ગુજરાત રાજ્યના શિક્ષણવિભાગના પત્ર-ક્રમાંક
મશબ / 1211 / 414 / છ, તા.15-9-2011-થી મંજૂર

# PHYSICS

## Standard 11

## (Semester II)

### PLEDGE

India is my country.
All Indians are my brothers and sisters.
I love my country, and I am proud of its rich and varied heritage.
I shall always strive to be worthy of it.
I shall respect my parents, teachers and all my elders and treat everyone with courtesy.
I pledge my devotion to my country and its people.
My happiness lies in their well-being and prosperity.

રાજ્ય સરકારની વિનામૂલ્યે યોજના હેઠળનું પુસ્તક

Gujarat State Board of School Textbooks
'Vidyayan', Sector 10-A, Gandhinagar-382 010

**Authors-Editors**

Dr. P. N. Gajjar (Convenor)
Dr. V. P. Patel
Prof. M. S. Rami
Dr. Arun P. Patel
Dr. Deepak H. Gadani
Shri Pankaj Chavda

**Reviewers**

Dr. Mukeshkumar M. Jotani
Dr. Pankaj B. Thakor
Dr. Nisarg K. Bhatt
Dr. Maheshkumar C. Patel
Dr. Hiren H. Rathod
Dr. Sanjay D. Jhala
Shri Jayesh D. Darji

**Language Correction**

Shri V. Balakrishnan

**Artist**

Shri G. V. Mevada

**Co-ordinator**

Shri Chirag H. Patel
(Subject Co-ordinator : Physics)

**Preparation and Planning**

Shri Haresh S. Limbachiya
(Dy. Director : Academic)

**Layout and Planning**

Shri Haresh S. Limbachiya
(Dy. Director : Production)

## PREFACE

The Gujarat State Secondary and Higher Secondary Education Board has prepared new syllabi in accordance with the new national syllabi prepared by N.C.E.R.T. based on N.C.F. 2005 and core-curriculum. These syllabi are sanctioned by the Government of Gujarat.

It is a pleasure for the Gujarat State Board of School Textbooks, to place before the students this textbook of **Physics, Standard 11, (Semester II)** prepared according to the new syllabus.

Before publishing the textbook, its manuscript has been fully reviewed by experts and teachers teaching at this level. Following suggestions given by teachers and experts, we have made necessary changes in the manuscript before publishing the textbook.

The Board has taken special care to ensure that this textbook is interesting, useful and free from errors. However, we welcome any suggestion from people interested in education, to improve the quality of the textbook.

| | |
|---|---|
| **Dr. Bharat Pandit** | **Sujit Gulati** IAS |
| Director | Executive President |
| Date : 05-08-2015 | Gandhinagar |

First Edition : 2011, Reprint : 2012, 2013, 2014, 2015

**Published by** : Bharat Pandit, Director, on behalf of Gujarat State Board of School Textbooks, 'Vidyayan', Sector 10-A, Gandhinagar

**Printed by** :

# FUNDAMENTAL DUTIES

It shall be the duty of every citizen of India : *

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practice heritages derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers and wild-life, and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(I) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;

(k) to provide opportunities for education by the parent or the gurdian, to his child or a ward between the age of 6 and 14 years as the case may be.

*Constitution of India : Section 51 A

# I N D E X



## About This Textbook...

We have pleasure in presenting this textbook of physics of Standard 11 to you. This book is on the syllabi based on the courses of National Curriculum Framework (NCF), Core-Curriculum and National Council of Educational Research and Training (NCERT) and has been sanctioned by the State Government keeping in view the National Education Policy.

The State Government has implemented the semester system in Standard 11. The semester system will reduce the educational load of the students and increase the interest towards study.

In this Textbook of Physics for Standard-11, Eight chapters are included in Semester I and Semester II each, looking into the depth of the topics, time which will be available for classroom teaching, etc...

The real understanding of the theories of physics is obtained only through solving related problems. Hence, for the new concept, solved problems are given. One of the positive sides of the book is that at the end of each chapter extended summary is given. On the basis of this one can see the whole contents of the chapter at a glance.

Keeping in view the formats of various entrance test conducted on all India basis, we have included MCQs, Short questions, objective questions and problems in this book. At the end of the book, Hints for solving the problems are also included so that students themselves can solve the problems. The Appendices given at the end of the book will also be very useful.

This book is published in quite a new look in four-colour printing so that the figures included in the book are much clear. It has been observed, generally, that students do not preserve old text-books, once they go to the higher standard. In the semester system, each semester has its own importance and the look of the book is also very nice so the students would like to preserve this book and it will become a reference book in future.

The previous textbook got excellent support from students, teachers and experts. So a substantial portion from that book is taken in this book either in its original form or with some changes. We are thankful to that team of authors. We are also thankful to the teachers who remained present in the Review workshop and gave their inputs to make this textbook error-free.

Proper care has been taken by authors, subject advisors and reviewers while preparing this book to see that it becomes error-free and concepts are properly developed. We welcome suggestions and comments for the importance of the textbook in future.

**Authors/Editors**

# CHAPTER 1

# DYNAMICS OF A SYSTEM OF PARTICLES

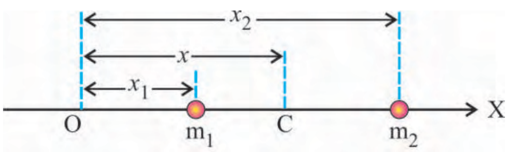


## 1.1 Introduction

In Semester-I, we have studied the linear motion of a particle. In this chapter, we will study about how to find out the centre of mass of a system of two particles, the centre of mass of a system of *n*-particles and the centre of mass of a rigid body. Further, we will study the kinetic theory of a system of particles in which the conservation of linear momentum is explained.

## 1.2 Centre of Mass of a System of Particles in One Dimension

As shown in Figure 1.1 consider two particles having mass $m_1$ and $m_2$ lying on X-axis at distances of $x_1$ and $x_2$ respectively from the origin (O).




Centre of mass of a system of two particles of masses

Figure 1.1

The centre of mass of this system is that point whose distance from origin O is given by

$$\therefore \; x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \qquad (1.2.1)$$

Here, $x$ is the mass-weighted average position of $x_1$ and $x_2$. If both particles are of the same mass, then $m_1 = m_2 = m$.

$$\therefore \quad x = \frac{mx_1 + mx_2}{m + m}$$

$$\therefore \quad x = \frac{x_1 + x_2}{2} \qquad (1.2.2)$$

Thus, **the centre of mass of the two particles of equal mass lies at the centre (on the line joining the two particles) between the two particles.**

Similarly, if $n$ particles of mass $m_1$, $m_2$,...., $m_n$ are lying on X-axis at distances $x_1$, $x_2$,...., $x_n$ respectively from the origin 'O', then the centre of mass of the system of $n$-particles is

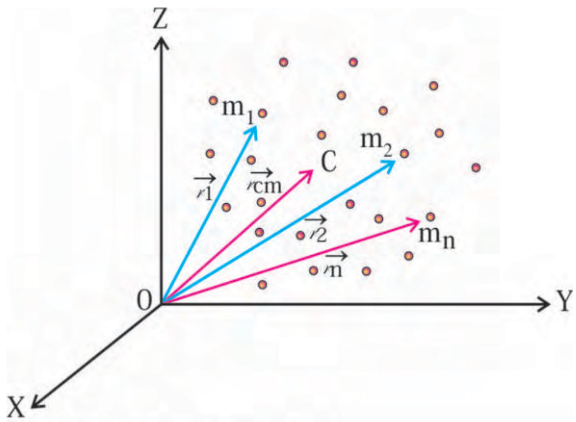
$$x = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n}$$

$$\therefore \quad x = \frac{\Sigma m_i x_i}{\Sigma m_i} \qquad (1.2.3)$$

$$\therefore \quad x = \frac{\Sigma m_i x_i}{M} \qquad (1.2.4)$$

Where $M = \Sigma m_i$ = total mass of the system of $n$-particles.

## 1.3 Centre of Mass of a System of *n*-Particles in Three Dimensions




System of *n*-particles in three dimensions

Figure 1.2

Figure 1.2 shows a system of $n$-particles in three dimensions. Let the position vectors of the particles of mass $m_1$, $m_2$, .... , $m_n$, with respect to the origin 'O' of the co-ordinate system are $\vec{r}_1$, $\vec{r}_2$, ...., $\vec{r}_n$, respectively. The position vector of centre of mass of the system is given by following equation.

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n}{m_1 + m_2 + \ldots + m_n} \qquad (1.3.1)$$

$$\therefore \quad \vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n}{M}$$

or

$$M\vec{r}_{cm} = m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n \qquad (1.3.2)$$

where,

$$M = m_1 + m_2 + \ldots + m_n$$

= total mass of system of $n$-particles

### 1.3.1 Motion of Centre of Mass and Newton's Second Law of Motion :

If the mass of each particle of the system of $n$-particles does not change with time, then differentiating equation (1.3.2) with respect to time.

$$M\frac{d\vec{r}_{cm}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \ldots + m_n\frac{d\vec{r}_n}{dt}$$

$$\therefore \quad M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_n\vec{v}_n \qquad (1.3.3)$$

Here, $\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt}$ is the velocity of centre of mass, and $\vec{v}_1, \vec{v}_2$...... are the velocities of respective particles.

$$\therefore \quad M\vec{v}_{cm} = \vec{P}_1 + \vec{P}_2 + \ldots + \vec{P}_n \qquad (1.3.4)$$

$$\therefore \quad M\vec{v}_{cm} = \vec{P} \qquad (1.3.5)$$

Where $\vec{P}_1$, $\vec{P}_2$, .... are the momenta of respective particles, and

$\vec{P} = \vec{P}_1 + \vec{P}_2 + \ldots + \vec{P}_n$ is the total momentum of the system of $n$ particles.

Equation (1.3.5) shows that **the total linear momentum of system of particles is equal to the product of total mass of the system and velocity of the centre of mass of the system.**

Differentiating equation (1.3.4) with respect to time.

$$M\frac{d\vec{v}_{cm}}{dt} = \frac{d\vec{P}_1}{dt} + \frac{d\vec{P}_2}{dt} + .... + \frac{d\vec{P}_n}{dt}$$

$$\therefore M\frac{d\vec{v}_{cm}}{dt} = \vec{F}_1 + \vec{F}_2 + .... + \vec{F}_n = \vec{F} \quad (1.3.6)$$

$$= m_1\vec{a}_1 + m_2\vec{a}_2 + ..... + m_n\vec{a}_n \quad (1.3.7)$$

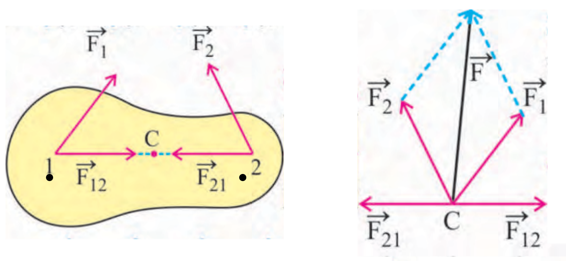
In equation (1.3.6) $\vec{F}_1$, $\vec{F}_2$, .... , $\vec{F}_n$ are the forces acting on the respective particles of the system and $\vec{F}$ is the resultant force. In equation (1.3.7), $\vec{a}_1$, $\vec{a}_2$, ...., $\vec{a}_n$ are the accelerations of the respective particles produced due to these forces.

From equation (1.3.5)

$$M\frac{d\vec{v}_{cm}}{dt} = \frac{d\vec{P}}{dt} = M\vec{a}_{cm} \quad (1.3.8)$$

The forces acting on the particles of a system are of two kinds :

(1) Internal forces prevailing among the particles of the system, and (2) External forces.




(*a*) (*b*)

Different types of forces acting on a system of two particles

Figure 1.3

For a system of two particles as shown in Figure 1.3, let the external forces acting on particles 1 and 2 are respectively $\vec{F}_1$ and $\vec{F}_2$, and the mutual forces of interaction acting between them are $\vec{F}_{12}$ and $\vec{F}_{21}$.

While discussing the overall motion of the system, all these forces may be considered to be acting on the centre of mass 'C' [See Figure 1.3(*b*)]. According to the Newton's Third Law of Motion, $\vec{F}_{12} = -\vec{F}_{21}$, and hence the resultant internal force becomes zero. Thus, in equation (1.3.6) the resultant force $\vec{F}$ acting on the system of particles is only the resultant external force. From equation (1.3.6) and (1.3.8).

$$M\frac{\vec{v}_{cm}}{dt} = M\vec{a}_{cm} = \vec{F} = \frac{d\vec{P}}{dt} \quad (1.3.9)$$

Equation (1.3.9) shows that **the resultant external force acting on a system is equal to the rate of change of total linear momentum of the system.** This is the Newton's Second Law of Motion for a system of particles. Further, equation (1.3.9) shows that **the centre of mass of the system moves under the influence of the resultant external force $\vec{F}$ as if the whole mass of the system is concentrated at its centre of mass.**

Newton's Second Law of Motion, for a particle, can be written without the help of the Third Law. But for a system of particles, the help of Newton's Third Law is required to obtain the Second Law. This fact is known as interdependence of Newton's Laws of Motion.

**Illustration 1 :** The particles of mass $m_1$, $m_2$ and $m_3$ are placed on the vertices of an equilateral triangle of sides '*a*'. Find the centre of mass of this system with respect to the position of particle of mass $m_1$.

**Solution :**

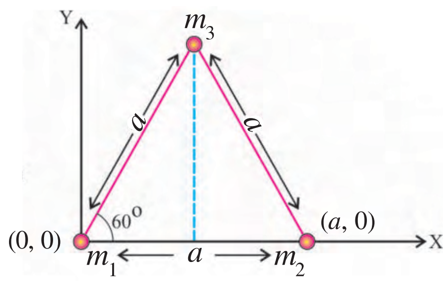



Figure 1.4

The angles of the three corners of an equilateral triangle are equal (60°). Hence, as shown in Figure (1.4) if we place the particle of mass $m_1$ at the origin (0, 0), and particle of mass $m_2$ along the X-axis at distance of '$a$' from the origin at ($a$, 0) position, then the co-ordinates of particle of mass $m_3$ are

$$(a\cos 60°, a\sin 60°) = \left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$$

Thus, the position vectors of the particles of mass $m_1$, $m_2$ and $m_3$ are respectively

$$\vec{r}_1 = (0, 0),\ \vec{r}_2 = (a, 0) \text{ and}$$

$$\vec{r}_3 = \left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$$

Hence, according to definition, the position vector of centre of mass of the system of three particles is

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3}$$

$$= \frac{m_1(0,0) + m_2(a,0) + m_3\left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)}{m_1 + m_2 + m_3}$$

$$= \frac{\left(m_2 a + \frac{m_3 a}{2}, \frac{\sqrt{3}}{2}m_3 a\right)}{m_1 + m_2 + m_3}$$

$$\therefore \vec{r}_{cm} = \left[\frac{\left(m_2 + \frac{m_3}{2}\right)a}{m_1 + m_2 + m_3}, \frac{\sqrt{3}\, m_3 \frac{a}{2}}{m_1 + m_2 + m_3}\right]$$

**Illustration 2 :** In a system of three particles, the linear momenta of the three particles are (1, 2, 3), (4, 5, 6) and (5, 6, 7). These components are in kg m s$^{-1}$. If the velocity of centre of mass of the system is (30, 39, 48) m s$^{-1}$, then find the total mass of the system.

**Solution :** Here $\vec{P}_1 = (1, 2, 3)$ kg m s$^{-1}$

$\vec{P}_2 = (4, 5, 6)$ kg m s$^{-1}$

$\vec{P}_3 = (5, 6, 7)$ kg m s$^{-1}$

$\vec{v}_{cm} = (30, 39, 48)$ m s$^{-1}$

Now, $M\vec{v}_{cm} = \vec{P} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3$

$\therefore$ M(30, 39, 48) = (1, 2, 3) + (4, 5, 6) + (5, 6, 7)

$\therefore$ (30 M, 39 M, 48 M) = (10, 13, 16)

Comparing respective co-efficients on both sides

$$30\text{ M} = 10 \Rightarrow \text{M} = \frac{1}{3}\text{ kg}$$

$$39\text{ M} = 13 \Rightarrow \text{M} = \frac{1}{3}\text{ kg}$$

$$48\text{ M} = 16 \Rightarrow \text{M} = \frac{1}{3}\text{ kg}$$

Thus, the total mass of the system is $\frac{1}{3}$ kg.

**Illustration 3 :** At time $t$ = 0, a stone of 0.1 kg is released freely from a high rise building. Another stone of 0.2 kg is released from the same position after 0.1 s.

(1) At $t$ = 0.3 s time, what will be the distance of centre of mass of the two stones from original position ? (Neither stone has yet reached the ground).

(2) How fast is the centre of mass of the system of two stones moving at that time ?

(3) What will be the momentum of the system of two stones at this time ?

**Solution :** Mass of stone 1 is $m_1$ = 0.1 kg

Mass of stone 2 is $m_2$ = 0.2 kg

Initial speed of stone 1 is $v_{01}$ = 0 m s$^{-1}$

Initial speed of stone 2 is $v_{02}$ = 0 m s$^{-1}$

(1) Here both the stones are moving in one direction so their velocity and momentum vectors can be regarded as scalers. At $t$ = 0.3 s time, the distance travelled by stone 1 is

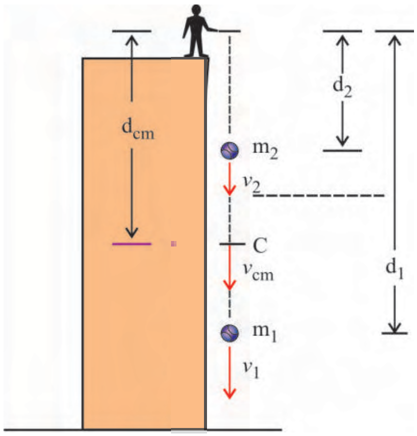



Figure 1.5

$$d_1 = v_{01}\,t + \frac{1}{2}\,g\,t^2$$

$$= 0 + \frac{1}{2}\,(9.8)\,(0.3)^2$$

$$d_1 = 0.441 \text{ m} \qquad (1)$$

Stone 2 is released after 0.1 s. Hence, at time $t = 0.3$ s, the time taken by stone 2 to fall down is $t' = 0.3$ s $- 0.1$ s $= 0.2$ s.

Hence, in time $t' = 0.2$ s (i.e. at $t = 0.3$ s), the distance travelled by stone 2 is

$$d_2 = v_{02}t' + \frac{1}{2}\,g\,t'^2$$

$$= 0 + \frac{1}{2}\,(9.8)\,(0.2)^2$$

$$d_2 = 0.196 \text{ m} \qquad (2)$$

Hence, at time $t = 0.3$ s, the distance of centre of mass of the system of the two stones is

$$d_{cm} = \frac{m_1d_1 + m_2d_2}{m_1 + m_2}$$

$$= \frac{(0.1)(0.441) + (0.2)(0.196)}{0.1 + 0.2}$$

$$\therefore\ d_{cm} = 0.277 \text{ m} \qquad (3)$$

(2) At time $t = 0.3$ s, the speed of stone 1 is

$$v_1 = v_{01} + gt = 0 + (9.8)(0.3)$$

$$\therefore\ v_1 = 2.94 \text{ m s}^{-1} \qquad (4)$$

At time $t = 0.3$ s, the time interval for fall of the stone 2 is $t' = 0.2$ s. Hence, after $t' = 0.2$ s time, the speed of stone 2 is

$$v_2 = v_{02} + gt' = 0 + (9.8)(0.2)$$

$$\therefore\ v_2 = 1.96 \text{ m s}^{-1} \qquad (5)$$

Hence at $t = 0.3$ s, the velocity of centre of mass of the system of two stones is

$$v_{cm} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$$

$$\therefore\ v_{cm} = \frac{(0.1)(2.94) + (0.2)(1.96)}{0.1 + 0.2}$$

$$\therefore\ v_{cm} = 2.29 \text{ m s}^{-1} \qquad (6)$$

(3) At time $t = 0.3$ s, the total momentum of the system of two stones

$$P = P_1 + P_2 = m_1v_1 + m_2v_2$$

$$\therefore\ P = (0.1)\,(2.94) + (0.2)\,(1.96)$$

$$\therefore\ P = 0.686 \text{ kg m s}^{-1}$$

$$= 0.69 \text{ kg m s}^{-1} \qquad (7)$$

**Illustration 4 :** Different forces acting on a lamina body (two dimensional) of mass 2 kg are shown in Figure 1.6. Calculate the linear acceleration of the centre of mass of the body.

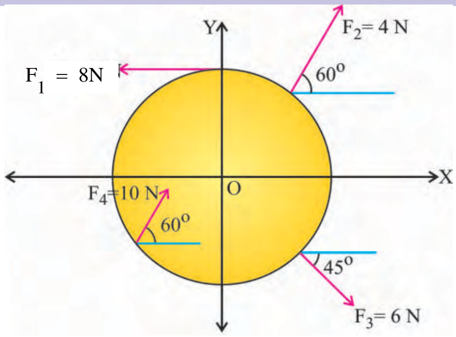



Figure 1.6

**Solution :** Writing all forces in the form of their components,

$$\vec{F}_1 = (-8,\ 0)\ \text{N}$$

$$\vec{F}_2 = (4\cos 60^o,\ 4\sin 60^o) = (2,\ 2\sqrt{3}\,)\text{N}$$

$$\vec{F}_3 = [6\cos(-45^o),\ 6\sin(-45^o)]$$

$$= (6\cos 45^o,\ -6\sin 45^o)$$

$$\therefore\ \vec{F}_3 = \left(\frac{6}{\sqrt{2}},\ \frac{-6}{\sqrt{2}}\right)\text{N}$$

$$\vec{F}_4 = (10\cos 60^o,\ 10\sin 60^o) = (5,\ 5\sqrt{3}\,)\text{N}$$

Now,

$$M\vec{a}_{cm} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4$$

Where M = 2 kg

$$\therefore \vec{a}_{cm} = \frac{1}{2}(\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4)$$

$$= \frac{1}{2}[(-8 + 2 + \frac{6}{\sqrt{2}} + 5), (2\sqrt{3} - \frac{6}{\sqrt{2}} + 5\sqrt{3})]$$

$$\therefore \vec{a}_{cm} = \frac{1}{2}[(-1 + \frac{6}{\sqrt{2}}), (7\sqrt{3} - \frac{6}{\sqrt{2}})] \text{m s}^{-2}$$

## 1.4 Law of Conservation of Linear Momentum

If the resultant force acting on a system is zero, then from equation (1.3.9)

$$\vec{F} = \frac{d\vec{P}}{dt} = 0 \qquad (1.4.1)$$

$$\therefore \vec{P} = \vec{P}_1 + \vec{P}_2 + ...... + \vec{P}_n = \text{constant} \qquad (1.4.2)$$

This shows that, **"if the resultant external force acting on a system is zero, then the total linear momentum of the system remains constant."** This statement is known as the **law of conservation of linear momentum.** In absence of resultant external force, the momenta of individual particles $\vec{P}_1$, $\vec{P}_2$, ..... may change, but these changes occur in such a way that the vector sum of changes in momenta always becomes zero. As the total change in the linear momentum is zero, the total momentum remains constant.

e.g. The gas molecules in a closed container move randomly in the container. During the inter-atomic collisions or the collision of the molecules with the wall of the container, their momentum changes individually. But the vector sum of the changes in momenta of all the particles is zero.

It means that their total momentum remains constant. (If the vector sum of the changes in momenta of the gas molecules were in a particular direction, then what would have happened ? Think).

**The law of conservation of linear momentum is fundamental and universal.** This law is equally true for the systems as big as that of planets and as small as that of tiny particles like electrons, protons, etc.

From equation (1.3.9)

$$\vec{F} = M\vec{a}_{cm} = M\frac{d\vec{v}_{cm}}{dt} = 0$$

$$\therefore \vec{a}_{cm} = 0 \text{ and } \vec{v}_{cm} = \text{constant}$$

Which shows that, **if the resultant external force is zero, then the acceleration of the centre of mass is zero. It means that the velocity of centre of mass remains constant.** Thus, in absence of external force the centre of mass of a system remains stationary if it was stationary or moves with constant velocity if it was in motion.

**Let us see the following illustration :**

Suppose, a chemical bomb is stationary. The initial momentum and kinetic energy of the bomb are zero. When the bomb explodes, its fragments are thrown in air. Though these fragments have different momenta in different directions, but the magnitudes and directions of these fragments would be such that

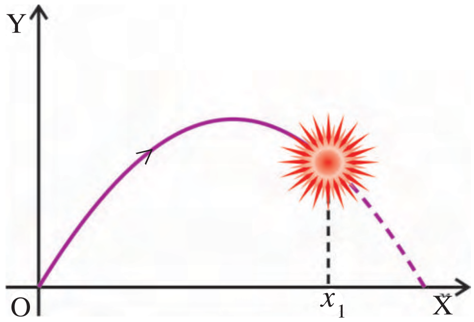
$$\vec{P}_1 + \vec{P}_2 + ...... + \vec{P}_n = 0$$

Here $\vec{P}_1$, $\vec{P}_2$, .... are the momenta of the fragments.

Here, the centre of mass of the system of fragments remains at the same point, where it was located before explosion of the bomb.

The kinetic energy of the bomb before explosion was zero, but the sum of kinetic energy of the fragments is not zero. Thus the kinetic energy of the system got changed. In the chapter of work, energy and power you came to know that the change in kinetic energy of the system is equal to the work done by the resultant external force. Here, resultant external force is zero. Then how does the kinetic energy of bomb change ? The fact is that chemical bomb possesses internal energy due to the chemical bonds between its complex molecules (and due to some additional reasons). When bomb explodes, the chemical bonds are broken and a part of the internal energy associated with them is converted into heat energy, and the remaining part in the form of kinetic energy of the fragments. Thus, in this case, the work is done at the cost of internal energy **which leads to the more general form of the work energy theorem.**

Here, the bomb was stationary. But, if the bomb was moving with constant velocity and explode during motion, then according to the law of conservation of linear momentum, its fragments would move in such directions, that the vector sum of their momenta become equal to the momentum of the bomb before explosion. The centre of mass would move such that its original velocity ($\vec{v}_{cm}$) is maintained (See Figure 1.7).




**Motion of Centre of Mass of Fragments of Bomb after Explision**

**Figure 1.7**

**Illustration 5 :** A bomb of mass 50 kg moving uniformly with a velocity of 10 m/s explodes spontaneously into two fragments of 40 kg and 10 kg. If the velocity of the smaller fragment is zero, then calculate the velocity of the smaller fragment.

**Solution :** Bomb is moving with uniform (constant) velocity. Hence, the external force acting on it is zero.

Initial linear momentum = Final linear momentum

$\therefore \; M\vec{v} = m_1\vec{v}_1 + m_2\vec{v}_2$

Where, M = total mass of the bomb = 50 kg

$m_1$ = mass of the larger fragment = 40 kg

$m_2$ = mass of the smaller fragment = 10 kg

$\vec{v}$ = velocity of the bomb = 10 m/s

$\vec{v}_1$ = velocity of larger fragment = 0

$\vec{v}_2$ = velocity of smaller fragment = ?

Hence,

$Mv = m_2\vec{v}_2$

$\therefore \; \vec{v}_2 = \frac{M}{m_2}\vec{v} = \frac{50}{10} \times 10 = 50$ m/s,

in the direction of $\vec{v}$

**Illustration 6 :** A sphere of mass 4 kg collides with a wall, at an angle of $30^o$ with the wall and rebounds in the direction making an angle of $60^o$ with its original direction of motion. Find the force on the wall if the ball remains in contact with the wall for 0.1 s. The initial and final velocities are the same, equal to 1 m $s^{-1}$.

**Solution :** The given situation is shown in the Figure 1.8.

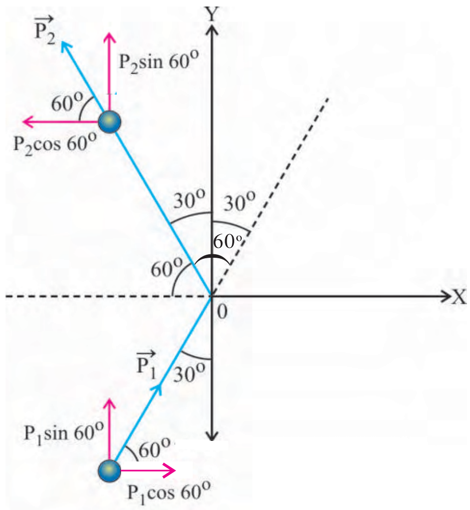



**Figure 1.8**

Here $\vec{P}_1$ = initial momentum of sphere

$= mv \cos 60\hat{i} + mv \sin 60\hat{j}$

$\vec{P}_2$ = final momentum of sphere

$= -mv \cos 60\hat{i} + mv \sin 60\hat{j}$

Hence, change in momentum of the sphere

$\Delta\vec{P} = \vec{P}_2 - \vec{P}_1$

$= -mv \cos 60\hat{i} + mv \sin 60\hat{j}$

$-mv \cos 60\hat{i} - mv \sin 60\hat{j}$

$= -2mv \cos 60\hat{i}$

$\therefore \; \vec{r} = -2 \times 4 \times 1 \times \frac{1}{2}\hat{i}$

$= -4\hat{i}$ kg m $s^{-1}$

Hence, momentum gained by the wall

$= 4\hat{i}$ kg m $s^{-1}$

$\therefore$ Force exerted on the wall

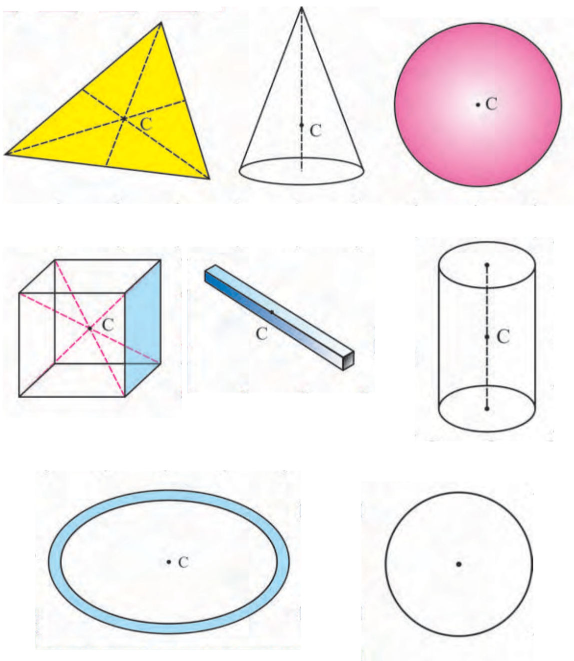
$$= \frac{\text{momentum gained by the wall}}{\text{time of contact}}$$

$$= \frac{4\hat{i}}{0.1} = 40\hat{i}\ \text{N}$$

Thus 40 N force acts on the wall in positive X-direction.

## 1.5 Centre of Mass of a Rigid Body

**A system of particles in which the relative positions of particles remain invariant is called a rigid body. The location of centre of mass of a rigid body depends on the distribution of mass in the body and the shape of the body. The centre of mass of a rigid body can be anywhere inside or outside the body.** For example, the centre of mass of a disc of uniform distribution of mass is at its geometric centre within the matter, whereas the centre of mass of a ring of uniform mass distribution lies at its geometrical centre which is outside of its matter. The centre of mass of a rod of uniform cross-section lies at its geometrical centre. The position of the centre of mass of symmetric bodies with uniform mass distribution can be easily obtained theoretically. The centre of mass 'C' of certain symmetric bodies are shown in Figure 1.9.




Centre of mass of some symmetric bodies

**Figure 1.9**

### 1.5.1 Theoretical Method for Estimation of the Centre of Mass of a Solid Body :

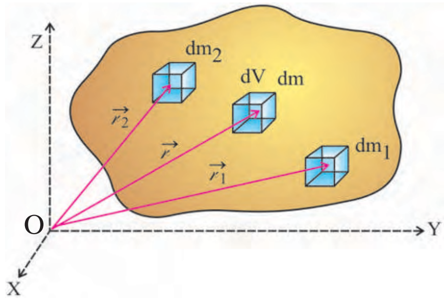



**Figure 1.10**

We know now that a solid body is made up of microscopic particles (molecules, ions or atoms) distributed continuously inside the body.

As shown in Figure 1.10, consider a small volume element $dV$, containing mass $dm$. Here $dm$ is called the mass element, whose position vector is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

This way the whole solid body can be considered to be made up of such small mass elements. Let the solid body is made up of small mass elements $dm_1$, $dm_2$, ....., $dm_n$ having position vectors $\vec{r}_1$, $\vec{r}_2$, ......, $\vec{r}_n$, respectively. Hence, according to definition, the position vector of centre of mass of the solid body is

$$\vec{r}_{cm} = \frac{dm_1\vec{r}_1 + dm_2\vec{r}_2 + \ldots\ldots + dm_n\vec{r}_n}{dm_1 + dm_2 + \ldots\ldots + dm_n} \quad (1.5.1)$$

As the mass distribution is continuous, the summation in equation (1.5.1) can be represented as an integration.

$$\therefore\ \vec{r}_{cm} = \frac{\int \vec{r}\,dm}{\int dm}$$

$$\therefore\ \vec{r}_{cm} = \frac{\int \vec{r}\,dm}{\text{M}} \quad (1.5.2)$$

Where, $\text{M} = \int dm$

= total mass of the solid body

Representing equation (1.5.2) in terms of its vector components,

$$(x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k}) \quad (1.5.3)$$

$$= \frac{1}{\text{M}}\int (x\hat{i} + y\hat{j} + z\hat{k})dm$$

$$\left.\begin{aligned} \therefore x_{cm} &= \frac{1}{M}\int x\,dm \\ y_{cm} &= \frac{1}{M}\int y\,dm \\ z_{cm} &= \frac{1}{M}\int z\,dm \end{aligned}\right\} \qquad (1.5.4)$$

### 1.5.2 Centre of mass of a solid body of uniform density and specific geometrical shape :

The centre of mass of a solid body of uniform mass density and specific geometrical shape can be calculated using the symmetry of the body. Using the law of symmetry we can easily prove that the centre of mass of such bodies lies at their geometric centre.

**Let us see the following Illustration :**

Suppose we have to find out the centre of mass of given triangular plate :

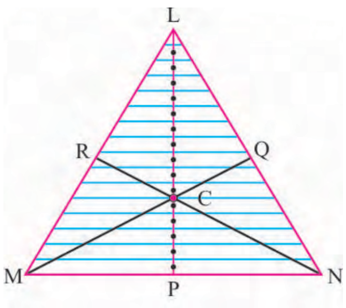
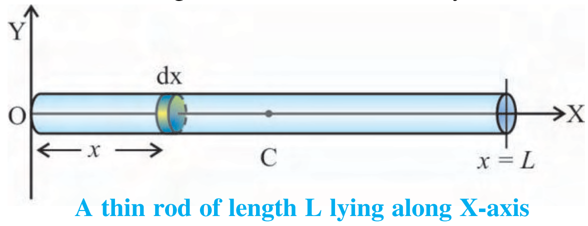



Figure 1.11

As shown in Figure 1.11 imagine the triangular plate to be divided into parallel thin stripes parallel to side MN of the triangle. According to the law of symmetry, the centre of mass of each stripe will be lying at its geometric centre. Thus the centre of mass of the triangular plate will be lying along the bisector LP. Similarly, considering the triangular plate to be divided into thin stripes parallel to the sides ML and LN of the triangle, we get bisectors NR and MQ, respectively. Then, the centre of mass of the triangular sheet will be lying at the intersection point 'C' of the three bisectors.

## 1.6 Centre of Mass of a Thin Rod of Uniform Density :

Figure 1.12 shows a thin rod of mass 'M' and length 'L' having uniform area of cross section and uniform linear mass density 'λ'. Put the rod along X-axis such that its one end lies at the origin of the co-ordinate system.




A thin rod of length L lying along X-axis

Figure 1.12

Consider a line element '$dx$' on the rod at a distance '$x$' from the origin.

The mass per unit length of the rod, $\lambda = \dfrac{M}{L}$

Hence, the mass of the line element $dx$ is,

$$dm = \lambda dx = \frac{M}{L}dx$$

According to definition, the centre of mass of the rod is

$$x_{cm} = \frac{1}{M}\int x\,dm$$

$$= \frac{1}{M}\int_0^L x \cdot \frac{M}{L}dx$$

$$= \frac{1}{L}\int_0^L x\,dx$$

$$= \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L$$

$$= \frac{1}{L}\left[\frac{L^2}{2} - 0\right]$$

$$\therefore x_{cm} = \frac{L}{2}$$

Thus, the centre of mass of the thin rod of uniform mass density lies at mid point of its length, i.e. at its geometric centre.

**Illustration 7 :**

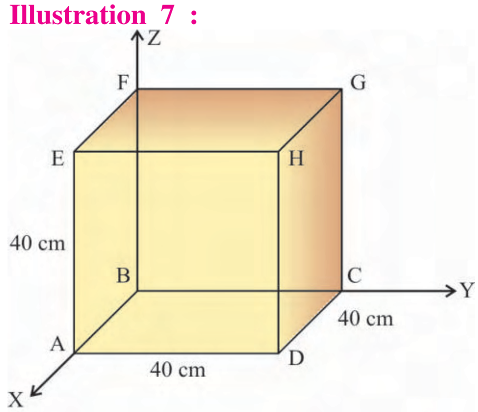



Figure 1.13

Figure 1.13 shows a cubical box made up of negligible thin metal sheet of uniform mass density. If the length of each side of the box is 40 cm, then

(*a*) Find out the co-ordinates $(x_{cm}, y_{cm}, z_{cm})$ of centre of mass of the box.

(*b*) If the box is open from upper side, (EFGH sheet is absent) then find out the co-ordinates $(x'_{cm}, y'_{cm}, z'_{cm})$ of the centre of mass of the box.

**Solution :** Each sheet of the box is negligibly thin and have uniform density. Hence, according to the law of symmetry, the centre of mass of each plate will be lying at its geometrical centre. Hence, calculating the centre of mass of each plate :

| Name of Plate | Co-ordinates of Centre |
|---|---|
| ABCD | (20, 20, 0) cm |
| EFGH | (20, 20, 40) cm |
| ABFE | (20, 0, 20) cm |
| DCGH | (20, 40, 20) cm |
| BCGF | (0, 20, 20) cm |
| ADHE | (40, 20, 20) cm |

(*a*) Considering that the whole mass (say M) of each sheet (plate) is concentrated at its centre of mass (The area and surface density of each plate is same, so the mass of each plate is also same, M = ρ × A), the position of centre of mass of such a system is

$$r_{cm} = (x_{cm}, y_{cm}, z_{cm})$$

$$= \frac{\left\{ \begin{array}{c} M(20, 20, 0) + M(20, 20, 40) \\ + M(20, 0, 20) + M(20, 40, 20) \\ + M(0, 20, 20) + M(40, 20, 20) \end{array} \right\}}{6M}$$

$$= \frac{M(120, 120, 120)}{6M}$$

$$\therefore \; r_{cm} = (20, 20, 20) \text{ cm}$$

(*b*) If the box is open from upper surface, then EFGH plate is absent, and hence the centre of mass of the remaining system is

$$r'_{cm} = (x'_{cm}, y'_{cm}, z'_{cm})$$

$$= \frac{\left\{ \begin{array}{c} M(20, 20, 0) + M(20, 0, 20) \\ + M(20, 40, 20) + M(0, 20, 20) \\ + M(40, 20, 20) \end{array} \right\}}{5M}$$

$$= \frac{M(100, 100, 80)}{5M}$$

$$= (20, 20, 16) \text{ cm}$$

## SUMMARY

1. **Centre of mass of a system of two particles :** The centre of mass of a system of two particles of masses $m_1$ and $m_2$ lying on X-axis at $x_1$ and $x_2$ distances respectively from origin is *a* point at a distance

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \text{ from the origin.}$$

2. **Centre of mass of a system of *n*–particles :** If there are *n*–particles in a system, and C is representing the centre of mass of the system, then 'C' is the point where the whole mass of system of *n*–particles can be considered to be concentrated. For a system of *n*–particles in a three dimensional space, if $m_1, m_2, ....., m_n$ are the masses of the particles and $\vec{r_1}, \vec{r_2}, ....., \vec{r_n}$ are respective position vectors of the particles, then the position vector of centre of mass of the system is

$$\vec{r}_{cm} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + .... + m_n\vec{r_n}}{m_1 + m_2 + .... + m_n}$$

3. The velocity of centre of mass of system of $n$–particles

$$\vec{v}_{cm} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + \dots + m_n \vec{v}_n}{M}$$

where, $M = m_1 + m_2 + \dots\dots + m_n$

4. Acceleration of centre of mass of a system of $n$–particles

$$\vec{a}_{cm} = \frac{m_1 \vec{a}_1 + m_2 \vec{a}_2 + \dots + m_n \vec{a}_n}{M}$$

5. Newton's Second Law of Motion for a system of particles :

$$\vec{F} = \frac{d\vec{P}}{dt} = M\frac{d\vec{v}_{cm}}{dt} = M\vec{a}_{cm}$$

6. **Conservation of linear momentum :** If the resultant external force acting on a system is zero, then the total linear momentum of the system remains constant. In the absence of resultant external force, the centre of mass of a system remains stationary if it was stationary and moves with constant velocity if it was in motion.
7. **Rigid body :** A system of particles in which the relative positions of particles remain invariant is called a rigid body.
8. **The centre of mass of a rigid body :** The location of centre of mass of a rigid body depends on the distribution of mass in the body and the shape of the body. The centre of mass of symmetric bodies lies at their geometric centre.
9. In general form, the co-ordinates of centre of mass of a rigid body are

$$x_{cm} = \frac{1}{M}\int x\,dm,\quad y_{cm} = \frac{1}{M}\int y\,dm,\quad z_{cm} = \frac{1}{M}\int z\,dm$$

## EXERCISES

**Choose the correct option from the given options :**

1. Suppose your mass is 50 kg. How fast should you run so that your linear momentum becomes equal to that of a bicycle rider of 100 kg moving along a straight road with a speed of 20 km/h ?

   (A) 40 m/s (B) 11.11 m/s (C) 20 km/h (D) 10 km/h

2. A bus of 2400 kg is moving on a straight road with a speed of 60 km/h. A car of 1600 kg is following the bus with a speed of 80 km/h. How fast is the centre of mass of the system of two vehicles moving ?

   (A) 70 km/h (B) 75 km/h (C) 72 km/h (D) 68 km/h

3. The momentum of a stone at time '$t$' is

   $[(0.5 \text{ kg m/s}^3)t^2 + (3.0 \text{ kg m/s})]\hat{i} + [1.5 \text{ kg m/s}^2]t\hat{j}]$. How much force is acting on it ?

   (A) $(t\hat{i} + 1.5\hat{j})$ N (B) $(0.5t\hat{i} + 1.5\hat{j})$ N

   (C) $[(0.5t + 3)\hat{i} + 1.5\hat{j}]$ N (D) $(0.5\hat{i} + 1.5\hat{j})$ N

**4.** A bird of 2 kg is flying with a constant velocity of $(2\hat{i} - 4\hat{j})$ m/s, and another bird of 3 kg with $(2\hat{i} + 6\hat{j})$ m/s. Then the velocity of centre of mass of the system of two birds is .......... m/s.

(A) $2\hat{i} + 5.2\hat{j}$ (B) $2\hat{i} + 2\hat{j}$ (C) $2\hat{i} - 2\hat{j}$ (D) $10\hat{i} + 10\hat{j}$

**5.** A quill of 0.100 g is falling with a velocity of $(-0.05\hat{j})$ m/s. When blown from lower side, its velocity changes to $(0.20\hat{i} + 0.15\hat{j})$ m/s. The change in its momentum will be .......... kg m/s.

(A) $2 \times 10^{-2}\hat{i} + 2 \times 10^{-2}\hat{j}$ (B) $2 \times 10^{-5}\hat{i} + 2 \times 10^{-5}\hat{j}$

(C) $2 \times 10^{-2}\hat{i} + 1 \times 10^{-2}\hat{j}$ (D) $2 \times 10^{-2}\hat{i} - 2 \times 10^{-2}\hat{j}$

**6.** A monkey sitting on a tree drops a 10 g seed of rose-apple on a crocodile, at rest below the tree. If the seed falls in the mouth of the crocodile in 2 s time and becomes stationary, then the momentum gained by the crocodile (in addition to the seed) is .......... kg m/s. ($g = 9.8$ m $s^{-2}$)

(A) 0.196 (B) −0.196 (C) 19.6 (D) −19.6

**7.** As shown in Figure (1.14), the stones of 30, 60, 90 and 120 g are placed at 3, 6, 9 and 12 hour symbols respectively of a weightless dial of clock having radius of 10 cm. Find the co-ordinates of centre of mass of this system.

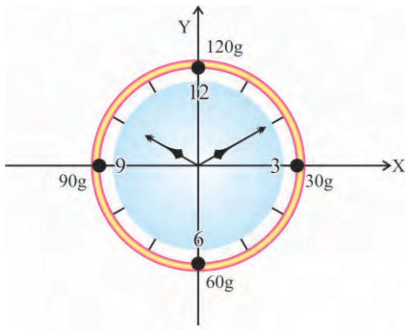


Figure 1.14

(A) (2, −2) cm (B) (0, 0) cm (C) (−2, 2) cm (D) (−4, 4) cm

**8.** In cricket match, a baller throws a ball of 0.5 kg with a speed of 20 m/s. When a batsman swings the bat, the ball strikes with the bat normal to it, and returns in opposite direction with speed of 30 m/s. If the time of contact of the ball with the bat is 0.1 s, then the force acting on the bat is .......... N.

(A) 250 (B) 25 (C) 50 (D) 125

**9.** A boy standing on the terrace of 10 storeyed building, drops four stones of different mass. At one moment, if the stone of 500 g is at $8^{th}$ floor, stone of 400 g is at $6^{th}$ floor, stone of 1 kg is at $3^{rd}$ floor and a stone of 600 g is reached at $1^{st}$ floor, then at that time, the centre of mass of the system of four stones is at .......... floor.

(A) $7^{th}$ (B) $5^{th}$ (C) $3^{rd}$ (D) $4^{th}$

10. As shown in Figure 1.15 the centre of mass of a thin metal sheet of uniform density is .......... cm.

(A) (10.00, 14.28)

(B) (11.67, 16.67)

(C) (8.75, 12.50)

(D) (7.78, 11.11)

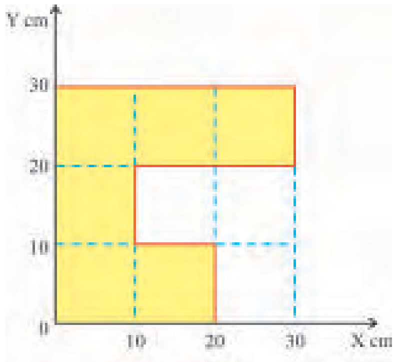



Figure 1.15

## ANSWERS

| | | | | |
|---|---|---|---|---|
| 1. (B) | 2. (D) | 3. (A) | 4. (B) | 5. (B) |
| 6. (A) | 7. (C) | 8. (A) | 9. (D) | 10. (B) |

## Answer the following questions in short :

1. What is the meaning of inter-dependence of Newton's Laws of Motion ?
2. Give definition of a rigid body.
3. Give two illustrations of rigid bodies in which the centre of mass lies in the matter of the body.
4. Where does the centre of mass of a thin rod of uniform mass density lie ?
5. What do you mean by the mass element *dm* of a solid body ?
6. When a stationary bomb explodes, then from where does its fragments get kinetic energy ?

## Answer the following questions :

1. Write down the expression for the centre of mass of a system of n-particles in three dimensions and obtain the expression for its velocity.
2. State the law of conservation of linear momentum and explain.
3. How does the illustration of chemical bomb lead to the more general form of work energy theorem ? Explain.
4. Write down the equation for the velocity of centre of mass of a system of *n*-particles, and derive the Newton's Second Law of Motion for it.
5. Explain the theoretical method for estimating the centre of mass of a solid body.
6. Obtain the position of centre of mass of a thin rod of uniform density with respect to the one end of the rod.

## Solve the following problems :

1. The distance between the centres of carbon and oxygen atoms in a carbon monoxide (CO) molecule is $1.130 \times 10^{-10}$ m. Find the position of centre of mass of CO molecule with respect to carbon atom.

   (Atomic mass of carbon = 12 g $mol^{-1}$, and atomic mass of oxygen = 16 g $mol^{-1}$)

   [**Ans. :** 0.64 Å]

2. The velocity vectors of three "particles" of masses 1 kg, 2 kg and 3 kg

are respectively (1, 2, 3), (3, 4, 5) and (6, 7, 8). The velocity vectors are in m $s^{-1}$. Find the velocity vector of centre of mass of this system of particles. [**Ans. :** $\frac{1}{6}$(25, 31, 37) m $s^{-1}$]

**3.** A car of 1000 kg is at rest at a traffic signal. At the instant, the light turns green, the car starts to move with a constant acceleration of 4.0 m $s^{-2}$ on a straight road. At the same instant, a truck of 2000 kg travelling at a constant speed of 8.0 m $s^{-1}$ overtakes and passes the car.

(*a*) How far will be the centre of mass of the car-truck system from the traffic light after 3 sec. ?

(*b*) What will be the speed of the centre of mass of the car-truck system then ?

[**Ans. :** (*a*) 22.0 m, (*b*) 9.33 m $s^{-1}$]

**4.** A dog having mass of 40 kg and a cat of 20 kg mass are standing on both sides of a roti at distance of 15–15 *m* each (See Figure 1.16). Both start to run at the same instant to eat the roti in such a way that the centre of mass of the system made up of the dog and the cat remains stationary. In the table, the position of the dog at different instants is represented, with respect to the origin lying at the roti. Calculate the position of the cat, the velocities, momenta and total momentum of both of them.

Which will reach to the roti first ? Dog or Cat ? Is the momentum conserved in this case ? Why ?

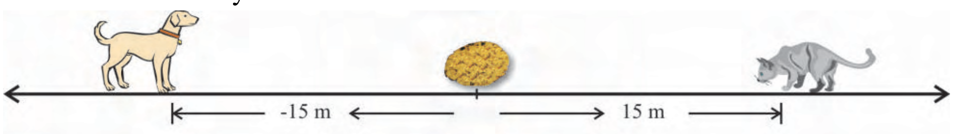



**Figure 1.16**

| Time | Distance from Roti | | Centre of mass of dog-cat | Velocity $ms^{-1}$ | | Momentum kg $ms^{-1}$ | | Total momentum |
|---|---|---|---|---|---|---|---|---|
| *t* sec | Dog $x_1(m)$ | Cat $x_2(m)$ | $x_{cm}(m)$ | Dog $v_1$ | Cat $v_2$ | Dog $P_1$ | Cat $P_2$ | $P = P_1 + P_2$ kg m $s^{-1}$ |
| 0 | –15.0 | 15 | .....(constant) | | | | | |
| 2 | –12.5 | ..... | ” | | | | | |
| 4 | –10.0 | ..... | ” | | | | | |
| 6 | –7.5 | ..... | ” | | | | | |

**Ans. :**

| Time *t* sec | Dog $x_1(m)$ | Cat $x_2(m)$ | Centre of mass $x_{cm}(m)$ | Dog $v_1$ | Cat $v_2$ | Dog $P_1$ kg $ms^{-1}$ | Cat $P_2$ kg $ms^{-1}$ | Total $P = P_1 + P_2$ kg $ms^{-1}$ |
|---|---|---|---|---|---|---|---|---|
| | | | $x_{cm}(m)$ | | | | | |
| 0 | –15.0 | 15.0 | –5.0 *m* (const.) | 0 | 0 | 0 | 0 | 0 |
| 2 | –12.5 | 10.0 | –5.0 *m* | 1.25 | –2.5 | 50 | –50 | 0 |
| 4 | –10.0 | 5.0 | –5.0 *m* | 1.25 | –2.5 | 50 | –50 | 0 |
| 6 | –7.5 | 0 | –5.0 *m* | 1.25 | –2.5 | 50 | –50 | 0 |

At $t = 6$ sec, $x_1 = -7.5\ m$, $x_2 = 0\ m$ and Roti is at origin $x = 0$.
Hence, cat will reach first.
Total momentum remains constant. Hence, momentum is conserved. It is due to the fact that centre of mass remains stationary (in this example).

5. The distance between two particles of masses $m_1$ and $m_2$ is $r$. If the distances of these particles from the centre of mass of the system are $r_1$ and $r_2$, respectively, they show that

$$r_1 = r\left[\frac{m_2}{m_1 + m_2}\right] \text{ and } r_2 = r\left[\frac{m_1}{m_1 + m_2}\right]$$

6. As shown in Figure 1.17 three indentical spheres 1, 2 and 3, each of radius R, are arranged on a horizontal surface so as to touch one another. The mass of each sphere is $m$. Determine the position of centre of mass of this system, taking the centre of sphere 1 as origin. Z-axis is in the direction perpendicular to the plane of the figure.

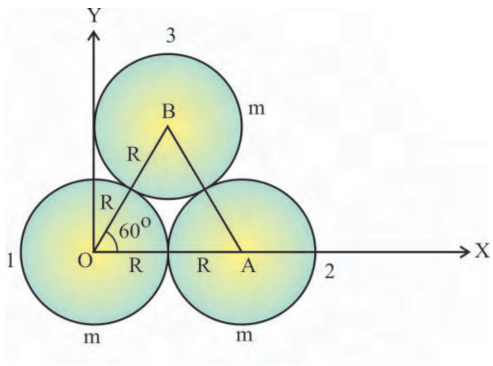



Figure 1.17

[**Ans. :** (R, $\frac{R}{\sqrt{3}}$, 0) $m$]

7. A small sphere of radius $a$ is cut from a homogeneous sphere of radius R as shown in Figure 1.18. Find the position of centre of mass of the remaining part with respect to the centre of mass of the original sphere.

[**Ans. :** $\left(\frac{-a^3 b}{R^3 - a^3}, 0, 0\right)$ ]

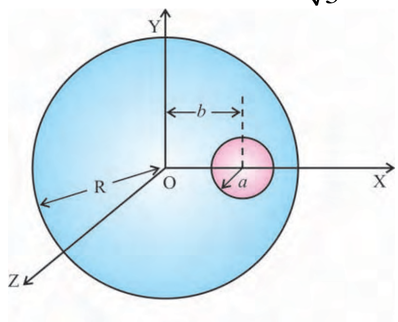



Figure 1.18

8. Figure shows the stationary positions of three "particles". Find out the co-ordinates of centre of mass for the system of particles. As shown in Figure 1.19, if the external forces $F_1 = 6.0$ N, $F_2 = 12.0$ N and $F_3 = 14.0$ N are acting on the particle then find out the acceleration and the direction of acceleration of the centre of mass.

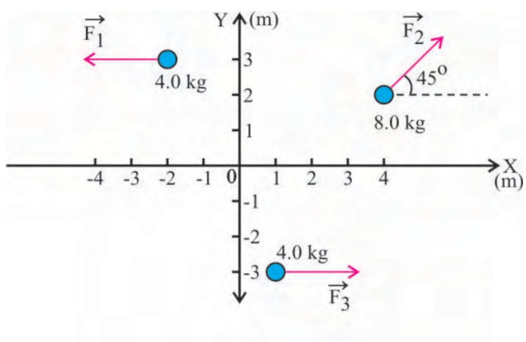



Figure 1.19

[**Ans. :** $\vec{r}_{cm} = (1.75, 1.00)$ m, $\vec{a}_{cm} = (1.03, 0.53)$ m s$^{-2}$,
$|\vec{a}| = a = 1.16$ m s$^{-2}$. The direction making an angle of $\theta = 27°$ with X-axis.]

**9.** Figure 1.20 shows an extremely thin disc of radius R, having uniform mass density of ρ. A small disc of radius $\frac{R}{2}$ is cut from it. Find the centre of mass of the remaining part of the disc with respect to the centre of mass of the original disc.

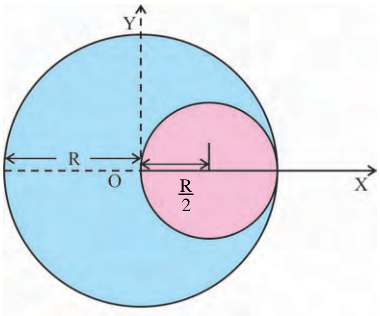



**Figure 1.20**

[**Ans. :** $(-\frac{R}{6}, 0)$]

•

**Prof. Satyendranath Bose (1894-1974)**

Satyendranath Bose was born on the 1st of January 1894 in Calcutta. He studied at the University of Calcutta, then taught there in 1916, taught at the University of Dacca (1921-45), and then returned to Calcutta (1945-56). He did important works in quantum theory, in particular on Planck's black body radiation law. Bose sent his work in Planck's Law and the Hypothesis of Light Quanta (1924) to Einstein. It was enthusiastically endorsed by Einstein. The paper was translated into German by Einstein. Bose also worked on statistical mechanics leading to the Bose-Einstein statistics.

Dirac gave the name boson to the particles obeying this statistics. Satyendranath Bose and Albert Einstein together published a series of papers on the physics of particles with integer spins (bosons). Satyendranath Bose passed away on February 4, 1974 at the age of 80.

## CHAPTER 2

# ROTATIONAL MOTION

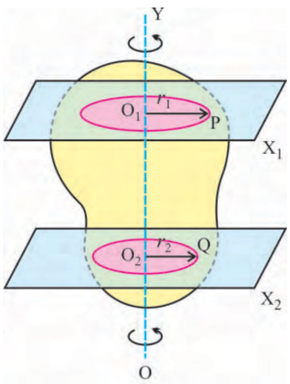


## 2.1 Introduction

Student Friends, you must have seen the motion of fan, top and also the motion of merry-go-round. You also know that. **the Earth is revolving round about its own axis.** In the present chapter we shall study such type of motion. This type of motion is called rotational motion. In the beginning we shall discuss the rotational motion of the rigid body about a fixed axis. At the last we shall discuss the motion of the rigid body, rolling without slipping.

**The system of particles in which the relative distance between the particles remain invariant is called the rigid body.** Rigid body is an ideal concept. From physics point of view a solid body and a rigid body do not mean the same. A solid body can be deformed while a rigid body cannot. But for many practical purposes a solid body can be treated as a rigid body.

## 2.2 Rotational Kinematics and Dynamics




Rotational motion of rigid body

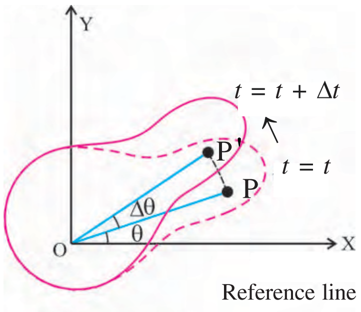
Figure 2.1

**If all the particles of a rigid body perform circular motion and the centres of these circles are steady on a definite straight line called axis of rotation it is a geometrical line and the motion of the rigid body is called the rotational motion.** In figure 2.1, two particles P and Q of a rigid body are shown. The rigid body rotates about the axis OY. $O_1$ and $r_1$ are the centre and the radius respectively of the circle on which particle P moves. Similarly $O_2$ and $r_2$ are the centre and the radius respectively of the circle on which the particle Q moves. The circular paths of particles P and Q are in the planes normal to the axis of rotation OY.

First, we shall describe the rotational motion without mentioning its causes. This branch of Physics is called the **rotational kinematics**, and the branch in which the rotational motion is described along with the causes of the rotational motion and the properties of the body is called **rotational dynamics.**

## 2.3 Relation between Variables of Rotational Motion And the Variables of Linear Motion

### (*a*) Angular Displacement :




Angular Displacement

Figure 2.2

Suppose a rigid body performs rotational motion about a fixed rotational axis OZ which is perpendicular to the plane of paper as shown in Figure 2.2.

The positions of the cross-sections of the rigid body with the plane of paper at time $t$ and $t + \Delta t$ are shown by dotted line and the continuous line respectively.

Consider a particle P of the rigid body. At any time the angle made by the line joining it to the centre of its circular path (O) (which is also the radius of its circular path) with a definite reference line (as shown in the figure) is called the angular position of that particle at that time. As shown in the figure, the particle P subtends an angle $\theta$ with the reference line OX, at time $t$. It is the angular position of that particle P at time $t$. The particle performs circular motion in the XY–plane and reaches from P to P', at time $t + \Delta t$, and its angular position is $\theta + \Delta\theta$ this time.

The change in the angular position of the particle is called its angular displacement. Thus the angular displacement of the particle P in time interval $\Delta t$ is $\Delta\theta$.

(Any line can be taken as a reference line. Generally, the positive X-axis is taken as the reference line). Since the relative distances between the particles of the rigid body remain invariant (unchanged), all particles experience equal angular displacement in the same time-interval. Hence the rotational motion of the rigid body can be described by the motion of some one representative particle out of its innumerable particles. Thus, in the above discussion the angular displacement $\Delta\theta$ is the angular displacement of the rigid body. Its SI unit is radian.

### (*b*) Angular speed and angular velocity :

According to the definition, the average angular speed during the time-interval $\Delta t$ is

$$<\omega> = \frac{\text{Angular displacement}}{\text{Time-interval}}$$

Here the angular displacement $\Delta\theta$ occurs in the time-interval $\Delta t$, hence

$$<\omega> = \frac{\Delta\theta}{\Delta t} \quad (2.3.1)$$

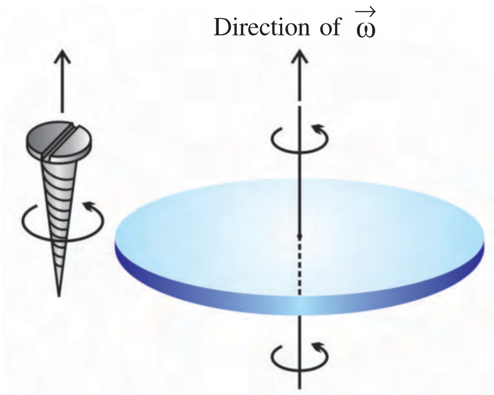
In the limit $\Delta t \rightarrow 0$, this ratio will become the instantaneous angular speed of the particle P at time $t$.

$$\therefore \ \omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t}$$

$$\therefore \ \omega = \frac{d\theta}{dt} \quad (2.3.2)$$

This is also the angular speed of the entire rigid body at time $t$ (From now onwards we will understand angular speed as instantaneous angular speed except specifically mentioned). The unit of $\omega$ is rad $s^{-1}$ or rotation $s^{-1}$. When a proper direction is linked with angular speed, it is called angular velocity. Conventionally the direction of angular velocity is determined from the right hand screw rule.

A right hand screw is adjusted parallel to the rotational axis as shown in the Figure 2.3, and is rotated in the same sense as the rotation of the body, the direction of shifting of the screw is taken as the direction of angular velocity $\vec{\omega}$.




Right hand screw rule

Figure 2.3

### (c) Scalar relation between angular velocity and linear velocity :

As shown in the Figure 2.2 the particle P, travels a linear distance equal to arc PP' in time-interval $\Delta t$. Hence, by definition average linear

speed $\langle v\rangle = \dfrac{\text{arc PP'}}{\text{time-interval } \Delta t}$

If the radius of the circular path of the particle P (the perpendicular distance of the particle P from the rotational axis) is r, then arc PP' $= r\ \Delta\theta$

$$\therefore\ \langle v\rangle = \frac{r\Delta\theta}{\Delta t}$$

$$= r\langle\omega\rangle \qquad (2.3.3)$$

In the limit $\Delta t \to 0$ the value of the above ratio gives the value of instantaneous linear velocity.

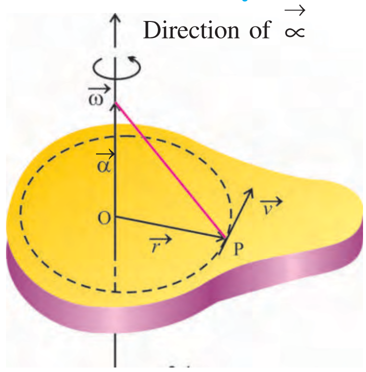
$$\therefore\ v = \lim_{\Delta t \to 0} \frac{r\Delta\theta}{\Delta t}$$

$$= r\frac{d\theta}{dt}$$

$$\therefore\ v = r\omega \qquad (2.3.4)$$

This shows scalar relation between linear velocity and angular velocity of a rigid body.

### (d) Vector relation between angular velocity and linear velocity :




Vector relation between the linear velocity and angular velocity

Figure 2.4(*a*)

The situation of the position vector $\vec{r}$ and the linear velocity $\vec{v}$ for a particle P of the rigid body with respect to the centre of its circular path in a plane perpendicular to the rotational axis, are as shown in the Figure 2.4(*a*). And the angular velocity $\vec{\omega}$ is according to right hand screw rule along the rotational axis (as shown in the figure).

**Linear velocity is a vector. In circular motion the direction of linear velocity at any point is along the tangent drawn to the circle at that point.** In the equation $v = r\omega$ the left hand side is the value of the linear velocity while on the right hand side $r$ and $\omega$ are the values of vector quantities $\vec{r}$ and $\vec{\omega}$. This fact suggests that we should take such a product of $\vec{r}$ and $\vec{\omega}$ that its product is also a vector, which is known as the vector product (cross product) of two vector products. Here direction of $\vec{\omega} \times \vec{r}$ according to right hand screw rule is in the direction of $\vec{v}$ and $\vec{\omega} \perp \vec{r}$. Hence, $\vec{\omega} \times \vec{r} = \omega r \sin 90 = \omega r =$ magnitude of $\vec{v}$.

Hence, we can write the vector relation between $\vec{v}$ and $\vec{\omega}$ as

$$\vec{v} = \vec{\omega} \times \vec{r} \qquad (2.3.5)$$

### (e) Angular acceleration :

Suppose instantaneous angular velocities of the particle P at time $t$ and $t + \Delta t$ are $\vec{\omega}$ and $\vec{\omega} + \Delta\vec{\omega}$ respectively.

Hence by definition,

Average angular acceleration

$$<\vec{\alpha}> = \frac{\Delta\vec{\omega}}{\Delta t} \qquad (2.3.6)$$

In the limit $\Delta t \to 0$, the value of the above ratio gives the instantaneous angular acceleration $\vec{\alpha}$ of the particle P at time $t$.

$$\therefore \; \vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t}$$

$$\therefore \; \vec{\alpha} = \frac{d\vec{\omega}}{dt} \qquad (2.3.7)$$

The direction of $\vec{\alpha}$ is in the direction of $\Delta\vec{\omega}$ (change in angular velocity). In case of the fixed rotational axis the direction of $\Delta\vec{\omega}$ is along the rotational axis, hence the direction of $\vec{\alpha}$ is also along the rotational axis. See Figure 2.4($a$).

The unit of $\vec{\alpha}$ is rad $s^{-2}$ or rotation $s^{-2}$.

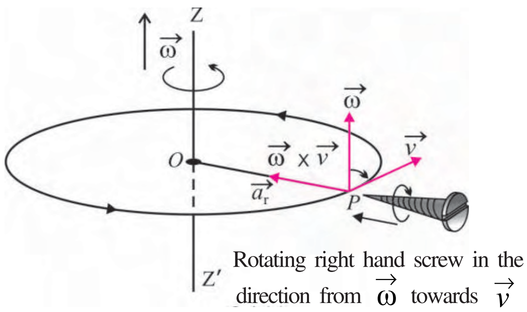
### ($f$) Relation between Linear Acceleration and Angular Acceleration :

The derivative of linear velocity with respect to time gives linear acceleration ($\vec{a}$). Differentiating equation (2.3.5) with respect to time, we get

$$\frac{d\vec{v}}{dt} = \vec{a} = \vec{\omega} \times \frac{d\vec{r}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{r}$$

Since $\frac{d\vec{r}}{dt} = \vec{v}$ and $\frac{d\vec{\omega}}{dt} = \vec{\alpha}$

$$\vec{a} = \vec{\omega} \times \vec{v} + \vec{\alpha} \times \vec{r} \qquad (2.3.8)$$




Radial component of linear acceleration

Figure 2.4($b$)

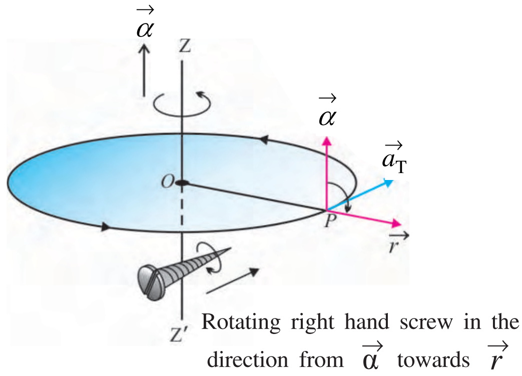
The two vector components of linear acceleration $\vec{a}$ are $\vec{\omega} \times \vec{v}$ and $\vec{\alpha} \times \vec{r}$.

According to Figure 2.4($b$), using right hand screw rule, the direction of $\vec{\omega} \times \vec{v}$ is found to be the radial direction towards the centre. Hence, $\vec{\omega} \times \vec{v}$ is called the **radial component** of linear acceleration $\vec{a}$. It is denoted by $\vec{a}_r$. Its magnitude is $\omega v \sin\frac{\pi}{2} = \omega v = \frac{v^2}{r} = r\omega^2$

$$...(\because \; v = r\omega)$$

Similarly the direction of $\vec{\alpha} \times \vec{r}$ is found to be along the tangent to the circular path. Hence it is called the **tangential** component of the linear acceleration. (See Figure 2.4 ($b$)). It is denoted as $\vec{a}_T$. Its magnitude is $\alpha r \sin \frac{\pi}{2} = \alpha r$

$$\therefore \; \vec{a} = \vec{a}_r + \vec{a}_T$$




Tangential component of linear acceleration

Figure 2.4 ($c$)

The radial component $\vec{a}_r$ and the tangential component $\vec{a}_T$ are mutually perpendicular. Hence the magnitude of $\vec{a}$ is

$$a = \sqrt{a_r^2 + a_T^2} = \sqrt{\omega^2 v^2 + \alpha^2 r^2} \qquad (2.3.9)$$

If the rigid body is rotating with constant angular velocity, that is, its angular acceleration $\alpha = 0$, then the tangential component of its linear acceleration becomes zero but the radial component remains non-zero. This condition is found in the uniform circular motion. You know very well that in uniform circular motion the centripetal acceleration is $\frac{v^2}{r}$.

In the above discussion we have seen that angular displacement ($\theta$), angular velocity ($\vec{\omega}$) and angular acceleration $\vec{\alpha}$ are equal for all particles of the rigid body. Thus $\theta$, $\vec{\omega}$ and $\vec{\alpha}$ are the characteristics of the rigid body and they are called the variables of the rotational kinematics.

Here, note that the description of motion of a particle of the rigid body rotating about a fixed axis can be made in respect of linear variables ($\vec{r}$, $\vec{v}$ and $\vec{a}$) and rotational variables ($\theta$, $\vec{\omega}$, $\vec{\alpha}$). But when all particles of the rigid body are to be considered, the rotational variables may be used so that the motion of the entire body is easily described.

**Illustration 1 :** The length of the second-hand of a clock is 20 cm. Find the values of (1) angular velocity (2) linear velocity (3) angular acceleration (4) radial acceleration (5) tangential acceleration (6) linear acceleration, for the particle at the tip of the second-hand.

**Solution :**

$r = 20$ cm

(1) The second-hand makes angular displacement of $2\pi$ radian in one minute (60 seconds.) $\therefore\ \omega = \frac{2\pi}{60} = \frac{\pi}{30}$ rad $s^{-1}$

(2) Linear velocity $v = \omega r = \frac{\pi}{30} \times 20 = \frac{2}{3}\pi$ cm $s^{-1}$

(3) The second-hand of a clock moves with constant angular speed. $\therefore\ \alpha = 0$ rad $s^{-1}$

(4) Radial acceleration $= a_r = \frac{v^2}{r}$

$$= \left(\frac{2\pi}{3}\right)^2 \left(\frac{1}{20}\right) = \frac{\pi^2}{45} \text{ cm s}^{-2}$$

(5) Tangential acceleration $= a_T = \alpha r = 0$

(6) Linear acceleration

$$a = \sqrt{a_r^2 + a_T^2} = a_r = \frac{\pi^2}{45} \text{ cm s}^{-2}$$

(Calculate these quantities for minute hand of length 15 cm and hour hand of length 10 cm by yourself.)

## 2.4 Equations of Rotational Motion with Constant Angular Acceleration

Suppose at $t = 0$ time the angular position of a particle of a rigid body is $\theta = 0$ and its angular velocity is $\omega_0$.

At $t = t$ time its angular position is $\theta = \theta$ and angular velocity $= \omega$.

If the rigid body is rotating about a fixed axis, then the directions of $\vec{\omega}_0$, $\vec{\omega}$ and its constant angular acceleration $\vec{\alpha}$ are all along the fixed axis. Hence relations of $\theta$, $\vec{\omega}$ and $\vec{\alpha}$ can be written in the scalar form : Since $\alpha$ is constant, according to definition

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_0}{t} \qquad (2.4.1)$$

OR $$\omega = \omega_0 + \alpha t \qquad (2.4.2)$$

This equation is similar to the equation $v = v_0 + at$ in linear motion.

Here, the angular acceleration is constant, hence using average angular velocity we can find the angular displacement.

$\therefore$ Angular displacement

$\theta$ = (average angular velocity) ($t$)

$$\therefore\ \theta = \left(\frac{\omega + \omega_0}{2}\right)t \qquad (2.4.3)$$

This equation is similar to the equation $x = \left(\frac{v + v_0}{2}\right)t$ in linear motion.

Substituting the value of $\omega$ from equation (2.4.2) in equation (2.3.3) we get

$$\theta = \left(\frac{\omega_0 + \alpha t + \omega_0}{2}\right) t$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \qquad (2.4.4)$$

This equation is similar to the equation $x = v_0 t + \frac{1}{2}at^2$ in linear motion.

Substituting the value of $t$ from equation (2.4.1) in equation (2.4.3), we get

$$\theta = \left(\frac{\omega + \omega_0}{2}\right)\left(\frac{\omega - \omega_0}{\alpha}\right)$$

$$\therefore\ 2\alpha\theta = \omega^2 - \omega_0^2 \qquad (2.4.5)$$

This equation is similar to the equation $2ax = v^2 - v_0^2$ in linear motion.

**Illustration 2 :** A mini–train in a children's park moving at a linear velocity of 18 km/h stops in 10 s due to constant angular deceleration produced in it. If the radius of the wheels of the mini-train is 30 cm, find the angular deceleration of the wheel.

**Solution :**

$v_0 = 18$ km/h $= 5$ m/s; $r = 30$ cm $= 0.3$ m

$$\omega_1 = \frac{v_1}{r} = \frac{5}{0.3} = \frac{50}{3} \text{ rad/s}$$

$\omega_2 = 0$, $t = 10$ s

$$\therefore\ \alpha = \frac{\omega_2 - \omega_1}{t} = \frac{0 - \frac{50}{3}}{10}$$

$$= \frac{-5}{3} = -1.666 \text{ rad/s}^2$$

**Illustration 3 :** A truck is moving at a speed of 54 km/h. The radius of its wheels is 50 cm. On applying the brakes the wheels stop after 20 rotation. What will be the linear distance travelled by the truck during this ? Also find the angular acceleration of the wheels.

**Solution :** Here, $v_1 = 54$ km/h $= 15$ m/s; $r = 50$ cm $= 0.5$ m, $\theta = 20$ rotations $= 20 \times 2\pi$ rad $= 40\pi$ rad; $d = ?$, $\alpha = ?$

$$v_1 = r\omega_1\ \therefore\ \omega_1 = \frac{v_1}{r} = \frac{15}{0.5} = 30 \text{ rad/s}$$

$$\omega_2 = 0;\ \alpha = \frac{\omega_2^2 - \omega_1^2}{2\theta} = \frac{0 - 900}{2 \times 40\pi}$$

$$= -3.58 \text{ rad/s}^2$$

Now, 1 rotation $= 2\pi r$ linear distance

$\therefore$ 20 rotations $= 20 \times 2\pi r$ distance.

$\therefore$ linear distance travelled by the truck

$d = 20 \times 2 \times 3.14 \times 0.5$

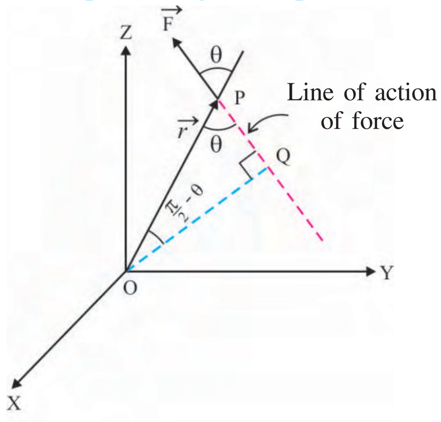
$= 62.8$ m

**2.5 Torque :** Up till now we have discussed rotational motion of rigid body without bothering the causes for it. Now we will think about the cause for it.

Torque is an important physical quantity of the rotational dynamics. Torque plays a similar role in relational motion as the force plays in the linear motion.

We will first discuss the torque acting on a particle and then will discuss the torque acting on the system of particles.

**(a) Torque acting on a particle :**




Torque acting on a particle

**Figure 2.5**

As shown in the Figure 2.5, suppose a force $\vec{F}$ acts on a particle P. The position vector of P with respect to origin O is $\vec{r}$. The angle between $\vec{r}$ and $\vec{F}$ is $\theta$. Here, the particle P is not necessarily be a particle of a rigid body.

The vector product of $\vec{r}$ and $\vec{F}$ is called the torque ($\vec{\tau}$) acting on the particle P, with respect to the point O.

$$\therefore\ \vec{\tau} = \vec{r} \times \vec{F} \qquad (2.5.1)$$

$$\therefore\ \tau = rF\sin\theta$$

From Figure 2.5, $r\sin\theta$ = OQ = the perpendicular distance of the line of action of force from O.

$\therefore\ \tau$ = (F) (perpendicular distance of line of action of force from O)

= Moment of force with respect to point O (by definition)

Thus, torque is the moment of force with respect to a given reference point. Its dimensional formula is $M^1 L^2 T^{-2}$ and its unit is N m.

Remember that,

(i) According to the right hand screw rule the direction of torque ( $\vec{\tau}$ ) is perpendicular to the plane formed by $\vec{r}$ and $\vec{F}$.

(ii) Since the value of torque ( $\vec{\tau}$ ) depends on the reference point, in defining the torque, the reference point must be mentioned.

**(*b*) Torque Acting on the System of Particles :**

The mutual internal forces between the particles of a system are equal and opposite, the resultant force and hence the torque produced due to them becomes zero. Hence we will not consider the internal forces in our discussion.

Suppose for a system of particles the position vector of different particles are $\vec{r_1}, \vec{r_2}, ...., \vec{r_n}$ and the respective forces acting on them are $\vec{F_1}, \vec{F_2}, ..., \vec{F_n}$. The resultant torque on the system means the vector sum of the torque acting on every particle of the system.

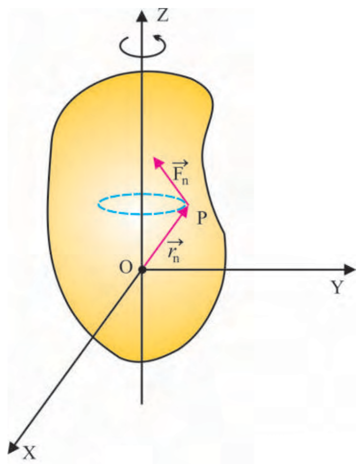
$$\vec{\tau} = \vec{\tau_1} + \vec{\tau_2} + ....... + \vec{\tau_n} \quad (2.5.2)$$

∴ Resultant torque

$$\vec{\tau} = (\vec{r_1} \times \vec{F_1}) + (\vec{r_2} \times \vec{F_2}) + ..... + (\vec{r_n} \times \vec{F_n})$$

$$= \sum_{i=1}^{n} (\vec{r_i} \times \vec{F_i}) \quad (2.5.3)$$

**(*c*) Torque acting on the rigid body :**




Torque acting on the rigid body

Figure 2.6

Suppose a rigid body rotates about a fixed axis OZ, as shown in the Figure 2.6. The forces acting on the particles with position vectors $\vec{r_1}, \vec{r_2}, ...., \vec{r_n}$ are $\vec{F_1}, \vec{F_2}, ..., \vec{F_n}$ respectively.

Considering the force $\vec{F_n}$ acting on the particle with position vector $\vec{r_n}$ the torque $\vec{\tau_n}$ acting on it is

$$\vec{\tau_n} = \vec{r_n} \times \vec{F_n}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_n & y_n & z_n \\ F_{nx} & F_{ny} & F_{nz} \end{vmatrix}$$

$$\therefore \vec{\tau_n} = (y_n F_{nz} - z_n F_{ny})\hat{i} + (z_n F_{nx} - x_n F_{nz})\hat{j} + (x_n F_{ny} - y_n F_{nx})\hat{k} \quad (2.5.4)$$

From equation (2.5.4) the torque acting on the entire body can be written as a vector sum of the torques acting on all particles, as follows :

$$\vec{\tau_n} = \sum_n (y_n F_{nz} - z_n F_{ny})\hat{i} + (z_n F_{nx} - x_n F_{nz})\hat{j} + (x_n F_{ny} - y_n F_{nx})\hat{k} \quad (2.5.5)$$

For the rotational motion of the rigid body about Z-aixs, only the Z–component of the above mentioned torque is responsible. For the rotational motion about X–axis the X–component and about Y–axis the Y–component of the torque is responsible. As a general case if the unit vector on the rotational axis is $\hat{n}$; the $\vec{\tau} \cdot \hat{n}$ component of the torque is responsible for the rotational motion, about that axis.

To produce the rotational motion of the rigid body external forces are required to be applied, but not on all the particles of it. As for example, we do not apply forces on all the particles of a door to open it or shut.

Since the relative distances between all the particles of a rigid body remain invariant, the torque produced by applying a force on any one particle becomes the torque on the entire rigid body. If the force acting on any one particle with position vector $\vec{r}$ is $\vec{F}$, then the torque on the rigid body can be taken as $\vec{\tau} = \vec{r} \times \vec{F}$.

**Illustration 4 :** The force acting on a particle $\vec{r} = (4, 6, 12)$ m of a rigid body is $\vec{F} = (6, 8, 10)$ N. Find the magnitude of the torque producing the rotational motion about an axis along which the unit vector is $\frac{1}{\sqrt{3}}(1, 1, 1)$.

**Solution :** $\vec{\tau} = \vec{r} \times \vec{F}$

The magnitude of the torque with respect to the axis on which the unit vector is $\hat{n}$, is

$$\tau_n = (\vec{r} \times \vec{F}) \cdot \hat{n}$$

$$\text{Now, } \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 6 & 12 \\ 6 & 8 & 10 \end{vmatrix}$$

$$= (-36)\hat{i} - (-32)\hat{j} + (-4)\hat{k}$$

$$\therefore \vec{\tau} = (-36, 32, -4) \text{ Nm}$$

Magnitude of the torque responsible for rotational motion

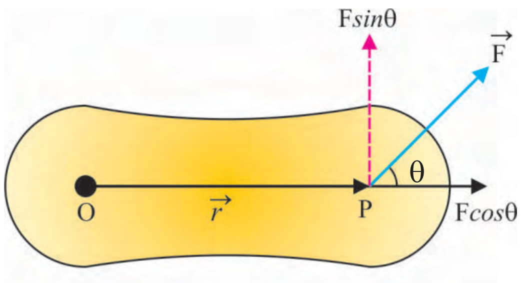
$$\text{Now, } (\vec{r} \times \vec{F}) \cdot \hat{n}$$

$$= (-36, 32, -4) \cdot \frac{1}{\sqrt{3}}(1, 1, 1)$$

$$= \frac{1}{\sqrt{3}}(-36 + 32 - 4)$$

$$= -\frac{8}{\sqrt{3}} \text{ N m} \quad \therefore \text{ The magnitude is } \frac{8}{\sqrt{3}} \text{ N m}$$

### (*d*) Physical interpretation of the definition of torque :




Effective component of the torque

Figure 2.7

Suppose a force $\vec{F}$ acts on the particle P of the rigid body as shown in the Figure 2.7. Here the force is taken in a plane perpendicular to the rotational axis, which is coming out from the plane of paper, from point O.

The position vector of point P with respect to the centre of its circular path is $\vec{r}$. The angle between $\vec{F}$ and $\vec{r}$ is $\theta$. To understand the effectiveness of $\vec{F}$ in producing the rotational motion, consider two components of $\vec{F}$.

(i) The component of $\vec{F}$ parallel to $\vec{r}$ is say $F_1 = F \cos\theta$. Hence $\vec{r} \times \vec{F_1} = 0$. This does not produce torque. Thus it does not produce the rotational motion.

(ii) The component of $\vec{F}$ perpendicular to $\vec{r}$ is $F_2 = F \sin\theta$. This component produces the rotational motion. If the magnitude of F and/or $\theta$ is more, then $\vec{F}$ becomes more effective. Moreover, our common experience tells us that if the position vector $\vec{r}$ of the point of action of $\vec{F}$ is more, then also $\vec{F}$ becomes more effective in producing rotation. Thus the quantity responsible for producing rotation is not only $\vec{F}$ but is r F sin$\theta$. This quantity is called the torque. Writing the above formula in the vector form

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad (2.5.6)$$

Remember that torque is the measure of the effectiveness of the force in producing the rotational motion.

**(*e*) Couple :** Two forces of equal magnitude and opposite directions which are not collinear form a couple. As shown in the Figure 2.8, forces $\vec{F_1}$ and $\vec{F_2}$ act on two particles P and Q of the rigid body having position vectors $\vec{r_1}$ and $\vec{r_2}$ respectively. Here, $|\vec{F_1}| = |\vec{F_2}|$ and the directions of $\vec{F_1}$ and $\vec{F_2}$ are mutually opposite. The resultant torque of the torques $\vec{\tau_1}$ and $\vec{\tau_2}$ produced due to the forces $\vec{F_1}$ and $\vec{F_2}$ is called the moment of the couple $(\vec{\tau})$.

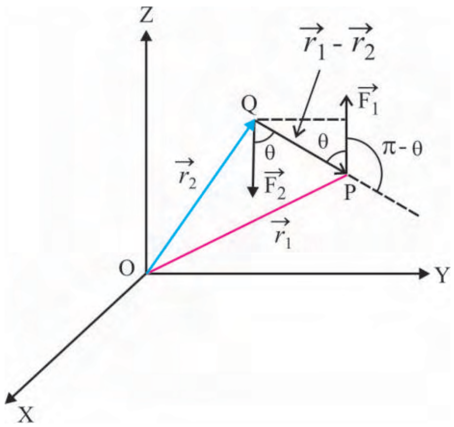



Figure 2.8

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2$$

$$\therefore \vec{\tau} = (\vec{r}_1 \times \vec{F}_1) + (\vec{r}_2 \times \vec{F}_2)$$

$$= (\vec{r}_1 \times \vec{F}_1) - (\vec{r}_2 \times \vec{F}_1)$$

$$(\because \vec{F}_2 = -\vec{F}_1)$$

$$\therefore \vec{\tau} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_1$$

$$= |\vec{r}_1 - \vec{r}_2|(F_1) \sin(\pi - \theta)$$

$$= |\vec{r}_1 - \vec{r}_2|(F_1) \sin\theta$$

Where $(\pi - \theta)$ is the angle between $(\vec{r}_1 - \vec{r}_2)$ and $\vec{F}_1$

From the figure $|\vec{r}_1 - \vec{r}_2| \sin\theta$ = perpendicular distance between the two forces.

$\therefore$ Moment of couple = $(F_1)$ (perpendicular distance between the two forces),

= (magnitude of any one of the two forces) (perpendicular distance between the two forces) (2.5.8)

Student friends, do you know that you are also using couple in practice ? When you are driving scooter or car, to turn the vehicle, the forces you apply on the steering, produce couple.

## (*f*) Equilibrium of a rigid body :

Now we shall discuss the equilibrium of the rigid body under the influence of many forces acting on it. If the external forces acting on the rigid body are $\vec{F}_1, \vec{F}_2, \ldots\ldots, \vec{F}_n$ and if resultant force $\vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots.. + \vec{F}_n = 0$ (2.5.9)

then the rigid body remains in translational equilibrium. Writing the above equation in the form of the components of forces, $\sum_i F_{xi} = 0$;

$$\sum_i F_{yi} = 0; \text{ and } \sum_i F_{zi} = 0 \qquad (2.5.9\ a)$$

If the torques produced by the above mentioned forces are $\vec{\tau}_1, \vec{\tau}_2, \ldots\ldots, \vec{\tau}_n$ then the rigid body remains in rotational equilibrium when

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + \ldots.. + \vec{\tau}_n = 0. \qquad (2.5.10)$$

That is, if rigid body is stationary, it will remain stationary and if it is performing rotational motion, it will continue rotational motion with constant angular velocity.

Writing this equation in the form of components of torques.

$$\sum_i \tau_{xi} = 0; \sum_i \tau_{yi} = 0; \text{ and } \sum_i \tau_{zi} = 0 \qquad (2.5.10\ a)$$

**Illustration 5 :** As shown in the figure a block of mass m moves with constant velocity under the influence of a force F acting in the direction making an angle θ with the horizontal. If the frictional force between the surface of the block and the horizontal surface is $f_k$, find the distance of line of action of normal reaction N from O. Length of the block is L and height is h.

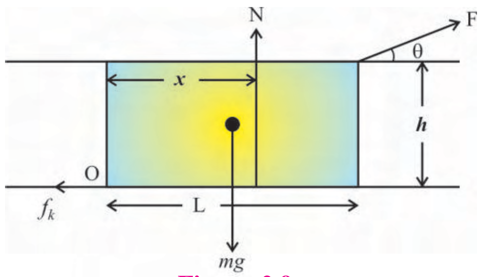



Figure 2.9

**Solution :** The block does not perform rotational motion in spite of being under the influence of various forces. Hence, it is in rotational equilibrium. In this condition the vector sum of the torques produced due to different forces should be zero. Taking all the torques with reference to the point O, we get, $\tau = f_k(0) - (mg)\left(\frac{L}{2}\right) + N(x) - (F\cos\theta)(h) + F\sin\theta(L) = 0.$

(Here the torque in the clockwise direction is taken negative and the torque in the anticlockwise direction is taken positive).

$$\therefore \quad N(x) = (mg)\left(\frac{L}{2}\right) + (F\cos\theta)(h) - F\sin\theta\,(L) \qquad (1)$$

Now, for translation equilibrium,

$mg = N + F\sin\theta$ and $F\cos\theta = f_k$

$\therefore\ N = mg - F\sin\theta$

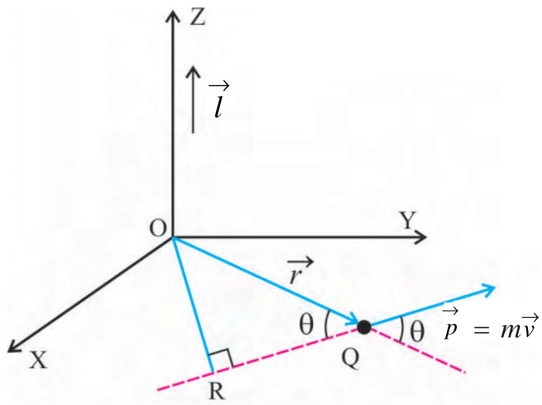
Substituting this value in equation (1), and making $x$ the subject of the formula we get,

$$x = \frac{(mg)\left(\frac{L}{2}\right) + (F\cos\theta)(h) - (F\sin\theta)(L)}{mg - F\sin\theta}$$

## 2.6 Angular Momentum

### (*a*) Angular Momentum of a Particle :

Suppose the position vector of a particle of mass $m$ is $\vec{OQ} = r$ in a Cartesian co-ordinate system as shown in the Figure 2.10. The linear velocity of this particle is $\vec{v}$ and its linear momentum is $\vec{p} = m\vec{v}$. Here, the particle Q is not necessarily a particle of a rigid body. Suppose the angle between $\vec{p}$ and $\vec{r}$ is $\theta$. We have taken the particle and its motion in the (X – Y) plane only for simplicity. The vector product of $\vec{r}$ and $\vec{p}$ is called the angular momentum $\vec{l}$ of the particle with reference to the point O.




Angular momentum

Figure 2.10

$$\vec{l} = \vec{r} \times \vec{p} \qquad (2.6.1)$$

The SI unit of $\vec{l}$ is kg $m^2s^{-1}$ or Js.

(i) The magnitude of $\vec{l}$ depends on the selection of the reference point, hence the reference point must be mentioned in its definition.

(ii) The direction of $\vec{l}$ is given by the right hand screw rule for the vector product. In the present case the direction of $\vec{l}$ is in OZ direction.

(iii) Now $|\vec{l}| = |\vec{r} \times \vec{p}| = r\,p\,\sin\theta$

But from Figure 2.10,

$r\sin\theta = OR$

$\therefore\quad l = (p)$ (distance OR)

Thus the angular momentum of the particle

= (linear momentum) (perpendicular distance of the vector of linear momentum from reference point)

= moment of linear momentum with reference to point O.

**Note :** Cartesian components of angular momentum of a particle :

By definition, the angular momentum is

$$\vec{l} = \vec{r} \times \vec{p}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$= (yp_z - zp_y)\hat{i} + (zp_x - xp_z)\hat{j} + (xp_y - yp_x)\hat{k}$$

$$\vec{l} = l_x\hat{i} + l_y\hat{j} + l_z\hat{k}$$

Here, $l_x$, $l_y$ and $l_z$ are the components of angular momentum with reference to X, Y and Z axes respectively.

### (*b*) The relation between angular momentum of a particle and torque acting on it :

Differentiating equation (2.6.1) with respect to time, we get

$$\frac{d\vec{l}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

But $\frac{d\vec{p}}{dt}$ = rate of change of linear momentum = $\vec{F}$ (force) and $\frac{d\vec{r}}{dt} = \vec{v}$ (velocity)