CBSE Test Paper 02

Chapter 5 Continuity and Differentiability

1. $\underset{x \to 0}{Lt} \frac{1-\cos x}{x^2}$ is equal to
    a. 1
    b. -1
    c. $\frac{1}{2}$
    d. 0
2. Let f(x) = $\begin{cases} e^{1/x}, x < 0 \\ x, x \geqslant 0 \end{cases}, then \underset{x \to 0}{Lt} f(x).$
    a. does not exist
    b. is equal to 0
    c. is equal to non – zero real number
    d. None of these
3. Let f and g be differentiable functions such that fog= I, the identity function. If g'(a) = 2 and g(a)=b, then f '(b)=.
    a. -2
    b. None of these
    c. 2
    d. $\frac{1}{2}$
4. $\frac{d^4}{dx^4}(\sin^3 x)$ is equal to
    a. $\frac{3}{4}\cos x - \frac{3^4 \cos 3x}{4}$
    b. None of these
    c. $\frac{3 \sin x - 3^4 \sin 3x}{4}$
    d. $\frac{3}{4}\sin x - \frac{3^4 \cos 3x}{4}$
5. The differential coefficient of log (|log x|) w.r.t. log x is
    a. $\frac{1}{x|\log x|}$

b. $\frac{1}{x \log x}$

c. None of these

d. $\frac{1}{\log x}$

6. The value of c in Rolle's Theorem for the function f(x) = e^x sin x, x $\in [0, \pi]$ is _______.
7. The set of points where the functions f given by f(x) = |x - 3|cos x is differentiable is _______.
8. The derivative of $\log_{10}$x w.r.t. x is _______.
9. Differentiate the following function with respect to x : $\sin(ax+b)$.
10. Differentiate the following function with respect to x: cos(log x + e^x), x > 0.
11. Find $\frac{dy}{dx}$ if $y = \frac{\sin(ax+b)}{\cos(cx+d)}$.
12. If $x = a\sec^3\theta$ and $y = a\tan^3\theta$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$.
13. Find $\frac{dy}{dx}$, if $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.
14. Examine the continuity of the function $f(x) = x^3 + 2x^2 - 1$ at x = 1.
15. Find the value of k so that the function f is continuous at the indicated point:

$$f(x) = \begin{cases} \frac{1-\cos kx}{x \sin x}, \ if\ x \neq 0 \\ \frac{1}{2}, \ if\ x = 0 \end{cases}$$ at x = 0.

16. If y = $\left(x + \sqrt{1+x^2}\right)^n$, then show that $\left(1+x^2\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2y$.
17. If $y^x = e^{y-x}$ prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$.
18. Find $\frac{dy}{dx}$, if $y = (x \cos x)^x + (x \sin x)^{\frac{1}{x}}$.

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Solution

1. c. $\frac{1}{2}$, **Explanation:** $\lim\limits_{x \to 0} \frac{1-\cos x}{x^2} = \lim\limits_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2}$ (Using L'hospital Rule).
2. b. is equal to 0, **Explanation:** $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} e^{\frac{1}{x}} = 0$
   $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} x = 0 \therefore \lim\limits_{x \to 0} f(x) = 0$
3. d. $\frac{1}{2}$, **Explanation:** $(fog)'(x) = 1 \forall x$
   $\Rightarrow (fog)'(a) = 1$
   $\Rightarrow f'(g(a))g'(a) = 1$
   $\Rightarrow f'(b)g'(a) = \frac{1}{2}$
4. c. $\frac{3\sin x - 3^4 \sin 3x}{4}$, **Explanation:** $\frac{d}{dx}(\sin^3 x) = 3\sin^2 x \cos x$
   $\frac{d^2}{dx^2}(\sin^3 x) = \frac{d}{dx}(3\sin^2 x \cos x) = 6\sin x \cos^2 x - 3\sin^3 x$
   $\frac{d^3}{dx^3}(\sin^3 x) = \frac{d}{dx}(6\sin^2 x \cos^2 x - 3\sin^3 x)$
   $= 6\cos^3 x - 12\sin^2 x \cos x - 9\sin^2 x \cos x = 6\cos^3 x - 21\sin^2 x \cos x$
   $\frac{d^4}{dx^4}(\sin^3 x) = \frac{d}{dx}(6\cos^3 x - 21\sin^2 x \cos x)$
   $= -18\cos^2 x \sin x - 42\sin x \cos^2 x + 21\sin^3 x$
   $= 60\sin x \cos^2 x + 21\sin^3 x = -60\sin x(1 - \sin^2 x) + 21\sin^3 x$
   $= -60\sin x + 60\sin^3 x + 21\sin^3 x = -60\sin x + 81\sin^3 x$
   $= -60\sin x + 81\left[\frac{3\sin x - \sin 3x}{4}\right] = \frac{3\sin z - 3^4 \sin 3z}{4}$
5. d. $\frac{1}{\log x}$, **Explanation:** Let y = log(|logx|) and z = logx, then ,we have;
   $\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \left(\frac{1}{\log x} \cdot \frac{1}{x}\right) / \left(\frac{1}{x}\right) \Rightarrow \frac{1}{\log x}$
6. $\frac{3\pi}{4}$
7. R - {3}
8. $(\log_{10}e)\frac{1}{x}$
9. Let $y = \sin(ax + b)$
   $\therefore \frac{dy}{dx} = \cos(ax + b)\frac{d}{dx}(ax + b)$
   $= \cos(ax + b)(a + 0)$

$= a\cos(ax + b)$

10. Let y = cos(logx + e^{x})

$\therefore \frac{dy}{dx} = -\sin(\log x + e^x)\frac{d}{dx}(\log x + e^x)$

$= -\sin(\log x + e^x).\left(\frac{1}{x} + e^x\right)$

11. $y = \frac{\sin(ax+b)}{\cos(cx+d)}$

$\frac{dy}{dx} = \frac{\cos(cx+d)\frac{d}{dx}\sin(ax+b) - \sin(ax+b)\frac{d}{dx}\cos(cx+d)}{\cos^2(cx+d)}$

$\frac{dy}{dx} = \frac{\cos(cx+d)\cos(ax+b).a + \sin(ax+b)\sin(cx+d).c}{\cos^2(cx+d)}$

12. We have $x = a\sec^3\theta$ and $y = a\tan^3\theta$

Differentiating w.r.t. θ , we get

$\frac{dx}{d\theta} = 3a\sec^2\theta\frac{d}{d\theta}(\sec\theta) = 3a\sec^3\theta\tan\theta$

and $\frac{dy}{d\theta} = 3a\tan^2\theta\frac{d}{d\theta}(\tan\theta) = 3a\tan^2\theta\sec^2\theta$

Thus $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta$

Hence, $\left(\frac{dy}{dx}\right)_{at\ \theta=\frac{\pi}{3}} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

13. Given: $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

To simplify the given Inverse Trigonometric function,we put, $x = \tan\theta$

$\Rightarrow y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$

$\Rightarrow y = 2\tan^{-1}x$

$\Rightarrow \frac{dy}{dx} = 2.\frac{1}{1+x^2} = \frac{2}{1+x^2}$

14. We have, $f(x) = x^3 + 2x^2 - 1$ at x = 1

$\therefore \lim\limits_{x\to 1^+} f(x) = \lim\limits_{h\to 0}(1+h)^3 + 2(1+h)^2 - 1 = 2$

and $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{h\to 0}(1-h)^3 + 2(1-h)^2 - 1 = 2$

$\therefore \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^-} f(x)$ and $f(1) = 1 + 2 - 1 = 2$

So, f(x) is continuous at x = 1.

15. We have, $f(x) = \begin{cases} \dfrac{1-\cos kx}{x\sin x},\ if\ x \neq 0 \\ \dfrac{1}{2},\ if\ x = 0 \end{cases}$

At x = 0, $LHL = \lim\limits_{x\to 0^-}\frac{1-\cos kx}{x\sin x} = \lim\limits_{h\to 0}\frac{1-\cos k(0-h)}{(0-h)\sin(0-h)}$

$= \lim_{h \to 0} \frac{1-\cos(-kh)}{-h\sin(-h)}$

$= \lim_{h \to 0} \frac{1-\cos kh}{h\sin\ h} [\because \cos(-\theta) = \cos\theta, \sin(-\theta) = -\sin\theta]$

$= \lim_{h \to 0} \frac{1-1+2\sin^2 \frac{kh}{2}}{h\sin\ h} \left[\because \cos\theta = 1 - 2\sin^2 \frac{\theta}{2}\right]$

$= \lim_{h \to 0} \frac{2\sin^2 \frac{kh}{2}}{h\sin\ h}$

$= \lim_{h \to 0} \frac{2\sin \frac{kh}{2}}{\frac{kh}{2}} \cdot \frac{\sin \frac{kh}{2}}{\frac{kh}{2}} \cdot \frac{1}{\frac{\sin\ h}{h}} \cdot \frac{k^2 h/4}{h}$

$= \frac{2k^2}{4} = \frac{k^2}{2} \left[\because \lim_{h \to 0} \frac{\text{sinh}}{h} = 1\right]$

Also, $f(0) = \frac{1}{2} \Rightarrow \frac{k^2}{2} = \frac{1}{2} \Rightarrow k = \pm 1$

16. According to the question, y = $\left(x + \sqrt{1+x^2}\right)^n$ ....(i)

Differentiating both sides w.r.t x,

$\Rightarrow \frac{dy}{dx} = n\left(x + \sqrt{1+x^2}\right)^{n-1} \left(1 + \frac{2x}{2\sqrt{1+x^2}}\right)$ [Using chain rule of derivative]

$\Rightarrow \quad \frac{dy}{dx} = n\left(x + \sqrt{1+x^2}\right)^{n-1} \left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)$

$\Rightarrow \quad \frac{dy}{dx} = \frac{n\left(x+\sqrt{1+x^2}\right)^n}{\sqrt{1+x^2}}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{ny}{\sqrt{1+x^2}}$ [From Equation(i)]

$\Rightarrow \sqrt{1+x^2} \frac{dy}{dx} = ny$.........(ii)

Differentiating both sides w.r.t x again,

$\Rightarrow \sqrt{1+x^2} \frac{d^2y}{dx^2} + \frac{2x}{2\sqrt{1+x^2}} \cdot \frac{dy}{dx} = n\frac{dy}{dx}$

$\Rightarrow \left(1+x^2\right) \frac{d^2y}{dx^2} + x\frac{dy}{dx} = n \cdot \sqrt{1+x^2} \frac{dy}{dx}$ [multiplying both sides by $\sqrt{1+x^2}$]

$\Rightarrow \left(1+x^2\right) \frac{d^2y}{dx^2} + x\frac{dy}{dx} = n\sqrt{1+x^2} \cdot \frac{ny}{\sqrt{1+x^2}}$ [From Equation(ii)]

$\therefore \left(1+x^2\right) \frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2 y$ Hence Proved

17. We have, $y^x = e^{y-x}$

$\Rightarrow \log y^x = \log e^{y-x}$

$\Rightarrow x\log y = (y-x). \log e = (y-x) [\because \log e = 1]$

$\Rightarrow \log y = \frac{(y-x)}{x}$ ...(i)

Now, differentiating w.r.t. x, we get

$\frac{d}{dx}\log y.\frac{dy}{dx} = \frac{d}{dx}\frac{(y-x)}{x}$

$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \frac{x.\frac{d}{dx}(y-x)-(y-x).\frac{d}{dx}.x}{x^2}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{x\left(\frac{dy}{dx}-1\right)-(y-x)}{x^2}$

$\Rightarrow \frac{x^2}{y}\cdot\frac{dy}{dx} = x\frac{dy}{dx} - x - y + x$

$\Rightarrow \frac{dy}{dx}\left(\frac{x^2}{y} - x\right) = -y$

$\therefore \frac{dy}{dx} = \frac{-y^2}{x^2-xy} = \frac{-y^2}{x(x-y)}$

$= \frac{y^2}{x^2(y-x)}.x = \frac{y^2}{x^2}\cdot\frac{1}{\frac{(y-x)}{x}}$

$= \frac{(1+\log y)^2}{\log y}\left[\because \log y = \frac{y-x}{x} \Rightarrow \log y = \frac{y}{x} - 1 \Rightarrow 1 + \log y = \frac{y}{x}\right]$

Hence Proved.

18. Let $y = u + v$

Where $u = (x\cos x)^x, v = (x.\sin x)^{\frac{1}{x}}$

$u = (x\cos x)^x$

Taking log both sides

$\log u = \log\,(x\cos x)^x$

$\log u = x.\log(x.\cos x)$

Differentiating both sides

$\frac{1}{u}\cdot\frac{du}{dx} = x.\frac{1}{x\cos x}(-x\sin x + \cos x.1) + \log(x\cos x).1$

$\frac{du}{dx} = u\,[-x\tan x + 1 + \log(x.\cos x)]$

$v = (x.\sin x)^{\frac{1}{x}}$

Taking log both side

$\log v = \log\,(x.\sin x)^{\frac{1}{x}}$

$\log v = \frac{1}{x}.\log(x.\sin x)$

Differentiate

$\frac{1}{v}\cdot\frac{dv}{dx} = \frac{1}{x}\cdot\frac{1}{x.\sin x}(x\cos x + \sin x.1) + \log(x.\sin x)\left(-\frac{1}{x^2}\right)$

$\frac{dv}{dx} = v\left[\frac{\cot x}{x} + \frac{1}{x^2} - \frac{\log(x\sin x)}{x^2}\right]$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$= (x\cos x)^x\,[-x.\tan x.1 + \log(x.\log x)] + (x.\sin x)^{\frac{1}{x}}\left[\frac{\cot x}{x} + \frac{1}{x^2} - \frac{\log(x.\sin x)}{x^2}\right]$