CBSE Test Paper 01

Chapter 7 Integrals

1. $\int \frac{1}{e^x+1} dx$ is equal to
   a. $\log\left(1+e^{-2x}\right)+C$
   b. $\log(e^{-2x}-2x)+C$
   c. – log(1 + e^{-x}) + C
   d. $\log(e^{3x}+x)+C$

2. The function $f(x)=\int\limits_0^x \log(t+\sqrt{1+t^2})dt$ is
   a. an odd function
   b. an even function
   c. Neither odd nor Even
   d. a periodic function

3. $\int\limits_0^{\pi/2} \log|\cos x|$ dx is equal to
   a. $-\frac{\pi}{2}\log 2$
   b. $\pi\ log\ 2$
   c. $\frac{\pi}{2}\ \log 3$
   d. $-\frac{\pi}{3}\ \log 3$

4. $\int\limits_a^b \frac{\log x}{x} dx$ is equal to
   a. $\frac{log\ (b-a)}{b-a}$
   b. log (a+b).log (b–a)
   c. $\log(ab).log(\frac{b}{a})$
   d. $\frac{1}{2}\log(ab).log\left(\frac{b}{a}\right)$

5. If $\int f(x)$ dx = f (x), then

a. f(x) = a^x
b. f (x) = x
c. f(x) = 0
d. $f(x) = e^x$

6. The function A(x) denotes the ________ function and is given by A (x) = $\int_a^x f(x)dx$.
7. The indefinite integral of $2x^{\frac{1}{2}}$ is ________.
8. The indefinite integral of $2x^2 + 3$ is ________.
9. Show that $\int \frac{2x+3}{x^2+3x}dx = \log\left|x^2+3x\right| + C$.
10. Evaluate $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$.
11. Evaluate $\int \sin^3 x dx$.
12. Evaluate the definite integral $\int_0^{\frac{\pi}{4}} \left(2\sec^2 x + x^3 + 2\right) dx$.
13. Integrate the following function $\frac{1}{\sqrt{7-6x-x^2}}$.
14. Integrate the function $(x^2 + 1) \log x$
15. $\int_0^{\pi/4} \frac{\sin x \cos x}{\cos^4 x+\sin^4 x}dx$.
16. Evaluate $\int \frac{\sqrt{1-\sin x}}{1+\cos x} e^{\frac{-x}{2}} dx$.
17. Evaluate $\int_0^1 x \log(1+2x)dx$
18. Evaluate $\int_1^3 \left(2x^2 + 5x\right) dx$ as a limit of a sum.

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Solution

1. c. –log(1 + e^{-x}) + C, **Explanation:**
   $\int \frac{e^{-x}}{1+e^{-x}} dx = -\int \frac{-e^{-x}}{1+e^{-x}} dx = -\log(1+e^{-x}) + C$
2. b. an even function, **Explanation:** t = -u
   f(-x) = $\int_0^x \log(-u+\sqrt{1+u^2})(-du)$
   = $-\int_0^\pi \log\left(\frac{1+u+u^2}{\sqrt{1+u^2+u}}\right)(-du)$
   = $\int_0^x \log(u+\sqrt{1+u^2})du$ = f(x)
   $\Rightarrow$ f(-x) = f(x) $\Rightarrow$ f is an even function
3. a. $-\frac{\pi}{2}\log 2$, **Explanation:** $\int_0^{\pi/2} \log|\cos x|dx$
   $= \int_0^{\pi/2} \log\left|\cos(\frac{\pi}{2}-x)\right|dx$
   $= \int_0^{\pi/2} \log|\sin x|dx$
   $= -\frac{\pi}{2}\ \log 2$ (standard result)
4. d. $\frac{1}{2}\log(ab).log\left(\frac{b}{a}\right)$, **Explanation:** $= \int_a^b (log\ x)^1 (\frac{1}{x})dx$
   $(let\ log\ x = t\ then\ \frac{1}{x}dx = dt)$
   $= \left[\frac{(log\ x)^2}{2}\right]_a^b$
   $= \frac{1}{2}\left[(log\ b)^2 - (log\ a)^2\right]$
   $= \frac{1}{2}(log\ b + log\ a)(log\ b - log\ a)$
   $= \frac{1}{2}\log(ab)\log\frac{b}{a}$
5. d. $f(x) = e^x$, m**Explanation:** $\frac{d}{dx}(f(x)) = f(x)$
   It implies that the function remains same after integrating or differentiating it.

So the function must be e^x

6. area
7. $\frac{4}{3}x^{\frac{3}{2}}+c$
8. $=\frac{4}{5}x^{\frac{5}{4}}+c$
9. Let $I=\int \frac{2x+3}{x^2+3x}dx$

   Put $x^2 + 3x = t$

   $\Rightarrow (2x + 3)dx = dt$

   $\therefore I=\int \frac{1}{t}dt=\log|t|+C$

   $= \log|(x^2 + 3x)| + C$
10. According to the question , $I=\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$

    $= \left[\sin^{-1}x\right]_0^1 \quad \left[\because \int_b^a \frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}(a)-\sin^{-1}(b)\right]$

    $=sin^{-1}(1)-sin^{-1}(0)$

    $=\sin^{-1}\left(\sin\frac{\pi}{2}\right)-\sin^{-1}(\sin 0)$

    $=\frac{\pi}{2}-0=\frac{\pi}{2}$
11. Let $I=\int \sin^3 x dx=\int \frac{3\sin x-\sin 3x}{4}dx\ \left[\because \sin 3x=3\sin x-4\sin^3 x\right]$

    $=\frac{1}{4}\int 3\sin x dx-\frac{1}{4}\int \sin 3x dx$

    $=\frac{1}{4}\left(-3\cos x+\frac{\cos 3x}{3}\right)+C$
12. $\int\limits_0^{\frac{\pi}{4}}\left(2\sec^2 x+x^3+2\right)dx$

    $=2\int\limits_0^{\frac{\pi}{4}}\sec^2 x dx+\int\limits_0^{\frac{\pi}{4}}x^3 dx+2\int\limits_0^{\frac{\pi}{4}}1dx$

    $=2(\tan x)_0^{\frac{\pi}{4}}+\left(\frac{x^4}{4}\right)_0^{\frac{\pi}{4}}+2(x)_0^{\frac{\pi}{4}}$

    $=2\left(\tan\frac{\pi}{4}-\tan 0^o\right)+\frac{\left(\frac{\pi}{4}\right)^4}{4}-0+2\left(\frac{\pi}{4}-0\right)$

    $=2(1-0)+\frac{\left(\frac{\pi^4}{256}\right)}{4}+\frac{2\pi}{4}$

    $=2+\frac{\pi^4}{1024}+\frac{\pi}{2}$

    $=\frac{\pi^4}{1024}+\frac{\pi}{2}+2$
13. $\int \frac{1}{\sqrt{7-6x-x^2}}dx$

$= \int \frac{1}{\sqrt{-x^2-6x+7}} dx$

$= \int \frac{1}{\sqrt{-(x^2+6x-7)}} dx$

$= \int \frac{1}{\sqrt{-(x^2+6x+9-9-7)}} dx$

$= \int \frac{1}{\sqrt{-\left\{(x+3)^2-16\right\}}} dx$

$= \int \frac{1}{\sqrt{(16)-(x+3)^2}} dx$

$= \int \frac{1}{\sqrt{(4)^2-(x+3)^2}} dx$

$= \sin^{-1}\left(\frac{x+3}{4}\right) + c$

$\left[\because \int \frac{1}{a^2-x^2} dx = \sin^{-1} \frac{x}{a}\right]$

14. $\int \left(x^2+1\right) \log x dx$

$= \int (\log x)\left(x^2+1\right) dx$

[Applying product rule]

$= \log x\left(\frac{x^3}{3}+x\right) - \int \frac{1}{x}\left(\frac{x^3}{3}+x\right) dx$

$= \left(\frac{x^3}{3}+x\right)\log x - \int\left(\frac{x^2}{3}+1\right) dx$

$= \left(\frac{x^3}{3}+x\right)\log x - \frac{1}{3}\int x^2 dx - \int 1 dx$

$= \left(\frac{x^3}{3}+x\right)\log x - \frac{1}{3}\frac{x^3}{3} - x + c$

$= \left(\frac{x^3}{3}+x\right)\log x - \frac{x^3}{9} - x + c$

15. $I = \int_0^{\pi/4} \frac{\sin x \cos x}{\cos^4 x+\sin^4 x} dx$

Dividing Numerator and Denominator by $\cos^4 x$

$= \int_0^{\pi/4} \frac{\frac{\sin x.\cos x}{\cos^4 x}}{\frac{\cos^4 x}{\cos^4 x}+\frac{\sin^4 x}{\cos^4 x}} dx$

$= \int_0^{\pi/4} \frac{\tan x.\sec^2 x}{1+\tan^4 x} dx$

$= \int_0^{\pi/4} \frac{\tan x.\sec^2 x}{1+\left(\tan^2 x\right)^2} dx$

put $\tan^2 x = t$

$2\tan x.\sec^2 x dx = dt$

when x=0, t=0 and when x=$\frac{\pi}{4}$ t=1

$\therefore I = \frac{1}{2}\int_0^1 \frac{dt}{1+t^2}$

$= \frac{1}{2}\left[\tan^{-1}t\right]_0^1$
$= \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}$

16. Given, $I = \int \frac{\sqrt{1-\sin x}}{1+\cos x}\cdot e^{\frac{-x}{2}}\,dx$
Let $\frac{-x}{2} = t \Rightarrow dx = -2dt$
$I = \int \frac{\sqrt{1-\sin(-2t)}}{1+\cos(-2t)} e^t(-2dt)\ [\because x = -2t]$
$= -2\int e^t \frac{\sqrt{1+\sin 2t}}{1+\cos 2t} dt[\because \sin(-\theta) = -\sin\theta \text{ and } \cos(-\theta) = \cos\theta]$
$= -2\int e^t \frac{\sqrt{cos^2x+sin^2x+2sinxcosx}}{1+\cos 2t} dt[\because cos^2x + sin^2x = 1, sin2x = 2sinxcosx]$
$= -2\int e^t \left(\frac{\sqrt{(\cos t+\sin t)^2}}{2\cos^2 t}\right) dt[\because (a+b)^2 = a^2+b^2+2ab]$
$= -2\int e^t \left(\frac{\cos t+\sin t}{2\cos^2 t}\right) dt$
$= -\int e^t(\sec t + \tan t\sec t)dt\,[\because \frac{1}{cosx} = secx, \frac{sinx}{cosx} = tanx]$
we know that , $\int e^t\left[f(t) + f'(t)\right]dt = e^t f(t) + C$
Now, consider $f(t) = sec\,t$
then $f'(t) = sec\,t\,tan\,t$
$\therefore I = e^t \sec t + C$
$= -e^{-x/2}\sec\frac{x}{2} + C\ \left[\because t = \frac{-x}{2} \text{ and } \sec(-\theta) = \sec\theta\right]$
$I = -e^{-x/2}\sec\frac{x}{2} + C$

17. $I = \int\limits_0^1 x\log(1+2x)dx$
$= \left[\log(1+2x)\frac{x^2}{2}\right]_0^1 - \int \frac{1}{1+2x}.2.\frac{x^2}{2}dx$
$= \frac{1}{2}\left[x^2\log(1+2x)\right]_0^1 - \int \frac{x^2}{1+2x}dx$
$= \frac{1}{2}[\log 3 - 0]_0^1 - \left[\int_0^1\left(\frac{x}{2} - \frac{\frac{x}{2}}{1+2x}\right)dx\right]$
$= \frac{1}{2}\log 3 - \frac{1}{2}\int_0^1 xdx + \frac{1}{2}\int_0^1 \frac{x}{1+2x}dx$
$= \frac{1}{2}\log 3 - \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\int_0^1 \frac{\frac{1}{2}(2x+1-1)}{(2x+1)}dx$
$= \frac{1}{2}\log 3 - \frac{1}{2}\left[\frac{1}{2} - 0\right] + \frac{1}{4}\int_0^1 dx - \frac{1}{4}\int_0^1 \frac{1}{1+2x}dx$
$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4}[x]_0^1 - \frac{1}{8}\left[\log|(1+2x)|\right]_0^1$
$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4} - \frac{1}{8}[\log 3 - \log 1]$
$= \frac{1}{2}\log 3 - \frac{1}{8}\log 3$
$= \frac{3}{8}\log 3$

18. According to the question , $I = \int_1^3 \left(2x^2 + 5x\right) dx$

On comparing the given integral with $\int_a^b f(x)dx$, we get

a = 1, b = 3 and f(x) =$2x^2$ + 5x

We know that ,

$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)\ ]$...(i)

where, $h = \frac{b-a}{n} \Rightarrow nh = b - a$

$$= 3 - 1 = 2$$

f(a) = f(1) = $2(1)^2$ + 5(1) = 2 + 5 = 7

f(a+h) = f(1 + h) = $2(1 + h)^2$ + 5(1 + h) = 2 + $2h^2$ + 4h + 5 + 5h = $2h^2$+ 9h + 7

f(a + 2h) = $2(1 + 2h)^2$ + 5(1 + 2h) = 2 + $8h^2$ + 8h + 5 +10h = $8h^2$ + 18h + 7

so on

f(a+(n-1)h) = f{1 + (n - 1)h} = $2\{1 + (n - 1)h\}^2$ + 5{1 + (n - 1)h}^{2} = 2 + 2(n - 1)$2h^2$ + 4(n - 1)h + 5 + 5(n - 1)h = 2(n - 1)$2h^2$ + 9(n - 1)h + 7

On putting all above values in (i), we get

$\int_1^3 \left(2x^2 + 5x\right)dx = \lim_{h\to 0}$ h[7 + ($2h^2$ + 9h + 7) +($8h^2$ + 18h + 7).......+2(n - 1)$2h^2$ + 9(n - 1)h + 7]

On rearranging terms , we get

$= \lim_{h\to 0}$h[7 + 7+ 7........+ 7] + $\lim_{h\to 0}$h[$2h^2$ + $8h^2$+.........+$2(n - 1)^2h^2$] + $\lim_{h\to 0}$h[9h + 18h +......+9(n -1)h]

$= \lim_{h\to 0}$7nh + $\lim_{h\to 0}$ $2h^3$[1^2 + 2^2 + 3^2 +.........+$(n -1)^2$] +$\lim_{h\to 0}$ $9h^2$[1+ 2 +........+(n - 1)]

$= \lim_{h\to 0} 7(2) + \lim_{h\to 0} \frac{2h^3 \cdot n(n-1)(2n-1)}{6} + \lim_{h\to 0} \frac{9h^2 \cdot n(n-1)}{2}$

$[\because \sum n = \frac{n(n+1)}{2}, \sum n^2 = \frac{n(n+1)(2n+1)}{6}]$

$= 14 + \lim_{h\to 0} \frac{nh(nh-h)(2nh-h)}{3} + \lim_{h\to 0} \frac{9}{2} \cdot nh(nh - h)$

$= 14 + \frac{2(2-0)(4-0)}{3} + \frac{9}{2} \cdot 2(2 - 0)$

$= 14 + \frac{16}{3} + 18$

$= \frac{42+16+54}{3}$

$= \frac{112}{3}$ sq units.