Chapter 7

# APPLICATIONS OF THE DEFINITE INTEGRAL

## § 7.1. Computing the Limits of Sums with the Aid of Definite Integrals

It is often necessary to compute the limit of a sum when the number of summands increases unlimitedly. In some cases such limits can be found with the aid of the definite integral if it is possible to transform the given sum into an integral sum.

For instance, considering the points $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$ as points of division of the interval $[0, 1]$ into $n$ equal parts of length $\Delta x = \frac{1}{n}$, for each continuous function $f(x)$, we have

$$\lim_{n \to \infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots+f\left(\frac{n}{n}\right)\right]=\int_0^1 f(x)\,dx.$$

**7.1.1.** Compute

$$\lim_{n \to \infty} \frac{\pi}{n}\left[\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\ldots+\sin\frac{(n-1)\pi}{n}\right].$$

*Solution.* The numbers in brackets represent the values of the function $f(x)=\sin x$ at the points

$$x_1=\frac{\pi}{n};\quad x_2=\frac{2\pi}{n};\quad \ldots;\quad x_{n-1}=\frac{(n-1)\pi}{n},$$

subdividing the interval $[0, \pi]$ into $n$ equal parts of length $\Delta x=\frac{\pi}{n}$. Therefore, if we add the summand $\sin\frac{n\pi}{n}=0$ to our sum, the latter will be the integral sum for the function $f(x)=\sin x$ on the interval $[0, \pi]$.

By definition, the limit of such an integral sum as $n \to \infty$ is the definite integral of the function $f(x)=\sin x$ from 0 to $\pi$:

$$\lim_{n \to \infty} \frac{\pi}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\ldots+\sin\frac{(n-1)\pi}{n}+\sin\frac{n\pi}{n}\right)=$$

$$=\int_0^{\pi}\sin x\,dx=-\cos x\Big|_0^{\pi}=2.$$

**7.1.2.** Compute the limit

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-2^2}}+\ldots+\frac{1}{\sqrt{4n^2-n^2}}\right).$$

*Solution.* Transform the sum in parentheses in the following way:

$$\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-2^2}}+\ldots+\frac{1}{\sqrt{4n^2-n^2}}=$$
$$=\frac{1}{n}\left(\frac{1}{\sqrt{4-\frac{1}{n^2}}}+\frac{1}{\sqrt{4-\left(\frac{2}{n}\right)^2}}+\ldots+\frac{1}{\sqrt{4-\left(\frac{n}{n}\right)^2}}\right).$$

The obtained sum is the integral sum for the function $f(x)=\frac{1}{\sqrt{4-x^2}}$ on the interval [0, 1] subdivided into $n$ equal parts. The limit of this sum as $n\to\infty$ is equal to the definite integral of this function from 0 to 1:

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-2^2}}+\ldots+\frac{1}{\sqrt{4n^2-n^2}}\right)=$$
$$=\int_0^1\frac{dx}{\sqrt{4-x^2}}=\arcsin\frac{x}{2}\Big|_0^1=\frac{\pi}{6}.$$

**7.1.3.** Compute

$$\lim_{n\to\infty}\frac{3}{n}\left[1+\sqrt{\frac{n}{n+3}}+\sqrt{\frac{n}{n+6}}+\sqrt{\frac{n}{n+9}}+\ldots+\sqrt{\frac{n}{n+3(n-1)}}\right].$$

*Solution.* Transform the given expression in the following way:

$$\frac{3}{n}\left[1+\sqrt{\frac{n}{n+3}}+\sqrt{\frac{n}{n+6}}+\ldots+\sqrt{\frac{n}{n+3(n-1)}}\right]=$$
$$=\frac{3}{n}\left[\sqrt{\frac{1}{1+0}}+\sqrt{\frac{1}{1+\frac{3}{n}}}+\sqrt{\frac{1}{1+\frac{6}{n}}}+\ldots+\sqrt{\frac{1}{1+\frac{3(n-1)}{n}}}\right].$$

The obtained sum is the integral sum for the function $f(x)=\sqrt{\frac{1}{1+x}}$ on the interval [0, 3]; therefore, by definition,

$$\lim_{n\to\infty}\frac{3}{n}\left(1+\sqrt{\frac{n}{n+3}}+\sqrt{\frac{n}{n+6}}+\ldots+\sqrt{\frac{n}{n+3(n-1)}}\right)=$$
$$=\int_0^3\sqrt{\frac{1}{1+x}}\,dx=\int_0^3(1+x)^{-\frac{1}{2}}\,dx=2\sqrt{1+x}\Big|_0^3=4-2=2.$$

**7.1.4.** Using the definite integral, compute the following limits:

(a) $\lim\limits_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+n}\right)$;

(b) $\lim\limits_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\ldots+\sqrt{1+\frac{n}{n}}\right)$;

(c) $\lim\limits_{n\to\infty}\frac{1+\sqrt[3]{2}+\sqrt[3]{3}+\ldots+\sqrt[3]{n}}{\sqrt[3]{n^4}}$;

(d) $\lim\limits_{n\to\infty}\frac{\pi}{2n}\left(1+\cos\frac{\pi}{2n}+\cos\frac{2\pi}{2n}+\ldots+\cos\frac{(n-1)\pi}{2n}\right)$;

(e) $\lim\limits_{n\to\infty} n\left[\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(2n)^2}\right]$.

**7.1.5.** Compute the limit $A=\lim\limits_{n\to\infty}\frac{\sqrt[n]{n!}}{n}$.

*Solution.* Let us take logarithms

$$\ln A=\lim_{n\to\infty}\ln\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\frac{1}{n}\left[\ln\frac{1}{n}+\ln\frac{2}{n}+\ldots+\ln\frac{n}{n}\right].$$

The expression in brackets is the integral sum for the integral

$$\int_0^1 \ln x\,dx=(x\ln x-x)\Big|_0^1=-1.$$

Consequently, $\ln A=-1$ and $\lim\limits_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=e^{-1}$.

## § 7.2. *Finding Average Values of a Function*

*The average value* of $f(x)$ over the interval $[a, b]$ is the number

$$\mu=\frac{1}{b-a}\int_a^b f(x)\,dx.$$

The square root $\left\{\frac{1}{b-a}\int_a^b [f(x)]^2\,dx\right\}^{\frac{1}{2}}$ of the average value of the square of the function is called the *root mean square* (rms) of the function $f(x)$ over $[a, b]$.

**7.2.1.** Find the average value $\mu$ of the function $f(x)=\sqrt[3]{x}$ over the interval $[0, 1]$.

*Solution.* In this case

$$\mu=\frac{1}{1-0}\int_0^1\sqrt[3]{x}\,dx=\frac{3x^{\frac{4}{3}}}{4}\Big|_0^1=\frac{3}{4}.$$

**7.2.2.** Find the average values of the functions:

(a) $f(x)=\sin^2 x$ over $[0, 2\pi]$;

(b) $f(x)=\frac{1}{e^x+1}$ over $[0, 2]$.

**7.2.3.** Determine the average length of all vertical chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ over the interval $a\leqslant x\leqslant 2a$.

*Solution.* The problem consists in finding the average value of the function $f(x)=2y=2\frac{b}{a}\sqrt{x^2-a^2}$ over the interval $[a, 2a]$:

$$\mu=2\frac{1}{a}\int_a^{2a}\frac{b}{a}\sqrt{x^2-a^2}\,dx=$$

$$=\frac{2b}{a^2}\left[\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right)\right]_a^{2a}=b\left[2\sqrt{3}-\ln\left(2+\sqrt{3}\right)\right].$$

**7.2.4.** Find the average ordinate of the sinusoid $y=\sin x$ over the interval $[0, \pi]$.

*Solution:*

$$\mu=\frac{1}{\pi}\int_0^{\pi}\sin x\,dx=-\frac{1}{\pi}\cos x\Big|_0^{\pi}=\frac{2}{\pi}\approx 0.637.$$

Rewrite the obtained result in the following way:

$$\mu\cdot\pi=\frac{2}{\pi}\cdot\pi=\int_0^{\pi}\sin x\,dx.$$

Using the geometric meaning of the definite integral, we can say that the area of the rectangle with the altitude $\mu=\frac{2}{\pi}$ and the base $\pi$ equals the area of a figure bounded by a half-wave of the sinusoid $y=\sin x$, $0\leqslant x\leqslant\pi$, and by the $x$-axis.

**7.2.5.** Find the average length of all positive ordinates of the circle $x^2+y^2=1$.

**7.2.6.** Show that the average value of the function $f(x)$, continuous on the interval $[a, b]$, is the limit of the arithmetic mean of the values of this function taken over equal intervals of the argument $x$.

*Solution.* Subdivide the interval $[a, b]$ into $n$ equal parts by the points $x_i=a+i\frac{b-a}{n}$ $(i=0, 1, 2, \ldots, n)$.

Form the arithmetic mean of the values of the function $f(x)$ at $n$ points of division $x_0, x_1, \ldots, x_{n-1}$:

$$\mu_n = \frac{f(x_0)+f(x_1)+\ldots+f(x_{n-1})}{n} = \frac{1}{n}\sum_{i=0}^{n-1} f(x_i).$$

This mean may be represented in the following form:

$$\mu_n = \frac{1}{b-a}\sum_{i=0}^{n-1} f(x_i)\,\Delta x_i,$$

where

$$\Delta x_i = \frac{b-a}{n}.$$

The latter sum is the integral sum for the function $f(x)$, therefore

$$\lim_{n\to\infty}\mu_n = \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\,\Delta x_i = \frac{1}{b-a}\int_a^b f(x)\,dx = \mu,$$

which completes the solution.

7.2.7. Find the average value of pressure ($p_m$) varying from 2 to 10 atm if the pressure $p$ and the volume $v$ are related as follows:

$$pv^{\frac{3}{2}} = 160.$$

*Solution.* As $p$ varies from 2 to 10 atm, $v$ traverses the interval $[4\sqrt[3]{4},\ 4\sqrt[3]{100}]$; hence

$$p_m = \frac{1}{4\left(\sqrt[3]{100}-\sqrt[3]{4}\right)}\int_{4\sqrt[3]{4}}^{4\sqrt[3]{100}} 160v^{-\frac{3}{2}}\,dv =$$

$$= -\frac{320}{4\left(\sqrt[3]{100}-\sqrt[3]{4}\right)}v^{-\frac{1}{2}}\Bigg|_{4\sqrt[3]{4}}^{4\sqrt[3]{100}} = \frac{40}{\sqrt[3]{20}\left(\sqrt[3]{10}+\sqrt[3]{2}\right)} \approx 4.32\text{ atm}.$$

7.2.8. In hydraulics there is Bazin's formula expressing the velocity $v$ of water flowing in a wide rectangular channel as a function of the depth $h$ at which the point under consideration is situated below the open surface,

$$v = v_0 - 20\sqrt{HL}\left(\frac{h}{H}\right)^2,$$

where $v_0$ is the velocity on the open surface, $H$ is the depth of the channel, $L$ its slope.

Find the average velocity $v_m$ of flow in the cross-section of the channel.

*Solution.* We have

$$v_m = \frac{1}{H}\int_0^H \left[v_0 - 20\sqrt{HL}\left(\frac{h}{H}\right)^2\right] dh = v_0 - \frac{20}{3}\sqrt{HL}.$$

**7.2.9.** Determine the average value of the electromotive force $E_m$ over one period, i.e. over the time from $t=0$ to $t=T$, if electromotive force is computed by the formula

$$E = E_0 \sin\frac{2\pi t}{T},$$

where $T$ is the duration of the period in seconds, $E_0$ the amplitude (the maximum value) of the electromotive force corresponding to the value $t=0.25T$. The fraction $\frac{2\pi t}{T}$ is called the *phase.*

*Solution.*

$$E_m = \frac{E_0}{T}\int_0^T \sin\frac{2\pi t}{T}\,dt = \frac{E_0 T}{T\cdot 2\pi}\left[-\cos\frac{2\pi t}{T}\right]_0^T = 0.$$

Thus, the average value of the electromotive force over one period equals zero.

**7.2.10.** Each of the two vertical poles $OA$ and $CD$ is equipped with an electric lamp of luminous intensity $i$ fixed at a height $h$. The distance between the poles is $d$. Find the average illumination of the straight line $OC$ connecting the bases of the poles.

**7.2.11.** Find the average value of the square of the electromotive force $(E^2)_m$ over the interval from $t=0$ to $t=\frac{T}{2}$ (see Problem 7.2.9).

*Solution.* Since

$$E = E_0 \sin\frac{2\pi t}{T},$$

we have

$$(E^2)_m = \frac{2}{T}E_0^2\int_0^{\frac{T}{2}} \sin^2\frac{2\pi t}{T}\,dt = \frac{2}{T}E_0^2\int_0^{\frac{T}{2}} \frac{1-\cos\frac{4\pi t}{T}}{2}\,dt =$$

$$= \frac{E_0^2}{T}\left[t - \frac{T}{4\pi}\sin\frac{4\pi t}{T}\right]_0^{\frac{T}{2}} = \frac{E_0^2}{2}.$$

**7.2.12.** If a function $f(x)$ is defined on an infinite interval $[0, \infty)$, then its average value will be

$$\mu = \lim_{b\to\infty}\frac{1}{b}\int_0^b f(x)\,dx,$$

if this limit exists. Find the average power consumption of an alternating-current circuit if the current intensity $I$ and voltage $u$ are expressed by the following formulas, respectively:

$$I = I_0 \cos(\omega t + \alpha);$$
$$u = u_0 \cos(\omega t + \alpha + \varphi),$$

where $\varphi$ is the constant phase shift of the voltage as compared with the current intensity (the parameters $\omega$ and $\alpha$ will not enter into the expression for the average power).

*Solution.* The average power consumption

$$w_m = \lim_{T \to \infty} \frac{1}{T} \int_0^T I_0 \cos(\omega t + \alpha)\, u_0 \cos(\omega t + \alpha + \varphi)\, dt.$$

Taking into consideration that

$$\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)],$$

we will get

$$w_m = \lim_{T \to \infty} \frac{I_0 u_0}{2T} \int_0^T [\cos(2\omega t + 2\alpha + \varphi) + \cos\varphi]\, dt =$$
$$= \lim_{T \to \infty} \left\{ \frac{I_0 u_0}{4\omega} \cdot \frac{\sin(2\omega T + 2\alpha + \varphi) - \sin(2\alpha + \varphi)}{T} + \frac{I_0 u_0}{2} \cos\varphi \right\} = \frac{I_0 u_0}{2} \cos\varphi.$$

Hence, it is clear why so much importance is attached to the quantity $\cos\varphi$ in electrical engineering.

**7.2.13.** Find the average value $\mu$ of the function $f(x)$ over the indicated intervals:

(a) $f(x) = 2x^2 + 1$ over $[0, 1]$;

(b) $f(x) = \frac{1}{x}$ over $[1, 2]$;

(c) $f(x) = 3^x - 2x + 3$ over $[0, 2]$.

**7.2.14.** A body falling to the ground from a state of rest acquires a velocity $v_1 = \sqrt{2gs_1}$ on covering a vertical path $s = s_1$. Show that the average velocity $v_m$ over this path is equal to $\frac{2v_1}{3}$.

**7.2.15.** The cross-section of the trough has the form of a parabolic segment with a base $a$ and depth $h$. Find the average depth of the trough.

**7.2.16.** Find the average value $I_m$ of alternating current intensity over time interval from 0 to $\frac{\pi}{\omega}$ (see Problem 7.2.12).

**7.2.17.** Prove that the average value of the focal radius of an ellipse $\rho=\frac{p}{1-\varepsilon\cos\varphi}$, where $p=\frac{b^2}{a}$; $a$, $b$ are the semi-axes and $\varepsilon$ is eccentricity, is equal to $b$.

**7.2.18.** On the segment $AB$ of length $a$ a point $P$ is taken at a distance $x$ from the end-point $A$. Show that the average value of the areas of the rectangles constructed on the segments $AP$ and $PB$ is equal to $\frac{a^2}{6}$.

**7.2.19.** Find the average value of the function

$$f(x)=\frac{\cos^2 x}{\sin^2 x+4\cos^2 x}$$

over the interval $\left[0, \frac{\pi}{2}\right]$. Check directly that this average, equal to $\frac{1}{6}$, is the value of the function $f(x)$ for a certain $x=\xi$ lying within the indicated interval.

## § 7.3. Computing Areas in Rectangular Coordinates

If a plane figure is bounded by the straight lines $x=a$, $x=b$ $(a<b)$ and the curves $y=y_1(x)$, $y=y_2(x)$, provided $y_1(x)\leqslant y_2(x)$ $(a\leqslant x\leqslant b)$, then its area is computed by the formula

$$S=\int_a^b [y_2(x)-y_1(x)]\,dx.$$

In certain cases the left boundary $x=a$ (or the right boundary $x=b$) can degenerate into a point of intersection of the curves

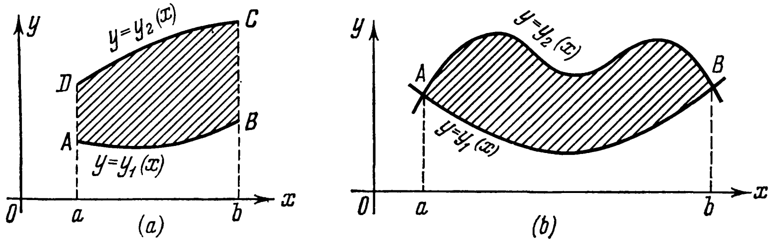


Fig. 65

$y=y_1(x)$ and $y=y_2(x)$. Then $a$ and $b$ are found as the abscissas of the points of intersection of the indicated curves (Fig. 65, $a$, $b$).

**7.3.1.** Compute the area of the figure bounded by the straight lines $x=0$, $x=2$ and the curves $y=2^x$, $y=2x-x^2$ (Fig. 66).

*Solution.* Since the maximum of the function $y=2x-x^2$ is attained at the point $x=1$ and is equal to 1, and the function $y=2^x \geqslant 1$ on the interval [0, 2], we have

$$S=\int_0^2 [2^x-(2x-x^2)]\,dx=\frac{2^x}{\ln 2}\Big|_0^2-\left(x^2-\frac{x^3}{3}\right)\Big|_0^2=\frac{3}{\ln 2}-\frac{4}{3}.$$

**7.3.2.** Compute the area of the figure bounded by the parabolas $x=-2y^2$, $x=1-3y^2$ (Fig. 67).

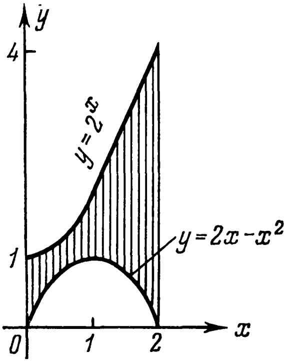


Fig. 66

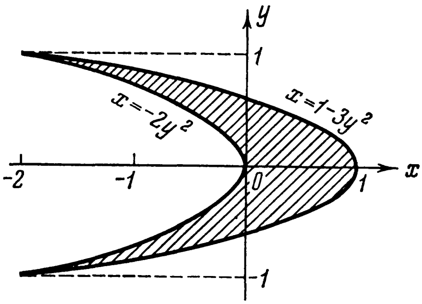


Fig. 67

*Solution.* Solving the system of equations

$$\begin{cases} x=2y^2; \\ x=1-3y^2, \end{cases}$$

find the ordinates of the points of intersection of the curves $y_1=-1$, $y_2=1$. Since $1-3y^2 \geqslant -2y^2$ for $-1 \leqslant y \leqslant 1$, then we have

$$S=\int_{-1}^{1} [(1-3y^2)-(-2y^2)]\,dy=2\left(y-\frac{y^3}{3}\right)\Big|_0^1=\frac{4}{3}.$$

**7.3.3.** Find the area of the figure contained between the parabola $x^2=4y$ and the witch of Agnesi $y=\frac{8}{x^2+4}$ (see Fig. 68).

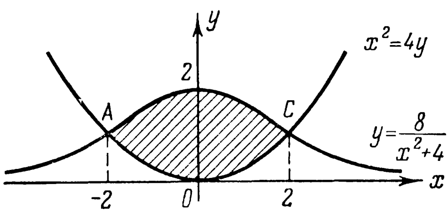


Fig. 68

*Solution.* Find the abscissas of the points $A$ and $C$ of intersection of the curves. For this purpose eliminate $y$ from the system of equations

$$\begin{cases} y=\dfrac{8}{x^2+4}, \\ y=\dfrac{x^2}{4}, \end{cases}$$

whence $\frac{8}{x^2+4}=\frac{x^2}{4}$, or $x^4+4x^2-32=0$.

The real roots of this equation are the points $x_1=-2$ and $x_2=2$. As is seen from the figure, $\frac{8}{x^2+4}\geqslant\frac{x^2}{4}$ on the interval $[-2, 2]$. (It is also possible to ascertain this by directly computing the values of these functions at any point inside the interval, for instance, at $x=0$.)

Consequently,

$$S=\int\limits_{-2}^{2}\left(\frac{8}{x^2+4}-\frac{x^2}{4}\right)dx=\left(4\arctan\frac{x}{2}-\frac{x^3}{12}\right)\Big|_{-2}^{2}=2\pi-\frac{4}{3}.$$

**7.3.4.** Find the area of the figure bounded by the parabola $y=x^2+1$ and the straight line $x+y=3$.

**7.3.5.** Compute the area of the figure which lies in the first quadrant inside the circle $x^2+y^2=3a^2$ and is bounded by the parabolas $x^2=2ay$ and $y^2=2ax\,(a>0)$ (Fig. 69).

*Solution.* Find the abscissa of the point $A$ of intersection of the parabola $y^2=2ax$ and the circle $x^2+y^2=3a^2$. Eliminating $y$ from the system of equations

$$\begin{cases} x^2+y^2=3a^2, \\ y^2=2ax, \end{cases}$$

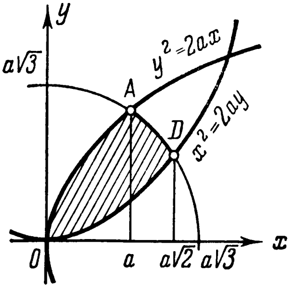

Fig. 69

we obtain $x^2+2ax-3a^2=0$, whence we get the only positive root: $x_A=a$. Analogously, we find the abscissa of the point $D$ of intersection of the circle $x^2+y^2=3a^2$ and the parabola $x^2=2ay$; $x_D=a\sqrt{2}$.

Thus, the sought-for area is equal to

$$S=\int\limits_{0}^{a\sqrt{2}}[y_2(x)-y_1(x)]\,dx,$$

where $y_1(x)=\frac{x^2}{2a}$, $y_2(x)=\begin{cases}\sqrt{2ax} & \text{for } 0\leqslant x\leqslant a, \\ \sqrt{3a^2-x^2} & \text{for } a<x\leqslant a\sqrt{2}.\end{cases}$

By the additivity property of the integral

$$S=\int_0^a\left(\sqrt{2ax}-\frac{x^2}{2a}\right)dx+\int_a^{a\sqrt{2}}\left(\sqrt{3a^2-x^2}-\frac{x^2}{2a}\right)dx=$$

$$=\left[\sqrt{2a}\cdot\frac{2}{3}x^{\frac{3}{2}}-\frac{x^3}{6a}\right]_0^a+\left[\frac{x}{2}\sqrt{3a^2-x^2}+\frac{3a^2}{2}\arcsin\frac{x}{a\sqrt{3}}-\frac{x^3}{6a}\right]_a^{a\sqrt{2}}=$$

$$=\frac{2\sqrt{2}}{3}a^2-\frac{a^2}{6}+\frac{3a^2}{2}\left(\arcsin\frac{\sqrt{2}}{\sqrt{3}}-\arcsin\frac{1}{\sqrt{3}}\right)-\frac{\sqrt{2}}{3}a^2+\frac{1}{6}a^2=$$

$$=\left(\frac{\sqrt{2}}{3}+\frac{3}{2}\arcsin\frac{1}{3}\right)a^2.$$

Here we make use of the trigonometric formula:

$$\arcsin\alpha-\arcsin\beta=\arcsin\left(\alpha\sqrt{1-\beta^2}-\beta\sqrt{1-\alpha^2}\right)\quad(\alpha\beta>0)$$

for transforming

$$\arcsin\sqrt{\frac{2}{3}}-\arcsin\frac{1}{\sqrt{3}}=\arcsin\left(\sqrt{\frac{2}{3}}\sqrt{\frac{2}{3}}-\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\right)=$$

$$=\arcsin\frac{1}{3}.$$

**7.3.6.** Compute the area of the figure lying in the first quadrant and bounded by the curves $y^2=4x$, $x^2=4y$ and $x^2+y^2=5$.

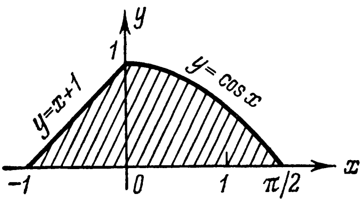

Fig. 70

**7.3.7.** Compute the area of the figure bounded by the lines $y=x+1$, $y=\cos x$ and the $x$-axis (Fig. 70).

*Solution.* The function

$$y=f(x)=\begin{cases}x+1 & \text{if } -1\leqslant x\leqslant 0,\\ \cos x & \text{if } \quad 0\leqslant x\leqslant\frac{\pi}{2}\end{cases}$$

is continuous on the interval $\left[-1,\frac{\pi}{2}\right]$. The area of the curvilinear trapezoid is equal to

$$S=\int_{-1}^{\frac{\pi}{2}}f(x)\,dx=\int_{-1}^{0}(x+1)\,dx+\int_0^{\frac{\pi}{2}}\cos x\,dx=\left.\frac{(x+1)^2}{2}\right|_{-1}^{0}+\sin x\Big|_0^{\frac{\pi}{2}}=\frac{3}{2}.$$

**7.3.8.** Find the area of the segment of the curve $y^2=x^3-x^2$ if the line $x=2$ is the chord determining the segment.

*Solution.* From the equality $y^2=x^2(x-1)$ it follows that $x^2(x-1)\geqslant 0$, therefore either $x=0$ or $x\geqslant 1$. In other words, the domain of definition of the implicit function $y^2=x^3-x^2$ consists of the point $x=0$ and the interval $[1, \infty)$. In computing the area the isolated point (0, 0) does not play any role, therefore, the interval of integration is [1, 2] (see Fig. 71).

Passing over to explicit representation $y=\pm x\sqrt{x-1}$, we see that the segment is bounded above by the curve $y=x\sqrt{x-1}$ and below by the curve $y=-x\sqrt{x-1}$. Hence,

$$S=\int_1^2 [x\sqrt{x-1}-(-x\sqrt{x-1})]\,dx=2\int_1^2 x\sqrt{x-1}\,dx.$$

Make the substitution

$$x-1=t^2, \quad dx=2t\,dt, \quad \begin{array}{c|c} x & t \\ \hline 1 & 0 \\ 2 & 1 \end{array}.$$

Then

$$S=4\int_0^1 (t^2+1)\,t^2\,dt=4\left[\frac{t^5}{5}+\frac{t^3}{3}\right]_0^1=\frac{32}{15}.$$

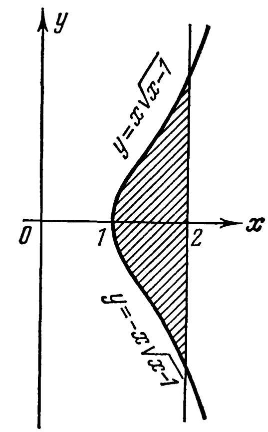

Fig. 71

**7.3.9.** Determine the area of the figure bounded by two branches of the curve $(y-x)^2=x^3$ and the straight line $x=1$.

*Solution.* Note first of all that $y$, as an implicit function of $x$, is defined only for $x\geqslant 0$; the left side of the equation is always non-negative. Now we find the equations of two branches of the curve $y=x-x\sqrt{x}$, $y=x+x\sqrt{x}$. Since $x\geqslant 0$, we have $x+x\sqrt{x}\geqslant x-x\sqrt{x}$, and therefore

$$S=\int_0^1 (x+x\sqrt{x}-x+x\sqrt{x})\,dx=2\int_0^1 x\sqrt{x}\,dx=\frac{4}{5}x^{\frac{5}{2}}\Big|_0^1=\frac{4}{5}.$$

**7.3.10.** Compute the area enclosed by the loop of the curve $y^2=x(x-1)^2$.

*Solution.* The domain of definition of the implicit function $y$ is the interval $0\leqslant x<+\infty$. Since the equation of the curve contains $y$ to the second power, the curve is symmetrical about the

$x$-axis. The positive branch $y_1(x)$ is given by the equation

$$y=y_1(x)=\sqrt{x}\,|x-1|=\begin{cases}\sqrt{x}(1-x), & 0\leqslant x\leqslant 1,\\ \sqrt{x}(x-1), & x>1.\end{cases}$$

The common points of the symmetrical branches $y_1(x)$ and $y_2(x)=-y_1(x)$ must lie on the $x$-axis. But $y_1(x)=\sqrt{x}\,|x-1|=0$ only at $x_1=0$ and at $x_2=1$.

Consequently, the loop is formed by the curves $y=\sqrt{x}(1-x)$ and $y=-\sqrt{x}(1-x)$, $0\leqslant x\leqslant 1$ (see Fig. 72), the area enclosed being

$$S=2\int_0^1\sqrt{x}(1-x)\,dx=2\int_0^1\left(x^{\frac{1}{2}}-x^{\frac{3}{2}}\right)dx=\frac{8}{15}.$$

**7.3.11.** Find the area enclosed by the loop of the curve $y^2=(x-1)(x-2)^2$.

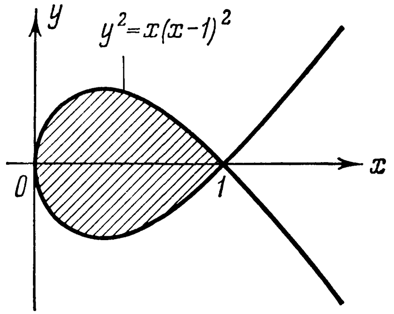


Fig. 72

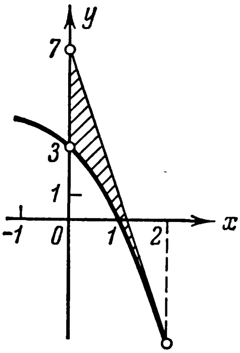


Fig. 73

**7.3.12.** Find the area of the figure bounded by the parabola $y=-x^2-2x+3$, the line tangent to it at the point $M(2, -5)$ and the $y$-axis.

*Solution.* The equation of the tangent at the point $M(2, -5)$ has the form $y+5=-6(x-2)$ or $y=7-6x$. Since the branches of the parabola are directed downward, the parabola lies below the tangent, i.e. $7-6x\geqslant -x^2-2x+3$ on the interval $[0, 2]$ (Fig. 73). Hence,

$$S=\int_0^2[7-6x-(-x^2-2x+3)]\,dx=\int_0^2(x^2-4x+4)\,dx=\frac{8}{3}.$$

**7.3.13.** Find the area bounded by the parabola $y=x^2-2x+2$, the line tangent to it at the point $M(3, 5)$ and the axis of ordinates.

**7.3.14.** We take on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (a>b)$$

a point $M(x, y)$ lying in the first quadrant.

Show that the sector of the ellipse bounded by its semi-major axis and the focal radius drawn to the point $M$ has an area

$$S=\frac{ab}{2}\arccos\frac{x}{a}.$$

With the aid of this result deduce a formula for computing the area of the entire ellipse.

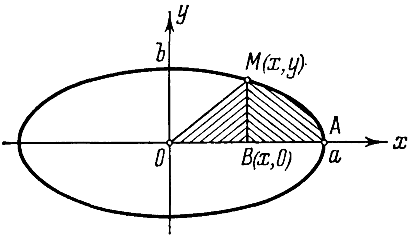


Fig. 74

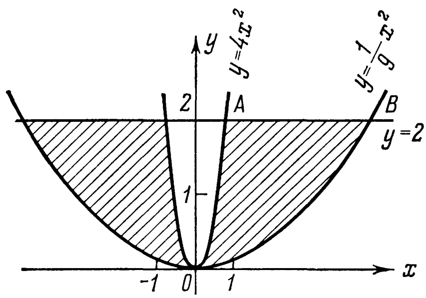


Fig. 75

*Solution.* We have (Fig. 74):

$$S_{OMAO}=S_{\Delta OMB}+S_{MABM}; \quad S_{\Delta OMB}=\frac{xy}{2}=\frac{b}{2a}x\sqrt{a^2-x^2};$$

$$S_{MABM}=\int_x^a y\,dx=\int_x^a \frac{b}{a}\sqrt{a^2-t^2}\,dt=\frac{b}{2a}\left(t\sqrt{a^2-t^2}+a^2\arcsin\frac{t}{a}\right)\Big|_x^a=$$

$$=\frac{b}{2a}\left[-x\sqrt{a^2-x^2}+a^2\left(\frac{\pi}{2}-\arcsin\frac{x}{a}\right)\right].$$

Since $\frac{\pi}{2}-\arcsin\frac{x}{a}=\arccos\frac{x}{a}$, we obtain

$$S_{MABM}=\frac{b}{2a}\left[-x\sqrt{a^2-x^2}+a^2\arccos\frac{x}{a}\right].$$

Hence

$$S_{OMAO}=S_{\Delta OMB}+S_{MABM}+\frac{ab}{2}\arccos\frac{x}{a}.$$

At $x=0$, the sector becomes a quarter of the ellipse, i.e.

$$\frac{1}{4}S_{\text{ellipse}}=\frac{ab}{2}\arccos 0=\frac{ab}{2}\cdot\frac{\pi}{2}=\frac{ab}{4}\pi,$$

and consequently, $S_{\text{ellipse}}=\pi ab$. At $a=b$ we get the area of a circle $S_{\text{circle}}=\pi a^2$.

**7.3.15.** Find the area bounded by the parabolas $y=4x^2$, $y=\frac{x^2}{9}$ and the straight line $y=2$.

*Solution.* In this case it is advisable to integrate with respect to $y$ and take advantage of the symmetry of the figure (see Fig. 75). Therefore, solving the equations of the parabolas for $x$, we have:

$$x=\pm\frac{\sqrt{y}}{2}, \quad x=\pm 3\sqrt{y}.$$

By symmetry of the figure about the $y$-axis the area sought is equal to the doubled area $S_{OABO}$:

$$S=2S_{OABO}=2\int_0^2\left(3\sqrt{y}-\frac{1}{2}\sqrt{y}\right)dy=5\int_0^2\sqrt{y}\,dy=\frac{20\sqrt{2}}{3}.$$

**7.3.16.** From an arbitrary point $M(x, y)$ of the curve $y=x^m$ $(m>0)$ perpendiculars $MN$ and $ML\,(x>0)$ are dropped onto the coordinate axes. What part of the area of the rectangle $ONML$ does the area $ONMO$ (Fig. 76) constitute?

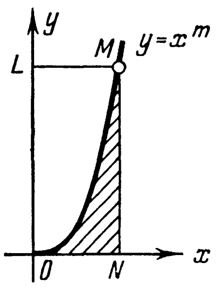

Fig. 76

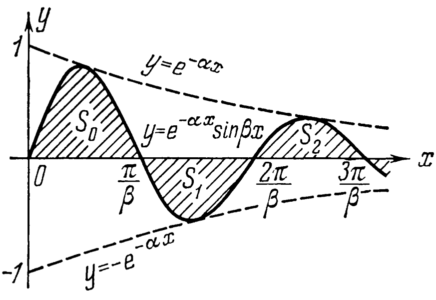

Fig. 77

**7.3.17.** Prove that the areas $S_0$, $S_1$, $S_2$, $S_3$, ..., bounded by the $x$-axis and half-waves of the curve $y=e^{-\alpha x}\sin\beta x$, $x\geqslant 0$, form a geometric progression with the common ratio $q=e^{-\frac{\alpha\pi}{\beta}}$.

*Solution.* The curve of Fig. 77 intersects the positive semi-axis $Ox$ at the points where $\sin\beta x=0$, whence

$$x_n=\frac{n\pi}{\beta}, \quad n=0, 1, 2, \ldots .$$

The function $y=e^{-\alpha x}\sin\beta x$ is positive in the intervals $(x_{2k}, x_{2k+1})$ and negative in $(x_{2k+1}, x_{2k+2})$, i.e. the sign of the function in the interval $(x_n, x_{n+1})$ coincides with that of the number $(-1)^n$. Therefore

$$S_n=\int\limits_{\frac{n\pi}{\beta}}^{\frac{(n+1)\pi}{\beta}} |y|\,dx=(-1)^n\int\limits_{\frac{n\pi}{\beta}}^{\frac{(n+1)\pi}{\beta}} e^{-\alpha x}\sin\beta x\,dx.$$

But the indefinite integral is equal to

$$\int e^{-\alpha x}\sin\beta x\,dx=-\frac{e^{-\alpha x}}{\alpha^2+\beta^2}(\alpha\sin\beta x+\beta\cos\beta x)+C.$$

Consequently,

$$S_n=(-1)^{n+1}\left[\frac{e^{-\alpha x}}{\alpha^2+\beta^2}(\alpha\sin\beta x+\beta\cos\beta x)\right]\Bigg|_{\frac{n\pi}{\beta}}^{\frac{(n+1)\pi}{\beta}}=$$

$$=\frac{(-1)^{n+1}}{\alpha^2+\beta^2}\left[e^{-\alpha(n+1)\pi/\beta}\beta(-1)^{n+1}-e^{\alpha n\pi/\beta}\beta(-1)^n\right]=$$

$$=\frac{\beta}{\alpha^2+\beta^2}e^{-\alpha n\pi/\beta}\left(1+e^{-\alpha\pi/\beta}\right).$$

Hence

$$q=\frac{S_{n+1}}{S_n}=\frac{e^{-\alpha(n+1)\pi/\beta}}{e^{-\alpha n\pi/\beta}}=e^{-\alpha\pi/\beta},$$

which completes the proof.

7.3.18. Find the areas enclosed between the circle $x^2+y^2-2x+4y-11=0$ and the parabola $y=-x^2+2x+1-2\sqrt{3}$.

*Solution.* Rewriting the equations of the curves, we have:

$$(x-1)^2+(y+2)^2=16,$$
$$y=-(x-1)^2-2\sqrt{3}+2.$$

Consequently, the centre of the circle lies at the point $C(1, -2)$ and the radius of the circle equals 4. The axis of the parabola coincides with the straight line $x=1$ and its vertex lies at the point $B(1,\ 2,\ -2\sqrt{3})$ (Fig. 78).

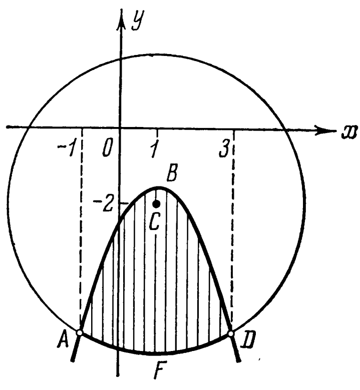

Fig. 78

The area $S_{ABDFA}$ of the smaller figure is found by the formula

$$S_{ABDFA}=\int\limits_{x_A}^{x_D}(y_{\text{par}}-y_{\text{circle}})\,dx,$$

where $x_A$ and $x_D$ are determined from the system of equations

$$\begin{cases} (x-1)^2+(y+2)^2=16, \\ y+2=-(x-1)^2-2\sqrt{3}+4, \end{cases}$$

whence $x_A=-1$, $x_D=3$.

Hence,

$$S_{ABDFA}=\int_{-1}^{3}[(-x^2+2x+1-2\sqrt{3})+(2+\sqrt{16-(x-1)^2})]\,dx=$$

$$=\left[-\frac{x^3}{3}+x^2+(3-2\sqrt{3})\,x+\frac{x-1}{2}\sqrt{16-(x-1)^2}+\right.$$

$$\left.+\frac{16}{2}\arcsin\frac{x-1}{4}\right]_{1}^{3}=\frac{32}{3}-8\sqrt{3}+2\sqrt{12}+16\arcsin\frac{1}{2}=$$

$$=\frac{32}{3}-4\sqrt{3}+\frac{8}{3}\pi.$$

The area of the second figure is easy to determine.

*Note.* The computation of the integral can be simplified by using the shift $x-1=z$ and taking advantage of the evenness of the integrand.

**7.3.19.** Compute the area bounded by the curves $y=(x-4)^2$, $y=16-x^2$ and the $x$-axis.

**7.3.20.** Compute the area enclosed between the parabolas

$$x=y^2; \qquad x=\frac{3}{4}y^2+1.$$

**7.3.21.** Compute the area of the portions cut off by the hyperbola $x^2-3y^2=1$ from the ellipse $x^2+4y^2=8$.

**7.3.22.** Compute the area enclosed by the curve $y^2=(1-x^2)^3$.

**7.3.23.** Compute the area enclosed by the loop of the curve $4(y^2-x^2)+x^3=0$.

**7.3.24.** Compute the area of the figure bounded by the curve $\sqrt{x}+\sqrt{y}=1$ and the straight line $x+y=1$.

**7.3.25.** Compute the area of the figure enclosed by the curve $y^2=x^2(1-x^2)$.

**7.3.26.** Compute the area enclosed by the loop of the curve $x^3+x^2-y^2=0$.

**7.3.27.** Compute the area bounded by the axis of ordinates and the curve $x=y^2(1-y)$.

**7.3.28.** Compute the area bounded by the curve $y=x^4-2x^3+x^2+3$, the axis of abscissas and two ordinates corresponding to the points of minimum of the function $y(x)$.

## § 7.4. Computing Areas with Parametrically Represented Boundaries

If the boundary of a figure is represented by parametric equations

$$x = x(t), \qquad y = y(t),$$

then the area of the figure is evaluated by one of the three formulas:

$$S = -\int_{\alpha}^{\beta} y(t)\, x'(t)\, dt; \quad S = \int_{\alpha}^{\beta} x(t)\, y'(t)\, dt; \quad S = \frac{1}{2}\int_{\alpha}^{\beta} (xy' - yx')\, dt,$$

where $\alpha$ and $\beta$ are the values of the parameter $t$ corresponding respectively to the beginning and the end of the traversal of the contour in the positive direction (the figure remains on the left).

**7.4.1.** Compute the area enclosed by the ellipse

$$x = a\cos t, \quad y = b\sin t \quad (0 \leqslant t \leqslant 2\pi).$$

*Solution.* Here it is convenient first to compute

$$xy' - yx' = a\cos t \times b\cos t + b\sin t \times a\sin t = ab.$$

Hence

$$S = \frac{1}{2}\int_0^{2\pi} (xy' - yx')\, dt = \frac{1}{2}\int_0^{2\pi} ab\, dt = \pi ab.$$

**7.4.2.** Find the area enclosed by the astroid $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{a}\right)^{\frac{2}{3}} = 1.$

*Solution.* Let us write the equation of the astroid in parametric form: $x = a\cos^3 t$, $y = a\sin^3 t$, $0 \leqslant t \leqslant 2\pi$. Here it is also convenient to evaluate first

$$xy' - yx' = a^2(\cos^3 t \cdot 3\sin^2 t\cos t + \sin^3 t \cdot 3\cos^2 t\sin t) =$$
$$= 3a^2\cos^2 t\sin^2 t.$$

Hence,

$$S = \frac{1}{2}\int_0^{2\pi} (xy' - yx')\, dt = \frac{3}{8}a^2\int_0^{2\pi} \sin^2 2t\, dt = \frac{3}{8}a^2\pi.$$

**7.4.3.** Find the area of the region bounded by an arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ and the $x$-axis.

*Solution.* Here the contour consists of an arc of the cycloid $(0 \leqslant t \leqslant 2\pi)$ and a segment of the $x$-axis $(0 \leqslant x \leqslant 2\pi a)$. Let us apply the formula $S = -\int_{\alpha}^{\beta} yx'\, dt$.

Since on the segment of the $x$-axis we have $y=0$, it only remains to compute the integral (taking into account the direction of a boundary traversal):

$$S=-\int_{2\pi}^{0} a(1-\cos t)\,a(1-\cos t)\,dt=a^2\int_{0}^{2\pi}(1-\cos t)^2\,dt=$$

$$=a^2\int_{0}^{2\pi}\left[1-2\cos t+\frac{1}{2}(1+\cos 2t)\right]dt=3\pi a^2.$$

**7.4.4.** Compute the area of the region enclosed by the curve $x=a\sin t$, $y=b\sin 2t$.

*Solution.* When constructing the curve one should bear in mind that it is symmetrical about the axes of coordinates. Indeed, if we substitute $\pi-t$ for $t$, the variable $x$ remains unchanged, while $y$ only changes its sign; consequently, the curve is symmetrical about the $x$-axis. When substituting $\pi+t$ for $t$ the variable $y$ remains unchanged, and $x$ only changes its sign, which means that the curve is symmetrical about the $y$-axis.

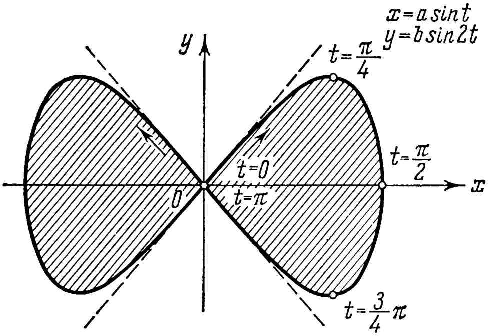


Fig. 79

Furthermore, since the functions $x=a\sin t$; $y=b\sin 2t$ have a common period $2\pi$, it is sufficient to confine ourselves to the following interval of variation of the parameter: $0\leqslant t\leqslant 2\pi$.

From the equations of the curve it readily follows that the variables $x$ and $y$ simultaneously retain non-negative values only when the parameter $t$ varies on the interval $\left[0, \frac{\pi}{2}\right]$, therefore at $0\leqslant t\leqslant\frac{\pi}{2}$ we obtain the portion of the curve situated in the first quadrant. The curve is shown in Fig. 79.

As is seen from the figure, it is sufficient to evaluate the area enclosed by one loop of the curve corresponding to the variation of the parameter $t$ from 0 to $\pi$ and then to double the result

$$S=2\int_0^{\pi} yx'\,dt=2\int_0^{\pi} b\sin 2t\times a\cos t\,dt=4ab\int_0^{\pi}\cos^2 t\sin t\,dt=$$

$$=-4ab\left(\frac{\cos^3 t}{3}\right)\Big|_0^{\pi}=\frac{8}{3}ab.$$

**7.4.5.** Find the area of the region enclosed by the loop of the curve

$$x=\frac{t}{3}(6-t);\qquad y=\frac{t^2}{8}(6-t).$$

*Solution.* Locate the points of self-intersection of the curve. Both functions $x(t)$ and $y(t)$ are defined throughout the entire number scale $-\infty<t<\infty$.

At the point of self-intersection the values of the abscissa (and ordinate) coincide at different values of the parameter. Since $x=3-\frac{1}{3}(t-3)^2$, the abscissas coincide at $t=3\pm\lambda$. For the function $y(t)$ to take on one and the same value at the same values of the parameter $t$, the equality $\frac{(3+\lambda)^2}{8}(3-\lambda)=\frac{(3-\lambda)^2}{8}(3+\lambda)$ must be fulfilled for $\lambda\neq 0$, whence $\lambda=\pm 3$.

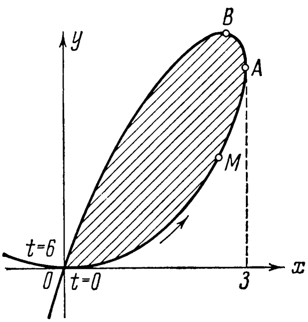


Fig. 80

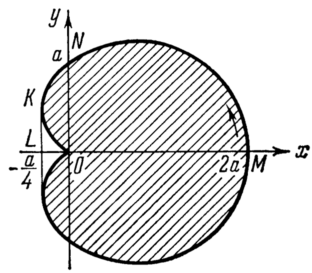


Fig. 81

Thus, at $t_1=0$ and at $t_2=6$ we have $x(t_1)=x(t_2)=0$, and $y(t_1)=y(t_2)=0$, i. e. the point (0, 0) is the only point of self-intersection. When $t$ changes from 0 to 6, the points of the curve are found in the first quadrant. As $t$ varies from 0 to 3, the point $M(x, y)$ describes the lower part of the loop, since in the indicated interval $x(t)$ and $y(t)=\frac{3tx}{8}$ increase, and then the function $x(t)$ begins to decrease, while $y(t)$ still keeps increasing. Figure 80 shows the traversal of the curve corresponding to increasing $t$ (the figure remains on the left).

In computing the area enclosed by the loop sought it is convenient to use the formula

$$S=\frac{1}{2}\int_0^6 (xy'-yx')\,dt=\frac{1}{2}\int_0^6 \frac{t^2(6-t)^2}{24}\,dt=\frac{27}{5}.$$

**7.4.6.** Find the area enclosed by the loop of the curve: $x=t^2$; $y=t-\frac{t^3}{3}$.

**7.4.7.** Compute the area enclosed by the cardioid: $x=a\cos t\,(1+\cos t)$; $y=a\sin t\,(1+\cos t)$.

*Solution.* Since $x(t)$ and $y(t)$ are periodic functions, it is sufficient to consider the interval $[-\pi, \pi]$. The curve is symmetrical about the $x$-axis, since on substituting $-t$ for $t$ the value of the variable $x$ remains unchanged, while $y$ only changes its sign, and $y\geqslant 0$ as $t$ varies from 0 to $\pi$.

As $t$ changes from 0 to $\pi$ the function $u=\cos t$ decreases from 1 to $-1$, and the abscissa $x=au(1+u)=a\left[-\frac{1}{4}+\left(u+\frac{1}{2}\right)^2\right]$ first decreases from $x\big|_{u=1}=2a$ to $x\big|_{u=-\frac{1}{2}}=-\frac{a}{4}$ and then increases to $x\big|_{u=-1}=0$. We can show that the ordinate $y$ increases on the interval $\left(0\leqslant t\leqslant\frac{\pi}{3}\right)$ and decreases on the interval $\left(\frac{\pi}{3}\leqslant t\leqslant\pi\right)$.

The curve is shown in Fig. 81, the arrow indicating the direction of its traversal as $t$ increases.

Consequently,

$$S=\frac{1}{2}\int_{-\pi}^{\pi}(xy'-yx')\,dt=a^2\int_0^{\pi}(1+\cos t)^2\,dt=\frac{3}{2}\pi a^2.$$

**7.4.8.** Compute the area of the region enclosed by the curve $x=\cos t$, $y=b\sin^3 t$.

**7.4.9.** Compute the areas enclosed by the loops of the curves:

(a) $x=t^2-1$, $y=t^3-t$;

(b) $x=2t-t^2$; $y=2t^2-t^3$;

(c) $x=t^2$; $y=\frac{t}{3}(3-t^2)$.

**7.4.10.** Compute the area of the region enclosed by the curve $x=a\cos t$; $y=b\sin t\cos^2 t$.

**7.4.11.** Compute the area enclosed by the evolute of the ellipse

$$x=\frac{c^2}{a}\cos^3 t;\quad y=-\frac{c^2}{b}\sin^3 t;\quad c^2=a^2-b^2.$$

## § 7.5. The Area of a Curvilinear Sector in Polar Coordinates

In polar coordinates the area of a sector bounded by the curve $\rho=\rho(\varphi)$ and the rays $\varphi_1=\alpha$ and $\varphi_2=\beta$ is expressed by the integral

$$S=\frac{1}{2}\int_{\alpha}^{\beta}\rho^2(\varphi)\,d\varphi.$$

**7.5.1.** Find the area of the region situated in the first quadrant and bounded by the parabola $y^2=4ax$ and the straight lines $y=x-a$ and $x=a$.

*Solution.* Let us introduce a polar system of coordinates by placing the pole at the focus $F$ of the parabola and directing the polar axis in the positive direction along the $x$-axis. Then the equation of the parabola will be $\rho=\frac{p}{1-\cos\varphi}$, where $p$ is the parameter of the parabola. In this case $p=2a$, and the focus $F$ has the coordinates $(a, 0)$. Hence, the equation of the parabola will acquire the form $\rho=\frac{2a}{1-\cos\varphi}$, and those of the straight lines will become $\varphi=\frac{\pi}{4}$ and $\varphi=\frac{\pi}{2}$ (Fig. 82). Therefore,

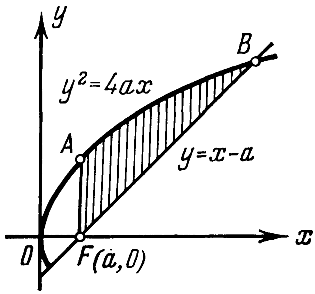


Fig. 82

$$S_{FABF}=\frac{1}{2}\int_{\pi/4}^{\pi/2}\frac{4a^2}{(1-\cos\varphi)^2}\,d\varphi=2a^2\int_{\pi/4}^{\pi/2}\frac{d\varphi}{4\sin^4\frac{\varphi}{2}}.$$

Changing the variable:

$$\cot\frac{\varphi}{2}=z,\qquad -\frac{d\varphi}{2\sin^2(\varphi/2)}=dz,$$

| $\varphi$ | $z$ |
|---|---|
| $\pi/4$ | $\cot(\pi/8)$ |
| $\pi/2$ | $1$ |

we obtain

$$S_{FABF}=a^2\int_{1}^{\cot(\pi/8)}(1+z^2)\,dz=a^2\left(\cot\frac{\pi}{8}+\frac{1}{3}\cot^3\frac{\pi}{8}-1-\frac{1}{3}\right)$$

or, taking into account that $\cot\frac{\pi}{8}=\frac{1+\cos(\pi/4)}{\sin(\pi/4)}=1+\sqrt{2}$,

$$S_{FABF}=2a^2\left(1+\frac{4}{3}\sqrt{2}\right).$$

**7.5.2.** Compute the area of the region enclosed by

(a) the cardioid $\rho = 1 + \cos\varphi$;
(b) the curve $\rho = a\cos\varphi$.

**7.5.3.** Find the area of the regions bounded by the curve $\rho = 2a\cos 3\varphi$ and the arcs of the circle $\rho = a$ and situated outside the circle.

*Solution.* Since the function $\rho = 2a\cos 3\varphi$ has a period $T = \frac{2\pi}{3}$, the radius vector describes three equal loops of the curve as $\varphi$ varies between $-\pi$ and $\pi$. Permissible values for $\varphi$ are those at which $\cos 3\varphi \geqslant 0$, whence

$$-\frac{\pi}{6} + \frac{2k\pi}{3} \leqslant \varphi \leqslant \frac{\pi}{6} + \frac{2k\pi}{3} \quad (k = 0, \pm 1, \pm 2, \ldots).$$

Consequently, one of the loops is described as $\varphi$ varies between $-\frac{\pi}{6}$ and $\frac{\pi}{6}$, and the other two loops as $\varphi$ varies between $\frac{\pi}{2}$ and $\frac{5\pi}{6}$, and between $\frac{7\pi}{6}$ and $\frac{3\pi}{2}$, respectively (Fig. 83). Cutting out the parts, belonging to the circle $\rho = a$, we get the figure whose area is sought. Clearly, it is equal to the triple area $S_{MLNM}$.

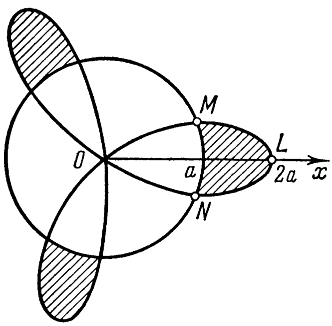

Fig. 83

Let us find the polar coordinates of the points of intersection $M$ and $N$. For this purpose solve the equation $2a\cos 3\varphi = a$, i. e. $\cos 3\varphi = \frac{1}{2}$. Between $-\frac{\pi}{6}$ and $\frac{\pi}{6}$ only the roots $-\frac{\pi}{9}$ and $\frac{\pi}{9}$ $(k=0)$ are found. Thus, the point $N$ is specified by the polar angle $\varphi_1 = -\frac{\pi}{9}$, and the point $M$ by $\varphi_2 = \frac{\pi}{9}$.

As is seen from the figure,

$$S_{MLNM} = S_{OMLNO} - S_{OMNO} =$$
$$= \frac{1}{2}\int_{-\pi/9}^{\pi/9} 4a^2\cos^2 3\varphi\, d\varphi - \frac{1}{2}\int_{-\pi/9}^{\pi/9} a^2\, d\varphi = a^2\left(\frac{\pi}{9} + \frac{\sqrt{3}}{6}\right).$$

**7.5.4.** Compute the area of the figure bounded by the circle $\rho = 3\sqrt{2}\, a\cos\varphi$ and $\rho = 3a\sin\varphi$.

*Solution.* The first circle lies in the right half-plane and passes through the pole $\rho = 0$, touching the vertical line. The second circle is situated in the upper half-plane and passes through the pole $\rho=0$, touching the horizontal line. Consequently, the pole is a point of intersection of the circles. The other point of intersection of the circles $B$ is found from the equation $3\sqrt{2}\,a\cos\varphi=3a\sin\varphi$, whence $B(\arctan\sqrt{2},\ a\sqrt{6})$. As is seen from Fig. 84, the sought-for area

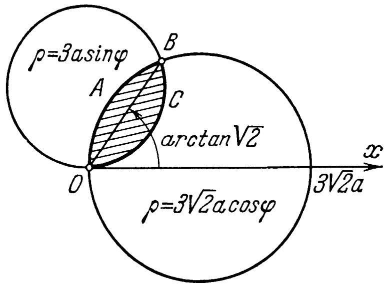

Fig. 84

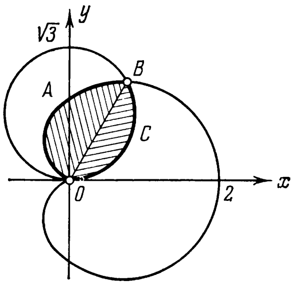

Fig. 85

$S$ is equal to the sum of the areas of the circular segments $OABO$ and $OCBO$ adjoining each other along the ray $\varphi=\arctan\sqrt{2}$. The arc $BAO$ is described by the end-point of the polar radius $\rho$ of the first circle for $\arctan\sqrt{2}\leqslant\varphi\leqslant\frac{\pi}{2}$, and the arc $OCB$ by the end-point of the polar radius $\rho$ of the second circle for $0\leqslant\varphi\leqslant\arctan\sqrt{2}$. Therefore

$$S_{OABO}=9a^2\int\limits_{\arctan\sqrt{2}}^{\frac{\pi}{2}}\cos^2\varphi\,d\varphi=\frac{9}{2}a^2\left(\frac{\pi}{2}-\arctan\sqrt{2}-\frac{\sqrt{2}}{3}\right),$$

$$S_{OCBO}=\frac{9}{2}a^2\int\limits_{0}^{\arctan\sqrt{2}}\sin^2\varphi\,d\varphi=\frac{9}{4}a^2\left(\arctan\sqrt{2}-\frac{\sqrt{2}}{3}\right).$$

Hence,

$$S_{OABO}+S_{OCBO}=2.25a^2(\pi-\arctan\sqrt{2}-\sqrt{2}).$$

**7.5.5.** Find the area of the figure cut out by the circle $\rho=\sqrt{3}\sin\varphi$ from the cardioid $\rho=1+\cos\varphi$ (Fig. 85).

*Solution.* Let us first find the points of intersection of these curves. To this end solve the system

$$\begin{cases}\rho=\sqrt{3}\sin\varphi, & 0\leqslant\varphi\leqslant\pi,\\ \rho=1+\cos\varphi,\end{cases}$$

whence $\varphi_1=\frac{\pi}{3}$, $\varphi_2=\pi$.

The sought-for area is the sum of two areas: one is a circular segment, the other a segment of the cardioid; the segments adjoin each other along the ray $\varphi=\frac{\pi}{3}$. The arc $BAO$ is described by the end-point of the polar radius $\rho$ of the cardioid as the polar angle $\varphi$ changes from $\frac{\pi}{3}$ to $\pi$, and the arc $OCB$ by the end-point of the polar radius $\rho$ of the circle for $0\leqslant\varphi\leqslant\frac{\pi}{3}$.

Therefore

$$S=\frac{1}{2}\int_0^{\frac{\pi}{3}} 3\sin^2\varphi\,d\varphi+\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}(1+\cos\varphi)^2d\varphi=$$

$$=\frac{3}{4}\left(\varphi-\frac{\sin 2\varphi}{2}\right)\Bigg|_0^{\frac{\pi}{3}}+\frac{1}{2}\left(\varphi+2\sin\varphi+\frac{\varphi}{2}+\frac{\sin 2\varphi}{4}\right)\Bigg|_{\frac{\pi}{3}}^{\pi}=$$

$$=\frac{3}{4}(\pi-\sqrt{3}).$$

**7.5.6.** Find the area of the figure bounded by the cardioid $\rho=a(1-\cos\varphi)$ and the circle $\rho=a$.

**7.5.7.** Find the area of the region enclosed by the loop of the folium of Descartes $x^3+y^3=3axy$.

*Solution.* Let us pass over to polar coordinates using the usual formulas $x=\rho\cos\varphi$, $y=\rho\sin\varphi$. Then the equation of the curve is:

$$\rho^3(\cos^3\varphi+\sin^3\varphi)=3a\rho^2\sin\varphi\cos\varphi,$$

or

$$\rho=\frac{3a\sin\varphi\cos\varphi}{\cos^3\varphi+\sin^3\varphi}=$$

$$=\frac{3a\sin 2\varphi}{(\sin\varphi+\cos\varphi)(2-\sin 2\varphi)}.$$

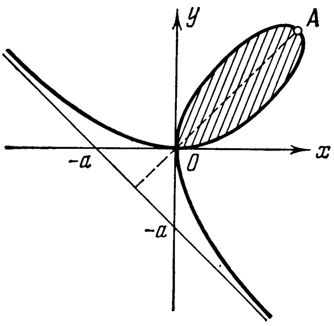

Fig. 86

It follows from this equation that, firstly, $\rho=0$ at $\varphi=0$ and at $\varphi=\frac{\pi}{2}$, and secondly, $\rho\to\infty$ as $\varphi\to\frac{3\pi}{4}$ and $\varphi\to-\frac{\pi}{4}$. The latter means that the folium of Descartes has an asymptote, whose equation $y=-x-a$ can be found in the usual way in rectangular coordinates.

Consequently, the loop of the folium of Descartes is described as $\varphi$ changes from 0 to $\frac{\pi}{2}$ and is situated in the first quadrant (see Fig. 86).

Thus, the sought-for area is equal to

$$S_{OAO}=\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{9a^2\cos^2\varphi\sin^2\varphi}{(\cos^3\varphi+\sin^3\varphi)^2}\,d\varphi.$$

Taking advantage of the curve's symmetry about the bisector $y=x$, i.e. about the ray $\varphi=\frac{\pi}{4}$, we can compute the area of half of the loop $\left(\text{from } \varphi=0 \text{ to } \varphi=\frac{\pi}{4}\right)$ and then double it. This enables us to apply the substitution

$$\tan\varphi=z,\qquad \frac{d\varphi}{\cos^2\varphi}=dz,$$

| $\varphi$ | $z$ |
|---|---|
| $0$ | $0$ |
| $\frac{\pi}{4}$ | $1$ |

which gives

$$S_{OAO}=9a^2\int_0^{\frac{\pi}{4}} \frac{\cos^2\varphi\sin^2\varphi}{(\cos^3\varphi+\sin^3\varphi)^2}\,d\varphi=9a^2\int_0^1\frac{z^2\,dz}{(1+z^3)^2}.$$

Still new substitution

$$1+z^3=v,\qquad 3z^2\,dz=dv,$$

| $z$ | $v$ |
|---|---|
| $0$ | $1$ |
| $1$ | $2$ |

leads to the integral

$$S_{OAO}=3a^2\int_1^2\frac{dv}{v^2}=\frac{3}{2}a^2.$$

**7.5.8.** Compute the area of the region enclosed by one loop of the curves:

(a) $\rho=a\cos 2\varphi$; (b) $\rho=a\sin 2\varphi$.

**7.5.9.** Compute the area enclosed by the portion of the cardioid $\rho=a(1-\cos\varphi)$ lying inside the circle $\rho=a\cos\varphi$.

**7.5.10.** Compute the area of the region enclosed by the curve $\rho=a\sin\varphi\cos^2\varphi$, $a>0$.

**7.5.11.** Compute the area of the region enclosed by the curve $\rho=a\cos^3\varphi$ $(a>0)$.

**7.5.12.** Compute the area of the portion $\left(\text{lying inside the circle } \rho = \frac{a}{\sqrt{2}}\right)$ of the figure bounded by the Bernoulli's lemniscate $\rho = a\sqrt{\cos 2\varphi}$.

**7.5.13.** Passing over to polar coordinates, compute the area of the region enclosed by the curve $(x^2+y^2)^3 = 4a^2x^2y^2$.

**7.5.14.** Passing over to polar coordinates, evaluate the area of the region enclosed by the curve $x^4+y^4 = a^2(x^2+y^2)$.

## § 7.6. Computing the Volume of a Solid

The volume of a solid is expressed by the integral

$$V = \int_a^b S(x)\,dx$$

where $S(x)$ is the area of the section of the solid by a plane perpendicular to the $x$-axis at the point with abscissa $x$; $a$ and $b$ are the left and right boundaries of variation of $x$. The function $S(x)$ is supposed to be known and continuously changing as $x$ varies between $a$ and $b$.

The volume $V_x$ of a solid generated by revolution about the $x$-axis of the curvilinear trapezoid bounded by the curve $y=f(x)$ $(f(x) \geqslant 0)$, the $x$-axis and the straight lines $x=a$ and $x=b$ $(a<b)$ is expressed by the integral

$$V_x = \pi \int_a^b y^2\,dx.$$

The volume $V_x$ of a solid obtained by revolving about the $x$-axis the figure bounded by the curves $y=y_1(x)$ and $y=y_2(x)$ $[0 \leqslant y_1(x) \leqslant y_2(x)]$ and the straight lines $x=a$, $x=b$ is expressed by the integral

$$V_x = \pi \int_a^b (y_2^2 - y_1^2)\,dx.$$

If the curve is represented parametrically or in polar coordinates, the appropriate change of the variable should be made in the above formulas.

**7.6.1.** Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

*Solution.* The section of the ellipsoid by the plane $x = \text{const}$ is an ellipse (Fig. 87)

$$\frac{y^2}{b^2\left(1-\frac{x^2}{a^2}\right)}+\frac{z^2}{c^2\left(1-\frac{x^2}{a^2}\right)}=1$$

with semi-axes $b\sqrt{1-\frac{x^2}{a^2}}$; $c\sqrt{1-\frac{x^2}{a^2}}$. Hence the area of the section (see Problem **7.4.1**)

$$S(x)=\pi b\sqrt{1-\frac{x^2}{a^2}}\times c\sqrt{1-\frac{x^2}{a^2}}=\pi bc\left(1-\frac{x^2}{a^2}\right)\quad(-a\leqslant x\leqslant a).$$

Therefore the volume $V$ of the ellipsoid is

$$V=\int_{-a}^{a}\pi bc\left(1-\frac{x^2}{a^2}\right)dx=\pi bc\left[x-\frac{x^3}{3a^2}\right]_{-a}^{a}=\frac{4}{3}\pi abc.$$

In the particular case $a=b=c$ the ellipsoid turns into a sphere, and we have $V_{\text{sphere}}=\frac{4}{3}\pi a^3$.

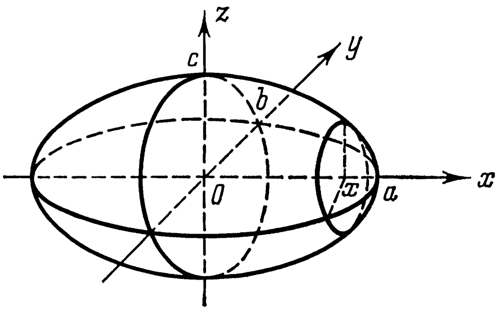


Fig. 87

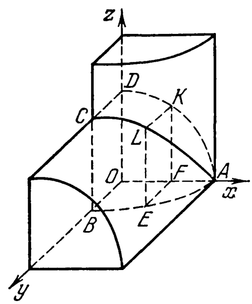


Fig. 88

**7.6.2.** Compute the volume of the solid spherical segment of two bases cut out by the planes $x=2$ and $x=3$ from the sphere $x^2+y^2+z^2=16$.

**7.6.3.** The axes of two identical cylinders with bases of radius $a$ intersect at right angles. Find the volume of the solid constituting the common portion of the two cylinders.

*Solution.* Take the axes of the cylinders to be the $y$- and $z$-axis (Fig. 88). The solid $OABCD$ constitutes one-eighth of the sought-for solid.

Let us cut this solid by a plane perpendicular to the $x$-axis at a distance $x$ from 0. In the section we get a square $EFKL$ with side $EF=\sqrt{a^2-x^2}$, therefore $S(x)=a^2-x^2$ and $V=8\int\limits_0^a (a^2-x^2)\,dx=$ $=\frac{16}{3}a^3$.

**7.6.4.** On all chords (parallel to one and the same direction) of a circle of radius $R$ symmetrical parabolic segments of the same altitude $h$ are constructed. The planes of the segments are perpendicular to the plane of the circle.

Find the volume of the solid thus obtained (Fig. 89).

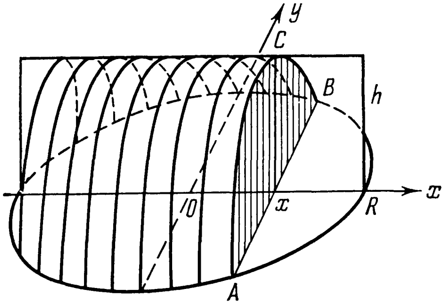

Fig. 89

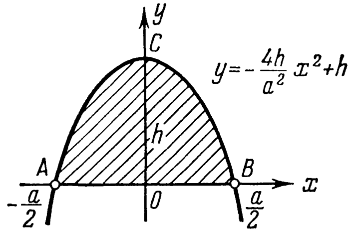

Fig. 90

*Solution.* First compute the area of the parabolic segment with base $a$ and altitude $h$. If we arrange the axes of coordinates as indicated in Fig. 90, then the equation of the parabola will be $y=\alpha x^2+h$.

Determine the parameter $\alpha$. Substituting the coordinates of the point $B\left(\frac{a}{2},\ 0\right)$, we get $0=\alpha\frac{a^2}{4}+h$, whence $\alpha=-\frac{4h}{a^2}$; hence the equation of the parabola is $y=-\frac{4h}{a^2}x^2+h$, and the desired area

$$S=2\int\limits_0^{\frac{a}{2}} y\,dx=2\int\limits_0^{\frac{a}{2}}\left(-\frac{4h}{a^2}x^2+h\right)dx=\frac{2}{3}ah.$$

Now find the volume of the solid. If the axes of coordinates are arranged as indicated in Fig. 89, then in the section of the solid by a plane perpendicular to the $x$-axis at the point with abscissa $x$ we obtain a parabolic segment of area $S=\frac{2}{3}ah$, where $a=2y=$ $=2\sqrt{R^2-x^2}$. Hence,

$$S(x)=\frac{4}{3}\sqrt{R^2-x^2}\,h \text{ and } V=\int\limits_{-R}^{R} S(x)\,dx=\frac{8}{3}h\int\limits_0^R \sqrt{R^2-x^2}\,dx=\frac{2}{3}\pi hR^2.$$

**7.6.5.** The plane of a moving triangle remains perpendicular to the fixed diameter of a circle of radius $a$: the base of the triangle is a chord of the circle, and its vertex lies on a straight line parallel to the fixed diameter at a distance $h$ from the plane of the circle. Find the volume of the solid generated by the movement of this triangle from one end of the diameter to the other.

**7.6.6.** Compute the volume of the solid generated by revolving about the $x$-axis the area bounded by the axes of coordinates and the parabola $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$.

*Solution.* Let us find the points of intersection of the curve and the axes of coordinates: at $x=0$ $y=a$, at $y=0$ $x=a$. Thus, we have the interval of integration $[0, a]$.

From the equation of the parabola we get $y=\left(a^{\frac{1}{2}}-x^{\frac{1}{2}}\right)^2$; therefore

$$V=\pi\int_0^a y^2\,dx=\pi\int_0^a\left(a^{\frac{1}{2}}-x^{\frac{1}{2}}\right)^4 dx=\pi\int_0^a\left(a^2-4a^{\frac{3}{2}}x^{\frac{1}{2}}+6ax-\right.$$

$$\left.-4a^{\frac{1}{2}}x^{\frac{3}{2}}+x^2\right)dx=\frac{1}{15}\pi a^3.$$

**7.6.7.** The figure bounded by an arc of the sinusoid $y=\sin x$, the axis of ordinates and the straight line $y=1$ revolves about the $y$-axis (Fig. 91).

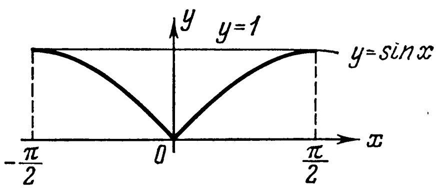


Fig. 91

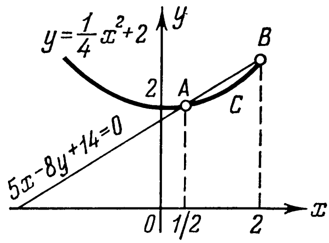


Fig. 92

Compute the volume $V$ of the solid of revolution thus generated.

*Solution.* The inverse function $x=\arcsin y$ is considered on the interval $[0, 1]$. Therefore

$$V=\pi\int_{y_1}^{y_2} x^2\,dy=\pi\int_0^1(\arcsin y)^2\,dy.$$

Apply the substitution $\arcsin y = t$. Hence

$$y = \sin t, \quad dy = \cos t\, dt, \quad \begin{array}{|c|c|} y & t \\ \hline 0 & 0 \\ 1 & \pi/2 \end{array}.$$

And so, $V = \pi \int_0^{\frac{\pi}{2}} t^2 \cos t\, dt$. Integrating by parts, we get $V = \frac{\pi(\pi^2 - 8)}{4}$.

**7.6.8.** Compute the volume of the solid generated by revolving about the $x$-axis the figure bounded by the parabola $y = 0.25x^2 + 2$ and the straight line $5x - 8y + 14 = 0$.

*Solution.* The solid is obtained by revolving the area $ABCA$ (Fig. 92) about the $x$-axis. To find the abscissas of the points $A$ and $B$ solve the system of equations:

$$\begin{cases} y = \frac{1}{4}x^2 + 2, \\ 5x - 8y + 14 = 0. \end{cases}$$

Whence $x_A = \frac{1}{2}$; $x_B = 2$. In our case $y_1(x) = \frac{1}{4}x^2 + 2$ and $y_2(x) = (5/8)x + 7/4$. Hence,

$$V = \pi \int_{1/2}^{2} \left[\frac{1}{16}\left(\frac{5}{2}x + 7\right)^2 - \left(\frac{1}{4}x^2 + 2\right)^2\right] dx = \frac{891}{1\,280}\pi.$$

**7.6.9.** Compute the volume of the solid generated by revolving about the $y$-axis the figure bounded by the parabolas $y = x^2$ and $8x = y^2$.

*Solution.* It is obvious that $x_2(y) = \sqrt{y} \geqslant x_1(y) = \frac{y^2}{8}$ on the interval from the origin of the coordinates to the point of intersection of the parabolas (Fig. 93). Let us find the ordinates of the points of intersection of the parabolas by excluding $x$ from the system of equations:

$$\begin{cases} y = x^2, \\ y^2 = 8x. \end{cases}$$

We obtain $y_1 = 0$, $y_2 = 4$. Hence, $V = \pi \int_0^4 \left(y - \frac{y^4}{64}\right) dy = \frac{24}{5}\pi$.

**7.6.10.** Compute the volume of the solid torus. The torus is a solid generated by revolving a circle of radius $a$ about an axis lying in its plane at a distance $b$ from the centre ($b \geqslant a$). (A tire, for example, has the form of the torus.)

**7.6.11.** Compute the volume of the solid obtained by revolving about the $x$-axis the figure bounded by two branches of the curve $(y-x)^2=x^3$ and the straight line $x=1$.

**7.6.12.** Find the volume of the solid generated by revolving about the line $y=-2a$ the figure bounded by the parabola $y^2=4ax$ and the straight line $x=a$ (Fig. 94).

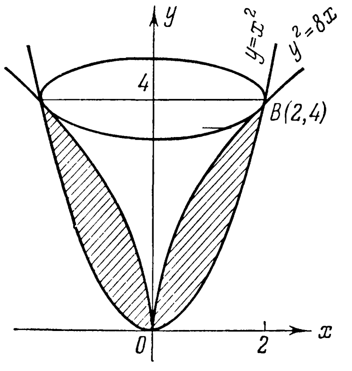


Fig. 93

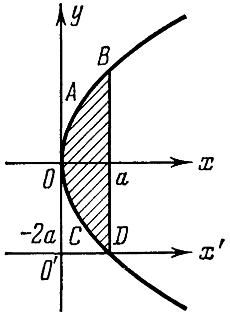


Fig. 94

*Solution.* If we transfer the origin of coordinates into the point $O'(0, -2a)$ retaining the direction of the axes, then in the new system of coordinates the equation of the parabola will be

$$(y'-2a)^2=4ax.$$

Hence $y_2=2a+\sqrt{4ax}$ (for the curve $OAB$), and $y_1=2a-\sqrt{4ax}$ (for the curve $OCD$). The sought-for volume is equal to

$$V=\pi\int_0^a (y_2^2-y_1^2)\,dx=\pi\int_0^a [(2a+2\sqrt{ax})^2-(2a-2\sqrt{ax})^2]\,dx=\frac{32}{3}\pi a^3.$$

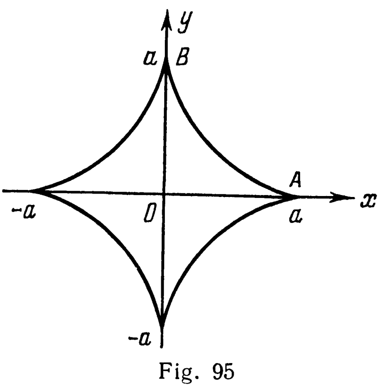


Fig. 95

**7.6.13.** Find the volume of the solid generated by revolving about the $x$-axis the figure enclosed by the astroid: $x=a\cos^3 t$; $y=a\sin^3 t$.

*Solution.* The sought-for volume $V$ is equal to double the volume obtained by revolving the figure $OAB$ (Fig. 95). Therefore,

$$V=2\pi\int_0^a y^2\,dx.$$

Change the variable

$$x = a\cos^3 t,\quad dx = -3a\cos^2 t\sin t\,dt,\quad y = a\sin^3 t,$$

| $x$ | $t$ |
|---|---|
| 0 | $\pi/2$ |
| $a$ | 0 |

Hence,

$$V = 2\pi\int_{\frac{\pi}{2}}^{0} a^2\sin^6 t\,(-3a\cos^2 t\sin t)\,dt =$$

$$= 6\pi a^3\left[\int_0^{\frac{\pi}{2}}\sin^7 t\,dt - \int_0^{\frac{\pi}{2}}\sin^9 t\,dt\right].$$

Using the formula from Problem **6.6.9** for computing the above integrals, we get

$$V = 6\pi a^3\left(\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}-\frac{8}{9}\times\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\right)=\frac{32}{105}\pi a^3.$$

**7.6.14.** Compute the volume of the solid generated by revolving one arc of the cycloid $x = a(t-\sin t)$, $y = a(1-\cos t)$ about the $x$-axis

**7.6.15.** Compute the volume of the solid obtained by revolving about the polar axis the cardioid $\rho = a(1+\cos\varphi)$ shown in Fig. 81.

*Solution.* The sought-for volume represents the difference between the volumes generated by revolving the figures $MNKLO$ and $OKLO$ about the $x$-axis (which is the polar axis at the same time).

As in the preceding problem, let us pass over to the parametric representation of the curve with the polar angle $\varphi$ as the parameter:

$$x = \rho\cos\varphi = a\cos\varphi\,(1+\cos\varphi),$$
$$y = \rho\sin\varphi = a\sin\varphi\,(1+\cos\varphi).$$

It is obvious that the abscissa of the point $M$ equals $2a$ (the value of $x$ at $\varphi = 0$), the abscissa of the point $K$ being the minimum of the function $x = a(1+\cos\varphi)\cos\varphi$.

Let us find this minimum:

$$\frac{dx}{d\varphi} = -a\sin\varphi\,(1+2\cos\varphi) = 0,$$

$$\varphi_1 = 0;\quad \varphi_2 = \frac{2}{3}\pi.$$

At $\varphi_1 = 0$ we obtain $x_M = 2a$, at $\varphi_2 = \frac{2}{3}\pi$, $x_K = -\frac{a}{4}$.

Hence, the sought-for volume is equal to

$$V=\pi\int_{-\frac{a}{4}}^{2a} y_2^2\,dx-\pi\int_{-\frac{a}{4}}^{0} y_1^2\,dx.$$

Changing the variable $x=a\cos\varphi(1+\cos\varphi)$, we get

$y^2=a^2(1+\cos\varphi)^2\sin^2\varphi$,
$dx=-a\sin\varphi(1+2\cos\varphi)\,d\varphi$,

| $x$ | $\varphi$ |
|---|---|
| $-a/4$ | $2\pi/3$ |
| $2a$ | $0$ |

,

| $x$ | $\varphi$ |
|---|---|
| $-a/4$ | $2\pi/3$ |
| $0$ | $\pi$ |

.

Thus,

$$V=\pi\int_{\frac{2}{3}\pi}^{0} a^2(1+\cos\varphi)^2\sin^2\varphi\,[-a\sin\varphi(1+2\cos\varphi)]\,d\varphi-$$

$$-\pi\int_{\frac{2}{3}\pi}^{\pi} a^2(1+\cos\varphi)^2\sin^2\varphi\,[-a\sin\varphi(1+2\cos\varphi)]\,d\varphi=$$

$$=\pi a^3\int_0^{\pi}\sin^3\varphi(1+\cos\varphi)^2(1+2\cos\varphi)\,d\varphi=$$

$$=\pi a^3\int_{-1}^{1}(1-u^2)(1+u)^2(1+2u)\,du=\frac{8}{3}\pi a^3\quad(u=\cos\varphi).$$

**7.6.16.** Compute the volume of the solid bounded by:

(a) the hyperboloid of one sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ and the planes $z=-1$ and $z=1$;

(b) the parabolic cylinder $z=4-y^2$, the planes of coordinates and the plane $x=a$;

(c) the elliptic paraboloid $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ and the plane $z=k$ $(k>0)$.

**7.6.17.** A wedge is cut off from a right circular cylinder of radius $a$ by a plane passing through the diameter of the cylinder base and inclined at an angle $\alpha$ to the base. Find the volume of the wedge.

**7.6.18.** Compute the volume of the solid generated by revolving the figure bounded by the following lines:

(a) $xy=4$, $x=1$, $x=4$, $y=0$ about the $x$-axis;
(b) $y=2x-x^2$, $y=0$ about the $x$-axis;

(c) $y=x^3$, $y=0$, $x=2$ about the $y$-axis;
(d) $y=\sin x$ (one wave), $y=0$ about the $x$-axis;
(e) $x^2-y^2=4$, $y=\pm 2$ about the $y$-axis;
(f) $(y-a)^2=ax$, $x=0$, $y=2a$ about the $x$-axis.

**7.6.19.** Find the volume of the solid obtained by revolving the curve $y^2=\frac{ax^3-x^4}{a^2}$ about the $x$-axis.

**7.6.20.** Compute the volume of the solid generated by revolving about the $x$-axis the figure bounded by the lines $y=\sin x$ and $y=\frac{2}{\pi}x$.

**7.6.21.** Compute the volume of the solid generated by revolving about the $x$-axis the curvilinear trapezoid bounded by the catenary $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)=a\cosh\frac{x}{a}$ and the straight lines $x_1=-c$, $x_2=c$ $(c>0)$.

**7.6.22.** Compute the volume of the solid generated by revolving about the $x$-axis the figure bounded by the cosine line $y=\cos x$ and the parabola $y=\frac{9}{2\pi^2}x^2$.

**7.6.23.** Compute the volume of the solid generated by revolving about the $x$-axis the figure bounded by the circle $x^2+y^2=1$ and the parabola $y^2=\frac{3}{2}x$.

**7.6.24.** On the curve $y=x^3$ take two points $A$ and $B$, whose abscissas are $a=1$ and $b=2$, respectively.

Find the volume of the solid generated by revolving the curvilinear trapezoid $aABb$ about the $x$-axis.

**7.6.25.** An arc of the evolute of the ellipse $x=a\cos t$; $y=b\sin t$ situated in the first quadrant revolves about the $x$-axis

Find the volume of the solid thus generated.

**7.6.26.** Compute the volume of the solid generated by revolving the region enclosed by the loop of the curve $x=at^2$, $y=a\left(t-\frac{t^3}{3}\right)$ about the $x$-axis.

**7.6.27.** Compute the volumes of the solids generated by revolving the region enclosed by the lemniscate $(x^2+y^2)^2=a^2(x^2-y^2)$ about the $x$- and $y$-axes.

**7.6.28.** Compute the volume of the solid generated by revolving the region enclosed by the curve $\rho=a\cos^2\varphi$ about the polar axis.

## § 7.7. The Arc Length of a Plane Curve in Rectangular Coordinates

If a plane curve is given by the equation $y=y(x)$ and the derivative $y'(x)$ is continuous, then the length of an arc of this curve is expressed by the integral

$$l=\int_a^b \sqrt{1+y'^2}\,dx$$

where $a$ and $b$ are the abscissas of the end-points of the given arc.

**7.7.1.** Compute the length of the arc of the semicubical parabola $y^2=x^3$ between the points (0, 0) and (4, 8) (Fig. 96).

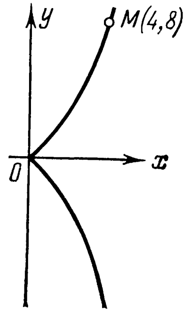

Fig. 96

*Solution.* The function $y(x)$ is defined for $x \geqslant 0$. Since the given points lie in the first quadrant, $y=x^{\frac{3}{2}}$. Hence,

$$y'=\frac{3}{2}\sqrt{x} \text{ and } \sqrt{1+y'^2}=\sqrt{1+\frac{9}{4}x}.$$

Consequently,

$$l=\int_0^4 \sqrt{1+\frac{9}{4}x}\,dx=\frac{4}{9}\cdot\frac{2}{3}\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}\Bigg|_0^4=\frac{8}{27}(10\sqrt{10}-1).$$

**7.7.2.** Compute the length of the arc cut off from the curve $y^2=x^3$ by the straight line $x=\frac{4}{3}$.

**7.7.3.** Compute the arc length of the curve $y=\ln\cos x$ between the points with the abscissas $x=0$, $x=\frac{\pi}{4}$.

*Solution.* Since $y'=-\tan x$, then $\sqrt{1+y'^2}=\sqrt{1+\tan^2 x}=\sec x$. Hence,

$$l=\int_0^{\frac{\pi}{4}} \sec x\,dx=\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\Bigg|_0^{\frac{\pi}{4}}=\ln\tan\frac{3\pi}{8}.$$

**7.7.4.** Compute the arc length of the curve $y=\ln\frac{e^x+1}{e^x-1}$ from $x_1=a$ to $x_2=b$ $(b>a)$.

**7.7.5.** Find the arc length of the curve $x=\frac{1}{4}y^2-\frac{1}{2}\ln y$ between the points with the ordinates $y=1$ and $y=2$.

*Solution.* Here it is convenient to adopt $y$ as the independent variable; then

$$x'=\frac{1}{2}y-\frac{1}{2y} \text{ and } \sqrt{1+x'^2}=\sqrt{\left(\frac{1}{2}y+\frac{1}{2y}\right)^2}=\frac{1}{2}y+\frac{1}{2y}.$$

Hence,

$$l=\int_1^2\sqrt{1+x'^2}\,dy=\int_1^2\left(\frac{1}{2}y+\frac{1}{2y}\right)dy=\frac{3}{4}+\frac{1}{2}\ln 2.$$

**7.7.6.** Find the length of the astroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

*Solution.* As is known, the astroid is symmetrical about the axes of coordinates and the bisectors of the coordinate angles. Therefore, it is sufficient to compute the arc length of the astroid between the bisector $y=x$ and the $x$-axis and multiply the result by 8.

In the first quadrant $y=\left(a^{\frac{2}{3}}-x^{\frac{2}{3}}\right)^{\frac{3}{2}}$ and $y=0$ at $x=a$, $y=x$ at $x=\frac{a}{2^{3/2}}$.

Further,

$$y'=\frac{3}{2}\left(a^{\frac{2}{3}}-x^{\frac{2}{3}}\right)^{\frac{1}{2}}\left(-\frac{2}{3}\right)x^{-\frac{1}{3}}=-x^{-\frac{1}{3}}\left(a^{\frac{2}{3}}-x^{\frac{2}{3}}\right)^{\frac{1}{2}}$$

and

$$\sqrt{1+y'^2}=\sqrt{1+x^{-\frac{2}{3}}\left(a^{\frac{2}{3}}-x^{\frac{2}{3}}\right)}=\left(\frac{a}{x}\right)^{\frac{1}{3}}.$$

Consequently,

$$l=8\int_{\frac{a}{2^{3/2}}}^{a}a^{\frac{1}{3}}x^{-\frac{1}{3}}\,dx=6a.$$

*Note.* If we compute the arc length of an astroid situated in the first quadrant, we get the integral

$$\int_0^a a^{\frac{1}{3}}x^{-\frac{1}{3}}\,dx,$$

whose integrand increases infinitely as $x\to 0$.

**7.7.7.** Compute the length of the path $OABCO$ consisting of portions of the curves $y^2=2x^3$ and $x^2+y^2=20$ (Fig. 97).

*Solution.* It is sufficient to compute the arc lengths $l_{\breve{OA}}$ and $l_{\breve{AB}}$ since by symmetry of the figure about the $x$-axis

$$l=2\,(l_{\breve{OA}}+l_{\breve{AB}}).$$

Solving the system of equations

$$\begin{cases} x^2+y^2=20, \\ y^2=2x^3, \end{cases}$$

we find the point $A\,(2,\ 4)$.

Find $l_{\breve{OA}}$. Here

$$y=\sqrt{2}\,x^{\frac{3}{2}},\quad y'=\frac{3}{2}\sqrt{2x},\quad \sqrt{1+y'^2}=\sqrt{1+\frac{9}{2}x}.$$

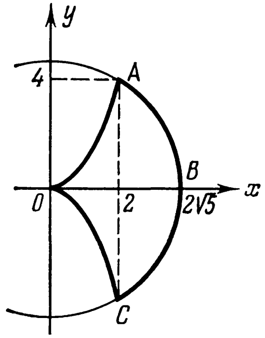


Fig. 97

Hence,

$$l_{\breve{OA}}=\int_0^2\sqrt{1+\frac{9}{2}x}\,dx=\frac{4}{27}(10\sqrt{10}-1).$$

Since on the circle of radius $\sqrt{20}$ $l_{\breve{AB}}$ is the length of an arc corresponding to the central angle arc tan 2,

$$l_{\breve{AB}}=\sqrt{20}\operatorname{arc}\tan 2.$$

Finally we have

$$l=\frac{8}{27}(10\sqrt{10}-1)+4\sqrt{5}\operatorname{arc}\tan 2.$$

**7.7.8.** Compute the arc length of the curve:

(a) $y=\frac{x^2}{2}-1$ cut off by the $x$-axis;

(b) $y=\ln(2\cos x)$ between the adjacent points of intersection with the $x$-axis.

(c) $3y^2=x(x-1)^2$ between the adjacent points of intersection with the $x$-axis (half the loop length).

**7.7.9.** Compute the arc length of the curve

$$y=\frac{1}{2}\left[x\sqrt{x^2-1}-\ln\left(x+\sqrt{x^2-1}\right)\right]$$

between

$$x=1 \text{ and } x=a+1.$$

**7.7.10.** Find the arc length of the path consisting of portions of the curves $x^2=(y+1)^3$ and $y=4$.

## § 7.8. The Arc Length of a Curve Represented Parametrically

If a curve is given by the equations in the parametric form $x=x(t)$, $y=y(t)$ and the derivatives $x'(t)$, $y'(t)$ are continuous on the interval $[t_1, t_2]$, then the arc length of the curve is expressed by the integral

$$l=\int_{t_1}^{t_2}\sqrt{x'^2(t)+y'^2(t)}\,dt,$$

where $t_1$ and $t_2$ are the values of the parameter $t$ corresponding to the end-points of the arc $(t_1<t_2)$.

**7.8.1.** Compute the arc length of the involute of a circle $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$ from $t=0$ to $t=2\pi$.

*Solution.* Differentiating with respect to $t$, we obtain

$$x'_t=at\cos t,\quad y'_t=at\sin t,$$

whence $\sqrt{x_t'^2+y_t'^2}=at$. Hence,

$$l=\int_0^{2\pi}at\,dt=\frac{at^2}{2}\Big|_0^{2\pi}=2a\pi^2.$$

**7.8.2.** Find the length of one arc of the cycloid:

$$x=a(t-\sin t),\quad y=a(1-\cos t).$$

**7.8.3.** Compute the length of the astroid: $x=a\cos^3 t$, $y=a\sin^3 t$.

*Solution.* Differentiating with respect to $t$, we obtain

$$x'_t=-3a\cos^2 t\sin t;$$
$$y'_t=3a\sin^2 t\cos t.$$

Hence

$$\sqrt{x_t'^2+y_t'^2}=\sqrt{9a^2\sin^2 t\cos^2 t}=3a\,|\sin t\cos t\,|=\frac{3a}{2}|\sin 2t\,|.$$

Since the function $|\sin 2t|$ has a period $\frac{\pi}{2}$,

$$l=4\times\frac{3a}{2}\int_0^{\frac{\pi}{2}}\sin 2t\,dt=6a.$$

*Note.* If we forget that we have to take the arithmetic value of the root and put $\sqrt{x_t'^2+y_t'^2}=3a\sin t\cos t$, we shall obtain the wrong result, since

$$3a\int_0^{2\pi}\sin t\cos t\,dt=\frac{3a}{2}\sin^2 t\Big|_0^{2\pi}=0.$$

**7.8.4.** Compute the length of the loop of the curve $x=\sqrt{3}t^2$, $y=t-t^3$.

*Solution.* Let us find the limits of integration. Both functions $x(t)$ and $y(t)$ are defined for all values of $t$. Since the function $x=\sqrt{3}t^2\geqslant 0$, the curve lies in the right half-plane. Since with a change in sign of the parameter $t$, $x(t)$ remains unchanged, while $y(t)$ changes sign, the curve is symmetrical about the $x$-axis. Furthermore, the function $x(t)$ takes on one and the same value not more than twice. Hence, it follows that the points of self-intersection of the curve lie on the $x$-axis, i.e., at $y=0$ (Fig. 98).

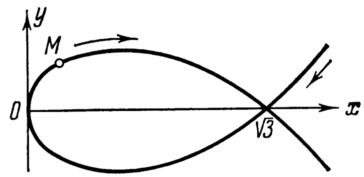

Fig. 98

The direction in which the moving point $M(x, y)$ runs along the curve as $t$ changes from $-\infty$ to $\infty$ is indicated by the arrows.

But $y=0$ at $t_1=0$, $t_{2,3}=\pm 1$. Since $x(t_2)=x(t_3)=\sqrt{3}$, the point $(\sqrt{3}, 0)$ is the only point of self-intersection of the curve. Consequently, we must integrate within the limits $t_2=-1$ and $t_3=1$.

Differentiating the parametric equations of the curve with respect to $t$, we get $x'_t=2\sqrt{3}t$, $y'_t=1-3t^2$, whence

$$\sqrt{x_t'^2+y_t'^2}=1+3t^2.$$

Consequently,

$$l=\int_{-1}^{1}(1+3t^2)\,dt=4.$$

**7.8.5.** Compute the arc length of the curve $x=\frac{t^6}{6}$, $y=2-\frac{t^4}{4}$ between the points of intersection with the axes of coordinates.

**7.8.6.** Compute the arc length of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

*Solution.* Let us pass over to the parametric representation of the ellipse

$$x=a\cos t,\quad y=b\sin t,\quad 0\leqslant t\leqslant 2\pi.$$

Differentiating with respect to $t$, we obtain

$$x'_t=-a\sin t;\quad y'_t=b\cos t,$$

whence

$$\sqrt{x_t'^2+y_t'^2}=\sqrt{a^2\sin^2 t+b^2\cos^2 t}=a\sqrt{1-\varepsilon^2\cos^2 t}$$

where $\varepsilon$ is the eccentricity of the ellipse,

$$\varepsilon=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}.$$

Thus

$$l=a\int_0^{2\pi}\sqrt{1-\varepsilon^2\cos^2 t}\,dt=4a\int_0^{\frac{\pi}{2}}\sqrt{1-\varepsilon^2\cos^2 t}\,dt.$$

The integral $\int_0^t\sqrt{1-\varepsilon^2\cos^2 t}\,dt$ is not taken in elementary functions; it is called the *elliptic integral of the second kind.* Putting $t=\frac{\pi}{2}-\tau$, we reduce the integral to the standard form:

$$\int_0^{\frac{\pi}{2}}\sqrt{1-\varepsilon^2\cos^2 t}\,dt=\int_0^{\frac{\pi}{2}}\sqrt{1-\varepsilon^2\sin^2 \tau}\,d\tau=E(\varepsilon),$$

where $E(\varepsilon)$ is the notation for the so-called *complete elliptic integral of the second kind.*

Consequently, for the arc length of an ellipse the formula $l=4aE(\varepsilon)$ holds good.

It is usual practice to put $\varepsilon=\sin\alpha$ and to use the tables of values for the function

$$E_1(\alpha)=E_1(\arcsin\varepsilon)=E(\varepsilon).$$

For instance, if $a=10$ and $b=6$, then

$$\varepsilon=\frac{\sqrt{10^2-6^2}}{10}=0.8=\sin 53^\circ.$$

Using the table of values of elliptic integrals of the second kind, we find $l=40E_1(53^\circ)=40\times 1.2776\approx 51.1$.

**7.8.7.** Compute the arc length of the curve

$$x=t^2,\quad y=\frac{t}{3}(t^2-3)$$

between the points of intersection with the $x$-axis.

**7.8.8.** Find the arc length of the cardioid:

$$x=a(2\cos t-\cos 2t),$$
$$y=a(2\sin t-\sin 2t).$$

**7.8.9.** Find the length of the closed curve

$$x = 4\sqrt{2}\,a\sin t; \quad y = a\sin 2t.$$

**7.8.10.** Find the arc length of the evolute of the ellipse

$$x = \frac{c^2}{a}\cos^3 t, \quad y = -\frac{c^2}{b}\sin^3 t, \quad c^2 = a^2 - b^2.$$

**7.8.11.** Compute the arc length of the curve

$$x = (t^2 - 2)\sin t + 2t\cos t,$$
$$y = (2 - t^2)\cos t + 2t\sin t$$

between $t_1 = 0$ and $t_2 = \pi$.

**7.8.12.** On the cycloid $x = a(t - \sin t)$; $y = a(1 - \cos t)$ find the point which divides the length of the first arc of the cycloid in the ratio 1:3.

## § 7.9. The Arc Length of a Curve in Polar Coordinates

If a smooth curve is given by the equation $\rho = \rho(\varphi)$ in polar coordinates, then the arc length of the curve is expressed by the integral:

$$l = \int_{\varphi_1}^{\varphi_2} \sqrt{\rho^2 + \rho_\varphi'^2}\, d\varphi,$$

where $\varphi_1$ and $\varphi_2$ are the values of the polar angle $\varphi$ at the end-points of the arc ($\varphi_1 < \varphi_2$).

**7.9.1.** Find the length of the first turn of the spiral of Archimedes $\rho = a\varphi$.

*Solution.* The first turn of the spiral is formed as the polar angle $\varphi$ changes from 0 to $2\pi$. Therefore

$$l = \int_0^{2\pi} \sqrt{a^2\varphi^2 + a^2}\, d\varphi = a\int_0^{2\pi} \sqrt{\varphi^2 + 1}\, d\varphi =$$
$$= a\left[\pi\sqrt{4\pi^2 + 1} + \frac{1}{2}\ln\left(2\pi + \sqrt{4\pi^2 + 1}\right)\right].$$

**7.9.2.** Find the length of the logarithmic spiral $\rho = ae^{m\varphi}$ between a certain point $(\rho_0, \varphi_0)$ and a moving point $(\rho, \varphi)$.

*Solution*. In this case (no matter which of the magnitudes, $\rho$ or $\rho_0$, is greater!)

$$l=\left|\int_{\varphi_0}^{\varphi}\sqrt{a^2e^{2m\varphi}+a^2m^2e^{2m\varphi}}\,d\varphi\right|=$$

$$=a\sqrt{1+m^2}\left|\int_{\varphi_0}^{\varphi}e^{m\varphi}\,d\varphi\right|=a\,\frac{\sqrt{1+m^2}}{m}\,|e^{m\varphi}-e^{m\varphi_0}|=$$

$$=\frac{\sqrt{1+m^2}}{m}\,|\rho-\rho_0|=\frac{\sqrt{1+m^2}}{m}\,|\Delta\rho|,$$

i.e. the length of the logarithmic spiral is proportional to the increment of the polar radius of the arc.

**7.9.3.** Find the arc length of the cardioid $\rho=a(1+\cos\varphi)$ $(a>0,\ 0\leqslant\varphi\leqslant 2\pi)$.

*Solution*. Here $\rho'_\varphi=-a\sin\varphi$,

$$\sqrt{\rho'^2_\varphi+\rho^2}=\sqrt{2a^2(1+\cos\varphi)}=\sqrt{4a^2\cos^2(\varphi/2)}=$$

$$=2a\,|\cos(\varphi/2)|=\begin{cases}2a\cos(\varphi/2), & 0\leqslant\varphi\leqslant\pi\\ -2a\cos(\varphi/2), & \pi\leqslant\varphi\leqslant 2\pi.\end{cases}$$

Hence, by virtue of symmetry

$$l=2a\int_0^{2\pi}\left|\cos\frac{\varphi}{2}\right|d\varphi=4a\int_0^{\pi}\cos\frac{\varphi}{2}\,d\varphi=8a.$$

**7.9.4.** Find the length of the lemniscate $\rho^2=2a^2\cos 2\varphi$ between the right-hand vertex corresponding to $\varphi=0$ and any point with a polar angle $\varphi<\frac{\pi}{4}$.

*Solution*. If $0\leqslant\varphi<\frac{\pi}{4}$, then $\cos 2\varphi>0$. Therefore

$$\rho=a\sqrt{2\cos 2\varphi};\quad \rho'_\varphi=-\frac{a\sqrt{2}\sin 2\varphi}{\sqrt{\cos 2\varphi}};$$

$$\sqrt{\rho^2+\rho'^2_\varphi}=\sqrt{2a^2\left(\cos 2\varphi+\frac{\sin^2 2\varphi}{\cos 2\varphi}\right)}=\frac{a\sqrt{2}}{\sqrt{\cos 2\varphi}}.$$

Hence,

$$l=a\sqrt{2}\int_0^{\varphi}\frac{d\varphi}{\sqrt{\cos 2\varphi}}=a\sqrt{2}\int_0^{\varphi}\frac{d\varphi}{\sqrt{1-2\sin^2\varphi}}.$$

The latter integral is called the *elliptic integral of the first kind*. It can be reduced to a form convenient for computing with the aid of special tables.

**7.9.5.** Find the arc length of the curve $\rho = a\sin^3\frac{\varphi}{3}$.

**7.9.6.** Compute the length of the segment of the straight line $\rho = a\sec\left(\varphi-\frac{\pi}{3}\right)$ between $\varphi=0$ and $\varphi=\frac{\pi}{2}$.

*Solution.* $\rho'_\varphi = a\sec\left(\varphi-\frac{\pi}{3}\right)\tan\left(\varphi-\frac{\pi}{3}\right)$;

$$\sqrt{\rho^2+\rho'^2_\varphi} = a\sec\left(\varphi-\frac{\pi}{3}\right)\sqrt{1+\tan^2\left(\varphi-\frac{\pi}{3}\right)} = a\sec^2\left(\varphi-\frac{\pi}{3}\right).$$

(The sign of the modulus in the function $\sec\left(\varphi-\frac{\pi}{3}\right)$ is omitted, since on the interval $\left[0, \frac{\pi}{2}\right]$ this function is positive.)

$$l = a\int_0^{\frac{\pi}{2}} \sec^2\left(\varphi-\frac{\pi}{3}\right)d\varphi = \frac{4\sqrt{3}}{3}a.$$

**7.9.7.** Find the length of the closed curve $\rho = a\sin^4\frac{\varphi}{4}$.

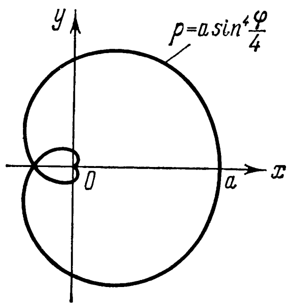


Fig. 99

*Solution.* Since the function $\rho = a\sin^4\frac{\varphi}{4}$ is even, the given curve is symmetrical about the polar axis. Since the function $\sin^4\frac{\varphi}{4}$ has a period $4\pi$, during half the period from 0 to $2\pi$ the polar radius increases from 0 to $a$, and will describe half the curve by virtue of its symmetry (Fig. 99).

Further, $\rho'_\varphi = a\sin^3(\varphi/4)\cos(\varphi/4)$ and

$$\sqrt{\rho^2+\rho'^2_\varphi} = \sqrt{a^2\sin^8(\varphi/4)+a^2\sin^6(\varphi/4)\cos^2(\varphi/4)} = a\sin^3(\varphi/4),$$

if $0\leqslant\varphi\leqslant 2\pi$.

Hence,

$$l = 2a\int_0^{2\pi}\sin^3(\varphi/4)\,d\varphi = 8a\int_0^{\pi/2}\sin^3 t\,dt = \frac{16}{3}a \quad (\varphi=4t).$$

**7.9.8.** Find the length of the curve $\varphi=\frac{1}{2}(\rho+1/\rho)$ between $\rho=2$ and $\rho=4$.

*Solution.* The differential of the arc $dl$ is equal to

$$dl = \sqrt{\rho^2+\rho'^2_\varphi}\,d\varphi = \sqrt{\rho^2\,d\varphi^2+d\rho^2} = \sqrt{\rho^2\left(\frac{d\varphi}{d\rho}\right)^2+1}\,d\rho.$$

From the equation of the curve we find $\frac{d\varphi}{d\rho}=\frac{1}{2}\left(1-\frac{1}{\rho^2}\right)$. Hence,

$$l=\int_2^4 \sqrt{\rho^2\cdot\frac{1}{4}\left(1-\frac{1}{\rho^2}\right)^2+1}\,d\rho=\int_2^4 \sqrt{\frac{1}{4}\left(\rho^2-2+\frac{1}{\rho^2}+4\right)}\,d\rho=$$

$$=\frac{1}{2}\int_2^4 \sqrt{\left(\rho+\frac{1}{\rho}\right)^2}\,d\rho=\frac{1}{2}\left(\frac{\rho^2}{2}+\ln\rho\right)\Big|_2^4=3+\frac{\ln 2}{2}.$$

**7.9.9.** Find the length of the hyperbolic spiral $\rho\varphi=1$ between $\varphi_1=\frac{3}{4}$ and $\varphi_2=\frac{4}{3}$.

**7.9.10.** Compute the length of the closed curve $\rho=2a(\sin\varphi+\cos\varphi)$.

**7.9.11.** Compute the arc length of the curve $\rho=\frac{p}{1+\cos\varphi}$ from $\varphi_1=-\frac{\pi}{2}$ to $\varphi_2=\frac{\pi}{2}$.

## *§ 7.10. Area of Surface of Revolution*

The area of the surface generated by revolving about the $x$-axis the arc $L$ of the curve $y=y(x)$ $(a\leqslant x\leqslant b)$ is expressed by the integral

$$P=2\pi\int_a^b y\sqrt{1+y'^2}\,dx.$$

It is more convenient to write this integral in the form $P=2\pi\int_L y\,dl$, where $dl$ is the differential of the arc length.

If a curve is represented parametrically or in polar coordinates, then it is sufficient to change the variable in the above formula, expressing appropriately the differential of the arc length (see §§ 7.8 and 7.9).

**7.10.1.** Find the area of the surface formed by revolving the astroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ about the $x$-axis.

*Solution.* Differentiating the equation of the astroid we get

$$\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}y'=0,$$

whence

$$y'=-\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}.$$

Then, $\sqrt{1+y'^2}=\sqrt{1+\frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}=\frac{a^{\frac{1}{3}}}{|x|^{\frac{1}{3}}}$. Since the astroid is symmetrical about the $y$-axis, in computing the area of the surface we may first assume $x \geqslant 0$, and then double the result. In other words, the desired area $P$ is equal to

$$P=2\times 2\pi\int_0^a y\sqrt{1+y'^2}\,dx=4\pi\int_0^a \left(a^{\frac{2}{3}}-x^{\frac{2}{3}}\right)^{\frac{3}{2}} a^{\frac{1}{3}} x^{-\frac{1}{3}}\,dx.$$

Make the substitution

$$a^{\frac{2}{3}}-x^{\frac{2}{3}}=t^2,$$

$$-\frac{2}{3}x^{-\frac{1}{3}}\,dx=2t\,dt,$$

| $x$ | $t$ |
|---|---|
| 0 | $a^{\frac{1}{3}}$ |
| $a$ | 0 |

Then $P=12\pi a^{\frac{1}{3}}\int_0^{a^{1/3}} t^4\,dt=\frac{12}{5}\pi a^2$.

**7.10.2.** Find the area of the surface generated by revolving about the $x$-axis a closed contour $OABCO$ formed by the curves $y=x^2$ and $x=y^2$ (Fig. 100).

*Solution.* It is easy to check that the given parabolas intersect at the points $O$ (0, 0) and $B$ (1, 1). The sought-for area $P=P_1+P_2$, where the area $P_1$ is formed by revolving the arc $OCB$, and $P_2$ by revolving the arc $OAB$.

Compute the area $P_1$. From the equation $x=y^2$ we get $y=\sqrt{x}$ and $y'=\frac{1}{2\sqrt{x}}$. Hence,

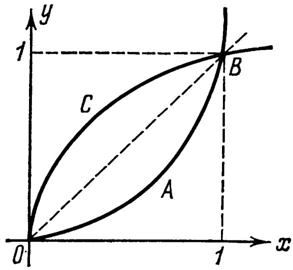

Fig. 100

$$P_1=2\pi\int_0^1 \sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx=2\pi\int_0^1 \frac{\sqrt{4x+1}}{2}\,dx=$$

$$=\frac{\pi}{6}(4x+1)^{\frac{3}{2}}\Big|_0^1=\frac{\pi}{6}(5\sqrt{5}-1).$$

Now compute the area $P_2$. We have $y=x^2$, $y'=2x$ and

$$P_2=2\pi\int_0^1 x^2\sqrt{1+4x^2}\,dx.$$

The substitution $x=\frac{1}{2}\sinh t$, $dx=\frac{1}{2}\cosh t\,dt$ gives

$$P_2=\frac{\pi}{4}\int\limits_0^{\operatorname{Arsinh} 2}\sinh^2 t\cosh^2 t\,dt=\frac{\pi}{32}\left(\frac{1}{4}\sinh 4t-t\right)\Big|_0^{\operatorname{Arsinh} 2}=$$

$$=\frac{9\sqrt{5}\pi}{16}-\frac{1}{32}\pi\ln(2+\sqrt{5}).$$

Thus,

$$P=P_1+P_2=\frac{(5\sqrt{5}-1)\pi}{6}+\frac{9\sqrt{5}\pi}{16}-\frac{1}{32}\pi\ln(2+\sqrt{5})=$$

$$=\frac{67\sqrt{5}\pi}{48}-\frac{\pi}{32}\ln(2+\sqrt{5})-\frac{\pi}{6}.$$

**7.10.3.** Compute the area of the surface generated by revolving:

(a) the portion of the curve $y=\frac{x^2}{2}$, cut off by the straight line $y=\frac{3}{2}$, about the $y$-axis;

(b) the portion of the curve $y^2=4+x$, cut off by the straight line $x=2$, about the $x$-axis.

**7.10.4.** Find the surface area of the ellipsoid formed by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the $x$-axis ($a>b$).

*Solution.* Solving the equation of the ellipse with respect to $y$ for $y\geqslant 0$, we get

$$y=\frac{b}{a}\sqrt{a^2-x^2};\qquad y'=-\frac{b}{a}\cdot\frac{x}{\sqrt{a^2-x^2}};$$

$$\sqrt{1+y'^2}=\sqrt{\frac{a^4-(a^2-b^2)x^2}{a^2(a^2-x^2)}}.$$

Hence

$$P=2\pi\int\limits_{-a}^{a}\frac{b}{a}\sqrt{a^2-x^2}\sqrt{\frac{a^4-(a^2-b^2)x^2}{a^2(a^2-x^2)}}\,dx=$$

$$=\frac{4\pi b}{a}\int\limits_0^a\sqrt{a^2-\varepsilon^2x^2}\,dx=2\pi ab\left(\sqrt{1-\varepsilon^2}+\frac{\arcsin\varepsilon}{\varepsilon}\right),$$

where the quantity $\varepsilon=\sqrt{\frac{a^2-b^2}{a^2}}=\frac{c}{a}$ is the eccentricity of the ellipse.

When $b\to a$ the eccentricity $\varepsilon$ tends to zero and

$$\lim_{\varepsilon\to 0}\frac{\arcsin\varepsilon}{\varepsilon}=1,$$

since the ellipse turns into a circle, in the limit we get the surface area of the sphere:

$$P=4\pi a^2.$$

**7.10.5.** Compute the area of the surface obtained by revolving the ellipse $4x^2+y^2=4$ about the $y$-axis.

**7.10.6.** An arc of the catenary

$$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)=a\cosh\frac{x}{a},$$

whose end-points have abscissas 0 and $x$, respectively, revolves about the $x$-axis.

Show that the surface area $P$ and the volume $V$ of the solid thus generated are related by the formula $P=\frac{2V}{a}$.

*Solution.* Since $y'=\sinh\frac{x}{a}$, we have $\sqrt{1+y'^2}=\cosh\frac{x}{a}$. Therefore

$$P=2\pi\int_0^x y\sqrt{1+y'^2}\,dx=2a\pi\int_0^x \cosh^2\frac{x}{a}\,dx=\frac{2}{a}\pi\int_0^x a^2\cosh^2\frac{x}{a}\,dx,$$

but

$$\pi\int_0^x a^2\cosh^2\frac{x}{a}\,dx=\pi\int_0^x y^2\,dx=V,$$

hence, $P=\frac{2V}{a}$.

**7.10.7.** Find the area of the surface obtained by revolving a loop of the curve $9ax^2=y(3a-y)^2$ about the $y$-axis.

*Solution.* The loop is described by a moving point as $y$ changes from 0 to $3a$. Differentiate with respect to $y$ both sides of the equation of the curve:

$$18axx'=(3a-y)^2-2y(3a-y)=3(3a-y)(a-y),$$

whence $xx'=\frac{(3a-y)(a-y)}{6a}$. Using the formula for computing the area of the surface of a solid of revolution about the $y$-axis, we have

$$P=2\pi\int_{y_1}^{y_2} x\sqrt{1+x'^2}\,dy=2\pi\int_{y_1}^{y_2}\sqrt{x^2+(xx')^2}\,dy=$$

$$=2\pi\int_0^{3a}\sqrt{\frac{y(3a-y)^2}{9a}+\frac{(3a-y)^2(a-y)^2}{36a^2}}\,dy=\frac{\pi}{3a}\int_0^{3a}(3a^2+2ay-$$

$$-y^2)\,dy=3\pi a^2.$$

**7.10.8.** Compute the area of the surface generated by revolving the curve $8y^2=x^2-x^4$ about the $x$-axis.

**7.10.9.** Compute the area of a surface generated by revolving about the $x$-axis an arc of the curve $x=t^2$; $y=\frac{t}{3}(t^2-3)$ between the points of intersection of the curve and the $x$-axis.

*Solution*. Putting $y=0$, find $t_1=0$ and $t_{2,3}=\pm\sqrt{3}$, and, hence, $x_1=0$ and $x_{2,3}=3$. Whence it follows that the curve intersects with the $x$-axis at two points: (0, 0) and (3, 0). When the parameter $t$ changes sign, the sign of the function $(x)t$ remains unchanged, and the function $y(t)$ changes its sign, which means that the curve is symmetrical about the $x$-axis (Fig. 101).

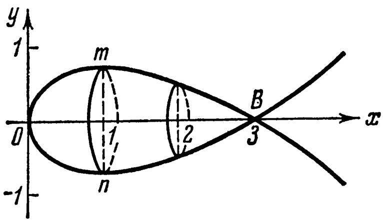


Fig. 101

To find the area of the surface it is sufficient to confine ourselves to the lower portion of the curve $OnB$ that corresponds to the variation of the parameter between 0 and $+\sqrt{3}$. Differentiating with respect to $t$, we find

$$x'_t=2t;\quad y'_t=t^2-1$$

and the linear element

$$dl=\sqrt{x_t'^2+y_t'^2}\,dt=(1+t^2)\,dt.$$

Hence,

$$P=2\pi\int_{t_1}^{t_2}|y(t)|\sqrt{x_t'^2+y_t'^2}\,dt=$$

$$=2\pi\int_0^{\sqrt{3}}-\frac{t}{3}(t^2-3)(1+t^2)\,dt=-\frac{2}{3}\pi\int_0^{\sqrt{3}}(t^5-2t^3-3t)\,dt=3\pi.$$

**7.10.10.** Compute the surface area of the torus generated by revolving the circle $x^2+(y-b)^2=r^2$ $(0<r<b)$ about the $x$-axis.

*Solution*. Let us represent the equation of the circle in parametric form: $x=r\cos t$; $y=b+r\sin t$.

Hence

$$x'_t=-r\sin t;\qquad y'_t=r\cos t.$$

The desired area is

$$P=2\pi\int_0^{2\pi}(b+r\sin t)\sqrt{(-r\sin t)^2+(r\cos t)^2}\,dt=$$

$$=2\pi r\int_0^{2\pi}(b+r\sin t)\,dt=4\pi^2br.$$

**7.10.11.** Compute the area of the surface formed by revolving the lemniscate $\rho=a\sqrt{\cos 2\varphi}$ about the polar axis.

*Solution*. Real values for $\rho$ are obtained for $\cos 2\varphi\geqslant 0$, i. e. for $-\frac{\pi}{4}\leqslant\varphi\leqslant\frac{\pi}{4}$ (the right-hand branch of the lemniscate), or for $\frac{3}{4}\pi\leqslant\varphi\leqslant\frac{5}{4}\pi$ (the left-hand branch of the lemniscate).

The linear element of the lemniscate is equal to

$$dl=\sqrt{\rho^2+\rho'^2}\,d\varphi=\sqrt{a^2\cos 2\varphi+\left(\frac{a\sin 2\varphi}{\sqrt{\cos 2\varphi}}\right)^2}\,d\varphi=\frac{a\,d\varphi}{\sqrt{\cos 2\varphi}}.$$

Besides, $y=\rho\sin\varphi=a\sin\varphi\sqrt{\cos 2\varphi}$.

The sought-for surface area $P$ is equal to double the area of the surface generated by revolving the right-hand branch. Therefore

$$P=2\times 2\pi\int_L y\,dl=4\pi a^2\int_0^{\frac{\pi}{4}}\frac{\sqrt{\cos 2\varphi}\sin\varphi\,d\varphi}{\sqrt{\cos 2\varphi}}=2\pi a^2(2-\sqrt{2}).$$

**7.10.12.** Compute the area of the surface formed by revolving about the straight line $x+y=a$ the quarter of the circle $x^2+y^2=a^2$ between $A(a, 0)$ and $B(0, a)$.

*Solution.* Find the distance $MN$ from the moving point $M(x, y)$, lying on the circle $x^2+y^2=a^2$, to the straight line $x+y=a$:

$$MN=\frac{|x+\sqrt{a^2-x^2}-a|}{\sqrt{2}}=\frac{x+\sqrt{a^2-x^2}-a}{\sqrt{2}},$$

since for the points of the circle that lie in the first quadrant $x+y\geqslant a$. Further,

$$dl=\sqrt{1+y'^2}\,dx=\sqrt{1+\left(\frac{x}{\sqrt{a^2-x^2}}\right)^2}\,dx=\frac{a\,dx}{\sqrt{a^2-x^2}}.$$

Hence,

$$P=2\pi\int_0^a\frac{x+\sqrt{a^2-x^2}-a}{\sqrt{2}}\cdot\frac{a\,dx}{\sqrt{a^2-x^2}}=$$

$$=\sqrt{2}\,\pi a\left[-\sqrt{a^2-x^2}+x-a\arcsin\frac{x}{a}\right]_0^a=\frac{\pi a^2}{\sqrt{2}}(4-\pi).$$

**7.10.13.** Compute the area of the surface formed by revolving one branch of the lemniscate $\rho=a\sqrt{\cos 2\varphi}$ about the straight line $\varphi=\frac{\pi}{4}$.

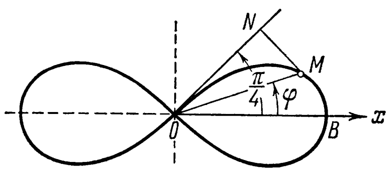

Fig. 102

*Solution.* From the triangle $OMN$ (Fig. 102) we find the distance $MN$ of an arbitrary point $M$ of the right-hand branch from the axis of revolution $\varphi=\frac{\pi}{4}$:

$$MN=\rho\sin\left(\frac{\pi}{4}-\varphi\right)=a\sqrt{\cos 2\varphi}\sin\left(\frac{\pi}{4}-\varphi\right);$$

then

$$dl = \frac{a\,d\varphi}{\sqrt{\cos 2\varphi}}.$$

Therefore $P = 2\pi \int\limits_{-\pi/4}^{\pi/4} a\sqrt{\cos 2\varphi}\,\sin\left(\frac{\pi}{4}-\varphi\right)\frac{a\,d\varphi}{\sqrt{\cos 2\varphi}} = 2\pi a^2.$

**7.10.14.** Compute the area of the surface formed by revolving about the $x$-axis the arc of the curve $y=\frac{x^3}{3}$ between $x=-2$ and $x=2$.

**7.10.15.** Compute the area of the surface generated by revolving one half-wave of the curve $y=\sin x$ about the $x$-axis.

**7.10.16.** Compute the area of the surface generated by revolving about the $y$-axis the arc of the parabola $x^2=4ay$ between the points of intersection of the curve and the straight line $y=3a$.

**7.10.17.** Find the area of the surface formed by revolving about the $x$-axis the arc of the curve $x=e^t\sin t$; $y=e^t\cos t$ between $t=0$ and $t=\frac{\pi}{2}$.

**7.10.18.** Compute the area of the surface obtained by revolving about the $x$-axis the arc of the curve $x=\frac{t^3}{3}$; $y=4-\frac{t^2}{2}$ between the points of its intersection with the axes of coordinates.

**7.10.19.** Compute the area of the surface generated by revolving the curve $\rho=2a\sin\varphi$ about the polar axis.

**7.10.20.** Compute the area of the surface formed by revolving about the $x$-axis the cardioid

$x=a(2\cos t-\cos 2t)$,
$y=a(2\sin t-\sin 2t)$.

## § 7.11. Geometrical Applications of the Definite Integral

**7.11.1.** Given: the cycloid (Fig. 103)

$$x=a(t-\sin t);\quad y=a(1-\cos t);\quad 0\leqslant t\leqslant 2\pi.$$

Compute:

(a) the areas of the surfaces formed by revolving the arc $OBA$ about the $x$- and $y$-axes;

(b) the volumes of the solids generated by revolving the figure $OBAO$ about the $y$-axis and the axis $BC$;

(c) the area of the surface generated by revolving the arc $BA$ about the axis $BC$;

(d) the volume of the solid generated by revolving the figure $ODBEABO$ about the tangent line $DE$ touching the figure at the vertex $B$;

(e) the area of the surface formed by revolving the arc of the cycloid [see item (d)].

*Solution.* (a) When revolving about the $x$-axis the arc $OBA$ generates a surface of area

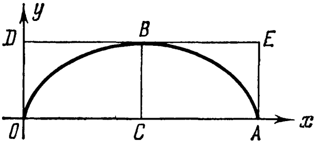


Fig. 103

$$P_x = 2\pi \int_L y\,dl = 2\pi \int_0^{2\pi} a(1-\cos t)\,2a \sin\frac{t}{2}\,dt =$$

$$= 8a^2\pi \int_0^{2\pi} \sin^3\frac{t}{2}\,dt = \frac{64\pi a^2}{3}.$$

When revolving about the $y$-axis the arc $OBA$ generates a surface of area

$$P_y = 2\pi \int_L x\,dl = 4\pi a^2 \int_0^{\pi} (t-\sin t)\sin\frac{t}{2}\,dt +$$

$$+ 4\pi a^2 \int_{\pi}^{2\pi} (t-\sin t)\sin\frac{t}{2}\,dt = 4\pi a^2 \int_0^{2\pi} (t-\sin t)\sin\frac{t}{2}\,dt = 16\pi^2 a^2.$$

(b) When revolving about the $y$-axis the figure $OBAO$ generates a solid of volume

$$V_y = \pi \int_0^{2a} (x_2^2 - x_1^2)\,dy = \pi \int_0^{2a} x_2^2\,dy - \pi \int_0^{2a} x_1^2\,dy,$$

where $x = x_1(y)$ is the equation of the curve $BA$, and $x = x_2(y)$ is the equation of the curve $OB$.

Making the substitution $y = a(1-\cos t)$, take into consideration that for the first integral $t$ varies between $2\pi$ and $\pi$, and for the second integral between 0 and $\pi$. Consequently,

$$V_y = \pi \int_{2\pi}^{\pi} a^2 (t-\sin t)^2 a \sin t\,dt - \pi \int_0^{\pi} a^2 (t-\sin t)^2 a \sin t\,dt =$$

$$= \pi a^3 \int_{2\pi}^{0} (t-\sin t)^2 \sin t\,dt =$$

$$= \pi a^3 \left[ \int_{2\pi}^{0} t^2 \sin t\,dt - \int_{2\pi}^{0} t(1-\cos 2t)\,dt + \int_{2\pi}^{0} \sin^3 t\,dt \right] = 6\pi^3 a^3.$$

For computing the volume of the solid obtained by revolving the figure $OBAO$ about the axis $BC$ it is convenient first to transfer the origin into the point $C$, which yields the following equations in the new system of coordinates

$$x'=a(t-\pi-\sin t); \quad y'=a(1-\cos t).$$

Taking into account only the arc $BA$, we get

$$V=\pi\int_0^{2a} x'^2\,dy'=\pi a^3\int_{2\pi}^{\pi}(t-\pi-\sin t)^2\sin t\,dt.$$

Putting $t-\pi=z$, we obtain

$$V=-\pi a^3\int_{\pi}^{0}(z+\sin z)^2\sin z\,dz=\pi a^3\int_0^{\pi}(z+\sin z)^2\sin z\,dz=$$
$$=\frac{\pi a^3}{6}(9\pi^2-16).$$

(c) Making the above-indicated shift of the origin, we get

$$dl=2a\sin\frac{t}{2}\,|dt|=-2a\sin\frac{t}{2}\,dt.$$

Therefore

$$P=\int_0^{2a}2\pi x\,dl=-4\pi a^2\int_{2\pi}^{\pi}(t-\pi-\sin t)\sin\frac{t}{2}\,dt=$$
$$=4\pi a^2\int_0^{\pi}(z+\sin z)\cos\frac{z}{2}\,dz=4\left(2\pi-\frac{8}{3}\right)\pi a^2.$$

(d) Transferring the origin into the point $B$ and changing the direction of the $y$-axis, we get

$$x'=a(t-\pi-\sin t), \quad y'=a(1+\cos t).$$

Putting $t-\pi=z$, we have

$$x'=a(z+\sin z), \quad y'=a(1-\cos z),$$

$z$ changing from $-\pi$ to $\pi$ for the arc $OBA$. Hence

$$V=\pi\int_{-\pi}^{\pi}a^3(1-\cos z)^2(1+\cos z)\,dz=\pi^2a^3.$$

(e) $$P=2\pi\int_{-\pi}^{\pi}y\,dl=4\pi a^2\int_{-\pi}^{\pi}(1-\cos z)\cos\frac{z}{2}\,dz=\frac{32}{3}\pi a^2.$$

**7.11.2.** Find the volume of the solid bounded by the surfaces $z^2=8(2-x)$ and $x^2+y^2=2x$.

*Solution*. The first surface is a parabolic cylinder with generatrices parallel to the $y$-axis and the directrix $z^2=8(2-x)$ in the plane $xOz$, and the second is a circular cylinder with generatrices parallel to the $z$-axis and the directrix $x^2+y^2=2x$ in the plane $xOy$.

The volume $V$ is computed by the formula $V=\int_0^2 S(x)\,dx$. $S(x)$ represents the area of a triangle whose base is equal to $2y$ and altitude to $2z$:

$$S(x)=2y\times 2z=4\sqrt{2x-x^2}\sqrt{8(2-x)}.$$

Hence,

$$V=\int_0^2 4\sqrt{x(2-x)8(2-x)}\,dx=4\sqrt{8}\int_0^2 (2-x)\sqrt{x}\,dx=$$

$$=4\sqrt{8}\left(\frac{2}{3}2\sqrt{x^3}-\frac{2}{5}\sqrt{x^5}\right)\Big|_0^2=\frac{256}{15}.$$

**7.11.3.** Prove that if the figure $S$ is bounded by a simple convex contour and is situated between the ordinates $y_1$ and $y_2$ (Fig. 104), then the volume of the solid generated by revolving this figure about the $x$-axis can be expressed by the formula

$$V=2\pi\int_{y_1}^{y_2} yh\,dy,$$

where

$$h=x_2(y)-x_1(y),$$

$x=x_1(y)$ being the equation of the left portion of the contour and $x=x_2(y)$ that of the right portion.

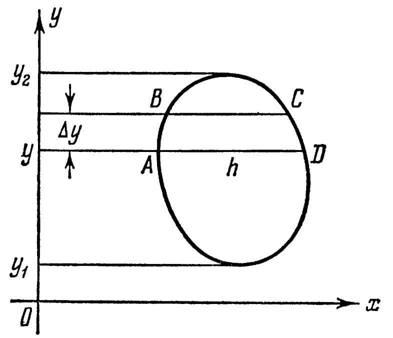

Fig. 104

*Solution.* Let the generating figure $S$ be bounded by a simple convex contour and contained between the ordinates $y_1$ and $y_2$. Subdivide the interval $[y_1, y_2]$ into parts and pass through the points of division straight lines parallel to the axis of revolution, thus cutting the figure $S$ into horizontal strips. Single out one strip and replace it by the rectangle $ABCD$, whose lower base is equal to the chord $AD=h$ specified by the ordinate $y$, its altitude $AB$ being equal to $\Delta y$. The solid generated by revolving the rectangle $ABCD$ about the $x$-axis is a hollow cylinder whose volume may be approximately taken for the element of volume

$$\Delta V\approx\pi(y+\Delta y)^2h-\pi y^2h=2\pi y\Delta yh+\pi h(\Delta y)^2.$$

Rejecting the infinitesimal of the second order with respect to $\Delta y$, we get the principal part or the differential of volume

$$dV=2\pi\,yh\,dy.$$

Knowing the differential of the volume, we get the volume proper through integration:

$$V = 2\pi \int_{y_1}^{y_2} yh\,dy.$$

Thus, we obtain one more formula for computing the volume of the solid of revolution.

**7.11.4.** The planar region bounded by the parabola $y = 2x^2 + 3$, the $x$-axis and the verticals $x = 0$ and $x = 1$ revolves about the $y$-axis. Compute the volume of the solid of revolution thus generated.

*Solution.* Divide the area of the figure into elementary strips by straight lines parallel to the $y$-axis. The volume $\Delta V$ of the elementary cylinder generated by revolving one strip is

$$\Delta V = \pi (x + \Delta x)^2 y - \pi x^2 y = 2\pi xy\,\Delta x + \pi y (\Delta x)^2,$$

where $\Delta x$ is the width of the strip.

Neglecting the infinitesimal of the second order with respect to $\Delta x$, we get the differential of the desired volume

$$dV = 2\pi xy\,dx.$$

Hence

$$V = \int_0^1 2\pi xy\,dx = 2\pi \int_0^1 x(2x^2 + 3)\,dx = 4\pi.$$

**7.11.5.** Compute the area of the portion of the cylinder surface $x^2 + y^2 = ax$ situated inside the sphere

$$x^2 + y^2 + z^2 = a^2.$$

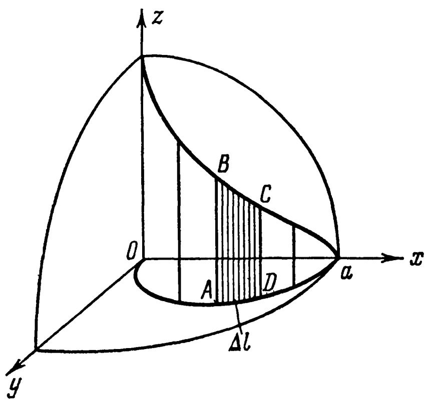

Fig. 105

*Solution.* The generatrices of the cylinder are parallel to the $z$-axis, the circle $\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$ serving as directrix (Fig. 105 shows a quarter of the sought-for surface).

Subdivide the portion of the circle shown in Fig. 105 into small arcs $\Delta l$. The generatrices passing through the points of division cut the cylinder surface into strips. If infinitesimals of higher order are neglected, the area of the strip $\hat{A}BCD$ is equal to $CD \cdot \Delta l$.

If $\rho$ and $\varphi$ are the polar coordinates of the point $D$, then $\rho = a\cos\varphi$ and $CD = \sqrt{a^2 - \rho^2} = a\sin\varphi$, and $\Delta l = a \cdot \Delta\varphi$, whence we find the element of area:

$$dP = a^2 \sin\varphi \, d\varphi.$$

Hence,

$$P = 4\int_0^{\frac{\pi}{2}} a^2 \sin\varphi \, d\varphi = 4a^2.$$

**7.11.6.** Find the area of the surface cut off from a right circular cylinder by a plane passing through the diameter of the base and inclined at an angle of 45° to the base.

*Solution.* Let the cylinder axis be the $z$-axis, and the given diameter the $x$-axis. Then the equation of the cylindrical surface will be $x^2+y^2=a^2$, and that of the plane forming an angle of 45° with the coordinate plane $xOy$ will be $y=z$.

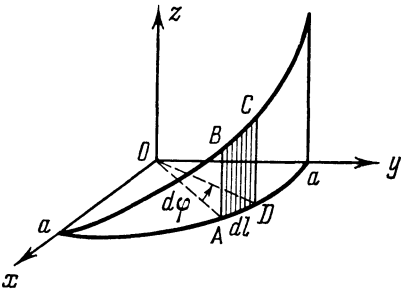


Fig. 106

The area of the infinitely narrow strip $ABCD$ (see Fig. 106) will be $dP=z\,dl$ (accurate to infinitesimals of a higher order), where $dl$ is the length of the elementary arc of the circumference of the base.

Introducing polar coordinates, we get

$$z=y=a\sin\varphi; \quad dl=a\,d\varphi.$$

Hence $dP=a^2\sin\varphi\,d\varphi$ and

$$P=a^2\int_0^{\pi}\sin\varphi\,d\varphi=a^2\left[-\cos\varphi\right]_0^{\pi}=2a^2.$$

**7.11.7.** The axes of two circular cylinders with equal bases intersect at right angles. Compute the surface area of the solid constituting the part common to both cylinders.

**7.11.8.** Compute the volume of the solid generated by revolving about the $y$-axis the figure bounded by the parabola $x^2=y-1$, the axis of abscissas and the straight lines $x=0$ and $x=1$.

**7.11.9.** Find the area $S$ of the ellipse given by the equation $Ax^2+2Bxy+Cy^2=1$ ($\delta=AC-B^2>0$; $C>0$).

*Solution.* Solving the equation with respect to $y$, we get

$$y_1=\frac{-Bx-\sqrt{C-\delta x^2}}{C}; \quad y_2=\frac{-Bx+\sqrt{C-\delta x^2}}{C},$$

where the values of $x$ must satisfy the inequality

$$C-\delta x^2 \geqslant 0.$$

Solving this inequality, we obtain the limits of integration:

$$-\sqrt{\frac{C}{\delta}} \leqslant x \leqslant \sqrt{\frac{C}{\delta}}.$$

Consequently, the sought-for area is equal to

$$S = \int_{-\sqrt{\frac{C}{\delta}}}^{\sqrt{\frac{C}{\delta}}} (y_2 - y_1)\, dx = \frac{4}{C} \int_0^{\sqrt{\frac{C}{\delta}}} \sqrt{C - \delta x^2}\, dx = \frac{\pi}{\sqrt{\delta}}.$$

**7.11.10.** Find the areas of the figures bounded by the curves represented parametrically:

(a) $x = 2t - t^2$; $y = 2t^2 - t^3$;

(b) $x = \frac{t^2}{1+t^2}$; $y = \frac{t(1-t^2)}{1+t^2}$.

**7.11.11.** Find the areas of the figures bounded by the curves given in polar coordinates:

(a) $\rho = a \sin 3\varphi$ (a three-leaved rose);

(b) $\rho = \frac{p}{1 - \cos\varphi}$ $\left[\frac{\pi}{4} \leqslant \varphi \leqslant \frac{\pi}{2}\right]$;

(c) $\rho = 3 \sin\varphi$ and $\rho = \sqrt{3} \cos\varphi$.

**7.11.12.** Find the arc length of the curve $y^2 = \frac{4}{9}(2 - x)^3$ cut off by the straight line $x = -1$.

**7.11.13.** Find the length of the arc $OA$ of the curve

$$y = a \ln \frac{a^2}{a^2 - x^2},$$

where $O(0, 0)$; $A\left(\frac{a}{2}, a \ln \frac{4}{3}\right)$.

**7.11.14.** Compute the arc length of the curve $y^2 = \frac{2}{3}(x - 1)^3$ contained inside the parabola $y^2 = \frac{x}{3}$.

**7.11.15.** Prove that the length of the ellipse

$$x = \sqrt{2} \sin t; \quad y = \cos t$$

is equal to the wavelength of the sinusoid $y = \sin x$.

**7.11.16.** Prove that the arc of the parabola $y = \frac{1}{2p} x^2$ corresponding to the interval $0 \leqslant x \leqslant a$ has the same length as the arc of the spiral $\rho = P\varphi$ corresponding to the interval $0 \leqslant \rho \leqslant a$.

**7.11.17.** Find the ratio of the area enclosed by the loop of the curve $y = \pm\left(\frac{1}{3} - x\right)\sqrt{x}$ to the area of a circle the circumference of which is equal to the length of the contour of this curve.

**7.11.18.** Find the volume of the segment cut off from the elliptical paraboloid $\frac{y^2}{2p}+\frac{z^2}{2q}=x$ by the plane $x=a$.

**7.11.19.** Compute the volume of the solid bounded by the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$ and the planes $z=c$ and $z=l>c$.

**7.11.20.** Find the volume of the right elliptical cone whose base is an ellipse with semi-axes $a$ and $b$, its altitude being equal to $h$.

**7.11.21.** Find the volume of the solid generated by revolving about the $x$-axis the figure bounded by the straight lines $y=x+1$; $y=2x+1$ and $x=2$.

**7.11.22.** Find the volume of the solid generated by revolving about the $x$-axis the figure bounded by the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the straight line $2ay-bx=0$ and the axis of abscissas.

**7.11.23.** Find the volume of the solid generated by revolving the curve $\rho=a\cos^2\varphi$ about the polar axis.

**7.11.24.** Find the areas of the surfaces generated by revolving the following curves:

(a) $y=\tan x\left(0\leqslant x\leqslant\frac{\pi}{4}\right)$ about the $x$-axis;

(b) $y=x\sqrt{\frac{x}{a}}\,(0\leqslant x\leqslant a)$ about the $x$-axis;

(c) $x^2+y^2-2rx=0$ about the $x$-axis between 0 and $h$.

## *§ 7.12. Computing Pressure, Work and Other Physical Quantities by the Definite Integrals*

I. To compute the force of liquid pressure we use Pascal's law, which states that the force of pressure of a liquid $P$ on an area $S$ at a depth of immersion $h$ is $P=\gamma hS$, where $\gamma$ is the specific weight of the liquid.

II. If a variable force $X=f(x)$ acts in the direction of the $x$-axis, then the work of this force over an interval $[x_1, x_2]$ is expressed by the integral

$$A=\int_{x_1}^{x_2} f(x)\,dx.$$

III. The kinetic energy of a material point of mass $m$ and velocity $v$ is defined as

$$K=\frac{mv^2}{2}.$$

IV. Electric charges repulse each other with a force $F=\frac{e_1e_2}{r^2}$, where $e_1$ and $e_2$ are the values of the charges, and $r$ is the distance between them.

*Note.* When solving practical problems we assume that all the data are expressed in one and the same system of units and omit the dimensions of the corresponding quantities.

**7.12.1.** Compute the force of pressure experienced by a vertical triangle with base $b$ and altitude $h$ submerged base downwards in water so that its vertex touches the surface of the water.

*Solution.* Introduce a system of coordinates as indicated in Fig. 107 and consider a horizontal strip of thickness $dx$ located at an arbitrary depth $x$.

Assuming this strip to be a rectangle, find the differential of area $dS=MN\,dx$. From the similarity of the triangles $BMN$ and $ABC$ we have $\frac{MN}{b}=\frac{x}{h}$, whence $MN=\frac{bx}{h}$ and $dS=\frac{bx}{h}dx$.

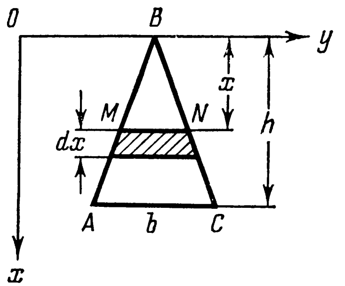


Fig. 107

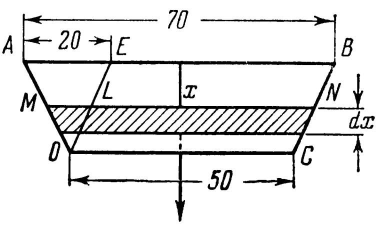


Fig. 108

The force of pressure experienced by this strip is equal to $dP=x\,dS$ accurate to infinitesimals of higher order (taking into consideration that the specific weight of water is unity). Consequently, the entire force of water pressure experienced by the triangle is equal to

$$P=\int_0^h x\,dS=\frac{b}{h}\int_0^h x^2\,dx=\frac{1}{3}bh^2.$$

**7.12.2.** Find the force of pressure experienced by a semicircle of radius $R$ submerged vertically in a liquid so that its diameter is flush with the liquid surface (the specific weight of the liquid is $\gamma$).

**7.12.3.** A vertical dam has the form of a trapezoid whose upper base is 70 m long, the lower one 50 m, and the altitude 20 m. Find the force of water pressure experienced by the dam (Fig. 108).

*Solution.* The differential $(dS)$ of area of the hatched figure is approximately equal to $dS=MN\,dx$. Taking into consideration the similarity of the triangles $OML$ and $OAE$, we find $\frac{ML}{20}=\frac{20-x}{20}$; whence $ML=20-x$, $MN=20-x+50=70-x$. Thus, $dS=MN\times\times dx=(70-x)\,dx$ and the differential of the force of water pressure is equal to

$$dP=x\,dS=x(70-x)\,dx.$$

Integrating with respect to $x$ from 0 to 20, we get

$$P=\int_0^{20}(70x-x^2)\,dx=11\,333\frac{1}{3}.$$

**7.12.4.** Calculate the work performed in pumping the water out of a semispherical boiler of radius $R$.

**7.12.5.** A rectangular vessel is filled with equal volumes of water and oil; water is twice as heavy as oil. Show that the force of pressure of the mixture on the wall will reduce by one fifth if the water is replaced by oil.

*Solution.* Let $h$ be the depth of the vessel and $l$ the length of the wall. Let us introduce a system of coordinates as shown in Fig. 109. Since the oil is situated above the water and occupies the upper half of the vessel, the force of the oil pressure experienced by the upper half of the wall is equal to

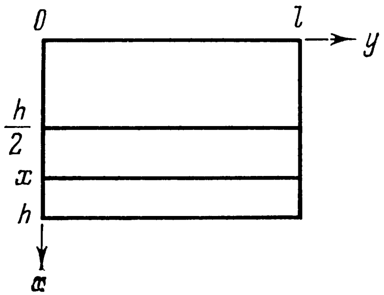

Fig. 109

$$P_1=\frac{1}{2}\int_0^{\frac{h}{2}}xl\,dx=\frac{lh^2}{16}.$$

The pressure at a depth $x>\frac{h}{2}$ is made up of the pressure of the oil column of height $\frac{h}{2}$ and that of the water column of height $x-\frac{h}{2}$, and therefore

$$dP_2=\left[\frac{h}{2}\times\frac{1}{2}+\left(x-\frac{h}{2}\right)\right]l\,dx=\left(x-\frac{h}{4}\right)l\,dx.$$

Consequently, the force of pressure of the mixture on the lower half of the wall is

$$P_2=\int_{\frac{h}{2}}^{h}l\left(x-\frac{h}{4}\right)dx=\frac{lh^2}{4}.$$

The entire pressure of the mixture on the wall is equal to

$$P = P_1 + P_2 = \frac{lh^2}{4} + \frac{lh^2}{16} = \frac{5}{16} lh^2.$$

If the vessel were filled only with oil, the force of pressure $\overline{P}$ on the same wall would be

$$\overline{P} = \frac{1}{2}\int_0^h xl\,dx = \frac{lh^2}{4}.$$

Hence,

$$P - \overline{P} = \frac{1}{16} lh^2 = \frac{1}{5} P.$$

**7.12.6.** The electric charge $E$ concentrated at the origin of coordinates repulses the charge $e$ from the point $(a, 0)$ to the point $(b, 0)$. Find the work $A$ of the repulsive force $F$.

*Solution.* The differential of the work of the force over displacement $dx$ is $dA = F\,dx = \frac{eE}{x^2}\,dx$.

Hence

$$A = eE\int_a^b \frac{dx}{x^2} = eE\left(\frac{1}{a} - \frac{1}{b}\right).$$

As $b \to \infty$ the work $A$ tends to $\frac{eE}{a}$.

**7.12.7.** Calculate the work performed in launching a rocket of weight $P$ from the ground vertically upwards to a height $h$.

*Solution.* Let us denote the force of attraction of the rocket by the Earth by $F$, the mass of the rocket by $m_R$, and the mass of the Earth by $m_E$. According to Newton's law

$$F = k\frac{m_R m_E}{x^2},$$

where $x$ is the distance between the rocket and the centre of the Earth. Putting $km_R m_E = K$, we get $F(x) = \frac{K}{x^2}$, $R \leqslant x \leqslant h + R$, $R$ being the radius of the Earth. At $x = R$ the force $F(R)$ will be the weight of the rocket $P$, i.e. $F(R) = P = \frac{K}{R^2}$, whence $K = PR^2$ and $F(x) = \frac{PR^2}{x^2}$.

Thus, the differential of the work is

$$dA = F(x)\,dx = \frac{PR^2}{x^2}\,dx.$$

Integrating, we obtain

$$A=\int_{R}^{R+h} F(x)\,dx=PR^2\int_{R}^{R+h}\frac{dx}{x^2}=\frac{PRh}{R+h}.$$

The limit $\lim\limits_{h\to\infty} A(h)=\lim\limits_{h\to\infty}\frac{PRh}{R+h}=PR$ is equal to the work performed by the rocket engine to achieve complete escape of the rocket from the Earth's gravity field (the Earth's motion is neglected).

**7.12.8.** Calculate the work that has to be done to stop an iron sphere of radius $R$ rotating about its diameter with an angular velocity $\omega$.

*Solution.* The amount of required work is equal to the kinetic energy of the sphere. To calculate this energy divide the sphere into concentric hollow cylinders of thickness $dx$; the velocity of the points of such a cylinder of radius $x$ is $\omega x$.

The element of volume of such a cylinder is $dV=4\pi x\sqrt{R^2-x^2}\,dx$, the element of mass $dM=\gamma\,dV$, where $\gamma$ is the density of iron, and the differential of kinetic energy $dK=2\pi\gamma\omega^2x^3\sqrt{R^2-x^2}\,dx$.

Hence,

$$K=2\pi\gamma\omega^2\int_0^R x^3\sqrt{R^2-x^2}\,dx=\frac{4\pi\gamma R^3}{3}\cdot\frac{\omega^2R^2}{5}=\frac{M\omega^2R^2}{5}.$$

**7.12.9.** Calculate the kinetic energy of a disk of mass $M$ and radius $R$ rotating with an angular velocity $\omega$ about an axis passing through its centre perpendicular to its plane.

**7.12.10.** Find the amount of heat released by an alternating sinusoidal current

$$I=I_0\sin\left(\frac{2\pi}{T}t-\varphi\right)$$

during a cycle $T$ in a conductor with resistance $R$.

*Solution.* For direct current the amount of heat released during a unit time is determined by the Joule-Lenz law

$$Q=0.24\,I^2R.$$

For alternating current the differential of amount of heat is $dQ=0.24\,I^2(t)\,R\,dt$, whence

$$Q=0.24R\int_{t_1}^{t_2} I^2\,dt.$$

In this case

$$Q=0.24\,RI_0^2\int_0^T\sin^2\left(\frac{2\pi}{T}t-\varphi\right)dt=$$

$$=0.12\,RI_0^2\left[t-\frac{T}{2\pi}\,\frac{\sin 2\left(\frac{2\pi}{T}t-\varphi\right)}{2}\right]\Bigg|_0^T=0.12RTI_0^2.$$

**7.12.11.** Find the pressure of a liquid of specific weight $d$ on a vertical ellipse with axes $2a$ and $2b$ whose centre is submerged in the liquid to a level $h\,(h \geqslant b)$.

**7.12.12.** Find the pressure of a liquid of specific weight $d$ on the wall of a circular cylinder of base radius $r$ and altitude $h$ if the cylinder is full of liquid.

**7.12.13.** Calculate the work performed to overcome the force of gravity in pumping the water out of a conical vessel with the vertex downwards; the radius of the cone base is $R$ and its altitude is $H$.

**7.12.14.** Compute the work required to stretch a spring by 6 cm, if a force of one kilogram is required to stretch it by 1 cm.

## § 7.13. *Computing Static Moments and Moments of Inertia. Determining Coordinates of the Centre of Gravity*

In all problems of this paragraph we will assume that the mass is distributed uniformly in a body (linear, two- and three-dimensional) and that its density is equal to unity.

1. For a plane curve $L$ the static moments $M_x$ and $M_y$ about the $x$- and $y$-axis are expressed by the formulas

$$M_x = \int_L y\,dl, \quad M_y = \int_L x\,dl.$$

The moment of inertia about the origin of coordinates

$$I_0 = \int_L (x^2 + y^2)\,dl.$$

If the curve $L$ is given by the explicit equation $y = y(x)\,(a \leqslant x \leqslant b)$, then $dl$ has to be replaced by $\sqrt{1 + y'^2}\,dx$ in the above formulas.

If the curve $L$ is given by the parametric equations $x = x(t)$, $y = y(t)\,(t_1 \leqslant t \leqslant t_2)$, then $dl$ should be replaced by $\sqrt{x'^2 + y'^2}\,dt$ in these formulas.

2. For the plane figure bounded by the curves $y = y_1(x)$, $y = y_2(x)$, $y_1(x) \leqslant y_2(x)$ and the straight lines $x = a$, $x = b$ $(a \leqslant x \leqslant b)$ the static moments are expressed by the formulas

$$M_x = \frac{1}{2}\int_a^b (y_2^2 - y_1^2)\,dx; \quad M_y = \int_a^b x\,(y_2 - y_1)\,dx.$$

3. The centre of gravity of a plane curve has the following coordinates: $x_c = \frac{M_y}{l}$, $y_c = \frac{M_x}{l}$, where $l$ is the length of the curve $L$.

The centre of gravity of a plane figure has the coordinates: $x_c=\frac{M_y}{S}$, $y_c=\frac{M_x}{S}$, where $S$ is the area of the figure.

**7.13.1.** Find the static moment of the upper portion of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

about the $x$-axis.

*Solution.* For the ellipse

$$y\,dl=y\sqrt{1+y'^2}\,dx=\sqrt{y^2+(yy')^2}\,dx;$$

since $y^2=b^2-\frac{b^2}{a^2}x^2$ and $yy'=-\frac{b^2}{a^2}x$, we have

$$y\,dl=\sqrt{b^2-\frac{b^2}{a^2}x^2+\frac{b^4}{a^4}x^2}\,dx=\frac{b}{a}\sqrt{a^2-\varepsilon^2x^2}\,dx,$$

where $\varepsilon$ is the eccentricity of the ellipse, $\varepsilon=\frac{\sqrt{a^2-b^2}}{a}$.

Integrating from $-a$ to $a$, we find

$$M_x=\frac{b}{a}\int_{-a}^{a}\sqrt{a^2-\varepsilon^2x^2}\,dx=\frac{2b}{a}\int_{0}^{a}\sqrt{a^2-\varepsilon^2x^2}\,dx=$$

$$=\frac{b}{a}\left(a\sqrt{a^2-\varepsilon^2a^2}+\frac{a^2}{\varepsilon}\arcsin\varepsilon\right)=b\left(b+\frac{a}{\varepsilon}\arcsin\varepsilon\right).$$

In the case of a circle, i. e. at $a=b$, we shall have $M_x=2a^2$, since $\varepsilon=0$ and $\lim\limits_{\varepsilon\to 0}\frac{\arcsin\varepsilon}{\varepsilon}=1$.

**7.13.2.** Find the moment of inertia of a rectangle with base $b$ and altitude $h$ about its base.

*Solution.* Let us consider an elementary strip of width $dy$ cut out from the rectangle and parallel to the base and situated at a distance $y$ from it. The mass of the strip is equal to its area $dS=b\,dy$, the distances from all its points to the base being equal to $y$ accurate to $dy$. Therefore, $dI_x=by^2\,dy$ and

$$I_x=\int_0^h by^2\,dy=\frac{bh^3}{3}.$$

**7.13.3.** Find the moment of inertia of an arc of the circle $x^2+y^2=R^2$ lying in the first quadrant about the $y$-axis.

**7.13.4.** Calculate the moment of inertia about the $y$-axis of the figure bounded by the parabola $y^2=4ax$ and the straight line $x=a$.

*Solution.* We have $dI_x=x^2\,dS$, where $dS$ is the area of a vertical strip situated at a distance $x$ from the $y$-axis (Fig. 110):

$$dS=2\,|y|\,dx=2\sqrt{4ax}\,dx.$$

Hence,

$$I_x = \int_0^a 4x^2 \sqrt{ax}\,dx = 4\sqrt{a}\int_0^a x^{\frac{5}{2}}\,dx = \frac{8}{7}a^4.$$

**7.13.5.** In designing wooden girder bridges we often have to deal with logs flattened on two opposite sides. Figure 111 shows the cross-section of such a log. Determine the moment of inertia of this cross-section about the horizontal centre line.

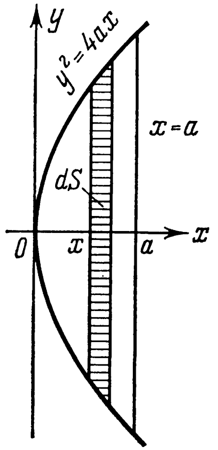


Fig. 110

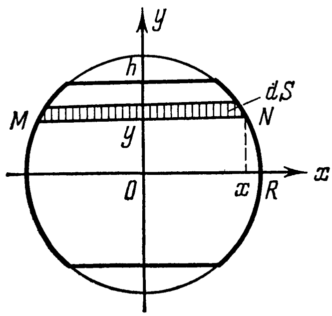


Fig. 111

*Solution.* Arrange the system of coordinates as is shown in the accompanying drawing. Then

$$dI_x = y^2\,dS, \text{ where } dS = MN\,dy = 2x\,dy = 2\sqrt{R^2-y^2}\,dy.$$

Whence

$$I_x = 2\int_{-h}^{h} y^2\sqrt{R^2-y^2}\,dy = 4\int_0^h y^2\sqrt{R^2-y^2}\,dy.$$

Substituting $y = R\sin t$; $dy = R\cos t\,dt$; $t_1 = 0$; $t_2 = \arcsin(h/R)$, we get

$$I_x = 4\int_0^h y^2\sqrt{R^2-y^2}\,dy = 4\int_0^{\arcsin(h/R)} R^2\sin^2 t\cdot R\cos t R\cos t\,dt =$$

$$= 4R^4\int_0^{\arcsin(h/R)} \sin^2 t\cos^2 t\,dt = \frac{R^4}{2}\int_0^{\arcsin(h/R)} (1-\cos 4t)\,dt =$$

$$= \frac{R^4}{2}\arcsin\frac{h}{R} + \frac{h}{R}(2h^2-R^2)\sqrt{R^2-h^2}.$$

When $h = R$, we obtain the moment of inertia of the circle about one of its diameters: $I_x = \frac{\pi R^4}{4}$.

**7.13.6.** Find the moment of inertia about the $x$-axis of the figure bounded by two parabolas with dimensions indicated in Fig. 112.

*Solution.* Arrange the system of coordinates as shown in Fig. 112 and write the equations of the parabolas.

The equation of the left parabola is: $y^2=\frac{b^2}{2a}\left(x+\frac{a}{2}\right)$, the equation of the right parabola, $y^2=\frac{b^2}{2a}\left(\frac{a}{2}-x\right)$.

For the hatched strip the moment of inertia is

$$dI_x=y^2\,dS=y^2\,|MN|\,dy,$$

where

$$|MN|=x_2-x_1=2\left(\frac{a}{2}-\frac{2a}{b^2}y^2\right)=$$
$$=a-\frac{4a}{b^2}y^2.$$

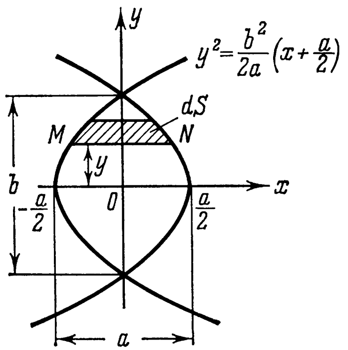


Fig. 112

Hence,

$$I_x=\int\limits_{-b/2}^{b/2} y^2\left(a-\frac{4a}{b^2}y^2\right)dy=2\int\limits_{0}^{b/2} y^2\left(a-\frac{4a}{b^2}y^2\right)dy=\frac{ab^3}{30}.$$

**7.13.7.** Find the static moments about the $x$- and $y$-axis of the arc of the parabola $y^2=2x$ between $x=0$ and $x=2$ $(y>0)$.

**7.13.8.** Find the static moments about the axes of coordinates of the line segment $\frac{x}{a}+\frac{y}{b}=1$ whose end-points lie on the coordinate axes.

**7.13.9.** Find the static moment about the $x$-axis of the arc of the curve $y=\cos x$ between $x_1=-\frac{\pi}{2}$ and $x_2=\frac{\pi}{2}$.

**7.13.10.** Find the static moment about the $x$-axis of the figure bounded by the lines $y=x^2$; $y=\sqrt{x}$.

**7.13.11.** Find the moments of inertia about the $x$- and $y$-axis of the triangle bounded by the lines $x=0$, $y=0$ and $\frac{x}{a}+\frac{y}{b}=1$ $(a>0,$ $b>0)$.

**7.13.12.** Find the moment of inertia of the trapezoid $ABCD$ about its base $AD$ if $AD=a$, $BC=b$ and the altitude of the trapezoid is equal to $h$.

**7.13.13.** Find the centre of gravity of the semicircle $x^2+y^2=a^2$ situated above the $x$-axis.

*Solution.* Since the arc of the semicircle is symmetrical about the $y$-axis, the centre of gravity of the arc lies on the $y$-axis, i. e. $x_c=0$.

To find the ordinate $y_c$, take advantage of the result of Problem **7.13.1**: $M_x = 2a^2$; therefore $y_c = \frac{2a^2}{\pi a} = \frac{2a}{\pi}$. Thus, $x_c = 0$, $y_c = \frac{2a}{\pi}$.

**7.13.14.** Find the coordinates of the centre of gravity of the catenary $y = \frac{1}{2}(e^x + e^{-x}) = \cosh x$ between $A(0, 1)$ and $B(a, \cosh a)$.

*Solution.* We have

$$dl = \sqrt{1 + y'^2}\, dx = \sqrt{1 + \sinh^2 x}\, dx = \cosh x\, dx$$

whence we find

$$l = \int_L dl = \int_0^a \cosh x\, dx = \sinh a.$$

Then

$$M_y = \int_L x\, dl = \int_0^a x \cosh x\, dx = x \sinh x \Big|_0^a - \int_0^a \sinh x\, dx =$$
$$= a \sinh a - \cosh a + 1.$$

Hence,

$$x_c = \frac{a \sinh a - (\cosh a - 1)}{\sinh a} = a - \frac{\cosh a - 1}{\sinh a} = a - \tanh \frac{a}{2}.$$

Analogously,

$$M_x = \int_L y\, dl = \int_0^a \cosh^2 x\, dx = \frac{1}{2} \int_0^a (1 + \cosh 2x)\, dx =$$
$$= \frac{1}{2}\left(x + \frac{\sinh 2x}{2}\right)\Big|_0^a = \frac{a}{2} + \frac{\sinh 2a}{4};$$

$$y_c = \frac{\frac{a}{2} + \frac{\sinh 2a}{4}}{\sinh a} = \frac{a}{2 \sinh a} + \frac{\cosh a}{2}.$$

**7.13.15.** Find the centre of gravity of the first arc of the cycloid: $x = a(t - \sin t)$, $y = a(1 - \cos t)$ $(0 \leqslant t \leqslant 2\pi)$.

*Solution.* The first arc of the cycloid is symmetrical about the straight line $x = \pi a$, therefore the centre of gravity of the arc of the cycloid lies on this straight line and $x_c = \pi a$. Since the length of the first arc of the cycloid $l = 8a$, we have

$$y_c = \frac{1}{l} \int_L y\, dl = \frac{1}{8a} 2a^2 \int_0^{2\pi} (1 - \cos t) \sin \frac{t}{2}\, dt = \frac{a}{2} \int_0^{2\pi} \sin^3 \frac{t}{2}\, dt = \frac{4}{3} a.$$

**7.13.16.** Determine the coordinates of the centre of gravity of the portion of the arc of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ situated in the first quadrant.

**7.13.17.** Find the Cartesian coordinates of the centre of gravity of the arc of the cardioid $\rho = a(1+\cos\varphi)$ between $\varphi = 0$ and $\varphi = \pi$.

*Solution.* Let us represent the equation of the cardioid in parametric form:

$$x = \rho\cos\varphi = a(1+\cos\varphi)\cos\varphi;$$
$$y = \rho\sin\varphi = a(1+\cos\varphi)\sin\varphi.$$

As the parameter $\varphi$ varies between 0 and $\pi$ the running point describes the upper portion of the curve. Since the length of the entire cardioid equals $8a$ and
$dl = \sqrt{(x'_\varphi)^2 + (y'_\varphi)^2}\,d\varphi = 2a\cos\frac{\varphi}{2}\,d\varphi$ (see Problem **7.9.3**), we have

$$x_c = \frac{1}{l}\int_L y\,dl = \frac{1}{4a}\int_0^{\pi} a\sin\varphi\,(1+\cos\varphi)\,2a\cos\frac{\varphi}{2}\,d\varphi =$$
$$= 2a\int_0^{\pi}\cos^4\frac{\varphi}{2}\sin\frac{\varphi}{2}\,d\varphi = -\frac{4}{5}a\cos^5\frac{\varphi}{2}\Big|_0^{\pi} = \frac{4}{5}a.$$

Analogously,

$$y_c = \frac{1}{4a}\int_L x\,dl = \frac{1}{4a}\int_0^{\pi} a\cos\varphi\,(1+\cos\varphi)\,2a\cos\frac{\varphi}{2}\,d\varphi =$$
$$= a\int_0^{\pi}\cos\varphi\cos^3\frac{\varphi}{2}\,d\varphi = a\int_0^{\pi}\left(2\cos^5\frac{\varphi}{2} - \cos^3\frac{\varphi}{2}\right)d\varphi.$$

Putting $\frac{\varphi}{2} = t$ we get (see Problem **6.6.9**)

$$y_c = 2a\int_0^{\frac{\pi}{2}}(2\cos^5 t - \cos^3 t)\,dt = 4a\frac{4\cdot 2}{5\cdot 3} - 2a\frac{2}{3} = \frac{4}{5}a.$$

And so, $x_c = y_c = \frac{4a}{5}$.

It is interesting to note that the centre of gravity of the above-considered half of the arc of the cardioid lies on the bisector of the first coordinate angle, though the arc itself is not symmetrical about this bisector.

**7.13.18.** Find the centre of gravity of the figure bounded by the ellipse $4x^2 + 9y^2 = 36$ and the circle $x^2 + y^2 = 9$ and situated in the first quadrant (Fig. 113).

*Solution.* Let us first calculate the static moments:

$$M_y=\int_0^3 x(y_2-y_1)\,dx=\int_0^3 x\left[\sqrt{9-x^2}-\frac{2}{3}\sqrt{9-x^2}\right]dx=$$
$$=\frac{1}{3}\int_0^3 x\sqrt{9-x^2}\,dx=3;$$

$$M_x=\frac{1}{2}\int_0^3 (y_2^2-y_1^2)\,dx=\frac{1}{2}\int_0^3\left[(9-x^2)-\frac{4}{9}(9-x^2)\right]dx=$$
$$=\frac{1}{2}\int_0^3\left(5-\frac{5}{9}x^2\right)dx=5.$$

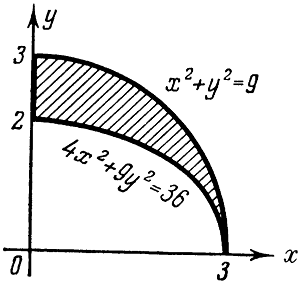


Fig. 113

The area of a quarter of a circle of radius 3 is equal to $\frac{9\pi}{4}$, and the area of a quarter of an ellipse with semi-axes $a=3$ and $b=2$ equals $\frac{3\pi}{2}$, therefore the area of the figure under consideration is $S=\frac{9\pi}{4}-\frac{3\pi}{2}=\frac{3\pi}{4}$.

Thus,

$$x_c=\frac{M_y}{S}=\frac{4}{\pi};\quad y_c=\frac{M_x}{S}=\frac{20}{3\pi}.$$

**7.13.19.** Find the centre of gravity of the figure bounded by the parabola $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$ and the axes of coordinates.

**7.13.20.** Find the Cartesian coordinates of the centre of gravity of the figure enclosed by the curve $\rho=a\cos^3\varphi$ $(a>0)$.

*Solution.* Since $\rho\geqslant 0$ in all cases, the given curve is traced when $\varphi$ changes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. By virtue of evenness of the function $\cos\varphi$ it is symmetrical about the polar axis and passes through the origin of coordinates at $\varphi=\pm\frac{\pi}{2}$.

Compute the area $S$ of the figure obtained:

$$S=2\times\frac{1}{2}\int_0^{\frac{\pi}{2}}\rho^2\,d\varphi=a^2\int_0^{\frac{\pi}{2}}\cos^6\varphi\,d\varphi=a^2\,\frac{1\times3\times5}{2\times4\times6}\times\frac{\pi}{2}=\frac{5}{32}\pi a^2.$$

Now arrange the axes of coordinates as shown in Fig. 114. Then the parametric equations of the curve are

$$x=\rho\cos\varphi=a\cos^4\varphi;$$
$$y=\rho\sin\varphi=a\sin\varphi\cos^3\varphi.$$

The centre of gravity of the figure lies on the $x$-axis, i.e. $y_c=0$ by virtue of symmetry about the $x$-axis. Finally, determine $x_c$:

$$x_c=\frac{2\int\limits_0^a xy\,dx}{S}=\frac{8a^3}{S}\int\limits_0^{\frac{\pi}{2}}\cos^{10}\varphi\sin^2\varphi\,d\varphi=\frac{8a^3}{S}\int\limits_0^{\frac{\pi}{2}}(\cos^{10}\varphi-\cos^{12}\varphi)\,d\varphi=$$
$$=\frac{8a^3}{(5/32)\pi a^2}\left(\frac{1\times3\times5\times7\times9}{2\times4\times6\times8\times10}-\frac{1\times3\times5\times7\times9\times11}{2\times4\times6\times8\times10\times12}\right)\frac{\pi}{2}=\frac{21}{40}a.$$

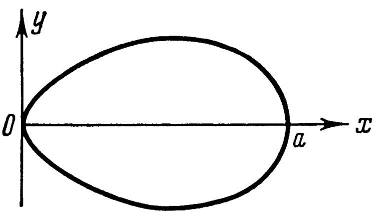


Fig. 114

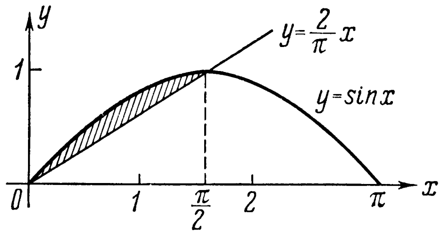


Fig. 115

**7.13.21.** Find the coordinates of the centre of gravity of the figure bounded by the straight line $y=\frac{2}{\pi}x$ and the sinusoid $y=\sin x$ $(x\geqslant 0)$ (Fig. 115).

*Solution*. The straight line $y=\frac{2}{\pi}x$ and the sine line $y=\sin x$ intersect at the points (0, 0) and $\left(\frac{\pi}{2},\ 1\right)$. The area of the figure bounded by these lines is

$$S=\int\limits_0^{\frac{\pi}{2}}\left(\sin x-\frac{2}{\pi}x\right)dx=\frac{4-\pi}{4}.$$

Hence,

$$x_c=\frac{\frac{1}{2}\int\limits_0^{\frac{\pi}{2}}\left(\sin^2 x-\frac{4}{\pi^2}x^2\right)dx}{\frac{4-\pi}{4}}=\frac{2}{4-\pi}\int\limits_0^{\frac{\pi}{2}}\left(\sin^2 x-\frac{4}{\pi^2}x^2\right)dx=$$

$$=\frac{2}{4-\pi}\left[\frac{1}{2}x-\frac{\sin 2x}{4}-\frac{4}{3\pi^2}x^3\right]\Bigg|_0^{\frac{\pi}{2}}=\frac{\pi}{6(4-\pi)};$$

$$y_c=\frac{\int\limits_0^{\frac{\pi}{2}}x\left(\sin x-\frac{2}{\pi}x\right)dx}{\frac{4-\pi}{4}}=\frac{4}{4-\pi}\int\limits_0^{\frac{\pi}{2}}x\sin x\,dx-$$

$$-\frac{8}{\pi(4-\pi)}\int\limits_0^{\frac{\pi}{2}}x^2\,dx=\frac{4}{4-\pi}-\frac{\pi^2}{3(4-\pi)}=\frac{12-\pi^2}{12-3\pi}.$$

**7.13.22.** Prove the following theorems (Guldin's theorems).

Theorem 1. *The area of a surface obtained by revolving an arc of a plane curve about some axis lying in the plane of the curve and not intersecting it is equal to the product of the length of the curve by the circumference of the circle described by the centre of gravity of the arc of the curve.*

Theorem 2. *The volume of a solid obtained by revolving a plane figure about some axis lying in the plane of the figure and not intersecting it is equal to the product of the area of this figure by the circumference of the circle described by the centre of gravity of the figure.*

*Proof.* (1) Compare the formula for the area of the surface of revolution of the curve $L$ about the $x$-axis (see § 7.10)

$$P=2\pi\int\limits_L y\,dl$$

with that for the ordinate of the centre of gravity of this curve

$$y_c=\frac{M_x}{l}=\frac{1}{l}\int\limits_L y\,dl.$$

Hence we conclude that

$$P=2\pi\, l y_c=l\cdot 2\pi y_c,$$

where $l$ is the length of the revolving arc, and $2\pi y_c$ is the length of a circle of radius $y_c$, i.e. the length of the circle described by the centre of gravity when revolving about the $x$-axis.

(2) Compare the formula for the volume of a solid generated by revolving a plane figure about the $x$-axis (see § 7.6)

$$V=\pi\int_a^b (y_2^2-y_1^2)\,dx$$

with that for the ordinate of the centre of gravity of this figure

$$y_c=\frac{M_x}{S}=\frac{1}{2S}\int_a^b (y_2^2-y_1^2)\,dx.$$

Hence we conclude that

$$V=\pi\cdot 2S\,y_c=S\cdot 2\pi y_c$$

where $S$ is the area of the revolving figure, and $2\pi y_c$ is the length of the circumference described by the centre of gravity when revolving about the $x$-axis.

**7.13.23.** Using the first Guldin theorem, find the centre of gravity of a semicircle of radius $a$.

*Solution.* Arrange the coordinate axes as shown in Fig. 116. By virtue of symmetry $x_c=0$. Now it remains to find $y_c$. If the semicircle revolves about the $x$-axis, then the surface $P$ of the solid of revolution is equal to $4\pi a^2$, and the arc length $l=\pi a$. Therefore, according to the first Guldin theorem,

$$4\pi a^2=\pi a\cdot 2\pi y_c;\quad y_c=2\frac{a}{\pi}.$$

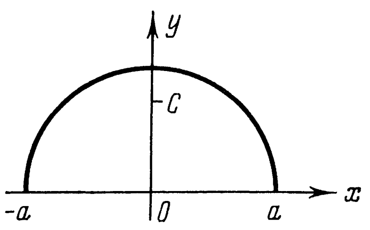

Fig. 116

**7.13.24.** Using the second Guldin theorem, find the coordinates of the centre of gravity of the figure bounded by the $x$-axis and one arc of the cycloid: $x=a(t-\sin t)$; $y=a(1-\cos t)$.

*Solution.* By virtue of the symmetry of the figure about the straight line $x=\pi a$ its centre of gravity lies on this straight line; hence, $x_c=\pi a$.

The volume $V$ obtained by revolving this figure about the $x$-axis is equal to $5\pi^2a^3$ (see Problem **7.6.14**), the area $S$ of the figure being equal to $3\pi a^2$ (see Problem **7.4.3**). Using the second Guldin theorem, we get

$$y_c=\frac{V}{2\pi S}=\frac{5\pi^2a^3}{2\pi\cdot 3\pi a^2}=\frac{5a}{6}.$$

**7.13.25.** An equilateral triangle with side $a$ revolves about an axis parallel to the base and situated at a distance $b>a$ from the base. Find the volume of the solid of revolution.

*Solution.* There are two possible ways of arranging the triangle with respect to the axis of revolution which are shown in Fig. 117, *a* and *b*.

The altitude of the equilateral triangle is $h=\frac{a\sqrt{3}}{2}$, the area $S=\frac{a^2\sqrt{3}}{4}$. The centre of gravity $O'$ is situated at the point of intersection of the medians and at a distance of $b-\frac{a\sqrt{3}}{6}$ from the axis of revolution in the first case, and $b+\frac{a\sqrt{3}}{6}$ in the second.

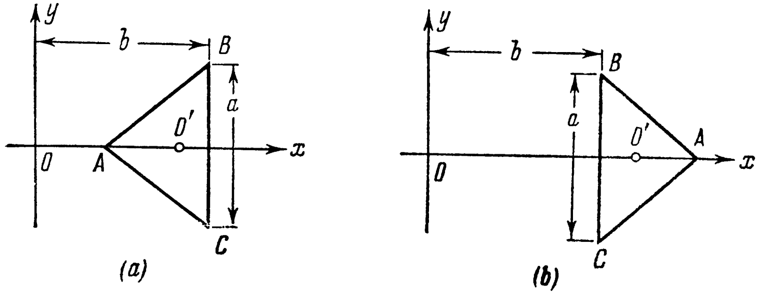


Fig. 117

By the second Guldin theorem

$$V_1=\frac{2\pi a^2\sqrt{3}}{4}\left(b-\frac{a\sqrt{3}}{6}\right)=\pi\left(\frac{a^2b\sqrt{3}}{2}-\frac{a^3}{4}\right),$$

$$V_2=\frac{2\pi a^2\sqrt{3}}{4}\left(b+\frac{a\sqrt{3}}{6}\right)=\pi\left(\frac{a^2b\sqrt{3}}{2}+\frac{a^3}{4}\right).$$

**7.13.26.** Find the centre of gravity of the arc of a circle of radius $R$ subtending a central angle $2\alpha$.

**7.13.27.** Find the centre of gravity of the figure bounded by the arc of the cosine line $y=\cos x$ between $x=-\frac{\pi}{3}$ and $x=\frac{\pi}{3}$ and the straight line $y=\frac{1}{2}$.

**7.13.28.** Find the coordinates of the centre of gravity of the figure enclosed by line $y^2=ax^3-x^4$.

**7.13.29.** Find the Cartesian coordinates of the centre of gravity of the arc of the logarithmic spiral $\rho=ae^{\varphi}$ from $\varphi_1=\frac{\pi}{2}$ to $\varphi_2=\pi$.

**7.13.30.** A regular hexagon with side $a$ revolves about one of its sides. Find the volume of the solid of revolution thus generated.

**7.13.31.** Using Guldin's theorem, find the centre of gravity of a semicircle of radius $R$.

## § 7.14. Additional Problems

**7.14.1.** Find the area of the portion of the figure bounded by the curves $y^m = x^n$ and $y^n = x^m$ ($m$ and $n$ positive integers) situated in the first quadrant. Consider the area of the entire figure depending on whether the numbers $m$ and $n$ are even or odd.

**7.14.2.** (a) Prove that the area of the curvilinear trapezoid bounded by the $x$-axis, straight lines $x=a$, $x=b$ and parabola $y=Ax^3+Bx^2+Cx+D$ can be computed using Chebyshev's formula

$$S=\frac{b-a}{3}\left[y\left(\frac{a+b}{2}-\frac{1}{\sqrt{2}}\frac{b-a}{2}\right)+y\left(\frac{a+b}{2}\right)+y\left(\frac{a+b}{2}+\frac{1}{\sqrt{2}}\frac{b-a}{2}\right)\right].$$

(b) Prove that an analogous area for a parabola of the fifth order

$$y=f(x)=Ax^5+Bx^4+Cx^3+Dx^2+Ex+F$$

can be computed using the Gauss formula

$$S=\frac{b-a}{9}\left[5f\left(\frac{a+b}{2}-\sqrt{\frac{3}{5}}\frac{b-a}{2}\right)+8f\left(\frac{a+b}{2}\right)+\right.$$
$$\left.+5f\left(\frac{a+b}{2}+\sqrt{\frac{3}{5}}\frac{b-a}{2}\right)\right].$$

**7.14.3.** Show that the area of a figure bounded by any two radius vectors of the logarithmic spiral $\rho=ae^{m\varphi}$ and its arc is proportional to the difference of the squares of these radii.

**7.14.4.** Prove that if two solids contained between parallel planes $P$ and $Q$ possess the property that on being cut by any plane $R$ parallel to these planes equivalent figures are obtained in their section, then the volumes of these solids are equal (Cavalieri's principle).

**7.14.5.** Prove that if the function $S(x)$ $(0\leqslant x\leqslant h)$ expressing the area of the section of a solid by a plane perpendicular to the $x$-axis is a polynomial of a degree not higher than three, then the volume of this solid is equal to $V=\frac{h}{6}\left[S(0)+4S\left(\frac{h}{2}\right)+S(h)\right]$. Using this formula, deduce formulas for computing the volume of a sphere, spherical segments of two and one bases, cone, frustrum of a cone, ellipsoid, and paraboloid of revolution.

**7.14.6.** Prove that the volume of a solid generated by revolving about the $y$-axis the figure $a\leqslant x\leqslant b$, $0\leqslant y\leqslant y(x)$, where $y(x)$ is a single-valued continuous function, is equal to

$$V=2\pi\int_a^b xy(x)\,dx.$$

**7.14.7.** Prove that the volume of the solid formed by revolving, about the polar axis, a figure $0 \leqslant \alpha \leqslant \varphi \leqslant \beta \leqslant \pi$, $0 \leqslant \rho \leqslant \rho(\varphi)$, is equal to

$$V = \frac{2\pi}{3} \int_{\alpha}^{\beta} \rho^3(\varphi) \sin\varphi \, d\varphi.$$

**7.14.8.** Prove that the arc length of the curve given by the parametric equations

$$\begin{aligned} x &= f''(t)\cos t + f'(t)\sin t, \\ y &= -f''(t)\sin t + f'(t)\cos t \end{aligned} \qquad (t_1 \leqslant t \leqslant t_2)$$

is equal to $[f(t) + f''(t)]_{t_1}^{t_2}$.

**7.14.9.** Find the arc length of the curve represented parametrically

$$x = \int_1^t \frac{\cos z}{z} dz, \qquad y = \int_1^t \frac{\sin z}{z} dz$$

between the origin and the nearest point from the vertical tangent line.

**7.14.10.** Deduce the formula for the arc length in polar coordinates proceeding from the definition without passing over from Cartesian coordinates to polar ones.

**7.14.11.** Prove that the arc length $l(x)$ of the catenary $y = \cosh x$ measured from the point $(0, 1)$ is expressed by the formula $l(x) = \sinh x$ and find parametric equations of this line, using the arc length as the parameter.

**7.14.12.** A flexible thread is suspended at the points $A$ and $B$ located at one and the same height. The distance between the points is $AB = 2b$, the deflection of the thread is $f$. Assuming the suspended thread to be a parabola, show that the length of the thread

$$l = 2b\left(1 + \frac{2}{3}\frac{f^2}{b^2}\right)$$

at a sufficiently small $\frac{f}{b}$.

**7.14.13.** Find the ratio of the area enclosed by the loop of the curve $y = \pm\left(\frac{1}{3} - x\right)\sqrt{x}$ to the area of the circle, whose circumference is equal in length to the contour of the curve.

**7.14.14.** Compute the length of the arc formed by the intersection of the parabolic cylinder

$$(y+z)^2 = 4ax$$

and the elliptic cone

$$\frac{4}{3}x^2+y^2-z^2=0,$$

between the origin and the point $M(x, y, z)$.

**7.14.15.** Prove that the area of the ellipse

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0 \quad (AC-B^2>0)$$

is equal to

$$S=-\frac{\pi\Delta}{(AC-B^2)^{3/2}}, \quad \text{where } \Delta=\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}.$$

**7.14.16.** Find: (a) the area $S$ of the figure bounded by the hyperbola $x^2-y^2=1$, the positive part of the $x$-axis and the radius vector connecting the origin of coordinates and the point $M(x, y)$ lying on this hyperbola.

(b) The area of the circular sector $Q$ bounded by the $x$-axis and the radius drawn from the centre to the point $N(x, y)$ lying on the circle $x^2+y^2=1$. Prove that the coordinates of the points $M$ and $N$ are expressed respectively through the areas $S$ and $Q$ by the formulas

$$x_M=\cosh 2S, \quad y_M=\sinh 2S, \quad x_N=\cos 2Q, \quad y_N=\sin 2Q.$$

**7.14.17.** Using Guldin's theorem, prove that the centre of gravity of a triangle is one third of the altitude distant from its base.

**7.14.18.** Let $\xi$ be the abscissa of the centre of gravity of a curvilinear trapezoid bounded by the continuous curve $y=f(x)$, the $x$-axis and the straight lines $x=a$ and $x=b$. Prove the validity of the following equality:

$$\int_a^b (ax+b)f(x)\,dx=(a\xi+b)\int_a^b f(x)\,dx$$

(Vereshchagin's rule).

**7.14.19.** Let a curvilinear sector be bounded by two radius vectors and a continuous curve $\rho=f(\varphi)$. Prove that the coordinates of the centre of gravity of this sector are expressed by the following formulas:

$$x_c=\frac{2}{3}\frac{\int_{\varphi_1}^{\varphi_2}\rho^3\cos\varphi\,d\varphi}{\int_{\varphi_1}^{\varphi_2}\rho^2\,d\varphi}; \quad y_c=\frac{2}{3}\frac{\int_{\varphi_1}^{\varphi_2}\rho^3\sin\varphi\,d\varphi}{\int_{\varphi_1}^{\varphi_2}\rho^2\,d\varphi}.$$

**7.14.20.** Prove that the Cartesian coordinates of the centre of gravity of an arc of the curve $\rho = f(\varphi)$ are expressed by the following formulas:

$$x_c = \frac{\int\limits_{\varphi_1}^{\varphi_2} \rho \cos\varphi \sqrt{\rho^2 + \rho'^2}\, d\varphi}{\int\limits_{\varphi_1}^{\varphi_2} \sqrt{\rho^2 + \rho'^2}\, d\varphi}; \quad y_c = \frac{\int\limits_{\varphi_1}^{\varphi_2} \rho \sin\varphi \sqrt{\rho^2 + \rho'^2}\, d\varphi}{\int\limits_{\varphi_1}^{\varphi_2} \sqrt{\rho^2 + \rho'^2}\, d\varphi}.$$