Lecture - 8

Time Varying Fields and Maxwell's Equation

Static Fields - Maxwell's Equations:-

Integral Form	Point form
i) $\oint D\cdot ds = Q$	$\nabla\cdot D = \rho_V$
ii) $\oint E\cdot dl = 0$	$\nabla\times E = 0$
iii) $\oint B\cdot ds = 0$	$\nabla\cdot B = 0$
iv) $\oint H\cdot dl = I$	$\nabla\times H = J$

Note:-

- The surface ~~and~~ Integral and Divergence expressions are consistently the same in static / time varying fields
- The line integral and curl expression are modified in time varying fields to explain AC voltages and currents

Open Integral's (Not Maxwell Equations):-

1. $\int D\cdot ds = \Psi_e = \text{coulombs}$
2. $\int E\cdot dl = V = \text{EMF} = \text{Volts}$
3. $\int B\cdot ds = \Psi_m = \text{Webers}$
4. $\int H\cdot dl = V_m = \text{MMF} = \text{Amps}$

time

Maxwell 2nd Equation and Faraday's Law:-

Faraday's Law Statement:-

EMF or voltage is induced even in a closed conductor when the magnetic flux crossing the surface changes with time.

i.e. Rate of change of magnetic flux is equal to induced EMF

$$\oint E\cdot dl = V = \overset{\text{due to Lenz's Law}}{-}\frac{d\psi_m}{dt} = -\frac{d}{dt}\int B\cdot ds$$

$$= \int -\frac{\partial B}{\partial t}\cdot ds$$

Lenz's Law :-

The induced EMF ~~allows~~ always opposes the basic changing flux (Cause)

$$\oint E\cdot dl = \int -\frac{\partial B}{\partial t}\cdot ds$$

$$\oint E\cdot dl = V = -\frac{d\psi_m}{dt} = -\frac{d}{dt}\int B\cdot ds = \int -\frac{\partial B}{\partial t}\cdot ds$$

Apply stoke's theorem $= \int \nabla \times E \cdot ds$

$$\boxed{\nabla \times E = -\frac{\partial B}{\partial t}}$$ Modified Maxwell's Second Equation

Note:-

Potential is unique at a time at a point in a space but changes with time and Hence the modification

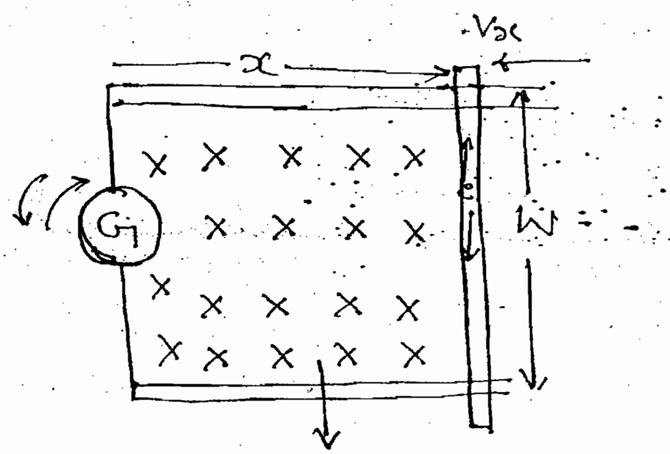
Sliding Rail Experiment :-

$$V = -\frac{d\psi_m}{dt}$$

$$\Rightarrow V = \frac{-d}{dt}(B\cdot A)$$

$$\Rightarrow \boxed{V = -B\cdot W \frac{dx}{dt}}$$

$$\Rightarrow \boxed{V = -B\cdot W\cdot V_x}$$



Static Uniform B field

Note:-

→ When a conducting rod is moving in a magnetic field a force is exist on the electron obeying Lorentz Law. This displaces the electrons towards one side. This accumulation is called as induced voltage

→ As the rod moves to and fro the voltage polarity changes. This is called as AC voltage.

→ The already displaced $\bar{e}$ opposes another further coming $\bar{e}$'s. This is called as Lenz's law

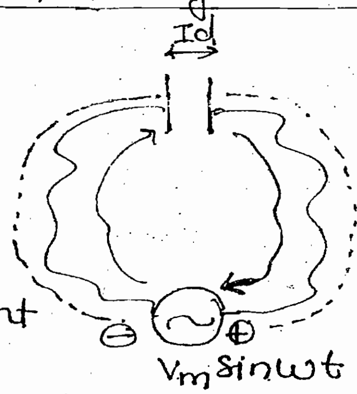
$$F_y = \cancel{q}\,(V_x \times B_z) = \cancel{q}E$$

$$\Rightarrow \frac{dx}{dt} B \cdot \sin\theta = \frac{V}{W} \quad \cancel{E}c$$

$$\Rightarrow V = \frac{d}{dt}(xBW) = \frac{d}{dt}(BA) \quad (\theta = 90^\circ)$$

$$\Rightarrow V = \frac{d\psi_m}{dt}$$

Maxwell's IV Equation and Inconsistency of Ampere's Law:-



→ When a capacitor is connected with AC harmonic voltage there is a current flowing in the wire and plates obeying ohm's law like any linear element

$$I = C\frac{dV}{dt}$$

$$I = C \cdot W \cdot V_m \sin(\omega t + 90^\circ)$$

$e^{j90} = j$ → Phase   Phase

$$I = j\omega C\, V_m \sin \omega t$$

$$V = \left(\frac{1}{j\omega C}\right) I$$

→ There is no current flowing b/w plates as their is dielectric Hence the circuit is not closed But Ampere's law state that current flows only

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closed circuits. Hence Maxwell's modified Ampere's law as

$$\oint H \cdot dl = I_c + I_d$$

$$\nabla \times H = J_c + J_d$$

Using equation of continuity

$$\nabla \cdot J_d = \frac{\partial \rho_v}{\partial t} = \frac{\partial (\nabla \cdot D)}{\partial t} = \nabla \cdot \frac{\partial D}{\partial t} \Rightarrow$$

$$\Rightarrow J_d = \frac{\partial D}{\partial t} \quad \& \quad I_d = \int \frac{\partial D}{\partial t} \cdot ds$$

Maxwell IV Equation is

$$\oint H \cdot dl = I_c + \int \frac{\partial D}{\partial t} \cdot ds$$

$$\Rightarrow \nabla \times H = J_c + \frac{\partial D}{\partial t}$$

where I_c = Conduction current

OR

= Moving $\bar{e}$'s

→ When an AC voltage applied to the capacitor plate the plates are alternatively charge and discharge. This continues takes place as the polarity changes. Hence this self establish continuity

→ As ρ_s on the plates changes with time, D b/w the plates changes with time. Hence $\frac{\partial D}{\partial t}$ is

$$\frac{C/m^2}{\text{second}} = Amp/m^2 = J_d$$

This is also a format of current

This is called as Displacement current density.

Summary 1:—

$$\oint E \cdot dl = \int -\frac{\partial B}{\partial t} \cdot ds$$

$$\oint H \cdot dl = \int \frac{\partial D}{\partial t} \cdot ds + J_c$$

→ A time varying magnetic flux is a cause of voltage

→ A time varying electric flux is a format of current

$$\boxed{\frac{\text{Weber}}{\text{second}} = \text{Volts}} \qquad \boxed{\frac{\text{Coulomb}}{\text{Sec}} = \text{Amp}}$$

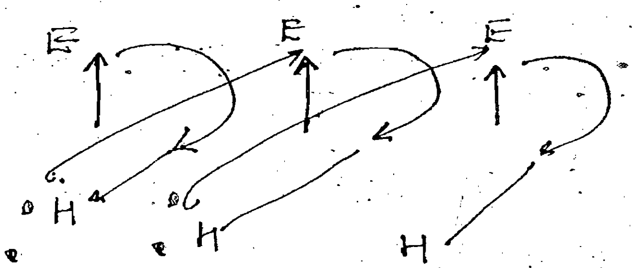
$$\begin{array}{ccc} \text{cause} & & \text{Effect} \\ \text{Voltage} & \longrightarrow & \text{Electric field} \\ \text{time} \uparrow & & \downarrow \text{time} \\ \text{H} \cdot \text{Field} & \longleftarrow & \text{current} \\ \text{Effect} & & \text{cause} \end{array}$$

Summary 2:–

Space varying Electric field ← $\underline{\nabla \times E} = -\frac{\partial B}{\partial t}$ → time varying B field

$$\nabla \times H = \frac{\partial D}{\partial t} + J_c$$

→ A time varying E field produces a space varying orthogonal H field and vice-versa



$$\text{H field} \underset{\text{space}}{\overset{\text{time}}{\rightleftarrows}} \text{E} \cdot \text{Field}$$

→ Accumulation leads to flow and flow leads to accumulation sustaining each other.

Summary 3:–

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

$$\nabla \times H = \epsilon \frac{\partial E}{\partial t} + \sigma E$$

→ μ = Henry/m , ϵ = Farad/m , σ = mho/m →

For E to transform to H and vice-versa the material and its permitting abilities are also important.

i.e. Material consists decides the E/H dynamics in the medium

→ $\vec{E}$ - Volts/m , $\vec{H}$ - amps/m , ∇ - perm

Note:-

→ For sustain oscillation or ever existing phenomenon are produc when the time derivative of E and H is always back the same function and such a function called as Harmonic function.

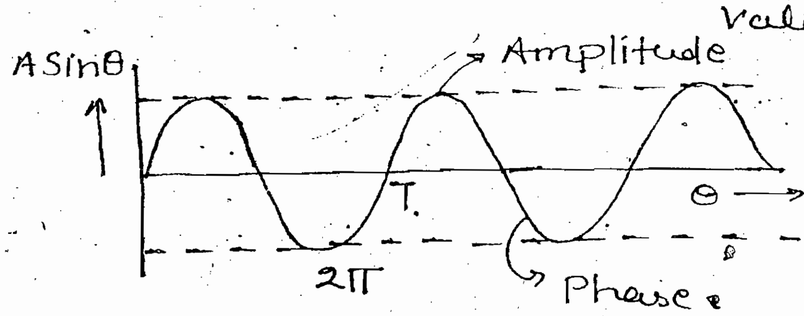
Harmonic Functions:-

They are 2 dimension quantities and having 3 formats

$$\left.\begin{array}{l} \to \ A\sin\theta \\ \to \ A\cos\theta \\ \to \ A e^{j\theta} \end{array}\right\}$$

Amplitude → Domain 1 → Peak value

Phase → Domain 2 → Instanteous value



Property 1:-

The phase should be a linear function of the variable

(I) $\theta \propto t$ t→time Harmonic

$\Rightarrow \theta = \omega t$ ω = Phase shift constant per unit time

$= \frac{2\pi}{T}$ rad/second

(II) $\theta \propto z$ → Space Harmonic

$\Rightarrow \theta = \beta z$ β = Phase ~~e~~ shift constant per unit length = $\frac{2\pi}{\lambda}$ rad/m

Property 2 :—

The derivative of every harmonic has to be back the same function shifted orthogonally by 90°

$$A \sin \omega t \xrightarrow{\text{I-derivative}} \omega A \sin(\omega t + 90^\circ) \xrightarrow[\text{der.}]{\text{II}} \omega^2 A \sin(\omega t + 180^\circ)$$

$$\xrightarrow{\text{III derivative}} \omega^3 A \sin(\omega t + 270)$$

$$A e^{j\omega t} \xrightarrow[\text{der}]{\text{I}} \omega A e^{j(\omega t + 90)} \xrightarrow[\text{der.}]{\text{II}} \omega^2 A e^{j(\omega t + 180)}$$

$$\xrightarrow{\text{III-der.}} \omega^3 A e^{j(\omega t + 270)}$$

$(\because j^3 = -j = e^{j270})$

$(-1 = e^{j180})$

$(\because j = e^{j90})$

→ All harmonics obey the basic property the second order derivative is back the same function

→ They satisfied the differential equation

$$D^2 - M^2 = 0$$

$$D^2 + M^2 = 0$$

eg:- E/H field equation in free space

Electro magnetic wave equation in free space

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

$$\nabla \times H = \epsilon \frac{\partial E}{\partial t}$$

Taking ∇ on both sides

$$\nabla \times (\nabla \times H) = \epsilon \frac{\partial}{\partial t}(\nabla \times E)$$

$$-\nabla^2 H = \epsilon \frac{\partial}{\partial t}\left(-\mu \frac{\partial H}{\partial t}\right)$$

$$\boxed{\begin{aligned} \nabla^2 H &= \epsilon \mu \frac{\partial^2 H}{\partial t^2} \\ \nabla^2 E &= \epsilon \mu \frac{\partial^2 E}{\partial t^2} \end{aligned}}$$

Space Harmonic — Time Harmonic

These are called as E/H wave equations in free space

<u>eg-(2)</u> :- V/I Equations in LC circuits

$$I = C\frac{dV}{dt}$$

$$V = -L\frac{dI}{dt} = -L\frac{d}{dt}\left(C\frac{dV}{dt}\right)$$

$$\Rightarrow \quad V = -LC\frac{d^2V}{dt^2}$$

$$\Rightarrow \quad \boxed{\frac{d^2V}{dt^2} = \frac{-1}{LC}V}$$

and $\boxed{\dfrac{d^2I}{dt^2} = \dfrac{-1}{LC}I}$

By comparison

$$\boxed{\omega = \frac{1}{\sqrt{LC}}}$$

<u>Types of Exponential Functions</u> :-

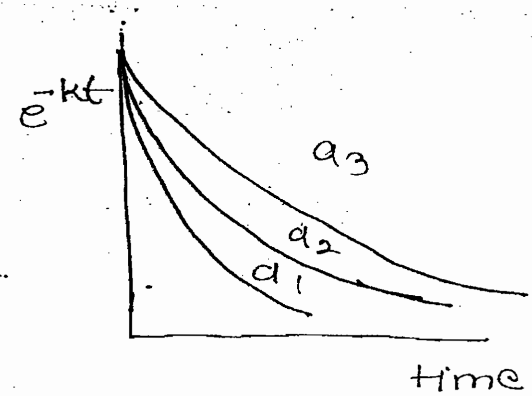
→ e^{-Kt} vs t

(1) <u>Case -(1)</u> :-

$K = a =$ +ve real no.

$a_1 > a_2 > a_3$

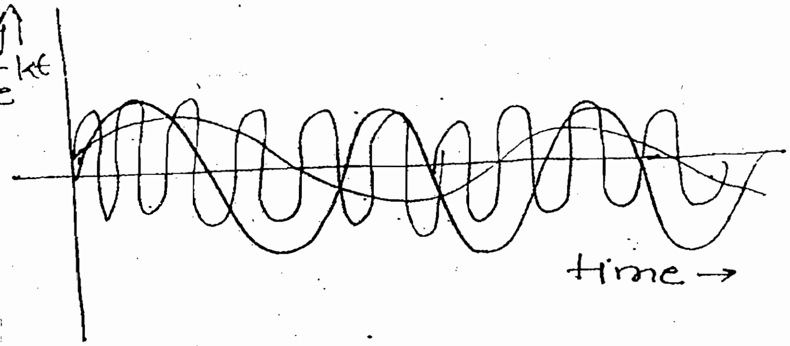
→ Real Exponential.



ii) Case-(II) :-

$k = j\omega$ = purely imaginary no.

ω = Phase Shift Constant

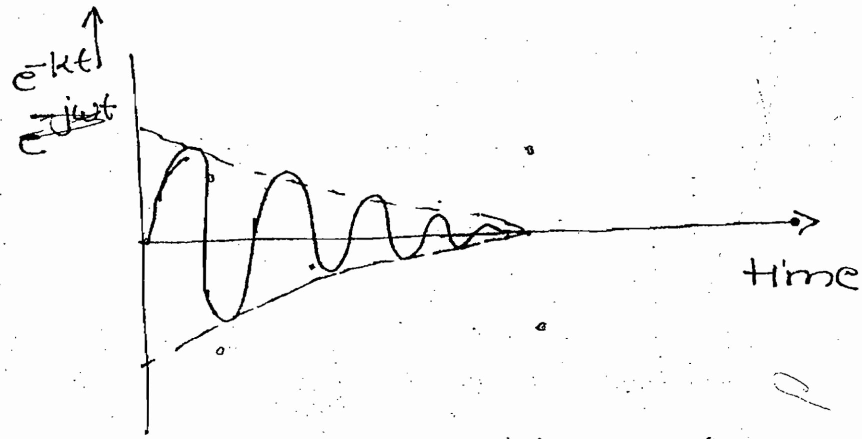


iii) Case-(III) :-

$k = a + j\omega$

$(e^{-at})\,(e^{-j\omega t})$

$A \cdot e^{j\theta}$



Note :-

→ j stands for an orthogonal shift from domain 1 to a second independent domain 2

eg:- (i) east and north displacements

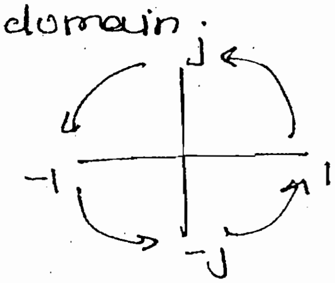
(ii) Resistance and reactance of a impedance

(iii) Amplitude and phase of a harmonic

→ A scaling of j in domain 1 is a shift of 90° in domain 2

eg:- (i) $j\omega\cancel{t}\, e^{-j\omega t} \longrightarrow \omega\, e^{j(\omega t + 90^\circ)}$

(ii) s-domain :



$1 \times j = j = 1\angle 90^\circ$

$j \times j = -1 = 1\angle 180^\circ$

$-1 \times j = -j = 1\angle 270^\circ$

$-j \times j = 1\angle 360^\circ$

EM Propogation in General Materials :- (σ, ϵ, μ)

Using Maxwell's Equation

$$\nabla \times H = \sigma E + \epsilon \frac{\partial E}{\partial t}$$

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

→ Every EM wave needs a time harmonic source at one end

$$\left.\begin{aligned} E_s &= E_0 e^{j\omega t} \\ H_s &= H_0 e^{j\omega t} \end{aligned}\right\} \quad \omega = \text{frequency in rad/s}$$

Range of ω :-

- f/ω → Hz/kHz → AF Wave
- → MHz → Radio frequency Wave
- → GHz/10^{12}Hz → Microwave
- → $10^{15} - 10^{18}$ Hz → Light wave
- → $10^{20} - 10^{22}$ Hz → X-rays / Gamma rays
- → 10^{25} Hz → Cosmic Rays

→ EM Wave Spectrum

$$\nabla \times H = \sigma E + \epsilon \frac{\partial E}{\partial t} = \sigma \underline{E_0 e^{j\omega t}} + \epsilon j\omega \underline{E_0 e^{j\omega t}}$$

$$\cancel{\nabla \times E = -\mu \frac{\partial H}{\partial t}} \qquad = (\sigma + j\omega\epsilon) E$$

$$\Rightarrow \quad \nabla \times H = (\sigma + j\omega\epsilon) E \quad \text{--- (1)}$$

$$\nabla \times E = -j\omega \mu H \quad \longrightarrow \text{(II)}$$

$$\nabla \times E = \mu \frac{\partial H}{\partial t}$$

$$= -\mu j\omega H_0 e^{j\omega t} = -j\omega\mu H$$

$$\nabla \times E = -j\omega\mu H \quad \text{--- (II)}$$

Eq-(I) & (II) are called as Maxwell Equations for time Harmonic source

Take H from (II) & put in (I)

$$H = \frac{\nabla \times E}{-j\omega H}$$

$$\Rightarrow \quad \nabla \times \left(\frac{\nabla \times E}{-j\omega\mu} \right) = (\sigma + j\omega\epsilon)E$$

$$\Rightarrow \quad \nabla \underbrace{(\nabla \cdot E)}_{=0} - \nabla^2 E = -j\omega\mu(\sigma + j\omega\epsilon)E$$

For a charge free region

$$\nabla^2 E = j\omega\mu(\sigma + j\omega\epsilon)E \quad \text{--- (III)}$$

Similarly

$$\underbrace{\nabla^2 H}_{\text{Space}} = j\omega\mu(\sigma + j\omega\epsilon)\underbrace{H}_{\text{time}} \quad \text{--- (IV)}$$

These are called as Helmholtz's Propogation Equations.

Note :-

→ If the source is the time harmonic then the effect is space harmonic propogation in the medium.