CBSE Test Paper 01

Chapter 8 Application of Integrals

1. The area bounded by the curves $y^2 = 20x$ and $x^2 = 16y$ is equal to
   a. $\frac{320}{3}\ sq.\ units$
   b. $80\pi\ sq.\ units$
   c. none of these
   d. $100\pi\ sq.\ units$
2. The area of the region bounded by the parabola (y - 2)^2 = x - 1, the tangent to the parabola at the point (2 , 3) and the x – axis is equal to
   a. none of these
   b. 6 sq. units
   c. 9 sq. units
   d. 12 sq. units
3. The area bounded by the curves $y = \sqrt{x}$, 2y + 3 = xand the x – axis in the first quadrant is
   a. 36
   b. 18
   c. 9
   d. none of these
4. If the area cut off from a parabola by any double ordinate is k times the corresponding rectangle contained by that double ordinate and its distance from the vertex, then k is equal to
   a. $\frac{2}{3}$
   b. 3
   c. $\frac{1}{3}$
   d. $\frac{3}{2}$
5. The area bounded by the curves y = cos x and y = sin x between the ordinates x = 0

and $x = \frac{\pi}{2}$ is equal to

a. $2(\sqrt{2}+1)$ sq. units
b. $2(\sqrt{2}-1)$ sq. units
c. $\left(4\sqrt{2}-1\right)$ sq. units
d. $\left(4\sqrt{2}+1\right)$ sq. units

6. The area of the bounded by the lines y = 2, x = 1, x = a and the curve y = f(x), which cuts the last two lines above the first line for all $a \geq 1$, is equal to $\frac{2}{3}\left[(2a)^{3/2} - 3a + 3 - 2\sqrt{2}\right]$. Find f(x)

7. Let f(x) be a continuous function such that the area bounded by the curve y=f(x), x-axis and the lines x=0 and x=a is $\frac{a^2}{2} + \frac{a}{2} sin\ a + \frac{\pi}{2}\ cos\ a$, then find $f\left(\frac{\pi}{2}\right)$.

8. Find the area of the region enclosed by the curves y = x , x = e, y = $\frac{1}{x}$ and the positive x-axis.

9. Calculate the area of the region enclosed between the circles: $x^2 + y^2 = 16$ and $(x + 4)^2 + y^2 = 16$.

10. Using integration, find the area of region bounded by the triangle whose vertices are (-1, 0), (1, 3) and (3, 2).

11. Find the area of the region $\left\{(x, y)\,;\, x^2 \leqslant y \leqslant x\right\}$.

12. Evaluate $\lim_{x\to\infty}\left(\frac{x^x}{x!}\right)^{1/x}$.

13. Evaluate $\lim_{x\to\infty}\left[\frac{1}{x} + \frac{x^2}{(x+1)^3} + \frac{x^2}{(x+2)^3} + \ldots\ldots\ldots + \frac{1}{8x}\right]$.

14. Find the area of the region enclosed by the parabola $x^2 = y$ and the line y=x + 2.

15. Using integration, find the area of the region enclosed between the two circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

Solution

1. (a) $\frac{320}{3}\ sq.\ units$
   Explanation: Eliminating y, we get: $x^4 = 256 \times 20x$
   $\Rightarrow x = 0, x = 8(10)^{\frac{1}{3}}$
   Required area:
   $= \int\limits_0^{8(10)^{\frac{1}{3}}} \left(\sqrt{20x} - \frac{x^2}{16}\right) dx$
   $= \frac{640}{3} - \frac{320}{3} = \frac{320}{3}$ sq units
2. (c) 9 sq. units
   Explanation: Given parabola is: $(y-2)^2 = x - 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2(y-2)}$
   When y= 3, x= 2
   $\therefore \frac{dy}{dx} = \frac{1}{2}$

   Therefore, tangent at (2, 3) is y – 3 = ½ (x – 2). i.e. x – 2y +4 = 0 . therefore required area is: $\int\limits_0^3 (y-2)^2 + 1.dy - \int\limits_0^3 (2y-4)dy = \left[\frac{(y-2)^3}{3} + y\right]_0^3 - \left[y^2 - 4y\right]_0^3 = 9$
3. (c) 9
   Explanation: Required area: $\int\limits_0^9 \sqrt{x}\, dx - \int\limits_3^9 \left(\frac{x-3}{2}\right) dx$
   $= \left[\frac{x^{\frac{3}{2}}}{3/2}\right]_0^9 - \frac{1}{2}\left[\frac{x^2}{2} - 3x\right]_3^9 = 9 sq. units$
4. (a) $\frac{2}{3}$
   Explanation: Required area: $2\int\limits_0^a \sqrt{4ax}\, dx$
   $= k\alpha(2\sqrt{4a\alpha})$
   $= \frac{8\sqrt{a}}{3}\alpha^{\frac{3}{2}}$
   $= 4\sqrt{a}k\alpha^{\frac{3}{2}} \Rightarrow k = \frac{2}{3}$
5. (b) $2(\sqrt{2} - 1)$sq. units

Explanation: Required area = $\int\limits_0^{\frac{\pi}{2}} |\sin x - \cos x|\, dx$

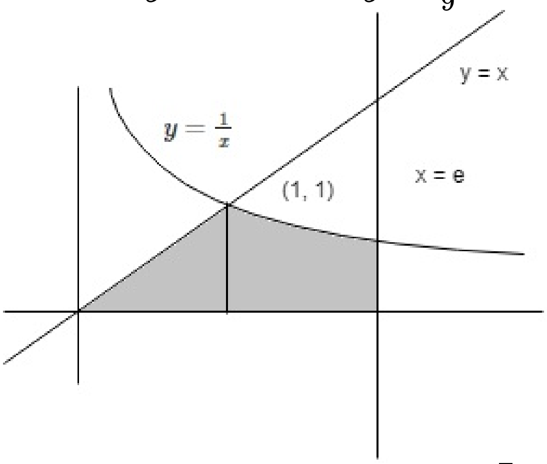
$= \int\limits_0^{\frac{\pi}{4}} (\cos x - \sin x) dx + \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} (sinx - \cos x) dx$

$= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-cosx - sinx]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - (0+1) - \left\{1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right\}$

$= \frac{4}{\sqrt{2}} - 2 = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$

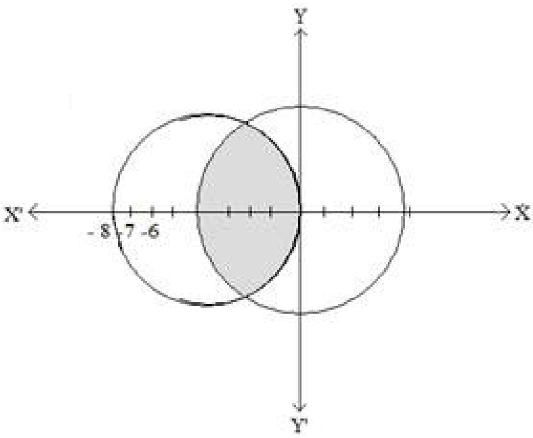
6. we are given,
   $\int_a^1 [f(x) - 2] dx = \frac{2}{3}\left[(2a)^{3/2} - 3a + 3 - 2\sqrt{2}\right]$
   Differentiating w.r.t a, we get
   f(a) - 2 $= \frac{2}{3}\left[\frac{3}{2}\sqrt{2a}.2 - 3\right]$
   f(a)= 2$\sqrt{2a}, a \geq 1$
   $\therefore\ f(x) = 2\sqrt{2x}, x \geq 1$
7. we have, $\int_0^a f(x) dx = \frac{a^2}{2} + \frac{a}{2} sin\ a + \frac{\pi}{2} cos\ a$
   Differentiating w.r.t a,we get,
   f(a)=a+ $\frac{1}{2}(sin\ a + acos\ a) - \frac{\pi}{2} sin\ a$
   put a=$\frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \frac{1}{2} - \frac{\pi}{2} = \frac{1}{2}$
8. We have $y = 4x^2\ and\ y = \frac{1}{9}x^2$



Required area =$2\int_0^2 \left(3\sqrt{y} - \frac{\sqrt{y}}{2}\right) dy$

$= 2\left(\frac{5y}{2}\frac{\sqrt{y}}{3/2}\right)_0^2$

$= 2.\frac{5}{3}2\sqrt{2} = \frac{20\sqrt{2}}{3}$

9. 

$x^2 + y^2 = 16$

$(x + 4)^2 + y^2 = 16$

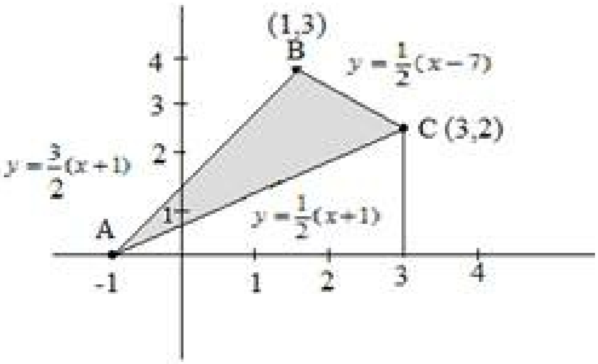
Intersecting at x = -2

Area$= 4\int_{-4}^{-2} \sqrt{16 - x^2}dx$

$= 4\left[\int_{-4}^{-2} \sqrt{4^2 - x^2}dx\right] = 4\left[\frac{x}{2}\sqrt{1 - x^2} + \frac{4^2}{2}sin^{-1}\frac{x}{4}\right]_{-4}^{-2}$

$= 4\left[(-2\sqrt{3} - \frac{4\pi}{3}) - (-4\pi)\right]$

$= \left(-8\sqrt{3} + \frac{32\pi}{3}\right)$

10. 

A (-1, 0) B (1, 3) C (3, 2)

Equation of AB

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$y - 0 = \frac{3-0}{1+1}(x + 1)$

$y = \frac{3}{2}(x + 1)$

Similarly,

Equation of BC $y = \frac{-1}{2}(x - 7)$

Equation of AC $= \frac{1}{2}(x + 1)$

Area $\Delta ABC = \int_{-1}^{1} \frac{3}{2}(x + 1)\,dx + \int_{1}^{3} \frac{1}{2}(x - 7)\,dx - \int_{-1}^{3} \frac{1}{2}(x + 1)\,dx$

$= \frac{3}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{1} + \frac{1}{2}\left[7x - \frac{x^2}{2}\right]_{1}^{3} - \left[\frac{x^2}{2}+x\right]_{-1}^{3}$

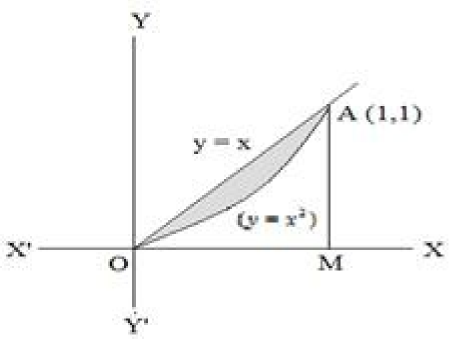
$= \frac{3}{2}\left[\left(\frac{1}{2}+1\right) - \left(\frac{1}{2}-1\right)\right] + \frac{1}{2}\left[\left(21-\frac{9}{2}\right) - \left(7-\frac{1}{2}\right]\right.$

$-\frac{1}{2}\left[\left(\frac{9}{2}+3\right) - \left(\frac{1}{2}-1\right)\right]$

$= \frac{3}{2}(2) + \frac{1}{2}(10) - \frac{1}{2}(8) = 3+5-4$

= 4 sq. units

11. y = x^2



y = x

$\Rightarrow$ x = 0, y = 0

x = 1, y = 1

Area $= \int_0^1 x dx - \int_0^1 x^2 dx$

$= \int_0^1 (x - x^2) dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

$= \frac{1}{2} - \frac{1}{3}$

$= \frac{1}{6}$ sq. units

12. Given $L = \lim_{x\to\infty}\left(\frac{x^x}{x!}\right)^{1/x}$

Taking logarithm on both sides

$log\ L = \lim_{x\to\infty} \frac{1}{x}\left(log\frac{x}{1} + log\frac{x}{2} + \ldots\ldots + log\frac{x}{x}\right)$

= $\lim_{x\to\infty} \frac{1}{x} \sum_{r=1}^{x} log\ \frac{x}{r}$

=$\lim_{x\to\infty} \frac{1}{x} \sum_{r=1}^{x} log\ \frac{1}{(r/x)}$

$= \int_0^1 log\frac{1}{x}\ dx$

$= -\int_0^1 log\ x\ dx$

$= -[xlog\ x + x]_0^1$

$= -[(1log\ 1 + 1) - (0\log 0 - 0)]$ = 1

$\therefore Log\ L = 1$

$\Rightarrow L = e$

$\Rightarrow \lim_{x\to\infty}\left(\frac{x^x}{x!}\right)^{1/x} = e$

13. Given, $\lim_{x\to\infty}\left[\frac{1}{x} + \frac{x^2}{(x+1)^3} + \frac{x^2}{(x+2)^3} + \ldots\ldots\ldots + \frac{1}{8x}\right]$

$= \lim_{x\to\infty}\sum_{r=0}^{x}\frac{x^2}{(x+r)^3}$

$= \lim_{x\to\infty}\sum_{r=0}^{x}\frac{1/x}{(1+r/x)^2}$

$= \int_0^1 \frac{dy}{(1+y)^3}$, replace $\frac{r}{x}$ by y and $\frac{1}{x}$ by dy

$= \left[\frac{-1}{2(1+y)^2}\right]_0^1$

$= \left[\frac{-1}{2(1+1^2)} - \frac{-1}{2(1+0^2)}\right]$

$= \left[\frac{-1}{2(2)} - \frac{-1}{2(1)}\right]$

$= \left[\frac{-1}{4} - \frac{-1}{2}\right] = \frac{1}{4}$

14. We have, x^2 = y and y = x + 2

$\Rightarrow x^2 = x + 2$

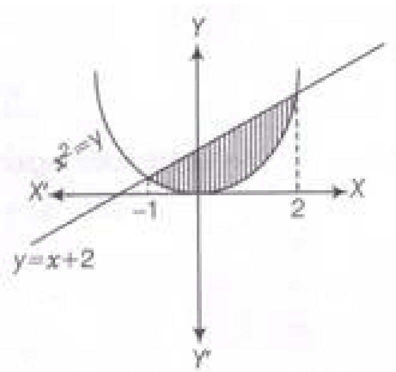
$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow x^2 - 2x + x - 2 = 0$

$\Rightarrow x(x-2) + 1(x-2) = 0$

$\Rightarrow (x+1)(x-2) = 0$

$\Rightarrow x = -1, 2$



$\therefore$ Required area of shaded region, $= \int_{-1}^{2}\left(x + 2 - x^2\right)dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$

$= \left(8 - 3 - \frac{1}{2}\right) = \frac{9}{2}$

15. Given circles are $x^2 + y^2 = 4$...(i)

$(x-2)^2 + y^2 = 4$...(ii)

Eq. (i) is a circle with centre origin and

Radius = 2.

Eq. (ii) is a circle with centre C (2, 0) and

Radius = 2.

On solving Eqs. (i) and (ii), we get

$(x-2)^2+y^2=x^2+y^2$

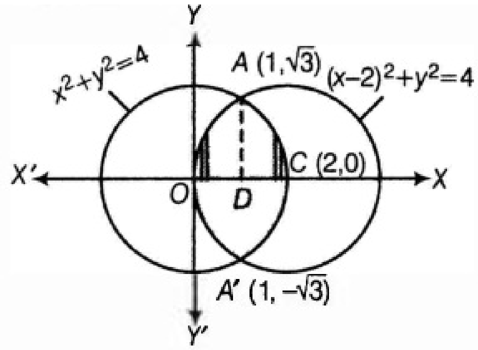
$\Rightarrow x^2 - 4x+4+y^2=x^2+y^2$

$\Rightarrow x=1$

On putting x = 1 in Eq. (i), we get

$y=\pm\sqrt{3}$

Thus, the points of intersection of the given circles are A (1, $\sqrt{3}$) and A'(1,-$\sqrt{3}$).



Clearly, required area= Area of the enclosed region OACA'O between circles

= 2 [Area of the region ODCAO]

=2 [Area of the region ODAO + Area of the region DCAD]

$=2\left[\int_0^1 y_2\,dx+\int_1^2 y_1\,dx\right]$

$=2\left[\int_0^1 \sqrt{4-(x-2)^2}dx+\int_1^2 \sqrt{4-x^2}dx\right]$

$=2\left[\frac{1}{2}(x-2)\sqrt{4-(x-2)^2}+\frac{1}{2}\times 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$

$+2\left[\frac{1}{2}x\sqrt{4-x^2}+\frac{1}{2}\times 4\sin^{-1}\frac{x}{2}\right]_1^2$

$=\left[(x-2)\sqrt{4-(x-2)^2}+4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1+\left[x\sqrt{4-x^2}+4\sin^{-1}\frac{x}{2}\right]_1^2$

$=\left[\left\{-\sqrt{3}+4\sin^{-1}\left(\frac{-1}{2}\right)\right\}-0-4\sin^{-1}(-1)\right]+\left[0+4\sin^{-1}1-\sqrt{3}-4\sin^{-1}\frac{1}{2}\right]$

$=\left[\left(-\sqrt{3}-4\times\frac{\pi}{6}\right)+4\times\frac{\pi}{2}\right]+\left[4\times\frac{\pi}{2}-\sqrt{3}-4\times\frac{\pi}{6}\right]$

$=\left(-\sqrt{3}-\frac{2\pi}{3}+2\pi\right)+\left(2\pi-\sqrt{3}-\frac{2\pi}{3}\right)$

$=\frac{8\pi}{3}-2\sqrt{3}$ sq units.