrac{yy_2}{b^2+\lambda} = 1$ ...(ii), be the equations of the tangents.

Then the equation of CP will be

$$x\left(\frac{x_1}{a^2} - \frac{x_2}{a^2+\lambda}\right) + y\left(\frac{y_1}{b^2} - \frac{y_2}{b^2+\lambda}\right) = 0,$$

which passes through the middle point of the chord if

$$(x_1 + x_2)\left(\frac{x_1}{a^2} - \frac{x_2}{a^2+\lambda}\right) + (y_1 + y_2)\left(\frac{y_1}{b^2} - \frac{y_2}{b^2+\lambda}\right) = 0,$$

i.e. if
$$\frac{x_1 x_2}{a^2(a^2+\lambda)} + \frac{y_1 y_2}{b^2(b^2+\lambda)} = 0,$$

which is true since (i) and (ii) are at right angles.

8. If $lx + my = 1$ be the pole of (h, k) with regard to $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1$, then $l = \dfrac{h}{a^2+\lambda}$, and $m = \dfrac{k}{b^2+\lambda}$.

Therefore from Art. 412, Cor. 1,

$$\frac{x}{a^2-b^2} = \frac{h(b^2+\lambda-k^2)}{h^2(b^2+\lambda)+k^2(a^2+\lambda)}$$

and
$$\frac{y}{a^2-b^2} = -\frac{k(a^2+\lambda-h^2)}{h^2(b^2+\lambda)+k^2(a^2+\lambda)};$$

$$\therefore\ \frac{hx-ky}{a^2-b^2} = 1 - \frac{2h^2k^2}{h^2(b^2+\lambda)+k^2(a^2+\lambda)},$$

and
$$\frac{kx+hy}{a^2-b^2} = -\frac{hk(a^2-b^2-h^2+k^2)}{h^2(b^2+\lambda)+k^2(a^2+\lambda)}.$$

$$\therefore\ \frac{hx-ky-a^2+b^2}{kx+hy} = \frac{2hk}{a^2-b^2-h^2+k^2},$$

which is a straight line.

9. The line $\dfrac{x}{\sqrt{a^2+\lambda}}\cos\alpha + \dfrac{y}{\sqrt{b^2+\lambda}}\sin\alpha = 1$ will touch $y^2 = 4cx$, (where $c^2 = a^2 - b^2$), if

$$c\sqrt{a^2+\lambda}\sin^2\alpha + \cos\alpha\,(b^2+\lambda) = 0. \quad \ldots\ldots\ldots\text{(i)}$$

Also the point whose locus is required is

$$x_1 = \sqrt{a^2+\lambda}\cos\alpha, \ldots\text{(ii)} \qquad y_1 = \sqrt{b^2+\lambda}\sin\alpha.\ldots\text{(iii)}$$

From (i) and (ii), $c\,(a^2+\lambda)\left\{1 - \dfrac{x_1^2}{a^2+\lambda}\right\} + x_1(b^2+\lambda) = 0,$

or $$c\,(a^2+\lambda) - cx_1^2 + x_1(a^2+\lambda-c^2) = 0.$$

$$\therefore\ (x_1+c)(a^2+\lambda - cx_1) = 0.$$

The first factor gives $x = -c$ as the locus of (x_1, y_1).

10. The normal at the point, whose eccentric angle is α, to the conic $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1$ passes through (h, k), if

$$\sqrt{a^2+\lambda}\,h\sec\alpha - \sqrt{b^2+\lambda}\,k\,\text{cosec}\,\alpha = a^2 - b^2.$$

The point (x_1, y_1) whose locus is required is given by

$$2x_1 = \sqrt{a^2+\lambda}\sec\alpha, \quad 2y_1 = \sqrt{b^2+\lambda}\,\text{cosec}\,\alpha.$$

Hence, on eliminating λ and α, we have

$$2hx_1 - 2ky_1 = a^2 - b^2. \quad \therefore \text{ etc.}$$

11. The pole of $lx + my = 1$ with respect to

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

is the point $\quad x = l\,(a^2+\lambda), \quad y = m\,(b^2+\lambda).$

$$\therefore\ \frac{x}{l} - \frac{y}{m} = a^2 - b^2,$$

which is perpendicular to $lx + my = 1$. Also the point of contact of this line with a confocal is clearly on the locus, since it is the pole of the line with regard to that confocal. Hence, etc.

12. The tangent at (x', y') to the conic

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

will be parallel to $y = x\tan\alpha$ if $\tan\alpha = -\frac{(b^2+\lambda)\,x'}{(a^2+\lambda)\,y'}$, whence

$$\lambda = -\frac{a^2y'\sin\alpha + b^2x'\cos\alpha}{x'\cos\alpha + y'\sin\alpha}.$$

$\therefore\ a^2+\lambda = \frac{(a^2-b^2)\,x'\cos\alpha}{x'\cos\alpha+y'\sin\alpha}$, and $b^2+\lambda=\frac{(b^2-a^2)\,y'\sin\alpha}{x'\cos\alpha+y'\sin\alpha}$.

Also
$$\frac{x'^2}{a^2+\lambda}+\frac{y'^2}{b^2+\lambda}=1.$$

Hence, substituting for λ, we have

$$x'^2 - y'^2 + x'y'(\tan\alpha - \cot\alpha) = a^2 - b^2,$$

so that the required locus is

$$x^2 - y^2 + xy(\tan\alpha - \cot\alpha) = a^2 - b^2 \text{..........(1)}$$

The axes are given by

$$\tan 2\theta = \frac{\tan\alpha - \cot\alpha}{2}. \quad \text{[Art. 349.].........(2)}$$

Also, in polar coordinates, (1) is

$$r^2\cos 2\theta + \tfrac{1}{2}r^2\sin 2\theta\,(\tan\alpha - \cot\alpha) = a^2 - b^2\text{....(3)}$$

For the locus of the vertices, we eliminate α between (2) and (3), and obtain

$$r^2\cos 2\theta + \tfrac{1}{2}r^2\sin 2\theta\,.\,2\tan 2\theta = a^2 - b^2,$$

i.e.
$$r^2 = (a^2 - b^2)\cos 2\theta.$$

13. The polar of (x', y') with regard to the circle $x^2 - 2cx + y^2 = r^2$, is $x(x'-c) + yy' = r^2 + cx'$, where

$$c = \sqrt{a^2 - b^2}.$$

If this is identical with

$$\frac{x}{\sqrt{a^2+\lambda}\sec\alpha} + \frac{y}{\sqrt{b^2+\lambda}\,\text{cosec}\,\alpha} = 1,$$

then $\sqrt{a^2+\lambda}\sec\alpha = \frac{r^2+cx'}{x'-c}$, and $\sqrt{b^2+\lambda}\,\text{cosec}\,\alpha = \frac{r^2+cx'}{y'}$.

Hence the locus of (x', y') is the circle

$$(b^2+\lambda)x^2+(b^2+\lambda)y^2-2cx(a^2+\lambda+r^2)+c^2(a^2+\lambda)-r^4=0,$$

since $\qquad c^2=a^2-b^2.$

Whatever be the value of λ, this always passes through the intersection of the circle

$$b^2(x^2+y^2)-2cx(a^2+r^2)+a^2c^2-r^4=0,$$

and the point-circle $\quad x^2+y^2-2cx+c^2=0.$

These meet in two imaginary points lying on the straight line $x=\dfrac{c^2-r^2}{2c}=\dfrac{a^2-b^2-r^2}{2\sqrt{a^2-b^2}}$.

Hence, etc.

14. See Art. 366. $(\alpha^2-\beta^2)^2$ must be the same for both conics.

$$\therefore \frac{(a-b)^2+4h^2}{(ab-h^2)^2}=\frac{(a'-b')^2+4h'^2}{(a'b'-h'^2)^2}.$$

Curvature.

420. Circle of Curvature. Def. If P, Q, and R be any three points on a conic section, one circle and only one circle can be drawn to pass through them. Also this circle is completely determined by the three points.

Let now the points Q and R move up to, and ultimately coincide with, the point P; then the limiting position of the above circle is called the circle of curvature at P; also the radius of this circle is called the radius of curvature at P, and its centre is called the centre of curvature at P.

421. Since the circle of curvature at P meets the conic in three coincident points at P, it will cut the curve in one other point P'. The line PP' which is the line joining P to the other point of intersection of the conic and the circle of curvature is called the common chord of curvature.

We shewed, in Art. 400, that, if a circle and a conic intersect in four points, the line joining one pair of points of intersection and the line joining the other pair are equally inclined to the axis. In our case, one pair of points is two of the coincident points at P, and the line joining them therefore the tangent at P; the other pair of points is the third point at P and the point P', and the line joining them the chord of curvature PP'. Hence *the tangent at P and the chord of curvature PP' are, in any conic, equally inclined to the axis.*

422. *To find the equation to the circle of curvature and the length of the radius of curvature at any point $(at^2, 2at)$ of the parabola $y^2 = 4ax$.*

If $S = 0$ be the equation to a conic, $T = 0$ the equation to the tangent at the point P, whose coordinates are at^2 and $2at$, and $L = 0$ the equation to any straight line passing through P, we know, by Art. 384, that $S + \lambda . L . T = 0$ is the equation to the conic section passing through three coincident points at P and through the other point in which $L = 0$ meets $S = 0$.

If λ and L be so chosen that this conic is a circle, it will be the circle of curvature at P, and, by the last article, we know that $L = 0$ will be equally inclined to the axis with $T = 0$.

In the case of a parabola

$$S \equiv y^2 - 4ax, \text{ and } T \equiv ty - x - at^2. \quad (\text{Art. } 229.)$$

Also the equation to a line through $(at^2, 2at)$ equally inclined with $T = 0$ to the axis is

$$t(y - 2at) + x - at^2 = 0,$$

so that

$$L \equiv ty + x - 3at^2.$$

The equation to the circle of curvature is therefore

$$y^2 - 4ax + \lambda (ty - x - at^2)(ty + x - 3at^2) = 0,$$

where

$$1 + \lambda t^2 = -\lambda, \quad i.e. \ \lambda = -\frac{1}{1 + t^2}.$$

On substituting this value of λ, we have, as the required equation,

$$x^2 + y^2 - 2ax(3t^2 + 2) + 4ayt^3 - 3a^2t^4 = 0,$$

i.e. $$[x - a(2 + 3t^2)]^2 + [y + 2at^3]^2 = 4a^2(1 + t^2)^3.$$

The circle of curvature has therefore its centre at the point $(2a + 3at^2, -2at^3)$ and its radius equal to

$$2a(1 + t^2)^{\frac{3}{2}}.$$

Cor. If S be the focus, we have SP equal to $a + at^2$, so that the radius of curvature is equal to $\dfrac{2 \,.\, SP^{\frac{3}{2}}}{\sqrt{a}}$.

423. *To find the equation to the circle of curvature at the point $P(a\cos\phi,\ b\sin\phi)$ of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.*

The tangent at the point P is

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1.$$

The straight line passing through P and equally inclined with this line to the axis is

$$\frac{\cos\phi}{a}(x - a\cos\phi) - \frac{\sin\phi}{b}(y - b\sin\phi) = 0,$$

i.e. $$\frac{x}{a}\cos\phi - \frac{y}{b}\sin\phi - \cos 2\phi = 0.$$

The equation to the circle of curvature is therefore of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \lambda\left[\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi - 1\right]\left[\frac{x}{a}\cos\phi - \frac{y}{b}\sin\phi - \cos 2\phi\right] = 0 \;\ldots\ldots\; (1).$$

Since it is a circle, the coefficients of x^2 and y^2 must be equal, so that

$$\frac{1}{a^2} + \lambda\frac{\cos^2\phi}{a^2} = \frac{1}{b^2} - \lambda\frac{\sin^2\phi}{b^2},$$

and therefore $$\lambda = \frac{a^2 - b^2}{b^2\cos^2\phi + a^2\sin^2\phi}.$$

On substitution in (1), the equation to the circle of curvature is

$$(b^2 \cos^2 \phi + a^2 \sin^2 \phi)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$
$$+ (a^2 - b^2)\left[\frac{x^2}{a^2}\cos^2\phi - \frac{y^2}{b^2}\sin^2\phi - \frac{x\cos\phi}{a}(1+\cos 2\phi) + \frac{y \sin\phi}{b}(1-\cos 2\phi) + \cos 2\phi\right] = 0,$$

i.e.
$$x^2 + y^2 - (a^2 - b^2)\left[\frac{2x\cos^3\phi}{a} - \frac{2y\sin^3\phi}{b}\right] + a^2(\cos^2\phi - 2\sin^2\phi) - b^2(2\cos^2\phi - \sin^2\phi) = 0.$$

The equation to the circle of curvature is then

$$\left\{x - \frac{a^2-b^2}{a}\cos^3\phi\right\}^2 + \left\{y + \frac{a^2-b^2}{b}\sin^3\phi\right\}^2$$
$$= (a^2-b^2)^2\left\{\frac{\cos^6\phi}{a^2} + \frac{\sin^6\phi}{b^2}\right\} - a^2\{\cos^2\phi - 2\sin^2\phi\} + b^2\{2\cos^2\phi - \sin^2\phi\}$$
$$= \frac{(a^2\sin^2\phi + b^2\cos^2\phi)^3}{a^2b^2}, \text{ after some reduction.}$$

The centre is therefore the point whose coordinates are $\left(\frac{a^2-b^2}{a}\cos^3\phi,\ -\frac{a^2-b^2}{b}\sin^3\phi\right)$ and whose radius is

$$\frac{(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{3}{2}}}{ab}.$$

Cor. 1. If CD be the semi-diameter which is conjugate to CP, then D is the point $(90° + \phi)$, so that its coordinates are $-a\sin\phi$ and $b\cos\phi$. (Art. 285.)

Hence $$CD^2 = a^2\sin^2\phi + b^2\cos^2\phi,$$

and therefore the radius of curvature $\rho = \frac{CD^3}{ab}$.

Cor. 2. If the point P have as coordinates x' and y' then, since $x' = a\cos\phi$ and $y' = b\sin\phi$, the equation to the circle of curvature is

$$\left(x - \frac{a^2-b^2}{a^4}x'^3\right)^2 + \left(y + \frac{a^2-b^2}{b^4}y'^3\right)^2 = \frac{(a^2+b^2-x'^2-y'^2)^3}{a^2b^2}.$$

Cor. 3. In a similar manner it may be shewn that the equation to the circle of curvature at any point (x', y') of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\left(x-\frac{a^2+b^2}{a^4}x'^3\right)^2+\left(y+\frac{a^2+b^2}{b^4}y'^3\right)^2=\frac{(a^2-b^2-x'^2-y'^2)^3}{-a^2b^2}$$
$$=\frac{(x'^2+y'^2-a^2+b^2)^3}{a^2b^2}.$$

424. If a circle and an ellipse intersect in four points, the sum of their eccentric angles is equal to an even multiple of π. [Page 235, Ex. 18.]

If then the circle of curvature at a point P, whose eccentric angle is θ, meet the curve again in Q, whose eccentric angle is ϕ, three of these four points coincide at P, so that three of these eccentric angles are equal to θ, whilst the fourth is equal to ϕ. We therefore have

$$3\theta+\phi=\text{an even multiple of } \pi=2n\pi.$$

Hence, if ϕ be supposed given, *i.e.* if Q be given, we have
$$\theta=\frac{2n\pi-\phi}{3}.$$

Giving n in succession the values 1, 2, and 3, we see that θ equals $\frac{2\pi-\phi}{3}$, $\frac{4\pi-\phi}{3}$, or $\frac{6\pi-\phi}{3}$.

Hence the circles of curvature at the points, whose eccentric angles are $\frac{2\pi-\phi}{3}$, $\frac{4\pi-\phi}{3}$, and $\frac{6\pi-\phi}{3}$, all pass through the point whose eccentric angle is ϕ.

Also since

$$\frac{2\pi-\phi}{3}+\frac{4\pi-\phi}{3}+\frac{6\pi-\phi}{3}+\phi=4\pi=\text{an even multiple of } \pi,$$

we see that the points $\frac{2\pi-\phi}{3}$, $\frac{4\pi-\phi}{3}$, $\frac{6\pi-\phi}{3}$, and ϕ all lie on a circle.

Hence *through any point Q on an ellipse can be drawn three circles which are the circles of curvature at three points* P_1, P_2, *and* P_3. *Also the four points* P_1, P_2, P_3, *and* Q *all lie on another circle.*

425. Evolute of a Curve. The locus of the centres of curvature at different points of a curve is called the evolute of the curve.

426. *Evolute of the parabola* $y^2 = 4ax$.

Let $(\bar{x}, \bar{y})$ be the centre of curvature at the point $(at^2, 2at)$ of this curve.

Then $\bar{x} = a(2 + 3t^2)$ and $\bar{y} = -2at^3$. (Art. 422.)

$$\therefore \ (\bar{x} - 2a)^3 = 27a^3t^6 = \tfrac{27}{4}a\bar{y}^2,$$

i.e. the locus of the centre of curvature is the curve

$$27ay^2 = 4(x - 2a)^3.$$

This curve meets the axis of x in the point $(2a, 0)$.

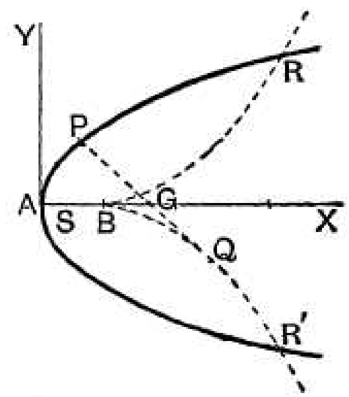
It also meets the parabola where

$$27a^2x = (x - 2a)^3,$$

i.e. where $x = 8a$,

and therefore

$$y = \pm 4\sqrt{2}a.$$



Hence it meets the parabola at the points

$$(8a, \ \pm 4\sqrt{2}a).$$

The curve is called a semi-cubical parabola and could be shewn to be of the shape of the dotted curve in the figure.

427. *Evolute of the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

If $(\bar{x}, \bar{y})$ be the centre of curvature corresponding to the point $(a\cos\phi, b\sin\phi)$ of the ellipse, we have

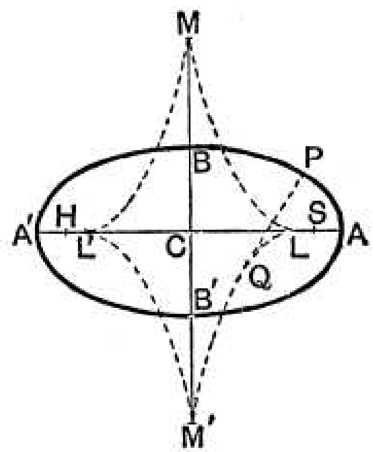
$$\bar{x} = \frac{a^2 - b^2}{a}\cos^3\phi \ \text{ and } \ \bar{y} = -\frac{a^2 - b^2}{b}\sin^3\phi.$$

Hence

$$(a\bar{x})^{\frac{2}{3}} + (b\bar{y})^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}} \{\cos^2 \phi + \sin^2 \phi\} = (a^2 - b^2)^{\frac{2}{3}}.$$

Hence the locus of the point $(\bar{x}, \bar{y})$ is the curve

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$



This curve could be shewn to be of the shape shewn in the figure where

$$CL = CL' = \frac{a^2 - b^2}{a},$$

and $$CM = CM' = \frac{a^2 - b^2}{b}.$$

The equation to the evolute of the hyperbola would be found to be

$$(ax)^{\frac{2}{3}} - (by)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}.$$

428. Contact of different orders. If two conics, or curves, touch, *i.e.* have two coincident points in common they are said to have contact of the first order. The tangent to a conic therefore has contact of the first order with it.

If two conics have three coincident points in common, they are said to have contact of the second order. The circle of curvature of a conic therefore has contact of the second order with it.

If two conics have four coincident points in common, they are said to have contact of the third order. No conics, which are not coincident, can have more than four coincident points; for a conic is completely determined if five points on it be given. Contact of the third order is therefore all that two conics can have, and then they are said to osculate one another.

Since a circle is completely determined when three points on it are given we cannot, in general, obtain a circle to have contact of a higher order than the second with a given conic. The circle of curvature is therefore often called the osculating circle.

In general, one curve osculates another when it has the highest possible order of contact with the second curve.

429. *Equation to a conic osculating another conic.*

If $S=0$ be the equation to a conic and $T=0$ the tangent at any point of it, the conic $S=\lambda T^2$ passes through four coincident points of $S=0$ at the point where $T=0$ touches it. (Art. 385, IV.)

Hence $S=\lambda T^2$ is the equation to the required osculating conic.

Ex. The equation of any conic osculating the conic

$$ax^2+2hxy+by^2-2fy=0 \quad \text{.........................(1)}$$

at the origin is

$$ax^2+2hxy+by^2-2fy+\lambda y^2=0 \quad \text{...................(2).}$$

For the tangent to (1) at the origin is $y=0$.

If (2) be a parabola, we have $h^2=a(b+\lambda)$, so that its equation is

$$(ax+hy)^2=2afy.$$

If (2) be a rectangular hyperbola, we have $a+b+\lambda=0$, and the equation to the osculating rectangular hyperbola is

$$a(x^2-y^2)+2hxy-2fy=0.$$

EXAMPLES XLVIII

1. If the normal at a point P of a parabola meet the directrix in L, prove that the radius of curvature at P is equal to $2PL$.

2. If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola, prove that

$$\rho_1^{-\frac{2}{3}}+\rho_2^{-\frac{2}{3}}=(2a)^{-\frac{2}{3}}.$$

3. PQ is the common chord of the parabola and its centre of curvature at P; prove that the ordinate of Q is three times that of P, and that the locus of the middle point of PQ is another parabola.

4. If ρ and ρ' be the radii of curvature at the ends, P and D, of conjugate diameters of the ellipse, prove that

$$\rho^{\frac{2}{3}}+\rho'^{\frac{2}{3}}=\frac{a^2+b^2}{(ab)^{\frac{2}{3}}},$$

and that the locus of the middle point of the line joining the centres of curvature at P and D is

$$(ax+by)^{\frac{2}{3}}+(ax-by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}.$$

5. O is the centre of curvature at any point of an ellipse, and Q and R are the feet of the other normals drawn from O; prove that the locus of the intersection of tangents at Q and R is $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$, and that the line QR is a normal to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{a^2b^2}{(a^2-b^2)^2}.$$

6. If four normals be drawn to an ellipse from any point on the evolute, prove that the locus of the centre of the rectangular hyperbola through their feet is the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$$

7. In general, prove that there are six points on an ellipse the circles of curvature at which pass through a given point O, not on the ellipse. If O be on the ellipse, why is the number of circles of curvature passing through it only four?

8. The circles of curvature at three points of an ellipse meet in a point P on the curve. Prove that (1) the normals at these three points meet on the normal drawn at the other end of the diameter through P, and (2) the locus of these points of intersection for different positions of P is the ellipse

$$4(a^2x^2+b^2y^2)=(a^2-b^2)^2.$$

9. Prove that the equation to the circle of curvature at any point (x', y') of the rectangular hyperbola $x^2-y^2=a^2$ is

$$a^2(x^2+y^2)-4xx'^3+4yy'^3+3a^2(x'^2+y'^2)=0.$$

10. Shew that the equation to the chord of curvature of the rectangular hyperbola $xy=c^2$ at the point "t" is $ty+t^3x=c(1+t^4)$, and that the centre of curvature is the point

$$\left(c\frac{1+3t^4}{2t^3},\ c\frac{3+t^4}{2t}\right).$$

Prove also that the locus of the pole of the chord of curvature is the curve $r^2=2c^2\sin 2\theta$.

11. PQ is the normal at any point of a rectangular hyperbola and meets the curve again in Q; the diameter through Q meets the curve again in R; shew that PR is the chord of curvature at P, and that PQ is equal to the diameter of curvature at P.

12. Prove that the equation to the circle of curvature of the conic $ax^2+2hxy+by^2=2y$ at the origin is

$$a(x^2+y^2)=2y.$$

13. If two confocal conics intersect, prove that the centre of curvature of either curve at a point of intersection is the pole of the tangent at that point with regard to the other curve.

14. Shew that the equation to the parabola, having contact of the third order with the rectangular hyperbola $xy=c^2$ at the point

$$\left(ct,\ \frac{c}{t}\right),$$

is
$$(x-yt^2)^2-4ct(x+yt^2)+8c^2t^2=0.$$

Prove also that its directrix bisects, and is perpendicular to, the radius vector of the hyperbola from the centre to the point of contact.

15. Prove that the equation to the parabola, which passes through the origin and has contact of the second order with the parabola $y^2=4ax$ at the point $(at^2,\ 2at)$, is

$$(4x-3ty)^2+4at^2(3x-2ty)=0.$$

16. Prove that the equation to the rectangular hyperbola, having contact of the third order with the parabola $y^2=4ax$ at the point $(at^2,\ 2at)$, is

$$x^2-2txy-y^2+2ax(2+3t^2)-2at^3y+a^2t^4=0.$$

Prove also that the locus of the centres of these hyperbolas is an equal parabola having the same axis and directrix as the original parabola.

17. Through every point of a circle is drawn the rectangular hyperbola of closest contact; prove that the centres of all these hyperbolas lie on a concentric circle of twice its radius.

18. A rectangular hyperbola is drawn to have contact of the third order with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; find its equation and prove that the locus of its centre is the curve

$$\left(\frac{x^2+y^2}{a^2+b^2}\right)^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}.$$

ANSWERS

5. Two of the normals drawn from O coincide, since it is a centre of curvature. The straight line $l_1x+m_1y=1$ (Art. 412) is therefore a tangent to the ellipse at some point ϕ and hence, by Art. 412, the equation to QR can be found in terms of ϕ.

SOLUTIONS/HINTS

1. Let P be the point t, so that $\tan P\hat{G}N = t$; then

$$2PL = 2 \,.\, XN \,.\, \sec P\hat{G}N = (2a + 2at^2)\sqrt{1+t^2}$$
$$= 2a\,(1+t^2)^{\frac{3}{2}} = \rho, \text{ by Art. 422.}$$

2. $\rho_1 = 2a\,(1+t^2)^{\frac{3}{2}}$, and $\rho_2 = 2a\left(1+\frac{1}{t^2}\right)^{\frac{3}{2}}$.

[Art. 233, Cor.]

$$\therefore\ \rho_1^{-\frac{2}{3}} + \rho_2^{-\frac{2}{3}} = (2a)^{-\frac{2}{3}}\left\{\frac{1}{1+t^2} + \frac{t^2}{1+t^2}\right\} = (2a)^{-\frac{2}{3}}.$$

3. Let P and Q be the points t_1 and t_2. Then since

$$t_1 y = x + at_1^2, \text{ and } y\,(t_1+t_2) = 2x + 2at_1t_2,$$

are equally inclined to the axis (Art. 421),

$$\therefore\ \frac{1}{t_1} + \frac{2}{t_1+t_2} = 0; \quad \therefore\ 3t_1 + t_2 = 0; \quad \therefore\ 3y_1 + y_2 = 0.$$

The coordinates of the middle point of PQ are

$$y = a\,(t_1+t_2), \quad 2x = a\,(t_1^2 + t_2^2).$$
$$\therefore\ y^2 - 2ax = 2a^2t_1t_2 = -\,6a^2t_1^2.$$

Also
$$y = \frac{y_1+y_2}{2} = -\,y_1 = -\,2at_1;$$
$$\therefore\ y^2 - 2ax = -\frac{6y^2}{4}, \text{ i.e. } 5y^2 = 4ax.$$

4. We have $ab\rho = CD^3$, and $ab\rho' = CP^3$;

$$\therefore\ (ab)^{\frac{2}{3}}\,(\rho^{\frac{2}{3}} + \rho'^{\frac{2}{3}}) = CP^2 + CD^2 = a^2 + b^2.$$

The coordinates of the middle point of the line joining the centres of curvature of P and D (*i.e.* the points ϕ and $90° + \phi$), are, by Art. 423,

$$2ax = (a^2 - b^2)\,(\cos^3\phi - \sin^3\phi),$$

and
$$-\,2by = (a^2 - b^2)\,(\sin^3\phi + \cos^3\phi).$$

$\therefore\ ax + by = -(a^2 - b^2)\sin^3\phi$, and $ax - by = (a^2 - b^2)\cos^3\phi$.
Hence, etc.

5. See Art. 412.

If $l_1x + m_1y = 1$ is the tangent at ϕ, then

$$l_1 = \frac{\cos\phi}{a}, \text{ and } m_1 = \frac{\sin\phi}{b}.$$

Therefore by equation (9), $l_2 = -\dfrac{1}{a\cos\phi}$, $m_2 = -\dfrac{1}{b\sin\phi}$.

The pole of the line $\dfrac{x}{a\cos\phi} + \dfrac{y}{b\sin\phi} = -1$, is

$$x = -\frac{a}{\cos\phi},\ y = -\frac{b}{\sin\phi}, \text{ whence } \frac{a^2}{x^2} + \frac{b^2}{y^2} = 1.$$

The equation of any normal to the curve

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = \frac{a^2b^2}{(a^2 - b^2)^2}$$

is

$$\frac{ab^2x}{(a^2 - b^2)\cos\alpha} - \frac{a^2by}{(a^2 - b^2)\sin\alpha} = -\frac{a^2b^2}{a^2 - b^2},$$

or

$$\frac{x}{a\cos\alpha} - \frac{y}{b\sin\alpha} = -1,$$

which becomes $\dfrac{x}{a\cos\phi} + \dfrac{y}{b\sin\phi} = -1$, on putting $\alpha = -\phi$,

i.e.

$$l_2x + m_2y = 1.$$

Hence, etc.

6. See Art. 414, Ex. 1, equation (2).

The centre of the curve is given by

$$(a^2 - b^2)\,y + b^2k = 0, \text{ and } (a^2 - b^2)\,x - a^2h = 0.$$

Also $(ah)^{\frac{2}{3}} + (bk)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$. [Art. 427.]

Eliminating h and k, we have $\left(\dfrac{x}{a}\right)^{\frac{2}{3}} + \left(\dfrac{y}{b}\right)^{\frac{2}{3}} = 1$.

7. If the circle of curvature at the point "ϕ" passes through the given point (x_1, y_1), the equation of Art. 423 gives, in general, six values of ϕ.

If a_1, a_2, a_3, a_4, a_5 be five *consecutive* points on the curve, then the circle of curvature at a_2 passes through a_1, a_2 and a_3; that at a_4 goes through a_3, a_4, a_5; hence three consecutive circles of curvature, viz. those at a_2, a_3 and a_4, go through a_3; and these three circles ultimately coincide with the circle of curvature at a_3 and so only count as one.

8. See Art. 424.

The normals at the points

$$\tfrac{1}{3}(2\pi-\phi),\ \tfrac{1}{3}(4\pi-\phi),\ \tfrac{1}{3}(6\pi-\phi) \text{ and } \pi+\phi$$

meet in a point, since the sum of these angles $=5\pi$.

[Art. 413.]

Since the eccentric angles of these points differ by $\frac{2\pi}{3}$, the equation of a chord joining any pair is of the form

$$\frac{2x\cos\theta}{a}+\frac{2y\sin\theta}{b}=1.$$

Then see Ex. XLVI, 10.

9. Putting $a^2=b^2$ in the equation of Art. 423, Cor. 3, we obtain

$$\left(x-\frac{2x'^3}{a^2}\right)^2+\left(y+\frac{2y'^3}{a^2}\right)^2=\frac{(x'^2+y'^2)^3}{a^4},$$

which reduces to the given equation.

10. If the tangent at t and the chord of curvature are inclined at ϕ and ϕ' to the axis of x; then, since they are equally inclined to the line $x=y$, we have

$$\phi+\phi'=\pi+2\frac{\pi}{4} \quad \text{(See Art. 72)} \quad =\frac{3\pi}{2}.$$

$$\therefore \tan\phi'=\tan\left(\frac{3\pi}{2}-\phi\right)=+\cot\phi=-t^2.$$

Therefore the equation to the chord of curvature is

$$y-\frac{c}{t}=-t^2(x-ct), \text{ or } ty+t^3x=c(1+t^4).$$

The equation of the circle of curvature is

$$\left(\frac{x}{t}+yt-2c\right)\{ty+t^3x-c(1+t^4)\}+\lambda(xy-c^2)=0,$$

where $$\lambda=-(1+t^4).$$

Hence the circle of curvature is

$$(x^2+y^2)t^2-xc\left(\frac{1}{t}+3t^3\right)-cy(3t+t^5)-3\lambda c^2=0,$$

so that $$2\bar{x}=c\frac{1+3t^4}{t^3},\ 2\bar{y}=\frac{3+t^4}{t}.$$

The lines $xk+yh=2c^2$, and $xt^3+yt=c(1+t^4)$ are identical if $\frac{k}{t^3}=\frac{h}{t}=\frac{2c}{1+t^4}$.

$$\therefore\ k=t^2h,\text{ and }\frac{hk}{t^4}=\frac{4c^2}{(1+t^4)^2};$$

$$\therefore\ hk\frac{h^2}{k^2}=\frac{4c^2h^4}{(k^2+h^2)^2}.$$

Therefore the equation of the required locus is $(x^2+y^2)^2=4c^2xy$, or in polar coordinates, $r^2=2c^2\sin 2\theta$.

11. Let P be the point t. Then Q is the point $-\frac{1}{t^3}$ (Ex. XXXVIII, 14), and therefore R is the point $\frac{1}{t^3}$.

Therefore the equation to PR is

$$\frac{x-ct}{c\frac{1}{t^3}-ct}=\frac{y-\frac{c}{t}}{ct^3-\frac{c}{t}},$$

or $t^3x+ty=c(1+t^4)$, and this, by the last example, is the chord of curvature at P.

Again $$PQ=c\left\{\left(t+\frac{1}{t^3}\right)^2+\left(\frac{1}{t}+t^3\right)^2\right\}^{\frac{1}{2}}=c\left(t^2+\frac{1}{t^2}\right)^{\frac{3}{2}},$$

and $$2\rho=2\frac{CP^3}{a^2}\text{ by Art. 423, Cor. 3,}$$

$$=2\frac{c^3\left(t^2+\frac{1}{t^2}\right)^{\frac{3}{2}}}{2c^2}\text{ (since }a^2=2c^2,)=PQ.$$

12. The inclination θ of the axis of the curve to the axis of x is given by

$$\tan 2\theta = \frac{2h}{a-b}. \quad [\text{Art. } 349.]$$

Let the inclination of the chord of curvature be θ_1; since it and the tangent at the origin, viz. the axis of x, are equally inclined to the axis of the curve, $\therefore \theta_1 = 2\theta$.

Therefore the chord of curvature is $y = x \tan \theta_1 = \frac{2hx}{a-b}$, i.e. $2hx - (a-b)y = 0$, and the circle of curvature is

$$y[2hx - (a-b)y] + \lambda[ax^2 + 2hxy + by^2 - 2y] = 0.$$

Since this is a circle, $\therefore \lambda = -1$, etc.

13. For the pole of $\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$, with regard to $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = a^2 - b^2$ [XLVII, Ex. 2], is given by

$$\frac{x'}{\cos^2\alpha} = (a^2-b^2)\frac{\cos\alpha}{a}, \text{ and } \frac{y'}{\sin^2\alpha} = -(a^2-b^2)\frac{\sin\alpha}{b};$$

and this, by Art. 423, is the centre of curvature of the first curve corresponding to the point α.

14. The conic $\left(\frac{x}{t} + yt - 2c\right)^2 + \lambda(xy - c^2) = 0$ (Art. 429) is a parabola if $\lambda = -4$. Hence, etc.

The equation of the directrix is (Art. 390)

$$2xt^3 + 2yt = c(1+t^4),$$

which is perpendicular to $x - yt^2 = 0$ and passes through

$$\left(\frac{ct}{2}, \frac{c}{2t}\right).$$

15. The conic

$$(x - ty + at^2)\{y - 2at + m(x - at^2)\} + \lambda(y^2 - 4ax) = 0$$

passes through (0, 0) if $m = -\frac{2}{t}$, and is a parabola if

$$(mt-1)^2 = 4m(\lambda - t).$$

Whence $$\lambda = -\frac{t}{8}.$$

Substitute for m and λ, etc.

16. The conic $(x - ty + at^2)^2 + \lambda(y^2 - 4ax) = 0$, is a rectangular hyperbola if $\lambda + 1 + t^2 = 0$.

Substitute for λ, etc.

The centre is given by

$$x - ty + a(2 + 3t^2) = 0, \quad \ldots\ldots\ldots\ldots\text{(i)}$$

and $$tx + y + at^3 = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Multiply (i) by t and (ii) by 3, and subtract;

$$\therefore\ t^2y + 2t(x - a) + 3y = 0. \quad \ldots\ldots\ldots\ldots\text{(iii)}$$

Eliminating t between (i) and (iii);

$$\therefore\ y^2(x - 7a)^2 + \{3y^2 + 2(x + 2a)(x - a)\}\{y^2 + 6a(x - a)\} = 0,$$

which reduces to $\{y^2 + (x - a)^2\}\{y^2 + 4a(x + 2a)\} = 0$.

The required locus is therefore $y^2 + 4a(x + 2a) = 0$.

17. The conic

$$(x\cos\theta + y\sin\theta - a)^2 + \lambda(x^2 + y^2 - a^2) = 0$$

is a rectangular hyperbola if $2\lambda + 1 = 0$.

Hence, substituting, we obtain

$$x^2\cos 2\theta + 2xy\sin 2\theta - y^2\cos 2\theta - 4ax\cos\theta - 4ay\sin\theta + 3a^2 = 0.$$

The centre is given by

$x\cos 2\theta + y\sin 2\theta = 2a\cos\theta$, and $x\sin 2\theta - y\cos 2\theta = 2a\sin\theta$.

Square and add; $\therefore\ x^2 + y^2 = 4a^2$.

18. The conic

$$\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + \left(\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi - 1\right)^2 = 0$$

is a rectangular hyperbola if

$$\lambda(a^2 + b^2) + b^2\cos^2\phi + a^2\sin^2\phi = 0.$$

On substituting for λ, the equations for the centre will be

$$\frac{x}{a^2+b^2}\cos 2\phi + \frac{y}{2ab}\sin 2\phi = \frac{\cos\phi}{a}, \quad \text{.........(i)}$$

and

$$-\frac{y}{a^2+b^2}\cos 2\phi + \frac{x}{2ab}\sin 2\phi = \frac{\sin\phi}{b}, \quad \text{......(ii)}$$

These give

$$\frac{x^2+y^2}{a^2+b^2}\cos 2\phi = \frac{x\cos\phi}{a} - \frac{y\sin\phi}{b}, \quad \text{......(iii)}$$

and

$$\frac{x^2+y^2}{2ab}\sin 2\phi = \frac{y\cos\phi}{a} + \frac{x\sin\phi}{b}. \quad \text{......(iv)}$$

On division, these give

$$\frac{\cos 2\phi}{a^2+b^2}(by\cos\phi + ax\sin\phi) = \frac{\sin 2\phi}{2ab}(bx\cos\phi - ay\sin\phi).$$

$$\therefore\ ay\left[a^2\sin\phi\sin 2\phi + b^2\cos\phi\cos 2\phi + b^2\cos\phi\right]$$
$$= bx\left[a^2\sin\phi - a^2\sin\phi\cos 2\phi + b^2\sin 2\phi\cos\phi\right].$$

$$\therefore\ ay\cos\phi = bx\sin\phi.$$

On substituting in (iii) and (iv), we have

$$\frac{x^2+y^2}{a^2+b^2}\cos\phi = \frac{x}{a}, \text{ and } \frac{x^2+y^2}{a^2+b^2}\sin\phi = \frac{y}{b}. \quad \therefore \text{ etc.}$$

ENVELOPES

430. Consider any point P on a circle whose centre is O and whose radius is a. The straight line through P at right angles to OP is a tangent to the circle at P. Conversely, if through O we draw any straight line OP of length a, and if through the end P we draw a straight line perpendicular to OP, this latter straight line touches, or envelopes, a circle of radius a and centre O, and this circle is said to be the envelope of the straight lines drawn in this manner.

Again, if S be the focus of a parabola, and PY be the tangent at any point P of it meeting the tangent at the vertex in the point Y, then we know (Art. 211, δ) that SYP is a right angle. Conversely, if S be joined to any point Y on a given line, and a straight line be drawn through Y perpendicular to SY, this line, so drawn, always touches, or envelopes, a parabola whose focus is S and such that the given line is the tangent at its vertex.

431. Envelope. Def. The curve which is touched by each of a series of lines, which are all drawn to satisfy some given condition, is called the Envelope of these lines.

As an example, consider the series of straight lines which are drawn so that each of them cuts off from a pair of fixed straight lines a triangle of constant area.

We know (Art. 330) that any tangent to a hyperbola always cuts off a triangle of constant area from its asymptotes.

Conversely, we conclude that, if a variable straight line cut off a constant area from two given straight lines, it always touches a hyperbola whose asymptotes are the two given straight lines, *i.e.* that its envelope is a hyperbola.

432. *If the equation to any curve involve a variable parameter, in the first degree only, the curve always passes through a fixed point or points.*

For if λ be the variable parameter, the equation to the curve can be written in the form $S + \lambda S' = 0$, and this equation is always satisfied by the points which satisfy $S = 0$ and $S' = 0$, *i.e.* the curve always passes through the point, or points, of intersection of $S = 0$ and $S' = 0$ [compare Art. 97].

433. *Curve touched by a variable straight line whose equation involves, in the second degree, a variable parameter.*

As an example, let us find the envelope of the straight lines given by the equation

$$m^2x - my + a = 0 \quad\text{.................. (1),}$$

where m is a quantity which, by its variation, gives the series of straight lines.

If (1) pass through the fixed point (h, k), we have

$$m^2h - mk + a = 0 \quad\text{.................. (2).}$$

This is an equation giving the values of m corresponding to the straight lines of the series which pass through the point (h, k). There can therefore be drawn two straight lines from (h, k) to touch the required envelope.

As (h, k) moves nearer and nearer to the required envelope these two tangents approach more and more nearly to coincidence, until, when (h, k) is taken on the envelope, the two tangents coincide.

Conversely, if the two tangents given by (2) coincide, the point (h, k) lies on the envelope.

Now the roots of (2) are equal if $k^2 = 4ah$,

so that the locus of (h, k), *i.e.* the required envelope, is the parabola $y^2 = 4ax$.

Hence, more simply, the envelope of the straight line (1) is the curve whose equation is obtained by writing down the condition that the equation (1), considered as a quadratic equation in m, may have equal roots.

By writing (1) in the form

$$y = mx + \frac{a}{m},$$

it is clear that it always touches the parabola $y^2 = 4ax$.

In the next article we shall apply this method to the general case.

434. *To find the envelope of a straight line whose equation involves, in the second degree, a variable parameter.*

The equation to the straight line is of the form

$$\lambda^2 P + \lambda Q + R = 0 \quad \text{.................. (1)},$$

where λ is a variable parameter and P, Q, and R are expressions of the first degree in x and y.

Equation (1) may be looked upon as an equation giving the two values of λ corresponding to any given point T.

Through this given point two straight lines to touch the required envelope may therefore be drawn.

If the point T be taken on the required envelope, the two tangents that can be drawn from it coalesce into the one tangent at T to the envelope.

Conversely, if the two straight lines given by (1) coincide, the resulting condition will give us the equation to the envelope.

But the condition that (1) shall have equal roots is

$$Q^2 = 4PR \qquad \text{.......................(2).}$$

This is therefore the equation to the required envelope.

Since P, Q, and R are all expressions of the first degree, the equation (2) is, in general, an equation of the second degree, and hence, in general, represents a conic section.

The envelope of any straight line, whose equation contains an arbitrary parameter and square thereof, is therefore always a conic.

435. The method of the previous article holds even if P, Q, and R be not necessarily linear expressions. It follows that the envelope of any family of curves, whose equation contains a variable parameter λ, in the second degree, is found by writing down the condition that the equation, considered as an equation in λ, may have equal roots.

436. Ex. 1. *Find the envelope of the straight line which cuts off from two given straight lines a triangle of constant area.*

Let the given straight lines be taken as the axes of coordinates and let them be inclined at an angle ω.

The equation to a straight line cutting off intercepts f and g from the axes is

$$\frac{x}{f} + \frac{y}{g} = 1 \qquad \text{.................................(1).}$$

If the area of the triangle cut off be constant, we have

$$\tfrac{1}{2} f \cdot g \cdot \sin \omega = \text{const.},$$

i.e.
$$fg = \text{const.} = K^2 \qquad \text{.............................(2).}$$

On substitution for g in (1), the equation to the straight line becomes
$$f^2 y - fK^2 + K^2 x = 0.$$

By the last article, the envelope of this line, for different values of f, is given by the equation

$$(-K^2)^2 = 4 \,.\, K^2 xy,$$

i.e.
$$xy = \frac{K^2}{4}.$$

The result is therefore a hyperbola whose asymptotes coincide with the axes of coordinates.

Ex. 2. *Find the envelope of the straight line which is such that the product of the perpendiculars drawn to it from two fixed points is constant.*

Take the middle point of the line joining the two fixed points as the origin, the line joining them as the axis of x, and let the two points be $(d, 0)$ and $(-d, 0)$.

Let the variable straight line have as equation

$$y = mx + c.$$

The condition then gives

$$\frac{md + c}{\sqrt{1+m^2}} \times \frac{-md + c}{\sqrt{1+m^2}} = \text{constant} = A,$$

so that
$$c^2 - m^2 d^2 = A\,(1+m^2).$$

The equation to the variable straight line is then

$$y - mx = c = \sqrt{(A + d^2)\, m^2 + A}.$$

Or, on squaring,

$$m^2 (x^2 - A - d^2) - 2mxy + (y^2 - A) = 0.$$

By Art. 435, the envelope of this is

$$(2xy)^2 = 4\,(x^2 - A - d^2)\,(y^2 - A),$$

i.e.
$$\frac{x^2}{A + d^2} + \frac{y^2}{A} = 1.$$

This is a conic section whose axes are the axes of coordinates and whose foci are the two given points.

Ex. 3. *Find the envelope of chords of an ellipse the tangents at the end of which intersect at right angles.*

Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

If the tangents intersect at right angles, their point of intersection P must lie on the director circle, and hence its coordinates must be of the form $(c\cos\theta,\ c\sin\theta)$, where $c = \sqrt{a^2 + b^2}$.

The chord is then the polar of P with respect to the ellipse, and hence its equation is

$$\frac{x \,.\, c\cos\theta}{a^2} + \frac{y \,.\, c\sin\theta}{b^2} = 1.$$

Let $t \equiv \tan\frac{\theta}{2}$. Then since

$$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{1-t^2}{1+t^2}, \text{ and } \sin\theta = \frac{2t}{1+t^2},$$

the equation to the line is

$$\frac{cx}{a^2}\frac{1-t^2}{1+t^2} + \frac{cy}{b^2}\frac{2t}{1+t^2} = 1,$$

i.e.
$$t^2\left[1+\frac{cx}{a^2}\right] - 2t\frac{cy}{b^2} + \left(1-\frac{cx}{a^2}\right) = 0.$$

The envelope of this is (Art. 434),

$$\left(-\frac{2cy}{b^2}\right)^2 = 4\left(1+\frac{cx}{a^2}\right)\left(1-\frac{cx}{a^2}\right),$$

i.e.
$$\frac{x^2c^2}{a^4} + \frac{y^2c^2}{b^4} = 1,$$

i.e.
$$\frac{x^2}{\frac{a^4}{a^2+b^2}} + \frac{y^2}{\frac{b^4}{a^2+b^2}} = 1.$$

Since $\frac{a^4}{a^2+b^2} - \frac{b^4}{a^2+b^2} = a^2 - b^2$, this equation represents a conic confocal with the given one.

Ex. 4. *The normals at four points of an ellipse meet in a point; if the line joining one pair of these points pass through a fixed point, prove that the line joining the other pair envelopes a parabola which touches the axes.*

Let the equation to the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad\ldots\ldots\ldots(1),$$

and let the equation to the two pairs of lines through the points be

$$lx + my = 1 \quad\ldots\ldots\ldots(2),$$

and
$$l_1x + m_1y = 1 \quad\ldots\ldots\ldots(3).$$

By Art. 412, Cor. (1), we then have

$$ll_1 = -\frac{1}{a^2} \text{ and } mm_1 = -\frac{1}{b^2} \quad\ldots\ldots\ldots(4).$$

If the straight line (3) pass through the fixed point (f, g), we have

$$l_1f + m_1g = 1,$$

so that, by (4),
$$-\frac{f}{a^2l} - \frac{g}{b^2m} = 1,$$

and therefore
$$l = -\frac{f}{a^2}\frac{mb^2}{mb^2+g}.$$

If this value of l be substituted in (2), it becomes

$$m^2a^2b^2y + m(a^2gy - b^2fx - a^2b^2) - a^2g = 0,$$

the envelope of which is

$$(a^2gy - b^2fx - a^2b^2)^2 = -4a^2g \cdot a^2b^2y,$$

i.e.

$$(a^2gy - b^2fx)^2 + 2a^2b^2(b^2fx + a^2gy) + a^4b^4 = 0 \quad\ldots\ldots\ldots\ldots(5).$$

This is a parabola since the terms of the second degree form a perfect square. Also, putting in succession x and y equal to zero, we get perfect squares, so that the parabola touches both axes.

437. *To find the envelope of the straight line*

$$lx + my + n = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where the quantities l, m, and n are connected by the relation

$$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0 \quad\ldots\ldots(2).$$

[Equation (1) contains two independent parameters $\frac{l}{n}$ and $\frac{m}{n}$, whilst (2) is an equation connecting them. We could therefore solve (2) to give $\frac{l}{n}$ in terms of $\frac{m}{n}$; on substituting in (1) we should then have an equation containing one independent parameter and its envelope could then be found.

It is easier, however, to proceed as follows.]

Eliminating n between (1) and (2), we see that the equation to the straight line may be written in the form

$$al^2 + bm^2 + c(lx + my)^2 - 2(fm + gl)(lx + my) + 2hlm = 0,$$

i.e.

$$(a - 2gx + cx^2)\left(\frac{l}{m}\right)^2 + 2(cxy - gy - fx + h)\frac{l}{m} + (b - 2fy + cy^2) = 0.$$

The envelope of this is, by Art. 435,

$$(cxy - gy - fx + h)^2 = (a - 2gx + cx^2)(b - 2fy + cy^2),$$

i.e., on reduction,

$$x^2(bc - f^2) + y^2(ca - g^2) + 2xy(fg - ch) + 2x(fh - bg) + 2y(gh - af) + ab - h^2 = 0.$$

The envelope is therefore a conic section.

Cor. The envelope is a parabola if

$$(fg-ch)^2=(bc-f^2)(ca-g^2),$$

i.e. if $\quad c=0$, or if $abc+2fgh-af^2-bg^2-ch^2=0$.

438. Ex. *Find the envelope of all chords of the parabola $y^2=4ax$ which subtend a given angle α at the vertex.*

Any straight line is

$$lx+my+n=0 \quad\quad\quad (1).$$

The lines joining the origin to its intersections with the parabola are, (by Art. 122), $\quad ny^2=-4ax(lx+my)$,

i.e. $\quad ny^2+4a\,mxy+4alx^2=0.$

If α be the angle between these lines, we have

$$\tan\alpha=\frac{2\sqrt{4a^2m^2-4aln}}{n+4al},$$

i.e. $\quad 16a^2l^2-16a^2\cot^2\alpha m^2+n^2+8aln(1+2\cot^2\alpha)=0.$

With this condition the envelope of (1) is, by the last article,

$$x^2(-16a^2\cot^2\alpha)+y^2[16a^2-(4a+8a\cot^2\alpha)^2]$$
$$+2x\,.\,16a^2\cot^2\alpha\,(4a+8a\cot^2\alpha)-256a^4\cot^2\alpha=0,$$

i.e. the ellipse

$$[x-4a(1+2\cot^2\alpha)]^2+4\operatorname{cosec}^2\alpha\,.\,y^2=64\cot^2\alpha\,.\operatorname{cosec}^2\alpha.$$

EXAMPLES XLIX

Find the envelope of the straight line $\frac{x}{\alpha}+\frac{y}{\beta}=1$ when

1. $a\alpha+b\beta=c$. **2.** $\alpha+\beta+\sqrt{\alpha^2+\beta^2}=c$.

3. $\frac{b^2}{\alpha^2}+\frac{a^2}{\beta^2}=1$.

Find the envelope of a straight line which moves so that

4. the sum of the intercepts made by it on two given straight lines is constant.

5. the sum of the squares of the perpendiculars drawn to it from two given points is constant.

6. the difference of these squares is constant.

7. Find the envelope of the straight line whose equation is

$$ax\cos\theta+by\sin\theta=c^2.$$

8. Circles are described touching each of two given straight lines; prove that the polars of a given point with respect to these circles all touch a parabola.

9. From any point P on a parabola perpendiculars PM and PN are drawn to the axis and tangent at the vertex; prove that the envelope of MN is another parabola.

10. Shew that the envelope of the chord which is common to the parabola $y^2=4ax$ and its circle of curvature is the parabola

$$y^2+12ax=0.$$

11. Perpendiculars are drawn to the tangents to the parabola $y^2=4ax$ at the points where they meet the straight line $x=b$; prove that they envelope another parabola having the same focus.

12. A variable tangent to a given parabola cuts a fixed tangent in the point A; prove that the envelope of the straight line through A perpendicular to the variable tangent is another parabola.

13. Shew that the envelope of chords of a parabola the tangents at the ends of which meet at a constant angle is, in general an ellipse.

14. A given parabola slides between two axes at right angles; prove that the envelope of its latus rectum is a fixed circle.

15. Prove that the envelope of chords of an ellipse which subtend a right angle at its centre is a concentric circle.

16. If the lines joining any point P on an ellipse to the foci meet the curve again in Q and R, prove that the envelope of the line QR is the concentric and coaxal ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\left(\frac{1+e^2}{1-e^2}\right)^2=1.$$

17. Prove that the envelope of chords of the rectangular hyperbola $xy=a^2$, which subtend a constant angle α at the point (x', y') on the curve, is the hyperbola

$$x^2x'^2+y^2y'^2=2a^2xy\,(1+2\cot^2\alpha)-4a^4\operatorname{cosec}^2\alpha.$$

18. Chords of a conic are drawn subtending a right angle at a fixed point O. Prove that their envelope is a conic whose focus is O and whose directrix is the polar of O with respect to the original conic.

19. Shew that the envelope of the polars of a fixed point O with respect to a system of confocal conics, whose centre is C, is a parabola having CO as directrix.

20. A given straight line meets one of a system of confocal conics in P and Q, and RS is the line joining the feet of the other two normals drawn from the point of intersection of the normals at P and Q; prove that the envelope of RS is a parabola touching the axes.

21. $ABCD$ is a rectangular sheet of paper, and it is folded over so that C lies on the side AB; prove that the envelope of the crease so formed is a parabola, whose focus is the initial position of C.

22. A circle, whose centre is A, is traced on a sheet of paper and any point B is taken on the paper. If the paper be folded so that the circumference of the circle passes through B, prove that the envelope of the crease so formed is a conic whose foci are A and B.

23. In the conic $\frac{l}{r} = 1 - e\cos\theta$ find the envelope of chords which subtend a constant angle 2α at the focus.

24. Circles are described on chords of the parabola $y^2 = 4ax$, which are parallel to the straight line $lx + my = 0$, as diameters; prove that they envelope the parabola

$$(ly + 2ma)^2 = 4a(l^2 + m^2)(x + a).$$

25. Prove that the envelope of the polar of any point on the circle $(x+a)^2 + (y+b)^2 = k^2$ with respect to the circle $x^2 + y^2 = c^2$ is the conic

$$k^2(x^2 + y^2) = (ax + by + c^2)^2.$$

26. Chords of the conic $\frac{l}{r} = 1 - e\cos\theta$ are drawn passing through the origin and on these circles as diameters circles are described. Shew that the envelope of these circles is the two circles

$$\frac{l}{r}\left(\frac{l}{r} + e\cos\theta\right) = 1 \pm e.$$

ANSWERS

1. $(by - ax - c)^2 = 4acx$. **2.** $x^2 + y^2 - c(x+y) + \frac{c^2}{4} = 0$.

3. $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$. **4.** A parabola touching each of the two lines.

5. A central conic. **6.** A parabola. **7.** $a^2x^2 + b^2y^2 = c^4$.

19. The line joining the foci is a particular case of the confocals and the polar of O with respect to it is the major axis; the minor axis is another particular case, so that two of the polars are lines through C at right angles; also the tangents at O to the confocals through it are two of the polars, and these are at right angles. Thus both C and O are on the directrix.

21. The crease is clearly the line bisecting at right angles the line joining the initial position of C to the position which C occupies when the paper is folded.

23. $\frac{l \cos \alpha}{r} = 1 - e \cos \alpha \cos \theta.$

SOLUTIONS/HINTS

1. Eliminating β, the equation to the straight line is

$$a\alpha^2 - \alpha(ax - by + c) + cx = 0;$$

therefore the required envelope is $(ax - by + c)^2 = 4acx$.

2. Since $(\alpha + \beta - c)^2 = \alpha^2 + \beta^2$, $\therefore \alpha(2\beta - 2c) = 2c\beta - c^2$.

Eliminating α from the equation to the straight line it becomes $2\beta^2(x - c) + \beta(2cy - 2cx + c^2) - c^2 y = 0$; therefore the required envelope is

$$(2cy - 2cx + c^2)^2 + 8y(x - c)c^2 = 0,$$

or

$$x^2 + y^2 - c(x + y) + \frac{c^2}{4} = 0.$$

3. The relation between α, β is satisfied if $\alpha = \frac{b}{\cos \theta}$ and $\beta = \frac{a}{\sin \theta}$. The equation to the straight line is then

$$\frac{x}{b}\cos\theta + \frac{y}{a}\sin\theta = 1,$$

which is always a tangent to the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.