Exercise 15.7

Chapter 15 Multiple Integrals 15.7 1E

Evaluate the integral $\iiint_B xyz^2\, dV$ with respect to y, then z, and then x.

Here B is the rectangular box as,

$$B = \{0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$$

Rewrite the iterated integral as follows:

$$\int_0^1 \int_0^3 \int_{-1}^2 xyz^2\, dy\,dz\,dx$$

First evaluate the inner integral with respect to y.

$$\begin{aligned}
\int_0^1 \int_0^3 \int_{-1}^2 xyz^2\, dy\,dz\,dx &= \int_0^1 \int_0^3 \left[\frac{xy^2z^2}{2}\right]_{y=-1}^{y=2} dz\,dx \\
&= \int_0^1 \int_0^3 \left[\frac{x(2)^2 z^2}{2} - \frac{x(-1)^2 z^2}{2}\right] dz\,dx \\
&= \int_0^1 \int_0^3 \left[2xz^2 - \frac{xz^2}{2}\right] dz\,dx \\
&= \int_0^1 \int_0^3 \frac{3xz^2}{2}\, dz\,dx
\end{aligned}$$

Evaluate the middle integral with respect to z.

$$\begin{aligned}
\int_0^1 \int_0^3 \frac{3xz^2}{2}\, dz\,dx &= \int_0^1 \left[\frac{xz^3}{2}\right]_{z=0}^{z=3} dx \\
&= \int_0^1 \left[\frac{x(3)^3}{2} - \frac{x(0)^3}{2}\right] dx \\
&= \int_0^1 \left[\frac{x(27)}{2} - 0\right] dx \\
&= \int_0^1 \frac{27x}{2}\, dx
\end{aligned}$$

Evaluate the outer integral with respect to x.

$$\begin{aligned}
\int_0^1 \frac{27x}{2}\, dx &= \left[\frac{27x^2}{4}\right]_{x=0}^{x=1} \\
&= \frac{27(1)^2}{4} - \frac{27(0)^2}{4} \\
&= \frac{27}{4} - 0 \\
&= \frac{27}{4}
\end{aligned}$$

Therefore, the value of the integral is $\boxed{\frac{27}{4}}$.

Chapter 15 Multiple Integrals 15.7 2E

Consider $\mathrm{E}=\left\{(x,y,z)\middle|\left(xy+z^2\right)\right\}dv$

Thus, we have

$$\int_0^3\int_0^1\int_0^2\left(xy+z^2\right)dxdydz=\int_0^3\int_0^1\left[\frac{x^2}{2}y+xz^2\right]_0^2dydz$$

$$=\int_0^3\int_0^1\left(\frac{4}{2}y+2z^2\right)dydz$$

$$=\int_0^3\left[\frac{2y^2}{2}+2yz^2\right]_0^1dz$$

$$=\int_0^3\left(1+2z^2\right)dz$$

$$=\left[z+\frac{2z^3}{3}\right]_0^3$$

$$=\left(3+\frac{2}{3}(3)^3\right)$$

$$=(3+18)$$

$$=21$$

Therefore; $\boxed{21}$

Using the second patteren,we get

$$\int_0^3\int_0^3\int_0^1\left(xy+z^2\right)dydzdx=\int_0^2\int_0^3\left[\frac{xy^2}{2}yz^2\right]_0^1dzdx$$

$$=\int_0^2\int_0^3\left(\frac{x}{2}+z^2\right)dzdx$$

$$=\int_0^2\left[\frac{x}{2}z+\frac{z^3}{3}\right]_0^3dx$$

$$=\int_0^2\left(\frac{3x}{2}+9\right)dx$$

$$=\left[\frac{3}{2}\frac{x^2}{2}+9x\right]_0^2$$

$$=\frac{3}{4}(4)+9(2)$$

$$=\boxed{21}$$

Using the third patteren,we get

$$\int_0^1\int_0^2\int_0^3 \left(xy+z^2\right)dzdxdy=\int_0^1\int_0^2\left[xyz+\frac{z^3}{3}\right]_0^2 dxdy$$

$$=\int_0^1\int_0^2(3xy+9)\,dxdy$$

$$=\int_0^3\left[\frac{3}{2}yx^2+9x\right]_0^2 dy$$

$$=\int_0^1\left[\frac{3}{2}y(4)+2(9)\right]dy$$

$$=\int_0^1(6y+18)\,dy$$

$$=\left[\frac{6y^2}{2}+18y\right]_0^1$$

$$=3(1)+18$$

$$=\boxed{21}$$

Chapter 15 Multiple Integrals 15.7 3E

Evaluate the iterated integral: $\int_0^2\int_0^{z^2}\int_0^{y-z}(2x-y)\,dx\,dy\,dz$

Consider,

$$\int_0^2\int_0^{z^2}\int_0^{y-z}(2x-y)\,dx\,dy\,dz=\int_0^2\int_0^{z^2}\left[\int_0^{y-z}(2x-y)\,dx\right]dy\,dz$$

$$=\int_0^2\int_0^{z^2}\left[x^2-xy\right]_0^{y-z}dy\,dz$$

$$=\int_0^2\left[\int_0^{z^2}\left((y-z)^2-(y-z)y\right)dy\right]dz$$

$$=\int_0^2\left[\int_0^{z^2}-z(y-z)\,dy\right]dz$$

Continue the above,

$$= \int_0^2 \left[-\frac{y^2 z}{2} + z^2 y \right]_0^{z^2} dz$$

$$= \int_0^2 \left[-\frac{(z^2)^2 z}{2} + z^2(z^2) \right] dz$$

$$= \int_0^2 \left[-\frac{z^5}{2} + z^4 \right] dz$$

$$= \left[-\frac{z^6}{12} + \frac{z^5}{5} \right]_0^2$$

$$= -\frac{2^6}{12} + \frac{2^5}{5}$$

$$= \frac{16}{15}$$

Chapter 15 Multiple Integrals 15.7 4E

$$\int_0^1 \int_x^{2x} \int_0^y 2xyz \; dz \; dy \; dx$$

$$= \int_0^1 \int_x^{2x} 2xy \frac{z^2}{2} \Big]_0^y dy \, dx$$

$$= \int_0^1 \int_x^{2x} xy\left(y^2 - 0\right) dy \, dx$$

$$= \int_0^1 \int_x^{2x} xy^3 \, dy \, dx$$

$$= \int_0^1 x . \frac{y^4}{4} \Big]_x^{2x} dx$$

i.e. $\int_0^1 \int_x^{2x} \int_0^y 2xyz \; dz \; dy \; dx = \int_0^1 \frac{x}{4}\left[16x^4 - x^4\right] dx$

$$= \int_0^1 \frac{x}{4}\left(15x^4\right) dx$$

$$= \frac{15}{4} \int_0^1 x^5 \, dx$$

$$= \frac{15}{4} . \frac{x^6}{6} \Big]_0^1$$

$$= \frac{15}{24}(1-0)$$

$$= \boxed{\frac{15}{24}}$$

Chapter 15 Multiple Integrals 15.7 5E

Consider the triple integral $\int_1^2 \int_0^{2z} \int_0^{\ln x} xe^{-y}\, dy\, dx\, dz$.

Let us start by removing the innermost integral.

$$\begin{aligned}\int_1^2 \int_0^{2z} \int_0^{\ln x} xe^{-y}\, dy\, dx\, dz &= \int_1^2 \int_0^{2z} x\left(-e^{-y}\right)_0^{\ln x} dx\, dz \\ &= \int_1^2 \int_0^{2z} x\left(-e^{-\ln x} + e^0\right) dx\, dz \\ &= \int_1^2 \int_0^{2z} x\left(-\frac{1}{x} + 1\right) dx\, dz \\ &= \int_1^2 \int_0^{2z} (x-1)\, dx\, dz\end{aligned}$$

The integral is simplified to $\int_1^2 \int_0^{2z} (x-1)\, dx\, dz$.

Now, let us evaluate the outer integral and apply the limits.

$$\begin{aligned}\int_1^2 \int_0^{2z} x - 1\, dx\, dz &= \int_1^2 \left(\frac{x^2}{2} - x\right)_0^{2z} dz \\ &= \int_1^2 \left[\frac{(2z)^2}{2} - 2z\right] - \left[\frac{(0)^2}{2} - 0\right] dz \\ &= \int_1^2 \left(2z^2 - 2z\right) dz\end{aligned}$$

We have the simplified form as $\int_1^2 \left(2z^2 - 2z\right) dz$.

Now evaluate the integral.

$$\begin{aligned}\int_1^2 2z^2 - 2z\, dz &= \left(\frac{2z^3}{3} - z^2\right)_1^2 \\ &= \left[\frac{2(2)^3}{3} - (2)^2\right] - \left[\frac{2(1)^3}{3} - (1)^2\right] \\ &= \frac{4}{3} + \frac{1}{3} \\ &= \frac{5}{3}\end{aligned}$$

Thus, $\int_1^2 \int_0^{2z} \int_0^{\ln x} xe^{-y}\, dy\, dx\, dz = \boxed{\frac{5}{3}}$.

Chapter 15 Multiple Integrals 15.7 6E

We have the triple integral $\int_0^1\int_0^1\int_0^{\sqrt{1-z^2}} \frac{z}{y+1}\,dx\,dz\,dy$.

Let us start by removing the innermost integral.

$$\begin{aligned}\int_0^1\int_0^1\int_0^{\sqrt{1-z^2}} \frac{z}{y+1}\,dx\,dz\,dy &= \int_0^1\int_0^1\left(\frac{zx}{y+1}\right)_0^{\sqrt{1-z^2}}dz\,dy\\ &= \int_0^1\int_0^1\left[\frac{z\left(\sqrt{1-z^2}\right)}{y+1}\right]-\left[\frac{z(0)}{y+1}\right]dz\,dy\\ &= \int_0^1\int_0^1\frac{z\sqrt{1-z^2}}{y+1}\,dz\,dy\end{aligned}$$

The integral is simplified to $\int_0^1\int_0^1\frac{z\sqrt{1-z^2}}{y+1}\,dz\,dy$.

Now, let us evaluate the outer integral and apply the limits.

$$\begin{aligned}\int_0^1\int_0^1\frac{z\sqrt{1-z^2}}{y+1}\,dz\,dy &= \int_0^1\left[-\frac{\left(1-z^2\right)^{3/2}}{3(y+1)}\right]_0^1 dy\\ &= \int_0^1\left[-\frac{\left(1-1^2\right)^{3/2}}{3(y+1)}\right]-\left[-\frac{\left(1-0^2\right)^{3/2}}{3(y+1)}\right]dy\\ &= \int_0^1\frac{1}{3(y+1)}\,dy\end{aligned}$$

We have the simplified form as $\int_0^1\frac{1}{3(y+1)}\,dy$.

Evaluate the integral.

$$\begin{aligned}\int_0^1\frac{1}{3(y+1)}\,dy &= \left(\frac{1}{3}\ln(y+1)\right)_0^1\\ &= \left[\frac{1}{3}\ln(1+1)\right]-\left[\frac{1}{3}\ln(0+1)\right]\\ &= \frac{1}{3}\ln 2\end{aligned}$$

Thus, the iterated integral evaluates to $\boxed{\frac{1}{3}\ln 2}$.

Chapter 15 Multiple Integrals 15.7 7E

We have

$$\int_0^{\frac{\pi}{2}} \int_0^{y} \int_0^{x} \cos(x+y+z)\, dz\,dx\,dy$$

Then

$$\int_0^{\frac{\pi}{2}} \int_0^{y} \int_0^{x} \cos(x+y+z)\, dz\,dx\,dy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{y} [\sin(x+y+z)]_0^{x}\, dx\, dy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{y} [\sin(x+y+x) - \sin(x+y)]\, dx\, dy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{y} [\sin(2x+y) - \sin(x+y)]\, dx\, dy$$

$$= \int_0^{\frac{\pi}{2}} \left[-\frac{\cos(2x+y)}{2} + \cos\left(x+y\right)\right]_0^{y} dy$$

$$= \int_0^{\frac{\pi}{2}} \left[-\frac{\cos(2y+y)}{2} + \cos\left(y+y\right) + \frac{\cos y}{2} - \cos y\right] dy$$

$$= \int_0^{\frac{\pi}{2}} \left[-\frac{\cos 3y}{2} + \cos 2y - \frac{\cos y}{2}\right] dy$$

$$= \left[-\frac{\sin 3y}{6} + \frac{\sin 2y}{2} - \frac{\sin y}{2} \right]_0^{\frac{\pi}{2}}$$

$$= \left[-\frac{1}{6}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{2}\right) - \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + 0 - 0 + 0 \right]$$

$$= \left[-\frac{1}{6}\left(-1\right) + \frac{1}{2}\left(0\right) - \frac{1}{2}\left(1\right) \right]$$

$$= \frac{1}{6} - \frac{1}{2}$$

$$= \frac{1-3}{6} = -\frac{1}{3}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \int_0^{y} \int_0^{x} \cos(x+y+z)\, d\,z\,dx\,dy = -\frac{1}{3}$$

Therefore the solution is $-\frac{1}{3}$

Chapter 15 Multiple Integrals 15.7 8E

We have

$$\int_0^{\sqrt{\pi}} \int_0^x \int_0^{xz} x^2 \sin y\,dy\,dz\,dx$$

Then

$$\int_0^{\sqrt{\pi}} \int_0^x \int_0^{xz} x^2 \sin y\,dy\,dz\,dx = \int_0^{\sqrt{\pi}} \int_0^x x^2 [-\cos y]_0^{xz}\,dz\,dx$$

$$= \int_0^{\sqrt{\pi}} \int_0^x x^2 \left[-\cos xz + 1\right] dz\,dx$$

$$= \int_0^{\sqrt{\pi}} x^2 \left[\frac{-\sin xz}{x} + z\right]_0^x dx$$

$$= \int_0^{\sqrt{\pi}} x^2 \left[\frac{-\sin x^2}{x} + x + 0 - 0\right] dx$$

$$= \int_0^{\sqrt{\pi}} x^2 \left[\frac{-\sin x^2}{x} + x\right] dx$$

$$= \int_0^{\sqrt{\pi}} \left[-x\sin x^2 + x^3\right] dx$$

$$= \int_0^{\sqrt{\pi}} -x\sin x^2\,dx + \int_0^{\sqrt{\pi}} x^3\,dx$$

$$= \int_0^{\sqrt{\pi}} -x\sin x^2\,dx + \left[\frac{x^4}{4}\right]_0^{\sqrt{\pi}}$$

$$= \int_0^{\sqrt{\pi}} -x\sin x^2\,dx + \frac{\pi^2}{4}$$

$$\Rightarrow \int_0^{\sqrt{\pi}} \int_0^x \int_0^{xz} x^2 \sin y\,dy\,dz\,dx = \int_0^{\sqrt{\pi}} -x\sin x^2\,dx + \frac{\pi^2}{4}$$

Let

$$A = \int_0^{\sqrt{\pi}} -x\sin x^2 \, \mathrm{d}\, x$$

Let $x^2 = t$

$$\Rightarrow 2\mathrm{xdx} = \mathrm{dt}$$

$$\Rightarrow \mathrm{xdx} = \frac{\mathrm{dt}}{2}$$

When $x = 0 \Rightarrow t = 0$

When $x = \sqrt{\pi} \Rightarrow t = \pi$

Thus the substitution rule gives

$$A = \int_0^{\pi} -\mathrm{sint} \cdot \frac{\mathrm{dt}}{2}$$

$$= \frac{-1}{2}\left[-\mathrm{cost}\right]_0^{\pi} = \frac{1}{2}\left[\cos\pi - \cos 0\right]$$

$$= \frac{1}{2}\left[-1-1\right] = -1$$

$$\Rightarrow A = -1$$

So

$$\int_0^{\sqrt{\pi}} \int_0^{x} \int_0^{xz} x^2 \,\mathrm{sinydydzdx} = \frac{\pi^2}{4} - 1$$

The solution is $\frac{\pi^2}{4} - 1$

Chapter 15 Multiple Integrals 15.7 9E

Consider the triple integral $\iiint_E y\,dV$

Where $E = \{(x,y,z) \mid 0 \le x \le 3, 0 \le y \le x, x-y \le z \le x+y\}$

If $E = \{(x,y,z) \mid a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x,y) \le z \le u_2(x,y)\}$ then

$$\iiint_E f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdydx$$

Therefore,

$$\iiint_E y\,dV = \int_0^3 \int_0^x \int_{x-y}^{x+y} y\,dz\,dy\,dx$$

$$= \int_0^3 \int_0^x y[z]_{x-y}^{x+y}\,dy\,dx$$

$$= \int_0^3 \int_0^x y[(x+y)-(x-y)]dy\,dx$$

Continuing the above step,

$$\iiint_E y\,dV = \int_0^3 \int_0^x 2y^2\,dy\,dx$$

$$= \int_0^3 \left[\frac{2y^3}{3}\right]_0^x dx$$

$$= \int_0^3 \left(\frac{2x^3}{3} - 0\right)dx$$

$$= \left[\frac{2x^4}{12}\right]_0^3$$

$$= \frac{81}{6} - 0$$

$$= \boxed{\frac{27}{2}}$$

Chapter 15 Multiple Integrals 15.7 10E

Consider the triple integral $\iiint\limits_E e^{z/y}\,dV$

Where

$$E=\{(x,y,z)\,|\,0\le y\le 1, y\le x\le 1, 0\le z\le xy\}.$$

If $E=\{(x,y,z)\,|\,c\le y\le d, h_1(y)\le x\le h_2(y), u_1(x,y)\le z\le u_2(x,y)\}$ then

$$\iiint\limits_E f(x,y,z)\,dV=\int_c^d\int_{h_1(y)}^{h_2(y)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

Therefore,

$$\iiint\limits_E e^{z/y}\,dV=\int_0^1\int_y^1\int_0^{xy} e^{z/y}\,dz\,dx\,dy$$

$$=\int_0^1\int_y^1\left[\frac{e^{z/y}}{1/y}\right]_0^{xy} dx\,dy$$

$$=\int_0^1\int_y^1 y\left[e^x-e^0\right]dx\,dy$$

$$=\int_0^1 y\left(e^x-x\right)_y^1 dy$$

$$=\int_0^1\left[y\left(e^1-1\right)-y\left(e^y-y\right)\right]dy$$

Continuing the above step,

$$\iiint\limits_E e^{z/y}\,dV=(e-1)\int_0^1 y\,dy+\int_0^1 y^2\,dy-\int_0^1 ye^y\,dy$$

$$=(e-1)\frac{y^2}{2}\bigg|_0^1+\frac{y^3}{3}\bigg|_0^1-(y-1)e^y\Big|_0^1$$

$$=\frac{e-1}{2}+\frac{1}{3}-(1-1)e^1+(0-1)e^0$$

$$=\frac{e}{2}-\frac{1}{2}+\frac{1}{3}-1$$

$$=\boxed{\frac{e}{2}-\frac{7}{6}}$$

Chapter 15 Multiple Integrals 15.7 11E

Consider the following integral:

$$\iiint \frac{z}{x^2+z^2}dV.$$

Here the region is $E=\{(x,y,z)\,|\,1\le y\le 4, y\le z\le 4, 0\le x\le z\}$.

The objective is to evaluate the triple integral.

If $E=\{(x,y,z)\,|\,c\le y\le d, h_1(y)\le z\le h_2(y), u_1(y,z)\le x\le u_2(y,z)\}$ then

$$\iiint_E f(x,y,z)dV=\int_c^d\int_{h_1(y)}^{h_2(y)}\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dxdzdy$$

Now, solve the integral as follows:

$$\begin{aligned}\int_1^4\int_y^4\int_0^z \frac{z}{x^2+z^2}\,dx\,dz\,dy &= \int_1^4\int_y^4\left(\left(\tan^{-1}\left(\frac{x}{z}\right)\right)_0^z\right)dz\,dy \quad \left[\int\frac{a}{x^2+a^2}dx=\tan^{-1}\left(\frac{x}{a}\right)+C\right]\\ &=\int_1^4\int_y^4\left(\tan^{-1}\left(\frac{z}{z}\right)-\tan^{-1}\left(\frac{0}{z}\right)\right)dz\,dy\\ &=\int_1^4\int_y^4\left(\tan^{-1}(1)\right)dz\,dy\\ &=\int_1^4\int_y^4\frac{\pi}{4}dz\,dy \quad \left[\tan^{-1}(1)=\frac{\pi}{4}\right]\end{aligned}$$

Continuation to the above steps:

$$\int_1^4\int_y^4\int_0^z \frac{z}{x^2+z^2}\,dx\,dz\,dy = \int_1^4 \frac{\pi}{4}\left([z]_y^4\right)dy$$

$$= \int_1^4 \frac{\pi}{4}(4-y)\,dy$$

$$= \frac{\pi}{4}\left(4y-\frac{y^2}{2}\right)_1^4$$

$$= \frac{\pi}{4}\left(4(4-1)-\left(\frac{4^2}{2}-\frac{1^2}{2}\right)\right)$$

$$= \frac{\pi}{4}\left(4(3)-\left(\frac{16}{2}-\frac{1}{2}\right)\right)$$

$$= \frac{\pi}{4}\left(12-\frac{15}{2}\right)$$

$$= \frac{\pi}{4}\left(\frac{24-15}{2}\right)$$

$$= \frac{\pi}{4}\left(\frac{9}{2}\right)$$

$$= \frac{9\pi}{8}$$

Thus, the integral evaluates to $\frac{9\pi}{8}$.

Chapter 15 Multiple Integrals 15.7 12E

Consider $\iiint_E \sin y dV$.

Here, E lies below the plane $z = x$ and above the triangular region with vertices $(0,0,0),(\pi,0,0)$ and $(0,\pi,0)$.

Thus, the region is defined as $E = \{(x,y,z) \mid 0 \le x \le \pi,\ 0 \le y \le \pi,\ \ 0 \le z \le x\}$.

The triple integral can be evaluated as follows:

$$\iiint_E \sin y dV = \int_0^\pi \int_0^\pi \int_0^x \sin y dz dy dx$$

$$= \int_0^\pi \int_0^\pi \sin y (z)_0^x dy dx$$

$$= \int_0^\pi \int_0^\pi x \sin y dy dx$$

$$= \int_0^\pi x(-\cos y)_0^\pi dx$$

$$= -\int_0^\pi x(\cos \pi - \cos 0) dx$$

$$= -\int_0^\pi x(-1-1) dx \qquad \text{Use } \cos \pi = -1 \text{ and } \cos 0 = 1$$

$$= 2\int_0^\pi x dx$$

$$= 2\left(\frac{x^2}{2}\right)_0^\pi$$

$$= (\pi^2 - 0)$$

$$= \pi^2$$

Therefore, $\iiint_E \sin y dV = \boxed{\pi^2}$.

Chapter 15 Multiple Integrals 15.7 13E

$$\iiint_E 6xy\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} 6\,xy\,[z]_0^{1+x+y}\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} 6\,xy\,(1+x+y)\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} \left(6xy+6x^2y+6xy^2\right)dy\,dx$$

$$= \int_0^1 \left[6x\frac{y^2}{2}+6x^2\frac{y^2}{2}+6x\frac{y^3}{3}\right]_{y=0}^{y=\sqrt{x}}\,dx$$

$$= \int_0^1 \left[3x\left(\sqrt{x}\right)^2+3x^2\left(\sqrt{x}\right)^2+2x\left(\sqrt{x}\right)^3\right]dx$$

i.e. $$\iiint_E 6xy\,dV = \int_0^1 \left[3x^2+3x^3+2x^{5/2}\right]dx$$

$$= \left[3.\frac{x^3}{3}+3.\frac{x^4}{4}+2.\frac{x^{7/2}}{\frac{7}{2}}\right]_0^1$$

$$= \left[x^3+\frac{3}{4}x^4+\frac{4}{7}x^{7/2}\right]_0^1$$

$$= \left(1+\frac{3}{4}+\frac{4}{7}\right)-0$$

$$= \frac{28+21+16}{28}$$

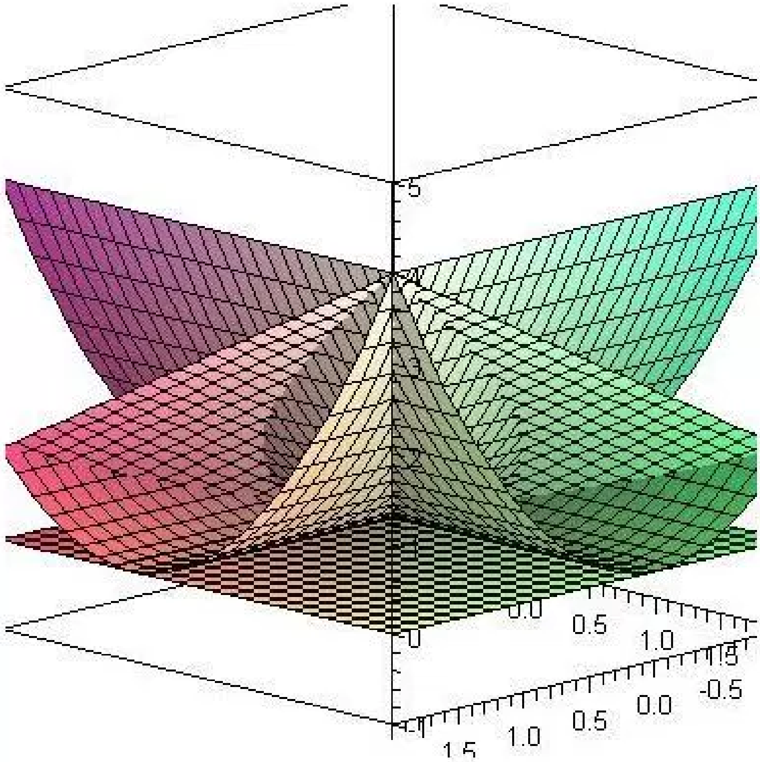
$$= \boxed{\frac{65}{28}}$$

Chapter 15 Multiple Integrals 15.7 14E

Consider the triple integration:

$$I = \iiint_E xy\,dV$$

The region enclosed by two curved surfaces $y = x^2$, $x = y^2$, the bottom surface $z = 0$ and the inclined surface $z = x + y$ is as follows:



The intersection of the two parabolic cylinders:

$$y = x^2$$

$$y^2 = x^4$$

And $x = y^2 \Rightarrow x = x^4$

$$x\left(1 - x^3\right) = 0$$

$$x = 0 \text{ or } x = 1$$

To get the limits, we solve the parabolic surfaces $y = x^2$ and $x = y^2$ which meet at $(0,0)$ and $(1,1)$.

Also, when $x = y = 1$, the lateral surface $z = x + y$ takes $z = 2$

So, we follow that z varies from $z = 0$ through $z = x + y$

y Varies on the curved lower surface x^2 to the upper surface $\sqrt{x}$.

x Varies from 0 through 1

The region is $E = \left\{(x, y, z) \mid 0 \le x \le 1, x^2 \le y \le \sqrt{x}, 0 \le z \le x + y\right\}$.

Using these details, the given integral as follows:

$$\iiint_E xy\,dV = \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}}\int_{z=0}^{x+y} xy\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{y=\sqrt{x}} xy\left[z\right]_0^{x+y} dy\,dx$$

$$= \int_0^1\int_{x^2}^{\sqrt{x}} xy(x+y)\,dy\,dx$$

$$= \int_0^1\int_{x^2}^{\sqrt{x}} \left(x^2y + xy^2\right)dy\,dx$$

$$= \int_0^1\left[x^2\frac{y^2}{2} + x\frac{y^3}{3}\right]_{x^2}^{\sqrt{x}} dx$$

$$= \frac{1}{2}\int_0^1\left(x^3 - x^6\right)dx + \frac{1}{3}\int_0^1\left(x^{\frac{5}{2}} - x^7\right)dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4} - \frac{x^7}{7}\right]_0^1 + \frac{1}{3}\left[\frac{2}{7}x^{\frac{7}{2}} - \frac{x^8}{8}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{4} - \frac{1}{7}\right) + \frac{1}{3}\left(\frac{2}{7} - \frac{1}{8}\right)$$

$$= \frac{1}{2}\left(\frac{7-4}{28}\right) + \frac{1}{3}\left(\frac{16-7}{56}\right)$$

$$= \frac{3}{56} + \frac{3}{56}$$

$$= \boxed{\frac{3}{28}}.$$

Hence, the value of the triple integral is $\iiint_E xy\,dV = \boxed{\frac{3}{28}}$.

Chapter 15 Multiple Integrals 15.7 15E

Consider the triple integral,

$$\iiint_T x^2 dV.$$

Here, T is the solid tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

The objective is to evaluate the triple integral.

Equation of the plane passing through the three vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ is

$$x+y+z=1.$$

This can be written as follows:

$$z=1-x-y.$$

The limits varies from $x=0$, $y=0$, $z=0$ to the plane $x+y+z=1$

So, the limits of z are $z=0$ to $z=1-x-y$.

When $z=0$, this plane becomes the line $x+y=1$

So, y varies from $y=0$ to $y=1-x$

Similarly, when $y=0$, the point $x=1$

Therefore, x varies from $x=0$ to $x=1$

Now, the description of T is as follows:

$$T=\{(x,y,z) \mid 0\le x\le 1, 0\le y\le 1-x, 0\le z\le 1-x-y\}$$

Rewrite the integral as the iterated integral as follows:

$$\begin{aligned}
I &= \iiint_T x^2 dV \\
&= \int_0^1\int_0^{1-x}\int_0^{1-x-y} x^2\, dz\, dy\, dx \\
&= \int_0^1\int_0^{1-x} x^2 [z]_0^{1-x-y}\, dydx \\
&= \int_0^1\int_0^{1-x} x^2 [1-x-y]\, dydx \\
&= \int_0^1\int_0^{1-x} \left[x^2-x^3-x^2y\right] dydx
\end{aligned}$$

The integral with respect to y is calculated as follows:

$$I = \int_0^1 \left(x^2 y - x^3 y - \frac{1}{2} x^2 y^2 \right|_0^{1-x} dx$$

$$= \int_0^1 \left[\left[x^2 (1-x) - x^3 (1-x) - \frac{1}{2} x^2 \left(1 - 2x + x^2\right) \right] - 0 \right] dx$$

$$= \int_0^1 \left[x^2 - x^3 - x^3 + x^4 - \frac{1}{2} x^2 + x^3 - \frac{1}{2} x^4 \right] dx$$

$$= \int_0^1 \left[\frac{1}{2} x^2 - x^3 + \frac{1}{2} x^4 \right] dx$$

$$= \left(\frac{1}{6} x^3 - \frac{1}{4} x^4 + \frac{1}{10} x^5 \right|_0^1$$

$$= \left(\frac{1}{6} - \frac{1}{4} + \frac{1}{10} - 0 \right)$$

$$= \frac{10 - 15 + 6}{60}$$

$$= \frac{1}{60}$$

Therefore, the value of the triple integral is,

$$\iiint_T x^2 dV = \boxed{\frac{1}{60}}.$$

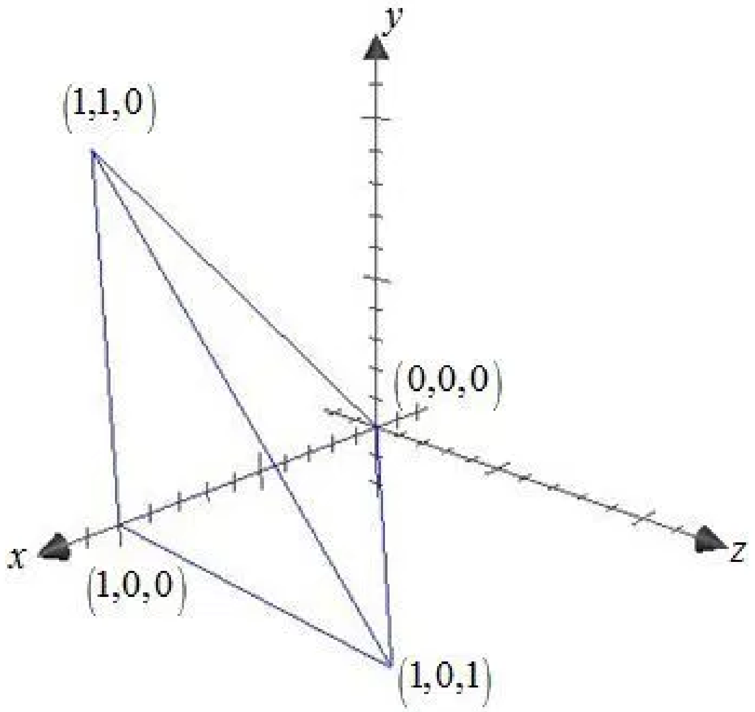
Chapter 15 Multiple Integrals 15.7 16E

Consider the triple integral $\iiint_T xyz\,dV$.

Here T is the solid tetrahedron with vertices $(0,0,0),(1,0,0),(1,1,0)$ and $(1,0,1)$.

The objective is to evaluate the above triple integral.

Draw a picture to help set up the integral.



Notice that $(0,0,0)$, $(1,1,0)$ lie in the *xy* plane, and the equation of the line between those two points in the *xy*-plane is $y = x$.

Now need to determine an equation of the plane containing $(0,0,0)$, $(1,0,1)$, and $(1,1,0)$.

Let $\mathbf{a} = (1,0,1)-(0,0,0) = (1,0,1)$ and $\mathbf{b} = (1,1,0)-(0,0,0) = (1,1,0)$. Then $\mathbf{a}\times\mathbf{b} = (-1,1,1)$ and the equation of the plane is $-1(x-0)+1(y-0)+1(z-0)=0$, or $-x+y+z=0$.

Describe this region by homing in on it one dimension at a time. To ensure that a point is inside the region first fix $0 \le x \le 1$. Then fix $0 \le y \le x$. Finally fix $0 \le z \le x-y$.

So, $T = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le x-y\}$

Use this description to set up the integral.

$$\iiint_T xyz\,dV = \int_0^1\int_0^x\int_0^{x-y} xyz\,dz\,dy\,dx$$

Evaluate the integral:

$$\iiint_T xyz\,dV = \int_0^1\int_0^x\int_0^{x-y} xyz\,dz\,dy\,dx$$

$$= \int_0^1\int_0^x xy\left(\frac{z^2}{2}\right)_0^{x-y} dy\,dx$$

$$= \int_0^1\int_0^x xy\left[\frac{1}{2}(x-y)^2 - 0\right] dy\,dx$$

$$= \int_0^1\int_0^x \left[\frac{1}{2}xy\left(x^2 - 2xy + y^2\right)\right] dy\,dx$$

$$= \int_0^1\int_0^x \left(\frac{1}{2}x^3y - x^2y^2 + \frac{1}{2}xy^3\right) dy\,dx$$

$$= \int_0^1 \left(\frac{1}{4}x^3y^2 - \frac{1}{3}x^2y^3 + \frac{1}{8}xy^4\right)_0^x dx$$

$$= \int_0^1 \left(\frac{1}{4}x^3\cdot x^2 - \frac{1}{3}x^2\cdot x^3 + \frac{1}{8}x\cdot x^4\right) dx$$

$$= \int_0^1 \left(\frac{1}{4}x^5 - \frac{1}{3}x^5 + \frac{1}{8}x^5\right) dx$$

$$= \int_0^1 \left(\frac{6-8+3}{24}\right) x^5\,dx$$

$$= \int_0^1 \left(\frac{1}{24}x^5\right) dx$$

$$= \left(\frac{1}{144}x^6\right)_0^1$$

$$= \frac{1}{144}\cdot 1 - 0$$

$$= \frac{1}{144}$$

Thus, the value of the integral is $\iiint_T xyz\,dV = \boxed{\frac{1}{144}}$.

Chapter 15 Multiple Integrals 15.7 17E

Consider the following paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$.

The objective is to find the triple $\iiint_E x\,dV$ where E is bounded by the paraboloid and the plane.

Notice that $x = 4y^2 + 4z^2$ and $x = 4$ intersect in the circle $y^2 + z^2 = 1$, or the region of the yz-plane.

Convert the rectangular coordinates into polar coordinates, use

$y = r\cos\theta, z = r\sin\theta$ and $y^2 + z^2 = r^2$.

Substitute $y^2 + z^2 = r^2$ in $y^2 + z^2 = 1$, then

$$\begin{aligned} y^2 + z^2 &= 1 \\ r^2 &= 1 \\ r &= \pm 1 \end{aligned}$$

Therefore, the region is $R = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$.

Substitute $y = r\cos\theta, z = r\sin\theta$ and $y^2 + z^2 = r^2$ in $x = 4y^2 + 4z^2$, then

$$\begin{aligned} x &= 4\left(y^2 + z^2\right) \\ x &= 4r^2 \end{aligned}$$

So that, the limits of x from $x = 4r^2$ to 4.

Thus, the solid by be described as

$$E = \{(r,\theta,x) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 4r^2 \le x \le 4\}$$

$4r^2 \le x \le 4$, which means that $x = 4r^2$ and $x = 4$ are the lower and upper limits of integration of x respectively, and $0 \le r \le 1$, which means that $r = 0$ and $r = 1$ are the are the lower and upper limits of integration of r respectively, and $0 \le \theta \le 2\pi$, which means that $\theta = 0$ and $\theta = 2\pi$ are the are the lower and upper limits of integration of θ respectively.

Setup the integral and also evaluate as follows:

$$\begin{aligned} \iiint_E x\, dV &= \int_0^{2\pi}\int_0^1\int_{4r^2}^4 x \cdot r\,dx\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 \left[\frac{x^2}{2}\right]_{4r^2}^4 r\,dr\,d\theta \qquad \text{Use } \int x^n dx = \frac{x^{n+1}}{n+1} + C \\ &= \int_0^{2\pi}\int_0^1 \left[\frac{4^2}{2} - \frac{\left(4r^2\right)^2}{2}\right] r\,dr\,d\theta \end{aligned}$$

Continuous to the above step,

$$= \int_0^{2\pi}\int_0^1 \left[\frac{16}{2} - \frac{16r^4}{2}\right] r\,dr\,d\theta$$

$$= \frac{16}{2}\int_0^{2\pi}\int_0^1 \left[1 - r^4\right] r\,dr\,d\theta$$

$$= 8\int_0^{2\pi}\int_0^1 \left[r - r^5\right] dr\,d\theta$$

$$= 8\int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^6}{6}\right]_0^1 d\theta$$

$$= 8\int_0^{2\pi}\left[\frac{1^2}{2} - \frac{1^6}{6} - \frac{0^2}{2} + \frac{0^6}{6}\right] d\theta$$

$$= 8\int_0^{2\pi}\left[\frac{1}{2} - \frac{1}{6} - 0 + 0\right] d\theta$$

Continuous to the above step,

$$= 8\int_0^{2\pi}\left[\frac{3}{6} - \frac{1}{6}\right] d\theta$$

$$= 8\int_0^{2\pi}\left[\frac{3-1}{6}\right] d\theta$$

$$= 8\cdot\frac{2}{6}\int_0^{2\pi} d\theta$$

$$= 8\cdot\frac{1}{3}\left[\theta\right]_0^{2\pi}$$

$$= \frac{8}{3}\left[2\pi - 0\right]$$

$$= \frac{16\pi}{3}$$

Hence, the required value of the triple integral $\iiint_E x\,dV$ is $\boxed{\frac{16\pi}{3}}$

Chapter 15 Multiple Integrals 15.7 18E

Consider *E* is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$ in the first octant.

The objective is to evaluate the triple integral

$$\iiint_T z\, dV$$

Since *E* lies in the first octant, we know its *x* and *y* values are bounded below by 0. Since *x* is at least 0 and $y = 3x$, $0 \le x \le \frac{1}{3}y$. In the *yz*-plane, $y^2 + z^2 = 9$ is the quarter circle that lies in the first quadrant, so $0 \le z \le \sqrt{9 - y^2}$. When *z* is 0, we know from the equation of the circle that *y* is at most 3, and $0 \le y \le 3$

$0 \le y \le 3$, which means that $y = 0$ and $x = 3$ are the lower and upper limits of integration of *y* respectively, and $0 \le x \le \frac{1}{3}y$, which means that $x = 0$ and $x = \frac{1}{3}y$ are the are the lower and upper limits of integration of *x* respectively, and $0 \le z \le \sqrt{9 - y^2}$, which means that $z = 0$ and $z = \sqrt{9 - y^2}$ are the are the lower and upper limits of integration of *z* respectively.

This allows as to rewrite the integral as the iterated integral $\int_0^3 \int_0^{\frac{1}{3}y} \int_0^{\sqrt{9-y^2}} z\, dz\, dx\, dy$

Notice that the integral $\int_0^3 \int_0^{\frac{1}{3}y} \int_0^{\sqrt{9-y^2}} z\, dz\, dx\, dy$ is of the form

$$\int_a^b \int_{\varphi_1(y)}^{\varphi_2(y)} \int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\, dz\, dx\, dy$$

where we first integrate with the function $f(x,y,z) = z$ respect to *z*, holding *x* and *y* constant, from $\phi_1(x,y) = 0$ to $\phi_2(x,y) = \sqrt{9 - y^2}$ first, integrate what we found from the first integration with respect to *x*, holding *y* constant, from $\varphi_1(y) = 0$ to $\varphi_2(y) = \frac{1}{3}y$, and lastly integrate what we found from the second integration with respect to *y* from 0 to 3.

First integrating the function $f(x,y,z) = z$ respect to *z*, holding *x* and *y* constant, from $\phi_1(x,y) = 0$ to $\phi_2(x,y) = \sqrt{9 - y^2}$

$$\begin{aligned}\int_{\phi_1(x,y)=0}^{\phi_2(x,y)=\sqrt{9-y^2}} z\, dz &= \left[\frac{1}{2}z^2\right]_{\phi_1(x,y)=0}^{\phi_2(x,y)=\sqrt{9-y^2}} \\ &= \left[\frac{1}{2}\left(\sqrt{9-y^2}\right)^2 - (0)\right] \\ &= \frac{1}{2}\left(9 - y^2\right)\end{aligned}$$

where we have used the integral rules for integrating polynomial functions (power rule, etc, in this case), leaving us with the function $f(x,y) = \frac{1}{2}\left(9 - y^2\right)$.

Now integrating the function $f(x,y)=\frac{1}{2}\left(9-y^2\right)$ with respect to *x*, holding *y* constant, from $\varphi_1(y)=0$ to $\varphi_2(y)=\frac{1}{3}y$, we have

$$\int_{\varphi_1(y)=0}^{\varphi_2(y)=\frac{1}{3}y}\frac{1}{2}\left(9-y^2\right)\,dx=\frac{1}{2}\left(9-y^2\right)[x]_{\varphi_1(y)=0}^{\varphi_2(y)=\frac{1}{3}y}$$

$$=\frac{1}{2}\left(9-y^2\right)\left[\frac{1}{3}y-0\right]$$

$$=\frac{1}{2}\left(9-y^2\right)\left(\frac{1}{3}y\right)$$

$$=\frac{3}{2}y-\frac{1}{6}y^3$$

where we have used the integral rules for integrating polynomial functions (power rule, etc, in this case), leaving us with the function $f(y)=\frac{3}{2}y-\frac{1}{6}y^3$.

Finally integrating the function $f(y)=\frac{3}{2}y-\frac{1}{6}y^3$ with respect to *y* from 0 to 3, we have

$$\int_0^3\frac{3}{2}y-\frac{1}{6}y^3\,dy=\left[\frac{3}{4}y^2-\frac{1}{24}y^4\right]_{y=0}^{y=3}$$

$$=\left[\frac{3}{4}(3)^2-\frac{1}{24}(3)^4\right]-[0]$$

$$=\boxed{\frac{27}{8}}$$

Chapter 15 Multiple Integrals 15.7 19E

Given plane is $2x+y+z=4$

$$\Rightarrow z=4-2x-y$$

$z=0$ Gives $y=4-2x$

$y=0$ Gives $x=2$

Thus volume $V=\iiint dz\,dy\,dx$

$$=\int_0^2\int_0^{4-2x}\int_0^{4-2x-y} dz\,dy\,dx$$

$$=\int_0^2\int_0^{4-2x} z\Big]_0^{4-2x-y} dy\,dx$$

$$=\int_0^2\int_0^{4-2x} (4-2x-y)\,dy\,dx$$

$$=\int_0^2 \left(4y-2xy-\frac{y^2}{2}\right)\Bigg]_0^{4-2x} dx$$

$$=\int_0^2 \left[4(4-2x)-2x(4-2x)-\frac{1}{2}(4-2x)^2\right]dx$$

$$=\int_0^2 \left[16-8x-8x+4x^2-\frac{1}{2}(16-16x+4x^2)\right]dx$$

i.e. $V=\int_0^2 \left(16-8x-8x+4x^2-8+8x-2x^2\right)dx$

$$=\int_0^2 \left(8-8x+2x^2\right)dx$$

$$=\left[8x-\frac{8x^2}{2}+\frac{2x^3}{3}\right]_0^2$$

$$=8(2-0)-4\left(2^2-0\right)+\frac{2}{3}\left(2^3-0\right)$$

$$=16-16+\frac{16}{3}$$

$$=\boxed{\frac{16}{3}}$$

Chapter 15 Multiple Integrals 15.7 20E

Consider the following equations of paraboloid and the plane:

$y = x^2 + z^2$ and $y = 8 - x^2 - z^2$

The objective is to find the volume of the solid enclosed by the paraboloids $y = x^2 + z^2$ and the plane $y = 8 - x^2 - z^2$.

The intersection of the paraboloid and the plane is as follows:

$$\begin{aligned} x^2 + z^2 &= 8 - x^2 - z^2 \\ 2\left(x^2 + z^2\right) &= 8 \\ x^2 + z^2 &= 4 \end{aligned}$$

Notice that $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$ intersect in the circle $x^2 + z^2 = 4$.

So, in polar coordinates the region of the *xz*-plane described by

$R = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$.

In polar coordinates the equation of the paraboloid as $y = r^2$ and the plane by $y = 8 - r^2$.

So, $0 \le r \le 2$, $r^2 \le y \le 8 - r^2$.

Thus the solid by be described as

$E = \{(r,\theta,y) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ r^2 \le y \le 8 - r^2\}$

This allows as to write the volume as the iterated integral as follows:

$$\begin{aligned} \text{Volume } &= \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} dy\ r\ dr\ d\theta \\ &= \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} r\ dy\ dr\ d\theta \\ &= \int_0^{2\pi}\int_0^2 r[y]_{y=r^2}^{y=8-r^2}\ dr d\theta \\ &= \int_0^{2\pi}\int_0^2 r\left(\left(8-r^2\right)-r^2\right) dr d\theta \end{aligned}$$

$$= \int_0^{2\pi}\int_0^2 \left(8r - 2r^3\right) dr d\theta$$

$$= \int_0^{2\pi}\left[4r^2 - \frac{1}{2}r^4\right]_0^2 d\theta$$

$$= \int_0^{2\pi}\left[16 - 8 - (0)\right] d\theta$$

$$= \int_0^{2\pi} 8 d\theta$$

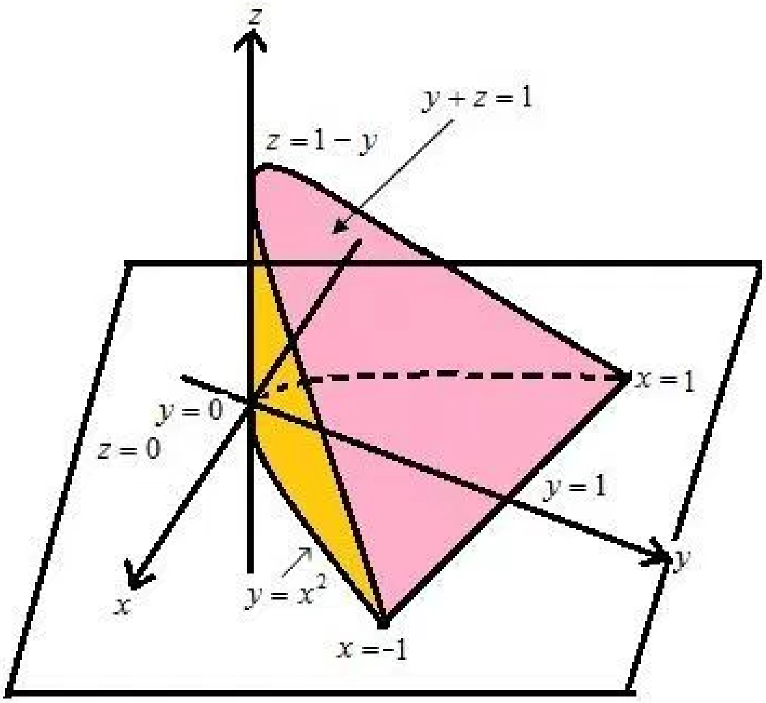
$$= 8(\theta)_0^{2\pi}$$

$$= 16\pi$$

Hence, the volume of the region is $\boxed{16\pi}$.

Chapter 15 Multiple Integrals 15.7 21E

The graph below shows the region of integration. The left boundary is the surface $y = x^2$. The upper plane is $z = 0$ and $y + z = 1$.



From the diagram it's clear that *z* ranges from 0 to $1 - y$.

The upper and lower planes intersect at $y = 1$. Thus *y* values range from x^2 to 1.

The left side boundary $y = x^2$ intersects $y = 1$ at $x = \pm 1$, so *x* ranges from -1 to 1.

The region is symmetric so the volume is double the volume of the portion in the first quadrant that is *x* can be taken to start at 0.

$$V = 2\int_0^1 \int_{x^2}^1 \int_0^{1-y} dz dy dx$$

Integrate with respect to *z*.

$$V = 2\int_0^1 \int_{x^2}^1 \left(z\Big|_0^{1-y} dy dx\right.$$

$$= 2\int_0^1 \int_{x^2}^1 \left[(1-y) - 0\right] dy dx$$

$$= 2\int_0^1 \int_{x^2}^1 (1-y) \, dy dx$$

Integrate with respect to *y*.

$$V = 2\int_0^1 \left(y - \frac{1}{2}y^2 \Big|_{x^2}^{1} \right) dx$$

$$= 2\int_0^1 \left[\left(1 - \frac{1}{2}\right) - \left(x^2 - \frac{1}{2}x^4\right) \right] dx$$

$$= 2\int_0^1 \left[\frac{1}{2} - \left(x^2 - \frac{1}{2}x^4\right) \right] dx$$

$$= 2\int_0^1 \left(\frac{1}{2} - x^2 + \frac{1}{2}x^4 \right) dx$$

Integrate with respect to *x*.

$$V = 2\left(\frac{1}{2}x - \frac{1}{3}x^3 + \frac{1}{10}x^5 \Big|_0^1 \right)$$

$$= 2\left[\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10} \right) - 0 \right]$$

$$= 2\left(\frac{8}{30} \right)$$

$$= \boxed{\frac{8}{15}}$$

Therefore, the volume of the solid region is $\boxed{\frac{8}{15}}$.

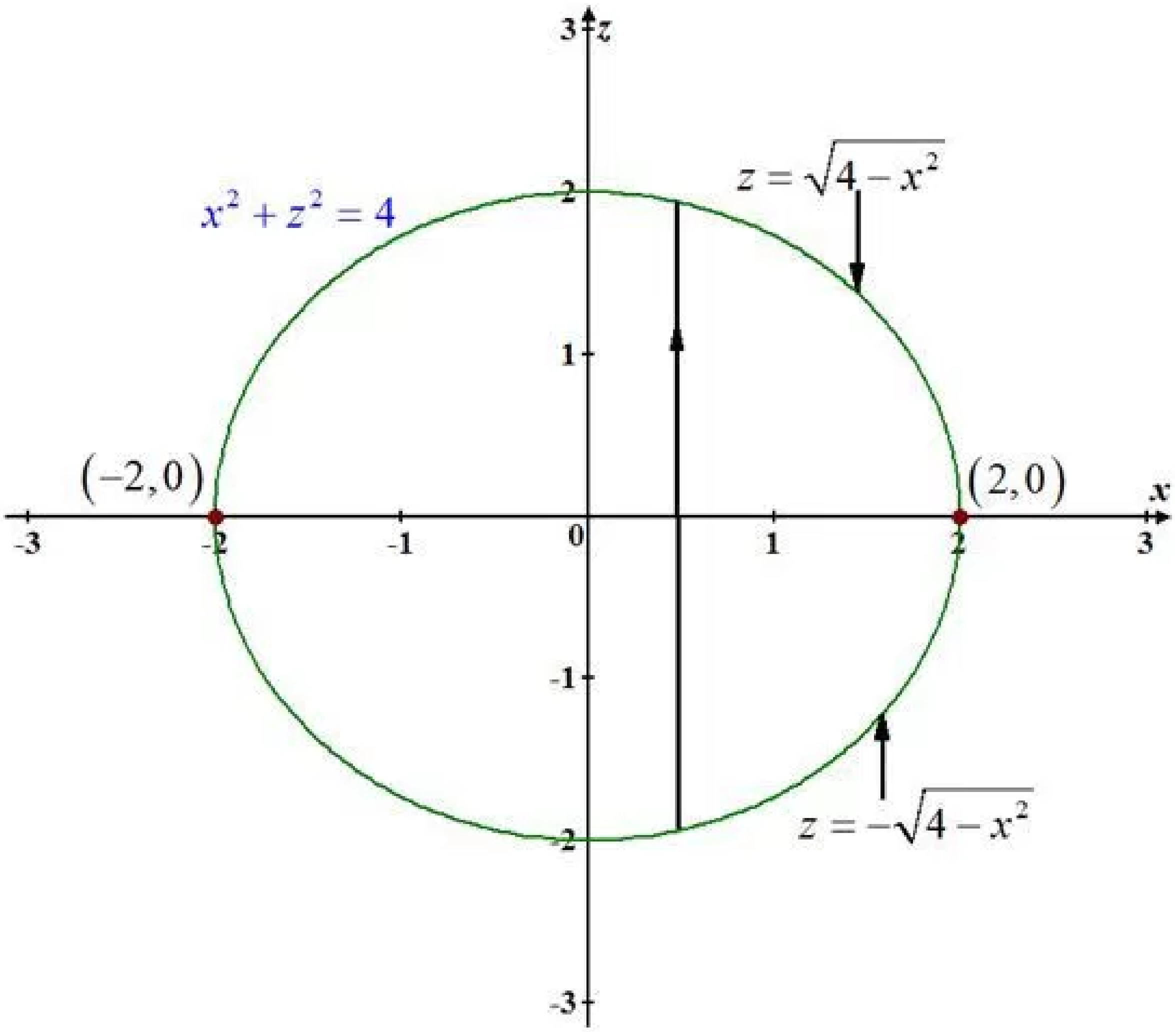
Chapter 15 Multiple Integrals 15.7 22E

Consider the cylinder $x^2+z^2=4$ and the planes $y=-1, y+z=4$.

The objective is to find the volume of the solid enclosed by the cylinder $x^2+z^2=4$ and the planes $y=-1$ and $y+z=4$.

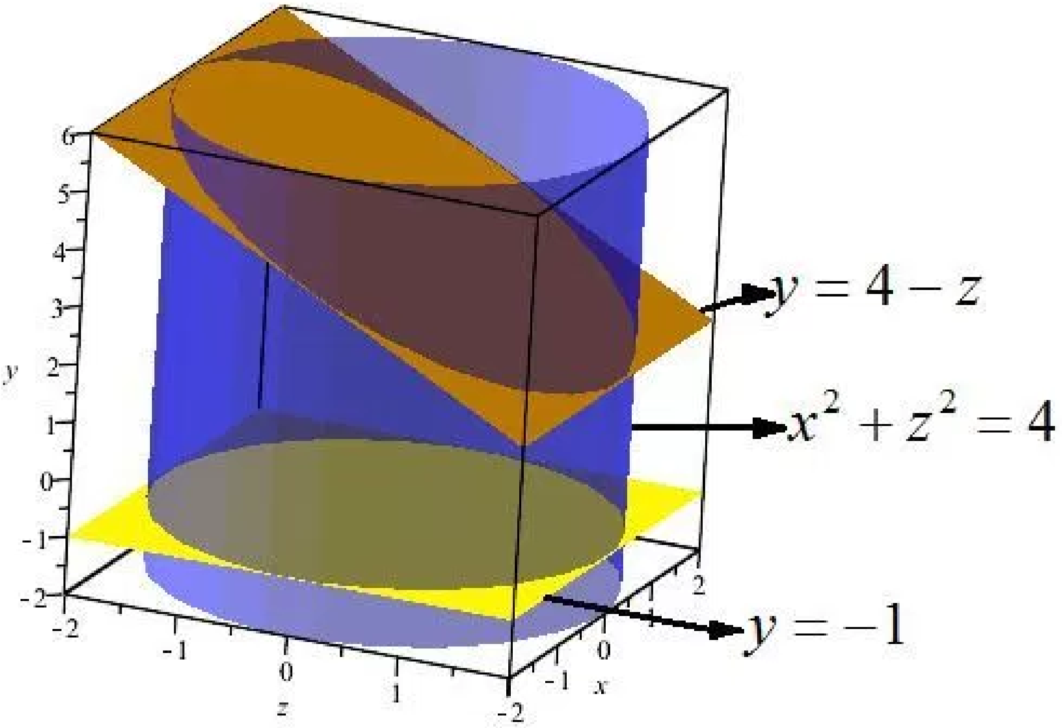
Observe that $x^2+z^2=4$ is a circle on *xz*-plane.

Sketch the circle and find the limits of *x* and *z* as follows:



From the figure, observe that *x* varies from -2 to 2 and *z* varies from $\sqrt{4-x^2}$ to $-\sqrt{4-x^2}$ (since $x^2+z^2=4 \Rightarrow z=\pm\sqrt{4-x^2}$).

The sketch of the solid enclosed by the cylinder $x^2+z^2=4$ and the planes $y=-1$ and $y+z=4$ is as follows:



From the figure observe that the lower boundary is the plane $y=-1$ and the upper boundary is the plane $y+z=4$.

That is, *y* varies from $y=-1$ to $y=4-z$.

Now the volume of the solid is given by $V=\iiint_E dxdydz$.

That is, here the volume of the solid is $V=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-1}^{4-z} dydzdx$.

Evaluate the above integral as follows:

Compute the integral with respect to *y*.

$$
\begin{aligned}
V &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (y)\Big|_{-1}^{4-z} dzdx \\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[(4-z)-(-1)\right] dzdx \\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4-z+1\right] dzdx \\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (5-z)\, dzdx
\end{aligned}
$$

Compute the integral with respect to z.

$$V=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(5-z)\,dz\,dx$$

$$=\int_{-2}^{2}\left(5z-\frac{1}{2}z^2\right)\Bigg|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx$$

$$=\int_{-2}^{2}\left[\left(5\sqrt{4-x^2}-\frac{1}{2}\left(\sqrt{4-x^2}\right)^2\right)-\left(-5\sqrt{4-x^2}-\frac{1}{2}\left(-\sqrt{4-x^2}\right)^2\right)\right]dx$$

$$=\int_{-2}^{2}\left[\left(5\sqrt{4-x^2}-\frac{1}{2}\left(4-x^2\right)+5\sqrt{4-x^2}+\frac{1}{2}\left(4-x^2\right)\right)\right]dx$$

$$=10\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

Compute the integral with respect to x.

$$V=10\int_{-2}^{2}\sqrt{2^2-x^2}\,dx$$

Use the following formula to solve the integral.

$$\int\sqrt{a^2-u^2}\,du=\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right).$$

Apply this formula to the volume. Put $a=2, u=x$.

$$V=10\left(\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right)\Bigg|_{-2}^{2}$$

$$=10\left[\left(\frac{2}{2}\sqrt{4-4}+\frac{4}{2}\sin^{-1}(1)\right)-\left(\frac{-2}{2}\sqrt{4-4}+\frac{4}{2}\sin^{-1}(-1)\right)\right]$$

$$=10\left[\left(0+\left(\frac{4}{2}\right)\left(\frac{\pi}{2}\right)\right)-\left(0+\left(\frac{4}{2}\right)\left(-\frac{\pi}{2}\right)\right)\right]$$

$$=10\left[\pi-(-\pi)\right]$$

$$=10(\pi+\pi)$$

$$=10(2\pi)$$

$$=\boxed{20\pi}$$

Therefore, the volume of solid enclosed by the cylinder $x^2+z^2=4$ and the planes $y=-1$ and $y+z=4$ is $\boxed{20\pi}$.

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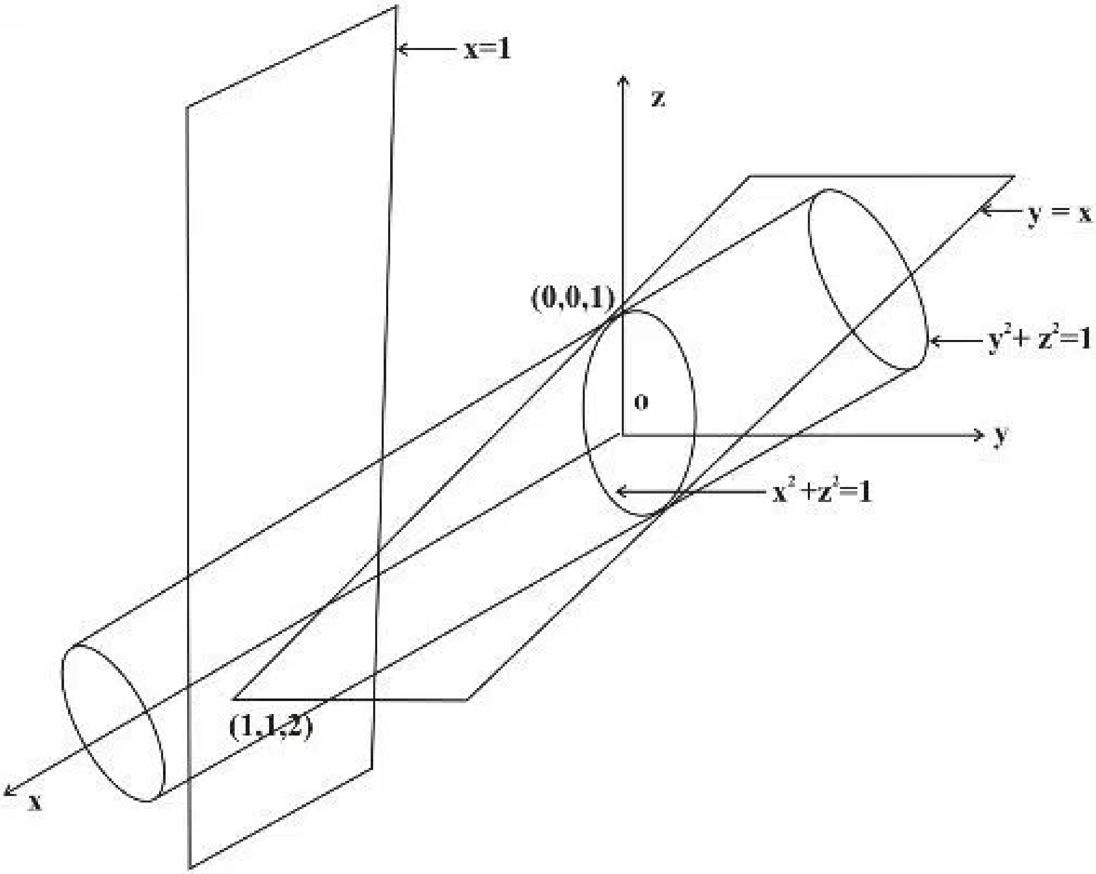

Now the cylinder $y^2 + z^2 = 1$ meets the plane $y = x$ in the circle $x^2 + z^2 = 1$

Then the region of integration is bounded by cylinder $y^2 + z^2 = 1$, planes $y = x$, $x = 1, y = 0, z = 0, x = 0$

Then the region is given by

$E = \left\{(x, y, z):\ \ 0 \le x \le 1,\ \ 0 \le y \le x,\ \ 0 \le z \le \sqrt{1 - y^2}\right\}$

And hence the volume of the solid is

(A)

$$v(E) = \iiint_E dv$$
$$= \int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\, dy\, dx$$

(B)

$$v(E)=\int_0^1\int_0^x\int_0^{\sqrt{1-y^2}} dz\,dy\,dx$$

$$=\int_0^1\int_0^x (z)_0^{\sqrt{1-y^2}}\,dy\,dx$$

$$=\int_0^1\int_0^x \sqrt{1-y^2}\,dy\,dx$$

$$=\int_0^1\left[\frac{y}{2}\sqrt{1-y^2}+\frac{1}{2}\sin^{-1}y\right]_0^x dx$$

$$=\frac{1}{2}\int_0^1\left[x\sqrt{1-x^2}+\sin^{-1}x\right]dx$$

$$=\frac{1}{2}\int_0^1 x\sqrt{1-x^2}\,dx+\frac{1}{2}\int_0^1\sin^{-1}x\,dx$$

$$=\frac{1}{2}\times\frac{-1}{2}\times\frac{2}{3}\left[\left(1-x^2\right)^{3/2}\right]_0^1+\frac{1}{2}\left[x\sin^{-1}x+\sqrt{1-x^2}\right]_0^1$$

i.e. $v(E)=-\frac{1}{6}[0-1]+\frac{1}{2}\left[\sin^{-1}1+0-0-1\right]$

$$=\frac{1}{6}+\frac{1}{2}\sin^{-1}(1)-\frac{1}{2}$$

$$=\frac{1}{2}\times\frac{\pi}{2}-\frac{3}{6}$$

$$=\frac{\pi}{4}-\frac{1}{3}$$

Hence $\boxed{v(E)=\frac{\pi}{4}-\frac{1}{3}}$

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It is need to approximate $\iiint_B \sqrt{x^2+y^2+z^2}\ dV$

The value of the triple integral by using a triple Riemann sum,

$$\iiint_B f(x,y,z)\ dV \approx \lim_{\max \Delta x_i \Delta y_j \Delta z_k \to 0} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(x_{ijk}, y_{ijk}, z_{ijk}\right) \Delta V_{ijk}$$

$$= \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(x_i, y_j, z_k\right) \Delta V$$

Where *l* represents the number of partitions along the *x* interval $[a,b]$

$$a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_l = b$$

And, *m* represents the number of partitions along the *y* interval $[c,d]$

$$c = y_0 < y_1 < \cdots < y_{i-1} < y_i < \cdots < y_m = d$$

Where *n* represents the number of partitions along the *z* interval $[r,s]$

$$r = z_0 < z_1 < \cdots < z_{i-1} < z_i < \cdots < z_n = s$$

And $\Delta V_{ijk} = \Delta x_{ijk} \Delta y_{ijk} \Delta z_{ijk}$.

To divide the solid into eight equal sub-boxes, divide each interval of *x*, *y*, and *z* into partitions with of the cube root of eight, which is two, so $l = m = n = 2$

In this particular problem it is known that the *x*, *y*, and *z* intervals are in $[0,4]$ and $\Delta x_{ijk} = \Delta y_{ijk} = \Delta z_{ijk} = 2$.

Additionally,

$$\Delta V_{ijk} = 2\cdot 2\cdot 2$$
$$= 8$$

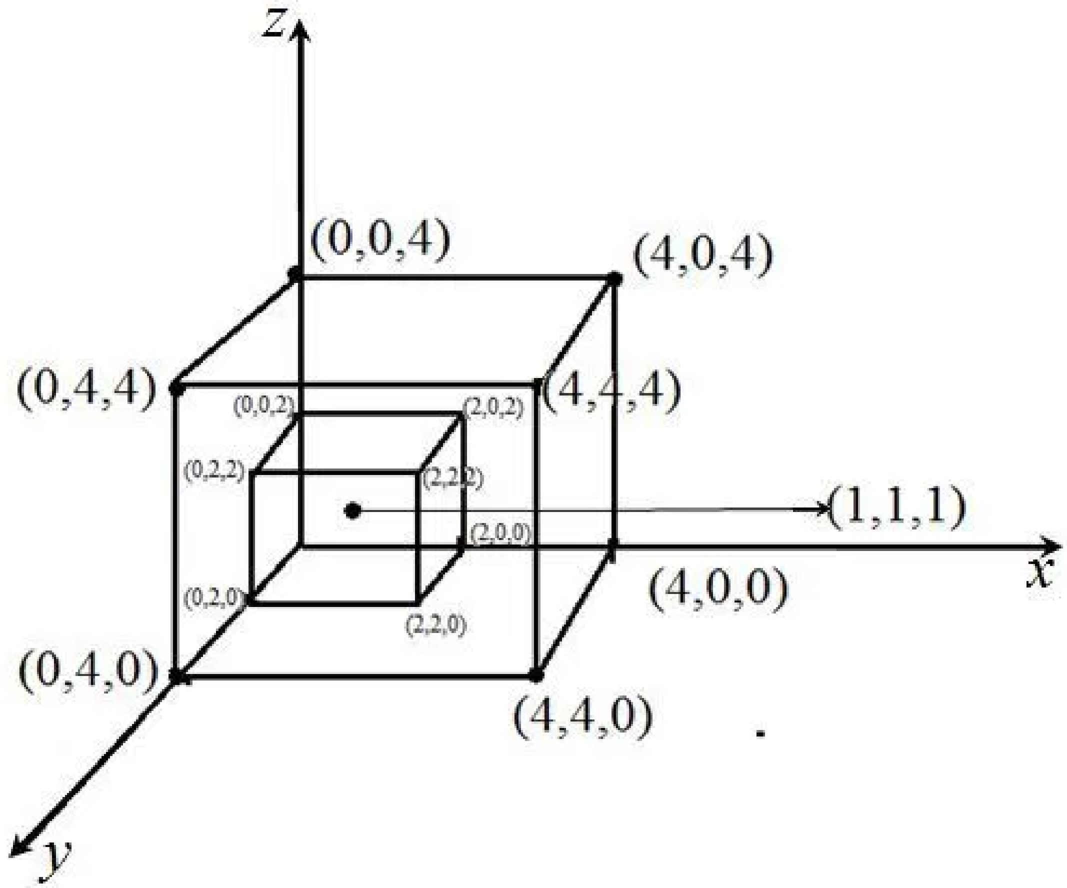
This allows us to write the integral as the triple Riemann sum

$$\iiint_B f(x,y,z)\ dV \approx \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f\left(x_{ijk}, y_{ijk}, z_{ijk}\right) 8$$

$$\approx 8\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f\left(x_{ijk}, y_{ijk}, z_{ijk}\right)$$

We can choose the sample points $f\left(x_{ijk}, y_{ijk}, z_{ijk}\right)$ in any manner.

The diagram of the cube is as follows



In this particular problem, choose the midpoints of each sub-box

$x_1 = 1,\ x_2 = 3$

$y_1 = 1,\ y_2 = 3$

$z_1 = 1,\ z_2 = 3$

This means the sample points are,

$$f(1,1,1) = \sqrt{3}, \quad f(3,3,3) = 3\sqrt{3},$$
$$f(1,3,3) = \sqrt{19}, \quad f(3,1,1) = \sqrt{11},$$
$$f(1,3,1) = \sqrt{11}, \quad f(3,1,3) = \sqrt{19},$$
$$f(1,1,3) = \sqrt{11}, \quad f(3,3,1) = \sqrt{19}$$

Now evaluate the triple Riemann sum (add all the values evaluated at each sample point and multiply by 8)

$$8\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f\left(x_{ijk}, y_{ijk}, z_{ijk}\right) = 8\left(\sqrt{3}+3\sqrt{11}+3\sqrt{19}+3\sqrt{3}\right)$$

$$\approx \boxed{239.638}$$

(b)

Use Maple, to find $\int_1^3\int_1^3\int_1^3 \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$

> $evalf\left(int\left(int\left(int\left(8\sqrt{x^2+y^2+z^2}, x=1..3\right), y=1..3\right), z=1..3\right)\right)$

227.8281039

Therefore,

$$\int_1^3\int_1^3\int_1^3 \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx \approx \boxed{228}$$

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The Midpoint Rule states that the triple integral of a function $f(x, y, z)$ over a box B can be approximated by dividing the box up into cubes, calculating the function value at the center of each cube, and then adding those values. Mathematically, we have

$$\iiint_B f(x,y,z)\,dV \approx \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(\bar{x}_i, \bar{y}_j, \bar{z}_k)\Delta V$$

Where $\bar{x}_i$ the midpoint of is $[x_{i-1}, x_i]$, $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$, and $\bar{z}_k$ is the midpoint of $[z_{k-1}, z_k]$.

Apply the Midpoint Rule with the cube $[0,1]\times[0,1]\times[0,1]$ divided into eight cubes. If B is to be divided into 8 equal cubes, that mean the x, y, and z dimensions of B should each be divided into 2 intervals.

$$\iiint_B f(x,y,z)\,dV \approx \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(\bar{x}_i, \bar{y}_j, \bar{z}_k)\Delta V$$

$$= \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f(\bar{x}_i, \bar{y}_j, \bar{z}_k)\Delta V$$

$$= f(\bar{x}_1,\bar{y}_1,\bar{z}_1)\Delta V + f(\bar{x}_1,\bar{y}_1,\bar{z}_2)\Delta V + f(\bar{x}_1,\bar{y}_2,\bar{z}_1)\Delta V + f(\bar{x}_1,\bar{y}_2,\bar{z}_2)\Delta V$$
$$+ f(\bar{x}_2,\bar{y}_1,\bar{z}_1)\Delta V + f(\bar{x}_2,\bar{y}_1,\bar{z}_2)\Delta V + f(\bar{x}_2,\bar{y}_2,\bar{z}_1)\Delta V + f(\bar{x}_2,\bar{y}_2,\bar{z}_2)\Delta V$$

Since the x, y, and z dimensions are all divided in half, the x-, y-, and z-intervals are all [0,1/2] and [1/2,1]. Since the midpoints of the intervals all happen at 1/4 and 3/4, the midpoints of the four regions occur at $\{(x,y,z) \mid x,y,z \in \{1/4, 3/4\}\}$ —in other words, every possible combination of 1/4 and 3/4. The volume of each cube is (1/2)(1/2)(1/2) = 1/8.

Plug all of this in with $f(x,y,z) = \cos(xyz)$:

$$\begin{aligned}\iiint_B \cos(xyz) \approx \cos&\left(\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\right)(1/8) + \cos\left(\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\right)(1/8)\\ &+\cos\left(\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\right)(1/8) + \cos\left(\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\right)(1/8)\\ &+\cos\left(\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\right)(1/8) + \cos\left(\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\right)(1/8)\\ &+\cos\left(\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\right)(1/8) + \cos\left(\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\right)(1/8)\\ &= (1/8)\left(\cos(1/64) + 3\cos(3/64) + 3\cos(9/64) + \cos(27/64)\right)\\ &= \boxed{.9849}\end{aligned}$$

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The volume of the solid is given to be

$$v = \int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\, dz\, dx$$

From this we see that the region of integration is

$$E = \{(x,y,z) : \ 0 \le x \le 1,\ 0 \le z \le 1-x,\ \ 0 \le y \le 2-2z\}$$

Therefore the solid is bounded by planes $x + z = 1$, $y + 2z = 2$, $x = 1$ and lies in the first octant (that is bounded by $x = 0, y = 0, z = 0$)

Now plane $x + z = 1$ meets x − axis in (1, 0, 0) and z − axis in (0, 0, 1). Also the plane $y + 2z = 2$ meets y − axis in (0, 2, 0) and z − axis in (0, 0, 1). And both the planes meet in straight line $y = 2x$. Also they meet in xy − plane at (1, 2, 0) (obtained by putting $z = 0$ in both the equation)

Hence the required solid is

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Consider the following iterated integral:

$$\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy$$

The objective is to sketch the solid whose volume is obtained by the integral.

The region of integration is,

$$E = \{(x, y, z): 0 \le y \le 2, 0 \le z \le 2 - y, 0 \le x \le 4 - y^2\}$$

The solid bounded by the three coordinate planes,

The planes are $z = 2 - y$ and the cylindrical surface $x = 4 - y^2$.

By using MAPLE, sketch the solid.

Here the ranges are $0 \le x \le 4 - y^2$, $0 \le z \le 2 - y$, $0 \le y \le 2$.

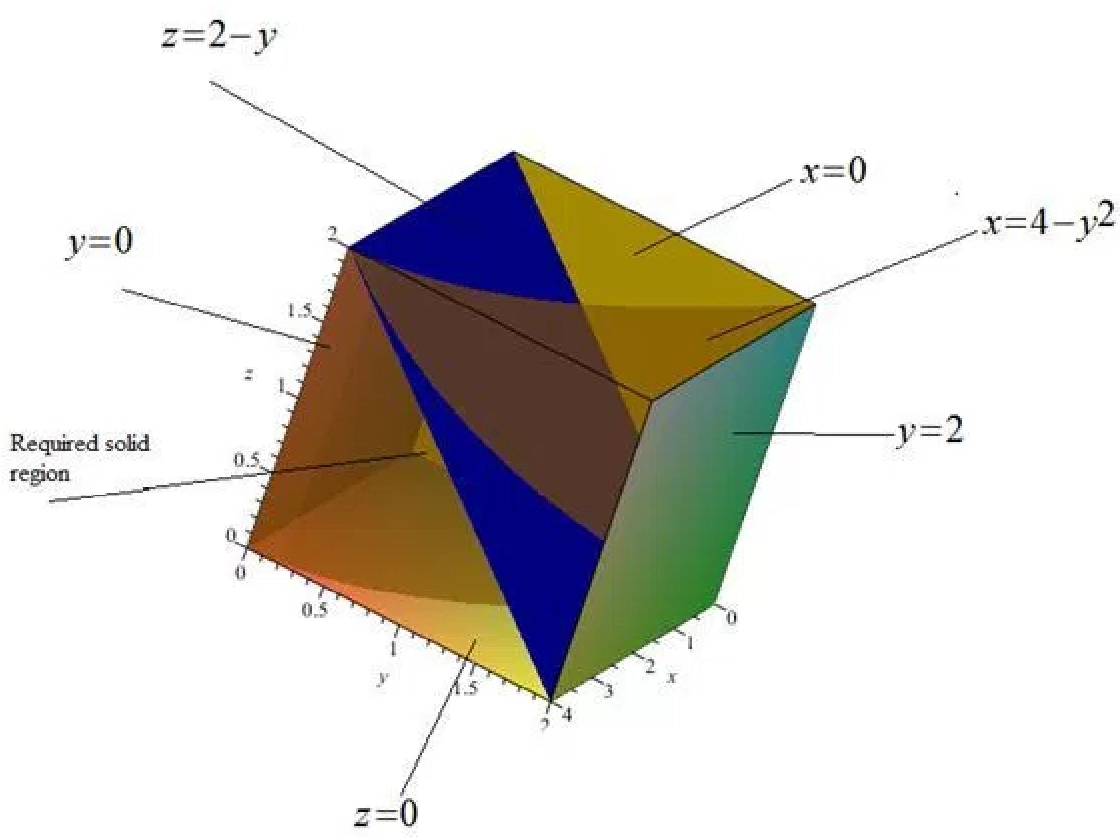
By entering the data into MAPLE

```
> x = 0, x = 4 - y^2, y = 0, y = 2, z = 0, z = 2 - y
```

$x = 0, x = -y^2 + 4, y = 0, y = 2, z = 0, z = 2 - y$

```
> plots[:-display](plots[:-implicitplot3d](x=0,x=0..4,y=0..2,z=0..2),plots[:-implicitplot3d
    ](x=-y^2+4,x=0..4,y=0..2,z=0..2),plots[:-implicitplot3d](y=0,x=0..4,y=0..2,
    z=0..2),plots[:-implicitplot3d](y=2,x=0..4,y=0..2,z=0..2),plots[:-implicitplot3d]
    (z=0,x=0..4,y=0..2,z=0..2),plots[:-implicitplot3d](z=2-y,x=0..4,y=0..2,z=0
    ..2))
```

The solid is as shown below:



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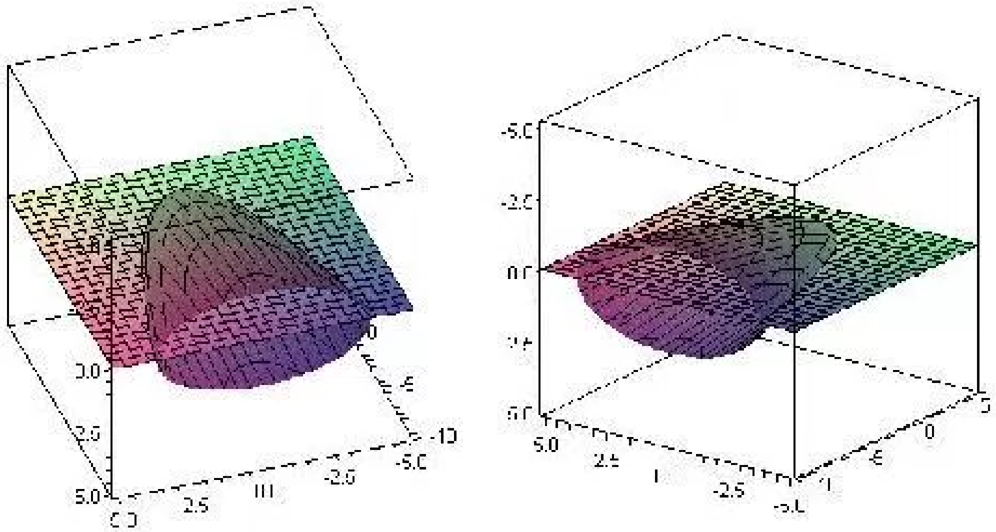
The paraboloid $y = 4 - x^2 - 4z^2$ when intersected by the plane $y = 0$ is seen as



To find the traces parallel to the xy-plane, set $z = k$:

$y = -x^2 + 4 - 4k^2$

These cross-sections are all parabolas opening in the negative y-direction.
To find the traces parallel to the xz - plane, set $y = k$:

$x^2 + 4z^2 = 4 - k$

These cross-sections are all ellipses as long as $k < 4$. If $k > 4$, no points (x, z) satisfy this equation; if $k = 4$, only the point with $x = 0$, $z = 0$ satisfies it. Therefore, this shape has a point at (0, 4, 0) and widens out into ellipses in the negative y-direction. It does not exist for $y > 4$.

To find the traces parallel to the yz-plane, set $x = k$:

$y = -4z^2 + 4 - k^2$

These cross-sections are all parabolas opening in the negative y-direction.
Since the surface has parabolic cross-sections in two dimensions that open in the negative y-direction, and ellipses that start at a point and then widen in the negative y-direction, this shape is an elliptic paraboloid with the y-axis as its axis and opening in the negative y-direction. Furthermore, its vertex is at (0, 4, 0). Here is a graph:

The volume in question for this problem is the solid inside the elliptic paraboloid but bounded on the open end of the paraboloid by the plane $y = 0$. We go through the six different ways of ordering the limits of integration.

First we examine doing x, then y, then z.

The limits of integration in x are between the two x-halves of the elliptic paraboloid surface, or $x=\sqrt{4-y-4z^2}$ and $x=-\sqrt{4-y-4z^2}$. The limits of y in terms of z must account for the "deepest" cross-sectional parabola parallel to the yz-plane, which occurs in the yz-plane when $x=0$, making the upper y limit the equation $y=4-4z^2$. The lower y limit is where the $y=0$ plane slices through the paraboloid and bounds the solid. The z limits are the most extreme values of z, which occur in the plane $y=0$ when $x=0$:

$0=4-0^2-4z^2$

$4z^2=4$

$z=\pm 1$

The first version of the iterated integral is therefore

$$\boxed{\int_{-1}^{1}\int_{0}^{4-4z^2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\,dx\,dy\,dz}$$

Next we do x, then z, then y. The limits in x are still the borders of the paraboloid in x, $x=\sqrt{4-y-4z^2}$ and $x=-\sqrt{4-y-4z^2}$. The limits in z are the upper and lower halves of the parabolic cross-sections parallel to the yz-plane expressed in terms of y. The most extreme z-values of these parabolic cross-sections occur at $x=0$, so we set $x=0$ and solve for z in the surface equation to find $y=4-4z^2$

$z=\pm\dfrac{\sqrt{4-y}}{2}$ as the limits in z. The limits in y are now just the extreme values of y, which are 0 and 4.

The second version of the iterated integral is therefore

$$\boxed{\int_{0}^{4}\int_{-\sqrt{4-y}/2}^{\sqrt{4-y}/2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\,dx\,dz\,dy}$$

Next we do y, then x, then z. The limits in y are the plane $y=0$ and the surface equation as given, $y=4-x^2-4z^2$. The limits in x become the two halves of the elliptic cross-section in x, the widest of which occurs at the widest opening of the elliptic paraboloid allowed by the solid, which is at $y=0$. Plugging $y=0$ into the surface equation and solving for x gives $x=\pm\sqrt{4-4z^2}$ as the limits in x. The z limits are the most extreme values of z, which occur in the plane $y=0$ when $x=0$:

$0=4-0^2-4z^2$

$4z^2=4$

$z=\pm 1$

The third version of the iterated integral is therefore

$$\boxed{\int_{-1}^{1}\int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}}\int_{0}^{4-x^2-4z^2} f(x,y,z)\,dy\,dx\,dz}$$

Next we do y, then z, then x. The limits in y are still the plane $y = 0$ and the surface equation as given, $y = 4 - x^2 - 4z^2$. The limits in z are the upper and lower halves of the elliptic paraboloid at the most extreme point, which happens when it is most open—in other words, when $y = 0$. Plugging $y = 0$ into the surface equation and solving for z gives $z = \pm\frac{\sqrt{4-x^2}}{2}$ as the limits in z. Finally, the limits in x are the most extreme values of x, which occur at the most open part of the parabola, when $y = 0$ and $z = 0$, or $x = \pm 2$.

The fourth version of the iterated integral is therefore

$$\boxed{\int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_{0}^{4-x^2-4z^2} f(x,y,z)\,dydzdx}$$

Next we do z, then x, then y.

The limits of integration in z are between the two z-halves of the elliptic paraboloid surface, or $z = \pm\frac{\sqrt{4-x^2-y}}{2}$. The limits in x are the upper and lower halves of the parabolic cross-section parallel to the xy-plane where it is most extreme, which occurs when $z = 0$. Plugging in $z = 0$ and solving for x gives $x = \pm\sqrt{4-y}$. The limits in y are now just the extreme values of y, which are 0 and 4.

The fifth version of the iterated integral is therefore

$$\boxed{\int_{0}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x,y,z)\,dzdxdy}$$

Finally we do z, then y, then x.

The limits of integration in z are still between the two z-halves of the elliptic paraboloid surface, or $z = \pm\frac{\sqrt{4-x^2-y}}{2}$. The limits in y are the plane $y = 0$ and the cross-section parallel to the xy-plane that is most extreme in the y, which happens at $z = 0$; plugging in $z = 0$ gives $y = 4 - x^2$ as the upper y limit. Finally, the limits in x are the most extreme values of x, which occur at the most open part of the parabola, when $y = 0$ and $z = 0$, or $x = \pm 2$.

The sixth version of the iterated integral is therefore

$$\boxed{\int_{-2}^{2}\int_{0}^{4-x^2}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x,y,z)\,dzdydx}$$

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Consider the surface:

$$y^2+z^2=9,\ x=-2,\ x=2$$

From this problem, the *x* is bounded by $-2 \le x \le 2$, *y* is bounded by $-\sqrt{9-z^2} \le y \le \sqrt{9-z^2}$, and *z* is bounded by $-\sqrt{9-y^2} \le y \le \sqrt{9-y^2}$.

Therefore, the all six integrals are equivalent.

First integral:

$$\boxed{\int_{-2}^{2}\int_{-3}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z)\,dydzdx}$$

Second integral:

$$\boxed{\int_{-2}^{2}\int_{-3}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z)\,dzdydx}$$

Third integral:

$$\boxed{\int_{-3}^{3}\int_{-2}^{2}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z)\,dydxdz}$$

Fourth integral:

$$\boxed{\int_{-3}^{3}\int_{-2}^{2}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z)\,dzdxdy}$$

Fifth integral:

$$\boxed{\int_{-3}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}}\int_{-2}^{2} f(x,y,z)\,dxdydz}$$

Sixth integral:

$$\boxed{\int_{-3}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}}\int_{-2}^{2} f(x,y,z)\,dxdzdy}$$

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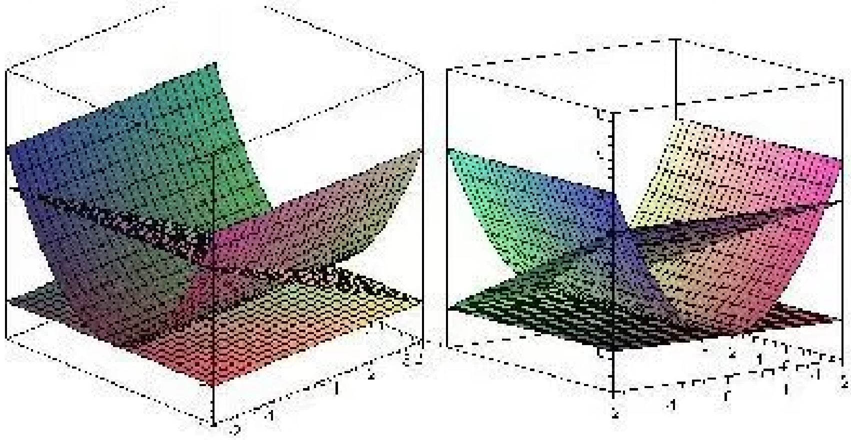
The region enclosed by the surfaces $y = x^2, z = 0$, and $y + 2z = 4$ is



It helps to have a clear image of the graph of the solid. The first boundary is a parabolic cylinder with the equation $y = x^2$ with traces that are parabolas parallel to the xy - plane. The horizontal plane $z = 0$ bounds the bottom of the solid; the plane $z = -y/2 + 2$ bounds the top, slanting downward diagonally.

The upper boundary intersects the parabolic cylinder at its vertex when $x = y = 0$. Plug $y = 0$ into the plane equation:

$z = 0 + 2$

$z = 2$

To see where the upper boundary slants down and intersects the plane $z = 0$, plug in $z = 0$:

$0 = -y/2 + 2$

$y = 4$

The line of intersection of the planes is also at the widest point of the parabola that is still part of the solid, so provides the most extreme x-values. Plug in $y = 4$ to the equation of the parabola to find that $x = \pm 2$ at these extreme corners of the solid.

First we integrate along x, then y, then z.

The limits of integration in x are between the two x-halves of the parabolic cylinder, which are $x = \sqrt{y}$ and $x = -\sqrt{y}$. The limits of y in terms of z go from $y = 0$, the vertex of the parabolic cylinder, to the slanted plane that is the upper boundary of the solid, or $y = 4 - 2z$. The limits in z are the most extreme values of z, or $z = 0$ and $z = 2$.

The first version of the iterated integral is therefore

$$\int_0^2 \int_0^{4-2z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\,dx\,dy\,dz$$

Next we do x, then z, then y.

The limits in x are still the two x-halves of the parabolic cylinder, which are $x = \sqrt{y}$ and $x = -\sqrt{y}$. The limits of z in terms of y go from the plane $z = 0$, the bottom of the solid, to the slanted plane that is the upper boundary of the solid, or $z = -y/2 + 2$. The limits in y are the most extreme values of y, or $y = 0$ and $y = 4$.

The second version of the iterated integral is therefore

$$\boxed{\int_0^4 \int_0^{-y/2+2} \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dxdzdy}$$

Next we do y, then x, then z.

The limits in y are the parabolic cylinder $y = x^2$ and the slanted plane $y = 4 - 2z$. To get x in terms of z, write the limits in x in terms of y: $x = -\sqrt{y}$ and $x = \sqrt{y}$. Then use the equation for the plane, $y = 4 - 2z$, to plug in and get $x = -\sqrt{4-2z}$ and $x = \sqrt{4-2z}$. The limits in z are the most extreme values of z, or $z = 0$ and $z = 2$.

The third version of the iterated integral is therefore

$$\boxed{\int_0^2 \int_{-\sqrt{4-2z}}^{\sqrt{4-2z}} \int_{x^2}^{4-2z} f(x,y,z)\,dydxdz}$$

Next we do y, then z, then x.

The limits in y are the parabolic cylinder $y = x^2$ and the slanted plane $y = 4 - 2z$. To get z in terms of x, write the limits in z in terms of y: $z = 0$ and $z = -y/2 + 2$. Then use the equation for the parabola, $y = x^2$, to plug in and get $z = -x^2/2 + 2$. The limits in x are the most extreme values of x, or $x = \pm 2$.

The fourth version of the iterated integral is therefore

$$\boxed{\int_{-2}^2 \int_0^{-x^2/2+2} \int_{x^2}^{4-2z} f(x,y,z)\,dydzdx}$$

Next we do z, then x, then y.

The limits of integration in z are the two planes $z = 0$, the bottom of the solid, and the slanted plane that is the upper boundary of the solid, or $z = -y/2 + 2$. The limits in x are the two x-halves of the parabolic cylinder, which are $x = -\sqrt{y}$ and $x = \sqrt{y}$. The limits in y are the most extreme values of y, $y = 0$ and $y = 4$.

The fifth version of the iterated integral is therefore

$$\boxed{\int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{-y/2+2} f(x,y,z)\,dzdxdy}$$

Finally we do z, then y, then x.

The limits of integration in z are still the two planes $z = 0$, the bottom of the solid, and the slanted plane that is the upper boundary of the solid, or $z = -y/2+2$. The limits in y in terms of x are the parabolic cylinder $y = x^2$ and, since the upper plane's cross-section doesn't change in terms of x, the most extreme value of y, or $y = 4$. The limits in x are the most extreme values of x, or $x = \pm 2$.
The sixth version of the iterated integral is therefore

$$\boxed{\int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{-y/2+2} f(x,y,z)\,dz\,dy\,dx}$$

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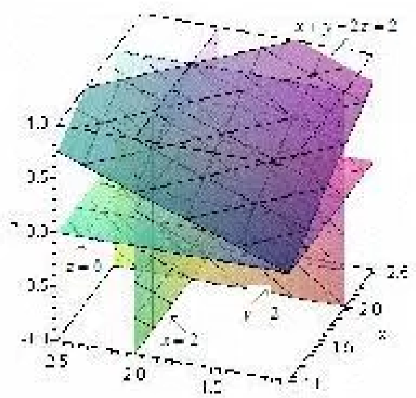

It helps to have a clear image of the graph of the solid. The solid is bordered by the plane $x = 2$, parallel to the yz-plane, the plane $y = 2$, parallel to the xz - plane, the $z = 0$ coordinate plane, and the inclined plane $x + y - 2z = 2$. We find the vertices of this solid bounded by the given four planes.

The three first planes form a corner when $x = y = 2$ and $z = 0$, at the point (2, 2, 0). If we plug $x = y = 2$ into the inclined plane equation, we get

$2 + 2 - 2z = 2$

$\Rightarrow z = 1$

so the inclined plane intersects $x=2$ and $y=2$ boundaries at $z=1$, forming a corner at (2,2,1) directly above the corner at (2,2,0). If we plug $x=2$, $z=0$ into the equation for the slanted plane, we get

$2+y-0=2$

$\Rightarrow y=0$

To give the point (2, 0, 0) as another corner of the solid. Finally, plug in $y=2$, $z=0$ to get

$x+2-0=2$

$x=0$

giving the point (0,2,0) as the fourth corner of the solid. The solid is therefore composed of four flat planes with corner points at (2,2,0), (2,2,1), (2,0,0), and (0,2,0). We can use these points to find the lines that border the solid in the projections on the coordinate planes.

In the xy - plane, the base of the solid is a triangle with vertices at points (2,2), (2,0), and (0,2); it is bounded by the lines $x=2$, $y=2$, and the line that connects (2,0) and (0,2), which has slope of -1 and y-intercept 2 and is therefore $y=-x+2$.

Parallel to the xz-plane at $y=2$, the solid has a triangular face with the xz-coordinates (2,0), (2,1), and (0,0). In the projection on the xz-plane we therefore have the bounds $z=0$, $x=2$, and the line with slope 1/2 and z-intercept 0, or $z=x/2$.

Parallel to the yz-plane at $x=2$, the solid has a triangular face with the yz-coordinates (2,0), (2,1), and (0,0). Just as in the xz-cross-section, this triangle has yz-equations of $z=0$, $y=2$, and the line $z=y/2$ in the projection on the yz-plane.

First we integrate along x, then y, and then z.

The x limits are the slanted face of the plane solved for x, which is $x=2-y+2z$, and the plane $x=2$. The limits of y in terms of z are the equations from the projection in the yz-plane solved for y, or $y=2z$, to $y=2$. The limits in z are the extreme values of z, or $z=0$ and $z=1$.

The first version of the iterated integral is therefore

$$\boxed{\int_0^1\int_{2z}^2\int_{2-y+2z}^2 f(x,y,z)\,dx\,dy\,dz}$$

Next we do x, then z, then y.

The x limits are still the slanted face of the plane solved for x, which is $x = 2 - y + 2z$, and the plane $x = 2$. The limits of z in terms of y are the equations from the projection in the yz-plane solved for z, or $z = 0$ to $z = y/2$. The limits in y are the extreme values of y, or $y = 0$ and $y = 2$.

The second version of the iterated integral is therefore

$$\boxed{\int_0^2 \int_0^{y/2} \int_{2-y+2z}^{2} f(x,y,z)\,dx\,dz\,dy}$$

Next we do y, then x, then z.

The y limits are the slanted face of the plane solved for y, which is $y = 2 - x + 2z$, and the plane $y = 2$. The limits of x in terms of z are the equations from the projection in the xz-plane solved for x, or $x = 2z$ to $z = 2$. The limits in z are the extreme values of z, or $z = 0$ and $z = 1$.

The third version of the iterated integral is therefore

$$\boxed{\int_0^1 \int_{2z}^{2} \int_{2-x+2z}^{2} f(x,y,z)\,dy\,dx\,dz}$$

Next we do y, then z, then x.

The y limits are still the slanted face of the plane solved for y, which is $y = 2 - x + 2z$, and the plane $y = 2$. The limits of z in terms of x are the equations from the projection in the xz-plane solved for z, or $z = 0$ to $z = x/2$. The limits in x are the extreme values of x, or $x = 0$ and $x = 2$.

The fourth version of the iterated integral is therefore

$$\boxed{\int_0^2 \int_0^{x/2} \int_{2-x+2z}^{2} f(x,y,z)\,dy\,dz\,dx}$$

Next we do z, then x, then y.

The z limits are the plane $z=0$ and the slanted face of the plane solved for z, which is $z=(x+y-2)/2$. The limits of x in terms of y are the equations from the projection in the xy-plane solved for x, or $x=2-y$ to $x=2$. The limits in y are the extreme values of y, or $y=0$ and $y=2$.

The fifth version of the iterated integral is therefore

$$\boxed{\int_0^2\int_{2-y}^2\int_0^{(x+y-2)/2} f(x,y,z)\,dz\,dx\,dy}$$

Finally we do z, then y, then x.

The y limits are the plane are still $z=0$ and the slanted face of the plane solved for z, which is $z=(x+y-2)/2$. The limits of y in terms of x are the equations from the projection in the xy-plane solved for y, or $y=2-x$ to $y=2$. The limits in x are the extreme values of x, or $x=0$ and $x=2$.

The sixth version of the iterated integral is therefore

$$\boxed{\int_0^2\int_{2-x}^2\int_0^{(x+y-2)/2} f(x,y,z)\,dz\,dy\,dx}$$

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Consider the following integral:

$$\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx.$$

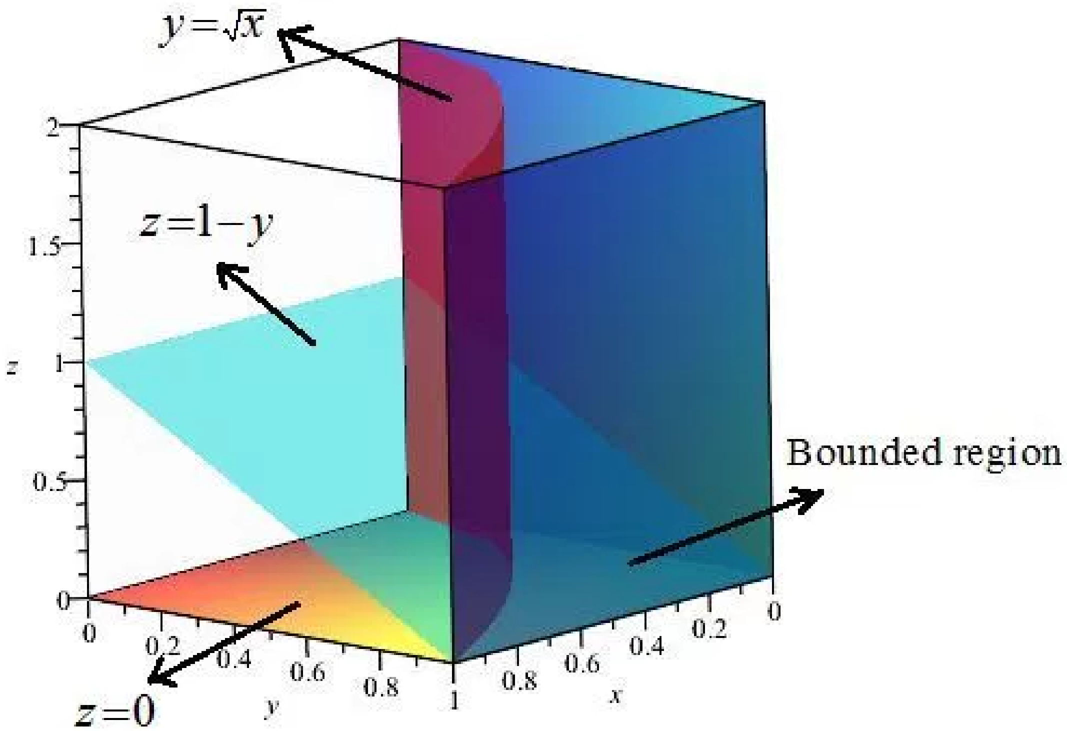
$dzdydx$:

From the given triple integral, notice that the variable z varies from 0 to $1-y$, y varies from $\sqrt{x}$ to 1 and the variable x varies from 0 to 1.

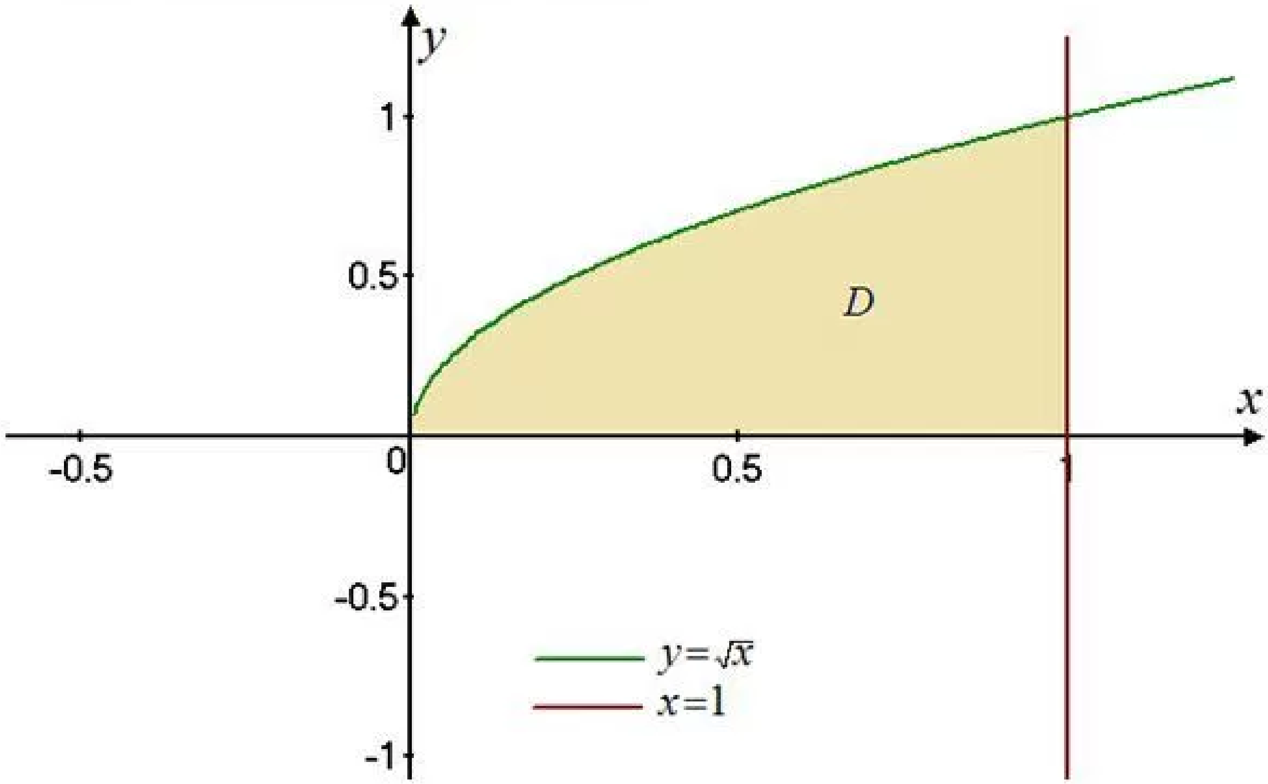
So the region E can be written as follows:

$$E=\left\{(x,y,z)\,|\,0\le z\le 1-y,\sqrt{x}\le y\le 1,0\le x\le 1\right\}.$$

The sketch of the region E is shown below:



The projection D of the region E on the xy plane is shown below:



$dzdxdy$:

Rewrite the equation $y=\sqrt{x}$ as follows:

$$y=\sqrt{x}$$
$$y^2=\left(\sqrt{x}\right)^2$$
$$x=y^2$$

The description of the solid *E* is $E=\left\{(x,y,z)\middle|0\le y\le 1, 0\le x\le y^2, 0\le z\le 1-y\right\}$.

$$\iiint_E dV=\int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dz\,dx\,dy$$

$dydzdx$:

Rewrite the equation $z=1-y$ as follows:

$$z=1-y$$
$$y=1-z$$

The description of the solid *E* is $E=\left\{(x,y,z)\middle|\sqrt{x}\le y\le 1-z, 0\le z\le 1-\sqrt{x}, 0\le x\le 1\right\}$.

$$\iiint_E dV=\int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx$$

$dydxdz$:

Rewrite the equation $z=1-y$ as follows:

$$z=1-y$$
$$y=1-z$$
$$\sqrt{x}=1-z \qquad \left(\text{Since } y=\sqrt{x}\right)$$
$$x=(1-z)^2$$

And the description of solid *E* is $E=\left\{(x,y,z)\middle|\sqrt{x}\le y\le 1-z, 0\le x\le (1-z)^2, 0\le z\le 1\right\}$.

$$\iiint_E dV=\int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz$$

$dxdydz$:

And the description of solid E is $E=\{(x,y,z)|0\le x\le y^2,0\le y\le 1-z,0\le z\le 1\}$.

$$\iiint_E dV=\int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz$$

$dxdzdy$:

The description of solid E is $E=\{(x,y,z)|0\le y\le 1,0\le x\le y^2,0\le z\le 1-y\}$.

$$\iiint_E dV=\int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy$$

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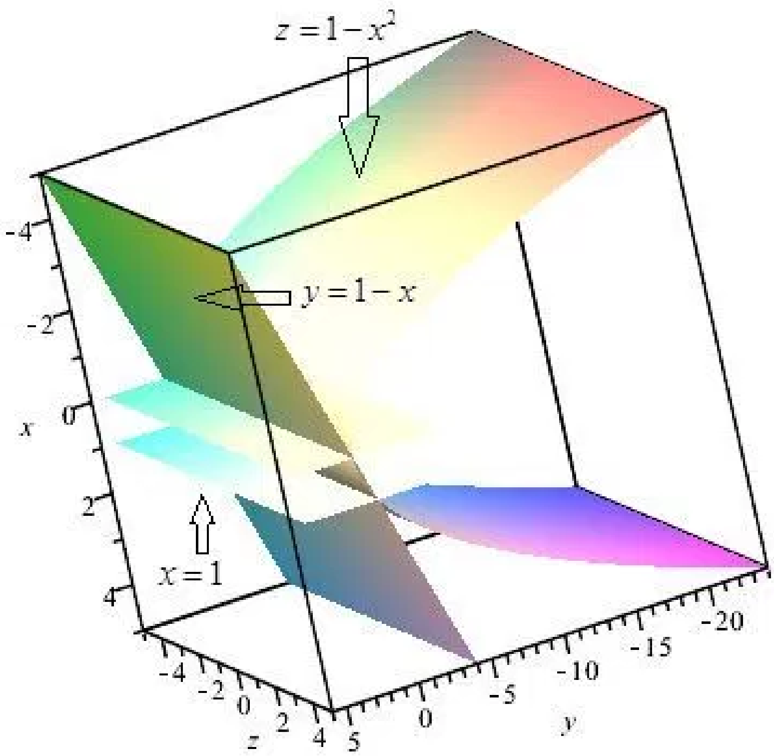
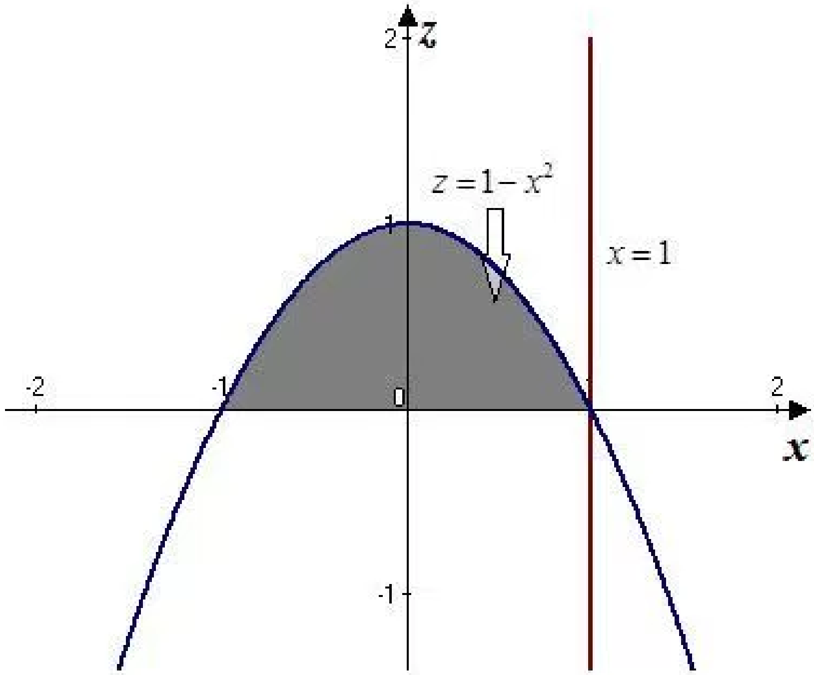
Consider the following integral:

$$\int_0^1\int_0^{1-x^2}\int_0^{1-x} f(x,y,z)\,dydzdx.$$

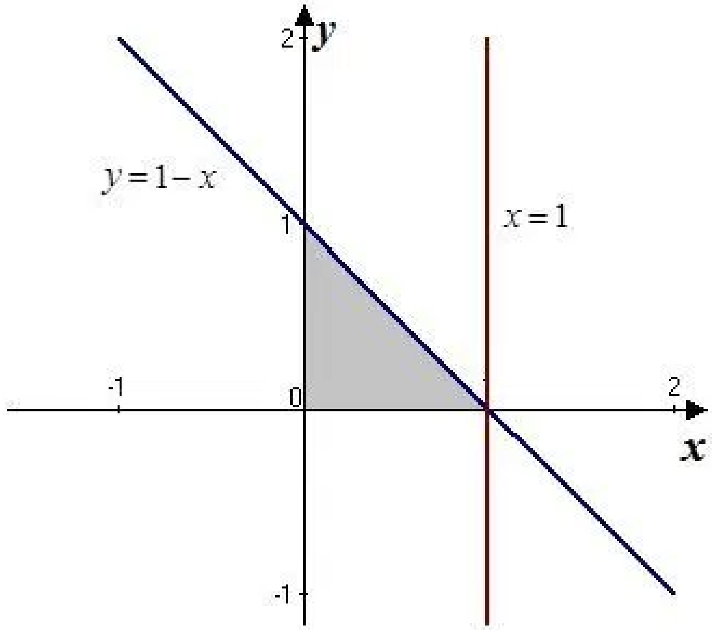
The sketch of the region E is shown below:



The projection D of the region E on the $xy-$plane is shown below:



The projection D of the region E on the $xz-$plane is shown below:



Rewrite the given integration as equivalent iterated integrals in the five other orders.

$dzdydx$:

From the given triple integral, notice that the variable y varies from 0 to $1-x$, the variable z varies from 0 to $1-x^2$ and the variable x varies from 0 to 1.

The description of the solid E is,

$$E=\{(x,y,z)\,|\,0\le y\le 1-x, 0\le z\le 1-x^2, 0\le x\le 1\}.$$

So, $\iiint_E dV=\int_0^1\int_0^{1-x}\int_0^{1-x^2} f(x,y,z)\,dzdydx.$

$dydxdz$:

Rewrite the equation $z=1-x^2$ as follows:

$$z=1-x^2$$
$$x^2=1-z$$
$$x=\sqrt{1-z}$$

The description of the solid *E* is,

$$E=\left\{(x,y,z)\middle|0\le z\le 1,0\le x\le\sqrt{1-z},0\le y\le 1-x\right\}.$$

So, $\iiint_E dV=\int_0^1\int_0^{\sqrt{1-z}}\int_0^{1-x} f(x,y,z)\,dydxdz.$

$dydzdx$:

Rewrite the equation $y=1-x$ as follows:

$$y=1-x$$
$$x=1-y$$

The description of the solid *E* is,

$$E=\left\{(x,y,z)\middle|0\le x\le 1-y,0\le z\le 1-x^2,0\le y\le 1\right\}.$$

So, $\iiint_E dV=\int_0^1\int_0^{1-y}\int_0^{1-x^2} f(x,y,z)\,dzdxdy.$

$dxdzdy$:

Rewrite the equation $y=1-x, z=1-x^2$ as follows:

$$\begin{aligned} z&=1-x^2\\ &=1-(1-y)^2\\ &=1-(y^2-2y+1)\\ &=2y-y^2 \end{aligned}$$

The description of the solid *E* is,

$$E=\left\{(x,y,z)\middle|0\le x\le 1,0\le z\le 2y-y^2,0\le y\le\sqrt{1-z}\right\}.$$

So, $\iiint_E dV=\int_0^1\int_0^{2y-y^2}\int_0^{\sqrt{1-z}} f(x,y,z)\,dxdzdy.$

$dxdydz$:

Rewrite the equation $y=1-x, z=1-x^2$ as follows:

$$\begin{aligned} y &= 1-x \\ &= 1-\sqrt{1-z} \end{aligned}$$

The description of the solid *E* is,

$$E=\left\{(x,y,z)\middle|0\le z\le 1, 0\le y\le 1-\sqrt{1-z}, 0\le x\le 1-y\right\}.$$

So, $\iiint_E dV=\int_0^1\int_0^{1-\sqrt{1-z}}\int_0^{1-y} f(x,y,z)\,dxdydz.$

Therefore, equivalent iterated integrals in the five orders are,

$$\int_0^1\int_0^{1-x^2}\int_0^{1-x} f(x,y,z)\,dy\,dz\,dx=\boxed{\int_0^1\int_0^{1-x}\int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx}$$

$$=\boxed{\int_0^1\int_0^{2y-y^2}\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy}$$

$$=\boxed{\int_0^1\int_0^{1-y}\int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy}$$

$$=\boxed{\int_0^1\int_0^{1-\sqrt{1-z}}\int_0^{1-y} f(x,y,z)\,dx\,dy\,dz}$$

$$=\boxed{\int_0^1\int_0^{\sqrt{1-z}}\int_0^{1-x} f(x,y,z)\,dy\,dx\,dz}$$

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Consider the following integral:

$$\int_0^1 \int_y^1 \int_0^y f(x, y, z)\, dz\, dx\, dy$$

The objective is to write five other iterated integral that are equal to the above integral.

The region bounded by the lines is as follows:

$z = 0$ and $z = y$

$x = y$ and $x = 1$

$y = 0$ and $y = 1$

To write the first iterated integral, change the limits as follows:

$x : x = y$ and $x = 1$

$y :\ y = z$ and $y = 1$

$z : z = 0$ and $z=1$

The iterated integral is $\int_0^1 \int_z^1 \int_y^1 f(x, y, z)\, dx\, dy\, dz$

To write the second iterated integral, change the limits as follows:

$y :\ y = z$ and $y = x$

$x : x = z$ and $x = 1$

$z : z = 0$ and $z=1$

The second iterated integral is $\int_0^1 \int_z^1 \int_z^x f(x, y, z)\, dy\, dx\, dz$.

To write the third iterated integral, change the limits as follows:

$z = 0$ and $z = y$

$y = 0$ and $y = x$

$x = y$ and $x = 1$

The third iterated integral is $\int_0^1 \int_0^x \int_0^y f(x, y, z)\, dz\, dy\, dx$.

To write the fourth iterated integral, change the limits as follows:

$x : x = y$ and $x = 1$

$z : z = 0$ and z=y

$y :\ y = 0$ and $y = 1$

The fourth iterated integral is $\int_0^1 \int_0^y \int_y^1 f(x, y, z)\, dx\, dz\, dy$.

To write the fifth iterated integral, change the limits as follows:

$y :\ y = z$ and $y = x$

$z : z = 0$ and z=x

$x : x = 0$ and $x = 1$

The fifth iterated integral is $\int_0^1 \int_0^x \int_z^x f(x, y, z)\, dy\, dz\, dx$.

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The limits of integration of this integral are $x = 0$ to $x = z$, $z = y$ to $z = 1$, and $y = 0$ to $y = 1$. The shape is bordered by these planes. It is a pyramidal shape, apex down, with its apex at the origin and bases at $z = 1$, with lateral sides the $x = 0$ plane, the $y = 0$ plane, the plane $z = y$, and the plane $z = x$. It has edges along the z-axis, along the line $z = y$ in the yz-plane, along the line $z = x$ in the xz-plane, and along the line of intersection of the $z = y$ and $z = x$ planes, which has the projection $x = y$ in the xy-plane. The vertical plane through this edge, the plane $x = y$, bisects the volume.

We integrate in terms of x first, then y, then z.
The x values vary from the $x = 0$ plane to $x = z$. The y values vary from the plane $y = 0$ to $y = z$. The z values range from 0 to 1. So the integral is:

$$\boxed{\int_0^1 \int_0^z \int_0^z f(x,y,z) dx dy dz}$$

We integrate in terms of y first, then x, then z.
The y values vary from the $y = 0$ plane to $y = z$. The x values vary from the plane $x = 0$ to $x = z$. The z values range from 0 to 1. So the integral is:

$$\boxed{\int_0^1 \int_0^z \int_0^z f(x,y,z) dy dx dz}$$

We integrate in terms of y first, then z, then x.
The y values vary from the $y=0$ plane to $y=z$. The z values vary from the plane $z=x$ to the plane $z=1$. The x values range from 0 to 1. So the integral is:

$$\boxed{\int_0^1\int_x^1\int_0^z f(x,y,z)\,dy\,dz\,dx}$$

We integrate in terms of z first, then x, then y.
This volume is difficult to integrate over the z first because both the planes $z=x$ and $z=y$ border it on the bottom. We therefore split it into two integrals. In the first, the z values vary from the $z=y$ plane to the $z=1$ plane. The x values vary from the plane $x=0$ to the plane $x=y$, which splits the volume where the $z=y$ and $z=x$ planes meet. The y values range from 0 to 1.

In the second integral, the z-values vary from the $z=x$ plane to the $z=1$ plane. The x values vary from the $x=y$ plane that splits the integral to $x=1$. The y values range from 0 to 1.
So the integral is:

$$\boxed{\int_0^1\int_0^y\int_y^1 f(x,y,z)\,dz\,dx\,dy+\int_0^1\int_y^1\int_x^1 f(x,y,z)\,dz\,dx\,dy}$$

We integrate in terms of z first, then y, then x.
Once again, this volume is difficult to integrate over the z first because both the planes $z=x$ and $z=y$ border it on the bottom. We therefore split it into two integrals. In the first, the z values vary from the $z=y$ plane to the $z=1$ plane. The y values vary from the plane $x=y$, which splits the volume where the $z=y$ and $z=x$ planes meet, to the $y=1$ boundary. The x values range from 0 to 1.

In the second integral, the z-values vary from the $z=x$ plane to the $z=1$ plane. The y values vary from the $y=0$ plane to the $x=y$ plane that splits the integral. The x values range from 0 to 1.

So the integral is:

$$\boxed{\int_0^1\int_x^1\int_y^1 f(x,y,z)\,dz\,dy\,dx+\int_0^1\int_0^x\int_x^1 f(x,y,z)\,dz\,dy\,dx}$$

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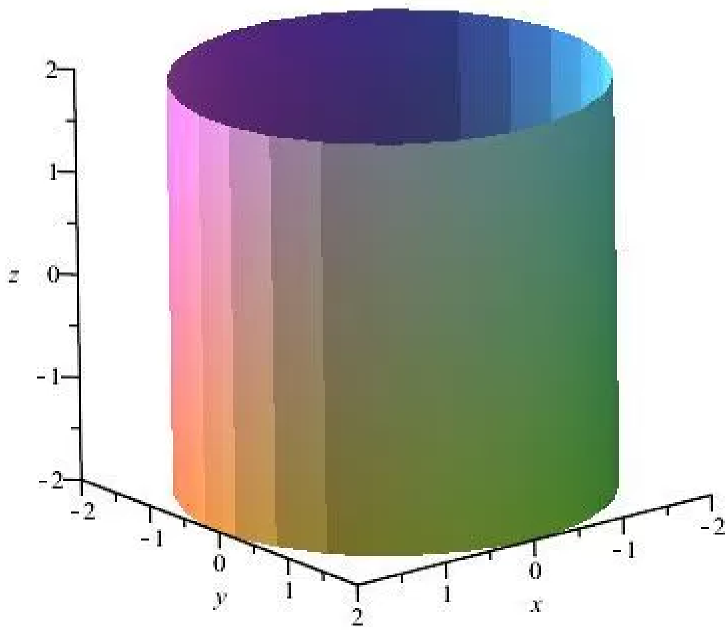
Consider the triple integral, $\iiint_C 4+5x^2yz^2\,dV$.

Here C is the cylindrical region $x^2+y^2\le 4, -2\le z\le 2$.

The objective is to evaluate the triple integral using only geometric interpretation and symmetry.

Since the region C is a cylinder of radius 2 and axis along the z-axis. It extends from $z=-2$ to $z=2$, a length of 4.

The graph of cylindrical region is shown below:



Break the integrand into two parts.

$$\iiint_C 4 + 5x^2yz^2dV = \iiint_C 4dV + \iiint_C 5x^2yz^2dV \quad \text{...... (1)}$$

Examine the first part first. Integrating a constant over a region results in the constant times the volume of the region.

Since the volume of a cylinder is the length 4times the area of the base.

The area of a circular base is $A = \pi r^2$.

Therefore,

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(2)^2 4 \\ &= 16\pi \end{aligned}$$

This first term is $4V = 64\pi$.

The second term has a factor of x^2 which is symmetric about $x = 0$ as is the region of integration. This term also has a factor of z^2 which is symmetric about $z = 0$ as is the region of integration. Finally this term has a factor of *y*, which is anti-symmetric about $y = 0$.

Therefore the second term makes a net contribution of zero to the integral.

From the equation (1), put the two terms together.

$$\iiint_C 4 + 5x^2yz^2dV = 64\pi + 0$$

$$= \boxed{64\pi}.$$

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The region E is given

$$E = \{(x, y, z):\ 0 \le x \le 1,\ 0 \le y \le \sqrt{x},\ 0 \le z \le 1 + x + y\}$$

Now $\rho(x, y, z) = 2$

Then mass of the solid is

$$m = \iiint_E \rho(x, y, z)\,dv$$

$$= \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2.\,dz\,dy\,dx$$

$$= 2\int_0^1 \int_0^{\sqrt{x}} (1 + x + y)\,dy\,dx$$

$$= 2\int_0^1 \left(y + xy + \frac{y^2}{2}\right)_{y=0}^{y=\sqrt{x}} dx$$

$$= 2\int_0^1 \left(\sqrt{x} + x^{3/2} + \frac{x}{2}\right) dx$$

i.e. $m = 2\left[\frac{2}{3}x^{3/2} + \frac{2}{5}x^{5/2} + \frac{x^2}{4}\right]_0^1$

$$= 2\left[\frac{2}{3} + \frac{2}{5} + \frac{1}{4}\right]$$

$$= 2 \times \frac{79}{60}$$

$$= \boxed{\frac{79}{30}}$$

Also $M_{yz} = \iiint_E x\,\rho(x, y, z)\,dv$

$$= 2\int_0^1 \int_0^{\sqrt{2}} \int_0^{1+x+y} x.\,dz\,dy\,dx$$

$$= 2\int_0^1 \int_0^{\sqrt{x}} x(1 + x + y)\,dy\,dx$$

$$= 2\int_0^1 x\left(y + xy + \frac{y^2}{2}\right)_{y=0}^{y=\sqrt{x}} dx$$

i.e. $M_{yz} = 2\int_0^1 \left(x^{3/2} + x^{5/2} + \frac{x^2}{2} \right) dx$

$$= 2\left[\frac{2}{5} x^{5/2} + \frac{2}{7} x^{7/2} + \frac{x^3}{6} \right]_0^1$$

$$= 2\left[\frac{2}{5} + \frac{2}{7} + \frac{1}{6} \right]$$

$$= 2 \times \frac{179}{210}$$

$$= \frac{358}{210}$$

And $M_{xz} = \iiint_E y\, \rho(x, y, z)\, dv$

$$= 2\int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} y\, dz\, dy\, dx$$

$$= 2\int_0^1 \int_0^{\sqrt{x}} y(1 + x + y)\, dy\, dx$$

$$= 2\int_0^1 \left(\frac{y^2}{2} + \frac{xy^2}{2} + \frac{y^3}{3} \right)_{y=0}^{y=\sqrt{x}} dx$$

i.e. $M_{xz} = 2\int_0^1 \left(\frac{x}{2} + \frac{x^2}{2} + \frac{x^{3/2}}{3} \right) dx$

$$= 2\left[\frac{x^2}{4} + \frac{x^3}{6} + \frac{2}{5} \frac{x^{5/2}}{3} \right]_0^1$$

$$= 2\left[\frac{1}{4} + \frac{1}{6} + \frac{2}{15} \right]$$

$$= 2 \times \frac{198}{360}$$

$$= \frac{396}{360}$$

$M_{xy} = \iiint_E z\, \rho(x, y, z)\, dv$

$$= 2\int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} z\, dz\, dy\, dx$$

$$= \frac{2}{2}\int_0^1 \int_0^{\sqrt{x}} \left(z^2 \right)_0^{1+x+y} dy\, dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} (1 + x + y)^2\, dy\, dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} \left(1 + x^2 + y^2 + 2x + 2y + 2xy \right) dy\, dx$$

$$= \int_0^1 \left[y + x^2 y + \frac{y^3}{3} + 2xy + y^2 + xy^2 \right]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \left[\sqrt{x} + x^{5/2} + \frac{x^{3/2}}{3} + 2x^{3/2} + x + x^2 \right] dx$$

$$= \left[\frac{2}{3} x^{3/2} + \frac{2}{7} x^{7/2} + \frac{2}{5} \cdot \frac{x^{5/2}}{3} + 2 \cdot \frac{2}{5} x^{5/2} + \frac{x^2}{2} + \frac{x^3}{3} \right]_0^1$$

$$= \frac{2}{3} + \frac{2}{7} + \frac{2}{15} + \frac{4}{5} + \frac{1}{2} + \frac{1}{3}$$

$$= \frac{571}{210}$$

Then $\bar{x} = \dfrac{M_{yz}}{m} = \dfrac{358}{210} \times \dfrac{30}{79} = \dfrac{358}{553}$

$\bar{y} = \dfrac{M_{xz}}{m} = \dfrac{396}{360} \times \dfrac{30}{79} = \dfrac{33}{79}$

$\bar{z} = \dfrac{M_{xy}}{m} = \dfrac{571}{210} \times \dfrac{30}{79} = \dfrac{571}{553}$

Hence center of mass is $\boxed{\left(\dfrac{358}{553}, \dfrac{33}{79}, \dfrac{571}{553} \right)}$

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The region E is given by

$$E = \{(x, y, z) : -1 \le y \le 1,\ 0 \le z \le 1 - y^2,\ 0 \le x \le 1 - z\}$$

Now it is given that $\rho(x,y,z)=4$

Then the mass of the solid will be

$$m=\iiint_E \rho(x,y,z)\,dv$$

$$=4\int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} dx\,dz\,dy$$

$$=4\int_{-1}^{1}\int_{0}^{1-y^2}(1-z)\,dz\,dy$$

$$=4\int_{-1}^{1}\left(z-\frac{z^2}{2}\right)_0^{1-y^2} dy$$

$$=4\int_{-1}^{1}\left(1-y^2-\frac{(1-y^2)^2}{2}\right)dy$$

$$=4\int_{-1}^{1}\left(1-y^2-\frac{(1-2y^2+y^4)}{2}\right)dy$$

$$=\int_{-1}^{1}\left(4-4y^2-2+4y^2-2y^4\right)dy$$

$$=\int_{-1}^{1}\left(2-2y^4\right)dy$$

$$=\left[2y-\frac{2y^5}{5}\right]_{-1}^{1}$$

$$=\left[4-\frac{4}{5}\right]$$

$$=\frac{16}{5}$$

$$M_{xz} = \iiint_E y\,\rho(x,y,z)\,dv$$

$$= 4\int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} y\,dx\,dz\,dy$$

$$= 4\int_{-1}^{1}\int_{0}^{1-y^2} \left[y\,x\right]_{x=0}^{x=1-z} dz\,dy$$

$$= 4\int_{-1}^{1}\int_{0}^{1-y^2} y(1-z)\,dz\,dy$$

$$= 4\int_{-1}^{1}\int_{0}^{1-y^2} (y-yz)\,dz\,dy$$

$$= 4\int_{-1}^{1}\left[yz-\frac{yz^2}{2}\right]_0^{1-y^2} dy$$

$$= 4\int_{-1}^{1}\left[y(1-y^2)-\frac{y(1-y^2)^2}{2}\right]dy$$

$$= 4\int_{-1}^{1}\left[(y-y^3)-\frac{(y-2y^3+y^5)}{2}\right]dy$$

$$= 2\int_{-1}^{1}\left[2(y-y^3)-(y-2y^3+y^5)\right]dy$$

$$= 2\int_{-1}^{1}\left[(2y-2y^3-y+2y^3-y^5)\right]dy$$

$$= 2\int_{-1}^{1}(y-y^5)\,dy$$

$$= 0 \qquad \text{[Since } (y-y^5) \text{ is an odd function]}$$

$$\text{Also } M_{yz} = \iiint_E x\,\rho(x,y,z)\,dv$$

$$= 4\int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} x\,dx\,dz\,dy$$

$$= 4\int_{-1}^{1}\int_{0}^{1-y^2}\left[\frac{x^2}{2}\right]_0^{1-z} dz\,dy$$

$$= 2\int_{-1}^{1}\int_{0}^{1-y^2}(1-z)^2\,dz\,dy$$

$$= 2\int_{-1}^{1}\int_{0}^{1-y^2}(1-2z+z^2)\,dz\,dy$$

$$= 2\int_{-1}^{1}\left[z-z^2+\frac{z^3}{3}\right]_0^{1-y^2} dy$$

$$= 2\int_{-1}^{1}\left[\left(1-y^2\right)-\left(1-y^2\right)^2+\frac{\left(1-y^2\right)^3}{3}\right]dy$$

$$=2\int_{-1}^{1}\left[\left(1-y^2\right)-\left(1-2y^2+y^4\right)+\frac{\left(1+3y^4-3y^2-y^6\right)}{3}\right]dy$$

$$=2\int_{-1}^{1}\left[\left(1-y^2-1+2y^2-y^4\right)+\frac{\left(1+3y^4-3y^2-y^6\right)}{3}\right]dy$$

$$=2\int_{-1}^{1}\left[\left(y^2-y^4\right)+\frac{\left(1+3y^4-3y^2-y^6\right)}{3}\right]dy$$

$$=\frac{2}{3}\int_{-1}^{1}\left[\left(3y^2-3y^4+1+3y^4-3y^2-y^6\right)\right]dy$$

$$=\frac{2}{3}\int_{-1}^{1}\left(1-y^6\right)dy$$

$$=\frac{2}{3}\left(y-\frac{y^7}{7}\right)_{-1}^{1}$$

$$=\frac{2}{3}\left(2-\frac{2}{7}\right)$$

$$=\frac{8}{7}$$

$$M_{xy} = \iiint_E z\,\rho(x,y,z)\,dv$$

$$= 4\int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} z\,dx\,dz\,dy$$

$$= 4\int_{-1}^{1}\int_{0}^{1-y^2}\left[zx\right]_0^{1-z}dz\,dy$$

$$= 4\int_{-1}^{1}\int_{0}^{1-y^2}\left[z(1-z)\right]dz\,dy$$

$$= 4\int_{-1}^{1}\left[\frac{z^2}{2}-\frac{z^3}{3}\right]_0^{1-y^2}dy$$

$$= \frac{4}{6}\int_{-1}^{1}\left[3\left(1-y^2\right)^2-2\left(1-y^2\right)^3\right]dy$$

$$= \frac{4}{6}\int_{-1}^{1}\left[3\left(1-2y^2+y^4\right)-2\left(1+3y^4-3y^2-y^6\right)\right]dy$$

$$= \frac{4}{6}\int_{-1}^{1}\left(3-6y^2+3y^4-2-6y^4+6y^2+2y^6\right)dy$$

$$= \frac{4}{6}\int_{-1}^{1}\left(1-3y^4+2y^6\right)dy$$

$$= \frac{4}{6}\left[y-\frac{3y^5}{5}+\frac{2y^7}{7}\right]_{-1}^{1}$$

$$= \frac{4}{6}\left[2-\frac{6}{5}+\frac{4}{7}\right]$$

$$= \frac{4}{6}\left[\frac{48}{35}\right]$$

$$= \frac{32}{35}$$

Now $\bar{x} = \dfrac{M_{yz}}{m}$

$$= \frac{8}{7}\times\frac{5}{16}$$

$$= \frac{5}{14}$$

$$\bar{y} = \frac{M_{xz}}{m}$$
$$= 0 \times \frac{5}{16}$$
$$= 0$$

$$\bar{z} = \frac{M_{xy}}{m}$$
$$= \frac{32}{35} \times \frac{5}{16}$$
$$= \frac{2}{7}$$

Hence the center of mass is $\boxed{\left(\frac{5}{14},\ 0,\ \frac{2}{7}\right)}$

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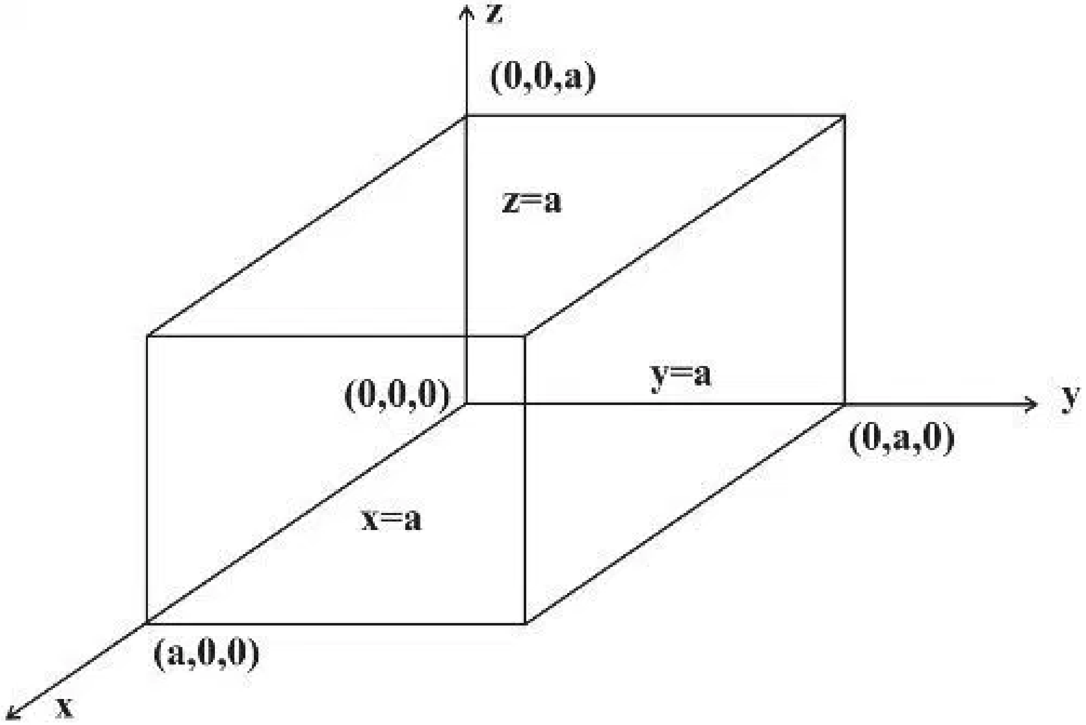

The region E is bounded by planes $x = 0, y = 0, z = 0, x = a, y = a, z = a$

i.e. $E = \{(x, y, z) : 0 \le x \le a,\ 0 \le y \le a,\ 0 \le z \le a\}$

Now $\rho(x,y,z) = x^2 + y^2 + z^2$

Then mass is

$$m = \iiint_E \rho(x,y,z)\,dv$$

$$= \int_0^a \int_0^a \int_0^a \left(x^2 + y^2 + z^2\right) dx\,dy\,dz$$

$$= \int_0^a \int_0^a \left(\frac{x^3}{3} + xy^2 + xz^2\right)_{x=0}^{x=a} dy\,dz$$

$$= \int_0^a \int_0^a \left[\frac{a^3}{3} + ay^2 + az^2\right] dy\,dz$$

$$= \int_0^a \left[\frac{a^3}{3}y + \frac{ay^3}{3} + az^2y\right]_{x=0}^{x=a} dz$$

i.e. $m = \int_0^a \left[\frac{a^3}{3} + \frac{a^4}{3} + a^2z^2\right] dz$

$$= \left[\frac{2}{3}a^4z + \frac{a^2x^2}{3}\right]_0^a$$

$$= \frac{2}{3}a^5 + \frac{1}{3}a^5$$

$$= \frac{3}{3}a^5$$

$$= \boxed{a^5}$$

Then $M_{yz} = \iiint_E x\,\rho(x,y,z)\,dv$

$$= \int_0^a \int_0^a \int_0^a \left(x^3 + xy^2 + xz^2\right) dx\,dy\,dz$$

$$= \int_0^a \int_0^a \left[\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^2z^2}{2}\right]_{x=0}^{x=a} dy\,dz$$

$$= \int_0^a \int_0^a \left[\frac{x^4}{4} + \frac{a^2y^2}{2} + \frac{a^2z^2}{2}\right] dy\,dz$$

$$= \int_o^a \left[\frac{a^4y}{4} + \frac{a^2y^3}{6} + \frac{a^2z^2y}{2}\right]_{y=0}^{y=a} dz$$

$$\text{i.e. } M_{yz} = \int_0^a \left[\frac{a^5}{4} + \frac{a^5}{6} + \frac{a^3 z^2}{2} \right] dz$$

$$= \left[\frac{a^5}{4} z + \frac{a^5 z}{6} + \frac{a^3 z^3}{6} \right]_0^a$$

$$= \frac{a^6}{4} + \frac{a^6}{6} + \frac{a^3}{6}$$

$$= \frac{7}{12} a^6$$

$$M_{zx} = \iiint_E y\, \rho(x, y, z)\, dv$$

$$= \int_0^a \int_0^a \int_0^a y\left(x^2 + y^2 + z^2\right) dx\, dy\, dz$$

$$= \int_0^a \int_0^a \left[\frac{x^3 y}{3} + xy^3 + xyz^2 \right]_{x=0}^{x=a} dy\, dz$$

$$= \int_0^a \int_0^a \left[\frac{a^3 y}{3} + ay^3 + ayz^2 \right] dy\, dz$$

$$= \int_0^a \left[\frac{a^3 y^2}{6} + \frac{ay^4}{4} + \frac{ay^2 z^2}{2} \right]_{y=0}^{y=a} dz$$

$$\text{i.e. } M_{zx} = \int_0^a \left[\frac{a^5}{6} + \frac{a^5}{4} + \frac{a^3 z^2}{2} \right] dz$$

$$= \left[\frac{a^5}{6} z + \frac{a^5}{4} z + \frac{a^3 z^3}{6} \right]_0^a$$

$$= \frac{a^6}{6} + \frac{a^6}{4} = \frac{a^6}{6}$$

$$= \frac{7}{12} a^6$$

$$M_{xy} = \iiint_E z\, \rho(x, y, z)\, dv$$

$$= \int_0^a \int_0^a \int_0^a z\left(x^2 + y^2 + z^2\right) dx\, dy\, dz$$

$$= \int_0^a \int_0^a \left[\frac{zx^3}{3} + xzy^2 + xz^3 \right]_{x=0}^{x=a} dy\, dz$$

$$= \int_0^a \int_0^a \left[\frac{a^3 z}{3} + azy^2 + az^3 \right] dy\, dz$$

i.e. $M_{xy} = \int_0^a \left[\frac{a^3 zy}{3} + \frac{azy^3}{3} + az^3 y \right]_{y=0}^{y=a} dz$

$$= \int_0^a \left[\frac{a^4 z}{3} + \frac{a^4 z}{3} + a^2 z^3 \right] dz$$

$$= \left[\frac{a^4 z^2}{6} + \frac{a^4 z^2}{6} + \frac{a^2 z^4}{4} \right]_0^a$$

$$= \frac{a^6}{6} + \frac{a^6}{6} + \frac{a^6}{4}$$

$$= \frac{7}{12} a^6$$

Then $\bar{x} = \frac{M_{yz}}{m}$

$$= \frac{7}{12} \frac{a^6}{a^5}$$

$$= \frac{7}{12} a$$

$$\bar{y} = \frac{M_{xz}}{m}$$

$$= \frac{7}{12} \frac{a^6}{a^5}$$

$$= \frac{7}{12} a$$

$$\bar{z} = \frac{M_{xy}}{m}$$

$$= \frac{7}{12} \frac{a^6}{a^5}$$

$$= \frac{7}{12} a$$

Hence center of mass of E is $\boxed{\left(\frac{7a}{12}, \frac{7a}{12}, \frac{7a}{12} \right)}$

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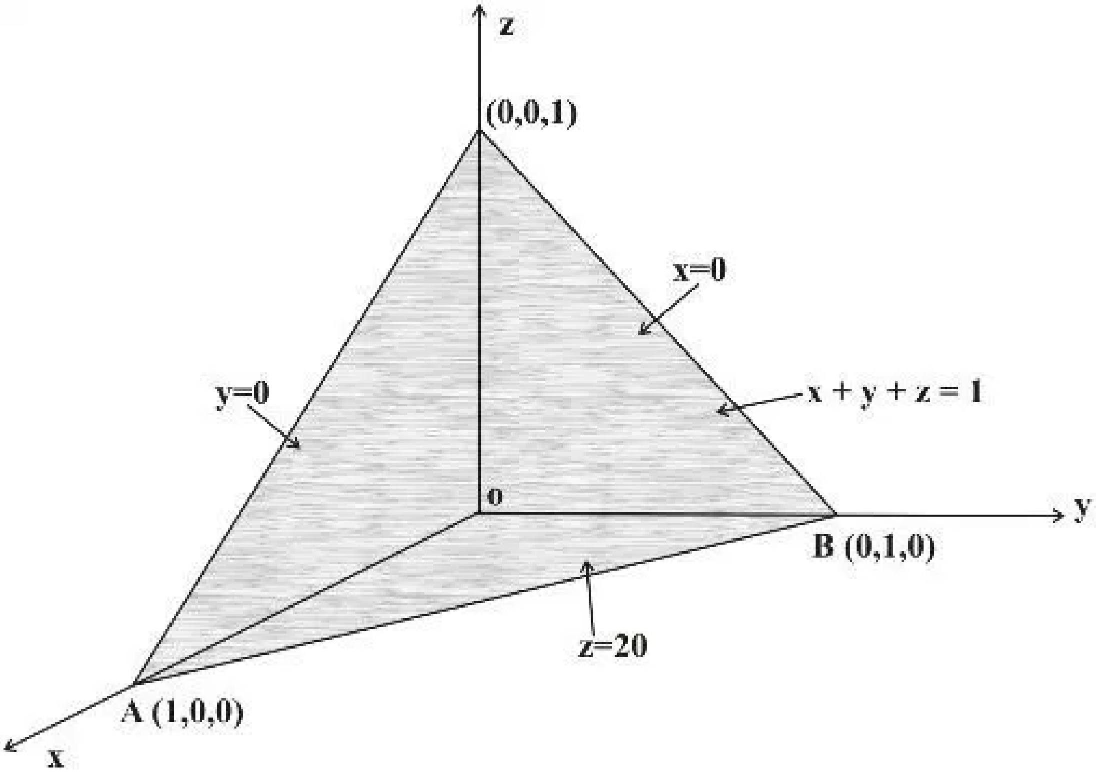

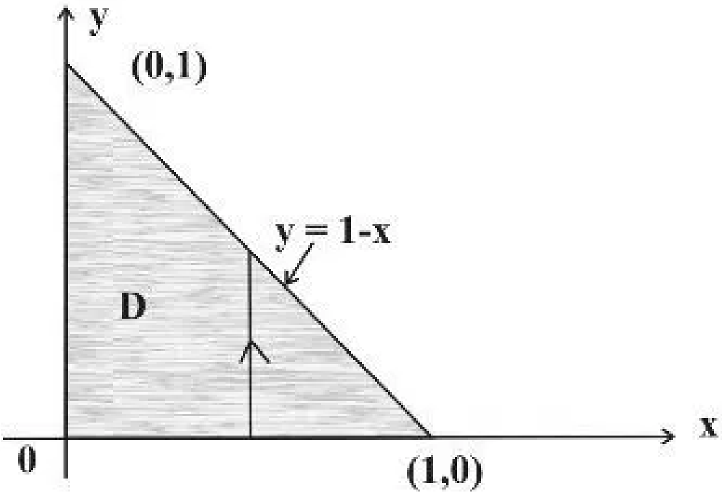

The tetrahedron E is and its projection D on xy − plane are shown in figures above. The lower boundary of E is the plane $z = 0$ and the upper boundary is the plane $x + y + z = 1$ i.e. $z = 1 - x - y$. Therefore E is given by

$$E = \{(x, y, z) : 0 \le x \le 1,\ \ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}$$

And $\quad \rho(x, y, z) = y$

Then the mass is given by

$$m = \iiint_E \rho(x,y,z)\,dv$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (y\,z)_{z=0}^{z=1-x-y}\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} y(1-x-y)\,dy\,dx$$

$$= \int_0^1 \left[\frac{y^2}{2}-\frac{xy^2}{2}-\frac{y^3}{3}\right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \left[\frac{(1-x)^2}{2}-\frac{x(1-x)^2}{2}-\frac{(1-x)^3}{3}\right]dx$$

$$= \int_0^1 \left[\frac{1-3x+3x^2-x^3}{2}-\frac{(1-x^3-3x+3x^2)}{3}\right]dx$$

i.e. m $= \frac{1}{6}\int_0^1 \left[3-9x+9x^2-3x^3-2+2x^3+6x-6x^2\right]dx$

$$= \frac{1}{6}\int_0^1 \left[1-3x+3x^2-x^3\right]dx$$

$$= \frac{1}{6}\left[x-\frac{3}{2}x^2+x^3-\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{6}\left[1-\frac{3}{2}+1-\frac{1}{4}\right]$$

$$= \frac{1}{6}\times\frac{1}{4}$$

$$= \frac{1}{24}$$

$$M_{yz} = \iiint_E x\,\rho(x,y,z)\,dv$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xy\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} xy(1-x-y)\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left(xy-x^2y-xy^2\right)dy\,dx$$

$$= \int_0^1 \left[\frac{xy^2}{2} - \frac{x^2y^2}{2} - \frac{xy^3}{3} \right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \left[\frac{\left(x - x^2\right)\left(1-x\right)^2}{2} - \frac{x\left(1-x\right)^3}{3} \right] dx$$

i.e. $M_{yz} = \int_0^1 \frac{x\left(1-x\right)^3}{6} dx$

$$= \frac{1}{6}\int_0^1 \left(x - x^4 - 3x^2 + 3x^3\right) dx$$

$$= \frac{1}{6}\left[\frac{x^2}{2} - \frac{x^5}{5} - x^6 + \frac{3}{4}x^4 \right]_0^1$$

$$= \frac{1}{6}\left[\frac{1}{2} - \frac{1}{5} - 1 + \frac{3}{4} \right]$$

$$= \frac{1}{6} \times \frac{1}{20}$$

$$= \frac{1}{120}$$

$$M_{xz} = \iiint_E y\, \rho\left(x, y, z\right) dv$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y^2\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} y^2 \left(1 - x - y\right) dy\, dx$$

$$= \int_0^1 \left[\frac{y^3}{3} - \frac{xy^3}{3} - \frac{y^4}{4} \right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \left[\frac{\left(1-x\right)^4}{12} \right] dx$$

i.e. $M_{xz} = \frac{1}{12}\int_0^1 \left(1 + x^4 - 4x^3 + 6x^2 - 4x\right) dx$

$$= \frac{1}{12}\left[x + \frac{x^5}{5} - x^4 + 2x^3 - 2x^2 \right]_0^1$$

$$= \frac{1}{12}\left[1 + \frac{1}{5} - 1 + 2 - 2 \right]$$

$$= \frac{1}{60}$$

$$M_{xy} = \iiint_E z\, \rho(x,y,z)\, dv$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} yz\, dz\, dy\, dz$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} \left(y z^2\right)_{z=0}^{z=1-x-y} dy\, dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} y(1-x-y)^2\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} \left(y + x^2 y + y^3 - 2xy - 2y^2 + 2xy^2\right) dy\, dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{y^2}{2} + \frac{x^2 y^2}{2} + \frac{y^4}{4} - xy^2 - \frac{2}{3}y^3 + \frac{2xy^3}{3}\right]_{y=0}^{y=1-x} dx$$

i.e. $$M_{xy} = \frac{1}{2}\int_0^1 \left[\frac{(1-x)^2}{2} y^2 - \frac{2}{3} y^3 (1-x) + \frac{1}{4} y^4\right]_{y=0}^{y=1-x} dx$$

$$= \frac{1}{2}\int_0^1 \frac{1}{12}(1-x)^4\, dx$$

$$= \frac{1}{24}\left[\frac{-(1-x)^5}{5}\right]_0^1$$

$$= \frac{1}{24} \times \frac{1}{5}$$

$$= \frac{1}{120}$$

Then $$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{120} \times 24 = \frac{1}{5}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{60} \times 24 = \frac{2}{5}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{120} \times 24 = \frac{1}{5}$$

Hence the mass is $m = \boxed{\frac{1}{24}}$

And center of mass is $\boxed{\left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}\right)}$

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To find the moments of inertia, use the following integrals with limits of integration determined by the solid.

$$I_x = \iiint_E \left(y^2+z^2\right)\rho(x,y,z)\ dV$$
$$I_y = \iiint_E \left(x^2+z^2\right)\rho(x,y,z)\ dV$$
$$I_z = \iiint_E \left(x^2+y^2\right)\rho(x,y,z)\ dV$$

Describe the cube as $E = \left\{(x,y,z)\,|\,0 \le x \le L,\ 0 \le y \le L,\ 0 \le z \le L\right\}$.

The solid is a cube of length *L*, with a vertex located at the origin and three edges that lie along the coordinate axis.

$0 \le x \le L$, which means that $x = 0$ and $x = L$ are the lower and upper limits of integration of *x* respectively.

$0 \le y \le L$, which means that $y = 0$ and $y = L$ are the lower and upper limits of integration of *y* respectively.

$0 \le z \le L$, which means that $z = 0$ and $z = L$ are the lower and upper limits of integration of *z* respectively.

Here the constant density of a cube is k.

Therefore, $\rho(x,y,z) = k$.

By symmetry of the cube and the density function, all the moments of inertia are to be equal.

Write the required moments of inertia as the iterated integral.

$$I_y = I_z = I_x = \iiint_E \left(y^2+z^2\right)\rho(x,y,z)\ dV$$
$$= \int_0^L \int_0^L \int_0^L \left(y^2+z^2\right)(k)\ dz\ dy\ dx$$
$$= k\int_0^L \int_0^L \int_0^L \left(y^2+z^2\right)\ dz\ dy\ dx$$

Compute the integrations.

$$k\int_0^L \int_0^L \int_0^L \left(y^2+z^2\right)\ dz\ dy\ dx = k\int_0^L \int_0^L \left[y^2 z + \frac{1}{3}z^3\right]_{z=0}^{z=L}\ dy\ dx$$
$$= k\int_0^L \int_0^L \left[y^2(L) + \frac{1}{3}(L)^3 - (0)\right] dy\ dx$$
$$= k\int_0^L \int_0^L \left(Ly^2 + \frac{1}{3}L^3\right) dy\ dx$$

Consider $k\int_0^L \int_0^L \left(Ly^2 + \frac{1}{3}L^3 \right) dy\, dx$.

$$k\int_0^L \int_0^L \left(Ly^2 + \frac{1}{3}L^3 \right) dy\, dx = k\int_0^L \left[\frac{1}{3}Ly^3 + \frac{1}{3}L^3 y \right]_{y=0}^{y=L} dx$$
$$= k\int_0^L \left[\frac{1}{3}L(L)^3 + \frac{1}{3}L^3(L) - (0) \right] dx$$
$$= k\int_0^L \frac{2}{3}L^4\, dx$$
$$= \frac{2}{3}kL^4 \int_0^L dx$$

Consider $\frac{2}{3}kL^4 \int_0^L dx$.

$$\frac{2}{3}kL^4 \int_0^L dx = \frac{2}{3}kL^4 [x]_0^L$$
$$= \frac{2}{3}kL^4 [L]$$
$$= \frac{2}{3}kL^5$$

Therefore, $k\int_0^L \int_0^L \int_0^L \left(y^2 + z^2 \right) dz\, dy\, dx = \boxed{\frac{2}{3}kL^5}$.

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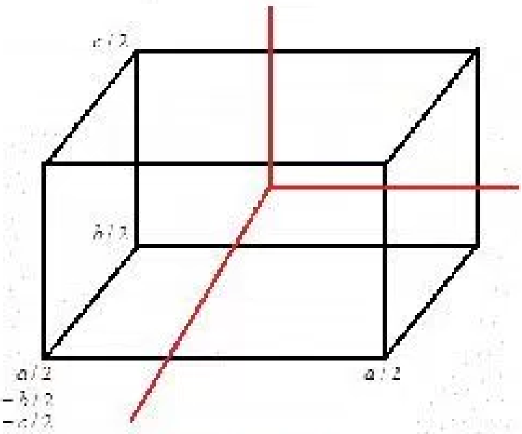
If the dimensions of a rectangular brick are a, b, and c, let the dimension of length a be parallel to the x-axis, the dimension of length b be parallel to the y-axis, and the dimension of length c be parallel to the z-axis. Since the brick is centered at the origin, it is bordered by the planes $x = \pm a/2$, $y = \pm b/2$, and $z = \pm c/2$.



The equations for the moments of inertia about the three coordinate axes for a solid E with density $\rho(x,y,z)$ are

$$I_x = \iiint_E (y^2+z^2)\rho(x,y,z)dV$$

$$I_y = \iiint_E (x^2+z^2)\rho(x,y,z)dV$$

$$I_z = \iiint_E (x^2+y^2)\rho(x,y,z)dV$$

Given that the mass of the brick is M.

Suppose the density of the brick is $\rho(x,y,z)=k$, then we have

$m = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} \rho(x,y,z)\,dxdydz$ can be written as $M = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} k\,dxdydz$

$$\Rightarrow M = k \times x\}_{-a/2}^{a/2} \times y\}_{-b/2}^{b/2} \times z\}_{-c/2}^{c/2}$$

$$\Rightarrow M = kabc$$

From this, we get $\rho(x,y,z) = k = \dfrac{M}{abc}$

Using this in the above formulae, we get

$$I_x = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\left((y^2+z^2)\frac{M}{abc}\right)dxdydz$$

$$= \frac{M}{abc}\int_{-c/2}^{c/2}\int_{-b/2}^{b/2} x\left(y^2+z^2\right)_{-a/2}^{a/2}dydz$$

$$= \frac{M}{abc}\int_{-c/2}^{c/2}\int_{-b/2}^{b/2} a\left(y^2+z^2\right)dydz$$

$$= \frac{M}{bc}\int_{-c/2}^{c/2} \frac{y^3}{3}+yz^2\}_{-b/2}^{b/2}dz$$

$$= \frac{M}{3bc}\int_{-c/2}^{c/2}\left(\frac{b^3}{8}+\frac{b^3}{8}\right)dz + \frac{M}{bc}\int_{-c/2}^{c/2}\left(\frac{b}{2}+\frac{b}{2}\right)z^2dz$$

$$= \frac{Mb^2}{12c}\times z\}_{-c/2}^{c/2} + \frac{M}{c}\times\frac{z^3}{3}\}_{-c/2}^{c/2}$$

$$= \frac{Mb^2}{12}+\frac{Mc^2}{12}$$

$$= \frac{M}{12}\left(b^2+c^2\right)$$

Since the formulae of moment of inertia are cyclic and the given function is cyclic, we get in the similar manner as above that

$$I_y = \iiint_E (x^2+z^2)\rho(x,y,z)dV$$

$$= \frac{M}{12}\left(a^2+c^2\right) \quad \text{and}$$

$$I_z = \iiint_E (x^2+y^2)\rho(x,y,z)dV$$

$$= \frac{M}{12}\left(a^2+b^2\right)$$

Thus, the required moment of inertia is

$$\boxed{I_x = \frac{M}{12}\left(b^2+c^2\right), I_y = \frac{M}{12}\left(a^2+c^2\right), I_z = \frac{M}{12}\left(a^2+b^2\right)}$$

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Convert to polar coordinates

$$0 \le z \le h,\ 0 \le \theta \le 2\pi,\ 0 \le r \le a$$

$$I_z = k\int_0^h \int_0^{2\pi} \int_0^a \left(r^2 \cos^2 + r^2 \sin^2\right) r \,\mathrm{d}\,r\,\mathrm{d}\,\theta\,\mathrm{d}\,z$$

$$= k\int_0^h \mathrm{d}\,z \int_0^{2\pi} \mathrm{d}\,\theta \left(\frac{r^4}{4}\right)_0^a = \frac{a^4}{4} k\int_0^h \mathrm{d}\,z \int_0^{2\pi} \mathrm{d}\,\theta$$

$$= \frac{a^4}{4} k\int_0^h \mathrm{d}\,z (\theta)_0^{2\pi} = \frac{\pi \mathrm{k} \mathrm{a}^4}{2} \int_0^h \mathrm{d}\,z = \frac{\pi k a^4}{2}(z)_0^h = \frac{\pi \mathrm{kh} a^4}{2}$$

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Suppose that $\rho(x,y,z)$ is the density function for a solid object that occupies a region E.

Then the moment of inertia of the solid about $z-\text{axis}$ is given by the integral,

$$I_z = \iiint_E \left(x^2+y^2\right)\rho(x,y,z)dV$$

Consider the cone $\sqrt{x^2+y^2} \le z \le h$.

Suppose that ρ is the constant density of the cone.

To evaluate the integral use rectangular coordinates.

Thus, $x = r\cos\theta$ and $y = r\sin\theta$ with $z = z$

And $r^2 = x^2 + y^2$

Therefore, the moment of inertia is given by the integral,

$$I_z = \iiint_E r^2 \rho r dr d\theta dz$$

Where E is the cone $\sqrt{x^2+y^2} \le z \le h$

The limits of integration are $0 \le z \le h,\ 0 \le r \le z,\ 0 \le \theta \le 2\pi$.

Evaluate the integral.

First integrate with respect to r.

$$\begin{aligned} I_z &= \int_{z=0}^{h}\int_{\theta=0}^{2\pi}\int_{r=0}^{z} r^2 \rho r dr d\theta dz \\ &= \rho \int_{z=0}^{h}\int_{\theta=0}^{2\pi}\int_{r=0}^{z} r^3 dr d\theta dz \\ &= \rho \int_{z=0}^{h}\int_{\theta=0}^{2\pi} \left[\frac{r^4}{4}\right]_0^z d\theta dz \\ &= \rho \int_{z=0}^{h}\int_{\theta=0}^{2\pi} \left[\frac{z^4}{4}\right] d\theta dz \end{aligned}$$

Now, integrate with respect to θ.

$$\begin{aligned} I_z &= \frac{\rho}{4}\int_{z=0}^{h} z^4 [\theta]_0^{2\pi} dz \\ &= \frac{\rho}{4}\int_{z=0}^{h} z^4 [2\pi - 0] dz \\ &= \frac{\rho}{4}\int_{z=0}^{h} z^4 [2\pi] dz \end{aligned}$$

Now, integrate with respect to z.

$$\begin{aligned} I_z &= \frac{\rho}{4}\int_{z=0}^{h} z^4 [2\pi] dz \\ &= \frac{\rho\pi}{2}\left[\frac{z^5}{5}\right]_0^h \\ &= \frac{\rho\pi h^5}{10} \end{aligned}$$

Hence, the moment of inertia of solid cone is $\boxed{I_z = \frac{\rho\pi h^5}{10}}$.

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Consider $\rho(x,y,z)=\sqrt{x^2+y^2}$

The solid enclosed by the cylinder $y=x^2$ and the planes $z=0$ and $y+z=1$

$z=1-y$, When $z=0,\ y=1$.

The region is

$$D=\{(x,y,z)|-1\le x\le 1,\ x^2\le y\le 1,\ 0\le z\le 1-y\}$$

$$Mass=m=\iiint \rho(x,y,z)\,dV$$

$$=\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y}\sqrt{x^2+y^2}\,dzdydx$$

The center of mass are

$$\bar{x}=\frac{M_{yz}}{m},\ \bar{y}=\frac{M_{xz}}{m},\ \bar{z}=\frac{M_{xy}}{m}$$

Thus,

$$M_{yz}=\iiint_E x\rho\,dV$$

$$=\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y}x\sqrt{(x^2+y^2)}\,dzdydx$$

$$M_{xz}=\iiint_E y\rho\,dV$$

$$=\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y}y\sqrt{(x^2+y^2)}\,dzdydx$$

$$M_{xy}=\iiint_E z\rho\,dV$$

$$=\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y}z\sqrt{(x^2+y^2)}\,dzdydx$$

Moments of inertia are

$$I_x=\iiint_E (y^2+z^2)\rho(x,y,z)\,dV$$

$$=\iiint_E (y^2+z^2)\sqrt{x^2+y^2}\,dV$$

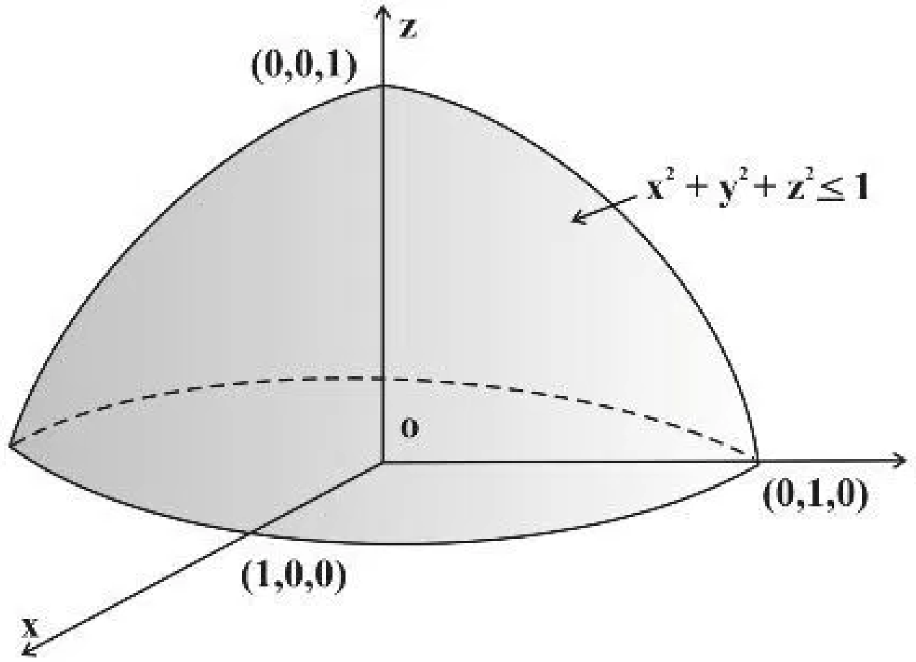
$$I_y=\iiint_E (x^2+z^2)\rho(x,y,z)\,dV$$

$$=\iiint_E (x^2+z^2)\sqrt{x^2+y^2}\,dV$$

$$I_z=\iiint_E (x^2+y^2)\rho(x,y,z)\,dV$$

$$=\iiint_E (x^2+y^2)\sqrt{x^2+y^2}\,dV$$

Chapter 15 Multiple Integrals 15.7 48E



Now the base of the hemisphere $x^2+y^2+z^2=1$ is a circle $x^2+y^2=1$

Then the region is given by

$$E=\left\{(x,y,z): -1\le x\le 1, -\sqrt{1-x^2}\le y\le \sqrt{1-x^2}, 0\le z\le \sqrt{1-x^2-y^2}\right\}$$

Now $\rho(x,y,z)=\sqrt{x^2+y^2+z^2}$

(A)

Then the mass is given by

$$m=\iiint_E \rho(x,y,z)\,dv$$

$$=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$

(B)

$$\bar{x}=\frac{1}{m}\iiint_E x\,\rho(x,y,z)\,dv$$

$$=\frac{1}{m}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} x\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$

$$\bar{y}=\frac{1}{m}\iiint_E y\,\rho(x,y,z)\,dv$$

$$=\frac{1}{m}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$

$$\bar{z}=\frac{1}{m}\iiint_E z\,\rho(x,y,z)\,dv$$

$$=\frac{1}{m}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$

(C)

$$I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dv$$
$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} (x^2+y^2)\sqrt{1+x^2+y^2}\,dz\,dy\,dx$$

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Consider E be the solid in the first octant bounded by the cylinder $x^2+y^2=1$ and the planes $y=z$, $x=0$, and $z=0$ with the density function $p(x,y,z)=1+x+y+z$.

Use a computer algebra system to find the exact values of the following quantities for E.

(a) The mass.

(b) The centre of mass.

(c) The moment of inertia about the z-axis.

Solution:

(a) The mass

To find the mass of the lamina, we integrate the given density function over the solid:

$$m = \iiint_E \rho(x,y,z)\,dV$$

Since z is lies between the planes $z=0$ and $z=y$, $0\le z\le y$. In the xy-plane, $x^2+y^2=1$ is the quarter circle described by $R=\left\{(x,y)\,|\,0\le x\le 1, 0\le y\le\sqrt{1-x^2}\right\}$

$0\le x\le 1$, which means that $x=0$ and $x=1$ are the lower and upper limits of integration of x respectively, $0\le y\le\sqrt{1-x^2}$, which means that $y=0$ and $y=\sqrt{1-x^2}$ are the are the lower and upper limits of integration of y respectively, and $0\le z\le y$, which means that $z=0$ and $z=y$ are the are the lower and upper limits of integration of z respectively.

This allows as to write the mass as the iterated integral

$$m=\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y 1+x+y+z\,dz\,dy\,dx$$

```
> m := int(int(int(1 + x + y + z, z = 0 ..y), y = 0 ..sqrt(1 - x^2)), x = 0
    ..1);
```

$$\frac{3}{32}\pi+\frac{11}{24}$$

Therefore

$$m=\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y 1+x+y+z\,dz\,dy\,dx$$
$$=\boxed{\frac{3}{32}\pi+\frac{11}{24}}$$

This is the mass of the region which is bounded by the cylinder $x^2+y^2=1$ and the planes $y=z$, $x=0$, and $z=0$.

(b) The center of mass

To find the center of mass, we use the following integrals with the same limits of integration that we used in calculating the mass:

$$\bar{x} = \frac{1}{m}\iiint_E x\rho(x,y,z)\, dV$$

$$\bar{y} = \frac{1}{m}\iiint_E y\rho(x,y,z)\, dV$$

$$\bar{z} = \frac{1}{m}\iiint_E z\rho(x,y,z)\, dV$$

where $m = \iiint_E \rho(x,y,z)\, dV$.

Using the limits of integration we previously found, the center of mass may be calculated by

$$\bar{x} = \frac{1}{m}\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y x(1+x+y+z)\ dz\ dy\ dx$$

$$\bar{y} = \frac{1}{m}\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y y(1+x+y+z)\ dz\ dy\ dx$$

$$\bar{z} = \frac{1}{m}\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y z(1+x+y+z)\ dz\ dy\ dx$$

Calculating the first coordinate of the centre of the mass is

$> xbar := \frac{1}{m}\cdot\left(int\left(int\left(int(x\cdot(1+x+y+z), z=0..y), y=0..\mathrm{sqrt}\left(1-x^2\right)\right), x=0..1\right)\right);$

$$\frac{7}{24\left(\frac{3}{32}\pi+\frac{11}{24}\right)}$$

simplify

$$\frac{28}{9\pi+44}$$

$$\frac{1}{\frac{3}{32}\pi\left(\frac{7}{24}\right)+\frac{11}{24}}$$

Calculating the y-coordinate of the centre of mass using the maple.

$> > ybar := \frac{1}{m}\cdot\left(int\left(int\left(int(y\cdot(1+x+y+z), z=0..y), y=0..\mathrm{sqrt}\left(1-x^2\right)\right), x=0..1\right)\right);$

$$\frac{\frac{4}{15}+\frac{1}{16}\pi}{\frac{3}{32}\pi+\frac{11}{24}}$$

simplify

$$\frac{2}{5}\,\frac{64+15\pi}{9\pi+44}$$

Calculating the z-coordinate of the centre of mass using the maple.

$$> > \quad zbar := \frac{1}{m}\cdot\left(int\left(int\left(int(z\cdot(1+x+y+z), z=0..y\right), y=0..\mathrm{sqrt}\left(1-x^2\right)\right), x=0..1\right)\right);$$

$$\frac{\frac{13}{90}+\frac{1}{32}\pi}{\frac{3}{32}\pi+\frac{11}{24}}$$

$$\underset{\text{simplify}}{=}$$

$$\frac{1}{15}\,\frac{208+45\,\pi}{9\,\pi+44}$$

Since we have calculated the individual coordinates, we can write the center of mass as

$$(\bar{x},\bar{y},\bar{z})=\boxed{\left(\frac{28}{44+9\pi},\frac{128+30\pi}{220+45\pi},\frac{208+45\pi}{660+135\pi}\right)}$$

(c) The moment of inertia about the z-axis

To find the moment of inertia about the z-axis, we use the following integral with the same limits of integration that we used in calculating the mass:

$$I_z=\iiint_E \left(x^2+y^2\right)\rho(x,y,z)\,dV$$

Using the limits of integration we previously found, the moment of inertia about the z-axis may be calculated by

$$I_z=\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^y \left(x^2+y^2\right)\left(1+x+y+z\right)\,dz\,dy\,dx$$

Calculating the moment of inertia about the z-axis using the maple.

$$> \quad int\left(int\left(int\left(\left(x^2+y^2\right)\cdot(1+x+y+z), z=0..y\right), y=0..\mathrm{sqrt}\left(1-x^2\right)\right), x=0..1\right);$$

$$\frac{1}{16}\pi+\frac{17}{60}$$

Chapter 15 Multiple Integrals 15.7 50E

Consider *E* be the solid in the first octant bounded by the cylinder $x^2+y^2=1$ and the planes $y=z$, $x=0$, and $z=0$ with the density function $\rho(x,y,z)=x^2+y^2$.

Since *E* lies in the first octant, *x* and *y* values are bounded below by 0.

Since *x* is at least 0 and $y=3x$, $0\le x\le \frac{1}{3}y$.

In the *yz*-plane, $y^2+z^2=9$ is the quarter circle that lies in the first quadrant, so $0\le z\le \sqrt{9-y^2}$.

When *z* is 0, we know from the equation of the circle $y^2+z^2=9$ that *y* is at most 3, and $0\le y\le 3$

Since $0\le y\le 3$, means that $y=0$ and $x=3$ are the lower and upper limits of integration of *y* respectively.

$0\le x\le \frac{1}{3}y$, means that $x=0$ and $x=\frac{1}{3}y$ are the lower and upper limits of integration of *x* respectively.

$0\le z\le \sqrt{9-y^2}$, means that $z=0$ and $z=\sqrt{9-y^2}$ are the lower and upper limits of integration of *z* respectively.

Write the mass as the iterate integral.

$$m=\int_0^3\int_0^{\frac{1}{3}y}\int_0^{\sqrt{9-y^2}}\left(x^2+y^2\right)\,dz\,dx\,dy$$

(a)

To find the mass of the lamina, integrate the density function over the solid:

$$m=\iiint_E \rho(x,y,z)\,dV$$

$$m=\int_0^3\int_0^{\frac{1}{3}y}\int_0^{\sqrt{9-y^2}}\left(x^2+y^2\right)\,dz\,dx\,dy$$

Use maple to find the value.

$$> \int_0^3\int_0^{\frac{y}{3}}\int_0^{\sqrt{9-y^2}}(x^2+y^2)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y;$$

$$\frac{56}{5}$$

Therefore $\int_0^3\int_0^{\frac{1}{3}y}\int_0^{\sqrt{9-y^2}}\left(x^2+y^2\right)\,dz\,dx\,dy=\boxed{\frac{56}{5}}$

(b)

To find the center of mass, use the following integrals.

$$\bar{x} = \frac{1}{m}\iiint_E x\rho(x,y,z)\ dV$$

$$\bar{y} = \frac{1}{m}\iiint_E y\rho(x,y,z)\ dV$$

$$\bar{z} = \frac{1}{m}\iiint_E z\rho(x,y,z)\ dV$$

Here $m = \iiint_E \rho(x,y,z)\ dV$.

$$= \frac{56}{5}$$
$$= 11.2$$

Consider $\bar{x} = \frac{1}{m}\int_0^3 \int_0^{\frac{1}{3}y} \int_0^{\sqrt{9-y^2}} x\left(x^2+y^2\right)\ dz\ dx\ dy$

> $\frac{1}{11.2}\int_0^3 \int_0^{\frac{y}{3}} \int_0^{\sqrt{9-y^2}} x\cdot(x^2+y^2)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y;$

$0.1192801339\,\pi$

$\xrightarrow{\text{at 5 digits}}$

0.37473

Therefore $\bar{x} = \frac{1}{m}\int_0^3 \int_0^{\frac{1}{3}y} \int_0^{\sqrt{9-y^2}} x\left(x^2+y^2\right)\ dz\ dx\ dy = \boxed{0.375}$

Consider $\bar{y} = \frac{1}{m}\iiint_E y\rho(x,y,z)\ dV$

> $\frac{1}{11.2}\int_0^3\int_0^{\frac{y}{3}}\int_0^{\sqrt{9-y^2}} y\cdot(x^2+y^2)\,dz\,dx\,dy;$

$0.7031250000\,\pi$

$\xrightarrow{\text{at 5 digits}}$

2.2089

Therefore $\bar{y} = \frac{1}{m}\int_0^3\int_0^{\frac{1}{3}y}\int_0^{\sqrt{9-y^2}} y\left(x^2+y^2\right)\ dz\ dx\ dy = \boxed{2.209}$

Consider $\bar{z} = \frac{1}{m}\iiint_E z\rho(x,y,z)\ dV$

> $\frac{1}{11.2}\int_0^3\int_0^{\frac{y}{3}}\int_0^{\sqrt{9-y^2}} z\cdot(x^2+y^2)\,dz\,dx\,dy;$

0.9375000000

Therefore $\bar{z} = \frac{1}{m}\int_0^3\int_0^{\frac{1}{3}y}\int_0^{\sqrt{9-y^2}} z\left(x^2+y^2\right)\ dz\ dx\ dy = \boxed{0.938}$

Therefore center of the mass is $(\bar{x},\bar{y},\bar{z}) = \boxed{(0.375, 2.209, 0.938)}$

(c)

Find the moment of inertia about the *z*-axis.

To find the moment of inertia about the *z*-axis, use the following integral.

$I_z = \iiint_E \left(x^2+y^2\right)\rho(x,y,z)\ dV$

$I_z = \int_0^3\int_0^{\frac{1}{3}y}\int_0^{\sqrt{9-y^2}} \left(x^2+y^2\right)\left(x^2+y^2\right)\ dz\ dx\ dy$

> $\int_0^3\int_0^{\frac{y}{3}}\int_0^{\sqrt{9-y^2}} (x^2+y^2)^2\,dz\,dx\,dy;$

$\frac{10464}{175}$

$\xrightarrow{\text{at 5 digits}}$

59.794

Therefore $I_z = \iiint_E \left(x^2+y^2\right)\rho(x,y,z)\ dV$

$= \boxed{59.794}$

Chapter 15 Multiple Integrals 15.7 51E

$f(x,y,z) = c\,xyz$, in E, where

$E = \{(x,y,z):\ 0 \le x \le 2,\ 0 \le y \le 2,\ 0 \le z \le 2\}$

And $f(x,y,z) = 0$, otherwise

(A)

Now $f(x,y,z)$ is the joint density function

Then $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx = 1$

i.e. $\int_0^2\int_0^2\int_0^2 c\,xyz\,dz\,dy\,dx = 1$

i.e. $c\int_0^2 x\,dx\int_0^2 y\,dy\int_0^2 z\,dz = 1$

i.e. $\frac{c}{8}\left(x^2\right)_0^2\left(y^2\right)_0^2\left(z^2\right)_0^2 = 1$

i.e. $\frac{c}{8}(4)(4)(4) = 1$

i.e. $c = \frac{8}{64} = \boxed{\frac{1}{8}}$

(B)

$$P(X \le 1, Y \le 1, Z \le 1) = \int_0^1\int_0^1\int_0^1 c\,xyz\,dz\,dy\,dx$$

$$= c\int_0^1 x\,dx\int_0^1 y\,dy\int_0^1 z\,dz$$

$$= \frac{c}{8}\left(x^2\right)_0^1\left(y^2\right)_0^1\left(z^2\right)_0^1$$

$$= \frac{c}{8}(1)(1)(1) = \boxed{\frac{1}{64}}$$

(As $c = 1/8$ from part (A))

(C)

$P(X+Y+Z \le 1)$

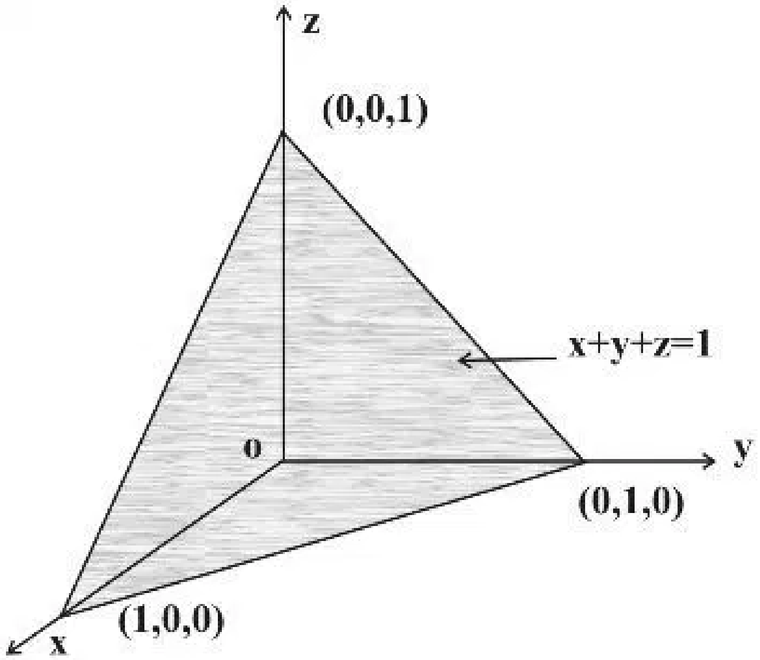
We need to find the probability such that

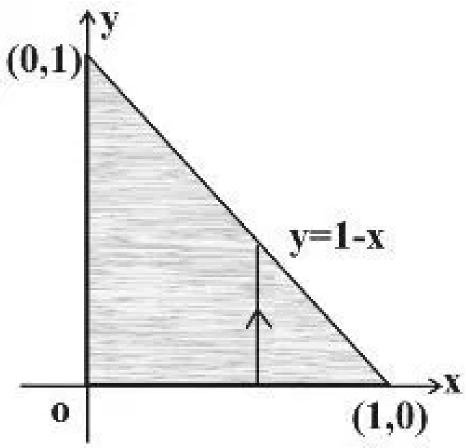
$X+Y+Z \le 1$

i.e. $P(X+Y+Z \le 1) = P((X,Y,Z)\,\varepsilon\,E)$

Where E is the tetrandron bounded by planes

$x = 0$, $y = 0$, $z = 0$, $x+y+z = 1$





Then $E=\{(x,y,z): 0\le x\le 1,\ 0\le y\le 1-x,\ 0\le z\le 1-x-y\}$

Therefore $P(X+Y+Z\le 1)$

$$=\iiint_E f(x,y,z)\,dv$$

$$=c\int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx$$

$$= \frac{c}{2}\int_0^1\int_0^{1-x} xy(1-x-y)^2\,dy\,dx$$

$$= \frac{c}{2}\int_0^1\int_0^{1-x} \left(xy+x^3y+xy^3-2x^2y-2xy^2+2x^2y^2\right)dy\,dx$$

$$= \frac{c}{2}\int_0^1\int_0^{1-x} \left(x(1-x)^2 y+2x(x-1)y^2+xy^3\right)dy\,dx$$

$$= \frac{c}{2}\int_0^1\left[x(1-x)^2\frac{y^2}{2}+2x(x-1)\frac{y^3}{3}+\frac{xy^4}{4}\right]_{y=0}^{y=1-2}dx$$

$$= \frac{c}{2\times 12}\int_0^1\left[6x(1-x)^4-8x(1-x)^4+3(1-x)^4x\right]dx$$

i.e. $P(X+Y+Z\le 1) = \frac{c}{24}\int_0^1 x(1-x)^4\,dx$

$$= \frac{c}{24}\int_0^1\left(x+x^5+6x^3-4x^4-4x^2\right)dx$$

$$= \frac{c}{24}\left(\frac{x^2}{2}+\frac{x^6}{6}+\frac{3}{2}x^4-\frac{4}{5}x^5-\frac{4}{3}x^3\right)_0^1$$

$$= \frac{c}{24}\left[\frac{1}{2}+\frac{1}{6}+\frac{3}{2}-\frac{4}{5}-\frac{4}{3}\right]$$

$$= \frac{c}{24\times 30} = \frac{1}{8\times 24\times 30}$$

$$= \frac{1}{5760}$$

i.e. $P(X+Y+Z\le 1) = \boxed{\frac{1}{5760}}$

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$f(x,y,z) = ce^{-(0.5x+0.2y+0.1z)}$, in E where

$E = \{(x,y,z):\ x\ge 0, y\ge 0, z\ge 0\}$

And $f(x,y,z)=0$, otherwise

(A)

Now $f(x,y,z)$ is the joint density function

Then $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx = 1$$

i.e. $$c\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-0.5x}.e^{-0.2y}.e^{-0.1z}\,dz\,dy\,dx = 1$$

i.e. $$c\int_0^{\infty} e^{-0.5x}dx\int_0^{\infty} e^{-0.2y}dy\int_0^{\infty} e^{-0.1z}dz = 1$$

i.e. $$\frac{c}{0.5}\left(e^{-0.5x}\right)_0^{\infty}.\frac{\left(e^{-0.2y}\right)_0^{\infty}}{-0.2}.\frac{\left(e^{-0.1z}\right)_0^{\infty}}{-0.1} = 1$$

i.e. $$\frac{c(0-1)}{(-0.5)}.\frac{(0-1)}{(-0.2)}.\frac{(0-1)}{(-0.1)} = 1$$

i.e. $$\frac{c}{(0.5)(0.2)(0.1)} = 1$$

i.e. $$\boxed{c = 0.01 = \frac{1}{100}}$$

(B)

$$P(X \le 1, Y \le 1) = \int_0^1\int_0^1\int_0^{\infty} c\,e^{-0.5x}.e^{-0.2y}.e^{-0.1z}\,dz\,dy\,dx$$

$$= c\int_0^1 e^{-0.5x}dx\int_0^1 e^{-0.2y}dy\int_0^{\infty} e^{-0.1z}dz$$

$$= c\frac{\left(e^{-0.5x}\right)_0^1}{(-0.5)}.\frac{\left(e^{-0.2y}\right)_0^1}{(-0.2)}.\frac{\left(e^{-0.1z}\right)_0^{\infty}}{(-0.1)}$$

$$= c\frac{\left(e^{-0.5}-1\right)}{(-0.5)}.\frac{\left(e^{-0.2}-1\right)}{(-0.2)}.\frac{(0-1)}{(-0.1)}$$

$$= c\frac{\left(1-e^{-0.5}\right)}{0.5}.\frac{\left(1-e^{-0.2}\right)}{0.2}.\frac{1}{0.1}$$

$$= \frac{0.01}{0.01}\times(0.3934)(0.18126)$$

$$= 0.07132$$

Hence $\boxed{P(X \le 1, Y \le 1) = 0.07132}$

(C)

$$P(X \le 1, Y \le 1, Z \le 1) = \int_0^1\int_0^1\int_0^1 c\,e^{-0.5x}.e^{-0.2y}.e^{-0.1z}.dz\,dy\,dx$$

$$= c\int_0^1 e^{-0.5x}dx\int_0^1 e^{-0.2y}dy\int_0^1 e^{-0.1z}dz$$

$$= c\,\frac{\left(e^{-0.5x}\right)_0^1}{(-0.5)}.\frac{\left(e^{-0.2y}\right)_0^1}{(-0.2)}.\frac{\left(e^{-0.1z}\right)_0^1}{(-0.1)}$$

$$= \frac{0.01}{(0.5)(0.2)(0.1)}\left(1-e^{-0.5}\right)\left(1-e^{-0.2}\right)\left(1-e^{-0.1}\right)$$

$$= 0.006787$$

Hence $\boxed{P(X \le 1, Y \le 1, Z \le 1) = 0.006787}$

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Now the region of integration is:

$$E = \{(x, y, z): 0 \le x \le L, 0 \le y \le L,\ 0 \le z \le L\}$$

And $f(x, y, z) = xyz$

Then
$$\iiint_E f(x, y, z)dv = \int_0^L\int_0^L\int_0^L xyz\,dz\,dy\,dx$$

$$= \int_0^L x\,dx\int_0^L y\,dy\int_0^L z\,dz$$

$$= \left(\frac{x^2}{2}\right)_0^L\left(\frac{y^2}{2}\right)_0^L\left(\frac{z^2}{2}\right)_0^L$$

$$= \frac{L^6}{8}$$

Now
$$v(E) = \int_0^L\int_0^L\int_0^L 1.dx\,dy\,dz$$

$$= (x)_0^L(y)_0^L(z)_0^L$$

$$= L^3$$

Then
$$f_{ave} = \frac{1}{v(E)}\iiint_E f(x, y, z)dv$$

$$= \frac{1}{L^3}\times\frac{L^6}{8}$$

$$= \boxed{\frac{L^3}{8}}$$

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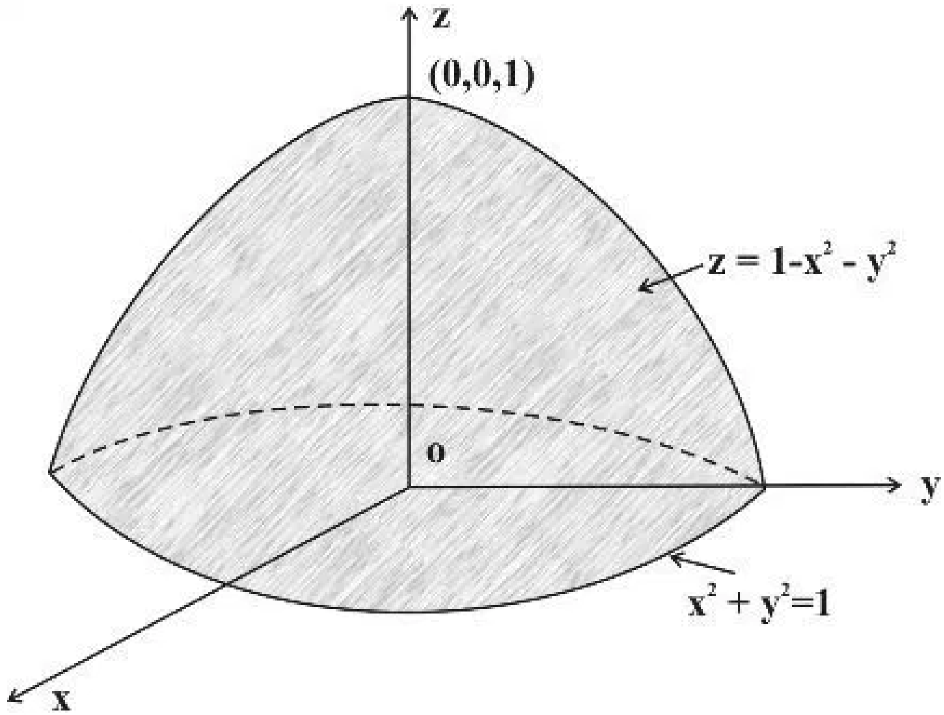

The paraboloid $z = 1 - x^2 - y^2$ meets $z = 0$ in a circle $x^2 + y^2 = 1$. Thus the region E is the solid under the parabolic $z = 1 - x^2 - y^2$ and above the circle $x^2 + y^2 = 1$

Now $f(x, y, z) = x^2 z + y^2 z$

$= z(x^2 + y^2)$

Then $\iiint_E f(x, y, z)\,dv = \iiint_E z(x^2 + y^2)\,dv$

It is easier to convert to polar co – ordinates in xy – plane. This gives

$$\iiint_E z(x^2 + y^2)\,dv$$

$$= \iint_D \left[(x^2 + y^2)\frac{z^2}{2}\right]_{z=0}^{z=1-x^2-y^2} dA$$

Where D is the circular disk given by

$$D = \{(r, \theta),\ 0 \le \theta \le 2\pi,\ 0 \le r \le 1\}$$

$$= \frac{1}{2}\iint \left(x^2+y^2\right)\left[1-\left(x^2+y^2\right)\right]^2 dA$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 r^2\left(1-r^2\right)^2 r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 r^3\left(1+r^4-2r^2\right)dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(r^7-2r^5+r^3\right)dr\,d\theta$$

$$= \frac{1}{2}\left[\frac{r^8}{8}-\frac{1}{5}r^6+\frac{1}{4}r^4\right]_0^1 (\theta)_0^{2\pi}$$

$$= \frac{1}{2}\left[\frac{1}{24}\right][2\pi]$$

$$= \frac{\pi}{24}$$

Also $v(E) = \iiint_E 1.dv$

$$= \iint_D (z)_0^{1-x^2-y^2}\,dA$$

$$= \iint_D \left[1-\left(x^2+y^2\right)\right]dA$$

$$= \int_0^{2\pi}\int_0^1 \left(1-r^2\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(r-r^3\right)dr\,d\theta$$

$$= \left(\frac{r^2}{2}-\frac{r^4}{4}\right)_0^1 (\theta)_0^{2\pi}$$

$$= \frac{1}{4}\times 2\pi = \frac{\pi}{2}$$

Thus $f_{ave} = \frac{1}{v(E)}\iiint_E f(x,y,z)\,dv$

$$= \frac{\pi}{24}\times\frac{2}{\pi}$$

$$= \frac{1}{12}$$

Hence $\boxed{f_{ave} = \frac{1}{12}}$

Consider $\iiint\limits_{E}\left(1-x^2-2y^2-3z^2\right)dV$

The region E is bounded by ellipsoid

$$x^2+2y^2+3z^2=1$$

So,

$$z^2=\frac{1}{3}\left(1-x^2-2y^2\right)$$
$$z=\pm\frac{1}{\sqrt{3}}\sqrt{1-x^2-2y^2}$$
$$y^2=\frac{1}{2}\left(\sqrt{1-x^2}\right)$$
$$y=\pm\sqrt{\frac{1-x^2}{2}}$$
$$x^2=\frac{1}{2}$$
$$x=\pm\frac{1}{2}$$

Thus, the region is

$$E=\left\{(x,y,z)\middle|-1\le x\le 1,\ -\sqrt{\frac{1-x^2}{2}}\le y\le\sqrt{\frac{1-x^2}{2}},\ -\sqrt{\frac{1-x^2-2y^2}{3}}\le z\le\sqrt{\frac{1-x^2-2y^2}{3}}\right\}$$

We have

$$\int_{-1}^{1}\int_{-\sqrt{\frac{1-x^2}{2}}}^{\sqrt{\frac{1-x^2}{2}}}\int_{-\sqrt{\frac{1-x^2-2y^2}{3}}}^{\sqrt{\frac{1-x^2-2y^2}{3}}}\left(1-x^2-2y^2-3z^2\right)dzdydx=2\int_{-1}^{1}\int_{-\sqrt{\frac{1-x^2}{2}}}^{\sqrt{\frac{1-x^2}{2}}}\left[\left(1-x^2-2y^2\right)z-z^3\right]_0^{\sqrt{\frac{1-x^2-2y^2}{3}}}dydx$$

$$=2\int_{-1}^{1}\int_{-\sqrt{\frac{1-x^2}{2}}}^{\sqrt{\frac{1-x^2}{2}}}\left[\left(1-x^2-2y^2\right)^{\frac{3}{2}}\sqrt{\frac{1-x^2-2y^2}{3}}-\left(\frac{1-x^2-2y^2}{3}\right)^{\frac{3}{2}}\right]dydx$$

$$=\frac{4}{3\sqrt{3}}\int_{-1}^{1}\int_{-\sqrt{\frac{1-x^2}{2}}}^{\sqrt{\frac{1-x^2}{2}}}\left(1-x^2-2y^2\right)^{\frac{3}{2}}dydx$$

$$=\frac{8}{3\sqrt{3}}\int_{-1}^{1}\int_{0}^{\sqrt{\frac{1-x^2}{2}}}\left(1-x^2-2y^2\right)^{\frac{3}{2}}dydx$$

By using computer algebra system, the maximum value if the triple integrant is $\frac{4\sqrt{6}\pi}{45}$