ENGINEERING MATHEMATICS

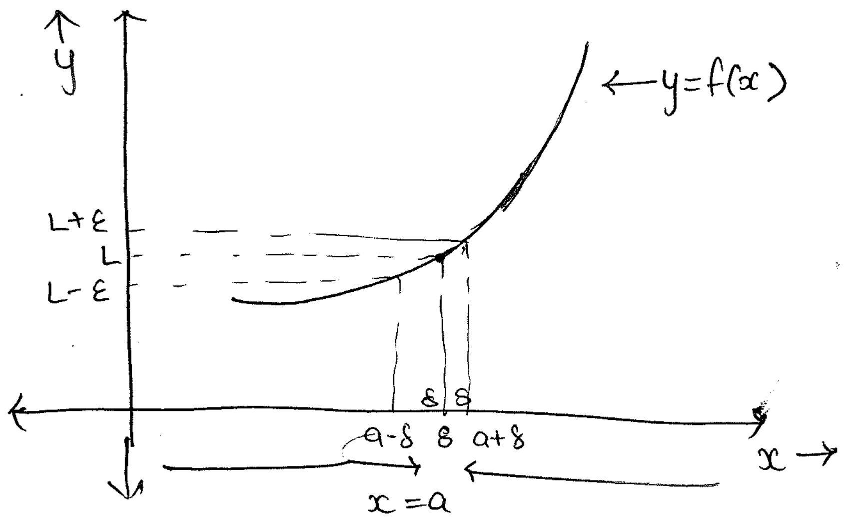
* Calculus *

1. Limit of a function
2. Continuity of a function
3. Differentiability
4. Mean value Theorem
5. Maxima and Minima
6. Definite and Indefinite integral
7. Vector Calculus

Limit :- A number 'l' is said to be the limit of a function $f(x)$ as $x \longrightarrow a$, $\forall x$, $\varepsilon > 0$ (however smal $\exists_a$ $\delta > 0$ such that $|f(x) - l| < \varepsilon$, $0 < |x - a| < \delta$.

$$\lim_{x \to a} f(x) = l$$

NOTE: A limit of a function can be real no. $-\infty$ and $+\infty$.



Left hand Limit (L.H.L.)

If i approach a from left hand side i.e. lesser value than a, then limit is known as left hand limit

$$\boxed{\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)}$$

$a-\delta < x < a$

Right hand limit (R.H.L.)

- If i approach a from right hand side i.e. greates value then a, then limit is known as right hand limit.

$$\boxed{\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)}$$

$a < x < a+\delta$

* Existence of a limit

- Limit of a function exists if left hand limit exists, R.H.L. exists and both are equal,

i.e. L.H.L. = R.H.L.

Q:- Check whether the limit of a function exists or not for given function $f(x) = \lim_{x\to a} \frac{1}{x-a}$

$$\text{L.H.L.} = \lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)$$

$$\text{L.H.L.} = \lim_{x\to a^-} f(x) = \lim_{h\to 0} \frac{1}{a-h-a} = -\infty$$

$$\text{R.H.L.} = \lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$$

$$= \lim_{h\to 0} \frac{1}{a+h-a} = +\infty$$

L.H.L. ≠ R.H.L.

Limit Doesn't exists.

Q:- $f(x)=\lim_{x\to 0} 2^{-1/x^2}$. Check existence of limit.

$L.H.L = \lim_{x\to 0^-} 2^{-1/0} = 2^{-\infty} = 0$

$R.H.L = \lim_{x\to 0^+} 2^{-1/0} = 2^{-\infty} = 0$

L.H.L. = R.H.L.

Limit will exists.

* FORMULAS *

① $\lim_{x\to a} \frac{x^n - a^n}{x-a} = n a^{n-1}$

② $\lim_{x\to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} \cdot a^{m-n}$

③ $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

④ $\lim_{x\to 0} \frac{e^{mx} - 1}{x} = m$

⑤ $\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$

⑥ $\lim_{x\to 0} \frac{a^x - b^x}{x} = \log(a/b)$

⑦ $\lim_{x\to 0} (1+x)^{1/x} = e$

⑧ $\lim_{x\to 0} (1+mx)^{1/x} = e^m$

⑨ $\lim_{x\to 0} (1+mx)^{n/x} = e^{m\cdot n}$

⑩ $\lim_{x\to 0} (1+mx)^{1/nx} = e^{m/n}$

⑪ $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\tan x}{x} = 1$

⑫ $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = \lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$

⑬ $\lim_{x\to 0} \frac{1-\cos mx}{1-\cos nx} = \frac{m^2}{n^2}$

⑭ $\lim_{x\to 0} \frac{\cos ax - \cos bx}{x^2} = \frac{b^2 - a^2}{2}$

⑮ $\lim_{x\to a} f(x)^{g(x)}$

$f(x)\to$ finite, $g(x)\to$ infinite

$e^{\lim_{x\to a} g(x)[f(x)-1]}$

⑯ $\lim_{x\to \infty} (1+1/x)^x = e$

⑰ $\lim_{x\to \infty} (1-1/x)^x = e^{-1}$

* L' Hôpital Rule *

- In-determinant form is $\frac{0}{0}, \frac{\infty}{\infty}$ then we will apply below rules.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)} = \dots$$

Other indeterminate form

$0 \times \infty$

$\uparrow \quad \uparrow$

$f(x) \quad g(x)$

$$\frac{f(x)}{1/g(x)} \Rightarrow \text{Make } \frac{\infty}{\infty} \text{ OR } \frac{0}{0}.$$

$1^{\infty}, 0^{\infty}$, etc.

$$y = \lim_{x\to a} f(x)$$

taking log on both sides

$$\log y = \lim_{x\to a} \log f(x)$$

$$y = e^{\lim_{x\to a}\log f(x)}$$

$\Rightarrow$ If $f(x)$ and $g(x)$ both are algebraic function and $x \to \infty$

Case:-1 Degree of $f(x) >$ Degree of $g(x)$ ANS: ∞

Eg:- $\lim_{x\to\infty}\frac{2x^3+3x^2+1}{2x^2+5x+1}$

$$\lim_{x\to\infty}\frac{x^3(2+3/x+1/x^3)}{x^2(2+5/x+1/x^2)}$$

$$\lim_{x\to\infty}\frac{x\cdot\cancel{2}}{\cancel{2}} = \infty$$

Case:2 Degree of $f(x) <$ Degree of $g(x)$ ANS: 0.

Eg:- $\lim_{x\to\infty}\frac{2x^2+1}{x^3+3x^2+1}$

$$\lim_{x\to\infty}\frac{x^2(2+1/x^2)}{x^3(1+3/x+1/x^3)}$$

$$\lim_{x\to\infty}\frac{(2+1/x^2)}{x(1+3/x+1/x^3)} = 0$$

Case: 3 Degree of $f(x)$ = Degree of $g(x)$

ANS: Co-efficient of highest degree

Eg: $\lim_{x\to\infty} \frac{2x^3+3x^2+1}{3x^3+5x^2+1}$

$= 2/3$

Q:- $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ $\left(\frac{0}{0}\right)$

$\lim_{x\to 0} \frac{0+\sin x}{2x}$ $\left(\frac{0}{0}\right)$

$\lim_{x\to 0} \frac{\cos x}{2}$

$= \boxed{1/2}$

Q:- $\lim_{x\to 0} \frac{1-\cos x}{x\sin x}$

$\lim_{x\to 0} \frac{1-\cos x}{x\left(\frac{\sin x}{x}\right)\cdot x}$

$\lim_{x\to 0} \frac{1-\cos x}{x^2}$

$= \boxed{1/2}$

★ Q:- $\lim_{x\to 0} \frac{\sinh x - \sin x}{x\sin^2 x}$

$\lim_{x\to 0} \frac{\sinh x - \sin x}{x\left(\frac{\sin^2 x}{x^2}\right)\cdot x^2}$

$\lim_{x\to 0} \frac{\sinh x - \sin x}{x^3}$ $\left(\frac{0}{0}\right)$

$\lim_{x\to 0} \frac{\cosh x - \cos x}{3x^2}$ $\left(\frac{0}{0}\right)$

$\lim_{x\to 0} \frac{\sinh x + \sin x}{6x}$ $\left(\frac{0}{0}\right)$

$\lim_{x\to 0} \frac{\cosh x + \cos x}{6}$

$= \frac{1+1}{6} = \boxed{\frac{1}{3}}$

NOTE: when both function is trigonometric try to avoid L' Hôpital Rule.

Q:- $\lim_{x\to\infty} (x)^{1/x}$

$y = \lim_{x\to\infty} (x)^{1/x}$

$\log y = \lim_{x\to\infty} \frac{1}{x}\log x$

$y = e^{\lim_{x\to\infty} \frac{\log x}{x}}$

$y = e^0 = \boxed{1}$

Q:- $\lim\limits_{x\to\infty} \frac{x+\sin x}{x}$

$\lim\limits_{x\to\infty} 1+\frac{\sin x}{x}$

$= 1 + \frac{[-1,1]}{\infty}$

$= 1 + 0$

$= \boxed{1}$

☆ Q:- $\lim\limits_{n\to\infty} \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots\cdots + \frac{1}{n(n+1)}$

$\lim\limits_{n\to\infty} \sum\limits_{n=1}^{n} \frac{1}{n(n+1)}$ $\qquad S_n = \Sigma T_n$

$\lim\limits_{n\to\infty} \sum\limits_{n=1}^{n} \frac{1}{n} - \frac{1}{n+1}$

$\lim\limits_{n\to\infty} \left[\left(1-\cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots\cdots \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)\right]$

$\lim\limits_{n\to\infty} 1 - 1/n+1$

$= \boxed{1}$

Q:- $\lim\limits_{x\to 0} \frac{e^x - (1+x+x^2/2)}{x^3}$ $\left(\frac{0}{0}\right)$

$\lim\limits_{x\to 0} \frac{e^x - (0+1+x)}{3x^2}$ $\left(\frac{0}{0}\right)$

$\lim\limits_{x\to 0} \frac{e^x - 1}{6x}$ $\left(\frac{0}{0}\right)$

$\lim\limits_{x\to 0} \frac{e^x}{6}$

$= \boxed{1/6}$

Q:- $\lim\limits_{x\to 8} \frac{x^{1/3} - 2}{x-8}$ $\left(\frac{0}{0}\right)$

$\lim\limits_{x\to 8} \frac{\frac{1}{3}x^{-2/3}}{1}$

$= \frac{1}{3}(8)^{-2/3}$

$= \frac{1}{3}(2)^{-2}$

$= \boxed{\frac{1}{12}}$

Q:- $\lim\limits_{x \to 0} \dfrac{-\sin x}{2\sin x + \cos x}$

$= \boxed{0}$

Q:- $\lim\limits_{n \to \infty} \left(\sqrt{n^2+n} - \sqrt{n^2+1}\right)$

$\lim\limits_{n \to \infty} \sqrt{n^2+n} - \sqrt{n^2+1} \times \dfrac{\left(\sqrt{n^2+n} + \sqrt{n^2+1}\right)}{\left(\sqrt{n^2+n} + \sqrt{n^2+1}\right)}$

$\lim\limits_{n \to \infty} \dfrac{\cancel{n^2}+n - (\cancel{n^2}+1)}{\sqrt{n^2+n} + \sqrt{n^2+1}}$

$\lim\limits_{n \to \infty} \dfrac{1n - 1}{1n\sqrt{1+1/n} - 1n\sqrt{1+1/n}}$

$= \dfrac{1}{1+1} = \boxed{\dfrac{1}{2}}$

Q:- $\lim\limits_{x \to \infty} \sqrt{x^2+x-1} - x$

$\lim\limits_{x \to \infty} \sqrt{x^2+x-1} - x \times \dfrac{\left(\sqrt{x^2+x-1} + x\right)}{\left(\sqrt{x^2+x-1} + x\right)}$

$\lim\limits_{x \to \infty} \dfrac{\cancel{x^2} + x - 1 - \cancel{x^2}}{\sqrt{x^2+x-1} + x}$

$\lim\limits_{x \to \infty} \dfrac{x-1}{\sqrt{x^2+x-1} + x} \quad \left(\dfrac{\infty}{\infty}\right)$

$\lim\limits_{x \to \infty} \dfrac{1}{\dfrac{1(2x+1)}{2\sqrt{x^2+x-1}} + 1} = \lim\limits_{x \to \infty} \dfrac{1}{\dfrac{2x+1}{2 \cdot x\sqrt{1+1/x - 1/x^2}} + 1}$

$= \dfrac{1}{1+1} = \boxed{\dfrac{1}{2}}$

Q:- $\lim_{x\to 0} \frac{\log_e(1+4x)}{e^{3x}-1}$ $\left(\frac{0}{0}\right)$

$\lim_{x\to 0} \frac{1\cdot 4}{(1+4x)\,3e^{3x}}$

$= \boxed{\frac{4}{3}}$

Q:- $\lim_{x\to 0} \frac{\tan x}{x^2-x}$

$\lim_{x\to 0} \frac{\tan x}{x(x-1)}$

$\lim_{x\to 0} \frac{1}{x-1}$

$= \boxed{-1}$

Q:- $\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{2n}$ (1^{∞})

$y = \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{2n}$

$\log y = \lim_{n\to\infty} 2n\log\left(1-\frac{1}{n}\right)$

$y = e^{\lim_{n\to\infty} 2n\log(1-1/n)}$

$\boxed{y = e^{-2}}$

$= e^{\lim_{n\to\infty} \frac{2n}{1/\log(1-1/n)}}$

$= e^{\lim_{n\to\infty} \frac{2}{-\frac{1}{1-1/n}+\frac{1}{n^2}}}$

$= e^{-2}$