Chapter 2

COORDINATES

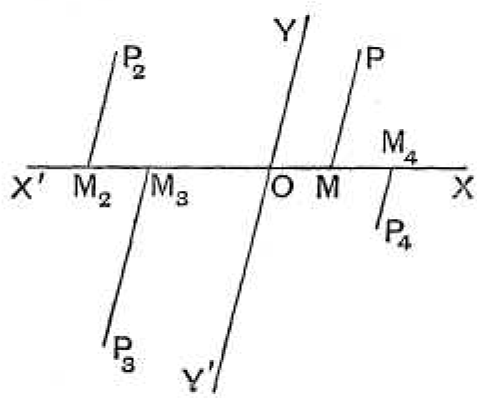
: LENGTHS OF STRAIGHT LINES AND AREAS OF TRIANGLES

15. Coordinates. Let OX and OY be two fixed straight lines in the plane of the paper. The line OX is called the axis of x, the line OY the axis of y, whilst the two together are called the axes of coordinates.

The point O is called the origin of coordinates or, more shortly, the origin.

From any point P in the plane draw a straight line parallel to OY to meet OX in M.

The distance OM is called the Abscissa, and the distance MP the Ordinate of the point P, whilst the abscissa and the ordinate together are called its Coordinates.



Distances measured parallel to OX are called x, with or without a suffix, (*e.g.* x_1, x_2... x', x'',...), and distances measured parallel to OY are called y, with or without a suffix, (*e.g.* y_1, y_2,... y', y'',...).

If the distances OM and MP be respectively x and y, the coordinates of P are, for brevity, denoted by the symbol (x, y).

Conversely, when we are given that the coordinates of a point P are (x, y) we know its position. For from O we have only to measure a distance OM $(=x)$ along OX and then from M measure a distance MP $(=y)$ parallel to OY and we arrive at the position of the point P. For example in the figure, if OM be equal to the unit of length and $MP=2OM$, then P is the point (1, 2).

16. Produce XO backwards to form the line OX' and YO backwards to become OY'. In Analytical Geometry we have the same rule as to signs that the student has already met with in Trigonometry.

Lines measured parallel to OX are positive whilst those measured parallel to OX' are negative; lines measured parallel to OY are positive and those parallel to OY' are negative.

If P_2 be in the quadrant YOX' and P_2M_2, drawn parallel to the axis of y, meet OX' in M_2, and if the numerical values of the quantities OM_2 and M_2P_2 be a and b, the coordinates of P are ($-a$ and b) and the position of P_2 is given by the symbol $(-a, b)$.

Similarly, if P_3 be in the third quadrant $X'OY'$, both of its coordinates are negative, and, if the numerical lengths of OM_3 and M_3P_3 be c and d, then P_3 is denoted by the symbol $(-c, -d)$.

Finally, if P_4 lie in the fourth quadrant its abscissa is positive and its ordinate is negative.

17. Ex. *Lay down on paper the position of the points*

(i) $(2, -1)$, (ii) $(-3, 2)$, and (iii) $(-2, -3)$.

To get the first point we measure a distance 2 along OX and then a distance 1 parallel to OY'; we thus arrive at the required point.

To get the second point, we measure a distance 3 along OX', and then 2 parallel to OY.

To get the third point, we measure 2 along OX' and then 3 parallel to OY'.

These three points are respectively the points P_4, P_2, and P_3 in the figure of Art. 15.

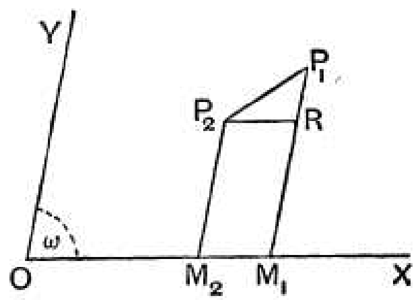
18. When the axes of coordinates are as in the figure of Art. 15, not at right angles, they are said to be Oblique Axes, and the angle between their two positive directions OX and OY, *i.e.* the angle XOY, is generally denoted by the Greek letter ω.

In general, it is however found to be more convenient to take the axes OX and OY at right angles. They are then said to be Rectangular Axes.

It may always be assumed throughout this book that the axes are rectangular unless it is otherwise stated.

19. The system of coordinates spoken of in the last few articles is known as the Cartesian System of Coordinates. It is so called because this system was first introduced by the philosopher Des Cartes. There are other systems of coordinates in use, but the Cartesian system is by far the most important.

20. *To find the distance between two points whose coordinates are given.*



Let P_1 and P_2 be the two given points, and let their coordinates be respectively (x_1, y_1) and (x_2, y_2).

Draw P_1M_1 and P_2M_2 parallel to OY, to meet OX in M_1 and M_2. Draw P_2R parallel to OX to meet M_1P_1 in R.

Then

$$P_2R = M_2M_1 = OM_1 - OM_2 = x_1 - x_2,$$

$$RP_1 = M_1P_1 - M_2P_2 = y_1 - y_2,$$

and $\qquad \angle P_2RP_1 = \angle OM_1P_1 = 180^\circ - P_1M_1X = 180^\circ - \omega.$

We therefore have [*Trigonometry*, Art. 164]

$$P_1P_2^2 = P_2R^2 + RP_1^2 - 2P_2R \cdot RP_1 \cos P_2RP_1$$

$$= (x_1 - x_2)^2 + (y_1 - y_2)^2 - 2(x_1 - x_2)(y_1 - y_2)\cos(180^\circ - \omega)$$

$$= \mathbf{(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos\omega} \ldots (1).$$

If the axes be, as is generally the case, at right angles, we have $\omega = 90^\circ$ and hence $\cos\omega = 0$.

The formula (1) then becomes

$$P_1P_2^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2,$$

so that in rectangular coordinates the distance between the two points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad \ldots\ldots\ldots\ldots\ldots (2).$$

Cor. The distance of the point (x_1, y_1) from the origin is $\sqrt{x_1^2 + y_1^2}$, the axes being rectangular. This follows from (2) by making both x_2 and y_2 equal to zero.

21. The formula of the previous article has been proved for the case when the coordinates of both the points are all positive.

Due regard being had to the signs of the coordinates, the formula will be found to be true for all points.

As a numerical example, let P_1 be the point (5, 6) and P_2 be the point (−7, −4), so that we have

$$x_1 = 5,\ y_1 = 6,\ x_2 = -7,$$

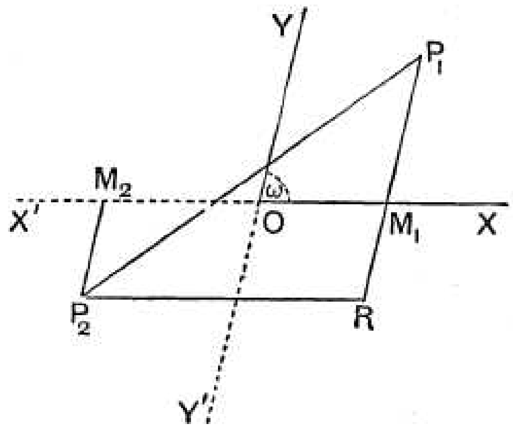
and $$y_2 = -4.$$

Then

$$P_2R = M_2O + OM_1 = 7 + 5 = -x_2 + x_1,$$

and

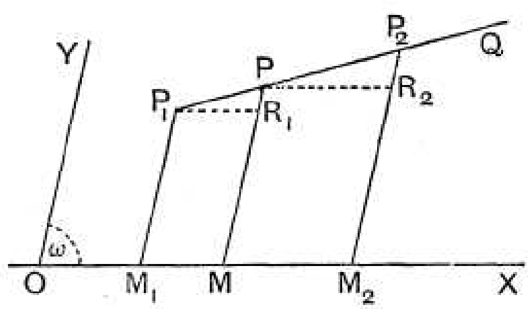
$$RP_1 = RM_1 + M_1P_1 = 4 + 6 = -y_2 + y_1.$$



The rest of the proof is as in the last article.

Similarly any other case could be considered.

22. *To find the coordinates of the point which divides in a given ratio $(m_1 : m_2)$ the line joining two given points (x_1, y_1) and (x_2, y_2).*



Let P_1 be the point (x_1, y_1), P_2 the point (x_2, y_2), and P the required point, so that we have

$$P_1P : PP_2 :: m_1 : m_2.$$

Let P be the point (x, y) so that if P_1M_1, PM, and P_2M_2 be drawn parallel to the axis of y to meet the axis of x in M_1, M, and M_2, we have

$$OM_1 = x_1,\ \ M_1P_1 = y_1,\ \ OM = x,\ \ MP = y,\ \ OM_2 = x_2,$$

and $$M_2P_2 = y_2.$$

Draw P_1R_1 and PR_2, parallel to OX, to meet MP and M_2P_2 in R_1 and R_2 respectively.

Then $$P_1R_1 = M_1M = OM - OM_1 = x - x_1,$$

$$PR_2 = MM_2 = OM_2 - OM = x_2 - x,$$

$$R_1P = MP - M_1P_1 = y - y_1,$$

and $$R_2P_2 = M_2P_2 - MP = y_2 - y.$$

From the similar triangles P_1R_1P and PR_2P_2 we have

$$\frac{m_1}{m_2} = \frac{P_1P}{PP_2} = \frac{P_1R_1}{PR_2} = \frac{x - x_1}{x_2 - x}.$$

$$\therefore\ m_1(x_2 - x) = m_2(x - x_1),$$

i.e. $$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}.$$

Again $$\frac{m_1}{m_2} = \frac{P_1P}{PP_2} = \frac{R_1P}{R_2P_2} = \frac{y - y_1}{y_2 - y},$$

so that $$m_1(y_2 - y) = m_2(y - y_1),$$

and hence $$y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}.$$

The coordinates of the point which divides P_1P_2 internally in the given ratio $m_1 : m_2$ are therefore

$$\mathbf{\frac{m_1x_2 + m_2x_1}{m_1 + m_2}} \text{ and } \mathbf{\frac{m_1y_2 + m_2y_1}{m_1 + m_2}}.$$

If the point Q divide the line P_1P_2 *externally* in the same ratio, *i.e.* so that $P_1Q : QP_2 :: m_1 : m_2$, its coordinates would be found to be

$$\mathbf{\frac{m_1x_2 - m_2x_1}{m_1 - m_2}} \text{ and } \mathbf{\frac{m_1y_2 - m_2y_1}{m_1 - m_2}}.$$

The proof of this statement is similar to that of the preceding article and is left as an exercise for the student.

Cor. The coordinates of the middle point of the line joining (x_1, y_1) to (x_2, y_2) are

$$\frac{x_1 + x_2}{2} \text{ and } \frac{y_1 + y_2}{2}.$$

23. Ex. 1. *In any triangle ABC prove that*

$$AB^2 + AC^2 = 2(AD^2 + DC^2),$$

where D is the middle point of BC.

Take B as origin, BC as the axis of x, and a line through B perpendicular to BC as the axis of y.

Let $BC = a$, so that C is the point (a, O), and let A be the point (x_1, y_1).

Then D is the point $\left(\frac{a}{2}, O\right)$.

Hence $$AD^2 = \left(x_1 - \frac{a}{2}\right)^2 + y_1^2, \text{ and } DC^2 = \left(\frac{a}{2}\right)^2.$$

Hence $$2(AD^2 + DC^2) = 2\left[x_1^2 + y_1^2 - ax_1 + \frac{a^2}{2}\right]$$
$$= 2x_1^2 + 2y_1^2 - 2ax_1 + a^2.$$

Also $$AC^2 = (x_1 - a)^2 + y_1^2,$$
and $$AB^2 = x_1^2 + y_1^2.$$

Therefore $$AB^2 + AC^2 = 2x_1^2 + 2y_1^2 - 2ax_1 + a^2.$$

Hence $$AB^2 + AC^2 = 2(AD^2 + DC^2).$$

This is the well-known theorem of Ptolemy.

Ex. 2. *ABC is a triangle and D, E, and F are the middle points of the sides BC, CA, and AB; prove that the point which divides AD internally in the ratio 2 : 1 also divides the lines BE and CF in the same ratio.*

Hence prove that the medians of a triangle meet in a point.

Let the coordinates of the vertices A, B, and C be (x_1, y_1), (x_2, y_2), and (x_3, y_3) respectively.

The coordinates of D are therefore $\frac{x_2 + x_3}{2}$ and $\frac{y_2 + y_3}{2}$.

Let G be the point that divides internally AD in the ratio 2 : 1, and let its coordinates be $\overline{x}$ and $\overline{y}$.

By the last article

$$\overline{x} = \frac{2 \times \frac{x_2 + x_3}{2} + 1 \times x_1}{2 + 1} = \frac{x_1 + x_2 + x_3}{3}.$$

So $$\overline{y} = \frac{y_1 + y_2 + y_3}{3}.$$

In the same manner we could shew that these are the coordinates of the points that divide BE and CF in the ratio 2 : 1.

Since the point whose coordinates are

$$\frac{x_1+x_2+x_3}{3} \text{ and } \frac{y_1+y_2+y_3}{3}$$

lies on each of the lines AD, BE, and CF, it follows that these three lines meet in a point.

This point is called the Centroid of the triangle.

EXAMPLES I

Find the distances between the following pairs of points.

1. (2, 3) and (5, 7). **2.** (4, −7) and (−1, 5).

3. (−3, −2) and (−6, 7), the axes being inclined at 60°.

4. (a, o) and (o, b). **5.** $(b+c,\ c+a)$ and $(c+a,\ a+b)$.

6. $(a\cos\alpha,\ a\sin\alpha)$ and $(a\cos\beta,\ a\sin\beta)$.

7. $(am_1^2,\ 2am_1)$ and $(am_2^2,\ 2am_2)$.

8. Lay down in a figure the positions of the points (1, −3) and (−2, 1), and prove that the distance between them is 5.

9. Find the value of x_1 if the distance between the points $(x_1, 2)$ and (3, 4) be 8.

10. A line is of length 10 and one end is at the point (2, −3); if the abscissa of the other end be 10, prove that its ordinate must be 3 or −9.

11. Prove that the points $(2a, 4a)$, $(2a, 6a)$, and $(2a+\sqrt{3}a, 5a)$ are the vertices of an equilateral triangle whose side is $2a$.

12. Prove that the points (−2, −1), (1, 0), (4, 3), and (1, 2) are at the vertices of a parallelogram.

13. Prove that the points (2, −2), (8, 4), (5, 7), and (−1, 1) are at the angular points of a rectangle.

14. Prove that the point $(-\frac{1}{14}, \frac{39}{14})$ is the centre of the circle circumscribing the triangle whose angular points are (1, 1), (2, 3), and (−2, 2).

Find the coordinates of the point which

15. divides the line joining the points (1, 3) and (2, 7) in the ratio 3 : 4.

16. divides the same line in the ratio 3 : −4.

17. divides, internally and externally, the line joining (−1, 2) to (4, −5) in the ratio 2 : 3.

18. divides, internally and externally, the line joining $(-3, -4)$ to $(-8, 7)$ in the ratio $7 : 5$.

19. The line joining the points $(1, -2)$ and $(-3, 4)$ is trisected; find the coordinates of the points of trisection.

20. The line joining the points $(-6, 8)$ and $(8, -6)$ is divided into four equal parts; find the coordinates of the points of section.

21. Find the coordinates of the points which divide, internally and externally, the line joining the point $(a+b, a-b)$ to the point $(a-b, a+b)$ in the ratio $a : b$.

22. The coordinates of the vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3). The line joining the first two is divided in the ratio $l : k$, and the line joining this point of division to the opposite angular point is then divided in the ratio $m : k+l$. Find the coordinates of the latter point of section.

23. Prove that the coordinates, x and y, of the middle point of the line joining the point $(2, 3)$ to the point $(3, 4)$ satisfy the equation

$$x - y + 1 = 0.$$

24. If G be the centroid of a triangle ABC and O be any other point, prove that

$$3(GA^2 + GB^2 + GC^2) = BC^2 + CA^2 + AB^2,$$

and $$OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2.$$

25. Prove that the lines joining the middle points of opposite sides of a quadrilateral and the line joining the middle points of its diagonals meet in a point and bisect one another.

26. $A, B, C, D \ldots$ are n points in a plane whose coordinates are (x_1, y_1), (x_2, y_2), (x_3, y_3), ... AB is bisected in the point G_1; G_1C is divided at G_2 in the ratio $1 : 2$; G_2D is divided at G_3 in the ratio $1 : 3$; G_3E at G_4 in the ratio $1 : 4$, and so on until all the points are exhausted. Shew that the coordinates of the final point so obtained are

$$\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} \text{ and } \frac{y_1 + y_2 + y_3 + \ldots + y_n}{n}.$$

[This point is called the **Centre of Mean Position** of the n given points.]

27. Prove that a point can be found which is at the same distance from each of the four points

$$\left(am_1, \frac{a}{m_1}\right), \left(am_2, \frac{a}{m_2}\right), \left(am_3, \frac{a}{m_3}\right), \text{ and } \left(\frac{a}{m_1 m_2 m_3}, am_1 m_2 m_3\right).$$

ANSWERS

1. 5. **2.** 13. **3.** $3\sqrt{7}$. **4.** $\sqrt{a^2+b^2}$.

5. $\sqrt{a^2+2b^2+c^2-2ab-2bc}$. **6.** $2a\sin\frac{\alpha-\beta}{2}$.

7. $a(m_1-m_2)\sqrt{(m_1+m_2)^2+4}$. **9.** $3 \pm 2\sqrt{15}$.

15. $(\frac{10}{7}, \frac{33}{7})$. **16.** $(-2, -9)$. **17.** $(1, -\frac{4}{5})$; $(-11, 16)$.

18. $(-5\frac{11}{12}, 2\frac{5}{12})$; $(-20\frac{1}{2}, 34\frac{1}{2})$. **19.** $(-\frac{1}{3}, 0)$; $(-\frac{5}{3}, 2)$.

20. $(-\frac{5}{2}, \frac{9}{2})$; $(1, 1)$; $(\frac{9}{2}, -\frac{5}{2})$.

21. $\left(\frac{a^2+b^2}{a+b}, \frac{a^2+2ab-b^2}{a+b}\right)$; $\left(\frac{a^2-2ab-b^2}{a-b}, \frac{a^2+b^2}{a-b}\right)$.

22. $\left(\frac{kx_1+lx_2+mx_3}{k+l+m}, \frac{ky_1+ly_2+my_3}{k+l+m}\right)$.

SOLUTIONS/HINTS

6. Distance $= a\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2}$

$$= 2a\sqrt{\sin^2\frac{\alpha+\beta}{2}\sin^2\frac{\alpha-\beta}{2}+\cos^2\frac{\alpha+\beta}{2}\sin^2\frac{\alpha-\beta}{2}}$$

$$= 2a\sin\frac{\alpha-\beta}{2}.$$

7. Distance $= a\sqrt{(m_1{}^2-m_2{}^2)^2+4(m_1-m_2)^2}$

$$= a(m_1-m_2)\sqrt{(m_1+m_2)^2+4}.$$

10. Let y be the ordinate; then $100=(10-2)^2+(y+3)^2$.

$$\therefore\ y^2+6y-27;\quad \therefore\ y=3 \text{ or } -9.$$

11. Let the points be A, B, C.

$BC^2=(\sqrt{3}a)^2+(-a)^2=4a^2$; $CA^2=(\sqrt{3}a)^2+a^2=4a^2$;

$$AB^2=0+(2a)^2=4a^2.$$

12. The middle point of one diagonal is

$$\left(\frac{4-2}{2}, \frac{3-1}{2}\right), \text{ i.e. } (1, 1).$$

The middle point of the other diagonal is

$$\left(\frac{1+1}{2}, \frac{2+0}{2}\right), \text{ i.e. } (1, 1).$$

13. Let the points be A, B, C, D.

$AB^2=(8-2)^2+(4+2)^2=72$, $BC^2=(8-5)^2+(4-7)^2=18$,
$CD^2=(5+1)^2+(7-1)^2=72$, $AD^2=(2+1)^2+(-2-1)^2=18$,
$AC^2=(5-2)^2+(7+2)^2=90$, $BD^2=(8+1)^2+(4-1)^2=90$.

$$\therefore\ AB=CD,\ BC=AD \text{ and } AC^2=AB^2+BC^2.$$

Hence the quadrilateral is a rectangle.

18. $\left(\frac{-7.8-5.3}{7+5},\ \frac{7.7-5.4}{7+5}\right)$, *i.e.* $(-5\frac{11}{12},\ 2\frac{5}{12})$.

$\left(\frac{-7.8+5.3}{7-5},\ \frac{7.7+5.4}{7-5}\right)$, *i.e.* $(-20\frac{1}{2},\ 34\frac{1}{2})$.

19. $\left(\frac{-1.3+2.1}{1+2},\ \frac{1.4-2.2}{1+2}\right)$, *i.e.* $\left(-\frac{1}{3},\ 0\right)$.

$\left(\frac{-2.3+1.1}{2+1},\ \frac{2.4-1.2}{2+1}\right)$, *i.e.* $\left(-\frac{5}{3},\ 2\right)$.

20. The middle point is $\left(\frac{8-6}{2},\ \frac{8-6}{2}\right)$, *i.e.* $(1,\ 1)$.

The other points are $\left(\frac{1-6}{2},\ \frac{1+8}{2}\right)$, *i.e.* $\left(-\frac{5}{2},\ \frac{9}{2}\right)$,

and $\left(\frac{1+8}{2},\ \frac{1-6}{2}\right)$, *i.e.* $\left(\frac{9}{2},\ -\frac{5}{2}\right)$.

21. $\left[\frac{a(a-b)+b(a+b)}{a+b},\ \frac{a(a+b)+b(a-b)}{a+b}\right]$,

i.e. $\left(\frac{a^2+b^2}{a+b},\ \frac{a^2+2ab-b^2}{a+b}\right)$.

$\left[\frac{a(a-b)-b(a+b)}{a-b},\ \frac{a(a+b)-b(a-b)}{a-b}\right]$,

i.e. $\left(\frac{a^2-2ab-b^2}{a-b},\ \frac{a^2+b^2}{a-b}\right)$.

22. The coordinates of the first point are

$$\left(\frac{lx_2+kx_1}{l+k},\ \frac{ly_2+ky_1}{l+k}\right);$$

the coordinates of the second point are

$$\left\{\frac{mx_3 + (k+l)\dfrac{lx_2 + kx_1}{k+l}}{m+k+l},\ \frac{my_3 + (k+l)\left(\dfrac{ly_2 + ky_1}{l+k}\right)}{m+l+k}\right\}.$$

23. The middle point is $\left(\frac{2+3}{2},\ \frac{3+4}{2}\right)$, *i.e.* $\left(\frac{5}{2},\ \frac{7}{2}\right)$,

and $$\tfrac{5}{2} - \tfrac{7}{2} + 1 = 0.$$

24. (1) See page 13, Ex. 2.

$$3(GA^2 + GB^2 + GC^2)$$

$$= 3\Sigma\left[\left(\frac{x_1+x_2+x_3}{3} - x_1\right)^2 + \left(\frac{y_1+y_2+y_3}{3} - y_1\right)^2\right]$$

$$= \tfrac{1}{3}\Sigma\left[(x_2 + x_3 - 2x_1)^2 + (y_2 + y_3 - 2y_1)^2\right]$$

$$= \tfrac{1}{3}\left[6\Sigma x_1^2 - 6\Sigma x_1x_2 + 6\Sigma y_1^2 - 6\Sigma y_1y_2\right]$$

$$= 2\Sigma x_1^2 - 2\Sigma x_1x_2 + 2\Sigma y_1^2 - 2\Sigma y_1y_2$$

$$= \Sigma\{(x_1 - x_2)^2 + (y_1 - y_2)^2\} = BC^2 + CA^2 + AB^2.$$

24. (2) Let (a, b) be the coordinates of O.

$$GA^2 + GB^2 + GC^2 + 3GO^2$$

$$= \tfrac{1}{3}\left[\Sigma(x_1 - x_2)^2 + \dots\right] + 3\left(\frac{x_1+x_2+x_3}{3} - a\right)^2 + \dots$$

$$= \tfrac{1}{3}\left[2\Sigma x_1^2 - 2\Sigma x_1x_2 + \dots + (x_1 + x_2 + x_3 - 3a)^2 + \dots\right]$$

$$= \tfrac{1}{3}\left[3\Sigma x_1^2 - 6a\Sigma x_1 + 9a^2 + \dots\right] = \Sigma x_1^2 - 2a\Sigma x_1 + 3a^2 + \dots]$$

$$= \Sigma\left[(a - x_1)^2 + (b - y_1)^2\right] = OA^2 + OB^2 + OC^2.$$

25. Let the angular points A, B, C, D of the quadrilateral be (x_1, y_1), (x_2, y_2), etc.

The middle point of AB is $\left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2}\right)$.

" " CD is $\left(\frac{x_3+x_4}{2},\ \frac{y_3+y_4}{2}\right)$.

∴ the middle point of the line joining these is

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}\right).$$

The symmetry shews that this is also the middle point of the lines joining the middle points of AD and BC, and of AC and BD.

26. Assume that the abscissa of G_{n-2} is $\frac{x_1+x_2+\dots x_{n-1}}{n-1}$.

Then abscissa of G_{n-1} will be

$$\frac{(n-1)\frac{x_1+x_2+\dots+x_{n-1}}{n-1}+x_n}{(n-1)+1}, \text{ i.e. } \frac{x_1+x_2+x_3+\dots+x_n}{n}.$$

But in Ex. 2, page 13, the abscissae of G_1 and G_2 were shewn to be $\frac{x_1+x_2}{2}$ and $\frac{x_1+x_2+x_3}{3}$.

Hence etc. by mathematical induction.

27. Let A, B, C, D be the four points.

If (x, y) be equidistant from A and D,

$$(x_1-am_1)^2+\left(y-\frac{a}{m_1}\right)^2=\left(x-\frac{a}{m_1m_2m_3}\right)^2+(y-am_1m_2m_3)^2,$$

which reduces to

$$2x-2ym_2m_3=a\left\{\frac{1+m_1^2m_2m_3-m_2^2m_3^2-m_1^2m_2^3m_3^3}{m_1m_2m_3}\right\}.$$

If (x, y) be equidistant from B and D, we have similarly

$$2x-2ym_3m_1=a\left\{\frac{1+m_1m_2^2m_3-m_3^2m_1^2-m_1^3m_2^2m_3^3}{m_1m_2m_3}\right\}.$$

Solving, we have

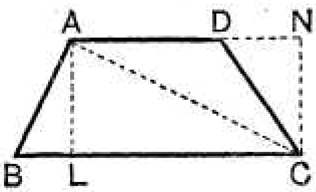
$$2x=a\left\{m_1+m_2+m_3+\frac{1}{m_1m_2m_3}\right\},$$

$$2y=a\left\{\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}+m_1m_2m_3\right\}.$$

This point is equidistant from A, B and D, and by symmetry is equidistant from B, C and D. Hence it is equidistant from the four points.

24. *To prove that the area of a trapezium,* i.e. *a quadrilateral having two sides parallel, is one half the sum of the two parallel sides multiplied by the perpendicular distance between them.*

Let $ABCD$ be the trapezium having the sides AD and BC parallel.

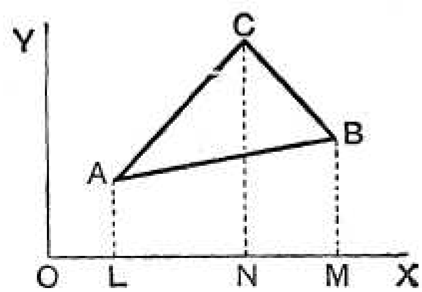


Join AC and draw AL perpendicular to BC and CN perpendicular to AD, produced if necessary.

Since the area of a triangle is one half the product of any side and the perpendicular drawn from the opposite angle, we have

$$\begin{aligned}\text{area } ABCD &= \Delta ABC + \Delta ACD \\ &= \tfrac{1}{2} . BC . AL + \tfrac{1}{2} . AD . CN \\ &= \tfrac{1}{2}(BC + AD) \times AL.\end{aligned}$$

25. *To find the area of the triangle, the coordinates of whose angular points are given, the axes being rectangular.*



Let ABC be the triangle and let the coordinates of its angular points A, B and C be (x_1, y_1), (x_2, y_2), and (x_3, y_3).

Draw AL, BM, and CN perpendicular to the axis of x, and let Δ denote the required area.

Then

$$\Delta = \text{trapezium } ALNC + \text{trapezium } CNMB - \text{trapezium } ALMB$$

$$= \tfrac{1}{2}LN(LA + NC) + \tfrac{1}{2}NM(NC + MB) - \tfrac{1}{2}LM(LA + MB),$$

by the last article,

$$= \tfrac{1}{2}\left[(x_3 - x_1)(y_1 + y_3) + (x_2 - x_3)(y_2 + y_3) - (x_2 - x_1)(y_1 + y_2)\right].$$

On simplifying we easily have

$$\Delta = \tfrac{1}{2}\left(\mathbf{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3}\right),$$

or the equivalent form

$$\Delta = \tfrac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right].$$

If we use the determinant notation this may be written (as in Art. 5)

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}.$$

Cor. The area of the triangle whose vertices are the origin (0, 0) and the points (x_1, y_1), (x_2, y_2) is $\frac{1}{2}(x_1y_2 - x_2y_1)$.

26. In the preceding article, if the axes be oblique, the perpendiculars AL, BM, and CN, are not equal to the ordinates y_1, y_2, and y_3, but are equal respectively to $y_1 \sin\omega$, $y_2 \sin\omega$, and $y_3 \sin\omega$.

The area of the triangle in this case becomes

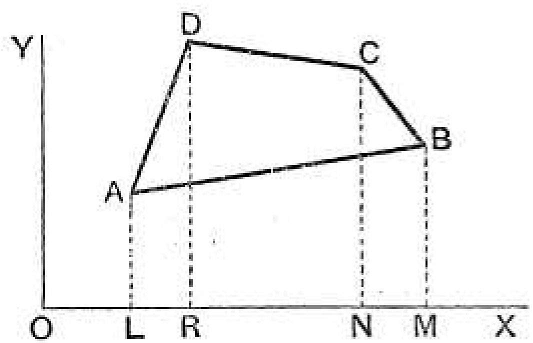
$$\frac{1}{2}\sin\omega\,\{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\},$$

i.e.

$$\frac{1}{2}\sin\omega \times \begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}.$$

27. In order that the expression for the area in Art. 25 may be a positive quantity (as all areas necessarily are) the points A, B, and C must be taken in the order in which they would be met by a person starting from A and walking round the triangle in such a manner that *the area of the triangle is always on his left hand.* Otherwise the expressions of Art. 25 would be found to be negative.

28. *To find the area of a quadrilateral the coordinates of whose angular points are given.*



Let the angular points of the quadrilateral, taken in order, be A, B, C, and D, and let their coordinates be respectively (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4).

Draw AL, BM, CN, and DR perpendicular to the axis of x.

Then the area of the quadrilateral

$$= \text{trapezium } ALRD + \text{trapezium } DRNC + \text{trapezium } CNMB - \text{trapezium } ALMB$$

$$= \tfrac{1}{2} LR\,(LA + RD) + \tfrac{1}{2} RN\,(RD + NC) + \tfrac{1}{2} NM\,(NC + MB) - \tfrac{1}{2} LM\,(LA + MB)$$

$$= \tfrac{1}{2}\{(x_4 - x_1)(y_1 + y_4) + (x_3 - x_4)(y_3 + y_4) + (x_2 - x_3)(y_3 + y_2) - (x_2 - x_1)(y_1 + y_2)\}$$

$$= \tfrac{1}{2}\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)\}.$$

29. The above formula may also be obtained by drawing the lines OA, OB, OC and OD. For the quadrilateral $ABCD$

$$= \triangle OBC + \triangle OCD - \triangle OBA - \triangle OAD.$$

But the coordinates of the vertices of the triangle OBC are $(0, 0)$, (x_2, y_2) and (x_3, y_3); hence, by Art. 25, its area is $\tfrac{1}{2}(x_2y_3 - x_3y_2)$.

So for the other triangles.

The required area therefore

$$= \tfrac{1}{2}[(x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) - (x_2y_1 - x_1y_2) - (x_1y_4 - x_4y_1)]$$

$$= \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)].$$

In a similar manner it may be shewn that the area of a polygon of n sides the coordinates of whose angular points, taken in order, are

$$(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3), \ldots (x_n, y_n)$$

is $\tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \ldots + (x_ny_1 - x_1y_n)]$.

EXAMPLES II

Find the areas of the triangles the coordinates of whose angular points are respectively

1. $(1, 3)$, $(-7, 6)$ and $(5, -1)$. **2.** $(0, 4)$, $(3, 6)$ and $(-8, -2)$.

3. $(5, 2)$, $(-9, -3)$ and $(-3, -5)$.

4. $(a, b+c)$, $(a, b-c)$ and $(-a, c)$.

5. $(a, c+a)$, (a, c) and $(-a, c-a)$.

6. $(a \cos\phi_1, b \sin\phi_1)$, $(a \cos\phi_2, b \sin\phi_2)$ and $(a \cos\phi_3, b \sin\phi_3)$.

7. $(am_1^2, 2am_1)$, $(am_2^2, 2am_2)$ and $(am_3^2, 2am_3)$.

8. $\{am_1m_2, a(m_1+m_2)\}$, $\{am_2m_3, a(m_2+m_3)\}$ and $\{am_3m_1, a(m_3+m_1)\}$.

9. $\left\{am_1, \frac{a}{m_1}\right\}$, $\left\{am_2, \frac{a}{m_2}\right\}$ and $\left\{am_3, \frac{a}{m_3}\right\}$.

Prove (by shewing that the area of the triangle formed by them is *zero*) that the following sets of three points are in a straight line:

10. $(1, 4)$, $(3, -2)$, and $(-3, 16)$.

11. $(-\frac{1}{2}, 3)$, $(-5, 6)$, and $(-8, 8)$.

12. $(a, b+c)$, $(b, c+a)$, and $(c, a+b)$.

Find the areas of the quadrilaterals the coordinates of whose angular points, taken in order, are

13. $(1, 1)$, $(3, 4)$, $(5, -2)$, and $(4, -7)$.

14. $(-1, 6)$, $(-3, -9)$, $(5, -8)$, and $(3, 9)$.

15. If O be the origin, and if the coordinates of any two points P_1 and P_2 be respectively (x_1, y_1) and (x_2, y_2), prove that

$$OP_1 \cdot OP_2 \cdot \cos P_1OP_2 = x_1x_2 + y_1y_2.$$

ANSWERS

1. 10. 2. 1. 3. 29. 4. $2ac$.

5. a^2. 6. $2ab \sin\frac{\phi_2-\phi_3}{2} \sin\frac{\phi_3-\phi_1}{2} \sin\frac{\phi_1-\phi_2}{2}$.

7. $a^2(m_2-m_3)(m_3-m_1)(m_1-m_2)$.

8. $\frac{1}{2}a^2(m_2-m_3)(m_3-m_1)(m_1-m_2)$.

9. $\frac{1}{2}a^2(m_2-m_3)(m_3-m_1)(m_1-m_2) \div m_1m_2m_3$.

13. $20\frac{1}{2}$. 14. 96.

SOLUTIONS/HINTS

1—3. Use the formula of Art. 25.

4. By Art. 25

$$\Delta = \tfrac{1}{2}\{a(b-c-c) + a(c-b-c) - a(b+c-b+c)\} = -2ac.$$

5. $\Delta = \frac{1}{2}\{a(c-c+a)+a(c-a-c-a)-a(c+a-c)\}$
$= -a^2.$

6. $\Delta = \frac{ab}{2}\{\Sigma \cos\phi_1(\sin\phi_2 - \sin\phi_3)\},$

$$= \frac{ab}{2}\{\sin(\phi_2-\phi_1)+\sin(\phi_3-\phi_2)+\sin(\phi_1-\phi_3)\}$$

$$= \frac{ab}{2}\left[2\sin\frac{\phi_3-\phi_1}{2}\cos\frac{2\phi_2-\phi_1-\phi_3}{2} + 2\sin\frac{\phi_1-\phi_3}{2}\cos\frac{\phi_1-\phi_3}{2}\right]$$

$$= ab\sin\frac{\phi_3-\phi_1}{2}\left[\cos\frac{2\phi_2-\phi_1-\phi_3}{2} - \cos\frac{\phi_3-\phi_1}{2}\right]$$

$$= 2ab\sin\frac{\phi_1-\phi_2}{2}\sin\frac{\phi_2-\phi_3}{2}\sin\frac{\phi_3-\phi_1}{2}.$$

7. $\Delta = \frac{1}{2}a^2\{\Sigma m_1^2(2m_2-2m_3)\} = a^2[m_1^2(m_2-m_3) + m_2^2(m_3-m_1)+m_3^2(m_1-m_2)]$
$= -a^2(m_2-m_3)(m_3-m_1)(m_1-m_2).$

8. $\Delta = \frac{1}{2}a^2\{\Sigma m_1m_2(\overline{m_2+m_3}-\overline{m_3+m_1})\}$
$= -\frac{1}{2}a^2(m_2-m_3)(m_3-m_1)(m_1-m_2).$

9. $\Delta = \frac{1}{2}a^2\Sigma\left\{m_1\left(\frac{1}{m_2}-\frac{1}{m_3}\right)\right\} = \frac{1}{2}a^2\Sigma\frac{m_1^2(m_3-m_2)}{m_1m_2m_3}$
$= \frac{1}{2}a^2(m_2-m_3)(m_3-m_1)(m_1-m_2) \div m_1m_2m_3.$

10, 11. Use the formula of Art. 25.

12. $\Delta = \frac{1}{2}\Sigma\{a(\overline{c+a}-\overline{a+b})\} = \frac{1}{2}\Sigma a(c-b) = 0.$

13, 14. Use the formula of Art. 28.

15. $OP_1 . OP_2 \sin P_1\hat{O}P_2 = \Delta P_1OP_2 = x_1y_2 - x_2y_1,$

and $OP_1^2 . OP_2^2 = (x_1^2+y_1^2)(x_2^2+y_2^2).$

$$\therefore OP_1^2 . OP_2^2(1-\sin^2 P_1\hat{O}P_2)$$

$$= (x_1^2+y_1^2)(x_2^2+y_2^2) - (x_1y_2-x_2y_1)^2 = (x_1x_2+y_1y_2)^2.$$

$$\therefore OP_1 . OP_2 \cos P_1\hat{O}P_2 = x_1x_2+y_1y_2.$$

30. Polar Coordinates. There is another method, which is often used, for determining the position of a point in a plane.

Suppose O to be a fixed point, called the **origin** or **pole**, and OX a fixed line, called the **initial line.**

Take any other point P in the plane of the paper and join OP. The position of P is clearly known when the angle XOP and the length OP are given.

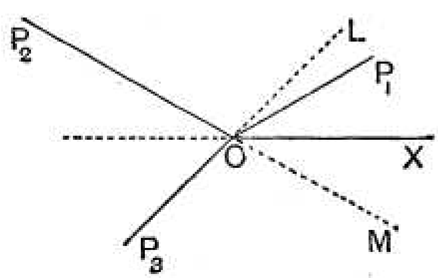
[For giving the angle XOP shews the direction in which OP is drawn, and giving the distance OP tells the distance of P along this direction.]

The angle XOP which would be traced out by the line OP in revolving from the initial line OX is called the vectorial angle of P and the length OP is called its radius vector. The two taken together are called the polar coordinates of P.

If the vectorial angle be θ and the radius vector be r, the position of P is denoted by the symbol (r, θ).

The radius vector is positive if it be measured from the origin O along the line bounding the vectorial angle; if measured in the opposite direction it is negative.

31. Ex. *Construct the positions of the points* (i) $(2, 30°)$, (ii) $(3, 150°)$, (iii) $(-2, 45°)$, (iv) $(-3, 330°)$, (v) $(3, -210°)$ *and* (vi) $(-3, -30°)$.



(i) To construct the first point, let the radius vector revolve from OX through an angle of $30°$, and then mark off along it a distance equal to two units of length. We thus obtain the point P_1.

(ii) For the second point, the radius vector revolves from OX through $150°$ and is then in the position OP_2; measuring a distance 3 along it we arrive at P_2.

(iii) For the third point, let the radius vector revolve from OX through 45° into the position OL. We have now to measure along OL a distance -2, *i.e.* we have to measure a distance 2 not along OL but in the *opposite* direction. Producing LO to P_3, so that OP_3 is 2 units of length, we have the required point P_3.

(iv) To get the fourth point, we let the radius vector rotate from OX through 330° into the position OM and measure on it a distance -3, *i.e.* 3 in the direction MO produced. We thus have the point P_2, which is the same as the point given by (ii).

(v) If the radius vector rotate through $-210°$, it will be in the position OP_2, and the point required is P_2.

(vi) For the sixth point, the radius vector, after rotating through $-30°$, is in the position OM. We then measure -3 along it, *i.e.* 3 in the direction MO produced, and once more arrive at the point P_2.

32. It will be observed that in the previous example the same point P_2 is denoted by each of the four sets of polar coordinates

$$(3,\ 150°),\ (-3,\ 330°),\ (3,\ -210°)\ \text{and}\ (-3,\ -30°).$$

In general it will be found that the same point is given by each of the polar coordinates

$$(r,\ \theta),\ (-r,\ 180°+\theta),\ \{r,\ -(360°-\theta)\}\ \text{and}\ \{-r,\ -(180°-\theta)\},$$

or, expressing the angles in radians, by each of the coordinates

$$(r,\ \theta),\ (-r,\ \pi+\theta),\ \{r,\ -(2\pi-\theta)\}\ \text{and}\ \{-r,\ -(\pi-\theta)\}.$$

It is also clear that adding 360° (or any multiple of 360°) to the vectorial angle does not alter the final position of the revolving line, so that $(r,\ \theta)$ is always the same point as $(r,\ \theta+n\,.\,360°)$, where n is an integer.

So, adding 180° or any odd multiple of 180° to the vectorial angle and changing the sign of the radius vector gives the same point as before. Thus the point

$$[-r,\ \theta+(2n+1)\,180°]$$

is the same point as $[-r,\ \theta+180°]$, *i.e.* is the point $[r,\ \theta]$.

33. *To find the length of the straight line joining two points whose polar coordinates are given.*

Let A and B be the two points and let their polar coordinates be $(r_1,\ \theta_1)$ and $(r_2,\ \theta_2)$ respectively, so that

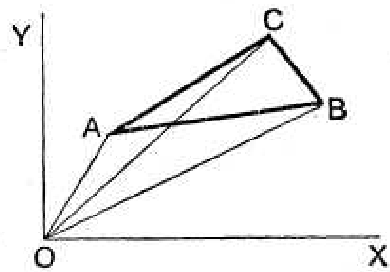
$$OA=r_1,\ OB=r_2,\ \angle XOA=\theta_1,\ \text{and}\ \angle XOB=\theta_2.$$

Then (*Trigonometry*, Art. 164)

$$AB^2 = OA^2 + OB^2 - 2OA \,.\, OB \cos AOB$$
$$= r_1{}^2 + r_2{}^2 - 2r_1r_2 \cos(\theta_1 - \theta_2).$$

34. *To find the area of a triangle the coordinates of whose angular points are given.*

Let ABC be the triangle and let (r_1, θ_1), (r_2, θ_2), and (r_3, θ_3) be the polar coordinates of its angular points.



We have

$$\triangle ABC = \triangle OBC + \triangle OCA - \triangle OBA \;\ldots\ldots(1).$$

Now

$$\triangle OBC = \tfrac{1}{2} OB \,.\, OC \sin BOC$$

[*Trigonometry*, Art. 198]

$$= \tfrac{1}{2} r_2 r_3 \sin(\theta_3 - \theta_2).$$

So $\triangle OCA = \frac{1}{2} OC \,.\, OA \sin COA = \frac{1}{2} r_3 r_1 \sin(\theta_1 - \theta_3)$,

and $\triangle OAB = \frac{1}{2} OA \,.\, OB \sin AOB = \frac{1}{2} r_1 r_2 \sin(\theta_1 - \theta_2)$

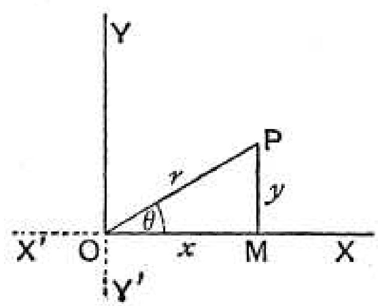
$$= -\tfrac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1).$$

Hence (1) gives

$$\triangle ABC = \tfrac{1}{2}\left[r_2 r_3 \sin(\theta_3 - \theta_2) + r_3 r_1 \sin(\theta_1 - \theta_3) + r_1 r_2 \sin(\theta_2 - \theta_1)\right].$$

35. *To change from Cartesian Coordinates to Polar Coordinates, and conversely.*

Let P be any point whose Cartesian coordinates, referred to rectangular axes, are x and y, and whose polar coordinates, referred to O as pole and OX as initial line, are (r, θ).



Draw PM perpendicular to OX so that we have

$OM = x$, $MP = y$, $\angle MOP = \theta$,

and $OP = r$.

From the triangle MOP we have

$$x = OM = OP \cos MOP = r \cos\theta \;\ldots\ldots\ldots(1),$$
$$y = MP = OP \sin MOP = r \sin\theta \;\ldots\ldots\ldots(2),$$
$$r = OP = \sqrt{OM^2 + MP^2} = \sqrt{x^2 + y^2} \;\ldots\ldots(3),$$

and $$\tan\theta = \frac{MP}{OM} = \frac{y}{x} \quad \text{.....................(4).}$$

Equations (1) and (2) express the Cartesian coordinates in terms of the polar coordinates.

Equations (3) and (4) express the polar in terms of the Cartesian coordinates.

The same relations will be found to hold if P be in any other of the quadrants into which the plane is divided by XOX' and YOY'.

Ex. *Change to Cartesian coordinates the equations*

$$(1)\ r = a\sin\theta, \text{ and } (2)\ r^{\frac{1}{2}} = a^{\frac{1}{2}}\cos\frac{\theta}{2}.$$

(1) Multiplying the equation by r, it becomes $r^2 = ar\sin\theta$, *i.e.* by equations (2) and (3), $x^2+y^2=ay$.

(2) Squaring the equation (2), it becomes

$$r = a\cos^2\frac{\theta}{2} = \frac{a}{2}(1+\cos\theta),$$

i.e. $$2r^2 = ar + ar\cos\theta,$$

i.e. $$2(x^2+y^2) = a\sqrt{x^2+y^2} + ax,$$

i.e. $$(2x^2+2y^2-ax)^2 = a^2(x^2+y^2).$$

EXAMPLES III

Lay down the positions of the points whose polar coordinates are

1. $(3, 45°)$. **2.** $(-2, -60°)$. **3.** $(4, 135°)$. **4.** $(2, 330°)$.

5. $(-1, -180°)$. **6.** $(1, -210°)$. **7.** $(5, -675°)$. **8.** $\left(a, \frac{\pi}{2}\right)$.

9. $\left(2a, -\frac{\pi}{2}\right)$. **10.** $\left(-a, \frac{\pi}{6}\right)$. **11.** $\left(-2a, -\frac{2\pi}{3}\right)$.

Find the lengths of the straight lines joining the pairs of points whose polar coordinates are

12. $(2, 30°)$ and $(4, 120°)$. **13.** $(-3, 45°)$ and $(7, 105°)$.

14. $\left(a, \frac{\pi}{2}\right)$ and $\left(3a, \frac{\pi}{6}\right)$.

15. Prove that the points $(0, 0)$, $\left(3, \frac{\pi}{2}\right)$, and $\left(3, \frac{\pi}{6}\right)$ form an equilateral triangle.

Find the areas of the triangles the coordinates of whose angular points are

16. $(1, 30°)$, $(2, 60°)$, and $(3, 90°)$.

17. $(-3, -30°)$, $(5, 150°)$, and $(7, 210°)$.

18. $\left(-a, \frac{\pi}{6}\right)$, $\left(a, \frac{\pi}{2}\right)$, and $\left(-2a, -\frac{2\pi}{3}\right)$.

Find the polar coordinates (drawing the figure in each case) of the points

19. $x=\sqrt{3}$, $y=1$. 20. $x=-\sqrt{3}$, $y=1$. 21. $x=-1$, $y=1$.

Find the Cartesian coordinates (drawing a figure in each case) of the points whose polar coordinates are

22. $\left(5, \frac{\pi}{4}\right)$. 23. $\left(-5, \frac{\pi}{3}\right)$. 24. $\left(5, -\frac{\pi}{4}\right)$.

Change to polar coordinates the equations

25. $x^2+y^2=a^2$. 26. $y=x\tan\alpha$. 27. $x^2+y^2=2ax$.

28. $x^2-y^2=2ay$. 29. $x^3=y^2(2a-x)$. 30. $(x^2+y^2)^2=a^2(x^2-y^2)$.

Transform to Cartesian coordinates the equations

31. $r=a$. 32. $\theta=\tan^{-1}m$. 33. $r=a\cos\theta$.

34. $r=a\sin 2\theta$. 35. $r^2=a^2\cos 2\theta$. 36. $r^2\sin 2\theta=2a^2$.

37. $r^2\cos 2\theta=a^2$. 38. $r^{\frac{1}{2}}\cos\frac{\theta}{2}=a^{\frac{1}{2}}$. 39. $r^{\frac{1}{2}}=a^{\frac{1}{2}}\sin\frac{\theta}{2}$.

40. $r(\cos 3\theta+\sin 3\theta)=5k\sin\theta\cos\theta$.

ANSWERS

12. $2\sqrt{5}$. 13. $\sqrt{79}$. 14. $\sqrt{7}a$. 16. $\frac{1}{4}(8-3\sqrt{3})$.

17. $\frac{7\sqrt{3}}{2}$. 18. $\frac{1}{4}a^2\sqrt{3}$. 25. $r^2=a^2$. 26. $\theta=\alpha$.

27. $r=2a\cos\theta$. 28. $r\cos 2\theta=2a\sin\theta$. 29. $r\cos\theta=2a\sin^2\theta$.

30. $r^2=a^2\cos 2\theta$. 31. $x^2+y^2=a^2$. 32. $y=mx$.

33. $x^2+y^2=ax$. 34. $(x^2+y^2)^3=4a^2x^2y^2$.

35. $(x^2+y^2)^2=a^2(x^2-y^2)$. 36. $xy=a^2$. 37. $x^2-y^2=a^2$.

38. $y^2+4ax=4a^2$. 39. $4(x^2+y^2)(x^2+y^2+ax)=a^2y^2$.

40. $x^3-3xy^2+3x^2y-y^3=5kxy$.

SOLUTIONS/HINTS

12—15. Use the formula of Art. 33.

16—18. Use the formula of Art. 34.

19—21. Use Art. 35. **25.** $r^2 = a^2$.

26. $r \sin\theta = r\cos\theta \,.\, \tan\alpha$; $\therefore \tan\theta = \tan\alpha$; $\therefore \theta = \alpha$.

27. $r^2 = 2ar\cos\theta$; $\therefore r = 2a\cos\theta$.

28. $r^2\cos^2\theta - r^2\sin^2\theta = 2ar\sin\theta$; $\therefore r\cos 2\theta = 2a\sin\theta$.

29. $r^3\cos^3\theta = r^2\sin^2\theta\,(2a - r\cos\theta)$.

$\therefore r\cos^3\theta = (1-\cos^2\theta)(2a - r\cos\theta)$

$= 2a - r\cos\theta - 2a\cos^2\theta + r\cos^3\theta$.

$\therefore r\cos\theta = 2a\sin^2\theta$.

30. $r^4 = a^2r^2(\cos^2\theta - \sin^2\theta)$; $\therefore r^2 = a^2\cos 2\theta$.

31. $r^2 = a^2$; $\therefore x^2 + y^2 = a^2$. **32.** $\tan\theta = m$. $\therefore y = mx$.

33. $r^2 = ar\cos\theta$; $\therefore x^2 + y^2 = ax$.

34. $r^3 = 2ar\cos\theta \,.\, r\sin\theta$; $\therefore (x^2+y^2)^{\frac{3}{2}} = 2axy$.

$\therefore (x^2+y^2)^3 = 4a^2x^2y^2$.

36. $2ra\cos\theta \,.\, r\sin\theta = 2a^2$; $\therefore xy = a^2$.

37. $r^2(\cos^2\theta - \sin^2\theta) = a^2$; $\therefore x^2 - y^2 = a^2$.

38. $2r\cos^2\dfrac{\theta}{2} = 2a$; $\therefore r(1+\cos\theta) = 2a$.

$\therefore \sqrt{x^2+y^2} + x = 2a$; $\therefore x^2 + y^2 = 4a^2 + x^2 - 4ax$.

$\therefore y^2 + 4ax = 4a^2$.

39. $2r = 2a\sin^2\dfrac{\theta}{2}$; $\therefore 2r = a(1-\cos\theta)$.

$\therefore 2r^2 = a(r - r\cos\theta)$; $\therefore 2(x^2+y^2) + ax = a\sqrt{x^2+y^2}$.

$\therefore 4(x^2+y^2)^2 + 4ax(x^2+y^2) = a^2y^2$.

40. $r^3(\cos^3\theta - 3\cos\theta\sin^2\theta + 3\sin\theta\cos^2\theta - \sin^3\theta)$

$= 5kr^2\sin\theta\cos\theta$.

$\therefore x^3 - 3xy^2 + 3x^2y - y^3 = 5kxy$.