Long Answer Questions-II

[6 Marks]

Q.1. Solve the following differential equation: $3\mathbf{e^x}\,tan\,\mathbf{y\ dx} + (2 - \mathbf{e^x})\sec^2\mathbf{y\ dy} = 0$**, given that when** $x = 0, y = \frac{\pi}{4}$

Ans.

Given, $3e^x \tan y\, dx + (2 - e^x) \sec^2 y\, dy = 0$

$\Rightarrow \quad (2 - e^x) \sec^2 y\, dy = -3e^x \tan y\, dx$

$\Rightarrow \quad \frac{\sec^2 y}{\tan y} dy = \frac{-3e^x}{2 - e^x} dx$

$\Rightarrow \quad \int \frac{\sec^2 y\, dy}{\tan y} = 3 \int \frac{-e^x\, dx}{2 - e^x}$

$\Rightarrow \quad \log |\tan y| = 3 \log |2 - e^x| + \log C$

$\Rightarrow \quad \log |\tan y| = \log |C. (2 - e^x)^3|$

$\Rightarrow \quad \tan y = C(2 - e^x)^3$

Putting $x = 0, y = \frac{\pi}{4}$, we get

$\Rightarrow \quad \tan\frac{\pi}{4} = C(2 - e^0)^3$

$\Rightarrow \quad 1 = C(2 - 1)^3$

$\Rightarrow \quad 1 = C$

Therefore, particular solution is

$\tan y = (2 - e^x)^3$.

Q.2. Solve: $x\, dy - y\, dx = \sqrt{x^2 + y^2}\, dx$

Ans.

The given differential equation can be written as

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\sqrt{x^2+y^2}+y}{x}, x \neq 0$$

Clearly, it is a homogeneous differential equation.

Putting $y = \mathbf{vx}$ **and** $\frac{\mathrm{dy}}{\mathrm{dx}} = v + x\frac{\mathrm{dv}}{\mathrm{dx}}$ in it, we get

$$v + x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\sqrt{x^2+v^2x^2}+\mathrm{vx}}{x}$$

$$\Rightarrow \quad v + x\frac{\mathrm{dv}}{\mathrm{dx}} = \sqrt{1+v^2} + v$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} = \sqrt{1+v^2}$$

$$\Rightarrow \quad \frac{\mathrm{dv}}{\sqrt{1+v^2}} = \frac{\mathrm{dx}}{x}$$

Integrating both sides, we get

$$\int \frac{1}{\sqrt{1+v^2}}\mathrm{dv} = \int \frac{1}{x}\mathrm{dx}$$

$$\Rightarrow \quad \log|v+\sqrt{1+v^2}| = \log|x| + \log C$$

$$\Rightarrow \quad |v+\sqrt{1+v^2}| = |\mathrm{Cx}|$$

$$\Rightarrow \quad \left|\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right| = |\mathrm{Cx}| \qquad [\because\ v = y/x]$$

$$\Rightarrow \quad \{y+\sqrt{x^2+y^2}\}^2 = C^2x^4 \qquad \text{[Squaring both sides]}$$

Hence, $\{y+\sqrt{x^2+y^2}\}^2 = C^2x^4$ gives the required solution.

Q.3. Find the particular solution of the differential equation $(1+x^3)\frac{\mathrm{dy}}{\mathrm{dx}} + 6x^2y = (1+x^2),$ **given that *y* = 1 when *x* = 1.**

Ans.

The given differential equation is

$$(1+x^3)\frac{dy}{dx}+6x^2y=(1+x^2)$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{6x^2}{(1+x^3)}y=\frac{1+x^2}{1+x^3}$$

It is in the form of $\frac{dy}{dx}+Py=Q$, where $P=\frac{6x^2}{(1+x^3)}, Q=\frac{1+x^2}{1+x^3}$

$$\therefore \quad IF=e^{\int P\ dx}=e^{\int \frac{6x^2}{1+x^3}\ dx}$$

$$=e^{2\int \frac{3x^2}{1+x^3}\ dx}=e^{2\int \frac{dt}{t}} \qquad [\text{Let } 1+x^3=t \Rightarrow 3x^2dx=dt]$$

$$=e^{2\log t}=e^{\log t^2}=t^2$$

$$=(1+x^3)^2$$

Therefore, general solution is

$$y.(1+x^3)^2=\int \frac{1+x^2}{1+x^3}\times(1+x^3)^2dx+C$$

$$=\int(1+x^2)(1+x^3)dx+C=\int(x^5+x^3+x^2+1)dx+C$$

$$y.(1+x^3)^2=\frac{x^6}{6}+\frac{x^4}{4}+\frac{x^3}{3}+x+C$$

Putting $y=1$, $x=1$, we get

$$\therefore \quad 4=\frac{1}{6}+\frac{1}{4}+\frac{1}{3}+1+C$$

$$\Rightarrow \quad C=4-\frac{1}{6}-\frac{1}{4}-\frac{1}{3}-1=\frac{9}{4}$$

Required particular solution is $y(1+x^3)^2=\frac{x^6}{6}+\frac{x^4}{4}+\frac{x^3}{3}+x+\frac{9}{4}$.

Q.4. Show that the differential equation $(xe^{\frac{y}{x}}+y)\,dx=x\,dy$ is homogeneous. Find the particular solution of this differential equation, given that *x* = 1 when *y* = 1.

Ans.

Given differential equation is,

$$\left(x.e^{\frac{y}{x}}+y\right)\mathrm{dx}=\mathrm{xdy}$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}}=\frac{x.e^{\frac{y}{x}}+y}{x} \qquad \ldots(i)$$

Let $F(x,y)=\frac{x.e^{\frac{y}{x}}+y}{x}$

$$\therefore \quad F(\lambda x,\lambda y)=\frac{\lambda x.e^{\frac{\lambda y}{\lambda x}}+\lambda y}{\lambda x}=\lambda^0\frac{x.e^{\frac{y}{x}}+y}{x}=\lambda^0F(x,y)$$

Hence, given differential equation (i) is homogenous.

Let $y=vx \quad \Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}}=v+x.\frac{\mathrm{dv}}{\mathrm{dx}}$

Now, given differential equation (i) would become

$$v+x\frac{\mathrm{dv}}{\mathrm{dx}}=\frac{x.e^{\frac{\mathrm{vx}}{x}}+\mathrm{vx}}{x}$$

$$\Rightarrow \quad v+x.\frac{\mathrm{dv}}{\mathrm{dx}}=e^v+v$$

$$\Rightarrow \quad x.\frac{\mathrm{dv}}{\mathrm{dx}}=e^v$$

$$\frac{\mathrm{dv}}{e^v}=\frac{\mathrm{dx}}{x}$$

$$\Rightarrow \quad \int e^{-v}\mathrm{dv}=\int\frac{\mathrm{dv}}{v}$$

$$\Rightarrow \quad \frac{e^{-v}}{-1}=\log x+C$$

$$-e^{-\frac{y}{x}}=\log x+C$$

$\Rightarrow \quad -\frac{1}{e^{\frac{y}{x}}} = \log x + C$

$\Rightarrow \quad e^{\frac{y}{x}} . \log x + \mathrm{C}e^{\frac{y}{x}} + 1 = 0$

Putting $x = 1$, $y = 1$, we get

$\therefore \quad e \log 1 + Ce + 1 = 0$

$\Rightarrow \quad C = -\frac{1}{e}$

$\therefore$ The required particular solution is

$$e^{\frac{y}{x}} . \log x - \frac{1}{e} e^{\frac{y}{x}} + 1 = 0 \quad \text{or} \quad e^{\frac{y}{x}} \log x - e^{\frac{y}{x} - 1} + 1 = 0$$

Q.5. Show that the differential equation $\left[x \sin^2\left(\frac{y}{x}\right) - y\right]\mathrm{dx} + x\,\mathrm{dy} = 0$ is homogeneous. Find the particular solution of this differential equation, given that $y = \frac{\pi}{4}$ when *x* = 1.

Ans.

Given differential equation is,

$$\left[x \sin^2\left(\frac{y}{x}\right) - y\right] \mathrm{dx} + x\,\mathrm{dy} = 0$$

$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y - x \sin^2\left(\frac{y}{x}\right)}{x} \qquad \ldots(i)$

Let $F(x,y) = \frac{y - x \sin^2\left(\frac{y}{x}\right)}{x}$

Then $F(\lambda x, \lambda y) = \frac{\lambda y - \lambda x \sin^2 \frac{\lambda y}{\lambda x}}{\lambda x} = \lambda^0 \frac{y - x \sin^2 \frac{y}{x}}{x} = \lambda^0 F(x,y)$

Hence, differential equation (i) is homogeneous.

Now, let $y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

Putting these value in (i), we get

$$\frac{dv}{dx} = \frac{vx - x\sin^2\frac{vx}{x}}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{x\{v - \sin^2 v\}}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v - \sin^2 v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \quad \frac{dv}{\sin^2 v} = -\frac{dx}{x}$$

Integrating both sides, we get

$$\Rightarrow \quad \int \operatorname{cosec}^2 v\, dv = -\int \frac{1}{x} dx$$

$$\Rightarrow \quad -\cot v = -\log x + C$$

$$\Rightarrow \quad \log x - \cot\left(\frac{y}{x}\right) = C \qquad \ldots(ii)$$

Putting $y = \frac{\pi}{4}$ and $x = 1$ in (ii), we get

$$\log 1 - \cot\frac{\pi}{4} = C$$

$$\Rightarrow \quad 0 - 1 = C$$

$$\Rightarrow \quad C = -1$$

Hence, particular solution is

$$\log x - \cot\left(\frac{y}{x}\right) = -1$$

$$\Rightarrow \quad \log x - \cot\left(\frac{y}{x}\right) + 1 = 0$$

Q.6. Find the differential equation of the family of curves $(x-h)^2+(y-k)^2=r^2$, where *h* and *k* are arbitrary constants.

Ans.

Given family of curve is:

$$(x-h)^2+(y-k)^2=r^2 \qquad \ldots(i)$$

Differentiating with respect to x, we get

$$\Rightarrow \quad 2(x-h)+2(y-k).\frac{dy}{dx}=x$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{x-h}{y-k} \qquad \ldots(ii)$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2}=-\left\{\frac{(y-k)-(x-h).\frac{dy}{dx}}{(y-k)^2}\right\}=-\left\{\frac{(y-k)+(x-h).\frac{x-h}{y-k}}{(y-k)^2}\right\} \qquad [\text{From } (\text{ii})]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-\left\{\frac{(y-k)^2+(x-h)^2}{(y-k)^3}\right\}=-\frac{r^2}{(y-k)^3} \qquad \ldots(\text{iii}) \qquad [\text{From } (i)]$$

From (ii) $\left(\frac{dy}{dx}\right)^2=\left(\frac{x-h}{y-k}\right)^2$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2=\frac{(x-h)^2}{(y-k)^2}$$

Adding 1 both the sides, we get

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2+1=\frac{(x-h)^2}{(y-k)^2}+1=\frac{(x-h)^2+(y-k)^2}{(y-k)^2}$$

Putting exponent (power) $\frac{3}{2}$ both sides, we get

$$\Rightarrow \quad \left[\left(\frac{dy}{dx}\right)^2+1\right]^{\frac{3}{2}}=\left[\frac{r^2}{(y-k)^2}\right]^{\frac{3}{2}}=\frac{r^3}{(y-k)^3}$$

$$\Rightarrow \quad \left[\left(\frac{dy}{dx}\right)^2+1\right]^{\frac{3}{2}}=r.\frac{r^2}{(y-k)^3}=-r\frac{d^2y}{dx^2} \qquad [\text{Using } (\text{iii})]$$

$$\Rightarrow \quad r\frac{d^2y}{dx^2}+\left[\left(\frac{dy}{dx}\right)^2+1\right]^{3/2}=0$$

Q.7. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x(2 \log x + 1)}{\sin y + y \cos y}$ given that $y = \frac{\pi}{2}$ when *x* = 1.

Ans.

Given differential equation is $\frac{dy}{dx} = \frac{x(2 \log x+1)}{\sin y+y \cos y}$

$\Rightarrow \quad (\sin y + y \cos y)\, dy = x\,(2 \log x + 1)\,dx$

$\Rightarrow \quad \int \sin y \,dy + \int y \cos y \,dy = 2\int x \log x \,dx + \int x \,dx$

$\Rightarrow \quad \int \sin y \,dy + [y \sin y - \int \sin y \,dy] = 2\left[\log x \frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}\,dx\right] + \int x \,dx$

$\Rightarrow \quad \int \sin y \,dy + y \sin y - \int \sin y \,dy = x^2 \log x - \int x \,dx + \int x \,dx + C$

$\Rightarrow \quad y \sin y = x^2 \log x + C \qquad \ldots (i)$

It is general solution.

For particular solution, we put $y = \frac{\pi}{2}$ when $x = 1$

(i) becomes $\frac{\pi}{2} \sin \frac{\pi}{2} = 1 \,.\, \log 1 + C$

$\frac{\pi}{2} = C \qquad [\because \log 1 = 0]$

Putting the value of C in (i), we get the required particular solution

$y \sin y = x^2 \log x + \frac{\pi}{2}$

Q.8. Show that the family of curves for which the slope of the tangent at any point (x, y) on it is $\frac{x^2+y^2}{2xy}$, is given by $x^2 - y^2 = Cx$.

Ans.

We know that the slope of the tangent at any point on a curve is $\frac{dy}{dx}$

Therefore, $\frac{dy}{dx}=\frac{x^2+y^2}{2\,xy}$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1+\frac{y^2}{x^2}}{\frac{2y}{x}} \qquad \ldots(i)$$

Clearly, equation (i) is a homogeneous differential equation. To solve it we make substitution

$$y=\mathrm{vx}$$

$$\Rightarrow \quad \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Putting the value of y and $\frac{dy}{dx}$ in equation (i), we get

$$v+x\frac{dv}{dx}=\frac{1+v^2}{2v}$$

or $\quad x\frac{dv}{dx}=\frac{1-v^2}{2v}$

$$\Rightarrow \quad \frac{2v}{1-v^2}\mathrm{dv}=\frac{dx}{x}$$

$$\Rightarrow \quad \frac{2v}{v^2-1}\mathrm{dv}=-\frac{dx}{x}$$

Integrating both sides, we get

$$\int\frac{2v}{v^2-1}\,\mathrm{dv}=-\int\frac{1}{x}\,\mathrm{dx}$$

or $\quad \log|v^2-1|=-\log|x|+\log|C_1|$

or $\quad \log|(v^2-1)\,(x)|=\log|C_1|$

or $\quad (v^2-1)\,x=\pm\,C_1$

Replacing v by $\frac{y}{x}$ we get

$$\left(\frac{y^2}{x^2}-1\right)x=\pm C_1$$

$$\Rightarrow \quad (y^2-x^2)=\pm C_1x$$

$$\Rightarrow \quad x^2-y^2=\mathrm{Cx} \qquad (\text{where } \pm C_1=C)$$