Appendix C
Rutherford Scattering

C.1 Classical Physics

In Chapter 1 we commented on the experiments of Geiger and Marsden that provided evidence for the existence of the nucleus. They scattered low-energy α-particles from thin gold foils and observed that sometimes the projectiles were scattered through large angles, in extreme cases close to 180°. If we start for the moment by ignoring the fact that there is a Coulomb interaction present, then it is easy to show that this behaviour is incompatible with scattering from light particles such as electrons.

Consider the non-relativistic elastic scattering of an α-particle of mass m_α and initial velocity $\mathbf{v}_\mathrm{i}$ from a target of mass m_t stationary in the laboratory. If the final velocities are $\mathbf{v}_\mathrm{f}$ and $\mathbf{v}_\mathrm{t}$, respectively, then we have the situation as shown in Figure C.1.

Conservation of linear momentum and kinetic energy are:

$$m_\alpha \mathbf{v}_\mathrm{i} = m_\alpha \mathbf{v}_\mathrm{f} + m_\mathrm{t} \mathbf{v}_\mathrm{t} \tag{C.1}$$

and

$$m_\alpha v_\mathrm{i}^2 = m_\alpha v_\mathrm{f}^2 + m_\mathrm{t} v_\mathrm{t}^2, \tag{C.2}$$

where $v_\mathrm{i} = |\mathbf{v}_\mathrm{i}|$ etc.. Squaring Equation (C.1) we obtain

$$m_\alpha v_\mathrm{i}^2 = m_\alpha v_\mathrm{f}^2 + \frac{m_\mathrm{t}^2}{m_\alpha} v_\mathrm{t}^2 + 2m_\mathrm{t}(\mathbf{v}_\mathrm{f} \cdot \mathbf{v}_\mathrm{t}) \tag{C.3}$$

and hence from Equation (C.2),

$$v_\mathrm{t}^2 \left(1 - \frac{m_\mathrm{t}}{m_\alpha}\right) = 2\mathbf{v}_\mathrm{f} \cdot \mathbf{v}_\mathrm{t}. \tag{C.4}$$

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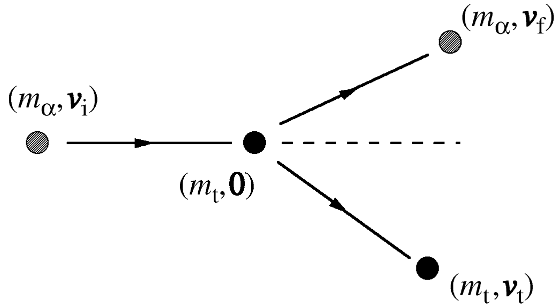

Figure C.1 Kinematics of the Geiger and Marsden experiment

Thus, if the target is an electron, with $m_t = m_e \ll m_\alpha$, the directions of motion of the outgoing α-particle and the recoiling target are essentially along the direction of the initial α-particle and no large-angle scatterings are possible. Such events could, in principle, be due to multiple small-angle scattering, but the thinness of the gold foil target rules this out.[1] If, however, $m_t = m_{Au} \gg m_\alpha$, then the left-hand side of Equation (C.4) will be negative and large scattering angles are possible.

The above only makes plausible the existence of a heavy nucleus, because it has ignored the existence of the Coulomb force, so we now have to take this into account. We will do this first using non-relativistic classical mechanics.

Consider the non-relativistic Coulomb scattering of a particle (the projectile) of mass m and electric charge ze from a target particle of mass M and electric charge Ze. The kinematics of this are shown in Figure C.2. The target mass is assumed to be sufficiently large that its recoil may be neglected. The initial velocity of the projectile is $\mathbf{v}$ and it is assumed that in the absence of any interaction it would travel in a straight line and pass the target at a distance b (called the *impact*

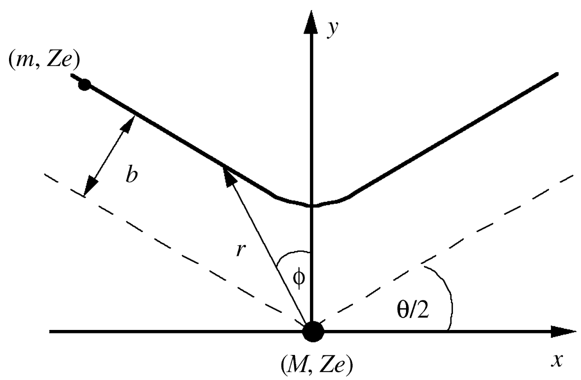

Figure C.2 Kinematics of Rutherford scattering

[1]For completeness one should also show that the observations cannot be due to scattering from the diffuse positive charge present. This was done by the authors of the original experiment.

parameter). The derivation follows from considering the implications of linear and angular momentum conservation.

Angular momentum conservation implies that

$$mvb = mr^2 \frac{d\phi}{dt}, \tag{C.5}$$

where $v = |\mathbf{v}|$. Since the scattering is symmetric about the y-axis, the component of linear momentum in the y-direction is initially $p = -mv\sin(\theta/2)$ and changes to $+mv\sin(\theta/2)$ after the interaction, i.e. the total change in momentum in the y-direction is

$$\Delta p = 2mv\sin(\theta/2). \tag{C.6}$$

The change in momentum may also be calculated by integrating the impulse in the y-direction due to the Coulomb force on the projectile. This gives

$$\Delta p = \int_{-\infty}^{+\infty} \frac{zZe^2}{4\pi\varepsilon_0 r^2} \cos\phi \, dt, \tag{C.7}$$

where we have taken $t = 0$ to coincide with the origin of the x-axis. Using Equation (C.5) to change variables, Equation (C.7) may be written

$$2mv\sin(\theta/2) = \frac{zZe^2}{4\pi\varepsilon_0}\left(\frac{1}{bv}\right)\int_{-\phi}^{+\phi} \cos\phi \; d\phi, \tag{C.8}$$

which, using $\phi = (\pi - \theta)/2$, yields

$$b = \frac{zZe^2}{8\pi\varepsilon_0} \cdot \frac{1}{E_{\text{kin}}} \cot(\theta/2), \tag{C.9}$$

where $E_{\text{kin}} = \frac{1}{2}mv^2$ is the kinetic energy of the projectile.

Finally, we need to calculate the differential cross-section. If the initial flux of projectile particles crossing a plane perpendicular to the beam direction is J, then the intensity of particles having impact parameters between b and $b + db$ is $2\pi b\,J db$ and this is equal to the rate dW at which particles are scattered into a solid angle $d\Omega = 2\pi \sin\,\theta d\theta$ between θ and $\theta + d\theta$. Thus

$$dW = 2\pi bJ\, db. \tag{C.10}$$

However, from Equation (1.47) and considering a single target particle,

$$dW = J\frac{d\sigma}{d\Omega}d\Omega = 2\pi J \sin\theta\, d\theta \frac{d\sigma}{d\Omega}, \tag{C.11}$$

i.e.

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta} \cdot \frac{db}{d\theta}. \tag{C.12}$$

The right-hand side of Equation (C.12) may be evaluated from Equation (C.9) and gives

$$\frac{d\sigma}{d\Omega} = \left(\frac{zZe^2}{16\pi\,\varepsilon_0\,E_{\rm kin}}\right)^2 \mathrm{cosec}^4\ (\theta/2). \tag{C.13}$$

This is the final form of the Rutherford differential cross-section for non-relativistic scattering.

C.2 Quantum Mechanics

While Equation (C.13) is adequate to describe the Geiger and Marsden experiments, in the case of electron scattering we need to take account of both relativity and quantum mechanics. This may be done using the general formalism for the differential cross-section in terms of the scattering potential that was derived in Chapter 1.

The starting point is Equation (1.55), which in the present notation is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2\hbar^4}\frac{p'^2}{vv'}\left|\mathcal{M}(\mathbf{q}^2)\right|^2, \tag{C.14}$$

where $\mathbf{v}$ and $\mathbf{p}$ are the velocity and momentum respectively of the projectile (which for convenience we take to have a unit negative charge) with $v = |\mathbf{v}|$, $p = |\mathbf{p}|$ and the primes refer to the final-state values. The matrix element is given by

$$\mathcal{M}(\mathbf{q}) = \int V(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}d\mathbf{x}, \tag{C.15}$$

where $\mathbf{q} = \mathbf{p} - \mathbf{p}'$ is the momentum transfer. $V(\mathbf{x})$ is the Coulomb potential

$$V(\mathbf{x}) = V_{\rm C}(\mathbf{x}) = -\frac{\alpha Z(\hbar c)}{r}, \tag{C.16}$$

where $r = |\mathbf{x}|$ and Ze is the charge of the target nucleus. Inspection of the integral in Equation (C.15) shows that it diverges at large r. However, in practice, charges are always screened at large distances by intervening matter and so we will interpret the integral as

$$\mathcal{M}_{\mathrm{C}}(q) = \underset{\lambda \to 0}{\mathrm{Lt}} \int \left(-\frac{Z\alpha(\hbar c)\mathrm{e}^{-\lambda r}}{r} \right) \mathrm{e}^{i\mathbf{q}\cdot\mathbf{x}/\hbar} \, \mathrm{d}^3\mathbf{x}. \tag{C.17}$$

To evaluate this, take $\mathbf{q}$ along the x-axis, so that in spherical polar coordinates $\mathbf{q} \cdot \mathbf{x} = qr\cos\theta$. The angular integration may then be done and yields

$$\mathcal{M}_{\mathrm{C}}(q) = -\frac{4\pi(\hbar c)Z\alpha\hbar}{q} \underset{\lambda \to 0}{\mathrm{Lt}} \int_0^\infty \mathrm{e}^{-\lambda r} \sin(qr/\hbar)\mathrm{d}r. \tag{C.18}$$

The remaining integral may be done by parts (twice) and taking the limit $\lambda \to 0$ gives

$$\mathcal{M}_{\mathrm{C}}(\mathbf{q}) = -\frac{4\pi(\hbar c)Z\alpha\hbar^2}{q^2}. \tag{C.19}$$

Finally, substituting Equation (C.19) into Equation (C.14) gives

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = 4Z^2\alpha^2(\hbar c)^2 \frac{p'^2}{vv'q^4}, \tag{C.20}$$

which is the general form of the Rutherford differential cross-section. To see that this is the same as Equation (C.13) in the non-relativistic limit, we may substitute the non-relativistic approximations

$$p^2 = p'^2 = 2mE_{\mathrm{kin}}, \quad \text{and} \quad v = v' = \sqrt{2E_{\mathrm{kin}}/m}, \tag{C.21}$$

together with the kinematic relation for the scattering angle

$$q = 2p\sin(\theta/2), \tag{C.22}$$

into Equation (C.20). The result in Equation (C.13) follows immediately.

Because we are assuming that the target mass is heavy so that its recoil may be neglected, to a good approximation $p = p'$ and $E = E'$, where E is the total energy of the electron. Also for relativistic electrons $v = v' \approx c$ and $E \approx pc$. Using these conditions together with Equation (C.22) in Equation (C.20), gives the relativistic result for the Rutherford differential cross-section in the convenient form:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Z^2\alpha^2(\hbar c)^2}{4E^2\sin^4(\theta/2)}, \tag{C.23}$$

which is the form used in Chapter 2 and elsewhere.

Problems

C.1 Calculate the differential cross-section in mb/sr for the scattering of a 20 MeV α-particle through an angle 20° by a nucleus $^{209}_{83}\mathrm{Bi}$, stating any assumptions made. Ignore spin and form factor effects.

C.2 Show that in Rutherford scattering at a fixed impact parameter b, the distance of closest approach d to the nucleus is given by $d = b[1 + \operatorname{cosec}(\theta/2)]/\operatorname{cosec}(\theta/2)$, where θ is the scattering angle.

C.3 Find an expression for the impact parameter b in the case of small-angle Rutherford scattering. A beam of protons with speed $v = 4 \times 10^7\,\mathrm{ms}^{-1}$ is incident normally on a thin foil of $^{194}_{78}\mathrm{Pt}$, thickness $10^{-5}\,\mathrm{m}$ (density $= 2.145 \times 10^4\,\mathrm{kg\,m}^{-3}$). Estimate the proportion of protons that experience double scattering, where each scattering angle is at least 5°.