Direct and Inverse Proportions

Direct Proportion

Suppose you go to a grocery shop to buy eggs. If the shopkeeper is selling 2 eggs for Rs 4, then what amount is required to buy 5 eggs?

We can solve this problem using the unitary method.

First, let us determine the cost of 1 egg.

It is given that cost of 2 eggs = Rs 4

$\therefore$ Cost of 1 egg = $\frac{\text{Rs } 4}{2} = \text{Rs } 2$

Hence, cost of 5 eggs = 5 × Rs 2 = Rs 10

What do we observe in the above example?

One simple thing that we observe is that as the number of eggs increases, its cost also increases. Such situations are examples of **direct variation** or **direct proportion**. In our day-to-day lives, we come across various such situations. For example, if a car moves with constant speed, then the distance covered by it is in direct proportion with the time taken to cover the distance.

Direct variation or **Direct proportion** can be defined as follows:

When two variable quantities increase or decrease simultaneously such that their ratio remains unchanged, then it is an example of direct variation and the quantities are said to be in direct proportion. It is said that one variable "varies directly" with the other.

Writing direct variation or direct proportion in the form of symbols:

If there is direct variation between two variables x and y, then it is represented as:

$x \propto y$

It is read as "***x* varies directly as *y***" or "***x* is directly proportional to *y***".

Also, in direct proportion or direct variation, the ratio of variables is always a constant value.

Thus, $\frac{x}{y} = k \text{ or } x = ky$, where *k* is known as the **constant of proportionality**.

So, it can be concluded that:

If x α y, then $\frac{x}{y} = k \text{ or } x = ky$, where k is a constant.
or
If $\frac{x}{y} = k \text{ or } x = ky$, then x α y, where k is a constant.

$\frac{x}{y} = k$ **is the equation of direct proportion.**

Now, suppose variables *x* and *y* are in direct proportion. If y_1 and y_2 are the values of *y* corresponding to the respective values x_1 and x_2 of *x,* then $\frac{x_1}{y_1} = k \text{ and } \frac{x_2}{y_2} = k$.

Therefore, we can write the equation as follows:

$$\boxed{\frac{x_1}{y_1} = \frac{x_2}{y_2}}$$

Now, let us consider the following situation to check whether the variables involved in it are in direct proportion or not.

Arnab goes to a stationery shop to buy some pens. If each pen costs Rs 5, then what amount is required to buy 4 such pens? Also, determine the amount that Arnab need to pay to buy 10 such pens?

It is given that cost of 1 pen = Rs 5

Hence, cost of 4 pens = 4 × Rs 5 = Rs 20

Similarly, cost of 10 pens = 10 × Rs 5 = Rs 50

This information can be represented in the tabular form, where the number of pens is denoted by variable *x* and their corresponding cost is denoted by variable *y*, as

Number of pens: x	1	4	10
Cost (Rs): y	5	20	50

If we observe the ratio of the corresponding values of x and y, then we see that

$$\frac{1}{5} = \frac{1}{5}$$
$$\frac{4}{20} = \frac{1}{5}$$
$$\frac{10}{50} = \frac{1}{5}$$

We, thus, observe that as the number of pens (x) increases, their cost (y) also increases in such a manner that their ratio $\left(\frac{x}{y}\right)$ remains constant, say k. Thus, in this case, the value of k is $\frac{1}{5}$.

So, this is an example of **direct variation.**

Hence, we say that ***x* and *y* are in direct proportion**, **if** $\frac{x}{y} = k \text{ or } x = ky$

Thus, to check whether the variables x and y are in direct proportion, we need to find the ratio $\frac{x}{y}$ for their corresponding values. If this ratio remains constant, then the variables are in direct proportion, otherwise they are not.

Let us now discuss some examples based on this concept.

Example 1:

The scale of a map is given as 1:10000. The distance between two buildings in a city on the map is 5 cm. What is the actual distance between the two buildings?

Solution:

Given situation is an example of direct variation.

So, it can be said that the distance on the map and the actual distance between the buildings are in direct proportion.

Let the actual distance between the buildings be x.

$$\therefore \frac{1}{10000} = \frac{5 \text{ cm}}{x}$$
$$\Rightarrow x = (10000 \times 5) \text{ cm}$$
$$x = 50000 \text{ cm}$$
$$x = \frac{50000}{100} \text{ m}$$
$$x = 500 \text{ m}$$

Thus, the actual distance between the buildings is 500 m.

Example 2:

If 1 kg 600 g of rice is sufficient for 20 people, then what quantity of rice will be sufficient for 27 people? Also calculate for how many people 2 kg 400 g of rice will be sufficient.

Solution:

Let *x* kg of rice be sufficient for 27 people and 2 kg 400 g of rice be sufficient for *y* number of people.

1 kg 600 g = 1.6 kg

2 kg 400 g = 2.4 kg

The given information can be represented by the following table:

Quantity of rice (kg): x	1.6	x	2.4
Number of people: y	20	27	y

In the given case, quantity of rice increases and decreases as the number of people increases and decreases respectively. So, this is an example of direct variation.

So, it can be said that the quantity of rice and the number of people are in direct proportion.

$$\therefore \frac{1.6}{20} = \frac{x}{27} \text{ and } \frac{1.6}{20} = \frac{2.4}{y}$$

$$x = \frac{27 \times 1.6}{20} \text{ and } y = \frac{2.4 \times 20}{1.6}$$

$$x = 2.16 \text{ kg and } y = 30 \text{ people}$$

$$x = 2 \text{ kg } 160 \text{ g and } y = 30 \text{ people}$$

Hence, 2 kg 160 g of rice will be sufficient for 27 people and 2 kg 400 g of rice will be sufficient for 30 people.

Example 3:

A car travels at a constant speed of 35 km/h. How far can it travel in 15 minutes?

Solution:

Speed of car = 35 km/h

This means that the car travels 35 km in 60 minutes.

Let the car travel *x* km in 15 minutes.

The given information can be represented by the following table:

Distance covered by the car (km): x	35	x
Time taken by the car (min): y	60	15

In the given case, the time taken by the car increases and decreases as the distance covered by the car increases and decreases respectively. So, this is an example of direct variation.

So, it can be said that the distance covered by the car and the time taken by it are in direct proportion.

$$\therefore \frac{35 \text{ km}}{60 \text{ min}} = \frac{x}{15 \text{ min}}$$

$$\Rightarrow x = \left(\frac{35}{60} \times 15\right) \text{ km}$$

$$\Rightarrow x = \left(\frac{35}{4}\right) \text{ km} = 8.75 \text{ km}$$

Thus, the car travels 8.75 km in 15 minutes.

Example 4:

If 12 machines can be assembled in 4 hours, then how many machines can be assembled in 8 hours?

Solution:

Let x machines be assembled in 8 hours.

The given information can be represented by the following table:

Number of machines: x	12	x
Time taken (hours): y	4	8

In the given case, the time taken increases and decreases as the number of machines increases and decreases respectively. So, this is an example of direct variation.

So, it can be said that the number of machines and the time taken to assemble them are in direct proportion.

$$\therefore \frac{12}{4} = \frac{x}{8}$$
$$x = \frac{12 \times 8}{4}$$
$$x = 24$$

Thus, 24 machines can be assembled in 8 hours.

Example 5:

At a particular time of the day, the length of the shadow of a 28 feet high tree is 1.4 feet. Determine the height of a tree that has a shadow of length 2.3 feet at the same time of the day?

Solution:

Let the height of the tree having shadow of length 2.3 feet at the given time be x feet.

The given information can be represented by the following table:

Height of the tree (ft): x	28	x
Length of the shadow (ft): y	1.4	2.3

In the given case, length of the shadow increases and decreases as height of the tree increases and decreases respectively. So, this is an example of direct variation.

So, it can be said that at a particular time of the day, the height of the tree and length of its shadow are in direct proportion.

$$\therefore \frac{28 \text{ ft}}{1.4 \text{ ft}} = \frac{x}{2.3 \text{ ft}}$$
$$x = \left(\frac{28}{1.4} \times 2.3\right) \text{ ft}$$
$$x = 46 \text{ ft}$$

Hence, the height of the second tree is 46 feet.

Example 6:

State whether the given situations involve two variables in direct proportion.

1. **Distance covered by a car and the quantity of petrol consumed by it**
2. **Number of workers and the time taken by them to complete a work**
3. **Speed of a person and the time taken by him/her to cover a fixed distance**
4. **Speed of a person and the distance covered by him/her in a fixed time**
5. **Time period and simple interest if the rate of interest is fixed**
6. **Time period and rate of interest if the simple interest is fixed**
7. **Area of cultivated land and the crop harvested**

Solution:

(a) The quantity of petrol consumed by a car increases if the distance covered by it also increases. Thus, the distance covered by a car is in direct proportion with the quantity of petrol consumed by it.

(b) More number of workers will take less time to complete a work. Thus, the number of workers and the time taken by them to complete the work are not in direct proportion.

(c) If a person travels at a higher speed, then he/she takes less time to cover a fixed distance. Thus, the speed of a person and the time taken by him/her to cover a fixed distance are not in direct proportion.

(d) If a person increases his/her speed, then he/she will cover more distance in a fixed time. Hence, the speed of a person and the distance covered by him/her in a fixed time are in direct proportion.

(e) For a fixed rate of interest, if the time period is more, then the simple interest will also be more. Hence, the time period and the simple interest are in direct proportion if the rate of interest is fixed.

(f) For a fixed simple interest, if the time period is more, then the rate of interest will be less. Hence, the time period and the rate of interest are not in direct proportion if the rate of interest is fixed.

(g) The more the area of land cultivated, the more will be the amount of crop harvested. Hence, the area of cultivated land and the crop harvested are in direct proportion.

Example 7:

The following table lists the distance covered by a person and the corresponding time taken by him to cover this distance. Check whether the distance covered changes in direct proportion with the time taken.

Distance covered	Time taken
4 km	16 minutes
6 km	24 minutes
12 km	48 minutes
20 km	72 minutes

Solution:

The above information can be represented in a table by taking the distance covered (km) as variable x and the corresponding time taken (minutes) as y.

Distance covered (km): x	4	6	12	20
Time taken (minutes): y	16	24	48	72

Thus, we can find the ratio $\frac{x}{y}$ for the corresponding values of *x* and *y* as

$$\frac{4}{16}=\frac{1}{4}$$
$$\frac{6}{24}=\frac{1}{4}$$
$$\frac{12}{48}=\frac{1}{4}$$
$$\frac{20}{72}=\frac{5}{18}$$

Since the ratio $\frac{x}{y}$ does not remain constant, the distance covered by the person does not change in direct proportion with the time taken.

Example 8:

Observe the following tables and find whether *x* and *y* are directly proportional.

(a)

x	2	9	14	15	17	19
y	6	27	42	45	51	57

(b)

x	50	45	37.5	30	25	5
y	40	36	30	24	20	4

(c)

x	50	48	42	39	30	17
y	40	38	32	29	20	7

Solution:

(a)

x	2	9	14	15	17	19

y	6	27	42	45	51	57

Thus, $\frac{2}{6} = \frac{1}{3}$

$\frac{9}{27} = \frac{1}{3}$

$\frac{14}{42} = \frac{1}{3}$

$\frac{15}{45} = \frac{1}{3}$

$\frac{17}{51} = \frac{1}{3}$

$\frac{19}{57} = \frac{1}{3}$

Since the ratio $\frac{x}{y}$ remains constant for the corresponding values of *x* and *y*, the variables *x* and *y* are directly proportional.

(b)

x	50	45	37.5	30	25	5
y	40	36	30	24	20	4

Thus, $\frac{50}{40} = \frac{5}{4}$

$\frac{45}{36} = \frac{5}{4}$

$\frac{37.5}{30} = \frac{5}{4}$

$\frac{30}{24} = \frac{5}{4}$

$\frac{25}{20} = \frac{5}{4}$

$\frac{5}{4} = \frac{5}{4}$

Since the ratio $\frac{x}{y}$ remains constant for the corresponding values of *x* and *y*, the variables *x* and *y* are directly proportional.

(c)

x	50	48	42	39	30	17
y	40	38	32	29	20	7

Thus, $\frac{50}{40} = \frac{5}{4}$

$\frac{48}{38} = \frac{24}{19}$

$\frac{42}{32} = \frac{21}{16}$

$\frac{39}{29} = \frac{39}{29}$

$\frac{30}{20} = \frac{3}{2}$

$\frac{17}{7} = \frac{17}{7}$

Since the ratio $\frac{x}{y}$ does not remain constant for the corresponding values of *x* and *y*, the variables *x* and *y* are **not** directly proportional.

Example 9:

If *a* varies directly as *b* and *b* = 56 when *a* = 49 then find the value of *a* when *b* = 64.

Solution:

We have

$a \propto b$

 $a = kb$ (Where *k* is the constant of proportionality)

 $49 = k \times 56$ ($b = 56$ when $a = 49$)

$$\Rightarrow k = \frac{7}{8}$$

On substituting the value of k, the equation of variation becomes

$$a = \frac{7}{8} \times b$$

$$\Rightarrow a = \frac{7}{8} \times 64 \qquad (\text{When } b = 64)$$

$$\Rightarrow a = 56$$

Example 10:

The area of a circle varies directly as the square of its radius. The area of a circle having diameter of 21 cm is 346.5 cm^2. Find the constant of proportionality. Also, find the radius of the circle having an area of 154 cm^2.

Solution:

Let the area, radius and diameter of the circle be A, r and d respectively.

It is given that

$A \propto r^2$

$\Rightarrow A = kr^2$ (Where k is the constant of proportionality)

$$\Rightarrow A = k \times \left(\frac{d}{2}\right)^2 \qquad \left(r = \frac{d}{2}\right)$$

$$\Rightarrow 346.5 = k \times \left(\frac{21}{2}\right)^2 \qquad (A = 346.5\,\text{cm}^2 \text{ when } d = 21\text{cm})$$

$$\Rightarrow 346.5 = k \times \frac{441}{4}$$

$$\Rightarrow 346.5 = k \times 110.25$$

$$\Rightarrow k = \frac{22}{7}$$

Substituting the value of k, the equation of variation becomes

$A = \frac{22}{7} r^2$

When $A = 154 \text{ cm}^2$, we have

$154 = \frac{22}{7} r^2$

$\Rightarrow r^2 = \frac{154 \times 7}{22}$

$\Rightarrow r^2 = 7 \times 7$

$\Rightarrow r = 7$ (Radius cannot be negative)

Thus, the value of constant of proportionality is $\frac{22}{7}$ and radius of the required circle 7 cm.

Example 11:

If the wages of 9 workers is Rs 1350, find the wages of 14 workers.

Answer:

Wages of 9 workers = Rs 1350

Let the wages of 14 workers be *x*.

When the number of workers increases, total wage will also increases. So, this is the case of direct proportion.

Therefore,

$9 : 14 :: 1350 : x$

$9x = 14 \times 1350$

$x = \frac{14 \times 1350}{9}$

$x = 2100$

Thus, the wages of 14 workers is Rs 2100.

Inverse Proportion

We have learnt to solve the problems where variables vary directly, but there are situations where variables can vary inversely.

When the first quantity increases with the decrease in the second quantity and when the first quantity decreases with the increase in the second quantity, such that their product remains unchanged, then the quantities are said to be in inverse proportion. It is said that one variable "varies inversely" with the other.

Let us consider such a problem.

6 men can whitewash a house in 20 days. Can we calculate the number of men required to whitewash that house in (i) 10 days (ii) 30 days?

Writing inverse variation or inverse proportion in the form of symbols:

If there is inverse variation between variables x and y, then it is represented as:

$$x \propto \frac{1}{y}$$

It is read as "***x* varies inversely as *y***" or "***x* is inversely proportional to *y***".

Also, in inverse variation or inverse proportion, the product of variables is always a constant value.
Thus, $xy = k$, where k is known as the **constant of proportionality.**

So, it can be concluded that:

If $xy = k$, then $x \propto \frac{1}{y}$, where k is the constant of proportionality.

or

If $x \propto \frac{1}{y}$, then $xy = k$, where k is the constant of proportionality.

$xy = k$ is the equation of inverse variation.

Now, if y_1 and y_2 are two values of y, and x_1 and x_2 are two values of x corresponding to y_1 and y_2 respectively, then we have

$x_1y_1 = k$ … (1)

$x_2y_2 = k$ … (2)

From the equations (1) and (2), we obtain

$$x_1y_1 = x_2y_2$$

We use this expression to solve problems when two variables are in inverse proportion.

Converting the proportion into inverse proportion:
While solving the problems related to inverse proportion, we should be careful about equating the ratios. This can be understood with the help of an example.

The water in the tank is sufficient for 5 members for 21 days. The same amount of water is sufficient to 15 people for 7 days.

In this case, if the number of people increases, the number of days decrease.
Thus, this is the case of inverse proportion.
Ratio of people = 5 : 15
Ratio of days = 21 : 7
We have, $5 : 15 \neq 21 : 7$
But when the proportion of the days is written in inverse proportion, we get
$5 : 15 = 7 : 21$
$\Rightarrow 5 \times 21 = 15 \times 7$

So, it can be concluded that when $p : q$ and $r : s$ are in inverse proportion, then $p : q = s : r$.

Now, let us learn to verify whether the given variables are in inverse proportion or not.

The following table shows the speed of a train and time taken to cover the same distance.

Speed (in km/hr)	Time taken (in hr)
20	6
30	4
40	3
60	2

Is the speed of the train and the time taken to cover the distance in inverse proportion?

We know that two variables x and y will be in inverse proportion if $xy = k$, where k is constant of proportionality.

Therefore, to check whether the two variables x and y of a given situation are in inverse proportion or not, we have to calculate the product of the value of variable x with its corresponding value of the variable y. If all these products are equal, then we can say that the variables x and y are in inverse proportion, otherwise not.

Using this concept, let us check whether the speed of train and time taken to cover the distance is in inverse proportion or not.

In the given table, there are two variables – speed of the train and time taken to cover the distance. For the first observation, the product of these variables is 20 × 6 = 120.

Similarly, for the other observations, the products of corresponding values of the variables are

30 × 4 = 120

40 × 3 = 120

60 × 2 = 120

Here, the products of the values of the variables for all observations are same.

Thus, the speed and time taken to cover the same distance are in inverse proportion.

Using this concept, we can check whether the variables of a given situation, given in a tabular form, are in inverse proportion or not.

Let us discuss one more example based on verification of inverse proportion.

Example 1:

Observe the following tables and check whether *x* and *y* are related inversely or not.

(i)

x	1	2	5	10
y	20	10	4	2

(ii)

x	7	8	9	10
y	27	20	12	7

Solution:

We know that two quantities x and y are in inverse proportion, if $x \times y$ = constant.

(i) Here, 1 × 20 = 20

$2 \times 10 = 20$

$5 \times 4 = 20$

$10 \times 2 = 20$

It can be seen that $x \times y = 20$, which is constant for each observation.

Therefore, x and y are in inverse proportion.

(ii) Here, $7 \times 27 = 189$

$8 \times 20 = 160$

$9 \times 12 = 108$

$10 \times 7 = 70$

It can be seen that $x \times y$ is not constant for each observation.

Therefore, x and y are not in inverse proportion.

Example 2:

In each of the following statements, find the situation where two variables are in inverse proportion.

1. **Distance covered by a car and the amount of petrol required**
2. **Number of workers and time taken by them to complete the work**
3. **Speed of a person and distance covered by him in a fixed time**
4. **Length and breadth of a rectangle to keep its area constant**

Solution:

(a) If we want to cover more distance, then we will require more amount of petrol. Hence, distance covered by a car and amount of petrol are in direct proportion, not in inverse proportion.

(b) If number of workers will be more, then they will take less time to complete the work. Hence, the number of workers and time taken by them to complete the work are in inverse proportion.

(c) If a person will increase his speed, then he will cover more distance in a fixed time. Hence, speed of a person and distance covered by him in a fixed time are not in inverse proportion.

(d) If we will increase the length of rectangle, then we will have to decrease its breadth in order to keep its area constant. Hence, to keep the area of a rectangle constant, its length and breadth should be in inverse proportion.

Example 3:

A packet of chocolates is to be distributed among 25 children such that each of them will get 4 chocolates. How many more chocolates would each of them get if the number of children is reduced by 5?

Solution:

Let each child get x more chocolates.

Hence, each child will get $(4 + x)$ chocolates.

There were 25 children.

If number of children is reduced by 5, then there are $25 - 5 = 20$ children

We can represent the given situation by constructing a table between the number of children and number of chocolates as shown below.

Number of children	25	20
Number of chocolates	4	4 + x

In the given case, the number of chocolates increases as the number of children decreases and the number of chocolates decreases as the number of children increases. So, this is an example of inverse variation.

Thus, it can be said that the number of children and the number of chocolates that each child got are in inverse proportion.

$\therefore 25 \times 4 = 20\,(4 + x)$

$$4 + x = \frac{25 \times 4}{20}$$

$4 + x = 5$

$x = 5 - 4 = 1$

Therefore, each child will get 1 more chocolate if 5 children are reduced.

Example 4:

In one full day in a school, there are 8 periods of 40 min each. The school management decided to increase the number of periods in a day so that more number of classes can take place in a single day. However, they want to keep the school duration the same. What will be the duration of new periods if the number of periods is increased to 10?

Solution:

Let the duration of new periods be x.

The given situation can be represented with the help of the following table.

Number of periods	8	10
Time duration (minutes)	40	x

In the given case, the time duration increases as the number of periods decreases and the time duration decreases as the number of periods increases. So, this is an example of inverse variation.

Here, the number of periods and time duration of each period are in inverse proportion.

$\therefore 40 \times 8 = x \times 10$

$$x = \frac{40 \times 8}{10} = 32$$

Therefore, the duration of new periods will be 32 minutes.

Example 5:

If a varies inversely as b and $b = 15$ when $a = 4$, then find the value of a when $b = 20$.

Solution:

Since a varies inversely as b, we have

$ab = k$ (Where k is constant of proportionality)

 $k = 4 \times 15$ ($b = 15$ when $a = 4$)

 $k = 60$

Substituting the value of k, the equation of variation becomes

$ab = 60$

 $a \times 20 = 60$ (When $b = 20$)

 $a = 3$

Example 6:

The time taken by a train to travel a particular distance is inversely proportional to its average speed. When the average speed of train is 60 km/hr, the journey can be completed in 8.5 hours, find the time taken by the train to finish the journey when its average speed is 85 km/hr.

Solution:

Let us denote the average speed and time taken by s and t respectively.

Then

$t \propto \frac{1}{s}$

⇒ $st = k$ (Where k is constant of proportionality)

 $k = 60 \times 8.5$ ($t = 8.5$ when $s = 60$)

 $k = 510$

Substituting the value of k, the equation of variation becomes

$st = 510$

 $85 \times t = 510$ (When $s = 85$)

 $t = 6$

Thus, the required time is 6 hours.

Example 7:

Six workers can do a job in 60 days. How many days will 10 workers take to do the same work?

Solution:

Number of days taken by 6 workers to complete the work = 60

Number of days taken by 10 workers to complete the work = x

Ratio of workers = 6 : 10

Ratio of days = 60 : x

Inverse of ratio of days = x : 60

Therefore,

$6 : 10 = x : 60$

$10x = 360$

$x = 36$

Thus, 10 workers will take 36 days to complete the work.