5

Differentiation

Contents :

5.1 Introduction

We have studied about functions in Standard 11. Let $f(x)$ be a function of x. Differentiation is a technique which is used for analyzing the way in which function $f(x)$ changes and how much does it change with a change in the value of x. That is, we can know how rapidly a function is changing at any point using differentiation. In real life, we have functions like production cost, revenue, profit, etc. and it is often important to know how rapidly these functions change with respect to change in produced units or sold units x.

Consider $y = f(x) = 2x^2 + 3$, a function of x. If the value of independent variable (x) is changed there will be a corresponding change in the dependent variable (y). If the value of x is 2 then the value of dependent variable y will be 11. Now we shall find the increase in y for a small increase in x. For a small increase in value of x, i.e. if we take values of x as 2.1, 2.01, 2.001, 2.0001, ... etc then we get corresponding value of y as 11.82, 11.082, 11.008, 11.0008, ..., etc. We denote the increase in x by δ_x and increase in y by δ_y. The ratio $\frac{\delta_y}{\delta_x}$ is termed as incrementary ratio. Let us observe this incrementary ratio for the above values of x and corresponding values of y.

x	δ_x	$y = f(x)$	δ_y	$\frac{\delta_y}{\delta_x}$
2.1	0.1	11.82	0.82	8.2
2.01	0.01	11.0802	0.0802	8.02
2.001	0.001	11.0080	0.0080	8.002
2.0001	0.0001	11.0008	0.0008	8.0002
. . .	. . .	. . .	. . .	. . .

We make the following observations from the above table :

(i) δ_y varies when δ_x varies

(ii) $\delta_y \to 0$ when $\delta_x \to 0$

(iii) The ratio $\frac{\delta_y}{\delta_x}$ tends to 8.

Hence, this example illustrates that $\delta_y \to 0$ when $\delta_x \to 0$ but $\frac{\delta_y}{\delta_x}$ tends to a finite value, not necessarily zero. The limit of $\frac{\delta_y}{\delta_x}$ is represented by $\frac{dy}{dx}$ and is called the derivative of y with respect to x.

In the above example $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta_y}{\delta_x} = 8$

In many business problems, we are interested in the rate of change of a function and, in particular, the range of values of independent variable for which the rate of change of a function may be positive or negative.

Differentiation is used in production, replacement, pricing and other management decision problems.

In short, differentiation is used to determine the rate of change in the dependent variable (function of independent variable) with respect to the independent variable.

5.2 Definition : Differentiation and Derivative

Let us consider a function $y = f(x)$.

When we take $x = a$, the value of the function will be $f(a)$. Now, when the value of x changes from a to $a + h$, the value of the function will change from $f(a)$ to $f(a+h)$. So, for a change of $(a+h) - a = h$ in the value of x, there is a change of $f(a+h) - f(a)$ in the value of $f(x)$. If there is a change of h in value of a then the relative change in the function will be $\frac{f(a+h)-f(a)}{h}$. If h is made very small then the limit of this relative change is called derivative of $f(x)$ with respect to x at $x = a$ and it is denoted by $f'(a)$.

Definition : Let $f : A \to R$ and $a \in A$, where A is an open interval of R. If $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists, then this limit of a function f is called **derivative** at $x = a$. It is denoted by $f'(a)$.

The process of obtaining derivative of a function is called **differentiation**.

Thus, $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.

For any value of x of the domain of f, we have $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. $f'(x)$ is called a derivative of $f(x)$ with respect to x.

If y is a function of x then its derivative is denoted by $\frac{dy}{dx}$.

We shall now find derivatives of some functions using this definition of derivative.

Illustration 1 : Obtain the derivative of $f(x) = x$ with the help of definition.

Here, $f(x) = x$

$\therefore \ f(x+h) = x + h$

Now, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{(x+h)-x}{h}$$

$$= \lim_{h\to 0} \frac{h}{h}$$

$$= 1 \qquad (\because \ h \neq 0)$$

Hence, if $f(x) = x$ then $f'(x) = 1$.

Illustration 2 : Obtain derivative of $f(x) = x^3$ with the help of definition.

Here, $f(x) = x^3$

$$\therefore \ f(x+h) = (x+h)^3$$

$$= x^3 + 3x^2h + 3xh^2 + h^3$$

Now, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2 \qquad (\because \ h \neq 0)$$

$$= 3x^2 + 3x(0) + (0)^2$$

$$= 3x^2$$

Hence, if $f(x) = x^3$ then $f'(x) = 3x^2$

Illustration 3 : Obtain derivative of $f(x) = x^n$ with the help of definition.

Here, $f(x) = x^n$

$$\therefore \ f(x+h) = (x+h)^n$$

Now, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

(Taking $x + h = t$, when $h \to 0$ then $t \to x$)

$$= \lim_{t \to x} \frac{t^n - x^n}{t - x} \qquad (\because \ x + h = t)$$

$$= nx^{n-1} \quad \left(\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right)$$

Hence, if $f(x) = x^n$ then $f'(x) = nx^{n-1}$

Illustration 4 : Obtain derivative of $f(x) = \sqrt{x}$ with the help of definition.

Here, $f(x) = \sqrt{x}$

$\therefore \ f(x+h) = \sqrt{x+h}$

Now, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

(Multiplying numerator and denominator by $\sqrt{x+h} + \sqrt{x}$)

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \qquad (\because \ h \neq 0)$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Hence, if $f(x) = \sqrt{x}$ then $f'(x) = \frac{1}{2\sqrt{x}}$

Illustration 5 : Obtain derivative of $f(x) = \frac{1}{x}$ with the help of definition.

Here, $f(x) = \frac{1}{x}$

$$\therefore \quad f(x+h) = \frac{1}{x+h}$$

Now, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{x - x - h}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-h}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)} \qquad (\because h \neq 0)$$

$$= \frac{-1}{x(x+0)}$$

$$= \frac{-1}{x^2}$$

Hence, if $f(x) = \frac{1}{x}$ then $f'(x) = \frac{-1}{x^2}$

Illustration 6 : Obtain derivative of $f(x) = k$ (k is constant) with the help of definition.

Here, $f(x) = k$

$$\therefore \quad f(x+h) = k$$

Now, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{k - k}{h}$$

$$= \lim_{h \to 0} \frac{0}{h}$$

$$= 0$$

Hence, if $f(x) = k$ then $f'(x) = 0$

EXERCISE 5.1

Obtain the derivatives of the following functions with the help of definition :

1. $f(x) = 2x + 3$

2. $f(x) = x^2$

3. $f(x) = x^7$

4. $f(x) = \frac{1}{x+1}, \quad x \neq -1$

5. $f(x) = \sqrt[3]{x}$

6. $f(x) = \frac{2}{3x-4}, \quad x \neq \frac{4}{3}$

7. $f(x) = 10$

*

5.3 Some Standard Derivatives

We shall use derivatives of following functions.

1. If $y = x^n$ (where $n \in R$ and $x \in R^+$)

then $\frac{dy}{dx} = nx^{n-1}$

2. If $y = k$ (where k is constant)

then $\frac{dy}{dx} = 0$

5.4 Working Rules for Differentiation

We shall accept certain working rules for differentiation without proof.

If u and v are differentiable functions of x then,

Rule 1 : If $y = u \pm v$ then

$$\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

Rule 2 : If $y = u \cdot v$ then

$$\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

Rule 3 : If $y = \frac{u}{v}$, $v \neq 0$ then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Rule 4 : **(Chain Rule)**

If y is a differentiable function of u and u is a differentiable function of x then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

We shall see some illustrations explaining the use of working rules for differentiation mentioned above.

Illustration 7 : Find $\frac{dy}{dx}$ for $y = x^4 - 3x^2 + 2x - 3$.

$$y = x^4 - 3x^2 + 2x - 3$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(x^4 - 3x^2 + 2x - 3\right)$$

$$= \frac{d}{dx}\left(x^4\right) - \frac{d}{dx}\left(3x^2\right) + \frac{d}{dx}(2x) - \frac{d}{dx}(3)$$

$$= \frac{d}{dx}\left(x^4\right) - 3\frac{d}{dx}\left(x^2\right) + 2\frac{d}{dx}(x) - \frac{d}{dx}(3)$$

$$= 4x^3 - 3(2x) + 2(1) - (0)$$

$$= 4x^3 - 6x + 2$$

Illustration 8 : Find $\frac{dy}{dx}$ for $y = x^3 + \sqrt{x} - \frac{4}{x} + \frac{1}{\sqrt[3]{x}} + \frac{1}{4}$.

$$y = x^3 + \sqrt{x} - \frac{4}{x} + \frac{1}{\sqrt[3]{x}} + \frac{1}{4}$$

$$= x^3 + x^{\frac{1}{2}} - 4x^{-1} + x^{-\frac{1}{3}} + \frac{1}{4}$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(x^{\frac{1}{2}}\right) - 4\frac{d}{dx}\left(x^{-1}\right) + \frac{d}{dx}\left(x^{-\frac{1}{3}}\right) + \frac{d}{dx}\left(\frac{1}{4}\right)$$

$$= 3x^2 + \frac{1}{2}x^{\frac{1}{2}-1} - 4\left(-1\,x^{-1-1}\right) + \left(\frac{-1}{3}\right)x^{\frac{-1}{3}-1} + 0$$

$$= 3x^2 + \frac{1}{2}x^{-\frac{1}{2}} + 4x^{-2} - \frac{1}{3}x^{-\frac{4}{3}}$$

$$= 3x^2 + \frac{1}{2x^{\frac{1}{2}}} + \frac{4}{x^2} - \frac{1}{3x^{\frac{4}{3}}}$$

Illustration 9 : If $y = \left(2x^2 + 3\right)(3x - 2)$ then find derivative of y with respect to x.

$$y = \left(2x^2 + 3\right)(3x - 2)$$

Take, $u = 2x^2 + 3$ and $v = 3x - 2$.

$$\therefore \quad \frac{du}{dx} = 4x \quad \text{and} \quad \frac{dv}{dx} = 3$$

Now, $y = u \cdot v$.

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= \left(2x^2 + 3\right)(3) + (3x - 2)(4x)$$

$$= 6x^2 + 9 + 12x^2 - 8x$$

$$= 18x^2 - 8x + 9$$

Note : Illustration 9 can also be solved using working rule 1 by simplifying y i.e. multiplying two terms of y.

Illustration 10 : Find $\frac{dy}{dx}$, $y = \frac{2x+3}{3x-2}$.

$$y = \frac{2x+3}{3x-2}$$

Take $u = 2x+3$ and $v = 3x-2$.

$$\therefore \quad \frac{du}{dx} = 2 \text{ and } \frac{dv}{dx} = 3$$

Now, $y = \frac{u}{v}$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(3x-2)(2) - (2x+3)(3)}{(3x-2)^2}$$

$$= \frac{(6x-4) - (6x+9)}{(3x-2)^2}$$

$$= \frac{6x-4-6x-9}{(3x-2)^2}$$

$$= \frac{-13}{(3x-2)^2}$$

Illustration 11 : If $y = \frac{3}{4x+5}$ then differentiate y with respect to x.

$$y = \frac{3}{4x+5}$$

Take $u = 3$ and $v = 4x+5$.

$$\therefore \quad \frac{du}{dx} = 0 \text{ and } \frac{dv}{dx} = 4$$

Now, $y = \frac{u}{v}$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(4x+5)(0) - 3(4)}{(4x+5)^2}$$

$$= \frac{0-12}{(4x+5)^2}$$

$$= \frac{-12}{(4x+5)^2}$$

Illustration 12 : If $y = \frac{2x^2 + 3x + 4}{x^2 + 5}$ then find $\frac{dy}{dx}$.

$$y = \frac{2x^2 + 3x + 4}{x^2 + 5}$$

Take $u = 2x^2 + 3x + 4$ and $v = x^2 + 5$.

$$\therefore \quad \frac{du}{dx} = 4x + 3 \text{ and } \frac{dv}{dx} = 2x$$

Now, $y = \frac{u}{v}$.

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\left(x^2 + 5\right)(4x + 3) - \left(2x^2 + 3x + 4\right)(2x)}{\left(x^2 + 5\right)^2}$$

$$= \frac{\left(4x^3 + 20x + 3x^2 + 15\right) - \left(4x^3 + 6x^2 + 8x\right)}{\left(x^2 + 5\right)^2}$$

$$= \frac{4x^3 + 20x + 3x^2 + 15 - 4x^3 - 6x^2 - 8x}{\left(x^2 + 5\right)^2}$$

$$= \frac{-3x^2 + 12x + 15}{\left(x^2 + 5\right)^2}$$

Illustration 13 : Differentiate $y = (3x + 7)^8$ with respect to x.

$$y = (3x + 7)^8$$

Taking $u = 3x + 7$, $y = u^8$

$$\therefore \quad \frac{du}{dx} = 3 \text{ and } \frac{dy}{du} = 8u^7$$

Now, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= \left(8u^7\right)(3)$$

$$= 24u^7$$

Putting value of u,

$$\frac{dy}{dx} = 24(3x + 7)^7$$

Illustration 14 : Find $\frac{dy}{dx}$, $y = \sqrt{x^2 + 3}$.

$$y = \sqrt{x^2 + 3}$$

Taking $u = x^2 + 3$, $y = \sqrt{u}$

$$\therefore \quad \frac{du}{dx} = 2x \text{ and } \frac{dy}{du} = \frac{1}{2\sqrt{u}}.$$

Now, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= \left(\frac{1}{2\sqrt{u}}\right) (2x)$$

$$= \frac{x}{\sqrt{u}}$$

Putting value of u,

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 3}}$$

Illustration 15 : Obtain derivative of $y = 1 + \frac{2}{3 + \frac{4}{x}}$ with respect to x.

$$y = 1 + \frac{2}{3 + \frac{4}{x}}$$

$$= 1 + \frac{2x}{3x + 4}$$

$$= \frac{(3x + 4) + 2x}{3x + 4}$$

$$\therefore \quad y = \frac{5x + 4}{3x + 4}$$

Here, take $u = 5x + 4$ and $v = 3x + 4$

$$\therefore \quad \frac{du}{dx} = 5 \text{ and } \frac{dv}{dx} = 3$$

Now, $y = \frac{u}{v}$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(3x + 4)(5) - (5x + 4)(3)}{(3x + 4)^2}$$

$$= \frac{(15x + 20) - (15x + 12)}{(3x + 4)^2}$$

$$= \frac{15x + 20 - 15x - 12}{(3x+4)^2}$$

$$= \frac{8}{(3x+4)^2}$$

Illustration 16 : If $2xy + 3x + y - 4 = 0$ then find $\frac{dy}{dx}$.

$$2xy + 3x + y - 4 = 0$$

$$\therefore \quad 2xy + y = 4 - 3x$$

$$\therefore \quad y(2x+1) = 4 - 3x$$

$$\therefore \quad y = \frac{4-3x}{2x+1}$$

Here, take $u = 4 - 3x$ and $v = 2x + 1$.

$$\therefore \quad \frac{du}{dx} = -3 \text{ and } \frac{dv}{dx} = 2$$

Now, $y = \frac{u}{v}$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(2x+1)(-3) - (4-3x)(2)}{(2x+1)^2}$$

$$= \frac{-6x - 3 - 8 + 6x}{(2x+1)^2}$$

$$= \frac{-11}{(2x+1)^2}$$

Illustration 17 : If $y = 2 + 3x + 4x^2 + \frac{5}{6-7x}$ then find $\frac{dy}{dx}$.

$$y = 2 + 3x + 4x^2 + \frac{5}{6-7x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left[2 + 3x + 4x^2 + \frac{5}{6-7x}\right]$$

$$= 0 + 3(1) + 4(2x) + \frac{d}{dx}\left(\frac{5}{6-7x}\right)$$

$$= 3 + 8x + \frac{(6-7x)(0) - 5(-7)}{(6-7x)^2} \qquad [\because \text{ Division rule}]$$

$$= 3 + 8x + \frac{35}{(6-7x)^2}$$

Illustration 18 : If $y = \left(x + \frac{6}{x+5}\right)\left(\frac{3x+2}{x^2+5x+6}\right)$ then find $\frac{dy}{dx}$.

$$y = \left(x + \frac{6}{x+5}\right)\left(\frac{3x+2}{x^2+5x+6}\right)$$

$$= \left[\frac{x(x+5)+6}{x+5}\right]\left(\frac{3x+2}{x^2+5x+6}\right)$$

$$= \left(\frac{x^2+5x+6}{x+5}\right)\left(\frac{3x+2}{x^2+5x+6}\right)$$

$$= \frac{3x+2}{x+5}$$

Here, take $u = 3x + 2$ and $v = x + 5$.

$$\therefore \quad \frac{du}{dx} = 3 \text{ and } \frac{dv}{dx} = 1$$

Now, $y = \frac{u}{v}$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x+5)(3) - (3x+2)(1)}{(x+5)^2}$$

$$= \frac{(3x+15) - (3x+2)}{(x+5)^2}$$

$$= \frac{3x+15-3x-2}{(x+5)^2}$$

$$= \frac{13}{(x+5)^2}$$

Illustration 19 : If $f(x) = 3x^2 + 2x + 1$ then find $f'(x)$ and hence obtain $f'(1)$.

Here, $f(x) = 3x^2 + 2x + 1$

$\therefore$ $f'(x) = 6x + 2$

$\therefore$ $f'(1) = 6(1) + 2$

$= 8$

Illustration 20 : If $f(x) = x^2 - x + 3$ then for which value of x, $f'(x) = 0$?

Here, $f(x) = x^2 - x + 3$

$\therefore \quad f'(x) = 2x - 1 + 0$

Now, $f'(x) = 0$ is given

$\therefore \quad 2x - 1 = 0$

$\therefore \quad 2x = 1$

$\therefore \quad x = \frac{1}{2}$

5.5 Second Order Differentiation

As seen in many of the previous illustrations, the derivative of a function of x is generally also a function of x. The derivative of $y = f(x)$ is denoted by $\frac{dy}{dx}$ or $f'(x)$. This derivative is called the first order derivative of the function. The second order derivative of the function means the derivative of the first order derivative of the function. It is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$. Second order derivative along with the first order derivative can be useful in maximization or minimization of a function. This can be applied to minimize cost function, maximize revenue function and maximize profit function.

We shall now see the method of obtaining second order derivative with few illustrations.

Illustration 21 : Obtain $\frac{dy}{dx}$ for $y = 3x^4 - 2x^3 + x^2 - 8x + 7$. Also obtain its value at $x = 1$.

$$y = 3x^4 - 2x^3 + x^2 - 8x + 7$$

$$\therefore \quad \frac{dy}{dx} = 12x^3 - 6x^2 + 2x - 8$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$$

$$= \frac{d}{dx}\left[12x^3 - 6x^2 + 2x - 8\right]$$

$$= 36x^2 - 12x + 2$$

Putting $x = 1$,

$$\frac{d^2y}{dx^2} = 36(1)^2 - 12(1) + 2$$

$$= 36 - 12 + 2$$

$$= 26$$

Illustration 22 : If $f(x) = 4x^3 + 2x^2 + 7x + 9$ then for which value of x, $f''(x) = 52$?

$$f(x) = 4x^3 + 2x^2 + 7x + 9$$

$$\therefore\ f'(x) = 12x^2 + 4x + 7$$

$$\therefore\ f''(x) = 24x + 4$$

Now, $f''(x) = 52$

$$\therefore\ 24x + 4 = 52$$

$$\therefore\ 24x = 48$$

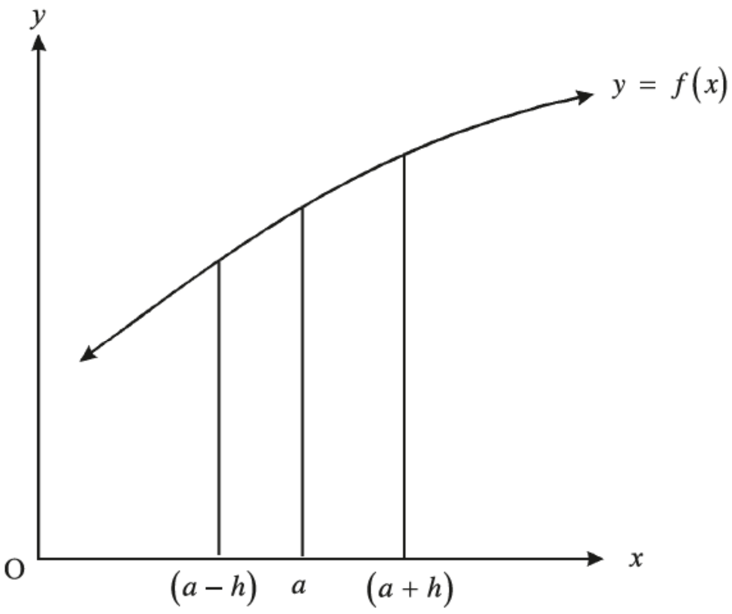
$$\therefore\ x = 2$$

5.6 Increasing Function and Decreasing Function

Increasing function

In the adjacent figure, the curve of the function $y = f(x)$ is drawn. The value of the function at $x = a$ is $y = f(a)$. If h is a very small positive number and if $f(a+h) > f(a)$ and also $f(a) > f(a-h)$ then $f(x)$ is said to be an increasing function at $x = a$.

If the function is increasing at $x = a$ then $f'(a) > 0$



Decreasing function

In the adjacent figure, the curve of the function $y = f(x)$ is drawn. The value of the function at $x = a$ is $y = f(a)$. If h is a very small positive number and if $f(a+h) < f(a)$ and also $f(a) < f(a-h)$, then $f(x)$ is said to be a decreasing function of $x = a$.

If the function is decreasing at $x = a$ then $f'(a) < 0$

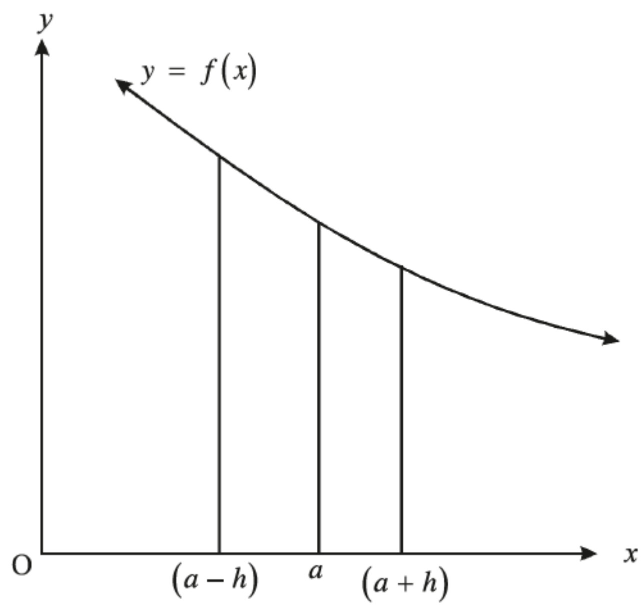


Illustration 23 : If $f(x) = x^2 - 4x$ then decide whether the function is increasing or decreasing at $x = -1$, $x = 0$ and $x = 3$.

$$f(x) = x^2 - 4x$$

$$\therefore \quad f'(x) = 2x - 4$$

At $x = -1$

$$f'(-1) = 2(-1) - 4$$

$$= -6 < 0$$

$\therefore$ Function is decreasing at $x = -1$.

At $x = 0$

$$f'(0) = 2(0) - 4$$

$$= -4 < 0$$

$\therefore$ Function is decreasing at $x = 0$.

At $x = 3$

$$f'(3) = 2(3) - 4$$

$$= 2 > 0$$

$\therefore$ Function is increasing at $x = 3$.

Illustration 24 : Decide whether the function $y = x^3 - 3x^2 + 7$ is increasing or decreasing at $x = 1$ and $x = 3$.

$$y = x^3 - 3x^2 + 7$$

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 6x$$

At $x = 1$

$$\frac{dy}{dx} = 3(1)^2 - 6(1)$$

$$= 3 - 6$$

$$= -3 < 0$$

$\therefore$ Function is decreasing at $x = 1$.

At $x = 3$

$$\frac{dy}{dx} = 3(3)^2 - 6(3)$$

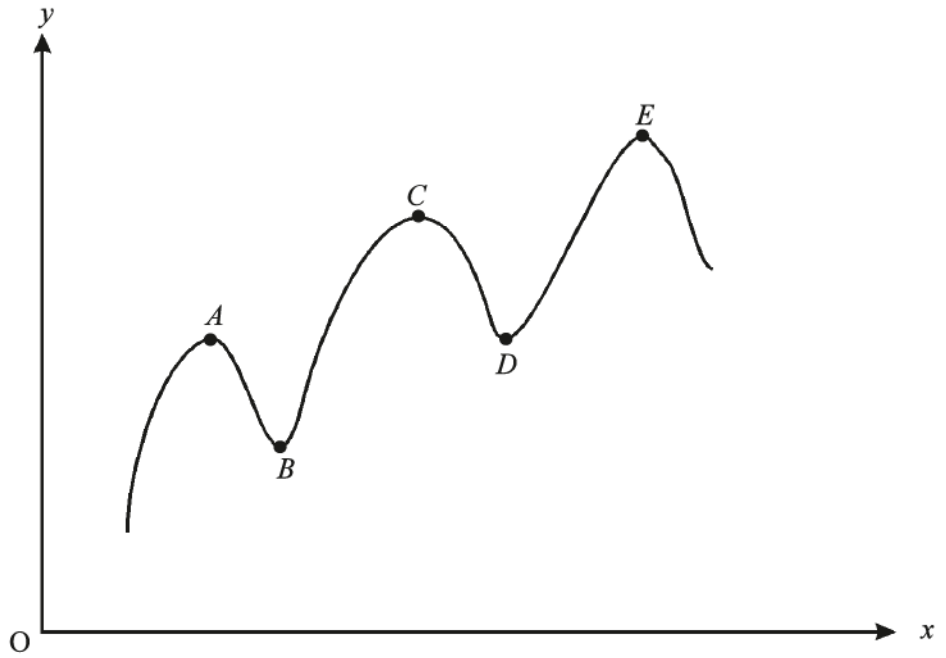
$$= 27 - 18$$

$$= 9 > 0$$

$\therefore$ Function is increasing at $x = 3$.

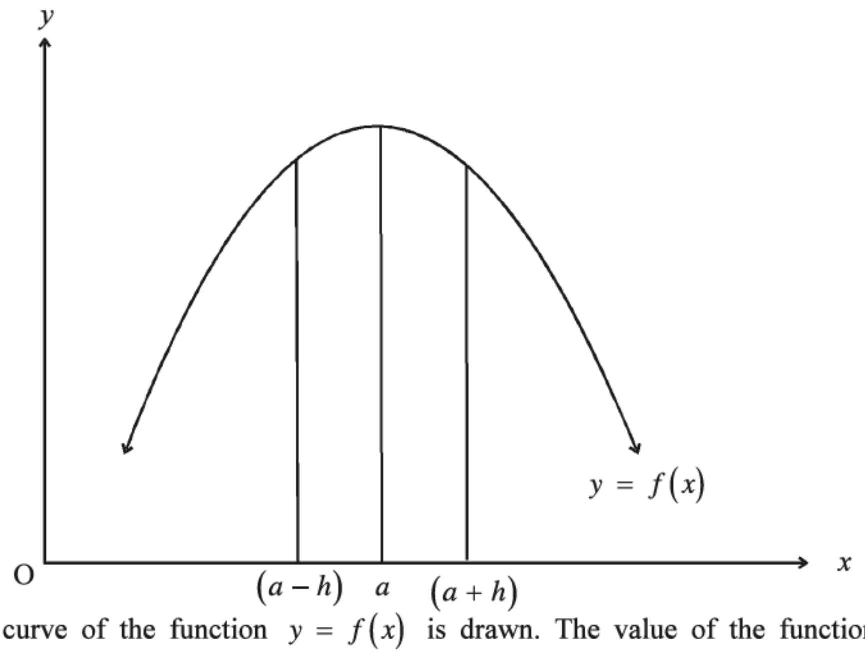
5.7 Maximum and Minimum Values of a Function

We discussed about increasing and decreasing function. Now, we shall study the method of obtaining maximum and minimum value of a function. Suppose the graph of a function $y = f(x)$ is obtained as follows.



It can be seen that the curve obtains maximum values at points *A, C* and *E* while its values are minimum at points *B* and *D.* Thus, the function may have more than one maximum or minimum values.

Maximum Value :

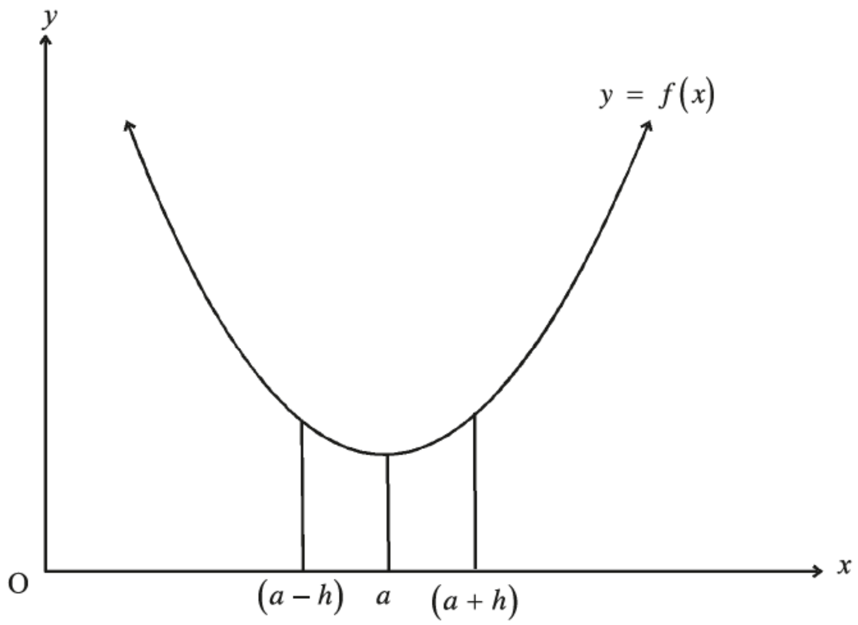


In the figure, curve of the function $y = f(x)$ is drawn. The value of the function at $x = a$ is $y = f(a)$. If h is a small positive number and if $f(a) > f(a+h)$ and also $f(a) > f(a-h)$ then $f(x)$ is said to be maximum at $x = a$.

The necessary and sufficient conditions for a function to be maximum at $x = a$ are as follows :

(i) $f'(a) = 0$ (ii) $f''(a) < 0$

Minimum Value :



In the figure, curve of the function $y = f(x)$ is drawn. The value of the function at $x = a$ is $y = f(a)$. If h is a small positive number and if $f(a) < f(a+h)$ and $f(a) < f(a-h)$ then $f(x)$ is said to be minimum at $x = a$.

The necessary and sufficient conditions for a function to be minimum at $x = a$ are as follows :

(i) $f'(a) = 0$ (ii) $f''(a) > 0$

The maximum and minimum values of a function are known as stationary maximum and stationary minimum values of function.

Maximum or minimum values do not mean the largest or the smallest value of a function. The function is maximum of $x = a$ only means that the value of the function is maximum in a small interval around $x = a$. Similarly, the function is minimum at $x = b$ only means that the value of the function is minimum in a small interval around $x = b$. The points where maximum or minimum values occur are known as **stationary points**. The necessary condition to obtain a stationary value is $\frac{dy}{dx} = 0$.

Method of obtaining maximum and minimum values of a function :

- Find the first derivative $\frac{dy}{dx} = f'(x)$ of the given function.
- Putting $\frac{dy}{dx} = 0$, solve the equation and obtain the values of x. These values of x give the stationary points.
- Find the second order derivative and put these values of x alternatively in the second derivative.
- The value of x at the stationary points for which the second order derivative is negative gives the maximum value of the function while the value of x at the stationary points for which the second order derivative is positive gives the minimum value of the function.
- The maximum and minimum values of a function are obtained by putting these values of x in the given function.

We shall now see the method of obtaining the maximum and minimum values of a function with a few illustrations.

Illustration 25 : Find the maximum and minimum values of $f(x) = 2x^3 + 3x^2 - 12x - 4$.

Here, $f(x) = 2x^3 + 3x^2 - 12x - 4$

$\therefore \quad f'(x) = 6x^2 + 6x - 12$

For stationary values, $f'(x) = 0$

$\therefore \quad 6x^2 + 6x - 12 = 0$

$\therefore \quad x^2 + x - 2 = 0$

$\therefore \quad (x+2)(x-1) = 0$

$\therefore \quad x = -2 \text{ or } x = 1$

Now, $f''(x) = 12x + 6$

At $x = -2$

$$f''(-2) = 12(-2) + 6$$
$$= -18 < 0$$

$\therefore$ We get the maximum value of the function at $x = -2$.

At $x = 1$

$$f''(1) = 12(1) + 6$$
$$= 18 > 0$$

$\therefore$ We get the minimum value of the function at $x = 1$.

Minimum value of $f(x)$

Putting $x = 1$ in the function $f(x)$,

$$f(1) = 2(1)^3 + 3(1)^2 - 12(1) - 4$$
$$= 2 + 3 - 12 - 4$$
$$= -11$$

Maximum value of $f(x)$

Putting $x = -2$ in the function $f(x)$,

$$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) - 4$$
$$= -16 + 12 + 24 - 4$$
$$= 16$$

Thus, the maximum value of $f(x)$ is 16 and the minimum value is -11.

Illustration 26 : Find the maximum and minimum values of $y = x^3 - 2x^2 - 4x - 1$.

Here, $y = x^3 - 2x^2 - 4x - 1$

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 4x - 4$$

For stationary values, $\frac{dy}{dx} = 0$

$$\therefore \quad 3x^2 - 4x - 4 = 0$$

$$\therefore \quad 3x^2 - 6x + 2x - 4 = 0$$

$$\therefore \quad 3x(x-2) + 2(x-2) = 0$$

$$\therefore \quad (x-2)(3x+2) = 0$$

$$\therefore \quad x = 2 \text{ or } x = -\frac{2}{3}$$

Now, $\frac{d^2y}{dx^2} = 6x - 4$

At $x = 2$

$$\frac{d^2y}{dx^2} = 6(2) - 4$$

$$= 8 > 0$$

$\therefore$ Function is minimum at $x = 2$.

At $x = -\frac{2}{3}$

$$\frac{d^2y}{dx^2} = 6\left(\frac{-2}{3}\right) - 4$$

$$= -4 - 4$$

$$= -8 < 0$$

$\therefore$ Function is maximum at $x = -\frac{2}{3}$.

Minimum value of function y

Putting $x = 2$ in the function y,

$$y = (2)^3 - 2(2)^2 - 4(2) - 1$$

$$= 8 - 8 - 8 - 1$$

$$= -9$$

Maximum value of function y

Putting $x = -\frac{2}{3}$ in the function y,

$$y = \left(-\frac{2}{3}\right)^3 - 2\left(-\frac{2}{3}\right)^2 - 4\left(-\frac{2}{3}\right) - 1$$

$$= \frac{-8}{27} - \frac{8}{9} + \frac{8}{3} - 1$$

$$= \frac{13}{27}$$

Thus, the maximum value of y is $\frac{13}{27}$ and the minimum value is –9.

5.8 Marginal Income and Marginal Cost

The differentiation is used to obtain solutions of economic and business problems. We have seen that the first and second order derivatives can be used to obtain the maximum and minimum values of a function.

First order derivative of a function can also be used to obtain marginal income and marginal cost.

In study of economics, the relation between price and demand of a commodity can be represented as a function. If we denote the price of a commodity by p and its demand by x then, we get the relation $x = f(p)$, which is called the **demand function**. If the income or revenue obtained by selling x units of a commodity is denoted by R then,

$$R = xp$$

Thus, revenue R is a function of demand x.

The change in revenue due to small change in demand is called **marginal revenue**.

Marginal revenue can be obtained by taking the derivative of revenue function with respect to x. Thus, when the demand is x then

$$\text{Marginal revenue} = \frac{dR}{dx}$$

If we denote the cost of producing x units by C then C can also be represented as function of x.

The change in cost due to small change in production is called **marginal cost**.

Marginal cost can be obtained by taking the derivative of cost function with respect to x. Thus, when the production is x then

$$\text{Marginal cost} = \frac{dC}{dx}$$

Illustration 27 : If the demand function of pizza is $p = 150 - 4x$ then find the marginal revenue when demand is of 3 pizzas and interpret it.

Here, demand function $p = 150 - 4x$

Now, revenue function $R = p \cdot x$

$$= (150 - 4x)\,x$$

$$\therefore \quad R = 150x - 4x^2$$

Marginal revenue $\frac{dR}{dx} = 150 - 8x$

When demand of pizza is $x = 3$ then

Marginal revenue $\frac{dR}{dx} = 150 - 8(3)$

$$= 126$$

Interpretation : Revenue for selling the 4th pizza is approximately ₹ 126.

Illustration 28 : If the demand function of a commodity is $x = \frac{50-p}{2}$ then find the marginal revenue when price is ₹ 30.

Demand function $x = \frac{50-p}{2}$

$\therefore \quad 2x = 50 - p$

$\therefore \quad p = 50 - 2x$

Now, revenue function $R = p \cdot x$

$= (50 - 2x)x$

$\therefore \quad R = 50x - 2x^2$

Marginal revenue $\frac{dR}{dx} = 50 - 4x$

When price $p = 30$ then

$x = \frac{50-30}{2}$

$\therefore \quad x = 10$

When demand $x = 10$ then

Marginal Revenue $= \frac{dR}{dx} = 50 - 4(10)$

$= 10$

Interpretation : Revenue for selling the 11th unit is approximately ₹ 10.

Illustration 29 : The cost function of a commodity is $C = 5x^2 + 6x + 2000$, where x is the number of units produced. Find marginal cost when production is 50 units.

Cost function $C = 5x^2 + 6x + 2000$

$\therefore$ Marginal Cost $\frac{dC}{dx} = 10x + 6$

When $x = 50$ then

Marginal Cost $\frac{dC}{dx} = 10(50) + 6$

$= 506$

Interpretation : The cost of producing the 51st unit is approximately ₹ 506.

5.9 Elasticity of Demand

Generally, a change in price of a commodity results in change in its demand in opposite direction. When the price of a commodity increases, its demand decreases and when the price of a commodity decreases, its demand increases. But these changes are not equal for all the commodities. For example, a sudden increase in price of luxury commodities results in a major decrease in its demand. While increase in the price of necessary commodities does not result in a major decrease in its demand. The change in demand for a commodity due to change in its price can be studied using elasticity of demand.

Definition : The ratio of percentage change in the demand of a commodity due to percentage change in its price is called elasticity of demand.

i.e.

$$\text{Elasticity of demand} = -\frac{\text{Percentage change in demand}}{\text{Percentage change in price}}$$

The ratio is negative as the changes in price and demand of a commodity is in opposite direction. For convenience, the value of elasticity of demand is taken positive and hence the negative sign is taken in the formula. If we denote the demand as x and price as p and the demand function $x = f(p)$ is given then

$$\text{Elasticity of demand} = -\frac{p}{x} \cdot \frac{dx}{dp}.$$

Illustration 30 : The demand function of a commodity is $x = 50 - 4p$. Find elasticity of demand when price is $p = 5$ and interpret it.

Demand function $x = 50 - 4p$

$$\therefore \quad \frac{dx}{dp} = 0 - 4(1)$$

$$= -4$$

$$\text{Now, elasticity of demand} = -\frac{p}{x} \cdot \frac{dx}{dp}$$

$$= \frac{-p}{(50-4p)} \times (-4)$$

$$= \frac{4p}{50-4p}$$

When price $p = 5$ then

$$\text{Elasticity of demand} = \frac{4(5)}{50-4(5)}$$

$$= \frac{20}{50-20}$$

$$= \frac{20}{30}$$

$$= 0.67$$

Interpretation : When the price changes by 1 percent, demand changes by 0.67 percent (in opposite direction) when the price is 5.

Illustration 31 : The demand function of a commodity is $p = 12 - \sqrt{x}$. Find the elasticity of demand when the price is 9 units and interpret it.

Demand function $p = 12 - \sqrt{x}$

$$\therefore \quad \frac{dp}{dx} = 0 - \frac{1}{2\sqrt{x}}$$

$$= -\frac{1}{2\sqrt{x}}$$

$$\therefore \quad \frac{dx}{dp} = -2\sqrt{x} \qquad \left[\because \frac{dx}{dp} = \frac{1}{\frac{dp}{dx}}\right]$$

$$\text{Now, elasticity of demand} = -\frac{p}{x} \cdot \frac{dx}{dp}$$

$$= \frac{-\left(12 - \sqrt{x}\right)}{x} \times \left(-2\sqrt{x}\right)$$

$$= \frac{\left(12 - \sqrt{x}\right)\left(2\sqrt{x}\right)}{x}$$

When demand is 9 units then

$$\text{Elasticity of demand} = \frac{\left(12 - \sqrt{9}\right)\left(2\sqrt{9}\right)}{9}$$

$$= \frac{(12 - 3)(2 \times 3)}{9}$$

$$= \frac{9 \times 6}{9}$$

$$= 6$$

Interpretation : When price changes by 1 percent, demand changes by 6 percent (in opposite direction) when demand is 9 units.

5.10 Minimization of cost function and maximization of Revenue function and Profit function

In practice, problems of minimizing the production cost of an item, maximizing the revenue by selling produced items and maximizing profits are to be solved. We know that the production cost C or revenue R by selling produced items and profit P can be represented as functions of x. Using the derivatives, we can decide when it will be maximum or minimum.

The conditions for minimizing the production cost function C are

$$\frac{dC}{dx} = 0 \text{ and } \frac{d^2C}{dx^2} > 0.$$

Similarly, the conditions for maximizing the revenue function R are

$$\frac{dR}{dx} = 0 \text{ and } \frac{d^2R}{dx^2} < 0.$$

And conditions for maximizing the profit function P are

$$\frac{dP}{dx} = 0 \text{ and } \frac{d^2P}{dx^2} < 0.$$

We shall now see the method of obtaining minimum cost, maximum revenue and maximum profit with few illustrations.

Illustration 32 : The daily cost of production for x tons of a commodity is $10x^2 - 1000x + 50000$. How many units should be produced for the minimum cost ? Also find the minimum cost.

Production cost function $C = 10x^2 - 1000x + 50000$

$$\therefore \quad \frac{dC}{dx} = 20x - 1000$$

Putting $\frac{dC}{dx} = 0$,

$$20x - 1000 = 0$$

$$\therefore \quad 20x = 1000$$

$$\therefore \quad x = 50$$

Now $\frac{d^2C}{dx^2} = 20$

Here, putting $x = 50$ in $\frac{d^2C}{dx^2}$,

$$\frac{d^2C}{dx^2} = 20 > 0$$

$\therefore$ Production cost is minimum at $x = 50$.

To find minimum cost, put $x = 50$ in the production cost function,

$$\text{Minimum Cost} = 10(50)^2 - 1000(50) + 50000$$

$$= 10(2500) - 50000 + 50000$$

$$= 25000$$

Illustration 33 : A factory produces x units and its production capacity is 60,000 units per day. Its daily total production cost is $C = 250000 + 0.08x + \frac{200000000}{x}$. How many units should be produced for minimum production cost ?

Production cost function $C = 250000 + 0.08x + \frac{200000000}{x}$

$$\therefore \quad \frac{dC}{dx} = 0.08 - \frac{200000000}{x^2}$$

Putting $\frac{dC}{dx} = 0$

$$0.08 - \frac{200000000}{x^2} = 0$$

$$\therefore \quad 0.08 = \frac{200000000}{x^2}$$

$$\therefore \quad 0.08\ x^2 = 200000000$$

$$\therefore \quad x^2 = 2500000000$$

$$\therefore \quad x = 50000 \quad \text{or} \quad x = -50000$$

Production cannot be negative, so we will take $x = 50000$.

Now $\frac{d^2C}{dx^2} = \frac{400000000}{x^3}$

Here, putting $x = 50000$ in $\frac{d^2C}{dx^2}$,

$$\frac{d^2C}{dx^2} = \frac{400000000}{(50000)^3} > 0$$

$\therefore$ Production cost is minimum at $x = 50000$.

Thus, 50,000 units should be produced so that the production cost is minimum.

Illustration 34 : The demand function of a watch is $p = 6000 - 2x$. Find the demand which maximizes the revenue and also find the corresponding price.

Demand function $p = 6000 - 2x$

Now, revenue function $R = p \cdot x$

$$= (6000 - 2x)x$$

$$\therefore \quad R = 6000x - 2x^2$$

$$\therefore \quad \frac{dR}{dx} = 6000 - 4x$$

Putting $\frac{dR}{dx} = 0$,

$$6000 - 4x = 0$$

$$\therefore \quad 6000 = 4x$$

$$\therefore \quad x = 1500$$

Now $\frac{d^2R}{dx^2} = 0 - 4$

$= -4$

Here, putting $x = 1500$ in $\frac{d^2R}{dx^2}$,

$\frac{d^2R}{dx^2} = -4 < 0$

$\therefore$ Revenue is maximum at $x = 1500$.

Now we shall find the corresponding price.

Putting $x = 1500$ in demand function $p = 6000 - 2x$,

Price $p = 6000 - 2(1500)$

$= 6000 - 3000$

$p = 3000$

Illustration 35 : If the production cost function for a producer is $C = 100 + 0.015x^2$ and revenue function is $R = 3x$ then find the profit function. How many units should be produced by the producer for maximum profit ?

Production cost function $C = 100 + 0.015x^2$ and revenue function $R = 3x$

Now, profit function $P = R - C$

$= 3x - (100 + 0.015x^2)$

$\therefore \quad P = 3x - 100 - 0.015x^2$

$\therefore \quad \frac{dP}{dx} = 3 - 0.015(2x)$

$= 3 - 0.03x$

Putting $\frac{dP}{dx} = 0$

$3 - 0.03x = 0$

$\therefore \quad 3 = 0.03x$

$\therefore \quad x = \frac{3}{0.03}$

$x = 100$

Now $\frac{d^2P}{dx^2} = 0 - 0.03(1)$

$= -0.03$

Here putting $x = 100$ in $\frac{d^2P}{dx^2}$,

$\frac{d^2P}{dx^2} = -0.03 < 0$

$\therefore$ At $x = 100$, profit is maximum.

Summary

- Derivative $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
- If $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$
- If $y = k$ (constant), $\frac{dy}{dx} = 0$
- If u and v are differentiable functions of x then,

  (1) If $y = u \pm v$ then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$

  (2) If $y = u \cdot v$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

  (3) If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

  (4) Chain Rule : $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
- If the function $f(x)$ is increasing at $x = a$ then $f'(a) > 0$.
- If the function $f(x)$ is decreasing at $x = a$ then $f'(a) < 0$.
- The necessary and sufficient conditions for a function to be maximum at $x = a$: $f'(a) = 0$ and $f''(a) < 0$.
- The necessary and sufficient conditions for a function to be minimum at $x = a$: $f'(a) = 0$ and $f''(a) > 0$.
- Marginal Cost $= \frac{dC}{dx}$
- Marginal Revenue $= \frac{dR}{dx}$
- Elasticity of demand $= -\frac{p}{x} \cdot \frac{dx}{dp}$
- The necessary and sufficient conditions for minimizing the production cost function C :

  $\frac{dC}{dx} = 0$ and $\frac{d^2C}{dx^2} > 0$.
- The necessary and sufficient conditions for maximizing the revenue function R :

  $\frac{dR}{dx} = 0$ and $\frac{d^2R}{dx^2} < 0$.
- The necessary and sufficient conditions for maximizing the profit function P :

  $\frac{dP}{dx} = 0$ and $\frac{d^2P}{dx^2} < 0$.

EXERCISE 5

Section A

Choose the correct option for the following multiple choice questions :

1. What is the formula for derivative of function $f(x)$?

(a) $\lim_{h\to x} \frac{f(x+h) - f(x)}{h}$ (b) $\lim_{h\to 0} \frac{f(x+h) + f(x)}{h}$

(c) $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ (d) $\lim_{h\to x} \frac{f(x) - f(x+h)}{h}$

2. What is $\frac{dy}{dx}$ if $y = ax^n$, a is a constant ?

(a) nx^{n-1} (b) $an\,x^{n-1}$ (c) 0 (d) $an\,x^{n+1}$

3. If $y = ax + b$, a and b are constant then what will be $\frac{dy}{dx}$?

(a) a (b) b (c) $a + b$ (d) 0

4. What is the derivative of $f(x) = \frac{4}{x^2}$?

(a) $\frac{4}{2x}$ (b) $-\frac{8}{x^3}$ (c) $\frac{8}{x^3}$ (d) 0

5. If u and v are two functions of x then what is the formula of derivative of their product ?

(a) $u\frac{du}{dx} + v\frac{dv}{dx}$ (b) $u\frac{dv}{dx} - v\frac{du}{dx}$ (c) $\frac{du}{dx} \times \frac{dv}{dx}$ (d) $u\frac{dv}{dx} + v\frac{du}{dx}$

6. If u and v are functions of x then what is the formula for derivative of $\frac{v}{u}$?

(a) $\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (b) $\frac{v\frac{du}{dx} + u\frac{dv}{dx}}{v^2}$ (c) $\frac{u\frac{dv}{dx} + v\frac{du}{dx}}{u^2}$ (d) $\frac{u\frac{dv}{dx} - v\frac{du}{dx}}{u^2}$

7. If the function $f(x)$ is increasing at $x = a$ then which is the correct option from the following ?

(a) $f'(a) < 0$ (b) $f'(a) > 0$ (c) $f'(a) = 0$ (d) $f''(a) > 0$

8. What are the necessary and sufficient conditions for a function to be minimum at $x = a$?

(a) $f'(a) = 0,\ f''(a) < 0$ (b) $f'(a) > 0,\ f''(a) > 0$

(c) $f'(a) = 0,\ f''(a) > 0$ (d) $f'(a) < 0,\ f''(a) > 0$

9. What is the formula for elasticity of demand ?

(a) $-\frac{p}{x} \cdot \frac{dx}{dp}$ (b) $\frac{p}{x} \cdot \frac{dx}{dp}$ (c) $-\frac{x}{p} \cdot \frac{dp}{dx}$ (d) $-\frac{p}{x} \cdot \frac{dp}{dx}$

10. What are the conditions of revenue function R to be maximum ?

(a) $\frac{dR}{dx} = 0, \frac{d^2R}{dx^2} < 0$ (b) $\frac{dR}{dx} = 0, \frac{d^2R}{dx^2} > 0$

(c) $\frac{dR}{dx} > 0, \frac{d^2R}{dx^2} < 0$ (d) $\frac{dR}{dx} > 0, \frac{d^2R}{dx^2} > 0$

Section B

Answer the following questions in one sentence:

1. Define differentiation.
2. Find $f'(x)$ for the function $f(x) = 50$.
3. Find $\frac{dy}{dx}$ if $y = a^n$, a is constant.
4. State the rule for derivative for product of two functions of x.
5. How will be the first order derivative of a function at $x = a$ if function is decreasing at $x = a$?
6. How will be the second order derivative of a function at $x = a$ if function is maximum at $x = a$?
7. What are the stationary points of a function ?
8. What is marginal revenue ?
9. Define marginal cost.
10. State the formula of elasticity of demand.
11. Find $f'(x)$ if $f(x) = 7x^2 - 6x + 5$.
12. Find $\frac{dy}{dx}$ if $y = 6x^3 + \frac{7}{2}x^2 + \frac{6}{5}x - 8$.

Section C

Answer the following questions :

1. Define derivative.
2. State the division rule of derivative.
3. State necessary and sufficient conditions for a function to be maximum at $x = a$.
4. Explain marginal cost and give its formula.
5. Define elasticity of demand.
6. What are the conditions for profit function P to be maximum ?
7. State the conditions for production cost function C to be minimum.
8. Find $f''(x)$ if $f(x) = \sqrt[4]{x}$.
9. Write the chain rule of differentiation.
10. Find $f''(0)$ if $f(x) = x^4 - 4x^3 + 3x^2 + x + 1$.
11. Find marginal revenue if revenue function is $90x - \frac{x^2}{2}$.

12. What is maximum value of a function ?

13. When can it be said that a function is decreasing at a point ?

14. Determine whether the function $y = 12 + 4x - 7x^2$ is increasing or decreasing at $x = 2$.

15. Find the derivative of $y = 4x^2 + 4x + 8$. For which value of x will the derivative be zero ?

16. $f(x) = x^3 + 5x^2 + 3x + 7$, prove that $f'(2) = 35$.

17. If $f(x) = 3x^2 + 3$ then for which value of x, $f'(x) = f(x)$?

18. Find $\frac{d^2y}{dx^2}$ if $y = 2x^3 + 5x^2 - 3 + \frac{4}{x^2} - \frac{5}{x^3}$.

19. Find $\frac{d^2y}{dx^2}$ if $y = \sqrt{x} + \frac{1}{\sqrt{x}}$.

20. Obtain marginal cost if the production cost function is $C = 0.0012x^2 - 0.18x + 25$.

Section D

Answer the following questions :

1. Find derivative of $y = ax + b$ $(a$ and b are constants$)$ using definition.

2. Find derivative of $f(x) = x^{10}$ using definition.

3. Find derivative of $\frac{2}{3 + 4x}$ using definition.

4. $y = x^3 - 3x^2 - 3x + 80$. For which value of x, $\frac{dy}{dx} = -6$?

5. Find $f'(2)$ if $f(x) = \frac{4x^5 + 3x^3 + 2x^2 + 24}{x^2}$.

6. Find the derivative of $y = (3x^2 + 4x - 2)(3x + 2)$ with respect to x.

7. Find $\frac{dy}{dx}$ if $y = \frac{ax + b}{bx + a}$ $(a$ and b are constants$)$.

8. Find the derivative of $y = 1 + \frac{1}{1 + \frac{1}{x}}$ with respect to x.

9. Find $\frac{dy}{dx}$ if $(2x + 3)(y + 2) = 15$.

10. Find $\frac{dy}{dx}$ if $y = 5 + \frac{6}{7x + 8}$.

11. Find $f'(x)$ if $f(x) = \sqrt{x^2 + 5}$.

12. Find the derivative of $(3x^3 - 2x^2 + 1)^{\frac{5}{2}}$ with respect to x.

13. Find $f'(x)$ if $f(x) = \left(x^2 + 3x + 4\right)^7$.

14. If $f(x) = 3x^2 + 4x + 5$ then for which value of x, $f'(x) = f''(x)$?

15. Find marginal revenue if demand function is $p = \frac{2500 - x^2}{100}$.

16. Determine whether the function $y = 3x^2 - 10x + 7$ is increasing or decreasing at $x = 1$ and $x = 2$.

17. Determine whether the function $y = 2x^3 - 7x^2 - 11x + 5$ is increasing or decreasing at $x = \frac{1}{2}$ and $x = 3$.

18. Determine whether the function $y = 3 + 2x - 7x^2$ is increasing or decreasing at $x = -4$ and $x = 4$.

19. Production cost of a factory producing sugar is $C = \frac{x^2}{10} + 5x + 200$. Find the marginal cost if the production is 100 units and interpret it.

20. The cost function of producing x units of a commodity is $C = 50 + 2x + \sqrt{x}$. Find the marginal cost if the production is 100 units and interpret it.

21. State the method of obtaining maximum or minimum value of a function.

Section E

Answer the following questions :

1. Give working rules for differentiation.
2. How can it be decided using derivative that the function is increasing or decreasing at a point ?
3. What is maximum value of a function ? State the conditions for maximum value.
4. What is minimum value of a function ? State the conditions for minimum value.
5. In a factory, production cost per hundred tons of steel is $\frac{1}{10}x^3 - 4x^2 + 50x + 300$. Determine the production for minimum cost.
6. The cost of producing x units of an item is $C = 1000 + 8x + \frac{5000}{x}$. What should be the production for minimum cost ? Also find the minimum cost.
7. Production cost function of a commodity is $C = 1500 + 0.05x - 2\sqrt{x}$. Prove that production is minimum when 400 units are produced.
8. The demand function of an item is $p = 30 - \frac{x^2}{10}$. Find the demand and price for maximum revenue.
9. In a market, demand law of rice is $x = 3(60 - p)$. Find the demand for maximum revenue. Also find the price and revenue for that demand.
10. If the demand function is $p = 75 - \frac{x^2}{2500}$ then at which demand is revenue maximum ? Also find the price for maximum revenue.

11. The profit function of a producer is $40x + 10000 - 0.1x^2$. At what production is the profit maximum ? Also find this maximum profit.

12. The profit function of a merchant is $5x - 100 - 0.01x^2$. How many units should be produced for maximum profit ?

Section F

Solve the following :

1. Find the values of x which maximize or minimize $y = 2x^3 - 15x^2 + 36x + 12$. Also find the maximum and minimum values of y.

2. Find the values of x which maximize or minimize $f(x) = 2x^3 + 3x^2 - 36x + 10$. Also find the maximum and minimum values of $f(x)$.

3. Find the maximum and minimum values of $f(x) = x^3 - x^2 - x + 2$.

4. A producer produces x units at cost $200x + 15x^2$. The demand function is $p = 1200 - 10x$. Find the profit function and how many units should be produced for maximum profit ?

5. The selling price of a refrigerator as determined by the company is ₹ 10,000. The total cost of the production for x refrigerator is $C = 0.1x^2 + 9000x + 100$ rupees. How many refrigerators should be manufactured for maximum profit ?

6. A toy is sold at ₹ 20. Total cost of producing x such toys is $C = 1000 + 16.5x + 0.001x^2$ rupees. How many toys should be produced for maximum profit ?

Gottfried Wilhelm Leibniz
(1646 - 1716)

Gottfried Leibniz was a German polymath and philosopher who occupies a prominent place in the history of mathematics and the history of philosophy, having developed differential and integral calculus independently of Isaac Newton. It was only in the 20th century that his Law of Continuity and Transcendental Law of Homogeneity found mathematical implementation (by means of non-standard analysis). He became one of the most prolific inventors in the field of mechanical calculators.

Leibniz made major contributions to physics and technology, and anticipated notions that surfaced much later in philosophy, probability theory, biology, medicine, geology, psychology, linguistics, and computer science.