DAY ELEVEN

Limits, Continuity and Differentiability

Learning & Revision for the Day

- Limits
- Important Results on Limit
- Methods to Evaluate Limits
- Continuity
- Differentiability

Limits

Let $y = f(x)$ be a function of x. If the value of $f(x)$ tend to a definite number as x tends to a, then the number so obtained is called the **limit** of $f(x)$ at $x = a$ and we write it as $\lim_{x\to a} f(x)$.

- If $f(x)$ approaches to l_1 as x approaches to 'a' from left, then l_1 is called the **left hand limit** of $f(x)$ at $x = a$ and symbolically we write it as $f(a-0)$ or $\lim_{x\to a^-} f(x)$ or $\lim_{h\to 0} f(a-h)$
- Similarly, **right hand limit** can be expressed as
$$f(a+0) \text{ or } \lim_{x\to a^+} f(x) \text{ or } \lim_{h\to 0} f(a+h)$$
- $\lim_{x\to a} f(x)$ exists iff $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and equal.

Fundamental Theorems on Limits

If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$ (where, l and m are real numbers), then

(i) $\lim_{x\to a} \{f(x) + g(x)\} = l + m$ [sum rule]

(ii) $\lim_{x\to a} \{f(x) - g(x)\} = l - m$ [difference rule]

(iii) $\lim_{x\to a} \{f(x) \cdot g(x)\} = l \cdot m$ [product rule]

(iv) $\lim_{x\to a} k \cdot f(x) = k \cdot l$ [constant multiple rule]

(v) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}, m \neq 0$ [quotient rule]

(vi) If $\lim_{x\to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x\to a} \dfrac{1}{f(x)} = 0$

(vii) $\lim_{x\to a} |f(x)| = \left|\lim_{x\to a} f(x)\right|$

(viii) $\lim_{x\to a} \log\{f(x)\} = \log\{\lim_{x\to a} f(x)\}$, provided $\lim_{x\to a} f(x) > 0$

(ix) If $f(x) \le g(x)$, $\forall x$, then $\lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a} g(x)$

(x) $\lim\limits_{x\to a}[f(x)]^{g(x)} = \{\lim\limits_{x\to a} f(x)\}^{\lim\limits_{x\to a} g(x)}$

(xi) $\lim\limits_{x\to a} f\{g(x)\} = f\{\lim\limits_{x\to a} g(x)\} = f(m)$ provided f is continuous at $\lim\limits_{x\to a} g(x) = m$.

(xii) **Sandwich Theorem** If $f(x) \le g(x) \le h(x) \forall x \in (\alpha, \beta) - \{a\}$ and $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = l$, then $\lim\limits_{x\to a} g(x) = l$ where $a \in (\alpha, \beta)$

Important Results on Limit

Some important results on limits are given below

1. Algebraic Limits

(i) $\lim\limits_{x\to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$, $n \in Q$, $a > 0$

(ii) $\lim\limits_{x\to \infty} \dfrac{1}{x^n} = 0$, $n \in N$

(iii) If m, n are positive integers and a_0, b_0 are non-zero real numbers, then

$$\lim_{x\to\infty} \frac{a_0 x^m + a_1 x^{m-1} + \ldots + a_{m-1} x + a_m}{b_0 x^n + b_1 x^{n-1} + \ldots + b_{n-1} x + b_n} = \begin{cases} \dfrac{a_0}{b_0} & \text{if } m = n \\ 0 & \text{if } m < n \\ \infty & \text{if } m > n, a_0 b_0 > 0 \\ -\infty & \text{if } m > n, a_0 b_0 < 0 \end{cases}$$

(iv) $\lim\limits_{x\to 0} \dfrac{(1+x)^n - 1}{x} = n$

(v) $\lim\limits_{x\to 0} \dfrac{(1+x)^m - 1}{(1+x)^n - 1} = \dfrac{m}{n}$

2. Trigonometric Limits

(i) $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1 = \lim\limits_{x\to 0} \dfrac{x}{\sin x}$

(ii) $\lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1 = \lim\limits_{x\to 0} \dfrac{x}{\tan x}$

(iii) $\lim\limits_{x\to 0} \dfrac{\sin^{-1} x}{x} = 1 = \lim\limits_{x\to 0} \dfrac{x}{\sin^{-1} x}$

(iv) $\lim\limits_{x\to 0} \dfrac{\tan^{-1} x}{x} = 1 = \lim\limits_{x\to 0} \dfrac{x}{\tan^{-1} x}$

(v) $\lim\limits_{x\to 0} \dfrac{\sin x^\circ}{x} = \dfrac{\pi}{180}$

(vi) $\lim\limits_{x\to \infty} \dfrac{\sin x}{x} = \lim\limits_{x\to\infty} \dfrac{\cos x}{x} = 0$

(vii) $\lim\limits_{x\to\infty} \sin x$ or $\lim\limits_{x\to\infty} \cos x$ oscillates between -1 to 1.

(viii) $\lim\limits_{x\to 0} \dfrac{\sin^P mx}{\sin^P nx} = \left(\dfrac{m}{n}\right)^P$

(ix) $\lim\limits_{x\to 0} \dfrac{\tan^P mx}{\tan^P nx} = \left(\dfrac{m}{n}\right)^P$

(x) $\lim\limits_{x\to 0} \dfrac{1 - \cos mx}{1 - \cos nx} = \dfrac{m^2}{n^2}$; $\lim\limits_{x\to 0} \dfrac{\cos ax - \cos bx}{\cos cx - \cos dx} = \dfrac{a^2 - b^2}{c^2 - d^2}$

(xi) $\lim\limits_{x\to 0} \dfrac{\cos mx - \cos nx}{x^2} = \dfrac{n^2 - m^2}{2}$

3. Logarithmic Limits

(i) $\lim\limits_{x\to 0} \dfrac{\log_a(1+x)}{x} = \log_a e$; $a > 0, \ne 1$

(ii) In particular, $\lim\limits_{x\to 0} \dfrac{\log_e(1+x)}{x} = 1$ and $\lim\limits_{x\to 0} \dfrac{\log_e(1-x)}{x} = -1$

4. Expotential Limits

(i) $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \log_e a$, $a > 0$

(ii) In particular, $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$ and $\lim\limits_{x\to 0} \dfrac{e^{\lambda x} - 1}{x} = \lambda$

(iii) $$\lim_{x\to\infty} a^x = \begin{cases} 0, & 0 \le a < 1 \\ 1, & a = 1 \\ \infty, & a > 1 \\ \text{Does not exist}, & a < 0 \end{cases}$$

5. 1^∞ Form Limits

(i) If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$, then $\lim\limits_{x\to a}\{1 + f(x)\}^{1/g(x)} = e^{\lim\limits_{x\to a} \frac{f(x)}{g(x)}}$

(ii) If $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} g(x) = \infty$, then $\lim\limits_{x\to a}\{f(x)\}^{g(x)} = e^{\lim\limits_{x\to a}\{f(x) - 1\} g(x)}$

In General Cases

(i) $\lim\limits_{x\to 0}(1+x)^{1/x} = e$

(ii) $\lim\limits_{x\to\infty}\left(1 + \dfrac{1}{x}\right)^x = e$

(iii) $\lim\limits_{x\to 0}(1+\lambda x)^{1/x} = e^\lambda$

(iv) $\lim\limits_{x\to\infty}\left(1 + \dfrac{\lambda}{x}\right)^x = e^\lambda$

(v) $\lim\limits_{x\to 0}(1 + ax)^{b/x} = \lim\limits_{x\to\infty}\left(1 + \dfrac{a}{x}\right)^{bx} = e^{ab}$

Methods To Evaluate Limits

To find $\lim\limits_{x\to a} f(x)$, we substitute $x = a$ in the function.

If $f(a)$ is finite, then $\lim\limits_{x\to a} f(x) = f(a)$.

If $f(a)$ leads to one of the following form $\dfrac{0}{0}$; $\dfrac{\infty}{\infty}$; $\infty - \infty$; $0 \times \infty$; 1^∞, 0 and ∞^0 (called indeterminate forms), then $\lim\limits_{x\to a} f(x)$ can be evaluated by using following methods

(i) **Factorization Method** This method is particularly used when on substituting the value of x, the expression take the form 0/0.

(ii) **Rationalization Method** This method is particularly used when either the numerator or the denominator or both involved square roots and on substituting the value of x, the expression take the form $\dfrac{0}{0}, \dfrac{\infty}{\infty}$.

NOTE To evaluate $x \to \infty$ type limits write the given expression in the form N/D and then divide both N and D by highest power of x occurring in both N and D to get a meaningful form.

L'Hospital's Rule

If $f(x)$ and $g(x)$ be two functions of x such that

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$.

(ii) both are continuous at $x = a$.

(iii) both are differentiable at $x = a$.

(iv) $f'(x)$ and $g'(x)$ are continuous at the point $x = a$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ provided that $g(a) \neq 0$.

Above rule is also applicable, if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$.

If $f'(x), g'(x)$ satisfy all the conditions embeded in L'Hospital's rule, then we can repeat the application of this rule on

$$\frac{f'(x)}{g'(x)} \text{ to get } \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}.$$

Sometimes, following expansions are useful in evaluating limits.

- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots + (-1 < x \le 1)$
- $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots (-1 < x < 1)$
- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$
- $a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \dots$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \dots$

Continuity

If the graph of a function has no break (or gap), then it is **continuous**. A function which is not continuous is called a **discontinuous** function. e.g. x^2 and e^x are continuous while $\frac{1}{x}$ and $[x]$, where $[\cdot]$ denotes the greatest integer function, are discontinuous.

Continuity of a Function at a Point

A function $f(x)$ is said to be continuous at a point $x = a$ of its domain if and only if it satisfies the following conditions

(i) $f(a)$ exists, where ('a' lies in the domain of f

(ii) $\lim_{x\to a} f(x)$ exist, i.e. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

or LHL = RHL

(iii) $\lim_{x\to a} f(x) = f(a)$, $f(x)$ is said to be

left continuous at $x = a$, if $\lim_{x\to a^-} f(x) = f(a)$

right continuous at $x = a$, if $\lim_{x\to a^+} f(x) = f(a)$

Continuity of a Function in an Interval

A function $f(x)$ is said to be continuous in (a,b) if it is continuous at every point of the interval (a,b). A function $f(x)$ is said to be continuous in $[a,b]$, if $f(x)$ is continuous in (a,b). Also, in addition $f(x)$ is continuous at $x = a$ from right and continuous at $x = b$ from left.

Results on Continuous Functions

(i) Sum, difference product and quotient of two continuous functions are always a continuous function. However, $r(x) = \frac{f(x)}{g(x)}$ is continuous at $x = a$ only if $g(a) \neq 0$.

(ii) Every polynomial is continuous at each point of real line.

(iii) Every rational function is continuous at each point where its denominator is different from zero.

(iv) Logarithmic functions, exponential functions, trigonometric functions, inverse circular functions and modulus function are continuous in their domain.

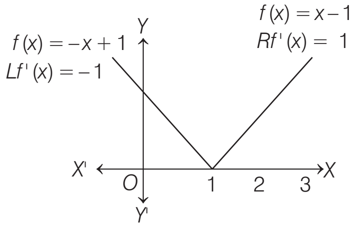
(v) $[x]$ is discontinuous when x is an integer.

(vi) If $g(x)$ is continuous at $x = a$ and f is continuous at $g(a)$, then fog is continuous at $x = a$.

(vii) $f(x)$ is a continuous function defined on $[a, b]$ such that $f(a)$ and $f(b)$ are of opposite signs, then there is atleast one value of x for which $f(x)$ vanishes, i.e. $f(a) > 0$, $f(b) < 0 \Rightarrow \exists\, c \in (a, b)$ such that $f(c) = 0$.

Differentiability

The function $f(x)$ is differentiable at a point P iff there exists a unique tangent at point P. In other words, $f(x)$ is differentiable at a point P iff the curve does not have P as a corner point, i.e. the function is not differentiable at those points where graph of the function has holes or sharp edges. Let us consider the function $f(x) = |x-1|$. It is not differentiable at $x = 1$. Since, $f(x)$ has sharp edge at $x = 1$.



(graph of $f(x)$ describe differentiability)

Differentiability of a Function at a Point

A function f is said to be differentiable at $x = c$, if **left hand** and **right hand** derivatives at c exist and are equal.

- **Right hand derivative** of $f(x)$ at $x = a$ denoted by $f'(a+0)$ or $f'(a^+)$ is $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$.

- **Left hand derivative** of $f(x)$ at $x = a$ denoted by $f'(a-0)$ or $f'(a^-)$ is $\lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$.
- Thus, f is said to be **differentiable** at $x = a$, if $f'(a+0) = f'(a-0) = \text{finite}$.
- The common limit is called the **derivative** of $f(x)$ at $x = a$ denoted by $f'(a)$. i.e. $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

Results on Differentiability

(i) Every polynomial, constant and exponential function is differentiable at each $x \in R$.

(ii) The logarithmic, trigonometric and inverse trigonometric function are differentiable in their domain.

(iii) The sum, difference, product and quotient of two differentiable functions is differentiable.

(iv) Every differentiable function is continuous but converse may or may not be true.

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 $\lim_{x \to 0} |x|^{[\cos x]}$, where [.] is the greatest integer function, is

(a) 1 (b) 0 (c) Does not exist (d) None of these

2 Let $f: R \to [0, \infty)$ be such that $\lim_{x \to 5} f(x)$ exists and $\lim_{x \to 5} \frac{[f(x)]^2 - 9}{\sqrt{|x-5|}} = 0$. Then, $\lim_{x \to 5} f(x)$ is equal to

(a) 3 (b) 0 (c) 1 (d) 2

3 If $\lim_{x \to \infty} \frac{2}{x}\left[\frac{x}{5}\right] = \frac{m}{n}$ (where [$\cdot$] denotes greatest integer function), then $m + n$ (where m, n are relatively prime) is

(a) 2 (b) 7 (c) 5 (d) 6

4 The value of the constant α and β such that $\lim_{x \to \infty} \left(\frac{x^2+1}{x+1} - \alpha x - \beta\right) = 0$ are respectively

(a) (1, 1) (b) (–1, 1) (c) (1, – 1) (d) (0, 1)

5 $\lim_{n \to \infty} \frac{3 \cdot 2^{n+1} - 4 \cdot 5^{n+1}}{5 \cdot 2^n + 7 \cdot 5^n}$ is equal to

(a) 0 (b) $\frac{3}{5}$ (c) $-\frac{4}{7}$ (d) $-\frac{20}{7}$

6 $\lim_{x \to \infty} \left[\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}\right]$ is equal to

(a) 0 (b) $\frac{1}{2}$ (c) $\log 2$ (d) e^4

7 The value of $\lim_{n \to \infty} \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{4}\right) \cos\left(\frac{x}{8}\right) \ldots \cos\left(\frac{x}{2^n}\right)$ is

(a) 1 (b) $\frac{\sin x}{x}$ (c) $\frac{x}{\sin x}$ (d) None of these

8 If f is periodic with period T and $f(x) > 0 \forall x \in R$, then $\lim_{n \to \infty} n\left(\frac{f(x+T) + 2f(x+2T) + \ldots + nf(x+nT)}{f(x+T) + 4f(x+4T) + \ldots + n^2 f(x+n^2T)}\right)$ is equal to

(a) 2 (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) None of these

9 $\lim_{x \to 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}$ is equal to → **JEE Mains 2015, 13**

(a) 4 (b) 3 (c) 2 (d) $\frac{1}{2}$

10 If $\lim_{x \to 0} \frac{[(a-n)nx - \tan x] \sin nx}{x^2} = 0$, where n is non-zero real number, then a is equal to

(a) 0 (b) $\frac{n+1}{n}$ (c) n (d) $n + \frac{1}{n}$

11 $\lim_{x \to 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ is equal to → **JEE Mains 2014**

(a) $\frac{\pi}{2}$ (b) 1 (c) $-\pi$ (d) π

12 If $f(x) = \begin{vmatrix} \sin x & \cos x & \tan x \\ x^3 & x^2 & x \\ 2x & 1 & 1 \end{vmatrix}$, then $\lim_{x \to 0} \frac{f(x)}{x^2}$ is equal to

(a) 3 (b) –1 (c) 0 (d) 1

13 The limit of the following is

$$\lim_{x \to 3} \frac{\sqrt{1 - \cos(x^2 - 10x + 21)}}{(x-3)}$$

(a) $-(2)^{3/2}$ (b) $(2)^{1/2}$ (c) $(2)^{-3/2}$ (d) 3

14 The value of $\lim_{x \to 0} \frac{1}{x}\left[\tan^{-1}\left(\frac{x+1}{2x+1}\right) - \frac{\pi}{4}\right]$ is → **JEE Mains 2013**

(a) 1 (b) $-\frac{1}{2}$ (c) 2 (d) 0

15 $\lim_{x \to \pi/2} \frac{\cot x - \cos x}{(\pi - 2x)^3}$ equals → **JEE Mains 2017**

(a) $\frac{1}{24}$ (b) $\frac{1}{16}$ (c) $\frac{1}{8}$ (d) $\frac{1}{4}$

16 If m and n are positive integers, then $\lim_{x\to 0} \frac{(\cos x)^{1/m} - (\cos x)^{1/n}}{x^2}$ equals to

(a) $m-n$ (b) $\frac{1}{n}-\frac{1}{m}$ (c) $\frac{m-n}{2mn}$ (d) None of these

17 $\lim_{x\to\infty} \frac{\cot^{-1}(\sqrt{x+1}-\sqrt{x})}{\sec^{-1}\left\{\left(\frac{2x+1}{x-1}\right)^x\right\}}$ is equal to

(a) 1 (b) 0 (c) $\frac{\pi}{2}$ (d) Does not exists

18 Let $p = \lim_{x\to 0} \frac{\ln(\cos 2x)}{3x^2}$, $q = \lim_{x\to 0} \frac{\sin^2 2x}{x(1-e^x)}$ and $r = \lim_{x\to 1} \frac{\sqrt{x}-x}{\ln x}$. Then p, q, r satisfy

(a) $p<q<r$ (b) $q<r<p$ (c) $p<r<q$ (d) $q<p<r$

19 Let $p = \lim_{x\to 0^+} (1+\tan^2\sqrt{x})^{1/2x}$, then $\log p$ is equal to **➜ JEE Mains 2016**

(a) 2 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

20 The value of $\lim_{x\to\infty} \left(\frac{3x-4}{3x+2}\right)^{\frac{x+1}{3}}$ is

(a) $e^{-1/3}$ (b) $e^{-2/3}$ (c) e^{-1} (d) e^{-2}

21 If $\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x} = e^2$, then the values of a and b are

(a) $a\in R, b\in R$ (b) $a=1, b\in R$
(c) $a\in R, b=2$ (d) $a=1, b=2$

22 If $f'(2)=6$ and $f'(1)=4$, then $\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$ is equal to

(a) 3 (b) $-3/2$ (c) $3/2$ (d) Does not exist

23 Let $f(a)=g(a)=k$ and their nth derivatives $f^n(a), g^n(a)$ exist and are not equal for some n. Further, if

$$\lim_{x\to a}\frac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)} = 4,$$

then the value of k is

(a) 4 (b) 2 (c) 1 (d) 0

24 If $f: R\to R$ be such that $f(1)=3$ and $f'(1)=6$. Then, $\lim_{x\to 0}\left(\frac{f(1+x)}{f(1)}\right)^{1/x}$ is equal to

(a) 1 (b) $e^{1/2}$ (c) e^2 (d) e^3

25 Let $f(x)=\begin{cases}\sin^2 x, & x \text{ is rational}\\ -\sin^2 x, & x \text{ is irrational}\end{cases}$, then set of points, where $f(x)$ is continuous, is

(a) $\left\{(2n+1)\frac{\pi}{2}, n\in I\right\}$ (b) a null set
(c) $\{n\pi, n\in I\}$ (d) set of all rational numbers

26 Let $f:[a, b]\to R$ be any function which is such that $f(x)$ is rational for irrational x and $f(x)$ is irrational for rational x. Then, in $[a,b]$

(a) f is discontinuous everywhere
(b) f is continuous only at $x=0$
(c) f is continuous for all irrational x and discontinuous for all rational x
(d) f is continuous for all rational x and discontinuous for all irrational x

27 Let $f(x)=1+|x-2|$ and $g(x)=1-|x|$, then the set of all points, where fog is discontinuous, is **➜ JEE Mains 2013**

(a) $\{0, 2\}$ (b) $\{0, 1, 2\}$ (c) $\{0\}$ (d) an empty set

28 If $f(x)=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$ and $g(x)=\sin x+\cos x$, then the points of discontinuity of $f\{g(x)\}$ in $(0, 2\pi)$ is

(a) $\left\{\frac{\pi}{2}, \frac{3\pi}{4}\right\}$ (b) $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$ (c) $\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$ (d) $\left\{\frac{5\pi}{4}, \frac{7\pi}{3}\right\}$

29 If $f(x)$ is differentiable at $x=1$ and $\lim_{h\to 0}\frac{1}{h}f(1+h)=5$, then $f'(1)$ is equal to

(a) 6 (b) 5 (c) 4 (d) 3

30 If $f(x)=3x^{10}-7x^8+5x^6-21x^3+3x^2-7$, then the value of $\lim_{h\to 0}\frac{f(1-h)-f(1)}{h^3+3h}$ is

(a) $\frac{53}{3}$ (b) $\frac{22}{3}$ (c) 13 (d) $\frac{22}{13}$

31 Let $f(2)=4$ and $f'(2)=4$. Then, $\lim_{x\to 2}\frac{xf(2)-2f(x)}{x-2}$ is given by

(a) 2 (b) -2 (c) -4 (d) 3

32 If f is a real-valued differentiable function satisfying $|f(x)-f(y)|\le (x-y)^2$; $x, y\in R$ and $f(0)=0$, then $f(1)$ is equal to

(a) 1 (b) 2 (c) 0 (d) -1

33 If $\lim_{x\to 0}\frac{\log(a+x)-\log a}{x}+k\lim_{x\to 0}\frac{\log x-1}{x-e}=1$, then

(a) $k=e\left(1-\frac{1}{a}\right)$ (b) $k=e(1+a)$
(c) $k=e(2-a)$ (d) Equality is not possible

34 The left hand derivative of $f(x)=[x]\sin(\pi x)$ at $x=k$, k is an integer, is

(a) $(-1)^k(k-1)\pi$ (b) $(-1)^{k-1}(k-1)\pi$
(c) $(-1)^k k\pi$ (d) $(-1)^{k-1}k\pi$

35 If $f(x)=\begin{cases}e^x, & x\le 0\\ |1-x|, & x>0\end{cases}$, then

(a) $f(x)$ is differentiable at $x=0$
(b) $f(x)$ is continuous at $x=0, 1$
(c) $f(x)$ is differentiable at $x=1$
(d) None of the above

36 If $f(x)=\begin{cases}xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x\ne 0\\ 0, & x=0\end{cases}$, then $f(x)$ is

(a) continuous as well as differentiable for all x
(b) continuous for all x but not differentiable at $x = 0$
(c) neither differentiable nor continuous at $x = 0$
(d) discontinuous everywhere

37 The set of points, where $f(x) = \dfrac{x}{1+|x|}$ is differentiable, is

(a) $(-\infty, -1) \cup (-1, \infty)$ (b) $(-\infty, \infty)$
(c) $(0, \infty)$ (d) $(-\infty, 0) \cup (0, \infty)$

38 Let $f(x) = \cos x$ and
$g(x) = \begin{cases} \min\{f(t): 0 \le t \le x\}, & x \in [0, \pi] \\ (\sin x) - 1, & x > \pi \end{cases}$ then

(a) $g(x)$ is discontinuous at $x = \pi$
(b) $g(x)$ is continuous for $x \in [0, \infty]$
(c) $g(x)$ is differentiable at $x = \pi$
(d) $g(x)$ is differentiable for $x \in [0, \infty]$

39 If the function $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx + 2, & 3 < x \le 5 \end{cases}$ is differentiable, then the value of $k + m$ is

(a) 2 (b) $\dfrac{16}{5}$ (c) $\dfrac{10}{5}$ (d) 4

40 If $f(x) = \begin{cases} \sin(\cos^{-1} x) + \cos(\sin^{-1} x), & x \le 0 \\ \sin(\cos^{-1} x) - \cos(\sin^{-1} x), & x > 0 \end{cases}$ then at $x = 0$

(a) $f(x)$ is continuous and differentiable
(b) $f(x)$ is continuous but not differentiable
(c) f is not continuous but differentiable
(d) f is neither continuous nor differentiable

41 If $f(x) = [\sin x] + [\cos x]$, $x \in [0, 2\pi]$, where $[.]$ denotes the greatest integer function. Then, the total number of points, where $f(x)$ is non-differentiable, is

(a) 2 (b) 3 (c) 5 (d) 4

42 If $f(x) = |\sin x|$, then

(a) f is everywhere differentiable
(b) f is everywhere continuous but not differentiable at $x = n\pi, n \in Z$
(c) f is everywhere continuous but not differentiable at $x = (2n+1)\dfrac{\pi}{2}, n \in Z$
(d) None of the above

43 **Statement I** $f(x) = |\log x|$ is differentiable at $x = 1$.
Statement II Both $\log x$ and $-\log x$ are differentiable at $x = 1$.

(a) Statement I is false, Statement II is true
(b) Statemnt I is true, Statement II is true; Statement II is a correct explanation of Statement I
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation of Statement I
(d) Statement I is true, Statement II is false

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 For each $t \in R$, let $[t]$ be the greatest integer less than or equal to t. Then, $\lim\limits_{x \to 0^+} x\left(\left[\dfrac{1}{x}\right] + \left[\dfrac{2}{x}\right] + \dots + \left[\dfrac{15}{x}\right]\right)$

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(a) is equal to 0 (b) is equal to 15
(c) is equal to 120 (d) does not exist (in R)

2 $\lim\limits_{x \to \infty} \left(\dfrac{x^2 + 5x + 3}{x^2 + x + 2}\right)^x$ is equal to

(a) e^4 (b) e^2 (c) e^3 (d) e

3 If α and β are the distinct roots of $ax^2 + bx + c = 0$, then $\lim\limits_{x \to \alpha} \dfrac{1 - \cos(ax^2 + bx + c)}{(x - \alpha)^2}$ is equal to

(a) $\dfrac{1}{2}(\alpha - \beta)^2$ (b) $-\dfrac{a^2}{2}(\alpha - \beta)^2$ (c) 0 (d) $\dfrac{a^2}{2}(\alpha - \beta)^2$

4 $\lim\limits_{n \to \infty} \sin[\pi\sqrt{n^2 + 1}]$ is equal to

(a) ∞ (b) 0
(c) Does not exist (d) None of these

5 If $x > 0$ and g is a bounded function, then $\lim\limits_{n \to \infty} \dfrac{f(x) \cdot e^{nx} + g(x)}{e^{nx} + 1}$ is equal to

(a) 0 (b) $f(x)$
(c) $g(x)$ (d) None of these

6 The value of $\lim\limits_{x \to 0} \left(\dfrac{a^x + b^x + c^x}{3}\right)^{2/x}$ (where $a, b, c > 0$) is

(a) $(abc)^3$ (b) abc
(c) $(abc)^{1/3}$ (d) None of these

7 If $f(x) = \cot^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$ and $g(x) = \cos^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right)$, then $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{g(x) - g(a)}$, where $0 < a < \dfrac{1}{2}$, is equal to

(a) $\dfrac{3}{2(1 + a^2)}$ (b) $\dfrac{3}{2(1 + x^2)}$ (c) $\dfrac{3}{2}$ (d) $-\dfrac{3}{2}$

8 If $f: R \to R$ is a function defined by $f(x) = [x] \cos\left(\dfrac{2x - 1}{2}\right)\pi$, where $[x]$ denotes the greatest integer function, then f is

(a) continuous for every real x
(b) discontinuous only at $x = 0$
(c) discontinuous only at non-zero integral values of x
(d) continuous only at $x = 0$

9 Let f be a composite function of x defined by

$f(u)=\dfrac{1}{u^2+u-2}$, $u(x)=\dfrac{1}{x-1}$.

Then, the number of points x, where f is discontinuous, is

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(a) 4 (b) 3 (c) 2 (d) 1

10 If $f:R \to R$ be a positive increasing function with $\lim\limits_{x\to\infty}\dfrac{f(3x)}{f(x)}=1$. Then, $\lim\limits_{x\to\infty}\dfrac{f(2x)}{f(x)}$ is equal to

(a) 1 (b) 2/3 (c) 3/2 (d) 3

11 Let $f(x)=\max\{\tan x, \sin x, \cos x\}$, where $x\in\left[-\dfrac{\pi}{2},\dfrac{3\pi}{2}\right]$.

Then, the number of points of non-differentiability is

(a) 1 (b) 3 (c) 0 (d) 2

12 Let $S=\{t\in R: f(x)=|x-\pi|\cdot(e^{|x|}-1)\}\sin|x|$ is not differentiable at $t\}$. Then, the set S is equal to

(a) ϕ (an empty set) (b) $\{0\}$ ➔ **JEE Mains 2018**

(c) $\{\pi\}$ (d) $\{0,\pi\}$

13 If $f(x)=\lim\limits_{n\to\infty}\dfrac{x^n+\left(\dfrac{\pi}{3}\right)^n}{x^{n-1}+\left(\dfrac{\pi}{3}\right)^{n-1}}$, where n is an even integer,

Then which of the following is incorrect?

(a) If $f:\left[\dfrac{\pi}{3},\infty\right)\to\left[\dfrac{\pi}{3},\infty\right)$, then f is both one-one and onto

(b) $f(x)=f(-x)$ has infinitely many solutions

(c) $f(x)$ is one-one for all $x\in R$ (d) None of these

14 Let $f(x)=\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n-1}\dfrac{x}{(kx+1)\{(k+1)x+1\}}$. Then,

(a) f is continuous but not differentiable at $x=0$

(b) f is differentiable at $x=0$

(c) f is neither continuous nor differentiable at $x=0$

(d) None of the above

15 If $x_1, x_2, x_3,\ldots,x_4$ are the roots of $x^n+ax+b=0$, then the value of $(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)$ is equal to

(a) nx_1+b (b) $nx_1^{n-1}+a$

(c) nx_1^{n-1} (d) nx_1^n

16 If $^*\lim\limits_{x\to\infty} x\ln\left(\begin{vmatrix}\dfrac{1}{x} & -b & c\\ 0 & \dfrac{1}{x} & -1\\ 1 & 0 & a/x\end{vmatrix}\right)=-4$, where a, b, c are real numbers, then

(a) $a=1, b\in R, c=-1$ (b) $a\in R, b=2, c=4$

(c) $a=1, b=1, c\in R$ (d) $a\in R, b=1, c=4$

17 If $\lim\limits_{x\to 0}\left[1+x+\dfrac{f(x)}{x}\right]^{1/x}=e^3$, then $\lim\limits_{x\to 0}\left[1+\dfrac{f(x)}{x}\right]^{1/x}$ is equal to

(a) e (b) e^2 (c) e^3 (d) None of these

18 The function $f(x)$ is discontinuous only at $x=0$ such that $f^2(x)=1\ \forall x\in R$. The total number of such function is

(a) 2 (b) 3

(c) 6 (d) None of these

19 Let $f(x)=x|x|$ and $g(x)=\sin x$

Statement I gof is differentiable at $x=0$ and its derivative is continuous at that point.

Statement II gof is twice differentiable at $x=0$.

(a) Statement I is false, Statement II is true

(b) Statement I is true, Statement II is true; Statement II is a correct explanation of Statement I

(c) Statement I is true, Statement II is true; Statement II is not a correct explanation of Statement I

(d) Statement I is true, Statement II is false

20 **Statement I** The function $f(x)=(3x-1)|4x^2-12x+5|\cos\pi x$ is differentiable at $x=\dfrac{1}{2}$ and $\dfrac{5}{2}$.

Statement II $\cos(2n+1)\dfrac{\pi}{2}=0, \forall n\in I$.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

21 Define $f(x)$ as the product of two real functions $f_1(x)=x$, $x\in IR$

and $f_2(x)=\begin{cases}\sin\dfrac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ as follows

$$f(x)=\begin{cases} f_1(x)\cdot f_2(x), & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

Statement I $f(x)$ is continuous on IR.

Statement II $f_1(x)$ and $f_2(x)$ are continuous on IR.

(a) Statement I is false, Statement II is true

(b) Statement I is true, Statement II is true; Statement II is correct explanation of Statement I.

(c) Statement I is true, Statement II is true; Statement II is not a correct explanation of Statement I

(d) Statement I is true, Statement II is false

22 Consider the function $f(x)=|x-2|+|x-5|, x\in R$.

Statement I $f'(4)=0$

Statement II f is continuous in $[2,5]$ and differentiable in $(2,5)$ and $f(2)=f(5)$.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not correct explanation for Statement I

(c) Statement I is true; Statement II is false

(d) Statement I is false; Statement II is true

ANSWERS

SESSION 1

1 (a)	**2** (a)	**3** (b)	**4** (c)	**5** (d)	**6** (b)	**7** (b)	**8** (c)	**9** (c)	**10** (d)
11 (d)	**12** (d)	**13** (a)	**14** (b)	**15** (b)	**16** (c)	**17** (a)	**18** (d)	**19** (c)	**20** (b)
21 (b)	**22** (a)	**23** (a)	**24** (c)	**25** (c)	**26** (a)	**27** (d)	**28** (b)	**29** (b)	**30** (a)
31 (c)	**32** (c)	**33** (a)	**34** (a)	**35** (b)	**36** (b)	**37** (b)	**38** (b)	**39** (a)	**40** (d)
41 (c)	**42** (b)	**43** (a)							

SESSION 2

1 (c)	**2** (a)	**3** (d)	**4** (b)	**5** (b)	**6** (d)	**7** (d)	**8** (a)	**9** (b)	**10** (a)
11 (b)	**12** (d)	**13** (c)	**14** (c)	**15** (b)	**16** (d)	**17** (b)	**18** (c)	**19** (b)	**20** (a)
21 (d)	**22** (b)								

Hints and Explanations

SESSION 1

1 RHL $= \lim_{x \to 0^+} (x)^0 = 1$

LHL $= \lim_{x \to 0^-} (-x)^0 = \lim_{x \to 0^-} 1 = 1$

$\because$ RHL = LHL

$\therefore \lim_{x \to 0} |x|^{[\cos x]} = 1$

2 Given, $\lim_{x \to 5} f(x)$ exists and

$$\lim_{x \to 5} \frac{[f(x)]^2 - 9}{\sqrt{|x-5|}} = 0$$

$\Rightarrow \lim_{x \to 5} [f(x)]^2 - 9 = 0$

$\Rightarrow \left(\lim_{x \to 5} [f(x)]\right)^2 = 9$

$\therefore \lim_{x \to 5} f(x) = 3, -3$

But $f : R \to [0, \infty)$

$\therefore$ Range of $f(x) \geq 0$

$\Rightarrow \lim_{x \to 5} f(x) = 3$

3 Clearly, $\lim_{x \to \infty} \frac{2}{x}\left[\frac{x}{5}\right] = \lim_{x \to \infty} \frac{2}{x}\left(\frac{x}{5} - \left\{\frac{x}{5}\right\}\right)$

$$= \frac{2}{5} - 0 = \frac{2}{5}$$

$\therefore m + n = 7$

4 $\lim_{x \to \infty}\left(\frac{x^2+1}{x+1} - \alpha x - \beta\right) = 0$

$$\Rightarrow \lim_{x \to \infty} \frac{x^2(1-\alpha) - x(\alpha+\beta) + 1 - \beta}{x+1} = 0$$

$\therefore 1 - \alpha = 0, \alpha + \beta = 0$

$\Rightarrow \alpha = 1, \beta = -1$

5 Clearly,

$$\lim_{n \to \infty} \frac{3 \cdot 2^{n+1} - 4 \cdot 5^{n+1}}{5 \cdot 2^n + 7 \cdot 5^n} = \lim_{n \to \infty} \frac{6 \cdot 2^n - 20 \cdot 5^n}{5 \cdot 2^n + 7 \cdot 5^n}$$

$$= \lim_{n \to \infty} \frac{6 \cdot \left(\frac{2}{5}\right)^n - 20}{5 \cdot \left(\frac{2}{5}\right)^n + 7} = \frac{0-20}{0+7} = -\frac{20}{7}$$

6 $\lim_{x \to \infty}\left[\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}\right]$

$$= \lim_{x \to \infty} \frac{x + \sqrt{x + \sqrt{x}} - x}{\sqrt{x + \sqrt{x + \sqrt{x}}} + \sqrt{x}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x + \sqrt{x}}}{\sqrt{x + \sqrt{x + \sqrt{x}}} + \sqrt{x}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{1 + x^{-1/2}}}{\sqrt{1 + \sqrt{x^{-1} + x^{-3/2}}} + 1} = \frac{1}{2}$$

7 We know that,
$\cos A \cdot \cos 2A \cdot \cos 4A \ldots \cos 2^{n-1}$

$$A = \frac{\sin 2^n A}{2^n \sin A}$$

Take $A = \frac{x}{2^n}$,

then $\cos\left(\frac{x}{2^n}\right) \cdot \cos\left(\frac{x}{2^{n-1}}\right) \ldots \cos\left(\frac{x}{4}\right) \cos\left(\frac{x}{2}\right)$

$$= \frac{\sin x}{2^n \sin\left(\frac{x}{2^n}\right)}$$

$\therefore \lim_{n \to \infty} \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{4}\right) \ldots \cos\left(\frac{x}{2^{n-1}}\right) \cdot \cos\left(\frac{x}{2^n}\right)$

$$= \lim_{n \to \infty} \frac{\sin x}{2^n \sin\left(\frac{x}{2^n}\right)}$$

$$= \lim_{n \to \infty} \frac{\sin x}{x} \cdot \frac{x/2^n}{\sin(x/2^n)}$$

$$= \frac{\sin x}{x}$$

8 Clearly, $f(x+T) = f(x+2T) = \ldots$

$= f(x+nT) = f(x)$

$$\therefore \lim_{n \to \infty} n \left(\frac{f(x+T) + 2f(x+2T) + \ldots + nf(x+nT)}{f(x+T) + 4f(x+4T) + \ldots + n^2 f(x+x^2T)}\right)$$

$$= \lim_{n \to \infty} \frac{nf(x)(1+2+3+\ldots+n)}{f(x)(1+2^2+3^2+\ldots+n^2)}$$

$$= \lim_{n \to \infty} \frac{n\left(\frac{n(n+1)}{2}\right)}{\frac{n(n+1)(2n+1)}{6}} = \frac{3}{2}$$

9 We have, $\lim_{x \to 0} \frac{(1-\cos 2x)(3+\cos x)}{x \tan 4x}$

$$= \lim_{x \to 0} \frac{2\sin^2 x(3+\cos x)}{x \times \frac{\tan 4x}{4x} \times 4x}$$

$$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2} \times \lim_{x \to 0} \frac{(3+\cos x)}{4} \times \frac{1}{\lim_{x \to 0} \frac{\tan 4x}{4x}}$$

$$= 2 \times \frac{4}{4} \times 1 = 2$$

$$\left[\because \lim_{x \to 2} \frac{\sin\theta}{\theta} = 1 \text{ and } \lim_{x \to 0} \frac{\tan\theta}{\theta} = 1\right]$$

10 Since,

$$\lim_{x \to 0}\left[(a-n)n - \frac{\tan x}{x}\right] \cdot \frac{\sin nx}{x} = 0$$

$$\Rightarrow \quad [(a-n)n-1]\, n = 0$$
$$\Rightarrow \quad (a-n)n = 1$$
$$\therefore \quad a = n + \frac{1}{n}$$

11 $$\lim_{x\to 0} \frac{\sin(\pi\cos^2 x)}{x^2} = \lim_{x\to 0} \frac{\sin(\pi - \pi\sin^2 x)}{x^2}$$
$$= \lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{x^2}$$
$$[\because \sin(\pi-\theta) = \sin\theta]$$
$$= \lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x} \times (\pi)\left(\frac{\sin^2 x}{x^2}\right)$$
$$= \pi \quad \left[\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right]$$

12 $$\because f(x) = x(x-1)\sin x - (x^3 - 2x^2)\cos x - x^3\tan x$$
$$= x^2\sin x - x^3\cos x - x^3\tan x + 2x^2\cos x - x\sin x$$
$$\therefore \lim_{x\to 0} \frac{f(x)}{x^2} = \lim_{x\to 0} \left(\sin x - x\cos x - x\tan x + 2\cos x - \frac{\sin x}{x}\right)$$
$$= 2 - 1 = 1$$

13 $$\lim_{x\to 3} \frac{\sqrt{1-\cos(x^2-10x+21)}}{(x-3)}$$
$$= \lim_{x\to 3} \frac{\sqrt{2}\sin\frac{(x-3)(x-7)}{2}}{(x-3)}$$
$$= \lim_{x\to 3} \frac{\sqrt{2}\sin\frac{(x-3)(x-7)}{2}}{\frac{(x-3)\cdot(x-7)}{2}} \cdot \frac{(x-7)}{2}$$
$$= \lim_{x\to 3} (x-7) \cdot \lim_{x\to 3} \frac{\sin\frac{(x-3)(x-7)}{2}}{\frac{(x-3)(x-7)}{2}} \times \frac{1}{\sqrt{2}}$$
$$= -(2)^{3/2}$$

14 $$\lim_{x\to 0} \frac{1}{x}\left\{\tan^{-1}\left(\frac{x+1}{2x+1}\right) - \frac{\pi}{4}\right\}$$
$$= \lim_{x\to 0} \left\{\tan^{-1}\left(\frac{x+1}{2x+1}\right) - \tan^{-1}(1)\right\}$$
$$= \lim_{x\to 0} \frac{1}{x}\cdot\tan^{-1}\left\{\frac{\frac{x+1}{2x+1} - 1}{1 + \frac{x+1}{2x+1}}\right\}$$
$$= \lim_{x\to 0} \frac{1}{x}\cdot\tan^{-1}\left(\frac{x+1-2x-1}{2x+1+x+1}\right)$$
$$= \lim_{x\to 0} \frac{1}{x}\cdot\tan^{-1}\left(\frac{-x}{4x+2}\right)\left[\text{form } \frac{0}{0}\right]$$
[using L' Hospital rule]
$$= \lim_{x\to 0} \frac{1}{1 + \frac{x^2}{(4x+2)^2}} \times \left[-\left(\frac{4x+2-4x}{(4x+2)^2}\right)\right]$$
$$= \lim_{x\to 0} - \frac{(+2)}{x^2 + (4x+2)^2}$$
$$= -\frac{(+2)}{0 + (0+2)^2} = \frac{-2}{4} = \frac{-1}{2}$$

15 $$\lim_{x\to \pi/2} \frac{\cot x - \cos x}{(\pi - 2x)^3}$$
$$= \lim_{x\to \pi/2} \frac{1}{8}\cdot\frac{\cos x(1-\sin x)}{\sin x\left(\frac{\pi}{2} - x\right)^3}$$
$$= \lim_{h\to 0} \frac{1}{8}\cdot\frac{\cos\left(\frac{\pi}{2} - h\right)\left[1 - \sin\left(\frac{\pi}{2} - h\right)\right]}{\sin\left(\frac{\pi}{2} - h\right)\left(\frac{\pi}{2} - \frac{\pi}{2} + h\right)^3}$$
$$= \frac{1}{8}\lim_{h\to 0} \frac{\sin h\,(1-\cos h)}{\cos h\cdot h^3}$$
$$= \frac{1}{8}\lim_{h\to 0} \frac{\sin h\left(2\sin^2\frac{h}{2}\right)}{\cos h\cdot h^3}$$
$$= \frac{1}{4}\lim_{h\to 0} \frac{\sin h\cdot\sin^2\left(\frac{h}{2}\right)}{h^3\cos h}$$
$$= \frac{1}{4}\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2\cdot\frac{1}{\cos h}\cdot\frac{1}{4}$$
$$= \frac{1}{4}\times\frac{1}{4} = \frac{1}{16}$$

16 $$\lim_{x\to 0} \frac{(\cos x)^{1/m} - (\cos x)^{1/n}}{x^2}$$
$$= \lim_{x\to 0} \frac{(\cos x)^{\frac{1}{m}-\frac{1}{n}} - 1}{x^2}\cdot\lim_{x\to 0}\frac{1}{(\cos x)^{1/n}}$$
$$= \lim_{x\to 0} \frac{\left(1 - 2\sin^2\frac{x}{2}\right)^{\frac{1}{m}-\frac{1}{n}} - 1}{x^2}$$
$$= \lim_{x\to 0} -2\left(\frac{1}{m} - \frac{1}{n}\right)\frac{\sin^2\frac{x}{2}}{x^2} = \frac{m-n}{2mn}$$

17 Clearly,
$$= \lim_{x\to\infty}(\sqrt{x+1} - \sqrt{x})$$
$$= \lim_{x\to\infty} \frac{1}{\sqrt{x+1} + \sqrt{x}} = 0$$
and
$$\lim_{x\to\infty}\left(\frac{2x+1}{x-1}\right)^x = \lim_{x\to\infty}\left(\frac{2+\frac{1}{x}}{1 - 1/x}\right)^x = \infty$$
$$\therefore \lim_{x\to\infty} \frac{\cot^{-1}(\sqrt{x+1} - \sqrt{x})}{\sec^{-1}\left(\frac{2x+1}{x-1}\right)^x} = \frac{\cot^{-1}(0)}{\sec^{-1}(\infty)}$$
$$= \frac{\pi/2}{\pi/2} = 1$$

18 Clearly, $$p = \lim_{x\to 0} \frac{\ln(1 + \cos 2x - 1)}{3x^2}$$
$$= \lim_{x\to 0} \frac{\ln(1+\cos 2x - 1)}{(\cos 2x - 1)}\cdot\frac{\cos 2x - 1}{3x^2}$$
$$= -\frac{2}{3}$$
$$q = \lim_{x\to 0} \frac{\sin^2 2x}{4x^2}\cdot\frac{4x^2}{x(1-e^x)} = -4$$
and $$r = \lim_{x\to 1} \frac{\sqrt{x} - x}{\ln(1 + x - 1)}$$
$$= \lim_{x\to 1} \frac{\sqrt{x}(1-\sqrt{x})}{\ln\left(\frac{1+x-1}{x-1}\right)\cdot(x-1)}$$
$$= \lim_{x\to 1} \frac{\sqrt{x}(1-x)}{\ln\left(\frac{1+(x-1)}{x-1}\right)\cdot(x-1)(1+\sqrt{x})}$$
$$= -\frac{1}{2}$$
Hence, $q < p < r$.

19 Given, $$p = \lim_{x\to 0^+}(1 + \tan^2\sqrt{x})^{\frac{1}{2x}}$$
$(1^\infty$ form)
$$= e^{\lim_{x\to 0^+}\frac{\tan^2\sqrt{x}}{2x}} = e^{\frac{1}{2}\lim_{x\to 0^+}\left(\frac{\tan\sqrt{x}}{\sqrt{x}}\right)^2}$$
$$= e^{\frac{1}{2}}$$
$$\therefore \log p = \log e^{\frac{1}{2}} = \frac{1}{2}$$

20 $$\lim_{x\to\infty}\left(\frac{3x-4}{3x+2}\right)^{\frac{x+1}{3}}$$
$$= \lim_{x\to\infty}\left(\frac{3x+2-6}{3x+2}\right)^{\frac{x+1}{3}}$$
$$= \lim_{x\to\infty}\left(1 - \frac{6}{3x+2}\right)^{\frac{x+1}{3}}$$
$$= \lim_{x\to\infty}\left[\left(1 - \frac{6}{3x+2}\right)^{\frac{3x+2}{-6}}\right]^{\frac{-6}{3x+2}\cdot\frac{x+1}{3}}$$
$$= \lim_{x\to\infty} e^{\frac{-2(x+1)}{3x+2}} = e^{-2/3}$$
$$\left[\because \lim_{x\to\infty}\frac{-2(x+1)}{3x+2} = \frac{-2}{3}\right]$$

21 Now, $$\lim_{x\to\infty}\left(1 + \frac{a}{x} + \frac{b}{x^2}\right)^{2x}$$
$$= \lim_{x\to\infty}\left(1 + \frac{a}{x} + \frac{b}{x^2}\right)^{2x\left(\frac{\frac{a}{x}+\frac{b}{x^2}}{\frac{a}{x}+\frac{b}{x^2}}\right)}$$
$$= e^{\lim_{x\to\infty} 2x\left(\frac{a}{x} + \frac{b}{x^2}\right)}$$
$$\left[\because \lim_{x\to\infty}(1+x)^{1/x} = e\right] = e^{2a}$$
But $$\lim_{x\to\infty}\left(1 + \frac{a}{x} + \frac{b}{x^2}\right)^{2x} = e^2$$
$$\Rightarrow \quad e^{2a} = e^2$$
$$\Rightarrow \quad a = 1$$
and $\quad b \in R$

22 $\lim_{h\to 0} \dfrac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$

$= \lim_{h\to 0} \dfrac{f'(2h+2+h^2)(2+2h)}{f'(h-h^2+1)(1-2h)}$

$= \dfrac{f'(2)\times 2}{f'(1)\times 1}$

$= \dfrac{6\times 2}{4\times 1} = 3$

23 $\lim_{x\to a} \dfrac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)} = 4$

Applying L'Hospital rule, we get

$\lim_{x\to a} \dfrac{f(a)g'(x)-g(a)f'(x)}{g'(x)-f'(x)} = 4$

$\Rightarrow \lim_{x\to a} \dfrac{kg'(x)-kf'(x)}{g'(x)-f'(x)} = 4$

$\therefore \quad k = 4$

24 Let $y = \left(\dfrac{f(1+x)}{f(1)}\right)^{1/x}$

$\Rightarrow \log y = \dfrac{1}{x}[\log f(1+x) - \log f(1)]$

$\Rightarrow \lim_{x\to 0} \log y = \lim_{x\to 0}\left[\dfrac{1}{f(1+x)} f'(1+x)\right]$

$\Rightarrow \lim_{x\to 0} \log y = \dfrac{f'(1)}{f(1)} = \dfrac{6}{3}$

$\therefore \quad \lim_{x\to 0} y = e^2$

25 Clearly, $f(x)$ is continuous only when $\sin^2 x = -\sin^2 x \Rightarrow 2\sin^2 x = 0$

$\Rightarrow \quad x = n\pi$

26 We have,

$f(x) = \begin{cases} \text{rational}, & \text{if } x \notin Q \quad \text{in}[a,b] \\ \text{irrational}, & \text{if } x \in Q \quad \text{in}[a,b] \end{cases}$

Let $C \in [a,b]$ and $c \in Q$. Then,

$f(c) = \text{irrational}$

and $\lim_{x\to c} f(x) = \lim_{h\to 0} f(c+h) =$ rational or irrational

Thus, f is discontinuous everywhere.

27 $g(x) = \begin{cases} 1+x, & x<0 \\ 1-x, & x\geq 0 \end{cases}$

$\therefore \ f\{g(x)\} = \begin{cases} 1+|x-1|, & x<0 \\ 1+|-x-1|, & x\geq 0 \end{cases}$

$= \begin{cases} 1+1-x, & x<0 \\ 1+x+1, & x\geq 0 \end{cases}$

$= \begin{cases} 2-x, & x<0 \\ 2+x, & x\geq 0 \end{cases}$

It is a polynomial function, so it is continuous in everywhere except at $x = 0$.

Now, LHL $= \lim_{x\to 0} 2-x = 2$,

RHL $= \lim_{x\to 0} 2+x = 2$

Also, $\quad f(0) = 2+0 = 2$

Hence, it is continuous everywhere.

28 $f\{g(x)\} = \begin{cases} 1, & 0<x<3\pi/4 \\ 0, & x = 3\pi/4, 7\pi/4 \\ -1, & 3\pi/4 < x < 7\pi/4 \\ & \text{or } 7\pi/4 < x < 2 \end{cases}$

Clearly, $[f\{g(x)\}]$ is not continuous at $x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$.

29 $f'(1) = \lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$

$= \lim_{h\to 0} \dfrac{f(1+h)}{h} - \lim_{h\to 0} \dfrac{f(1)}{h}$

Since, $\lim_{h\to 0} \dfrac{f(1+h)}{h} = 5$, so $\lim_{h\to 0} \dfrac{f(1)}{h}$ must be finite as $f'(1)$ exists and $\lim_{h\to 0} \dfrac{f(1)}{h}$ can be finite only, if $f(1) = 0$ and $\lim_{h\to 0} \dfrac{f(1)}{h} = 0$.

$\therefore \quad f'(1) = \lim_{h\to 0} \dfrac{f(1+h)}{h} = 5$

30 $\lim_{h\to 0} \dfrac{f(1-h)-f(1)}{h^3+3h}$

$= \lim_{h\to 0} \dfrac{f(1-h)-f(1)}{-h} \cdot \dfrac{-1}{h^2+3}$

$= f'(1)\cdot\left(\dfrac{-1}{3}\right) = \dfrac{53}{3}$

31 $\lim_{x\to 2} \dfrac{xf(2)-2f(x)}{x-2}$

$= \lim_{x\to 2} \dfrac{xf(2)-2f(2)+2f(2)-2f(x)}{x-2}$

$= \lim_{x\to 2} \dfrac{f(2)(x-2)-2\{f(x)-f(2)\}}{x-2}$

$= f(2) - 2\lim_{x\to 2} \dfrac{f(x)-f(2)}{x-2}$

$\left[\because f'(x) = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}\right]$

$= f(2) - 2f'(2)$

$= 4 - 2\times 4 = -4$

32 $\because \quad |f(x)-f(y)| \leq (x-y)^2$

$\therefore \quad \lim_{x\to y} \dfrac{|f(x)-f(y)|}{|x-y|} \leq \lim_{x\to y} |x-y|$

$\Rightarrow \quad |f'(y)| \leq 0 \quad \Rightarrow \quad f'(y) = 0$

$\Rightarrow \quad f(y) = \text{Constant}$

$\Rightarrow \quad f(y) = 0 \qquad [\because f(0) = 0, \text{given}]$

$\Rightarrow \quad f(1) = 0$

33 Let $f(x) = \log x$

$\Rightarrow \quad f'(x) = \dfrac{1}{x}$

Therefore, given function

$= f'(a) + kf'(e) = 1$

$\Rightarrow \quad \dfrac{1}{a} + \dfrac{k}{e} = 1$

$\Rightarrow \quad k = e\left(\dfrac{a-1}{a}\right)$

34 If x is just less than k, then $[x] = k-1$

$\therefore \quad f(x) = (k-1)\sin \pi x$

LHD of $\quad f(x) = \lim_{x\to k} \dfrac{(k-1)\sin \pi x - k\sin \pi k}{x-k}$

$= \lim_{x\to k} \dfrac{(k-1)\sin \pi x}{x-k}$,

where $x = k-h$

$= \lim_{h\to 0} \dfrac{(k-1)\sin \pi(k-h)}{-h}$

$= (k-1)(-1)^k \pi$

35 $f(x) = \begin{cases} e^x, & x\leq 0 \\ 1-x, & 0<x\leq 1 \\ x-1, & x>1 \end{cases}$

$Rf'(0) = \lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0} \dfrac{1-h-1}{h} = -1$

$Lf'(0) = \lim_{h\to 0} \dfrac{f(0-h)-f(0)}{-h}$

$= \lim_{h\to 0} \dfrac{e^{-h}-1}{-h} = 1$

So, it is not differentiable at $x = 0$.

Similarly, it is not differentiable at $x = 1$ but it is continuous at $x = 0$ and 1.

36 RHL $= \lim_{h\to 0} (0+h)e^{-2/h} = \lim_{h\to 0} \dfrac{h}{e^{2/h}} = 0$

LHL $= \lim_{h\to 0} (0-h)e^{-\left(\frac{1}{h}-\frac{1}{h}\right)} = 0$

Hence, $f(x)$ is continuous at $x = 0$.

Now, $Rf'(x) = \lim_{h\to 0} \dfrac{(0+h)e^{-\left(\frac{1}{h}+\frac{1}{h}\right)} - 0}{h}$

$= \lim_{h\to 0} e^{-2/h} = \infty$

and $Lf'(x) = \lim_{h\to 0} \dfrac{(0-h)e^{-\left(\frac{1}{h}-\frac{1}{h}\right)} - 0}{-h}$

$= \lim_{h\to 0} e^{-0} = 1$

$\therefore \quad Lf'(x) \neq Rf'(x)$

Hence, $f(x)$ is not differentiable at $x = 0$.

37 Since, $\quad f(x) = \dfrac{x}{1+|x|} = \dfrac{g(x)}{h(x)} \qquad$ [say]

It is clear that $g(x)$ and $h(x)$ are differentiable on $(-\infty, \infty)$ and $(-\infty, 0) \cup (0, \infty)$.

Now,

$\lim_{x\to 0} \dfrac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \dfrac{\dfrac{x}{1+|x|} - 0}{x} = 1$

Hence, $f(x)$ is differentiable on $(-\infty, \infty)$.

38 Clearly, $g(x) = \begin{cases} \cos x, & x \in [0,\pi] \\ \sin x - 1, & x > \pi \end{cases}$

Also, $g(\pi^-) = g(\pi) = g(\pi^+) = -1$

and $\quad g'(\pi^-) \neq g'(\pi^+)$

$\therefore \quad g$ is continuous at $x = \pi$

but not differentiable at $x = \pi$

39 Since, $g(x)$ is differentiable $\Rightarrow g(x)$ must be continuous.

$$g(x)=\begin{cases}k\sqrt{x+1}, & 0\le x\le 3\\ mx+2, & 3<x\le 5\end{cases}$$

At, $x=3$, RHL $=3m+2$

and at $x=3$, LHL $=2k$

$\therefore \quad 2k=3m+2$

Also, $g'(x)=\begin{cases}\dfrac{k}{2\sqrt{x+1}}, & 0\le x<3\\ m, & 3<x\le 5\end{cases}$

$\therefore \quad L(g'(3))=\dfrac{k}{4}$

and $R\{g'(3)\}=m \Rightarrow \dfrac{k}{4}=m$

i.e. $\quad k=4m$

On solving Eqs (i) and (ii), we get

$$k=\frac{8}{5}, m=\frac{2}{5}$$

$\Rightarrow \quad k+m=2$

40 Clearly, $f(x)=\begin{cases}2\sqrt{1-x^2}, & x\le 0\\ 0, & x>0\end{cases}$

$\therefore f(x)$ is discontinuous and hence non-differentiable at $x=0$.

41 $[\sin x]$ is non-differentiable at

$x=\dfrac{\pi}{2}, \pi, 2\pi$ and $[\cos x]$ is non-differentiable at

$x=0, \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ and 2π.

Thus, $f(x)$ is definitely non-differentiable at $x=\pi, \dfrac{3\pi}{2}, 0$.

Also,

$f\left(\dfrac{\pi}{2}\right)=1, f\left(\dfrac{\pi}{2}-0\right)=0,$

$f(2\pi)=1, \qquad f(2\pi-0)=-1$

Thus, $f(x)$ is also non-differentiable at $x=\dfrac{\pi}{2}$ and 2π.

42 Let $u(x)=\sin x$

$v(x)=|x|$

$\therefore f(x)=vou(x)=v(u(x))=v(\sin x)=|\sin x|$

$\because \quad u(x)=\sin x$ is a continuous function and $v(x)=|x|$ is a continuous function.

$\therefore f(x)=vou(x)$ is also continuous everywhere but $v(x)$ is not differentiable at $x=0$

$\Rightarrow f(x)$ is not differentiable where $\sin x=0$

$\Rightarrow \quad x=n\pi, n\in Z$

Hence, $f(x)$ is continuous everywhere but not differentiable at

$x=n\pi, n\in Z$.

43 $f(x)=\begin{cases}-\log x, x<1\\ \log x, x\ge 1\end{cases}$

$f'(x)=\begin{cases}-1/x, & x<1\\ \dfrac{1}{x}, & x>1\end{cases}$

$\therefore f'(1^-)=-1$ and $f'(1^+)=1$

Hence, $f(x)$ is not differentiable.

SESSION 2

1 We have,

$$\lim_{x\to 0^+} x\left(\left[\frac{1}{x}\right]+\left[\frac{1}{x}\right]+\ldots+\left[\frac{15}{x}\right]\right)$$

We know, $[x]=x-\{x\}$

$\therefore \quad \left[\dfrac{1}{x}\right]=\dfrac{1}{x}-\left\{\dfrac{1}{x}\right\}$

Similarly, $\left[\dfrac{n}{x}\right]=\dfrac{n}{x}-\left\{\dfrac{n}{x}\right\}$

$\therefore$ Given limit

$$=\lim_{x\to 0^+} x\left(\frac{1}{x}-\left\{\frac{1}{x}\right\}+\frac{2}{x}-\left\{\frac{2}{x}\right\}+\ldots\frac{15}{x}-\left\{\frac{15}{x}\right\}\right)$$

$$=\lim_{x\to 0^+}(1+2+3+\ldots+15)-x\left(\left\{\frac{1}{x}\right\}+\left\{\frac{2}{x}\right\}+\ldots+\left\{\frac{15}{x}\right\}\right)$$

$=120-0=120$

$$\left[\begin{array}{l}\because 0\le\left\{\dfrac{n}{x}\right\}<1, \text{therefore}\\ 0\le x\left\{\dfrac{n}{x}\right\}<x\Rightarrow \displaystyle\lim_{x\to 0^+} x\left\{\dfrac{n}{x}\right\}=0\end{array}\right]$$

2 Now, $\displaystyle\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x$

$$=\lim_{x\to\infty}\left(1+\frac{4x+1}{x^2+x+2}\right)^x$$

$$=\lim_{x\to\infty}\left[\left(1+\frac{4x+1}{x^2+x+2}\right)^{1\Big/\frac{(4x+1)}{x^2+x+2}}\right]^{\frac{(4x+1)x}{x^2+x+2}}$$

$$=e^{\lim\limits_{x\to\infty}\frac{\left(4+\frac{1}{x}\right)}{1+\frac{1}{x}+\frac{2}{x^2}}}=e^4$$

$$\left[\because \lim_{x\to 0}\left(1+\frac{1}{x}\right)^x=e\right]$$

3 Now, $\displaystyle\lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$

$$=\lim_{x\to\alpha}\frac{2\sin^2\left(\dfrac{ax^2+bx+c}{2}\right)}{(x-\alpha)^2}$$

$$=\lim_{x\to\alpha}\frac{2\sin^2\left(\dfrac{a}{2}(x-\alpha)(x-\beta)\right)}{\left(\dfrac{a}{2}\right)^2(x-\alpha)^2(x-\beta)^2}\left(\frac{a}{2}\right)^2(x-\beta)^2$$

$$=\lim_{x\to\alpha}\frac{a^2}{2}(x-\beta)^2 \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=\frac{a^2}{2}(\alpha-\beta)^2$$

4 $\displaystyle\lim_{n\to\infty}\sin\left\{n\pi\left(1+\frac{1}{n^2}\right)^{1/2}\right\}$

$$=\lim_{n\to\infty}\sin\left\{n\pi\left(1+\frac{1}{2n^2}-\frac{1}{8n^4}+\ldots\infty\right)\right\}$$

$$=\lim_{n\to\infty}\sin\left\{n\pi+\frac{\pi}{2n}-\frac{\pi}{8n^3}+\ldots\infty\right\}$$

$$=\lim_{n\to\infty}(-1)^n\sin\pi\left(\frac{1}{2n}-\frac{1}{8n^3}+\ldots\infty\right)$$

$=0$

5 Given, $x>0$ and g is a bounded function.

Then, $\displaystyle\lim_{n\to\infty}\frac{f(x)\cdot e^{nx}+g(x)}{e^{nx}+1}$

$$=\lim_{n\to\infty}\left[\frac{f(x)}{1+\left(\dfrac{1}{e^{nx}}\right)}+\frac{g(x)}{e^{nx}+1}\right]$$

$$=\frac{f(x)}{1+0}+\frac{\text{Finite}}{\infty}=f(x)$$

6 Let $\quad y=\displaystyle\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{2/x}$

$$\Rightarrow \log y=\lim_{x\to 0}\frac{2}{x}\log\left(\frac{a^x+b^x+c^x}{3}\right)$$

$$=2\lim_{x\to 0}\frac{\log(a^x+b^x+c^x)-\log 3}{x}$$

Apply L'Hospital's rule,

$$=2\lim_{x\to 0}\frac{\dfrac{a^x\log a+b^x\log b+c^x\log c}{a^x+b^x+c^x}}{1}$$

$\log y=\log(abc)^{2/3}$

$\Rightarrow \quad y=(abc)^{2/3}$

7 $f(x)=\cot^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

and $g(x)=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

On putting $x=\tan\theta$ in both equations, we get

$$f(\theta)=\cot^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$\Rightarrow f(\theta)=\cot^{-1}(\tan 3\theta)$

$\Rightarrow f(\theta)=\cot^{-1}\left\{\cot\left(\dfrac{\pi}{2}-3\theta\right)\right\}$

$=\dfrac{\pi}{2}-3\theta$

$\therefore \quad f'(\theta)=-3 \qquad \ldots$ (i)

and $g(\theta)=\cos^{-1}\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$=\cos^{-1}(\cos 2\theta)=2\theta$

$\therefore \quad g'(\theta) = 2$

Now,

$$\lim_{x\to a}\left(\frac{f(x)-f(a)}{g(x)-g(a)}\right) = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right) \times \frac{1}{\lim_{x\to a}\left(\frac{g(x)-g(a)}{x-a}\right)}$$

$$= f'(a)\cdot\frac{1}{g'(a)} = -3\times\frac{1}{2} = -\frac{3}{2}$$

8 Now, $\cos x$ is continuous, $\forall x \in R$

$\Rightarrow \cos \pi\left(x - \frac{1}{2}\right)$ is also continuous, $\forall\ x \in R$.

Hence, the continuity of f depends upon the continuity of $[x]$, which is discontinuous, $\forall\ x \in I$.

So, we should check the continuity of f at $x = n$, $\forall\ n \in I$

LHL at $x = n$ is given by

$$f(n^-) = \lim_{x\to n^-} f(x) = \lim_{x\to n^-} [x] \cos \pi\left(x - \frac{1}{2}\right) = (n-1)\cos\frac{(2n-1)\pi}{2} = 0$$

RHL at $x = n$ is given by

$$f(n^+) = \lim_{x\to n^+} f(x) = \lim_{x\to n^+} [x] \cos\ \pi\left(x - \frac{1}{2}\right) = (n)\cos\frac{(2n-1)\pi}{2} = 0$$

Also, value of the function at $x = n$ is

$$f(n) = [n]\cos\pi\left(n - \frac{1}{2}\right) = (n)\cos\frac{(2n-1)\pi}{2} = 0$$

$\therefore \quad f(n^+) = f(n^-) = f(n)$

Hence, f is continuous at $x = n$, $\forall\ n \in I$.

9 The function $u(x) = \dfrac{1}{x-1}$ is discontinuous at the point $x = 1$.

The function $y = f(u)$

$$= \frac{1}{u^2 + u - 2} = \frac{1}{(u+2)(u-1)}$$

is discontinuous at $u = -2$ and $u = 1$.

When $\quad u = -2$

$\Rightarrow \quad \dfrac{1}{x-1} = -2$

$\Rightarrow \quad x = \dfrac{1}{2}$

When $u = 1$

$\Rightarrow \quad \dfrac{1}{x-1} = 1$

$\Rightarrow \quad x = 2$

Hence, the composite function $y = f(x)$ is discontinuous at three points, $x = \frac{1}{2}$, $x = 1$ and $x = 2$.

10 Since, $f(x)$ is a positive increasing function.

$\therefore \quad 0 < f(x) < f(2x) < f(3x)$

$\Rightarrow \quad 0 < 1 < \dfrac{f(2x)}{f(x)} < \dfrac{f(3x)}{f(x)}$

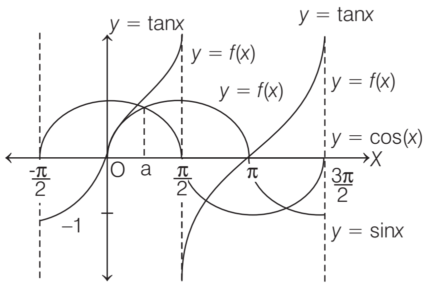
$$\Rightarrow \quad \lim_{x\to\infty} 1 \le \lim_{x\to\infty}\frac{f(2x)}{f(x)} \le \lim_{x\to\infty}\frac{f(3x)}{f(x)}$$

By Sandwich theorem,

$$\lim_{x\to\infty}\frac{f(2x)}{f(x)} = 1$$

11 We have, $f(x) = \max\{\tan x, \sin x, \cos x\}$, where $x \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

Let us draw the graph of $y = \tan x$, $y = \sin x$ and $y = \cos x$ in $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$



From the graph, it clear that $f(x)$ is non-differentiable at $x = a, \frac{\pi}{2}$ and π.

12 We have,

$$f(x) = |x-\pi|(e^{|x|}-1)\sin|x|$$

$$f(x) = \begin{cases} (x-\pi)(e^{-x}-1)\sin x, & x < 0 \\ -(x-\pi)(e^{x}-1)\sin x, & 0 \le x < \pi \\ (x-\pi)(e^{x}-1)\sin x, & x \ge \pi \end{cases}$$

We check the differentiability at $x = 0$ and π.

We have,

$$f'(x) = \begin{cases} (x-\pi)(e^{-x}-1)\cos x + (e^{x}-1)\sin x \\ \quad + (x-\pi)\sin x e^{-x}(-1), x < 0 \\ -[(x-\pi)(e^{x}-1)\cos x + (e^{x}-1)\sin x \\ \quad + (x-\pi)\sin x e^{x}], 0 < x < \pi \\ (x-\pi)(e^{x}-1)\cos x + (e^{x}-1)\sin x \\ \quad + (x-\pi)\sin x e^{x}, x > \pi \end{cases}$$

Clearly, $\lim_{x\to 0^-} f'(x) = 0 = \lim_{x\to 0^+} f'(x)$

and $\quad \lim_{x\to\pi^-} f'(x) = 0 = \lim_{x\to\pi^+} f'(x)$

$\therefore$ f is differentiable at $x = 0$ and $x = \pi$

Hence, f is equal to the set $\{0, \pi\}$.

13 We have, $f(x) = \lim_{n\to\infty}\dfrac{x^n + \left(\frac{\pi}{3}\right)^n}{x^{n-1} + \left(\frac{\pi}{3}\right)^{n-1}}$

$$= \lim_{n\to\infty}\frac{x\left(1 + \left(\frac{\pi}{3x}\right)^n\right)}{1 + \left(\frac{\pi}{3x}\right)^{n-1}} = x, \text{ if } x > \frac{\pi}{3}$$

Also, $f(x) = \lim_{n\to\infty}\dfrac{\pi}{3}\dfrac{\left(\left(\frac{3x}{\pi}\right)^n + 1\right)}{\left(\left(\frac{3x}{\pi}\right)^{n-1} + 1\right)} = \dfrac{\pi}{3}$,

if $x < \pi/3$

Note that $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3}$

$$\therefore \quad f(x) = \begin{cases} x, & \text{if } \ x \ge \frac{\pi}{3} \\ \frac{\pi}{3}, & \text{if } \ x < \frac{\pi}{3} \end{cases}$$

From the given options, it is clear that option (c) is incorrect.

14 Let $a_{k+1} = \dfrac{x}{(kx+1)\{(k+1)x+1\}}$

$$= \left\{\frac{1}{kx+1} - \frac{1}{(k+1)x+1}\right\}$$

$$\therefore f(x) = \lim_{n\to\infty}\sum_{k=0}^{n-1}\left\{\frac{1}{kx+1} - \frac{1}{(k+1)x+1}\right\}$$

$$= \lim_{n\to\infty}\left[1 - \frac{1}{nx+1}\right]$$

$$= \begin{cases} 1, & \text{if } \ x \ne 0 \\ 0, & \text{if } \ x = 0 \end{cases}$$

Clearly, $f(x)$ is neither continuous nor differentiable at $x = 0$.

15 Clearly, $x^n + xa + b = (x - x_1)(x - x_2)\ldots(x - x_n)$

$$\Rightarrow \quad \frac{x^n + xa + b}{x - x_1} = (x - x_2)(x - x_3)\ldots(x - x_n)$$

$$\Rightarrow \quad \lim_{x\to x_1}\frac{x^n + ax + b}{x - x_1} = (x_1 - x_2)(x_1 - x_3)\ldots(x_1 - x_n)$$

$$\Rightarrow \quad \lim_{x\to x_1}\frac{nx^{n-1} + a}{1} = (x_1 - x_2)(x_1 - x_3)\ldots(x_1 - x_n)$$

[using L'Hospital rule]

$$\Rightarrow \quad nx_1^{n-1} + a = (x_1 - x_2)(x_1 - x_3)\ldots(x_1 - x_n).$$

16 Let $L = \lim_{x\to\infty} x \ln \left(\begin{vmatrix} \frac{1}{x} & -b & c \\ 0 & \frac{1}{x} & -1 \\ 1 & 0 & \frac{a}{x} \end{vmatrix} \right)$

$$= \lim_{x\to\infty} x \ln\left(\frac{a}{x^3} + b - \frac{c}{x}\right)$$

Clearly, for limit to be exist, $b = 1$.

Thus, $L = \lim_{x\to\infty} x \ln\left(1 + \frac{a}{x^3} - \frac{c}{x}\right)$

$$= \lim_{x\to\infty} x\left(\frac{a}{x^3} - \frac{c}{x}\right)$$

$$\left[\because \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right]$$

$$= -c$$

$\because \quad L = -4$

$\therefore \quad c = 4$

Hence, $a \in R, b = 1$ and $c = 4$.

17 We have, $\lim_{x\to 0}\left[1 + x + \frac{f(x)}{x}\right]^{1/x} = e^3$

$\Rightarrow \quad e^{\lim_{x\to 0}\left\{1+x+\frac{f(x)}{x}-1\right\}\frac{1}{x}} = e^3$

$\Rightarrow \quad e^{\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right]} = e^3$

$\Rightarrow \quad \lim_{x\to 0}\frac{f(x)}{x^2} = 2$

Now,

$$\lim_{x\to 0}\left[1 + \frac{f(x)}{x}\right]^{1/x} = e^{\lim_{x\to 0}\left[1+\frac{f(x)}{x}-1\right]\frac{1}{x}}$$

$$= e^{\lim_{x\to 0}\frac{f(x)}{x^2}} = e^2$$

18 We have, $f^2(x) = 1 \ \forall x \in R$

$\therefore \quad f$ can take values +1 or −1

Since f is discontinuous only at $x = 0$

$\therefore f$ may be one of the followings

(i) $f(x) = \begin{cases} 1, & x \le 0 \\ -1, & x > 0 \end{cases}$

(ii) $f(x) = \begin{cases} 1, & x < 0 \\ -1, & x \ge 0 \end{cases}$

(iii) $f(x) = \begin{cases} -1, & x \le 0 \\ 1, & x > 0 \end{cases}$

(iv) $f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \ge 0 \end{cases}$

(v) $f(x) = \begin{cases} 1, & x > 0 \\ 1, & x < 0 \\ -1 & x = 0 \end{cases}$

(vi) $f(x) = \begin{cases} -1, & x > 0 \\ -1, & x < 0 \\ 1, & x = 0 \end{cases}$

19 $f(x) = x\,|x|$ and $g(x) = \sin x$

$gof\,(x) = \sin(x\,|x|) = \begin{cases} -\sin x^2, & x < 0 \\ \sin x^2, & x \ge 0 \end{cases} \therefore$

$(gof)'(x) = \begin{cases} -2x\cos x^2, & x < 0 \\ 2x\cos x^2, & x \ge 0 \end{cases}$

Clearly, $L\,(gof)'(0) = 0 = R\,(gof)'(0)$

So, gof is differentiable at $x = 0$ and also its derivative is continuous at $x = 0$.

Now,

$$(gof)''(x) = \begin{cases} -2\cos x^2 + 4x^2\sin x^2, \\ \qquad x < 0 \\ 2\cos x^2 - 4x^2\sin x^2, \\ \qquad x \ge 0 \end{cases}$$

$\therefore L\,(gof)''(0) = -2$ and $R\,(gof)''(0) = 2$

$\therefore \quad L\,(gof)''(0) \ne R\,(gof)''(0)$

Hence, $gof(x)$ is not twice differentiable at $x = 0$.

Therefore, Statement I is true, Statement II is false.

20 Statement I is correct as though $|4x^2 - 12x + 5|$ is non-differentiable at $x = \frac{1}{2}$ and $\frac{5}{2}$ but $\cos \pi x = 0$ those points.

So, $f'\left(\frac{1}{2}\right)$ and $f'\left(\frac{5}{2}\right)$ exists.

21 Here, $f(x) = \begin{cases} x\cdot\sin\left(\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$

To check continuity at $x = 0$,

$$\text{LHL} = \lim_{h\to 0}\left\{(-h)\sin\left(-\frac{1}{h}\right)\right\} = 0$$

$$\text{RHL} = \lim_{h\to 0}\left\{h\sin\left(\frac{1}{h}\right)\right\} = 0$$

$$f(0) = 0$$

So, $f(x)$ is continuous at $x = 0$.

Hence, Statement I is correct.

$$f_2(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$$

Here, $\lim_{x\to 0} f_2(x) = \lim_{x\to 0}\sin\left(\frac{1}{x}\right)$

which does not exist.

So, $f_2(x)$ is not continuous at $x = 0$.

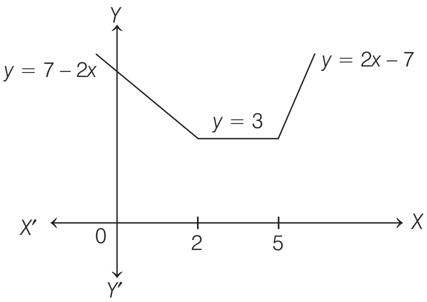
Hence, Statement II is false.

22 $\therefore f(x) = |x-2| + |x-5|$

$$= \begin{cases} (2-x)+(5-x), & x < 2 \\ (x-2)+(5-x), & 2 \le x \le 5 \\ (x-2)+(x-5), & x > 5 \end{cases}$$

$$= \begin{cases} 7-2x, & x < 2 \\ 3, & 2 \le x \le 5 \\ 2x-7, & x > 5 \end{cases}$$

Now, we can draw the graph of f very easily.



Statement I $f'(4) = 0$

It is obviously clear that, f is constant around $x = 4$, hence $f'(4) = 0$. Hence, Statement I is correct.

Statement II It can be clearly seen that

(i) f is continuous, $\forall\ x \in [2, 5]$

(ii) f is differentiable, $\forall\ x \in (2, 5)$

(iii) $f(2) = f(5) = 3$

Hence, Statement II is also correct but obviously not a correct explanation of Statement I.