Electromagnetic Induction Maxwell's Equations

(Part - 1)

Q. 288. A wire bent as a parabola y = ax^2 is located in a uniform magnetic field of induction B, the vector B being perpendicular to the plane x, y. At the moment t = 0 a connector starts sliding translation wise from the parabola apex with a constant acceleration w (Fig. 3.78). Find the emf of electromagnetic induction in the loop thus formed as a function of y.

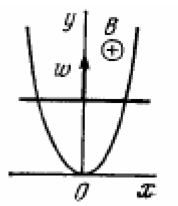
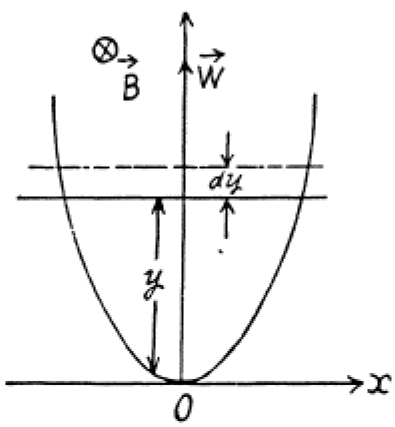


Fig. 3.78.

Solution. 288. Obviously, from Lenz's law, the induced current and hence the induced e.m.f. in the loop is anticlockwise.



From Faraday's law of electromagnetic induction,

$$\xi_{in} = \left|\frac{d\Phi}{dt}\right|$$

Here, $d\Phi = \vec{B} \cdot d\vec{S} = -2B\,x\,dy,$

and from $y = ax^2, x = \sqrt{\frac{y}{a}}$

Hence, $\xi_{in} = 2B\sqrt{\frac{y}{a}}\frac{dy}{dt}$

$$= By\sqrt{\frac{8w}{a}}, \text{ using } \frac{dy}{dt} = \sqrt{2wy}$$

Q. 289. A rectangular loop with a sliding connector of length l is located in a uniform magnetic field perpendicular to the loop plane (Fig. 3.79). The magnetic induction is equal to B. The connector has an electric resistance R, the sides AB and CD have resistances R_1 and R_2 respectively. Neglecting the self-inductance of the loop, find the current flowing in the connector during its motion with a constant velocity v.

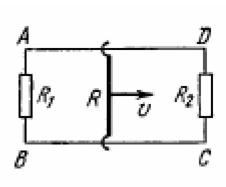
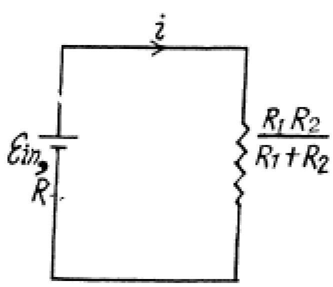


Fig. 3.79.

Solution. 289. Let us assume, $\vec{B}$ is directed into the plane of the loop. Then the motional e.m.f.



$$\xi_{in} = \left| \int -(\vec{v}\times\vec{B})\cdot d\vec{l} \right| = vBl$$

And directed in the same of $(\vec{v}\times\vec{B})$ (Fig.)

$$\text{So,} \quad i = \frac{\xi_{in}}{R + \frac{R_1 R_2}{R_1 + R_2}} = \frac{Bvl}{R + R_u}$$

As R_1 and R_2 are in parallel connections.

Q. 290. A metal disc of radius a = 25 cm rotates with a constant angular velocity ω = 130 rad/s about its axis. Find the potential difference between the centre and the rim of the disc if

(a) the external magnetic field is absent;
(b) the external uniform magnetic field of induction B = 5.0 mT is directed perpendicular to the disc.

Solution. 290. (a) As the metal disc rotates, any free electron also rotates with it with same angular velocity ω, and that's why an electron must have an acceleration $\omega^2 r$ directed towards the disc's centre, where r is separation of the electron from the centre of the disc. We know from Newton's second law that if a particle has some acceleration then there must be a net effective force on it in the direction of acceleration.

We also know that a charged particle can be influenced by two fields electric and magnetic. In our problem magnetic field is absent hence we reach at the conclusion that there is an electric field near any electron and is directed opposite to the acceleration of the electron.

If E be the electric field strength at a distance r from the centre of the disc, we have from Newton's second law.

$$F_n = m w_n$$

$$eE = m r \omega^2, \text{ or, } E = \frac{m \omega^2 r}{e},$$

And the potential difference

$$\varphi_{cen} - \varphi_{rim} = \int_0^a \vec{E} \cdot d\vec{r} = \int_0^a \frac{m \omega^2 r}{e} dr, \text{ as } \vec{E} \uparrow\downarrow d\vec{r}$$

Thus $\varphi_{cen} - \varphi_{rim} = \Delta \varphi = \dfrac{m \omega^2}{e} \dfrac{a^2}{2} = 3{\cdot}0 \text{ nV}$

(b) When field $\vec{B}$ is present, by definition, of motional e.m.f.

$$\varphi_1 - \varphi_2 = \int_1^2 -(\vec{v} \times \vec{B}) \cdot d\vec{r}$$

Hence the sought potential difference,

$$\varphi_{cen} - \varphi_{rim} = \int_0^a -v B\, dr = \int_0^a -\omega r B\, dr, \text{ (as } v = \omega r)$$

Thus $\varphi_{rim} - \varphi_{cen} = \varphi = \dfrac{1}{2} \omega B a^2 = 20 \text{ mV}$

(In general $\omega < \frac{eB}{m}$ so we can neglect the effect discussed in (1) here).

Q. 291. A thin wire AC shaped as a semi-circle of diameter d = 20 cm rotates with a constant angular velocity ω = 100 rad/s in a uniform magnetic field of induction B = 5.0 mT, with $\boldsymbol{\omega} \uparrow\uparrow \mathbf{B}$. The rotation axis passes through the end A of the wire and is perpendicular to the diameter AC. Find the value of a line integral $\int \mathbf{E}\, d\mathbf{r}$ along the wire from point A to point C. Generalize the obtained result.

Solution. 291. By definition,

$$\vec{E} = -(\vec{v} \times \vec{B})$$

So, $$\int_A^C \vec{E} \cdot d\vec{r} = \int_A^C -(\vec{v} \times \vec{B}) \cdot d\vec{r} = \int_0^d - v B\, dr$$

But, v = ωr, where r is the perpendicular distance of the point from A.

Hence, $$\int_A^C \vec{E} \cdot d\vec{r} = \int_0^d -\omega B r\, dr = -\frac{1}{2}\omega B d^2 = -10 \text{ mV}$$

This result can be generalized to a wire AC of arbitary planar shape. We have

$$\int_A^C \vec{E} \cdot d\vec{r} = -\int_A^C (\vec{v} \times \vec{B}) \cdot dr = -\int_A^C ((\omega \times \vec{r}) \times \vec{B}) \cdot d\vec{r}$$

$$= -\int_A^C (\vec{B} \cdot \vec{r}\, \vec{\omega} - \vec{B} \cdot \vec{\omega}\, \vec{r}) \cdot d\vec{r}$$

$$= -\frac{1}{2} B\omega d^2,$$

d being AC and $\vec{r}$ being measured from A.

Q. 292. A wire loop enclosing a semi-circle of radius a is located on the boundary

of a uniform magnetic field of induction B (Fig. 3.80). At the moment t = 0 the loop is set into rotation with a constant angular acceleration p about an axis 0 coinciding with a line of vector B on the boundary. Find the emf induced in the loop as a function of time t. Draw the approximate plot of this function. The arrow in the figure shows the emf direction taken to be positive.

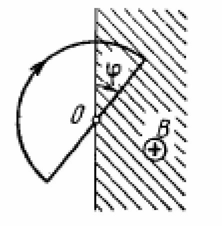


Fig. 3.80.

Solution. 292. Flux at any moment of time,

$$|\Phi_t| = |\vec{B} \cdot d\vec{S}| = B\left(\frac{1}{2}R^2\varphi\right)$$

Where φ is the sector angle, enclosed by the field.

Now, magnitude of induced e.m.f. is given by,

$$\xi_{in} = \left|\frac{d\Phi_t}{dt}\right| = \left|\frac{BR^2}{2}\frac{d\varphi}{dt}\right| = \frac{BR^2}{2}\omega,$$

Where ω is the angular velocity of the disc. But as it starts rotating from rest at t = 0 with an angular acceleration p its angular velocity ω (f) = βt. So,

$$\xi_{in} = \frac{BR^2}{2}\beta t.$$

According to Lenz law the first half cycle current in the loop is in anticlockwise sense, and in subsequent half cycle it is in clockwise sense.

Thus in general, $\xi_{in} = (-1)^n \frac{BR^2}{2}\beta t,$ where n in number of half revolutions.

The plot $\xi_{in}(t)$, where $t_n = \sqrt{2\pi n/\beta}$ is shown in the answer sheet.

Q. 293. A long straight wire carrying a current I and a II-shaped conductor with sliding connector are located in the same plane as shown in Fig. 3.81. The connector of length l and resistance R slides to the right with a constant velocity v.

Find the current induced in the loop as a function of separation r between the connector and the straight wire. The resistance of the H-shaped conductor and the self-inductance of the loop are assumed to be negligible.

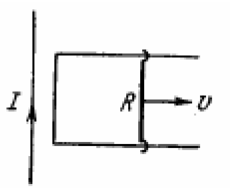


Fig. 3.81.

Solution. 293. Field, due to the current carrying wire in the region, right to it, is directed into the plane of the paper and its magnitude is given by,

$B = \frac{\mu_0}{2\pi}\frac{i}{r}$ Where r is the perpendicular distance from the wire.

As B is same along the length of the rod thus motional e.m.f.

$$\xi_{in} = \left| -\int_1^2 (\vec{v} \times \vec{B}) \cdot d\vec{l} \right| = vBl$$

And it is directed in the sense of $(\vec{v} \times \vec{B})$

So, current (induced) in the loop,

$$i_{in} = \frac{\xi_{in}}{R} = \frac{1}{2}\frac{\mu_0 I vi}{\pi R r}$$

Q. 294. A square frame with side a and a long straight wire carrying a current I are located in the same plane as shown in Fig. 3.82. The frame translates to the right with a constant velocity v. Find the emf induced in the frame as a function of distance x.

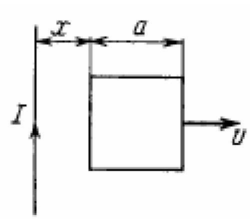


Fig. 3.82.

Solution. 294. Field, due to the current carrying wire, at a perpendicular distance x from it is given by,

$$B(x) = \frac{\mu_0}{2\pi}\frac{i}{x}$$

Motional e.m.f is given by $\left|\int -(\vec{v} \times \vec{B}) \cdot d\vec{l}\right|$

There will be no induced e.m.f. in the segments (2) and (4)

as, $\vec{v} \uparrow\uparrow d\vec{l}$ and magnitude of e.m.f. induced in 1 and 3, will be

$$\xi_1 = v\left(\frac{\mu_0}{2\pi}\frac{i}{x}\right)a \text{ and } \xi_2 = v\left(\frac{\mu_0}{2\pi}\frac{i}{(a+x)}\right)a,$$

Respectively, and their sense will be in the direction of $(\vec{v} \times \vec{B})$.

So, e.m .f., induced in the network $= \xi_1 - \xi_2$ [as $\xi_1 > \xi_2$]

$$= \frac{a\, v\, \mu_0\, i}{2\pi}\left[\frac{1}{x} - \frac{1}{a+x}\right] = \frac{v\, a^2\, \mu_0\, i}{2\,\pi\, x\,(a+x)}$$

Q. 295. A metal rod of mass m can rotate about a horizontal axis O, sliding along a circular conductor of radius a (Fig. 3.83). The arrangement is located in a uniform magnetic field of induction B directed perpendicular to the ring plane. The axis and the ring are connected to an emf source to form a circuit of resistance R. Neglecting the friction, circuit inductance, and ring resistance, find the law according to which the source emf must vary to make the rod rotate with a constant angular velocity ω.

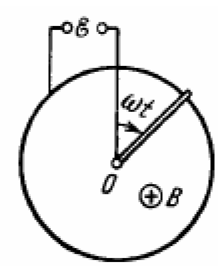


Fig. 3.83.

Solution. 295. As the rod rotates, an emf.

$$\frac{d}{dt}\frac{1}{2}a^2\theta \cdot B = \frac{1}{2}a^2 B\,\omega$$

is induced in it The net current in the conductor is then

$$\frac{\xi(t) - \frac{1}{2}a^2 B\,\omega}{R}$$

A magnetic force will then act on the conductor of magnitude BI per unit length. Its direction will be normal to B and the rod and its torque will be

$$\int_0^a \left(\frac{\xi(t) - \frac{1}{2} a^2 B \omega}{R} \right) dx\, B\, x$$

Obviously both magnetic and mechanical torque acting on the C.M. of the rod must be equal but opposite in sense. Then

for equilibrium at constant ω

$$\frac{\xi(t) - \frac{1}{2} a^2 B \omega}{R} \cdot \frac{B a^2}{2} = \frac{1}{2} mga \sin \omega t$$

or, $\xi(t) = \frac{1}{2} a^2 B \omega + \frac{mg R}{aB} \sin \omega t = \frac{1}{2\, aB} (a^3 B^2 \omega + 2\, mg R \sin \omega t)$

(The answer given in the book is incorrect dimensionally.)

Q. 296. A copper connector of mass m slides down two smooth copper bars, set at an angle a to the horizontal, due to gravity (Fig. 3.84). At the top the bars are interconnected through a resistance R. The separation between the bars is equal to l. The system is located in a uniform magnetic field of induction B, perpendicular to the plane in which the connector slides. The resistances of the bars, the connector and the sliding contacts, as well as the self-inductance of the loop, are assumed to be negligible. Find the steady-state velocity of the connector.

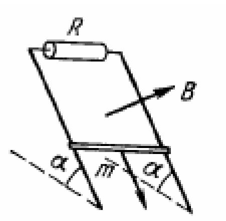
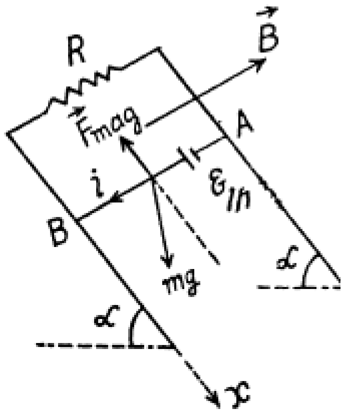


Fig. 3.84.

Solution. 296. From Lenz's law, the current through the connector is directed form A to B. Here $\xi_{in} = vBl$ between A and B

where v is the velocity of the rod at any moment.



For the rod, from $F_x = mw_x$

or, $mg \sin \alpha - i l B = mw$

For steady state, acceleration of the rod must be equal to zero.

Hence, $mg \sin \alpha = i l B$ (1)

But, $i = \frac{\xi_{in}}{R} = \frac{v B l}{R}$

From (1) and (2) $v = \frac{mg \sin \alpha \, R}{B^2 l^2}$

Q. 297. The system differs from the one examined in the foregoing problem (Fig. 3.84) by a capacitor of capacitance C replacing the resistance R. Find the acceleration of the connector.

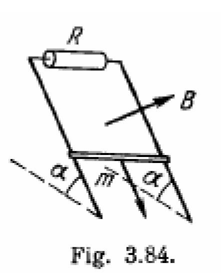
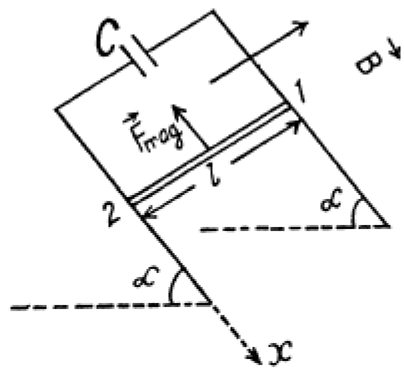


Fig. 3.84.

Solution. 297. From Lenz's 1 aw, the current through the copper bar is directed from 1 to 2 or in other words, the induced current in the circuit is in clockwise sense.



Potential difference across the capacitor plates,

$$\frac{q}{C} = \xi_{in} \quad \text{or,} \quad q = C\,\xi_{in}$$

Hence, the induced current in the loop,

$$i = \frac{dq}{dt} = C\frac{d\,\xi_{in}}{dt}$$

But the variation of magnetic flux through the loop is caused by the movement of the bar. So, the induced e.m.f. $\xi_{in} = B\,lv$

And, $\frac{d\,\xi_{in}}{dt} = Bl\frac{dv}{dt} = Blw$

Hence, $i = C\frac{d\,\xi}{dt} = C\,B\,l\,w$

Now, the forces acting on the bars are the weight and the Ampere's force, where

$$F_{amp} = i\,l\,B\,(CB\,l\,w)\,B = C\,l^2 B^2 w.$$

From Newton's second law, for the rod, $F_x = mw_x$

or, $mg\sin\alpha - C\,l^2 B^2 w = mw$

Hence

$$w = \frac{mg\sin\alpha}{C\,l^2 B^2 + m} = \frac{g\sin\alpha}{1 + \frac{l^2 B^2 C}{m}}$$

Q. 298. A wire shaped as a semi-circle of radius a rotates about an axis OO' with an angular velocity ω in a uniform magnetic field of induction B (Fig. 3.85). The rotation axis is perpendicular to the field direction. The total resistance of the circuit is equal to R. Neglecting the magnetic field of the induced current, find the mean amount of thermal power being generated in the loop during a rotation period.

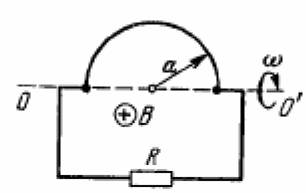


Fig. 3.85.

Solution. 298. Flux of $\vec{B},$ at an arbitrary moment of time t :

$$\Phi_t = \vec{B} \cdot \vec{S} = B \frac{\pi a^2}{2} \cos \omega t,$$

From Faraday's law, induced e.m.f., $\xi_{in} = -\frac{d\Phi}{dt}$

$$= -\frac{d\left(B \pi \frac{a^2}{2} \cos \omega t\right)}{dt} = \frac{B \pi a^2 \omega}{2} \sin \omega t.$$

And induced current, $i_{in} = \frac{\xi_{in}}{R} = \frac{B \pi a^2}{2R} \omega \sin \omega t.$

Now thermal power, generated in the circuit, at the moment t = t :

$$P(t) = \xi_{in} \times i_{in} = \left(\frac{B \pi a^2 \omega}{2}\right)^2 \frac{1}{R} \sin^2 \omega t$$

And mean thermal power generated,

$$< P > = \frac{\left[\frac{B \pi a^2 \omega}{2}\right]^2 \frac{1}{R} \int_0^T \sin^2 \omega t \, dt}{\int_0^T dt} = \frac{1}{2R \left(\frac{\pi \omega a^2 B}{2}\right)^2}$$

Note : The calculation of ξ_{in} which can also be checked by using motional emf is correct even though the conductor is not a closed semicircle , for the flux linked to the rectangular part containing the resistance R is not changing. The answer given in the book is off by a factor 1/4.

Q. 299. A small coil is introduced between the poles of an electromagnet so that its axis coincides with the magnetic field direction. The cross-sectional area of the coil is equal to S = 3.0 mm^2, the number of turns is N = 60. When the coil turns through 180° about its diameter, a ballistic galvanometer connected to the coil indicates a charge q = 4.5 µC flowing through it. Find the magnetic induction magnitude between the poles provided the total resistance of the electric circuit equals R = 40 Ω.

Solution. 299. The flux through the coil changes sign. Initially it is BS per turn. Finally it is - BS per turn. Now if flux is Φ at an intermediate state then the current at that moment will be

$$i = \frac{-N\frac{d\Phi}{dt}}{R}$$

So charge that flows during a sudden turning of the coil is

$$q = \int i\,dt = -\frac{N}{R}[\Phi - (-\Phi)] = 2NBS/R$$

Hence, $B = \frac{1}{2}\frac{qR}{NS} = 0{\cdot}5\text{ T}$ on putting the values.

Q. 300. A square wire frame with side a and a straight conductor carrying a constant current I are located in the same plane (Fig. 3.86). The inductance and the resistance of the frame are equal to L and R respectively. The frame was turned through 180° about the axis OO' separated from the current-carrying conductor by a distance b. Find the electric charge having flown through the frame.

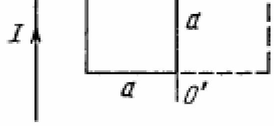


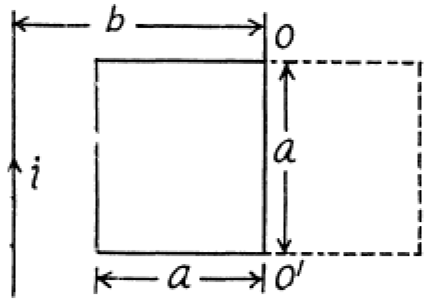
Fig. 3.86.

Solution. 300. According to Ohm's law and Faraday's law of induction, the current i_0 appearing in the frame, during its rotation, is determined by the formula, $i_0 = -\frac{d\Phi}{dt} = -\frac{L\,di_0}{dt}$

Hence, the required amount of electricity (charge) is,

$$q = \int i_0\,dt = -\frac{1}{R}\int (d\Phi + L\,di_0) = -\frac{1}{R}(\Delta\Phi + L\,\Delta i_0)$$

Since the frame has been stopped after rotation, the current in it vanishes, and hence $\Delta i_0 = 0$. It remains for us to find the increment of the flux $\Delta\Phi$ through the frame $(\Delta\Phi = \Phi_2 - \Phi_1)$.



Let us choose the normal $\vec{n}$ to the plane of the frame, for instance, so that in the final position, $\vec{n}$ is directed behind the plane of the figure (along $\vec{B}$).

Then it can be easily seen that in the final position, $\Phi_2 > 0$, while in the initial position, $\Phi_1 < 0$ (th e normal is opposite to $\vec{B}$), and $\Delta\Phi$ turns out to be simply equal to the flux through the surface bounded by the final and initial positions of the frame :

$$\Delta\Phi = \Phi_2 + |\Phi_1| = \int_{b-a}^{b+a} B\,a\,dr,$$

Where B is a function of r, whose form can be easily found with the help of the theorem of circulation. Finally omitting the minus sign, we obtain,

$$q = \frac{\Delta\Phi}{R} = \frac{\mu_0 a\,i}{2\pi R}\ln\frac{b+a}{b-a}$$

Q. 301. A long straight wire carries a current I_0. At distances a and b from it there are two other wires, parallel to the former one, which are interconnected by a resistance R (Fig. 3.87). A connector slides without friction along the wires with a constant velocity v. Assuming the resistances of the wires, the connector, the sliding contacts, and the self-inductance of the frame to be negligible, find:
(a) the magnitude and the direction of the current induced in the connector;
(b) the force required to maintain the connector's velocity constant.

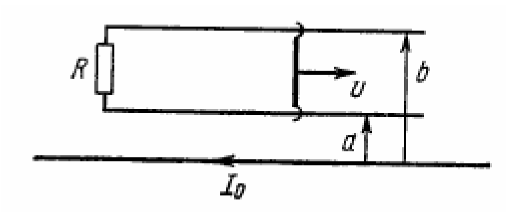
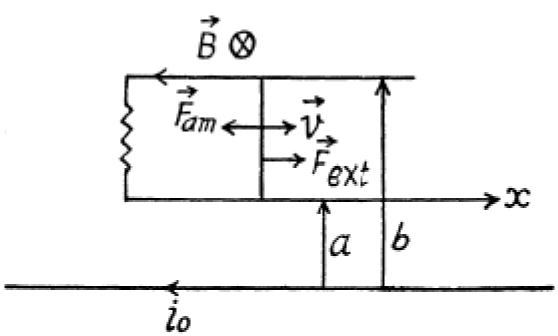


Fig. 3.87.

Solution. 301. As $\vec{B}$, due to the straight current carrying wire, varies along the rod (connector) and enters linearly so, to make the calculations simple $\vec{B}$ is made constant by taking its average value in the range [a, b].

$$< B > = \frac{\int_a^b B\,dr}{\int_a^b dr} = \frac{\int_a^b \frac{\mu_0}{2\pi}\frac{i_0}{r}\,dr}{\int_a^b dr}$$

or, $$< B > = \frac{\mu_0}{2\pi}\frac{i_0}{(b-a)}\ln\frac{b}{a}$$



(a) The flux of $\vec{B}$ changes through the loop due to the movement of the connector. According to Lenz's law, the current in the loop will be anticlockwise. The magnitude of motional e.m.f.

$$\xi_{in} = v < B > (b - a)$$
$$= \frac{\mu_0}{2\pi} \frac{i_0}{(b-a)} \ln\frac{b}{a} (b-a) \frac{dx}{dt} = \frac{\mu_0}{2\pi} i_0 \ln\frac{b}{a} v$$

So, induced current

$$i_{in} = \frac{\xi_{in}}{R} = \frac{\mu_0}{2\pi} \frac{i_0 v}{R} \ln\frac{b}{a}$$

(b) The force required to maintain the constant velocity of the connector must be the magnitude equal to that of Ampere's acting on the connector, but in opposite direction.

So, $$F_{ext} = i_{in}\, l < B > = \left(\frac{\mu_0}{2\pi} \frac{i_0}{R} v \ln\frac{b}{a}\right)(b-a)\left(\frac{\mu_0}{2\pi} \frac{i_0}{(b-a)} \ln\frac{b}{a}\right)$$

$$= \frac{v}{R}\left(\frac{\mu_0}{2\pi} i_0 \ln\frac{b}{a}\right)^2,$$ And will be directed as shown in the (Fig.)

Q. 302. A conducting rod AB of mass m slides without friction over two long conducting rails separated by a distance l (Fig. 3.88). At the left end the rails are interconnected by a resistance R. The system is located in a uniform magnetic field perpendicular to the plane of the loop. At the moment t = 0 the rod AB starts moving to the right with an initial velocity vo. Neglecting the resistances of the rails and the rod AB, as well as the self-inductance, find:
(a) the distance covered by the rod until it comes to a standstill;
(b) the amount of heat generated in the resistance R during this process.

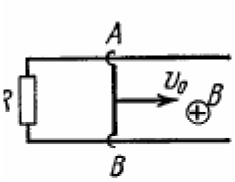
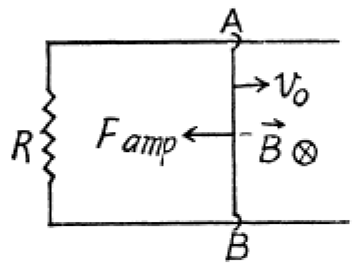


Fig. 3.88.

Solution. 302. (a) The flux through the loop changes due to the movement of the rod AB. Recording to Lenz's law current should be anticlockwise in sense as we have assumed $\vec{B}$ is directed into the plane of the loop. The motion e.m.f $\xi_{in}(t) = B l v$

And, induced current $i_{in} = \frac{v B l}{R}$



From Newton's law in projection form $f_x = mw_x$

$$-F_{amp} = m\frac{v\,dv}{dx}$$

But $\quad F_{amp} = i_{in}\,l\,B = \frac{v B^2 l^2}{R}$

So, $\quad -\frac{v B^2 l^2}{R} = m v \frac{dv}{dx}$

or, $\quad \int_0^x dx = -\frac{mR}{B^2 l^2}\int_{v_0}^0 dv \quad$ or, $\quad x = \frac{m R v_0}{B^2 l^2}$

(b) From equation of energy conservation; $E_f - E_i$ + Heat liberated = $A_{cell} + A_{ext}$

$$\left[0 - \frac{1}{2} m v_0^2\right] + \text{Heat liberated} = 0 + 0$$

So, heat liberated $= \frac{1}{2} m v_0^2$

Q. 303. A connector AB can slide without friction along a Shaped conductor located in a horizontal plane (Fig. 3.89). The connector has a length l, mass m, and resistance R. The whole system is located in a uniform magnetic field of induction B directed vertically. At the moment t = 0 a constant horizontal force F starts acting on the connector shifting it translation wise to the right. Find how the velocity of the connector varies with time t. The inductance of the loop and the resistance of the II-shaped conductor are assumed to be negligible.

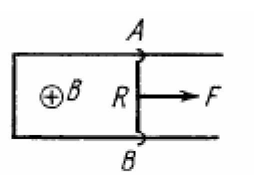
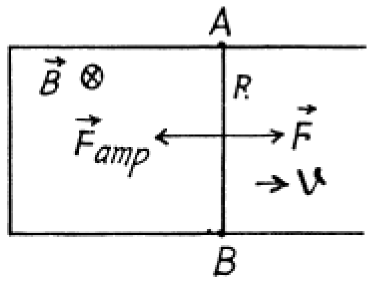


Fig. 3.89.

Solution. 303. With the help of the calculation, done in the previous problem, Ampere's force on the connector,



$\vec{F}_{amp} = \frac{v B^2 l^2}{R}$ directed towards left.

Now from Newton's second law,

$$F - F_{amp} = m \frac{dv}{dt}$$

So, $$F = \frac{v B^2 l^2}{R} + m \frac{dv}{dt}$$

or, $$\int_0^t dt = m \int_0^v \frac{dv}{F - \frac{v B^2 l^2}{R}}$$

or, $$\frac{t}{m} = -\frac{R}{B^2 l^2} \ln \left(\frac{F - \frac{v B^2 l^2}{R}}{F} \right)$$

Thus $$v = \left(1 - e^{\frac{-t B^2 l^2}{Rm}} \right) \frac{RF}{B^2 l^2}$$

Q. 304. Fig. 3.90 illustrates plane figures made of thin conductors which are located in a uniform magnetic field directed away from a reader beyond the plane of the drawing. The magnetic induction starts diminishing. Find how the currents induced in these loops are directed.

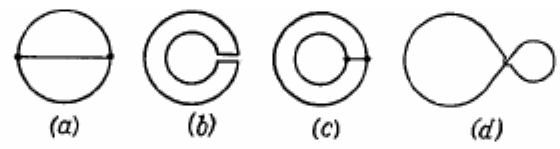
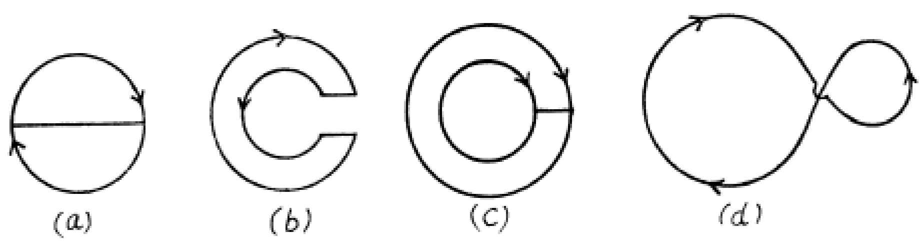


Fig. 3.90.

Solution. 304. According to Lenz, the sense of induced e.m.f. is such that it opposes the cause of change of flux. In our problem, magnetic field is directed away from the reader and is diminishing.



So, in figure (a), in the round conductor, it is clockwise and there is no current in the connector

In figure (b) in the outside conductor, clockwise.

In figure (c) in both the conductor, clockwise; and there is no current in the connector to obey the charge conservation.

In figure (d) in the left side of the figure, clockwise.

Q. 305. A plane loop shown in Fig. 3.91 is shaped as two squares with sides a = 20 cm and b = 10 cm and is introduced into a uniform magnetic field at right angles to the loop's plane. The magnetic induction varies with time as $B = B_0 \sin \omega t$, where $B_0 = 10$ mT and $\omega = 100\ s^{-1}$. Find the amplitude of the current induced in the loop if its resis- tance per unit length is equal to p = 50 mΩ/m. The inductance of the loop is to be neglected.

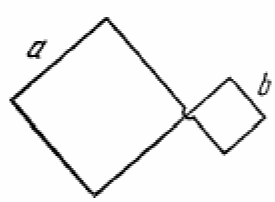


Fig. 3.91.

Solution. 305. The loops are connected in such a way that if the current is clockwise in one, it is anticlockwise in the other. Hence the e.m.f. in loop b opposes the e.m.f. in loop a.

e.m.f. in loop $a = \frac{d}{dt}(a^2 B) = a^2 \frac{d}{dt}(B_0 \sin \omega t)$

Similarly, e.m.f. in loop $b = b^2 B_0 \omega \cos \omega t$.

Hence, net e.m.f. in the circuit $= (a^2 - b^2) B_0 \omega \cos \omega t$, as both the e.m.f's are in opposite sense, and resistance of the circuit = 4 (a + b) p

Therefore, the amplitude of the current $= \frac{(a^2 - b^2) B_0 \omega}{4(a+b)\rho} = 0{\cdot}5$ A.

Q. 306. A plane spiral with a great num- ber N of turns wound tightly to one another is located in a uniform magnetic field perpendicular to the spiral's plane. The outside radius of the spiral's turns is equal to a. The magnetic induction varies with time as $B = B_0 \sin \omega t$, where B_0 and ω are constants. Find the amplitude of emf induced in the spiral

Solution. 306. The flat shape is made up of concentric loops, having different radii, varying from 0 to a.

Let us consider an elementary loop of radius r, then e.m.f. induced due to this

loop $= \frac{-d(\vec{B}\cdot\vec{S})}{dt} = \pi r^2 B_0 \omega \cos \omega t$.

And the total induced e.m.f.,

$$\xi_{ind} = \int_0^a (\pi r^2 B_0 \omega \cos \omega t)\, dN, \quad (1)$$

Where $\pi r^2 \omega \cos \omega t$ is the contribution of one turn of radius r and dN is the number of turns in the interval [r, r + dr].

So, $dN = \left(\frac{N}{a}\right) dr$ (2)

From (1) and (2), $\xi = \int_0^a -(\pi r^2 B_0 \omega \cos \omega t) \frac{N}{a} dr = \frac{\pi B_0 \omega \cos \omega t N a^2}{3}$

Maximum value of e.m.f. amplitude $\xi_{max} = \frac{1}{3}\pi B_0 \omega N a^2$

Q. 307. A H-shaped conductor is located in a uniform magnetic field perpendicular to the plane of the conductor and varying with time at the rate $\dot{B} = 0.10$ T/s. A conducting connector starts moving with an acceleration w = 10 cm/s^2 along the parallel bars of the conductor. The length of the connector is equal to l = 20 cm. Find the emf induced in the loop t = 2.0 s after the beginning of the motion, if at the moment t = 0 the loop area and the magnetic induction are equal to zero. The inductance of the loop is to be neglected.

Solution. 307. The flux through the loop changes due to the variation in $\vec{B}$ with time and also due to the movement of the connector.

So , $\xi_{in} = \left|\frac{d(\vec{B}\cdot\vec{S})}{dt}\right| = \left|\frac{d(BS)}{dt}\right|$ as $\vec{S}$ and $\vec{B}$ are collinicar

But, B, after t sec. of beginning of motion - Bt, and S becomes $= l\frac{1}{2}wt^2$, as

connector starts moving from rest with a constant acceleration w.

So, $\xi_{ind} = \frac{3}{2}B\,l\,w\,t^2$

Q. 308. In a long straight solenoid with cross-sectional radius a and number of turns per unit length n a current varies with a constant velocity $\dot{I}$ A/s. . Find the magnitude of the eddy current field strength as a function of the distance r from the solenoid axis. Draw the approximate plot of this function.

Solution. 308. We use $B = \mu_0 n I$

Then, from the law of electromagnetic induction

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt}$$

So, for r < a

$$E_\varphi 2\pi r = -\pi r^2 \mu_0 n \dot{I} \text{ or, } E_\varphi = -\frac{1}{2}\mu_0 n r \dot{I}. \text{ (where } \dot{I} = dI/dt)$$

For r>a

$E_\varphi 2\pi r = -\pi a^2 \mu_0 n \dot{I}$ or, $E_\varphi = -\mu_0 n \dot{I} a^2/2r$

The meaning of minus sign can be deduced from Lenz's law.

Q. 309. A long straight solenoid of cross-sectional diameter d = 5 cm and with n = 20 turns per one cm of its length has a round turn of copper wire of cross-sectional area S = 1.0 mm^2 tightly put on its winding. Find the current flowing in the turn if the current in the solenoid winding is increased with a constant velocity $\dot{I} =$ 100 A/s. The inductance of the turn is to be neglected.

Solution. 309. The e.m.f. induced in the turn is $\mu_0 n \dot{I} \pi \frac{d^2}{4}$

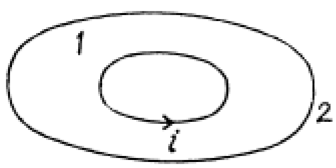
The resistance is $\frac{\pi d}{S} \rho.$

So, the current is $\frac{\mu_0 n \dot{I} S d}{4\rho} = 2\text{ mA},$ where p is the resistivity of copper.

Electromagnetic Induction Maxwell's Equations (Part - 3)

Q. 331. Two thin concentric wires shaped as circles with radii a and b lie in the same plane. Allowing for a ≪ b, find:
(a) their mutual inductance;
(b) the magnetic flux through the surface enclosed by the outside wire, when the inside wire carries a current I.

Solution. 331. The direct calculation of the flux Φ_2 is a rather complicated problem, since the configuration of the field itself is complicated. However, the application of the reciprocity theorem simplifies the solution of the problem. Indeed, let the same current i flow through loop 2. Then the magnetic flux created by this current through loop 1 can be easily found.



Magnetic induction at the centre of the loop, $: B = \frac{\mu_0 i}{2b}$

So, flux through loop $1, : \Phi_{12} = \pi a^2 \frac{\mu_0 i}{2b}$

And from reciprocity theorem,

$$\varphi_{12} = \Phi_{21}, \ \Phi_{21} = \frac{\mu_0 \pi a^2 i}{2b}$$

$$So, \ L_{12} = \frac{\Phi_{21}}{i} = \frac{1}{2} \mu_0 \pi a^2 / b$$

Q. 332. A small cylindrical magnet M (Fig. 3.95) is placed in the centre of a thin coil of radius a consisting of N turns. The coil is connected to a ballistic galvanometer. The active resistance of the whole circuit is equal to R. Find the magnetic moment of the magnet if its removal from the coil results in a charge q flowing through the galvanometer.

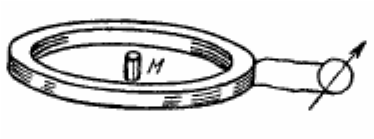


Fig. 3.95.

Solution. 332. Let $\vec{p}_m$ be be the magnetic moment of the magnet Af. Then the magnetic field due to this magnet is,

$$\frac{\mu_0}{4\pi}\left[\frac{3(\vec{p}_m\cdot\vec{r})\vec{r}}{r^5}-\frac{\vec{p}_m}{r^3}\right].$$

The flux associated with this, when the magnet is along the axis at a distance x from the centre, is

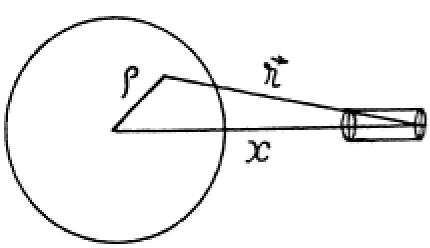
$$\Phi=\frac{\mu_0}{4\pi}\int\left[\frac{3(\vec{p}_m\cdot\vec{r})\vec{r}}{r^5}-\frac{\vec{p}_m}{r^3}\right]\cdot d\vec{S}=\Phi_1-\Phi_2.$$

$$\text{where, }\Phi_2=\frac{\mu_0}{4\pi}p_m\int_0^a\frac{2\pi\rho\,d\rho}{(x^2+\rho^2)^{3/2}}=\frac{\mu_0 p_m}{2}\left(\frac{1}{x}-\frac{1}{\sqrt{x^2+a^2}}\right)$$

$$\text{and }\Phi_1=\frac{3\mu_0 p_m x^2}{4\pi}\int_0^a\frac{2\pi\rho\,d\rho}{(x^2+\rho^2)^{5/2}}$$

$$=\frac{\mu_0 p_m x^2}{2}\left(\frac{1}{x^3}-\frac{1}{(x^2+a^2)^{3/2}}\right)$$

$$\text{So, }\Phi=\frac{-\mu_0 p_m a^2}{2(x^2+a^2)^{3/2}}$$



When the flux changes, an e.m.f. $-N\frac{d\Phi}{dt}$ is Induced and a current $-N\frac{d\Phi}{dt}$ flows.
The total charge q, flowing, as the magnet is removed to infinity from x = 0 is,

$$q = \frac{N}{R}\Phi(x = 0) = \frac{N}{R}\cdot\frac{\mu_0 p_m}{2a}$$

Or, $p_m = \frac{2aqR}{N\mu_0}$

Q. 333. Find the approximate formula expressing the mutual inductance of two thin coaxial loops of the same radius a if their centres are separated by a distance l, with l ≫ a.

Solution. 333. If a current l flows in one of the coils, the magnetic field at the centre of the other coil is,

$$B = \frac{\mu_0 a^2 I}{2(l^2 + a^2)^{3/2}} = \frac{\mu_0 a^2 I}{2 l^3}, \text{ as } l >> a.$$

The flux associated with the second coil is then approximately $\mu_0 \pi a^4 I / 2 l^3$

Hence, $L_{12} = \frac{\mu_0 \pi a^4}{2 l^3}$

Q. 334. There are two stationary loops with mutual inductance L_{12}. The current in one of the loops starts to be varied as $I_1 = \alpha t$, where α is a constant, t is time. Find the time dependence $I_2(t)$ of the current in the other loop whose inductance is L_2 and resistance R.

Solution. 334. When the current in one of the loop is $I_1 = \alpha t$, an e.m.f. $L_{12}\frac{dI_1}{dt} = L_{12}\alpha$, is

induced in the other loop. Then if the current in the other loop is I_2 we must have,

$$L_2\frac{dI_2}{dt} + RI_2 = L_{12}\alpha$$

This familiar equation has the solution,

$I_2 = \frac{L_{12}\alpha}{R}\left(1 - e^{\frac{-tR}{L_2}}\right)$ Which is the required current

Q. 335. A coil of inductance L = 2.0 μH and resistance R = 1.0 Ω is connected to a source of constant emf $\mathcal{E}$ = 3.0 V (Fig. 3.96). A resistance R_0 = 2.0 Ω is connected in parallel with the coil. Find the amount of heat generated in the coil after the switch Sw is disconnected. The internal resistance of the source is negligible.

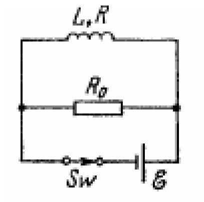
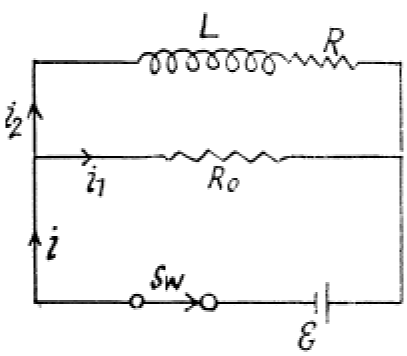


Fig. 3.96.

Solution. 335. Initially, after a steady current is set up, the current is flowing as shown.

In steady condition $i_{20} = \frac{\xi}{R},\ i_{10} = \frac{\xi}{R_0}.$



When the switch is disconnected, the current through R_0 changes from i_{10} to the right, to i_{20} to the left. (The current in the inductance cannot change suddenly.). We then have the equation,

$$L\,\frac{di_2}{dt} + (R + R_0)\,i_2 = 0.$$

This equation has the solution $i_2 = i_{20}\,e^{-t(R+R_0)/L}$

The heat dissipated in the coil is,

$$Q = \int_0^\infty i_2^2 R\,dt = i_{20}^2 R \int_0^\infty e^{-2t(R+R_0)/L}\,dt$$

$$= R\,i_{20}^2 \times \frac{L}{2(R+R_0)} = \frac{L\,\xi^2}{2R(R+R_0)} = 3\ \mu\mathrm{J}$$

Q. 336. An iron tore supports N = 500 turns. Find the magnetic field energy if a current I = 2.0 A produces a magnetic flux across the tore's cross-section equal to $\Phi = 1.0$ mWb.

Solution. 336. To find the magnetic field energy we recall that the flux varies linearly with current Thus, when the flux is Φ for current i, we can write $\phi = A\,i.$ The total energy enclosed in the field, when the current is /, is

$$W = \int \xi i\, dt = \int N \frac{d\Phi}{dt} i\, dt$$

$$= \int N\, d\Phi\, i = \int_0^I N A\, i\, di = \frac{1}{2} N A I^2 = \frac{1}{2} N \Phi I$$

The characteristic factor 1/2 appears in this way.

Q. 337. An iron core shaped as a doughnut with round cross-section of radius a = 3.0 cm carries a winding of N = 1000 turns through which a current I = 1.0 A flows. The mean radius of the doughnut is b = 32 cm. Using the plot in Fig. 3.76, find the magnetic energy stored up in the core. A field strength H is supposed to be the same throughout the cross-section and equal to its magnitude in the centre of the cross-section.

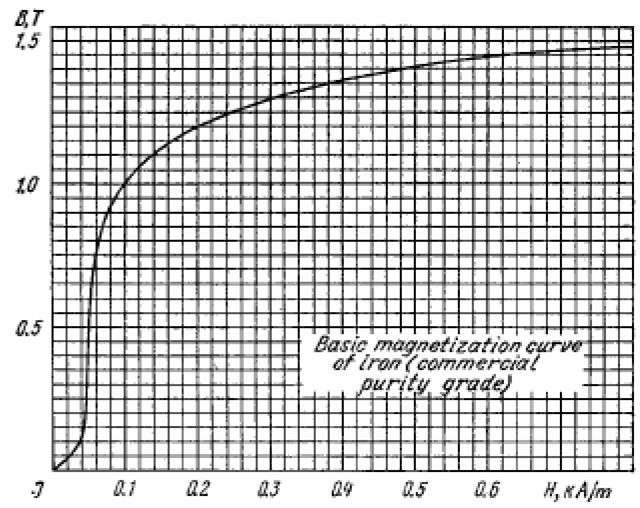


Fig. 3.76.

Solution. 337. We apply circulation theorem

$H \cdot 2\pi b = NI$, or, $H = NI/2\pi b$.

Thus the total energy,

$$W = \frac{1}{2} BH \cdot 2\pi b \cdot \pi a^2 = \pi^2 a^2\, b\, BH.$$

Given N , l , b we know H , and can find out B from the B - H curve. Then W can be calculated.

Q. 338. A thin ring made of a magnetic has a mean diameter d = 30 cm and supports a winding of N = 800 turns. The cross-sectional area of the ring is equal to S = 5.0 cm^2. The ring has a cross-cut of width b = 2.0 mm. When the winding carries a certain current, the permeability of the magnetic equals μ = 1400. Neglecting the dissipation of magnetic flux at the gap edges, find:
(a) the ratio of magnetic energies in the gap and in the magnetic;
(b) the inductance of the system; do it in two ways: using the flux and using the energy of the field.

Solution. 338. From From $\oint \vec{H} \cdot d\vec{r} = NI$,

$$H \cdot \pi d + \frac{B}{\mu_0} \cdot b = NI, (d >> b)$$

Also, $B = \mu \mu_0 H$. Thus, $H = \frac{NI}{\pi d + \mu b}$.

Since B is continuous across the gap, B is given by,

$B = \mu \mu_0 \frac{NI}{\pi d + \mu b}$, Both in the magnetic and the gap.

$$\text{(a)} \quad \frac{W_{gap}}{W_{magnetic}} = \frac{\frac{B^2}{2\mu_0} \times S \times b}{\frac{B^2}{2\mu\mu_0} \times S \times \pi d} = \frac{\mu b}{\pi d}.$$

$$\text{(b) The flux is } N \int \vec{B} \cdot d\vec{S} = N \mu\mu_0 \frac{NI}{\pi d + \mu b} \cdot S = \mu_0 \frac{SN^2 I}{b + \frac{\pi d}{\mu}},$$

So, $L = \frac{\mu_0 S N^2}{b + \frac{\pi d}{\mu}}$.

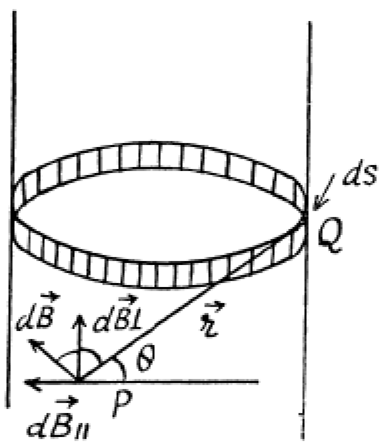
Energy wise; total energy

$$= \frac{B^2}{2\mu_0}\left(\frac{\pi d}{\mu} + b\right) S = \frac{1}{2} \frac{\mu_0 N^2 S}{b + \frac{\pi d}{\mu}} \cdot I^2 = \frac{1}{2} L I^2$$

The L, found in the one way, agrees with that, found in the other way. Note that, in calculating the flux, we do not consider the field in the gap, since it is not linked to the winding. But the total energy includes that of the gap.

Q. 339. A long cylinder of radius a carrying a uniform surface charge rotates about its axis with an angular velocity ω. Find the magnetic field energy per unit length of the cylinder if the linear charge density equals λ, and μ, = 1.

Solution. 339. When the cylinder with a linear charge density λ rotates with a circular frequency co, a surface current density (charge / length x time) of $i = \frac{\lambda\omega}{2\pi}$ is set up.



The direction of the surface current is normal to the plane of paper at Q and the contribution of this current to the magnetic field at P is $d\vec{B} = \frac{\mu_0}{4\pi}\frac{i(\vec{e}\times\vec{r})}{r^3}dS$ where $\vec{e}$ is the direction of the current. In magnitude, $|\vec{e}\times\vec{r}| = r$, since $\vec{e}$ is normal to $\vec{r}$ and the direction of $d\vec{B}$ is shown.

It's component, $d\vec{B}_{\parallel}$ cancels out by cylindrical symmetry. The component that survives is,

$$|\vec{B}_\perp| = \frac{\mu_0}{4\pi}\int\frac{i\,dS}{r^2}\cos\theta = \frac{\mu_0 i}{4\pi}\int d\Omega = \mu_0 i,$$

Where we have used $\frac{dS\cos\theta}{r^2} = d\Omega$ and $\int d\Omega = 4\pi,$ the total solid angle around any point.

The magnetic field vanishes outside the cylinder by similar argument.

The total energy per unit length of the cylinder is,

$$W_1 = \frac{1}{2\mu_0}\mu_0^2\left(\frac{\lambda\omega}{2\pi}\right)^2 \times \pi a^2 = \frac{\mu_0}{8\pi}a^2\lambda^2\omega^2$$

Q. 340. At what magnitude of the electric field strength in vacuum the volume energy density of this field is the same as that of the magnetic field with induction B = 1.0 T (also in vacuum).

Solution. 340. $w_E = \frac{1}{2}\varepsilon_0 E^2,$ for the electric field,

$w_B = \frac{1}{2\mu_0}B^2$ For the magnetic field.

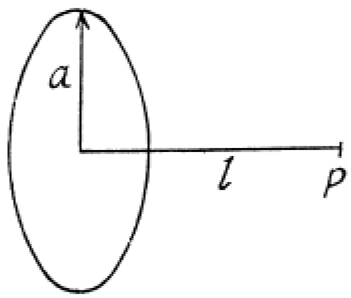
Thus, $\frac{1}{2\mu_0}B^2 = \frac{1}{2}\varepsilon_0 E^2,$

When $E = \frac{B}{\sqrt{\varepsilon_0\mu_0}} = 3 \times 10^8$ V/m

Q. 341. A thin uniformly charged ring of radius a = 10 cm rotates about its axis with an angular velocity ω = 100 rad/s. Find the ratio of volume energy densities of magnetic and electric fields on the axis of the ring at a point removed from its centre by a distance l = a.

Solution. 341. The electric field at P is,

$$E_p = \frac{ql}{4\pi\varepsilon_0(a^2+l^2)^{3/2}}$$



To get the magnetic field, note that the rotating ring constitutes a current i - q ω/2 π, and the corresponding magnetic field at P is,

$$B_p = \frac{\mu_0 a^2 i}{2(a^2+l^2)^{3/2}}$$

$$\text{Thus, } \frac{w_E}{w_M} = \frac{\varepsilon_0 \mu_0 E^2}{B^2} = \varepsilon_0 \mu_0 \left(\frac{ql \times 2}{4\pi \varepsilon_0 \mu_0 a^2 i}\right)^2$$

$$= \frac{1}{\varepsilon_0 \mu_0}\left(\frac{l}{a^2 \omega}\right)^2$$

$$\text{or, } \qquad \frac{w_M}{w_E} = \varepsilon_0 \mu_0 \omega^2 a^4/l^2$$

Q. 342. Using the expression for volume density of magnetic energy, demonstrate that the amount of work contributed to magnetization of a unit volume of para- or diamagnetic, is equal to A = — JB/2.

Solution. 342. The total eneigy of the magnetic field is,

$$\frac{1}{2}\int (\vec{B}\cdot\vec{H})\,dV = \frac{1}{2}\int \vec{B}\cdot\left(\frac{\vec{B}}{\mu_0}-\vec{J}\right)dV$$
$$= \frac{1}{2\mu_0}\int \vec{B}\cdot\vec{B}\,dV - \frac{1}{2}\int \vec{J}\cdot\vec{B}\,dV.$$

The second term can be interpreted as the energy of magnetization, and has the density $-\frac{1}{2}\vec{J}\cdot\vec{B}.$

Q. 343. Two identical coils, each of inductance L, are interconnected (a) in series, (b) in parallel. Assuming the mutual inductance of the coils to be negligible, find the inductance of the system in both cases.

Solution. 343. (a) In series, the current I flows through both coils, and the total e.m.f. induced, when the current changes is,

$$-2L\frac{dI}{dt} = -L'\frac{dI}{dt}$$

or, $L' = 2L$

(b) In parallel, the current flowing through either coil is $\frac{I}{2}$ and the e.m.f. induced is $-L\left(\frac{1}{2}\frac{dI}{dt}\right)$.

Equating this to $-L'\frac{dI}{dt}$, we find $L' = \frac{1}{2}L$.

Q. 344. Two solenoids of equal length and almost equal cross-sectional area are fully inserted into one another. Find their mutual inductance if their inductances are equal to L_1 and L_2.

Solution. 344. We use $L_1 = \mu_0 n_1^2 V, L_2 = \mu_0 n_2^2 V$

So, $L_{12} = \mu_0 n_1 n_2 V = \sqrt{L_1 L_2}$

Q. 345. Demonstrate that the magnetic energy of interaction of two current-carrying loops located in vacuum can be represented as $W_{ia} = (1/\mu_0)\int \mathbf{B}_1\mathbf{B}_2\,dV$, where $\mathbf{B}_1$ and $\mathbf{B}_2$ are the magnetic inductions within a volume element dV, produced individually by the currents of the first and the second loop respectively.

Solution. 345. The interaction energy is

$$\frac{1}{2\mu_0}\int\left|\vec{B}_1+\vec{B}_2\right|^2 dV - \frac{1}{2\mu_0}\int\left|\vec{B}_1\right|^2 dV - \frac{1}{2\mu_0}\int\left|\vec{B}_2\right|^2 dV$$

$$= \frac{1}{\mu_0}\int \vec{B}_1\cdot\vec{B}_2\,dV$$

Here, if $\vec{B}_1$ is the magnetic field produced by the first of the current carrying loops,

and $\vec{B}_2$, that of the second one, then the magnetic field due to both the loops will be $\vec{B}_1+\vec{B}_2$.

Q. 346. Find the interaction energy of two loops carrying currents I_1 and I_2 if both loops are shaped as circles of radii a and b, with a ≪ b. The loops' centres are located at the same point and their planes form an angle θ between them.

Solution. 346. We can think of the smaller coil as constituting a magnet of dipole moment,

$$p_m = \pi a^2 I_1$$

Its direction is normal to the loop and makes an angle θ with the direction of the

magnetic field, due to the bigger loop. This magnetic Geld is,

$$B_2 = \frac{\mu_0 I_2}{2b}$$

The interaction energy has the magnitude,

$$|W| = \frac{\mu_0 I_1 I_2}{2b} \pi a^2 \cos\theta$$

Its sign depends on the sense of the currents.

Q. 347. The space between two concentric metallic spheres is filled up with a uniform poorly conducting medium of resistivity p and permittivity ε. At the moment t = 0 the inside sphere obtains a certain charge. Find:
(a) the relation between the vectors of displacement current density and conduction current density at an arbitrary point of the medium at the same moment of time;
(b) the displacement current across an arbitrary closed surface wholly located in the medium and enclosing the internal sphere, if at the given moment of time the charge of that sphere is equal to q.

Solution. 347. (a) There is a radial outward conduction current Let Q be the

instantaneous charge on the inner sphere, then,

$$j \times 4\pi r^2 = -\frac{dQ}{dt} \quad \text{or,} \quad \vec{j} = -\frac{1}{4\pi r^2}\frac{dQ}{dt}\hat{r}.$$

On the other hand $\vec{j_d} = \frac{\partial \vec{D}}{\partial t} = \frac{d}{dt}\left(\frac{Q}{4\pi r^2}\hat{r}\right) = -\vec{j}$

(b) At the given moment, $\vec{E} = \frac{q}{4\pi\varepsilon_0 \varepsilon r^2}\hat{r}$

and by Ohm's law, $\vec{j} = \frac{\vec{E}}{\rho} = \frac{q}{4\pi\varepsilon_0\varepsilon\rho r^2}\hat{r}$

Then, $\vec{j_d} = -\frac{q}{4\pi\varepsilon_0\varepsilon\rho r^2}\hat{r}$

And $\oint \vec{j_d}\cdot d\vec{S} = -\frac{q}{4\pi\varepsilon_0\varepsilon\rho}\int\frac{dS\cos\theta}{r^2} = -\frac{q}{\varepsilon_0\varepsilon\rho}.$

The surface integral must be - ve because $\vec{j_d}$, being opposite of $\vec{j}$, is inward.

Q. 348. A parallel-plate capacitor is formed by two discs with a uniform poorly conducting medium between them. The capacitor was initially charged and then disconnected from a voltage source. Neglecting the edge effects, show that there is no magnetic field between capacitor plates.

Solution. 348. Here also we see that neglecting edge effects, $\vec{j_d} = -\vec{j}.$ Thus Maxwell's equations reduce to ,div $\vec{B} = 0$, Curl $\vec{H} = 0$, $\vec{B} = \mu\vec{H}$

A general solution of this equation is $\vec{B} = \text{constant} = \vec{B_0} \cdot \vec{B_0}$ can be thought of as

an extraneous magnetic field. If it is zero, $\vec{B} = 0.$

Q. 349. A parallel-plate air condenser whose each plate has an area S = 100 cm^2 is connected in series to an ac circuit. Find the electric field strength amplitude in the capacitor if the sinusoidal current amplitude in lead wires is equal to I$_m$. = 1.0 mA and the current frequency equals ω = 1.6-10^7 s^{-1}.

Solution. 349. Given $I = I_m \sin \omega t$. We see that

$$j = \frac{I_m}{S}\sin\omega t = -j_d = -\frac{\partial D}{\partial t}$$

or, $D = \frac{I_m}{\omega S}\cos\omega t$, so, $E_m = \frac{I_m}{\varepsilon_0\omega S}$ is the amplitude of the electric field and is 7V / cm

Q. 350. The space between the electrodes of a parallel-plate capacitor is filled with a uniform poorly conducting medium of conductivity σ and permittivity ε. The capacitor plates shaped as round discs are separated by a distance d. Neglecting the edge effects, find the magnetic field strength between the plates .at a distance r from their axis if an ac voltage V = V_m, cos cot is applied to the capacitor.

Solution. 350. The electric field between the plates can be written as,

$$E = Re\frac{V_m}{d}e^{i\omega t}, \text{ instead of } \frac{V_m}{d}\cos\omega t.$$

This gives rise to a conduction current,

$$j_c = \sigma E = \text{Re}\frac{\sigma}{d}V_m e^{i\omega t}$$

and a displacement current,

$$j_d = \frac{\partial D}{\partial t} = \text{Re}\, \varepsilon_0 \varepsilon\, i\omega\frac{V_m}{d}e^{i\omega t}$$

The total current is,

$$j_T = \frac{V_m}{d}\sqrt{\sigma^2 + (\varepsilon_0 \varepsilon \omega)^2}\cos(\omega t + \alpha)$$

where, $\tan\alpha = \frac{\sigma}{\varepsilon_0 \varepsilon \omega}$ on taking the real part of the resultant.

The corresponding magnetic field is obtained by using circulation theorem,

$$H \cdot 2\pi r = \pi r^2 j_T$$

or, H = H_m cos (ωt + α), where, $H_m = \frac{rV_m}{2d}\sqrt{\sigma^2 + (\varepsilon_0 \varepsilon\ \omega)^2}$

Q. 351. A long straight solenoid has n turns per unit length. An alternating current I = I_m sin ωt flows through it. Find the displacement current density as a function of the distance r from the solenoid axis. The cross-sectional radius of the solenoid equals R.

Solution. 351. Inside the solenoid, there is a magnetic field,

$B = \mu_0 n I_m \sin \omega t.$

Since this varies in time there is an associated electric field. This is obtained by using,

$$\oint_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S}$$

For $r < R,\ 2\pi r E = -\dot{B} \cdot \pi r^2,\ \text{or,}\ \ E = -\frac{\dot{B} r}{2}$

For $r > R,\ E = -\frac{\dot{B} R^2}{2r}$

The associated displacement current density is,

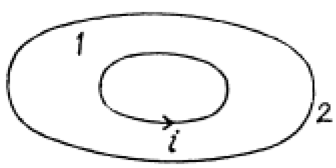
$$j_d = \varepsilon_0 \frac{\partial E}{\partial t} = \begin{cases} -\varepsilon_0 \ddot{B}\, r/2 \\ -\varepsilon_0 \ddot{B}\, R^2/2r \end{cases}$$

The answer, given in the book, is dimensionally incorrect without the factor ε_0.

Electromagnetic Induction Maxwell's Equations (Part - 3)

Q. 331. Two thin concentric wires shaped as circles with radii a and b lie in the same plane. Allowing for a ≪ b, find:
(a) their mutual inductance;
(b) the magnetic flux through the surface enclosed by the outside wire, when the inside wire carries a current I.

Solution. 331. The direct calculation of the flux Φ_2 is a rather complicated problem, since the configuration of the field itself is complicated. However, the application of the reciprocity theorem simplifies the solution of the problem. Indeed, let the same current i flow through loop 2. Then the magnetic flux created by this current through loop 1 can be easily found.



Magnetic induction at the centre of the loop, $: B = \frac{\mu_0 i}{2b}$

So, flux through loop $1, : \Phi_{12} = \pi a^2 \frac{\mu_0 i}{2b}$

And from reciprocity theorem,

$$\varphi_{12} = \Phi_{21}\,,\ \Phi_{21} = \frac{\mu_0 \pi a^2 i}{2b}$$

$$So,\ L_{12} = \frac{\Phi_{21}}{i} = \frac{1}{2}\mu_0 \pi a^2/b$$

Q. 332. A small cylindrical magnet M (Fig. 3.95) is placed in the centre of a thin coil of radius a consisting of N turns. The coil is connected to a ballistic galvanometer. The active resistance of the whole circuit is equal to R. Find the magnetic moment of the magnet if its removal from the coil results in a charge q flowing through the galvanometer.

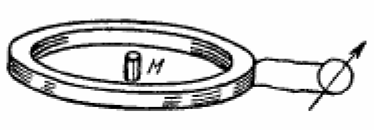


Fig. 3.95.

Solution. 332. Let $\vec{p}_m$ be be the magnetic moment of the magnet Af. Then the magnetic field due to this magnet is,

$$\frac{\mu_0}{4\pi}\left[\frac{3(\vec{p}_m\cdot\vec{r})\vec{r}}{r^5}-\frac{\vec{p}_m}{r^3}\right].$$

The flux associated with this, when the magnet is along the axis at a distance x from the centre, is

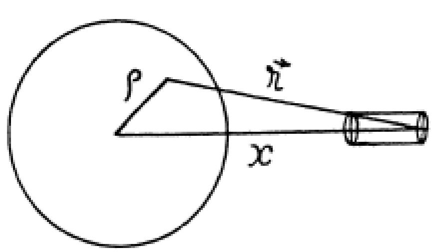
$$\Phi=\frac{\mu_0}{4\pi}\int\left[\frac{3(\vec{p}_m\cdot\vec{r})\vec{r}}{r^5}-\frac{\vec{p}_m}{r^3}\right]\cdot d\vec{S}=\Phi_1-\Phi_2,$$

$$\text{where, }\Phi_2=\frac{\mu_0}{4\pi}p_m\int_0^a\frac{2\pi\rho\,d\rho}{(x^2+\rho^2)^{3/2}}=\frac{\mu_0 p_m}{2}\left(\frac{1}{x}-\frac{1}{\sqrt{x^2+a^2}}\right)$$

$$\text{and }\quad \Phi_1=\frac{3\mu_0 p_m x^2}{4\pi}\int_0^a\frac{2\pi\rho\,d\rho}{(x^2+\rho^2)^{5/2}}$$

$$=\frac{\mu_0 p_m x^2}{2}\left(\frac{1}{x^3}-\frac{1}{(x^2+a^2)^{3/2}}\right)$$

$$\text{So, }\quad \Phi=\frac{-\mu_0 p_m a^2}{2(x^2+a^2)^{3/2}}$$



When the flux changes, an e.m.f. $-N\frac{d\Phi}{dt}$ is induced and a current $-N\frac{d\Phi}{dt}$ flows. The total charge q, flowing, as the magnet is removed to infinity from x = 0 is,

$$q = \frac{N}{R}\Phi(x=0) = \frac{N}{R}\cdot\frac{\mu_0 p_m}{2a}$$

or, $p_m = \frac{2aqR}{N\mu_0}$

Q. 333. Find the approximate formula expressing the mutual inductance of two thin coaxial loops of the same radius a if their centres are separated by a distance l, with l ≫ a.

Solution. 333. If a current l flows in one of the coils, the magnetic field at the centre of the other coil is,

$$B = \frac{\mu_0 a^2 I}{2(l^2+a^2)^{3/2}} = \frac{\mu_0 a^2 I}{2l^3}, \text{ as } l >> a.$$

The flux associated with the second coil is then approximately $\mu_0 \pi a^4 I/2l^3$

Hence, $L_{12} = \frac{\mu_0 \pi a^4}{2l^3}$

Q. 334. There are two stationary loops with mutual inductance L_{12}. The current in one of the loops starts to be varied as $I_1 = \alpha t$, where α is a constant, t is time. Find the time dependence I_2 (t) of the current in the other loop whose inductance is L_2 and resistance R.

Solution. 334. When the current in one of the loop is $I_1 = \alpha t$, an e.m.f. $L_{12}\frac{dI_1}{dt} = L_{12}\alpha$, is induced in the other loop. Then if the current in the other loop is I_2 we must have,

$$L_2\frac{dI_2}{dt} + RI_2 = L_{12}\alpha$$

This familiar equation has the solution,

$I_2 = \frac{L_{12}\alpha}{R}\left(1 - e^{\frac{-tR}{L_2}}\right)$ Which is the required current

Q. 335. A coil of inductance L = 2.0 μH and resistance R = 1.0 Ω is connected to a source of constant emf $\mathscr{E}$ = 3.0 V (Fig. 3.96). A resistance R_0 = 2.0 Ω is connected in parallel with the coil. Find the amount of heat generated in the coil after the switch Sw is disconnected. The internal resistance of the source is negligible.

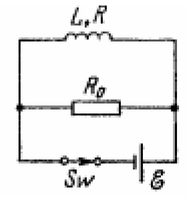
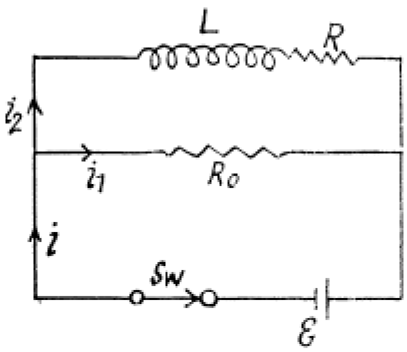


Fig. 3.96.

Solution. 335. Initially, after a steady current is set up, the current is flowing as shown.

In steady condition $i_{20} = \frac{\xi}{R}, \; i_{10} = \frac{\xi}{R_0}.$



When the switch is disconnected, the current through R_0 changes from i_{10} to the right, to i_{20} to the left. (The current in the inductance cannot change suddenly.). We then have the equation,

$$L \frac{di_2}{dt} + (R + R_0)\, i_2 = 0.$$

This equation has the solution $i_2 = i_{20}\, e^{-t(R+R_0)/L}$

The heat dissipated in the coil is,

$$Q = \int_0^\infty i_2^2 R\, dt = i_{20}^2 R \int_0^\infty e^{-2t(R+R_0)/L}\, dt$$

$$= R\, i_{20}^2 \times \frac{L}{2(R+R_0)} = \frac{L\, \xi^2}{2R(R+R_0)} = 3\ \mu\text{J}$$

Q. 336. An iron tore supports N = 500 turns. Find the magnetic field energy if a current I = 2.0 A produces a magnetic flux across the tore's cross-section equal to $\Phi = 1.0$ mWb.

Solution. 336. To find the magnetic field energy we recall that the flux varies linearly with current Thus, when the flux is Φ for current i, we can write $\phi = A\,i.$ The total energy enclosed in the field, when the current is /, is

$$W = \int \xi i\,dt = \int N\,\frac{d\Phi}{dt}\,i\,dt$$

$$= \int N\,d\Phi\,i = \int_0^I N A\,i\,di = \frac{1}{2} N A\,I^2 = \frac{1}{2} N\,\Phi\,I$$

The characteristic factor 1/2 appears in this way.

Q. 337. An iron core shaped as a doughnut with round cross-section of radius a = 3.0 cm carries a winding of N = 1000 turns through which a current I = 1.0 A flows. The mean radius of the doughnut is b = 32 cm. Using the plot in Fig. 3.76, find the magnetic energy stored up in the core. A field strength H is supposed to be the same throughout the cross-section and equal to its magnitude in the centre of the cross-section.

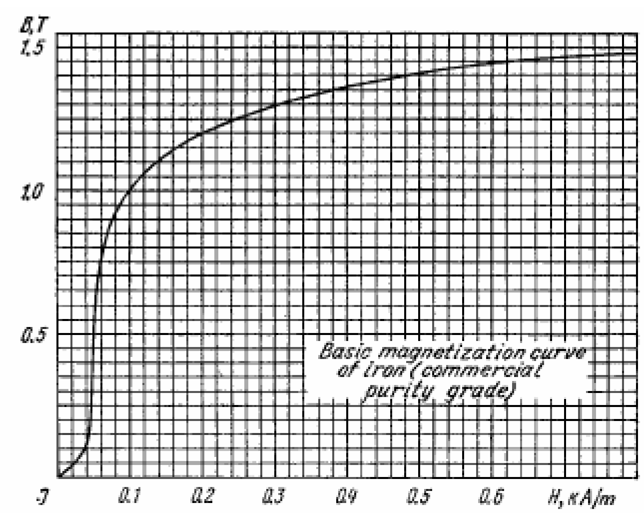


Fig. 3.76.

Solution. 337. We apply circulation theorem

$H \cdot 2\pi b = NI,$ or, $H = NI/2\pi b.$

Thus the total energy,

$$W = \frac{1}{2} BH \cdot 2\pi b \cdot \pi a^2 = \pi^2 a^2\, b\, BH.$$

Given N , l , b we know H , and can find out B from the B - H curve. Then W can be calculated.

Q. 338. A thin ring made of a magnetic has a mean diameter d = 30 cm and supports a winding of N = 800 turns. The cross-sectional area of the ring is equal to S = 5.0 cm². The ring has a cross-cut of width b = 2.0 mm. When the winding carries a certain current, the permeability of the magnetic equals μ = 1400. Neglecting the dissipation of magnetic flux at the gap edges, find:
(a) the ratio of magnetic energies in the gap and in the magnetic;
(b) the inductance of the system; do it in two ways: using the flux and using the energy of the field.

Solution. 338. From From $\oint \vec{H} \cdot d\vec{r} = NI,$

$$H \cdot \pi d + \frac{B}{\mu_0} \cdot b = NI, (d >> b)$$

Also, $B = \mu \mu_0 H$. Thus, $H = \frac{NI}{\pi d + \mu b}$.

Since B is continuous across the gap, B is given by,

$B = \mu \mu_0 \frac{NI}{\pi d + \mu b}$, both in the magnetic and the gap.

$$\text{(a)} \quad \frac{W_{gap}}{W_{magnetic}} = \frac{\frac{B^2}{2\mu_0} \times S \times b}{\frac{B^2}{2\mu\mu_0} \times S \times \pi d} = \frac{\mu b}{\pi d}.$$

$$\text{(b) The flux is } N \int \vec{B} \cdot d\vec{S} = N \mu\mu_0 \frac{NI}{\pi d + \mu b} \cdot S = \mu_0 \frac{SN^2 I}{b + \frac{\pi d}{\mu}},$$

So, $L = \frac{\mu_0 S N^2}{b + \frac{\pi d}{\mu}}$.

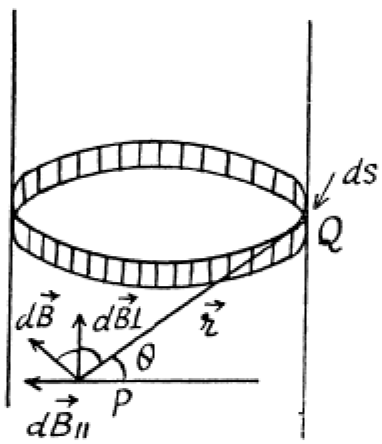
Energy wise; total energy

$$= \frac{B^2}{2\mu_0}\left(\frac{\pi d}{\mu}+b\right)S = \frac{1}{2}\frac{\mu_0 N^2 S}{b+\frac{\pi d}{\mu}}\cdot I^2 = \frac{1}{2}L I^2$$

The L, found in the one way, agrees with that, found in the other way. Note that, in calculating the flux, we do not consider the field in the gap, since it is not linked to the winding. But the total energy includes that of the gap.

Q. 339. A long cylinder of radius a carrying a uniform surface charge rotates about its axis with an angular velocity ω. Find the magnetic field energy per unit length of the cylinder if the linear charge density equals λ, and μ, = 1.

Solution. 339. When the cylinder with a linear charge density λ rotates with a circular frequency co, a surface current density (charge / length x time) of $i = \frac{\lambda\omega}{2\pi}$ is set up.



The direction of the surface current is normal to the plane of paper at Q and the contribution of this current to the magnetic field at P is $d\vec{B} = \frac{\mu_0}{4\pi}\frac{i(\vec{e}\times\vec{r})}{r^3}dS$ where $\vec{e}$ is the direction of the current. In magnitude, $|\vec{e}\times\vec{r}| = r$, since $\vec{e}$ is normal to $\vec{r}$; and the direction of $d\vec{B}$ is shown.

It's component, $dB_\parallel$ cancels out by cylindrical symmetry. The component that survives is,

$$|\vec{B}_\perp| = \frac{\mu_0}{4\pi}\int \frac{i\,dS}{r^2}\cos\theta = \frac{\mu_0 i}{4\pi}\int d\Omega = \mu_0 i,$$

Where we have used $\frac{dS\cos\theta}{r^2} = d\Omega$ and $\int d\Omega = 4\pi,$ the total solid angle around any point. The magnetic field vanishes outside the cylinder by similar argument.
The total energy per unit length of the cylinder is,

$$W_1 = \frac{1}{2\mu_0}\mu_0^2\left(\frac{\lambda\omega}{2\pi}\right)^2 \times \pi a^2 = \frac{\mu_0}{8\pi}a^2\lambda^2\omega^2$$

Q. 340. At what magnitude of the electric field strength in vacuum the volume energy density of this field is the same as that of the magnetic field with induction B = 1.0 T (also in vacuum).

Solution. 340. $w_E = \frac{1}{2}\varepsilon_0 E^2,$ for the electric field,

$w_B = \frac{1}{2\mu_0}B^2$ for the magnetic field.

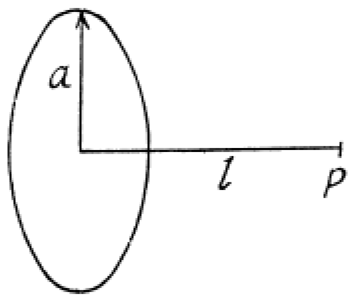
Thus, $\frac{1}{2\mu_0}B^2 = \frac{1}{2}\varepsilon_0 E^2,$

When $E = \frac{B}{\sqrt{\varepsilon_0\mu_0}} = 3 \times 10^8 \text{ V/m}$

Q. 341. A thin uniformly charged ring of radius a = 10 cm rotates about its axis with an angular velocity ω = 100 rad/s. Find the ratio of volume energy densities of magnetic and electric fields on the axis of the ring at a point removed from its centre by a distance l = a.

Solution. 341. The electric field at P is,

$$E_p = \frac{ql}{4\pi\varepsilon_0(a^2 + l^2)^{3/2}}$$



To get the magnetic field, note that the rotating ring constitutes a current i - q ω/2 π, and the corresponding magnetic field at P is,

$$B_p = \frac{\mu_0 a^2 i}{2(a^2 + l^2)^{3/2}}$$

$$\text{Thus, } \frac{w_E}{w_M} = \frac{\varepsilon_0 \mu_0 E^2}{B^2} = \varepsilon_0 \mu_0 \left(\frac{ql \times 2}{4\pi \varepsilon_0 \mu_0 a^2 i}\right)^2$$

$$= \frac{1}{\varepsilon_0 \mu_0}\left(\frac{l}{a^2 \omega}\right)^2$$

$$\text{or, } \quad \frac{w_M}{w_E} = \varepsilon_0 \mu_0 \omega^2 a^4 / l^2$$

Q. 342. Using the expression for volume density of magnetic energy, demonstrate that the amount of work contributed to magnetization of a unit volume of para- or diamagnetic, is equal to A = — JB/2.

Solution. 342. The total eneigy of the magnetic field is,

$$\frac{1}{2}\int (\vec{B}\cdot\vec{H})\, dV = \frac{1}{2}\int \vec{B}\cdot\left(\frac{\vec{B}}{\mu_0} - \vec{J}\right) dV$$
$$= \frac{1}{2\mu_0}\int \vec{B}\cdot\vec{B}\, dV - \frac{1}{2}\int \vec{J}\cdot\vec{B}\, dV.$$

The second term can be interpreted as the energy of magnetization, and has the density $-\frac{1}{2}\vec{J}\cdot\vec{B}.$

Q. 343. Two identical coils, each of inductance L, are interconnected (a) in series, (b) in parallel. Assuming the mutual inductance of the coils to be negligible, find the inductance of the system in both cases.

Solution. 343. (a) In series, the current I flows through both coils, and the total e.m.f. induced, when the current changes is,

$$-2L\frac{dI}{dt} = -L'\frac{dI}{dt}$$

or, L' = 2L

(b) In parallel, the current flowing through either coil is $\frac{I}{2}$ and the e.m.f. induced is

$$-L\left(\frac{1}{2}\frac{dI}{dt}\right).$$

Equating this to $-L'\frac{dI}{dt}$, we find $L' = \frac{1}{2}L$.

Q. 344. Two solenoids of equal length and almost equal crosssectional area are fully inserted into one another. Find their mutual inductance if their inductances are equal to L_1 and L_2.

Solution. 344. We use $L_1 = \mu_0 n_1^2 V, L_2 = \mu_0 n_2^2 V$

So, $L_{12} = \mu_0 n_1 n_2 V = \sqrt{L_1 L_2}$

Q. 345. Demonstrate that the magnetic energy of interaction of two current-carrying loops located in vacuum can be represented as $W_{ia} = (1/\mu_0)\int \mathbf{B}_1\mathbf{B}_2\, dV$, where $\mathbf{B}_1$ and $\mathbf{B}_2$ are the magnetic inductions within a volume element dV, produced individually by the currents of the first and the second loop respectively.

Solution. 345. The interaction energy is

$$\frac{1}{2\mu_0}\int \left|\vec{B}_1 + \vec{B}_2\right|^2 dV - \frac{1}{2\mu_0}\int \left|\vec{B}_1\right|^2 dV - \frac{1}{2\mu_0}\int \left|\vec{B}_2\right|^2 dV$$

$$= \frac{1}{\mu_0}\int \vec{B}_1 \cdot \vec{B}_2\, dV$$

Here, if $\vec{B}_1$ is the magnetic field produced by the first of the current carrying loops,

and $\vec{B}_2$, that of the second one, then the magnetic field due to both the loops will be $\vec{B}_1 + \vec{B}_2$.

Q. 346. Find the interaction energy of two loops carrying currents I_1 and I_2 if both loops are shaped as circles of radii a and b, with a ≪ b. The loops' centres are located at the same point and their planes form an angle θ between them.

Solution. 346. We can think of the smaller coil as constituting a magnet of dipole moment,

$$p_m = \pi a^2 I_1$$

Its direction is normal to the loop and makes an angle θ with the direction of the

magnetic field, due to the bigger loop. This magnetic Geld is,

$$B_2 = \frac{\mu_0 I_2}{2b}$$

The interaction energy has the magnitude,

$$|W| = \frac{\mu_0 I_1 I_2}{2b} \pi a^2 \cos\theta$$

Its sign depends on the sense of the currents.

Q. 347. The space between two concentric metallic spheres is filled up with a uniform poorly conducting medium of resistivity p and permittivity ε. At the moment t = 0 the inside sphere obtains a certain charge. Find:
(a) the relation between the vectors of displacement current density and conduction current density at an arbitrary point of the medium at the same moment of time;
(b) the displacement current across an arbitrary closed surface wholly located in the medium and enclosing the internal sphere, if at the given moment of time the charge of that sphere is equal to q.

Solution. 347. (a) There is a radial outward conduction current Let Q be the instantaneous charge on the inner sphere, then,

$$j \times 4\pi r^2 = -\frac{dQ}{dt} \text{ or, } \vec{j} = -\frac{1}{4\pi r^2}\frac{dQ}{dt}\hat{r}.$$

On the other hand $\vec{j_d} = \frac{\partial \vec{D}}{\partial t} = \frac{d}{dt}\left(\frac{Q}{4\pi r^2}\hat{r}\right) = -\vec{j}$

(b) At the given moment, $\vec{E} = \frac{q}{4\pi\varepsilon_0\varepsilon r^2}\hat{r}$

and by Ohm's law, $\vec{j} = \frac{\vec{E}}{\rho} = \frac{q}{4\pi\varepsilon_0\varepsilon\rho r^2}\hat{r}$

Then, $\vec{j_d} = -\frac{q}{4\pi\varepsilon_0\varepsilon\rho r^2}\hat{r}$

And $\oint \vec{j_d}\cdot d\vec{S} = -\frac{q}{4\pi\varepsilon_0\varepsilon\rho}\int\frac{dS\cos\theta}{r^2} = -\frac{q}{\varepsilon_0\varepsilon\rho}$.

The surface integral must be - ve because $\vec{j_d}$, being opposite of $\vec{j}$, is inward.

Q. 348. A parallel-plate capacitor is formed by two discs with a uniform poorly conducting medium between them. The capacitor was initially charged and then disconnected from a voltage source. Neglecting the edge effects, show that there is no magnetic field between capacitor plates.

Solution. 348. Here also we see that neglecting edge effects, $\vec{j_d} = -\vec{j}$. Thus Maxwell's

equations reduce to ,div $\vec{B} = 0$, Curl $\vec{H} = 0$, $\vec{B} = \mu\vec{H}$

A general solution of this equation is $\vec{B} =$ constant $= \vec{B_0}$ · $\vec{B_0}$ can be thought of as

an extraneous magnetic field. If it is zero, $\vec{B} = 0$.

Q. 349. A parallel-plate air condenser whose each plate has an area S = 100 cm^2 is connected in series to an ac circuit. Find the electric field strength amplitude in the capacitor if the sinusoidal current amplitude in lead wires is equal to I_m. = 1.0 mA and the current frequency equals ω = 1.6-10^7 s^{-1}.

Solution. 349. Given I = I_m sin ωt. We see that

$$j = \frac{I_m}{S}\sin\omega t = -j_d = -\frac{\partial D}{\partial t}$$

or, $D = \frac{I_m}{\omega S} \cos \omega t$, so, $E_m = \frac{I_m}{\varepsilon_0 \omega S}$ is the amplitude of the electric field and is 7V / cm

Q. 350. The space between the electrodes of a parallel-plate capacitor is filled with a uniform poorly conducting medium of conductivity σ and permittivity ε. The capacitor plates shaped as round discs are separated by a distance d. Neglecting the edge effects, find the magnetic field strength between the plates .at a distance r from their axis if an ac voltage V = V_m, cos cot is applied to the capacitor.

Solution. 350. The electric field between the plates can be written as,

$$E = Re \frac{V_m}{d} e^{i \omega t}, \text{ instead of } \frac{V_m}{d} \cos \omega t.$$

This gives rise to a conduction current,

$$j_c = \sigma E = \mathrm{Re} \frac{\sigma}{d} V_m e^{i \omega t}$$

And a displacement current,

$$j_d = \frac{\partial D}{\partial t} = \mathrm{Re}\, \varepsilon_0 \varepsilon\, i \omega \frac{V_m}{d} e^{i \omega t}$$

The total current is,

$$j_T = \frac{V_m}{d} \sqrt{\sigma^2 + (\varepsilon_0 \varepsilon \omega)^2} \cos (\omega t + \alpha)$$

Where, $\tan \alpha = \frac{\sigma}{\varepsilon_0 \varepsilon \omega}$ on taking the real part of the resultant.

The corresponding magnetic field is obtained by using circulation theorem,

$$H \cdot 2 \pi r = \pi r^2 j_T$$

Or, H = H_m cos (ωt + α), where, $H_m = \frac{r V_m}{2d} \sqrt{\sigma^2 + (\varepsilon_0 \varepsilon \omega)^2}$

Q. 351. A long straight solenoid has n turns per unit length. An alternating current I = I_m sin ωt flows through it. Find the displacement current density as a function of the distance r from the solenoid axis. The cross-sectional radius of the solenoid equals R.

Solution. 351. Inside the solenoid, there is a magnetic field,

$$B = \mu_0 n I_m \sin \omega t.$$

Since this varies in time there is an associated electric field. This is obtained by using,

$$\oint_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S}$$

For $r < R$, $2\pi r E = -\dot{B} \cdot \pi r^2$, or, $E = -\frac{\dot{B} r}{2}$

For $r > R$, $E = -\frac{\dot{B} R^2}{2r}$

The associated displacement current density is,

$$j_d = \varepsilon_0 \frac{\partial E}{\partial t} = \begin{cases} -\varepsilon_0 \ddot{B}\, r/2 \\ -\varepsilon_0 \ddot{B}\, R^2/2r \end{cases}$$

The answer, given in the book, is dimensionally incorrect without the factor ε_0.

Electromagnetic Induction Maxwell's Equations (Part - 4)

Q. 352. A point charge q moves with a non-relativistic velocity v = const. Find the displacement current density j_d at a point located at a distance r from the charge on a straight line
(a) coinciding with the charge path;
(b) perpendicular to the path and passing through the charge.

Solution. 352. In the non-relativistic limit.

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^3}\vec{r}$$

(a) On a straight line coinciding with the chaige path,

$$\vec{j_d} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{q}{4\pi}\left[\frac{-\vec{V}}{r^3} - \frac{3\,\vec{r}\,\dot{r}}{r^4}\right], \left(\text{using, } \frac{d\vec{r}}{dt} = -\vec{v}\right)$$

But in this case, $\dot{r} = -v$ and $v\frac{\vec{r}}{r} = \vec{v}$, so, $j_d = \frac{2q\vec{v}}{4\pi r^3}$

(b) In this case, $\dot{r} = 0$, as, $\vec{r} \perp \vec{v}$. Thus,

$$j_d = -\frac{q\vec{v}}{4\pi r^3}$$

Q. 353. A thin wire ring of radius a carrying a charge q approaches the observation point P so that its centre moves rectilinearly with a constant velocity v. The plane of the ring remains perpendicular to the motion direction. At what distance x_m, from the point P will the ring be located at the moment when the displacement current density at the point P becomes maximum? What is the magnitude of this maximum density?

Solution. 353. We have, $E_p = \frac{qx}{4\pi\varepsilon_0 (a^2+x^2)^{3/2}}$

Then $j_d = \frac{\partial D}{\partial t} = \varepsilon_0 \frac{\partial E}{\partial t} = \frac{qv}{4\pi (a^2+x^2)^{5/2}}(a^2 - 2x^2)$

This is maximum, when $x = x_m = 0$, and minimum at some other value. The maximum displacement current density is

$$(j_d)_{max} = \frac{qv}{4\pi a^3}$$

To check this we calculate $\frac{\partial j_d}{\partial x}$;

$$\frac{\partial j_d}{\partial x} = \frac{qv}{4\pi}[(-4x(a^2+x^2) - 5x(a^2-2x^2))]$$

This vanishes for x = 0 and for $x = \sqrt{\frac{3}{2}}\, a.$ The latter is easily shown to be a

smaller local minimum (negative maximum).

Q. 354. A point charge q moves with a non-relativistic velocity v = const. Applying the theorem for the circulation of the vector H around the dotted circle shown in Fig. 3.97, find H at the point A as a function of a radius vector r and velocity v of the charge.

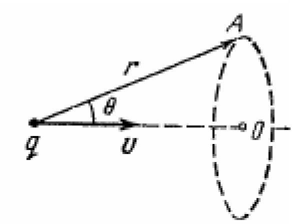


Fig. 3.97.

Solution. 354. We use Maxwell's equations in the form,

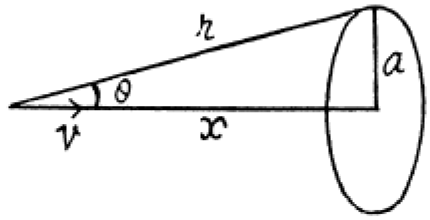
$$\oint \vec{B}\cdot d\vec{r} = \varepsilon_0 \mu_0 \frac{\partial}{\partial t}\int \vec{E}\cdot d\vec{S},$$

When the conduction current vanishes at the site.

We know that,

$$\int \vec{E}\cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0}\int \frac{d\vec{S}\cdot\hat{r}}{r^2}$$

$$= \frac{q}{4\pi\varepsilon_0}\int d\Omega = \frac{q}{4\pi\varepsilon_0} 2\pi(1-\cos\theta),$$



Where, 2k (1 - cos θ) is the solid angle, formed by the disc like surface, at the charge.

Thus, $\oint \vec{B} \cdot d\vec{r} = 2\pi a B = \frac{1}{2}\mu_0 q \cdot \sin\theta \cdot \dot{\theta}$

On the other hand $x = a \cot\theta$

Differentiating and using $\frac{dx}{dt} = -v,$
$v = a \operatorname{cosec}^2\theta\, \dot{\theta}$

Thus, $B = \frac{\mu_0 q v r \sin\theta}{4\pi r^3}$

This can be written as, $\vec{B} = \frac{\mu_0 q (\vec{v} \times \vec{r})}{4\pi r^3}$

And $\vec{H} = \frac{q}{4\pi}\frac{\vec{v} \times \vec{r}}{r^3}$ (The sense has to be checked independently.)

Q. 355. Using Maxwell's equations, show that
(a) a time-dependent magnetic field cannot exist without an electric field;
(b) a uniform electric field cannot exist in the presence of a time dependent magnetic field;
(c) inside an empty cavity a uniform electric (or magnetic) field can be time-dependent.

Solution. 355.

(a) If $\vec{B} = \vec{B}(t)$, then,

Curl $\vec{E} = \frac{-\partial \vec{B}}{\partial t} \neq 0.$

So, $\vec{E}$ cannot vanish.

(b) Here also, curl $\vec{E} \neq 0$, so $\vec{E}$ cannot be uniform.

(c) Suppose for instance, $\vec{E} = \vec{a} f(t)$

Where $\vec{a}$ is spatially and temporally fixed vector.

Then $-\frac{\partial \vec{B}}{dt} =$ curl $\vec{E} = 0.$ Generally speaking this contradicts the other equation

curl $\vec{H} = \frac{\partial \vec{D}}{\partial t} \neq 0$ for in this case the left hand side is time independent but RHS. Depends on time. The only exception is when f (r) is linear function. Then a uniform field $\vec{E}$ can be time dependent.

Q. 356. Demonstrate that the law of electric charge conservation, i.e. $\nabla \cdot \mathbf{j} = -\partial\rho/\partial t$, follows from Maxwell's equations.

Solution. 356. from the equation Curl $\vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}$

We get on taking divergence of both sides $-\frac{\partial}{\partial t}$ div $\vec{D} =$ div $\vec{j}$

But div $\vec{D} = \rho$ and hence div $\vec{j} + \frac{\partial \rho}{\partial t} = 0$

Q. 357. Demonstrate that Maxwell's equations $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ and $\nabla \cdot \mathbf{B} = 0$ are compatible, i.e. the first one does not contradict the second one.

Solution. 357. From $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

We get on taking divergence

$$0 = -\frac{\partial}{\partial t} \operatorname{div} \vec{B}$$

This is compatible with div $\vec{B} = 0$

Q. 358. In a certain region of the inertial reference frame there is magnetic field with induction B rotating with angular velocity ω. Find $\nabla \times \mathrm{E}$ in this region as a function of vectors ω and B.

Solution. 358. A rotating magnetic field can be represented by,

$B_x = B_0 \cos \omega t ; B_y = B_0 \sin \omega t$ and $B_z = B_{z0}$

Then curl, $\vec{E} = -\frac{\partial \vec{B}}{\partial t}$.

So, $-(\text{Curl}\,\vec{E})_x = -\omega B_0 \sin \omega t = -\omega B_y$

$-(\text{Curl}\,\vec{E})_y = \omega B_0 \cos \omega t = \omega B_x$ and $-(\text{Curl}\,\vec{E})_z = 0$

Hence, $\text{Curl}\ \vec{E} = -\vec{\omega} \times \vec{B}$,

Where, $\vec{\omega} = \vec{e}_3 \omega$.

Q. 359. n the inertial reference frame K there is a uniform magnetic field with induction B. Find the electric field strength in the frame K' which moves relative to the frame K with a non-relativistic velocity v, with v⊥B. To solve this problem, consider the forces acting on an imaginary charge in both reference frames at the moment when the velocity of the charge in the frame K' is equal to zero.

Solution. 359. Consider a particle with charge ey moving with velocity $\vec{v}$, in frame K. It experiences a force $\vec{F} = e\vec{v} \times \vec{B}$

In the frame K', moving with velocity $\vec{v}$, relative to K, the particle is at rest. This means

that there must be an electric field $\vec{E}$ in K', so that the particle experinces a force,

$\vec{F}' = e\vec{E}' = \vec{F} = e\,\vec{v} \times \vec{B}$

Thus, $\vec{E}' = \vec{v} \times \vec{B}$

Q. 360. A large plate of non-ferromagnetic material moves with a constant velocity v = 90 cm/s in a uniform magnetic field with induction B = 50 mT as shown in Fig. 3.98. Find the surface density of electric charges appearing on the plate as a result of its motion.

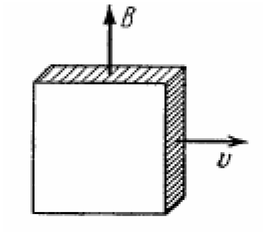


Fig. 3.98.

Solution. 360. Within the plate, there will appear a $(\vec{v} \times \vec{B})$ force, which will cause charges inside the plate to drift, until a countervailing electric field is set up. This electric field is related to By by E = eB, since v & B are mutually perpendicular, and E is perpendicular to both. The charge density ± σ, on the force of the plate, producing this electric field, is given by

$$E = \frac{\sigma}{\varepsilon_0} \text{ or } \sigma = \varepsilon_0 v B = 0{\cdot}40 \text{ pC/m}^2$$

Q. 361. A long solid aluminum cylinder of radius a = 5.0 cm rotates about its axis in a uniform magnetic field with induction B = 10 mT. The angular velocity of rotation equals ω = 45 rad/s, with $\omega \uparrow\uparrow \mathbf{B}$. Neglecting the magnetic field of appearing charges, find their space and surface densities.

Solution. 361. Choose $\vec{\omega} \uparrow\uparrow \vec{B}$ along the z-axis, and choose $\vec{r}$, as the cylindrical polar radius vector of a reference point (perpendicular distance from the axis). This point has the velocity,

$$\vec{v} = \vec{\omega} \times \vec{r},$$

and experiences a $(\vec{v} \times \vec{B})$ force, which must be counterbalanced by an electric field,

$$\vec{E} = -(\vec{\omega} \times \vec{r}) \times \vec{B} = -(\vec{\omega} \cdot \vec{B})\,\vec{r}.$$

There must appear a space charge density,

$$\rho = \varepsilon_0 \operatorname{div} \vec{E} = -2\,\varepsilon_0\,\vec{\omega} \cdot \vec{B} = -8 \text{ pC/m}^3$$

Since the cylinder, as a whole is electrically neutral, the surface of the cylinder must acquire a positive charge of surface density,

$$\sigma = +\frac{2\,\varepsilon_0\,(\vec{\omega} \cdot \vec{B})\,\pi a^2}{2\pi a} = \varepsilon_0\, a\, \vec{\omega} \cdot \vec{B} = +{\cdot}2 \text{ pC/m}^2$$

Q. 362. A non-relativistic point charge q moves with a constant velocity v. Using the field transformation formulas, find the magnetic induction B produced by this charge at the point whose position relative to the charge is determined by the radius vector r.

Solution. 362. In the reference frame K', moving with the particle,

$$\vec{E'} = \vec{E} + \vec{v_0} \times \vec{B} = \frac{q\vec{r}}{4\pi\varepsilon_0 r^3}$$

$$\vec{B'} = \vec{B} - \vec{v_0} \times \vec{E}/c^2 = 0.$$

Here, $\vec{v_0} =$ velocity of K', relative to the K frame, in which the particle has velocity $\vec{v}$.

Clearly, $\vec{v_0} = \vec{v}$. From the second equation,

$$\vec{B} = \frac{\vec{v} \times \vec{E}}{c^2} = \varepsilon_0 \mu_0 \times \frac{q}{4\pi\varepsilon_0} \frac{\vec{v} \times \vec{r}}{r^3} = \frac{\mu_0}{4\pi} \frac{q(\vec{v} \times \vec{r})}{r^3}$$

Q. 363. Using Eqs. (3.6h), demonstrate that if in the inertial reference frame K there is only electric or only magnetic field, in any other inertial frame K' both electric and magnetic fields will coexist simultaneously, with E' ⊥ B'

Solution. 363. Suppose, there is only electric field $\vec{E}$, in K. Then in K', considering

nonrelativistic velocity $\vec{v}$, $\vec{E'} = \vec{E}$, $\vec{B} = -\frac{\vec{v} \times \vec{E}}{c^2}$,

So, $\vec{E'} \cdot \vec{B'} = 0$

In the relativistic case,

$$\left.\begin{aligned} \vec{E'}_{\parallel} &= \vec{E}_{\parallel} \\ E'_{\perp} &= \frac{\vec{E}\perp}{\sqrt{1 - v^2/c^2}} \end{aligned}\right\} \begin{aligned} \vec{B'}_{\parallel} &= \vec{B}_{\parallel} = 0 \\ \vec{B'}_{\perp} &= \frac{-\vec{v} \times \vec{E}/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

Now, $\vec{E'} \cdot \vec{B'} = \vec{E'}_{\parallel} \cdot \vec{B'}_{\parallel} + \vec{E'}_{\perp} \cdot \vec{B'}_{\perp} = 0$, since

$$E'_\perp \cdot B'_\perp = -\vec{E}_\perp \cdot (\vec{v} \times \vec{E})/(1 - v^2/c^2) = -\vec{E}_\perp \cdot (\vec{v} \times \vec{E}_\perp)\Big/\left(1 - \frac{v^2}{c^2}\right) = 0$$

Q. 364. In an inertial reference frame K there is only magnetic field with induction B = b (yi — xj)/ ($x^2 + y^2$), where b is a constant, i and j are the unit vectors of the x and y axes. Find the electric field strength E' in the frame K' moving relative to the frame K with a constant non-relativistic velocity v = vk; k is the unit vector of the z axis. The z' axis is assumed to coincide with the z axis. What is the shape of the field E'

Solution. 364. In $K, \vec{B} = b\,\dfrac{y\hat{i} - x\hat{j}}{x^2 + y^2}$, $b =$ constant.

In $K', \ \vec{E'} = \vec{v} \times \vec{B} = bv\,\dfrac{y\hat{j} - x\hat{i}}{x^2 + y^2} = bv\,\dfrac{\vec{r}}{r^2}$

The electric field is radial $(\vec{r} = x\hat{i} + y\hat{j})$.

Q. 365. In an inertial reference frame K there is only electric field of strength E = a (xi + yj)/($x^2 + y^2$), where a is a constant, i and j are the unit vectors of the x and y axes. Find the magnetic induction B' in the frame K' moving relative to the frame K with a constant non-relativistic velocity v = vk; k is the unit vector of the z axis. The z' axis is assumed to coincide with the z axis. What is the shape of the magnetic induction B'?

Solution. 365. In $K, \vec{E} = a\,\dfrac{\vec{r}}{r^2}$, $\vec{r} = (x\hat{i} + y\hat{j})$

In $K', \vec{B'} = -\dfrac{\vec{v} \times \vec{E}}{c^2} = \dfrac{a\,\vec{r} \times \vec{v}}{c^2 r^2}$

The magnetic lines are circular.

Q. 366. Demonstrate that the transformation formulas (3.6h) follow from the formulas (3.6i) at $v_0 \ll c$.

Solution. 366. In the non-relativistic limit, we neglect v^2/ c^2 and write,

$$\left.\begin{array}{l}\vec{E}'_{\parallel} = \vec{E}_{\parallel} \\ \vec{E}'_{\perp} = \vec{E}_{\perp} + \vec{v}\times\vec{B}\end{array}\right\} \begin{array}{l}\vec{B}'_{\parallel} = \vec{B}_{\parallel} \\ \vec{B}'_{\perp} = \vec{B}_{\perp} - \vec{v}\times\vec{E}/c^2\end{array}$$

These two equations can be combined to give,

$$\vec{E}' = \vec{E} + \vec{v}\times\vec{B}, \ \vec{B}' = \vec{B} - \vec{v}\times\vec{E}/c^2$$

Q. 367. In an inertial reference frame K there is only a uniform electric field E = 8 kV/m in strength. Find the modulus and direction.
(a) Of the vector E', (b) of the vector B' in the inertial reference frame K' moving with a constant velocity v relative to the frame K at an angle α = 45° to the vector E. The velocity of the frame K' is equal to a β = 0.60 fraction of the velocity of light.

Solution. 367. Choose $\vec{E}$ in he direction of the z-axis, $\vec{E} = (0, 0, E)$. The frame K' is moving with velocity $\vec{v} = (v\sin\alpha, 0, v\cos\alpha)$ in the x - z plane. Then in the frame K',

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} \ B'_{\parallel} = 0$$

$$\vec{E}'_{\perp} = \frac{\vec{E}_{\perp}}{\sqrt{1 - v^2/c^2}} \ \vec{B}'_{\perp} = \frac{-\vec{v}\times\vec{E}/c^2}{\sqrt{1 - v^2/c^2}}$$

The vector along $\vec{v}$ is $\vec{e} = (\sin\alpha, 0, \cos\alpha)$ and the perpendicular vector in the x - z plane is

$$\vec{f} = (-\cos\alpha, 0, \sin\alpha),$$

(a) Thus using $\vec{E} = E\cos\alpha\ \vec{e} + E\sin\alpha\ \vec{f},$

$$E'_{\parallel} = E\cos\alpha \ \text{ and } \ E'_{\parallel} = \frac{E\sin\alpha}{\sqrt{1 - v^2/c^2}},$$

So $$E' = E\sqrt{\frac{1 - \beta^2\cos^2\alpha}{1 - \beta^2}} \ \text{ and } \ \tan\alpha' = \frac{\tan\alpha}{\sqrt{1 - v^2/c^2}}$$

$$\text{(b)} \ B'_{\parallel} = 0, \ \vec{B}'_{\perp} = \frac{\vec{v}\times\vec{E}/c^2}{\sqrt{1 - v^2/c^2}}$$

$$B' = \frac{\beta E \sin\alpha}{c\sqrt{1-\beta^2}}$$

Q. 368. Solve a problem differing from the foregoing one by a magnetic field with induction B = 0.8 T replacing the electric field.

Solution. 368. Choose $\vec{B}$ in the z direction, and the velocity $\vec{v} = (v\sin\alpha, 0, v\cos\alpha)$ in the x - z plane, then in the K! Frame,

$$\vec{E'}_{\parallel} = \vec{E}_{\parallel} = 0 \qquad \vec{B'}_{\parallel} = \vec{B}_{\parallel}$$

$$\vec{E'}_{\perp} = \frac{\vec{v}\times\vec{B}}{\sqrt{1-v^2/c^2}} \qquad \vec{B'}_{\perp} = \frac{\vec{B}_{\perp}}{\sqrt{1-v^2/c^2}}$$

We find similarly, $E' = \dfrac{c\,\beta B \sin\alpha}{\sqrt{1-\beta^2}}$

$$B' = B\sqrt{\frac{1-\beta^2\cos^2\alpha}{1-\beta^2}} \quad \tan\alpha' = \frac{\tan\alpha}{\sqrt{1-\beta^2}}$$

Q. 369. Electromagnetic field has two invariant quantities. Using the transformation formulas (3.6i), demonstrate that these quantities are (a) EB; (b) E^2 — c^2B^2.

Solution. 369. (a) We see that, $\vec{E'}\cdot\vec{B'} = \vec{E'}_{\parallel}\cdot\vec{B'}_{\parallel} + \vec{E'}_{\perp}\cdot\vec{B'}_{\perp}$

$$= \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \frac{(\vec{E}_{\perp} + \vec{v}\times\vec{B})\cdot\left(\vec{B}_{\perp} - \dfrac{\vec{v}\times\vec{E}}{c^2}\right)}{1-\dfrac{v^2}{c^2}}$$

$$= \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \frac{\vec{E}_{\perp}\cdot\vec{B}_{\perp} - (\vec{v}\times\vec{B})\cdot(\vec{v}\times\vec{E})/c^2}{1-v^2/c^2}$$

$$= \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \frac{\vec{E}_{\perp}\cdot\vec{B}_{\perp} - (\vec{v}\times\vec{B}_{\perp})\cdot(\vec{v}\times\vec{E}_{\perp})/c^2}{1-\dfrac{v^2}{c^2}}$$

But, $\vec{A}\times\vec{B}\cdot\vec{C}\times\vec{D} = A\cdot C\,B\cdot D - A\cdot D\,B\cdot C,$

$$\vec{E}'\cdot\vec{B}' = \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \vec{E}_{\perp}\cdot\vec{B}_{\perp}\frac{\left(1-\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}} = \vec{E}\cdot\vec{B}$$

so,

(b) $E'^2 - c^2B'^2 = E'^2_{\parallel} - c^2B'^2_{\parallel} + E'^2_{\perp} - c^2B'^2_{\perp}$

$$= E^2_{\parallel} - c^2B^2_{\parallel} + \frac{1}{1-\frac{v^2}{c^2}}\left[(\vec{E}_{\perp} + \vec{v}\times\vec{B})^2 - c^2\left(\vec{B}_{\perp} - \frac{\vec{v}\times\vec{E}}{c^2}\right)^2\right]$$

$$= E^2_{\parallel} - c^2B^2_{\parallel} + \frac{1}{1-\frac{v^2}{c^2}}\left[E^2_{\perp} - c^2B^2_{\perp} + (\vec{v}\times\vec{B}_{\perp})^2 - \frac{1}{c^2}(\vec{v}\times E_1)^2\right]$$

$$= E^2_{\parallel} - c^2B^2_{\parallel} + \frac{1}{1-\frac{v^2}{c^2}}[E^2_{\perp} - c^2B^2_{\perp}]\left(1-\frac{v^2}{c^2}\right) = E^2 - c^2B^2,$$

since, $(\vec{v}\times\vec{A}_{\perp})^2 = v^2A^2_{\perp}$

Q. 370. In an inertial reference frame K there are two uniform mutually perpendicular fields: an electric field of strength E = 40 kV/m and a magnetic field induction B = 0.20 mT. Find the electric strength E' (or the magnetic induction B') in the reference frame K' where only one field, electric or magnetic, is observed. Instruction. Make use of the field invariants cited in the foregoing problem.

Solution. 370. In this case, $\vec{E}\cdot\vec{B} = 0,$ as the fields are mutually perpendicular. Also, $E^2 - c^2B^2 = -20\times10^8\left(\frac{\text{V}}{\text{m}}\right)^2$ is $-ve.$

Thus, we can find a frame, in which E' = 0, and

$$B' = \frac{1}{c}\sqrt{c^2B^2 - E^2} = B\sqrt{1-\frac{E^2}{c^2B^2}} = 0{\cdot}20\sqrt{1-\left(\frac{4\times10^4}{3\times10^8\times2\times10^{-4}}\right)^2} = 0{\cdot}15\text{ mT}$$

Q. 371. A point charge q moves uniformly and rectilinearly with a relativistic velocity equal to a β fraction of the velocity of light (β = v/c). Find the electric field strength E produced by the charge at the point whose radius vector relative to the charge is equal to r and forms an angle θ with its velocity vector.

Solution. 371. Suppose the charge qr moves in the positive direction of the x-axis of the frame K. Let us go over to the moving frame K', at whose origin the charge is at rest. We take the x and x' axes of the two frames to be coincident, and the y & y' axes, to be parallel.

In the K' frame, $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q\vec{r}}{r'^3}$,

And this has the following components,

$$E'_x = \frac{1}{4\pi\varepsilon_0}\frac{qx'}{r'^3},\ E_y{}' = \frac{1}{4\pi\varepsilon_0}\frac{qy'}{r'^3}$$

Now let us go back to the frame K. At the moment, when the origins of the two frames coincide, we take t * 0. Then,

$$x = r\cos\theta = x'\sqrt{1-\frac{v^2}{c^2}},\ y = r\sin\theta = y'$$

Also, $E_x = E_x{}',\ E_y = E_y{}' \big/ \sqrt{1 - v^2/c^2}$

From these equations, $r'^2 = \frac{r^2\,(1-\beta^2\sin^2\theta)}{1-\beta^2}$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^3(1-\beta^2\sin^2\theta)^{3/2}}\left[(1-\beta^2)^{3/2}\left(x'\hat{i} + \frac{y'}{\sqrt{1-\beta^2}}\hat{j}\right)\right]$$

$$= \frac{q\vec{r}(1-\beta^2)}{4\pi\varepsilon_0 r^3(1-\beta^2\sin^2\theta)^{3/2}}$$