CHAPTER 03

Quadratic Equation In One Variable

Introduction

A person has some land and she wants to make a garden of area 800 square meters in one part such that length of the garden is twice its breadth. What should be the length and breadth of the garden?

If breadth of garden is taken as x meter then its length will be $2x$ meter.

Since garden is rectangular,

So, area of garden = length of garden × breadth of garden

$$800 = 2x.x$$

Or $$\frac{800}{2} = x^2$$

Or $$x^2 = 400$$

Or $$x^2 = 20^2$$

Or $$x^2 - 20^2 = 0 \quad \text{.....(i)}$$

Those values of x for which both sides of equation (i) are equal, give us the breadth of the garden.

Once we know the breadth, then we can also find length.

Writing a Problem in Algebraic Form

We saw above that we can find the value of x from $x^2 - 20^2 = 0$. This equation is actually the algebraic description of the conditions given in the problem.

Let us discuss some more situations and look at their algebraic representations.

Naresh wants to make a garden bed (*kyari)* in front of his house. The garden bed is in the shape of a right-angled triangle of area 500 square meter, in which he wants the length of the base of the triangle to be 30 meter longer than its perpendicular height.

To find the size of the garden-bed, we should take its perpendicular as x meter and base as $x + 30$ meter.

Since area of right angle triangle = $\frac{1}{2}$ × length of perpendicular × length of base

$$\Rightarrow \quad 500 = \frac{1}{2} \times x(x+30)$$

$$\Rightarrow \quad 500 \times 2 = x^2 + 30x$$

$$\Rightarrow \quad x^2 + 30x - 1000 = 0 \quad \text{..... (ii)}$$

Equation (ii) is algebraic representation of our question and the values of x for which both sides of equation (ii) become equal give us the perpendicular height of the right-angle flower bed.

Quadratic Equations:- The maximum degree of x is 2 in both of the above algebraic representations. If this quadratic polynomial is included we get an algebraic equation in one variable which can be solved to get the solution. Now, we consider some more examples of the same type.

Think about the following statement:-

The product of two consecutive numbers is zero.

If first number is x, then second number will be $x + 1$.

$$\therefore \quad x(x+1) = 0$$

$$x^2 + x = 0 \quad \text{..... (iii)}$$

In each of the equations (i), (ii), (iii) there is only one variable and their maximum degree is two. In each of these equations, there necessarily exist one such term whose degree is two. Thus, all these are quadratic equations in one variable.

You know that $ax^2 + bx + c$ is a two degree polynomial in one variable (where a, b, c are real numbers and $a \neq 0$). It is known as quadratic polynomial. If we equate this quadratic polynomial equal zero it becomes a quadratic equation.

That is, $ax^2 + bx + c = 0$

Because there is only one variable in the equation and maximum degree of the variable is two, so it is called a quadratic polynomial. This is the standard form of a quadratic equation, also known as square equation.

Some more quadratic equations are given below:-

(i) $x^2 - 2x = 0$ (ii) $(x+1)(x+2) = 0$ (iii) $x^2 = 0$

(iv) $x^2 - 9 = 0$ (v) $z^2 + 3 = 0$ (vi) $x^2 - \sqrt{5}x + 6 = 0$

(vii) $3y^2 + 6y + 6 = 0$ (viii) $(x-2)^2 = 0$ (ix) $3m - 2m^2 + 5 = 0$

$x^2 - 5\sqrt{x} + 3 = 0$ is not a quadratic (square) equation because left hand side of this equation is not a polynomial.

Try These

Select quadratic (square) equations in one variable from the following-

(i)	$x^2 - 3x = 0$	(ii)	$-3x^2 - 2^2 = 0$	(iii)	$x + 2 = 0$
(iv)	$x^2 + y = 9$	(v)	$x^2 + 9 = 0$	(vi)	$x + 5y = 0$
(vii)	$(x-1)(x+2) = 0$	(viii)	$x^2 + 2\sqrt{x} - 1 = 0$	(ix)	$(x-3)^2 = 0$
(x)	$x(x-5) = 0$	(xi)	$x^2 + \sqrt{5}x + 3 = 0$	(xi)	$y^2 - z^2 + 3 = 0$
(xiii)	$x^2 - 3\sqrt{x} + 2 = 0$	(xiv)	$x^2 - \sqrt{3}x + 2 = 0$	(xv)	$(x+1)(x+5) = 0$

Exercise - 1

1. Select square equations from the following equations:-

(i)	$x^2 + 3x - 2 = 0$	(ii)	$x^2 + \frac{1}{x} = 1$
(iii)	$9x^2 - 100x - 20 = 0$	(iv)	$x^2 - 3\sqrt{x} + 2 = 0$
(v)	$x - \frac{2}{x} = -x$	(vi)	$\sqrt{5}x^2 - 3x + \frac{1}{2} = 0$
(vii)	$x^2 - 10x = 0$	(viii)	$x + y = 10$
(ix)	$x + 5 = 7$	(x)	$x(x-8) = 0$

Roots of Quadratic Equation

$p(x) = x^2 - 3x + 2$ is a quadratic polynomial. Zeroes of this polynomial are those value of x for which $p(x)$ will be zero. To find the zeroes we have to find factors of $x^2 - 3x + 2$.

$$x^2 - 3x + 2 = x^2 - 2x - x + 2$$
$$= (x^2 - 2x) - 1(x - 2)$$
$$= x(x-2) - 1(x-2)$$
$$= (x-2)(x-1)$$

Value of the polynomial $x^2 - 3x + 2$ will be zero if $(x-2)(x-1) = 0$

$\Rightarrow \quad (x-2)=0 \quad$ Or $\quad (x-1)=0$

$\Rightarrow \quad x-2=0 \quad$ Or $\quad x-1=0$

$\therefore\ x=2 \quad$ Or $\quad x=1$

So 2 and 1 are zeroes of polynomial x^2-3x+2.

Now, let us find such values of x in square equation $x^2-3x+2=0$ for which both sides of the equation will be equal. We call the values roots of the equation.

$$x^2-3x+2=0$$
$$(x-2)(x-1)=0$$

$\Rightarrow \quad (x-2)=0 \quad$ Or $\quad (x-1)=0$

$\Rightarrow \quad x-2=0 \quad$ Or $\quad x-1=0$

$\therefore\ x=2 \quad$ Or $\quad x=1$

Here, we find that the equation is satisfied when $x = 2, 1$. The same values of x are also zeroes of polynomial x^2-3x+2. So we can say that zeroes of the polynomial are roots of the equation made by its factors.

How to find whether given values are roots of polynomial or not?

It is very easy to know whether a given value is root of a square equation or not. If we put a value in an equation and both sides of the equation become equal then these values are roots of the equation, otherwise not.

Let us learn to verify roots with the help of some examples.

Example-1. Verify whether $x = 1$ and $x = -1$ are roots of the square polynomial $x^2-x+1=0$ or not?

Solution : In the given equation $x^2-x+1=0$ left hand side is x^2-x+1 and right hand side is 0. On putting $x = 1$ in the left hand side-

$$=1^2-1+1$$
$$=1$$

Clearly, LHS $\neq$ RHS

So, $x = 1$, is not a zero of the given square polynomial.

Similarly, on putting $x = -1$

$$=(-1)^2-(-1)+1$$
$$=1+1+1$$
$$=3$$

Clearly, LHS ≠ RHS

So $x = -1$ is not a root of the given square polynomial $x^2 - x + 1 = 0$.

Example-2. Verify whether $x = 2$, $x = 3$ are roots of the square equation $x^2 - 5x + 6 = 0$ or not?

Solution : In given equation $x^2 - 5x + 6 = 0$ left hand side is $x^2 - 5x + 6$ and right hand side is 0. On putting $x = 2$ in LHS

$x^2 - 5x + 6$

$= (2)^2 - 5(2) + 6$

$= 4 - 10 + 6$

$= 0$

Clearly, LHS = RHS

Similarly, on putting $x = 3$

$= (3)^2 - 5(3) + 6$

$= 9 - 15 + 6$

$= 0$

Clearly, LHS = RHS

Thus $x = 2$, $x = 3$ are roots of equation $x^2 - 5x + 6 = 0$.

Try These

Verify whether given values of x are roots of the equation or not

(i)	$x^2 + 6x + 5 = 0$	;	$x = -5, x = -1$
(ii)	$9x^2 - 3x - 2 = 0$	;	$x = \frac{2}{3}, x = -\frac{1}{3}$
(iii)	$x^2 + x + 1 = 0$	;	$x = 0, x = 1$

Methods of Solving Quadratic Equations

So far we have looked at how make square equations. Now we will discuss the methods of solving them.

$x^2 - 7x = 0$ is a square equation.

Can we determine the value of x in the given equation?

$\because \quad x^2 - 7x = 0$

$\Rightarrow \quad x(x-7) = 0$

$\Rightarrow \quad x = 0$ Or $x - 7 = 0$

Then, $x = 0$ Or $x = 7$

Since the values of x satisfy the given equation, so these will be the solution of the equation $x^2 - 7x = 0$.

Similarly, let us try to solve $(x–2)(x+1) = 0$

$\Rightarrow \quad x + 1 = 0$ Or $x - 2 = 0$

$\therefore \quad x = -1$ Or $x = 2$

Since the values of x satisfy the given equation, thus $x = –1$ and $x = 2$ will be the roots of $(x+1)(x-2) = 0$.

Try These

Solve the following equations-

(i) $x^2 - 11x = 0$ (ii) $(x-1)^2 = 0$ (iii) $(x+3)^2 = 0$

(iv) $(x-2)(x+3) = 0$ (v) $x(x-1) = 0$

Solving Quadratic Equations by Factorization Method

At the beginning of this chapter, different situations were described and we formed equations (i), (ii) and (iii) on their basis. We will now solve them.

Equation (i) : On comparing $x^2 – 20^2 = 0$ with the identify $a^2 - b^2 = (a-b)(a+b)$.

$$x^2 - 20^2 = (x-20)(x+20)$$

$\therefore \quad (x-20)(x+20) = 0$

$\Rightarrow \quad (x-20) = 0$ Or $(x+20) = 0$

$\Rightarrow \quad x - 20 = 0$ Or $x + 20 = 0$

$\Rightarrow \quad x = 20$ Or $x = -20$

Because length and breadth can't be negative therefore x can't be –20. In this context x has been taken as breadth of the rectangular garden.

$\therefore$ Breath of rectangular garden is = 20 meter.

and length will be $2x = 2 \times 20 = 40$ meter

Equation (ii) : For the given flower bed, we have to find perpendicular height and length of base in $x^2 + 30x - 1000 = 0$.

Because, left hand side of equation (ii) is a quadratic polynomial, so we will factorize it.

$$x^2 + 50x - 20x - 1000 = 0 \qquad (\because 50x - 20x = 30x)$$

$\Rightarrow \quad x(x+50) - 20(x+50) = 0$

$\Rightarrow \quad (x+50)(x-20) = 0$

$\Rightarrow \quad (x+50) = 0$ Or $(x-20) = 0$

$\Rightarrow \quad x+50 = 0$ Or $x-20 = 0$

$\Rightarrow \quad x = -50$ Or $x = 20$

Thus, perpendicular height of right-angle flower bed = 20 meters and length of its base should be $x + 30 = 20 + 30 = 50$ meters.

Similarly, we can get the values by finding factors of equation (iii)

In examples given below we will get solutions by factorization method.

Example-3. Solve the following equations by factorization-

(i) $8x^2 - 22x - 21 = 0$ (ii) $x^2 + 2\sqrt{2}x - 6 = 0$

(iii) $\sqrt{3}x^2 + 10x + 7\sqrt{3} = 0$ (iv) $\frac{x+3}{x-2} - \frac{1-x}{x} = \frac{17}{4}, x \neq 0$

Solution (i) : $8x^2 - 22x - 21 = 0$

$\Rightarrow \quad 8x^2 - 28x + 6x - 21 = 0$

$\Rightarrow \quad 4x(2x-7) + 3(2x-7) = 0$

$\Rightarrow \quad (2x-7)(4x+3) = 0$

$\Rightarrow \quad (2x-7) = 0$ Or $(4x+3) = 0$

$\Rightarrow \quad 2x-7 = 0$ Or $4x+3 = 0$

$\Rightarrow \quad 2x = 7$ Or $4x = -3$

$\Rightarrow \quad x = \frac{7}{2}$ Or $x = -\frac{3}{4}$

Thus $x = \frac{7}{2}, -\frac{3}{4}$ are two roots of the given equation.

Solution (ii) : $x^2 + 2\sqrt{2}x - 6 = 0$

$\Rightarrow \quad x^2 + 3\sqrt{2}x - \sqrt{2}x - 6 = 0$

$\Rightarrow \quad x(x+3\sqrt{2}) - \sqrt{2}(x+3\sqrt{2}) = 0$

$\Rightarrow \quad (x+3\sqrt{2})(x-\sqrt{2}) = 0$

$\Rightarrow \quad \left(x+3\sqrt{2}\right)=0 \quad \text{Or} \quad \left(x-\sqrt{2}\right)=0$

$\Rightarrow \quad x+3\sqrt{2}=0 \quad \text{Or} \quad x-\sqrt{2}=0$

$\Rightarrow \quad x=-3\sqrt{2} \quad \text{Or} \quad x=\sqrt{2}$

Thus, $x=-3\sqrt{2},\sqrt{2}$ are two roots of the given equation.

Solution (iii) : $\sqrt{3}x^2+10x+7\sqrt{3}=0$

$\Rightarrow \quad \sqrt{3}x^2+3x+7x+7\sqrt{3}=0$

$\Rightarrow \quad \sqrt{3}x\left(x+\sqrt{3}\right)+7\left(x+\sqrt{3}\right)=0$

$\Rightarrow \quad \left(x+\sqrt{3}\right)\left(\sqrt{3}x+7\right)=0$

$\Rightarrow \quad \left(x+\sqrt{3}\right)=0 \quad \text{Or} \quad \left(\sqrt{3}x+7\right)=0$

$\Rightarrow \quad x+\sqrt{3}=0 \quad \text{Or} \quad \sqrt{3}x+7=0$

$\Rightarrow \quad x=-\sqrt{3} \quad \text{Or} \quad x=-\dfrac{7}{\sqrt{3}}$

Thus, $x=-\sqrt{3},-\dfrac{7}{\sqrt{3}}$ are two roots of the given equation.

Solution (iv) : $\dfrac{x+3}{x-2}-\dfrac{1-x}{x}=\dfrac{17}{4}$ $\quad , x\neq 0$

$\Rightarrow \quad \dfrac{x(x+3)-(1-x)(x-2)}{x(x-2)}=\dfrac{17}{4}$

$\Rightarrow \quad \dfrac{x^2+3x-\left(x-x^2-2+2x\right)}{x^2-2x}=\dfrac{17}{4}$

$\Rightarrow \quad \dfrac{x^2+3x-x+x^2+2-2x}{x^2-2x}=\dfrac{17}{4}$

$\Rightarrow \quad \dfrac{2x^2+2}{x^2-2x}=\dfrac{17}{4}$

$\Rightarrow \quad 4\left(2x^2+2\right)=17\left(x^2-2x\right)$

$\Rightarrow \quad 8x^2+8=17x^2-34x$

$\Rightarrow \quad 17x^2-8x^2-34x-8=0$

$\Rightarrow \quad 9x^2-34x-8=0$

$\Rightarrow \quad 9x^2-36x+2x-8=0$

$\Rightarrow \quad 9x(x-4)+2(x-4)=0$

$\Rightarrow \quad (x-4)(9x+2)=0$

$\Rightarrow \quad (x-4)=0$ Or $(9x+2)=0$

$\Rightarrow \quad x-4=0$ Or $9x+2=0$

$\Rightarrow \quad x=4$ Or $x=\frac{-2}{9}$

Thus, $x=4, \frac{-2}{9}$ are two roots of the given equation.

Exercise - 2

1. Verify whether values are roots of equation or not-

   (i) $2x^2+x-6=0;$ $\quad x=2, x=-\frac{3}{2}$

   (ii) $x^2-4x+4=0;$ $\quad x=2, x=-3$

   (iii) $6x^2-6x-12=0;$ $\quad x=-3, x=4$

   (iv) $4x^2-9x=0;$ $\quad x=0, x=\frac{9}{4}$

   (v) $x^2-3\sqrt{3}x+6=0;$ $\quad x=\sqrt{3}, x=-2\sqrt{3}$

2. Find the roots of the given quadratic equation-

   (i) $(x-4)(x-2)=0$ $\quad$ (ii) $(2x+3)(3x-7)=0$

   (iii) $(x-7)^2=0$ $\quad$ (iv) $x^2-11x=0$

   (v) $(x+12)^2=0$ $\quad$ (vi) $x(x+1)=0$

3. Is $\sqrt{2}$ a root of equation $x^2+2x-4=0$?

4. Find the roots of the following square equations by factorization:-

   (i) $9x^2-3x-2=0$ $\quad$ (ii) $4x^2+5x=0$

   (iii) $3x^2-11x+10=0$ $\quad$ (iv) $5x^2+3x-2=0$

   (v) $6x^2+7x+2=0$ $\quad$ (vi) $4\sqrt{3}x^2+5x-2\sqrt{3}=0$

   (vii) $10x-\frac{1}{x}=3$ $\quad$ (viii) $x^2-4\sqrt{2}x+6=0$

   (ix) $abx^2+(b^2-ac)x-bc=0$

   (x) $\frac{x+1}{x-1}-\frac{x-1}{x+1}=\frac{5}{6};$ $\quad x \neq 1,-1$

Applications of quadratic equations

We often encounter many situations in our daily life which we solve by forming quadratic equations around them. We will now look at some examples from day to day where we can form quadratic equations and find their solutions.

Example-4. If the sum of a number and its reciprocal is $2\frac{1}{30}$ then find the number.

Solution: Let the number be x and then its reciprocal will be $\frac{1}{x}$.

$\therefore \quad x+\frac{1}{x}=2\frac{1}{30}$

$\Rightarrow \quad \frac{x^2+1}{x}=\frac{61}{30}$

$\Rightarrow \quad 30\left(x^2+1\right)=61x$

$\Rightarrow \quad 30x^2-61x+30=0$

$\Rightarrow \quad 30x^2-36x-25x+30=0$

$\Rightarrow \quad 6x(5x-6)-5(5x-6)=0$

$\Rightarrow \quad (5x-6)(6x-5)=0$

$\Rightarrow \quad 5x-6=0$ Or $6x-5=0$

$\Rightarrow \quad x=\frac{6}{5}$ Or $x=\frac{5}{6}$

Therefore, the numbers are $x=\frac{6}{5}$ and $x=\frac{5}{6}$.

Example-5. A chessboard contains 64 equal squares and the area of each square is 6.25 cm^2. A 2 *cm* wide border surrounds the board on all sides. Find the length of one side of the chess board.

Solution : Let the length of chessboard with border be x *cm*.

Border is 2 *cm* wide

Addition to length of board due to border = 2 + 2 = 4 *cm*

Area of chessboard without border = $(x-4)^2$

Chessboard contains 64 equal squares,

and area of each square is 6.25 cm^2

$\therefore \quad (x-4)^2=64\times 6.25$

$\Rightarrow \quad x^2-8x+16=400$

$\Rightarrow \quad x^2-8x+16-400=0$

$\Rightarrow \quad x^2 - 8x - 384 = 0$

$\Rightarrow \quad x^2 - 24x + 16x - 384 = 0$

$\Rightarrow \quad x(x-24) + 16(x-24) = 0$

$\Rightarrow \quad (x-24)(x+16) = 0$

$\Rightarrow \quad x - 24 = 0 \quad$ Or $\quad x + 16 = 0$

$\Rightarrow \quad x = 24 \quad$ Or $\quad x = -16$

The length of side of chessboard cannot be negative.

Thus, the length of one side of chessboard will be 24 *cm*

Example-6. The sum of marks obtained by Mohan in mathematics and science in a class test is 28. If his marks in mathematic are increased by 3 and marks in science decreased by 4, then the product of marks obtained in both subjects is 180. Find the marks obtained by Mohan in mathematics and science.

Solution: If Mohan got x marks in mathematics, then his marks in science will be $= 28 - x$

When Mohan gets 3 more marks in mathematics,

then his marks in mathematics $= x + 3$

And when his marks in science are reduced by 4, then his marks in science $= 28 - x - 4$

$\because$ Product of his marks in mathematics and science = 180 (given)

$\therefore \quad (x+3)(28-x-4) = 180$

$\Rightarrow \quad (x+3)(24-x) = 180$

$\Rightarrow \quad -x^2 - 3x + 24x + 72 = 180$

$\Rightarrow \quad -x^2 + 21x + 72 - 180 = 0$

$\Rightarrow \quad -x^2 + 21x - 108 = 0$

$\Rightarrow \quad x^2 - 12x - 9x + 108 = 0$

$\Rightarrow \quad x(x-12) - 9(x-12) = 0$

$\Rightarrow \quad (x-12)(x-9) = 0$

$\Rightarrow \quad x - 12 = 0 \quad$ Or $\quad x - 9 = 0$

$\Rightarrow \quad x = 12 \quad$ Or $\quad x = 9$

When $x = 12 \Rightarrow 28 - x = 28 - 12 = 16$

When $x = 9 \Rightarrow 28 - x = 28 - 9 = 19$

Thus, if Mohan got 12 marks in mathematics, then he got 16 marks in science else if he got 9 marks in mathematics, then he got 19 marks in science.

Example-7. Present age of a man is equal to the square of present age of his son. If 1 year ago man's age was 8 times that of his son's age then find the present age of both.

Solution : Let the present age of son = x years.

Then, present age of man = x^2 year

Age of son one year ago = $x - 1$ years

and age of man one year ago = $x^2 - 1$ years

$\because$ 1 year ago age of man was 8 times that of his son

$\therefore$ $x^2 - 1 = 8(x - 1)$

$\Rightarrow$ $x^2 - 1 = 8x - 8$

$\Rightarrow$ $x^2 - 8x + 7 = 0$

$\Rightarrow$ $x^2 - 7x - x + 7 = 0$

$\Rightarrow$ $x(x - 7) - 1(x - 7) = 0$

$\Rightarrow$ $(x - 7)(x - 1) = 0$

$\Rightarrow$ $x - 7 = 0$ Or $x - 1 = 0$

$\Rightarrow$ $x = 7$ Or $x = 1$

For $x = 1$, age of son and man both would be 1 year, which is not possible.

So, on taking $x = 7$ years

Present age of son $x = 7$ years

Present age of man $x^2 = (7)^2 = 49$ years

Example-8. Perimeter of a rectangular field is 82 meter and its area is 400 square meter. Find the breadth of the field.

Solution : Let the breadth of rectangular field be x meter

Perimeter of rectangular field = 82 meter

$\Rightarrow$ 2 (length + breadth) = 82 meter

$\Rightarrow$ (length + breadth) = 41 meter

$\Rightarrow$ (length + x) = 41 meter

$\Rightarrow$ length = 41 – x meter

$\because$ Area of rectangular field = 400 square meter

$\Rightarrow$ length × breadth = 400 square meter

$\Rightarrow$ $(41 - x)x = 400$

$\Rightarrow \quad 41x - x^2 = 400$

$\Rightarrow \quad -x^2 + 41x - 400 = 0$

$\Rightarrow \quad x^2 - 41x + 400 = 0$

$\Rightarrow \quad x^2 - 25x - 16x + 400 = 0$

$\Rightarrow \quad x(x-25) - 16(x-25) = 0$

$\Rightarrow \quad (x-25)(x-16) = 0$

$\Rightarrow \quad x - 25 = 0$ Or $x - 16 = 0$

$\Rightarrow \quad x = 25$ Or $x = 16$

Thus, breadth of rectangular field will be 16 meter and length will be 25 meter. Of the two values of x, one will represent breadth and the other length.

Example-9. Some students planned a picnic. The budget for food was Rs. 500. But five of these failed to go and thus the cost of food for each member increased by Rs. 5. How many students attended the picnic?

Solution : Let the number of students who planned for picnic be x

$\because$ Total amount given by x students for food expenditure = Rs. 500

$\therefore$ Amount given by one student for food expenditure = Rs. $\dfrac{500}{x}$

But five students failed to go

then, number of students going $= x - 5$

Amount given by $x - 5$ students for food expenditure = Rs. 500

Amount given by one student for food expenditure = Rs. $\dfrac{500}{x-5}$

According to question, 5 students failed to go and due to this the cost of food for each member increased by Rs. 5

$$\therefore \quad \frac{500}{x-5} - \frac{500}{x} = 5$$

$$\Rightarrow \quad \frac{500x - 500(x-5)}{x(x-5)} = 5$$

$$\Rightarrow \quad \frac{500\{x-(x-5)\}}{x(x-5)} = 5$$

$$\Rightarrow \quad \frac{500(x-x+5)}{x(x-5)} = 5$$

$$\Rightarrow \quad \frac{500 \times 5}{5} = x^2 - 5x$$

$$\Rightarrow \quad x^2 - 5x - 500 = 0$$

$$\Rightarrow \quad x^2 - 25x + 20x - 500 = 0$$

$$\Rightarrow \quad x(x-25) + 20(x-25) = 0$$

$$\Rightarrow \quad (x-25)(x+20) = 0$$

$\Rightarrow \quad x - 25 = 0$ Or $x + 20 = 0$

$\Rightarrow \quad x = 25$ Or $x = -20$

$\because$ Number of students can't be negative, thus $x = 25$

$\therefore$ Number of students who attended the picnic $= x - 5 = 20$.

Example-10. A person buys a number of books for Rs. 80. If he had bought 4 more books for the same amount, each book would have cost Rs. 1 less. How many books did he buy?

Solution : Let the number of books bought be x.

$\because$ Cost of x books = Rs. 80

$\therefore$ Cost of one book = Rs. $\frac{80}{x}$

If he had bought 4 more books then number of books $= x + 4$

$\because$ Cost of x + 4 books = Rs. 80

$\therefore$ Cost of 1 book = Rs. $\frac{80}{x+4}$

According to questions Rs. $\frac{80}{x+4}$ is Rs. 1 less than Rs. $\frac{80}{x}$ –

$$\therefore \quad \frac{80}{x} - 1 = \frac{80}{x+4}$$

$$\Rightarrow \quad \frac{80}{x} - \frac{80}{x+4} = 1$$

$$\Rightarrow \quad 80\left[\frac{1}{x} - \frac{1}{x+4}\right] = 1$$

$$\Rightarrow \quad 80\left[\frac{x+4-x}{x(x+4)}\right] = 1$$

$$\Rightarrow \quad \frac{80 \times 4}{(x^2 + 4x)} = 1$$

$$\Rightarrow \quad 320 = x^2 + 4x$$

$$\Rightarrow \quad x^2 + 4x - 320 = 0$$

$$\Rightarrow \quad x^2 + 20x - 16x - 320 = 0$$

$$\Rightarrow \quad x(x+20) - 16(x+20) = 0$$

$$\Rightarrow \quad (x+20)(x-16) = 0$$

$$\Rightarrow \quad (x+20) = 0 \quad \text{Or} \quad (x-16) = 0$$

$$\Rightarrow \quad x = -20 \ \text{(Unacceptable) or } \ x = 16$$

Thus, number of books = 16.

Solution of a quadratic equation by completing the square

In the previous section, we learnt factorization method of obtaining the roots of a quadratic equation. In this section, we shall study another method. In this method we convert the equation into $(x - a)^2$ or $(x + a)^2$ form. To do this we need to add certain terms to both sides of the equation.

This method is used in the following example-

Example-11. Solve the square equation $x^2 + 6x = 0$ by the method of completing the square.

Solution : $x^2 + 6x = 0$

$$\Rightarrow \quad x^2 + 2x \times 3 = 0$$

To make the equation perfect square we need to add square of coefficient of $2x$, which is square of 3 or 9 to both sides of the equation.

$$\Rightarrow \quad x^2 + 2x \times 3 + 3^2 = 3^2$$

$$\Rightarrow \quad x^2 + 2 \times x \times 3 + 3^2 = 9$$

$$\Rightarrow \quad (x+3)^2 = 9 \qquad \text{(By using the identity } x^2 + 2xa + a^2 = (x+a)^2\text{)}$$

$$\Rightarrow \quad x + 3 = \pm\sqrt{9}$$

$$\Rightarrow \quad x + 3 = \pm 3$$

On taking (+) sign	On taking (–) sign
$\Rightarrow \quad x+3=+3$	$x+3=-3$
$\Rightarrow \quad x=3-3$	$x=-3-3$
$\Rightarrow \quad x=0$	$x=-6$

Because x is common in $x^2 + 6x$, therefore we can write the equations as $x(x + 6) = 0$. Now, it is clear that $x = 0$ or $x = -6$. We can see that the solution of problem is not affected by the method used.

Example-12. Solve the quadratic equation $x^2 - 6x + 5 = 0$ by the method of completing the squares.

Solution : $x^2-6x+5=0$

$\Rightarrow \quad x^2-2x\times 3+5=0$

$\Rightarrow \quad x^2-2x\times 3=-5$

By adding square of coefficient of $2x$, that is square of 3 to both sides of equation.

$\Rightarrow \quad x^2-2x\times 3+3^2=-5+3^2$

$\Rightarrow \quad (x-3)^2=-5+9 \qquad \left[\because x^2-2xa+a^2=(x-a)^2\right]$

$\Rightarrow \quad (x-3)^2=4$

$\Rightarrow \quad x-3=\pm\sqrt{4}$

$\Rightarrow \quad x-3=\pm 2$

On taking (+) sign	On taking (–) sign
$\Rightarrow \quad x-3=+2$	$x-3=-2$
$\Rightarrow \quad x=2+3$	$x=-2+3$
$\Rightarrow \quad x=5$	$x=1$

On splitting the middle term of polynomial, we find that $(x - 5)(x - 1) = 0$ is the equation and thus the roots are $x = 5$ and $x = 1$. But here there is no common term.

Example-13. Solve the quadratic equation $x^2-\frac{5}{2}x-3=0$.

Solution : $x^2-\frac{5}{2}x-3=0$

$\Rightarrow$

$x^2-\frac{5}{2}x=3$

$\Rightarrow \quad x^2-2x\times\frac{1}{2}\times\frac{5}{2}=3$

$\Rightarrow \quad x^2 - 2x \times \frac{5}{4} = 3$

By adding square of $\frac{5}{4}$ which is coefficient of $2x$ to both sides of the equation,

$\Rightarrow \quad x^2 - 2x \times \frac{5}{4} + \left(\frac{5}{4}\right)^2 = 3 + \left(\frac{5}{4}\right)^2$

$\Rightarrow \quad \left(x - \frac{5}{4}\right)^2 = \frac{25}{16} + 3$

$\Rightarrow \quad \left(x - \frac{5}{4}\right)^2 = \frac{25+48}{16}$

$\Rightarrow \quad \left(x - \frac{5}{4}\right)^2 = \frac{73}{16}$

$\Rightarrow \quad x - \frac{5}{4} = \pm\sqrt{\frac{73}{16}}$

$\Rightarrow \quad x = \frac{5}{4} \pm \sqrt{\frac{73}{16}}$

On taking (+) sign	**On taking (–) sign**
$\Rightarrow \quad x = \frac{5}{4} + \sqrt{\frac{73}{16}}$	$x = \frac{5}{4} - \sqrt{\frac{73}{16}}$
$\Rightarrow \quad x = \frac{5}{4} + \frac{\sqrt{73}}{4}$	$x = \frac{5}{4} - \frac{\sqrt{73}}{4}$
$\Rightarrow \quad x = \frac{5+\sqrt{73}}{4}$	$x = \frac{5-\sqrt{73}}{4}$

Can we find the solution of the polynomial by taking common terms or splitting the middle term?

Example-14. Solve the quadratic equation $2x^2 - 7x + 3 = 0$

Solution : $2x^2 - 7x + 3 = 0$

$\Rightarrow \quad 2x^2 - 7x = -3$

On dividing both sides of equation by 2

$\Rightarrow \quad x^2 - \frac{7}{2}x = -\frac{3}{2}$

$$\Rightarrow \quad x^2 - 2x\times\frac{1}{2}\times\frac{7}{2} = -\frac{3}{2} \qquad \left[\because \frac{7}{2} = 2\times\frac{1}{2}\times\frac{7}{2} = 2\times\frac{7}{4}\right]$$

$$\Rightarrow \quad x^2 - 2x\times\frac{7}{4}+\left(\frac{7}{4}\right)^2 = -\frac{3}{2}+\left(\frac{7}{4}\right)^2$$

(Adding the square of $\frac{7}{4}$ which is the coefficient of $2x$ to both sides)

$$\Rightarrow \quad \left(x-\frac{7}{4}\right)^2 = -\frac{3}{2}+\frac{49}{16} \qquad \left[\because x^2-2xa+a^2=(x-a)^2\right]$$

$$\Rightarrow \quad \left(x-\frac{7}{4}\right)^2 = \frac{-24+49}{16}$$

$$\Rightarrow \quad \left(x-\frac{7}{4}\right)^2 = \frac{25}{16}$$

$$\Rightarrow \quad x-\frac{7}{4} = \pm\sqrt{\frac{25}{16}}$$

$$\Rightarrow \quad x-\frac{7}{4} = \pm\frac{5}{4}$$

Taking (+) sign	**Taking (–) sign**
$\Rightarrow \quad x-\frac{7}{4}=+\frac{5}{4}$	$x-\frac{7}{4}=-\frac{5}{4}$
$\Rightarrow \quad x=\frac{5}{4}+\frac{7}{4}$	$x=-\frac{5}{4}+\frac{7}{4}$
$\Rightarrow \quad x=\frac{12}{4}$	$x=\frac{2}{4}$
$\Rightarrow \quad x=3$	$x=\frac{1}{2}$

Can we solve this sum by splitting the middle term? Try.

Try These

Solve the following quadratic equations by the method of completing the square.

(i) $x^2-\frac{3}{4}x+3=0$ (ii) $2x^2+5x+3=0$

(iii) $9x^2-15x+6=0$

Example-15. Solve $\sqrt{6+\sqrt{6+\sqrt{6+\ldots\ldots\ldots\ldots\ldots\ldots}}}$.

Solution : Let $x=\sqrt{6+\sqrt{6+\sqrt{6+\ldots\ldots\ldots\ldots\ldots\ldots}}}$

$\Rightarrow \quad x=\sqrt{6+x}$ (On squaring both sides)

$\Rightarrow \quad x^2=\left(\sqrt{6+x}\right)^2$

$\Rightarrow \quad x^2=6+x$

$\Rightarrow \quad x^2-x-6=0$

$\Rightarrow \quad x^2-3x+2x-6=0$

$\Rightarrow \quad x(x-3)+2(x-3)=0$

$\Rightarrow \quad (x-3)(x+2)=0$

$\Rightarrow \quad (x-3)=0$ Or $(x+2)=0$

$\Rightarrow \quad x=3$ Or $x=-2$

Thus $x=3,-2$

Exercise - 3

1. Solve the following quadratic equations by completing the squares method.

   (i) $2x^2+x-4=0$ (ii) $3x^2+11x+10=0$

   (iii) $5x^2-6x-2=0$ (iv) $x^2-4\sqrt{2}x+6=0$

   (v) $3x^2+2x-1=0$ (vi) $x^2-4x+3=0$

2. Find the solution of $\sqrt{7+\sqrt{7+\sqrt{7+\sqrt{7\ldots\ldots\ldots}}}}$.

3. Find two consecutive positive natural numbers, sum of whose squares is 365.

4. The product of two consecutive natural numbers is 20. Find the numbers.

5. Find two numbers whose sum is 48 and product is 432.

6. Area of a right angle triangle is 165 square meter. If the height of the right triangle is 7 *cm* more than its base, then find the height of the triangle.

7. Perimeter and area of a flower bed are 76 meter and 357 square meters respectively. Find the length and breadth of the flower bed.

8. Area of a rectangular park is 100 square meter. Length of the park is 15 meter more than the breadth. If someone wants to fence along the sides of the park and cost of fencing one meter is Rs. 5, then find the total cost of fencing all sides of the park.

9. The sum of present ages of a man and his son is 45 years. 5 years ago product of their ages was 4 times that of age of man. Find their present ages.

10. The product of Neelmani's age 5 years ago and 8 years ago is 40. Find Neelmani's present age.

11. Some students planned a picnic. The budget for food was Rs. 480. But 8 of them failed to go and thus the cost of food for each member increased by Rs. 10. How many attended the picnic?

12. The sum of marks obtained by Kamal in English and mathematics is 40 in class test of grade 10. If the marks which he got in mathematics are increased by 3 and marks which he got in English decreased by 4 then the product of marks is 360. Find the marks obtained by him in mathematics and English.

Indian Mathematician - Sridharacharya

Sridharacharya was the first Indian mathematician to solve quadratic (square) equations involving one variable. He had worked in the field of Arithmetic, Geometry, Square roots, Cube roots etc. Sridharacharya was a famous mathematician between the era of Brahmagupta (628 AD) and Bhaskaracharya (1150 AD).

It is said that from the Himalayas in the north to the Malwas in the south and from the west to the east sea coasts, there was no mathmematician who was better than Sridharacharya.

चतुराहत वर्ग समै रूपैः पक्ष द्वयं गुणयेत्।
अव्यक्त वर्ग रूपैर्युक्तौ पक्षौततो मूलम्।।

From the book, "Paati Ganit Avam Ganit Ke Itihas"
Authors : Venugopal and Dr. Herrer

Formula to Solve Quadratic Equations

Standard form of quadratic (square) is $ax^2 + bx + c = 0$ ....., where a, b, c are real numbers and $a \neq 0$

On dividing both sides by the coefficient of x^2 -

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\Rightarrow \qquad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$\Rightarrow \quad x^2 + 2.\frac{b}{2a}x = -\frac{c}{a}$ (On squaring the coefficient $2x$ and adding to both sides)

$\Rightarrow \quad x^2 + 2.\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$

$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{-c}{a} + \frac{b^2}{4a^2}$

$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$

$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{(b^2 - 4ac)}{4a^2}$

$\Rightarrow \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$ (Taking square root)

$\Rightarrow \quad x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Think and Discuss

Mention the method by which $3x^2 + 7x + 1 = 0$ can be solved? Will the solution be same for all methods? Why are completing the square and splitting the middle term methods not so easy?

Let us solve some quadratic equations.

Example-16. Solve the quadratic equation $(x-1)(2x-1) = -2$.

Solution : $(x-1)(2x-1) = -2$

$\Rightarrow \quad 2x^2 - 2x - x + 1 = 0$

$\Rightarrow \quad 2x^2 - 3x + 1 = 0$

On comparing the equation with the standard form $ax^2 + bx + c = 0$

$a = 2, b = -3, c = 1$

On putting the values of a, b and c in the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 2 \times 1}}{2(2)}$$

$$x = \frac{3 \pm \sqrt{9-8}}{4}$$

$$x = \frac{3 \pm 1}{4}$$

On taking (+) sign	**On taking (–) sign**
$x = \dfrac{3+1}{4} = \dfrac{4}{4}$ $x = 1$	$x = \dfrac{3-1}{4} = \dfrac{2}{4}$ $x = \dfrac{1}{2}$

Thus, the roots of the equation are $x = 1, \dfrac{1}{2}$.

Try These

Solve the following equations:-

(i) $3x^2 - 2x + 2 = 0$ (ii) $x^2 - 2x + 1 = 0$

Discriminant of Quadratic Equation

In the quadratic equation $ax^2 + bx + c = 0$, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. In the formula $b^2 - 4ac$ is said to be discriminant of the quadratic equation. It can also written as $D = b^2 - 4ac$. It is called discriminant because it discriminates between the two values of the quadratic equation. If it is zero then the both values are equal.

Let us try to find the discriminant of some quadratic equations:-

Example-17. Find the discriminant of the quadratic equation $4x^2 - 4x + 1 = 0$.

Solution : On comparing $4x^2 - 4x + 1 = 0$ with the standard form, $ax^2 + bx + c = 0$ of quadratic equation we find that:

Here $a = 4, b = -4, c = 1$

Discriminant $D = b^2 - 4ac$

$D = (-4)^2 - 4 \times 4 \times 1$

$D = 16 - 16$

$D = 0$

Example-18. Find the discriminant of the quadratic equation $2x^2 + 5x + 5 = 0$.

Solution : On comparing $2x^2 + 5x + 5 = 0$ with the standard form, $ax^2 + bx + c = 0$ of quadratic equation we find that:

Here $a = 2, b = 5, c = 5$

$\because \quad D = b^2 - 4ac$

$D = 5^2 - 4 \times 2 \times 5$

$D = 25 - 40$

$D = -15$

Example-19. Find the discriminant of the quadratic equation $3x^2 - 2\sqrt{8}x + 2 = 0$.

Solution : On comparing $3x^2 - 2\sqrt{8}x + 2 = 0$ with the standard form, $ax^2 + bx + c = 0$ of quadratic equation we find that:

$a = 3, b = -2\sqrt{8}, c = 2$

$\because \quad D = b^2 - 4ac$

$D = \left(-2\sqrt{8}\right)^2 - 4 \times 3 \times 2$

$D = 4 \times 8 - 24$

$D = 32 - 24$

$D = 8$

Try These

Find the discriminant of the following quadratic equations:

(i) $2x^2 - 2\sqrt{2}x + 1 = 0$ (ii) $16x^2 + 24x + 9 = 0$

(iii) $9x^2 - 10x + 15 = 0$ (iv) $x^2 + 16x + 64 = 0$

Nature of Roots of Quadratic Equation

In the above examples we found the discriminant of various quadratic equations by using the formula $D = b^2 - 4ac$ In the examples the values of D where respectively 0, –15, 8. These values of D were zero, negative or positive numbers. This means that discriminant can be zero, negative or positive. Does this give us some important information about the roots of quadratic equations? Let us try to find out.

$\because$ The formula to find the roots $ax^2 + bx + c = 0$ where a, b, c are real numbers and $a \neq 0$ is given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $b^2 - 4ac = D$

thus,

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

D = b^2– 4ac discriminates (differentiates) between the nature of roots. So it is called discriminant.

Let us discuss the following three cases.

Case-1. If D = 0 then

$$x = \frac{-b \pm \sqrt{0}}{2a}$$

$$x = \frac{-b \pm 0}{2a}$$

On taking (+) value	On taking (–) value
$x = \frac{-b+0}{2a} = \frac{-b}{2a}$	$x = \frac{-b-0}{2a} = \frac{-b}{2a}$

Here, both values of x are equal and real.

Conclusion: If D = 0 then both roots of the quadratic equations are real and equal.

Case-2. If D = any positive number, then

Let D = 49

$$x = \frac{-b \pm \sqrt{49}}{2a}$$

$$x = \frac{-b \pm 7}{2a}$$

On taking (+) sign	On taking (–) sign
$x = \frac{-b+7}{2a}$	$x = \frac{-b-7}{2a}$

Here, the values of x are distinct and both are real numbers.

Conclusion: If D > 0 i.e. positive then the roots of quadratic equation are distinct and real.

Case-3. If D = any negative number, then

Say, D = – 81

$$x = \frac{-b \pm \sqrt{-81}}{2a}$$

The roots of a negative number are imaginary.

On taking (+) sign	On taking (–) sign
$x = \frac{-b+\sqrt{-81}}{2a}$	$x = \frac{-b-\sqrt{-81}}{2a}$

Here, both the values of x are distinct but being the square root of a negative number they are imaginary.

Conclusion : If D<0 i.e. negative, the roots are distinct but imaginary, i.e. there are no real roots.

Some examples of identifying the nature of roots

Let us try to see some examples of the nature of roots of quadratic equations.

Example-20. Find the nature of roots of the quadratic equation $x^2 - 4x + 4 = 0$.

Solution : On comparing $x^2 - 4x + 4 = 0$ with $ax^2 + bx + c = 0$, we find that:

$a = 1, b = -4, c = 4$

On putting the values in discriminant, $D = b^2 - 4ac$

$$D = (-4)^2 - 4 \times 1 \times 4$$

$$D = 16 - 16$$

$$D = 0$$

$\because$ Discriminant of given equation is zero, thus both roots of this equation are real and equal.

Example-21. Find the nature of roots of the quadratic equation $x^2 - 5x + 6 = 0$.

Solution : On comparing $x^2 - 4x + 4 = 0$ with $ax^2 + bx + c = 0$, we find that:

$a = 1, b = -5, c = 6$

On putting the values in discriminant, $D = b^2 - 4ac$

$$D = (-5)^2 - 4 \times 1 \times 6$$

$$D = 25 - 24$$

$$D = 1$$

D > 0 (Positive)

Since, the discriminant of the given quadratic equation is positive, so the two roots are real and distinct.

Example-22. Find the nature of roots of the quadratic equation $4x^2 - x + 1 = 0$.

Solution : On comparing $4x^2 - x + 1 = 0$ with $ax^2 + bx + c = 0$, we find that:

$a = 4, b = -1, c = 1$

On putting the values in discriminant, $D = b^2 - 4ac$

$$D = (-1)^2 - 4 \times 4 \times 1$$

$$D = 1 - 16$$

D = – 15 (negative)

Since, the discriminant of the given quadratic equation is negative, so the two roots are distinct but imaginary.

Try These

Find the nature of roots of the following equations:-

(i) $x^2 + x + 2 = 0$ (ii) $2x^2 + x - 1 = 0$

(iii) $2x^2 + 5x + 5 = 0$ (iv) $2y^2 - 2\sqrt{6}y + 3 = 0$

Finding unknown constant coefficients

On the basis of nature of roots of a quadratic equation we can find the value of unknown coefficient of the variable term. Let us try to understand by the following example-

Example-23. Find the value of k in the quadratic equation $9x^2 + 3kx + 4 = 0$ given that the roots of the quadratic equation are equal.

Solution : On comparing the quadratic equation $9x^2 + 3kx + 4 = 0$ with the standard from $ax^2 + bx + c = 0$, we find that $a = 9, b = 3k, c = 4$

Putting these values in $D = b^2 - 4ac$

$$D = (3k)^2 - 4 \times 9 \times 4$$

$$D = 9k^2 - 144$$

When roots are equal, then $D = 0$

Thus, $9k^2 - 144 = 0$

$$9k^2 = 144$$

$$k^2 = \frac{144}{9}$$

$$k^2 = 16$$

$$k = \pm 4$$

Suppose the roots were distinct and real then what could we about the value of k?

Try These

Find the value of k in the following quadratic equations given that the roots of the quadratic equation are real and equal

(i) $16x^2 + kx + 9 = 0$ (ii) $3x^2 - 2\sqrt{8}x + k = 0$

(iii) If roots are imaginary in both questions then what can you say about k?

Exercise-4

1. Find the discriminant of the following equations-

(i) $x^2 - 4x + 2 = 0$ (ii) $(x-1)(2x-1) = 0$

(iii) $\sqrt{3}x^2 + 2\sqrt{2}x - 2\sqrt{3} = 0$ (iv) $x^2 - 4x + a = 0$

(v) $x^2 + px + qx = 0$

2. Find the nature of roots of the following quadratic equations.

(i) $x^2 - 4x + 4 = 0$ (ii) $2x^2 + 2x + 2 = 0$

(iii) $3x^2 - 2\sqrt{6}x + 2 = 0$ (iv) $x^2 + 2\sqrt{5}x - 1 = 0$

(v) $\frac{3}{5}x^2 - \frac{2}{3}x + 1 = 0$

3. Find the value of k if the roots of the given equations are real and equal.

(i) $2x^2 - 10x + k = 0$ (ii) $kx^2 - 5x + k = 0$

(iii) $2x^2 + kx + \frac{9}{8} = 0$ (iv) $9x^2 - kx + 16 = 0$

(vi) $kx^2 + 4x + 1 = 0$

4. Solve the following quadratic equations with the help of formula.

(i) $9x^2 + 7x - 2 = 0$ (ii) $6x^2 + x - 2 = 0$

(iii) $6x^2 + 7x - 10 = 0$ (iv) $2x^2 - 9x + 7 = 0$

(v) $x^2 - 7x - 5 = 0$ (vi) $4 - 11x = 3x^2$

(vii) $9x^2 - 4 = 0$ (viii) $\sqrt{3}x^2 - 10x - 8\sqrt{3} = 0$

(ix) $2x^2 + x - 6 = 0$ (x) $2x^2 - 2\sqrt{6}x + 3 = 0$

Relation between roots and coefficients of quadratic equations

We saw the relation between zeroes and coefficients of quadratic polynomials, the same relation can also be seen between the roots and coefficients of quadratic equations.

Let the roots of equation $ax^2 + bx + c = 0$ be α anda β where a, b, c are real numbers and $a \neq 0$ then $(x-\alpha)(x-\beta) = 0$ or $x^2 - (\alpha+\beta)x + \alpha\beta = 0$ quadratic equations......(1)

Quadratic equation $ax^2 + bx + c = 0$ can also be written in the following form

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad ......(2)$$

We find that equations (1) and (2) are two different forms of the same equation and thus, on comparing the two we get,

Sum of roots $\alpha + \beta = \frac{-b}{a}$

and product of roots $\alpha\beta = \frac{c}{a}$

Example-24. Find the sum and product of roots of $x^2 - 5x - 24 = 0$

Solution : On comparing equation $x^2 - 5x - 24 = 0$ with $ax^2 + bx + c = 0$, we find that

$$a = 1, b = -5, c = -24$$

$\because$ Sum of roots $= \frac{-b}{a}$

$\therefore$ Sum of roots $= \frac{-(-5)}{1}$

$= \frac{5}{1}$

$= 5$

$\because$ Product of roots $= \frac{c}{a}$

$\therefore$ Product of roots $= \frac{-24}{1}$

$= -24$

Example-25. Find the sum and product of roots of $3x^2 + 2x + 7 = 0$

Solution : On comparing equation $3x^2 + 2x + 7 = 0$ with $ax^2 + bx + c = 0$, we find that

$a = 3, b = 2, c = 7$

$\because$ Sum of roots $= \frac{-b}{a}$

$\therefore$ Sum of roots $= \frac{-2}{3}$

$\because$ Product of roots $= \frac{c}{a}$

$\therefore$ Product of roots $= \frac{7}{3}$

Think and Discuss

If roots are given then can we form a quadratic equation? On the basis of the concepts studied in the chapter on "polynomials" and the relations between roots and coefficient, discuss with your friends the possibility of forming a quadratic polynomial.

Forming quadratic equations if roots are known

We have learnt to find roots of quadratic equations. If roots of a quadratic equation are given then is it possible to form that quadratic equation?

Yes, we can form quadratic equations with the help of sum of roots and product of roots. Thus, if roots are given then we can form quadratic equation.

If the roots of a quadratic equation are α and β then that quadratic equation will be

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

So, x^2 – (Sum of roots)x + Product of roots = 0

Example-26. Find the quadratic equation whose roots are 3 and –8.

Solution : If the roots are given then the quadratic equation is

$\Rightarrow$ x^2 – (Sum of roots) x + Product of roots = 0

$\Rightarrow$ $x^2 - [3 + (-8)]x + 3 \times (-8) = 0$

$\Rightarrow$ $x^2 - (-5)x + (-24) = 0$

$\Rightarrow$ $x^2 + 5x - 24 = 0$

Example-27. Find the quadratic equation whose roots are $\frac{4}{3}$ and $\frac{7}{3}$.

Solution : If the roots are given then the quadratic equation is

$\Rightarrow$ x^2 – (Sum of roots) x + Product of roots = 0

$\Rightarrow$ $x^2 - \left(\frac{4}{3} + \frac{7}{3}\right)x + \frac{4}{3} \times \frac{7}{3} = 0$

$\Rightarrow$ $x^2 - \frac{11}{3}x + \frac{28}{9} = 0$

$\Rightarrow$ $9x^2 - 33x + 28 = 0$

Example-28. Find the quadratic equation whose roots are $\left(2 + \sqrt{7}\right)$ and $\left(2 - \sqrt{7}\right)$.

Solution : If the roots are given then the quadratic equation is

$\Rightarrow$ x^2 – (Sum of roots) x + Product of roots = 0

$\Rightarrow$ $x^2 - \left[2 + \sqrt{7} + 2 - \sqrt{7}\right]x + \left(2 + \sqrt{7}\right)\left(2 - \sqrt{7}\right) = 0$

$\Rightarrow$ $x^2 - 4x + 4 - 7 = 0$ $\quad \because (2)^2 - \left(\sqrt{7}\right)^2 = 4 - 7$

$\Rightarrow$ $x^2 - 4x - 3 = 0$

Example-29. It the sum and product of the roots of a quadratic equation are –8 and 4 respectively then find the quadratic equation.

Solution : If roots are given then the quadratic equation can be formed using

$\Rightarrow$ x^2 – (Sum of roots) x + Product of roots = 0

$\Rightarrow$ $x^2 - (-8)x + 4 = 0$

$\Rightarrow$ $x^2 + 8x + 4 = 0$

Try These

Form quadratic equations whose roots are

(i) $2, 3$ (ii) $-5, -3$ (iii) $\sqrt{5}, \sqrt{3}$

Exercise-5

1. Find the quadratic equations whose sum and product of roots are as following:-

   (i) Sum of roots = – 4 Product of roots = 12

   (ii) Sum of roots = 6 Product of roots = 9

   (iii) Sum of roots = $= 2\sqrt{7}$ Product of roots = 8

   (iv) Sum of roots = $= \frac{4}{9}$ Product of roots = 1

2. Find the quadratic equation whose roots are as following:

   (i) $7, 4$ (ii) $-5, -11$ (iii) $-2, 4$ (iv) $12, -24$

   (v) $\frac{4}{5}, \frac{-3}{5}$ (vi) $4, 4$ (vii) $\frac{-1}{3}, \frac{2}{5}$ (viii) $8, 3$

   (ix) $\sqrt{3}-7, \sqrt{3}+7$ (x) $6+\sqrt{5}, 6-\sqrt{5}$

3. Find the sum and product of roots of the following quadratic equations:

   (i) $3x^2 + 7x + 1 = 0$ (ii) $2x^2 - 2x + 3 = 0$

   (iii) $3x^2 - 5x - 2 = 0$ (iv) $2x^2 - 2\sqrt{6}x + 3 = 0$

   (v) $x^2 + 6x - 6 = 0$

What We Have Learnt

1. Polynomial ax^2+bx+c, is a one variable polynomial in degree 2, where a, b, c are real numbers and $a \neq 0$ gSA It is said to be a quadratic polynomial. On equating this quadratic polynomial to zero it becomes a quadratic equation $ax^2 + bx + c = 0$ Because there is only one variable in the equation and the maximum degree of the variable is 2 so it is said to be a quadratic equation in one variable.

2. Quadratic equations are also known as square equations.

3. The standard form of a quadratic equation in the variable x is $ax^2 + bx + c = 0$ where a, b, c are real number and $a \neq 0$.

4. A quadratic equation which is in the form $ax^2 + bx + c = 0$ has only two roots.

5. We can find the roots of a quadratic equation involving one variable x by factorization and we can also find roots by putting the values of a, b, c in the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

6. In the formula of quadratic equations, D = b^2- $4ac$ is discriminant and we can know about the nature of roots if we know the value of D.

7. If α and β are two roots of a quadratic (square) equation then the quadratic equation in variable $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ (here, we can take any variable such as y, z etc.)

8. If α and β are roots of a quadratic equation $ax^2 + bx + c = 0$ then the following relation holds between their roots and coefficient.

   Sum of roots $\alpha + \beta = \frac{-b}{a}$

   Product of roots $\alpha\beta = \frac{c}{a}$

9. When D = $b^2 - 4ac > 0$ i.e. value of D is positive then both roots of the quadratic equation are real and distinct.

10. When D = $b^2 - 4ac = 0$ i.e. value of D is 0 then the roots of the quadratic equation are real and equal.

11. When D = $b^2 - 4ac < 0$ i.e. value of D is negative then roots of the quadratic equation are imaginary and distinct

ANSWER KEY

Exercise - 1

1. (i), (iii), (v), (vi), (vii), (x) are square equations.

Exercise - 2

1. (i) $x = 2, x =$ $x = 2, x = -\frac{3}{2}$ are not roots of equation.

   (ii) $x = 2$ is a root of equation but $x = -3$ is not a root of equation.

   (iii) $x = -3, x = 4$ are not roots of equation.

   (iv) $x = 0, x = \frac{9}{4}$ are roots of equation.

   (v) $x = \sqrt{3}$ is a root of equation but $x = -2\sqrt{3}$ is not a root of equation.

2. (i) $x = 4, x = 2$ (ii) $x = -\frac{3}{2}, x = \frac{7}{3}$ (iii) $x = 7, x = 7$

(iv) $x = 0, x = 11$ (v) $x = -12, x = -12$ (vi) $x = 0, x = -1$

3. $x = \sqrt{2}$ is not a root of equation.

4. (i) $x = \frac{2}{3}, x = -\frac{1}{3}$ (ii) $x = 0, x = -\frac{5}{4}$ (iii) $x = \frac{5}{3}, x = 2$

(iv) $x = \frac{2}{5}, x = -1$ (v) $x = -\frac{2}{3}, x = -\frac{1}{2}$ (vi) $x = -\frac{2}{\sqrt{3}}, x = \frac{\sqrt{3}}{4}$

(vii) $x = \frac{1}{2}, x = -\frac{1}{5}$ (viii) $x = 3\sqrt{2}, x = \sqrt{2}$

(ix) $x = -\frac{b}{a}, x = \frac{c}{b}$ (x) $x = 5, x = -\frac{1}{5}$

Exercise - 3

1. (i) $\frac{-1+\sqrt{33}}{4}, \frac{-1-\sqrt{33}}{4}$ (ii) $\frac{-5}{3}, -2$ (iii) $\frac{3+\sqrt{19}}{5}, \frac{3-\sqrt{19}}{5}$

(iv) $\sqrt{2}, 3\sqrt{2}$ (v) $\frac{1}{3}, -1$ (vi) 3,1

2. $\frac{1+\sqrt{29}}{2}, \frac{1-\sqrt{29}}{2}$

3. Consecutive natural numbers 6, 7

4. Consecutive natural numbers 4, 5

5. Numbers 36, 12

6. Length of base = 15 meter, length of perpendicular = 22 meter

7. 21 meter, 17 meter

8. Rupees 250

9. 36 years, 9 years

10. 13 years

11. 16 students

12. 12, 28 or 21, 19

Exercise - 4

1. (i) 8 (ii) 1 (iii) 32 (iv) $16-4a$ (v) p^2-4q

2. (i) Real and equal roots (ii) Roots are not real

(iii) Real and equal roots (iv) Real and different roots

(v) Roots not real

3. (i) $k=\frac{25}{2}$ (ii) $k=\pm\frac{5}{2}$ (iii) $k=\pm 3$

(iv) $k=\pm 24$ (vi) $k=4$

4. (i) $\frac{2}{9},-1$ (ii) $\frac{1}{2},-\frac{2}{3}$ (iii) $-2,\frac{5}{6}$ (iv) $\frac{7}{2},1$

(v) $\frac{1}{2}\left(7+\sqrt{69}\right),\frac{1}{2}\left(7-\sqrt{69}\right)$ (vi) $-4,\frac{1}{3}$ (vii) $\frac{2}{3},\frac{-2}{3}$

(viii) $\frac{12}{\sqrt{3}},\frac{-2}{\sqrt{3}}$ (ix) $-2,\frac{3}{2}$ (x) $\frac{\sqrt{6}}{2}$

Exercise - 5

1. (i) $x^2+4x-12=0$ (ii) $x^2-6x-9=0$

(iii) $x^2-2\sqrt{7}x+8=0$ (iv) $9x^2-4x+9=0$

2. (i) $x^2-11x+28=0$ (ii) $x^2-16x+55=0$

(iii) $x^2-2x-8=0$ (iv) $x^2+12x-288=0$

(v) $x^2-\frac{1}{5}x-\frac{12}{25}=0$ (vi) $x^2-8x+16=0$

(vii) $15x^2-x-2=0$ (viii) $x^2-11x+24=0$

(ix) $x^2-2\sqrt{3}x-46=0$ (x) $x^2-12x+31=0$

3. (i) $\frac{-7}{3},\frac{1}{3}$ (ii) $1,\frac{3}{2}$ (iii) $\frac{5}{3},\frac{-2}{3}$

(iv) $\sqrt{6},\frac{3}{2}$ (v) $-6,-6$