Electric Oscillations (Part - 1)

Q.94. Due to a certain cause the free electrons in a plane copper plate shifted over a small distance x at right angles to its surface. As a result, a surface charge and a corresponding restoring force emerged, giving rise to so-called plasma oscillations. Find the angular frequency of these oscillations if the free electron concentration in copper is n, = 0.85.10^{29} m^{-1}.

Ans. I f the electron (charge of each electron = - e) are shifted by a same distance x, a net + ve charge density (per unit area) is induced on the surface. This will result in an electric field $E = n e x / \varepsilon_0$ in the direction of x and a restoring force on an electron of

$$- \frac{n e^2 x}{\varepsilon_0},$$

Thus $$m \ddot{x} = - \frac{n e^2 x}{\varepsilon_0}$$

or $$\ddot{x} + \frac{n e^2}{m \varepsilon_0} x = 0$$

This gives $$\omega_p = \sqrt{\frac{n e^2}{m \varepsilon_0}} = 1.645 \times 10^{16} \text{ s}^{-1}.$$

As the plasma frequency for the problem.

Q.95. An oscillating circuit consisting of a capacitor with capacitance C and a coil of inductance L maintains free undamped oscillations with voltage amplitude across the capacitor equal to V_m. For an arbitrary moment of time find the relation between the current I in the circuit and the voltage V across the capacitor. Solve this problem using Ohm's law and then the energy conservation law.

Ans. Since there are no sources of emf in the circuit, Ohm’s 1 law reads

$$\frac{q}{c} = -L \frac{dI}{dt}$$

Where q = change on the capacitor, $I = \frac{dq}{dt} =$ current through the coil. Then

$$\frac{d^2 q}{dt^2} + \omega_0^2 q = 0, \ \omega_0^2 = \frac{1}{LC}.$$

The solution of this equation is

$$q = q_m \cos(\omega_0 t + \alpha)$$

From the problem $V_m = \frac{q_m}{C}$ Then

$$I = -\omega_0 C V_m \sin(\omega_0 t + \alpha)$$

and $$V = V_m \cos(\omega_0 t - \alpha)$$

$$V^2 + \frac{I^2}{\omega_0^2 C^2} = V_m^2$$

or $$V^2 + \frac{L I^2}{C} = V_m^2 .$$

By energy conservation $\frac{1}{2} L I^2 + \frac{q^2}{2C} = \text{constant}$

When the P.D. across the capacitor takes its maximum value V_m, the current I must be

zero. Thus "constant" $= \frac{1}{2} C V_m^2$

Hence $\frac{L I^2}{C} + V^2 = V_m^2$ once again.

Q.96. An oscillating circuit consists of a capacitor with capacitance C, a coil of inductance L with negligible resistance, and a switch. With the switch disconnected, the capacitor was charged to a voltage V_m, and then at the moment t = 0 the switch was closed. Find:
(a) the current I (t) in the circuit as a function of time;
(b) the emf of self-inductance in the coil at the moments when the electric energy of the capacitor is equal to that of the current in the coil.

Ans. After the switch was closed, the circuit satisfies

$$-L \frac{dI}{dt} = \frac{q}{C}$$

or $$\frac{d^2 q}{dt^2} + \omega_0^2 q = 0 \Rightarrow q = C V_m \cos \omega_0 t$$

Where we have used the fact that when the switch is dosed we must have

$$V = \frac{q}{C} = V_m, \; I = \frac{dq}{dt} = 0 \text{ at } t = 0.$$

Thus (a)

$$I = \frac{dq}{dt} = -C V_m \omega_0 \sin \omega_0 t$$

$$= -V_m \sqrt{\frac{C}{L}} \sin \omega_0 t$$

(b) The electrical energy of the capacitor is $\frac{q^2}{2C} \propto \cos^2 \omega_0 t$ and of the inductor is $\frac{1}{2} L I^2 \propto \sin^2 \omega_0 t$.

The two are equal when

$$\omega_0 t = \frac{\pi}{4}$$

At that instant the emf of the self-inductance is

$$-L \frac{di}{dt} = V_m \cos \omega_0 t = V_m / \sqrt{2}$$

Q.97. In an oscillating circuit consisting of a parallel-plate capacitor and an inductance coil with negligible active resistance the oscillations with energy W are sustained. The capacitor plates were slowly drawn apart to increase the oscillation frequency η-fold. What work was done in the process?

Ans. In the oscillating circuit, let

$$q = q_m \cos \omega t$$

be the change on the condenser where

$\omega^2 = \frac{1}{LC}$ and C and C is the instantaneous capacity of the condenser (S = area of plates)

$$C = \frac{\varepsilon_0 S}{y}$$

y = distance between the plates. Since the oscillation frequency increases η fold, the quantity

$$\omega^2 = \frac{y}{\varepsilon_0 S L}$$

changes η^2 fold and so does y i.e. changes from y_0 initially to $\eta^2 y_0$ finally. Now the

P.D. Across the condenser is

$$V = \frac{q_m}{C}\cos\omega t = \frac{y\, q_m}{\varepsilon_0 S}\cos\omega t$$

And hence the electric field between the plates is

$$E = \frac{q_m}{\varepsilon_0 S}\cos\omega t$$

Thus, the chaige on the plate being $q_m \cos\omega t$, the force on the plate is

$$F = \frac{q_m^2}{\varepsilon_0 S}\cos^2\omega t$$

Since this force is always positive and the plate is pulled slowly we can use the average force

$$\bar{F} = \frac{q_m^2}{2\,\varepsilon_0 S}$$

And work done is $A = \bar{F}(\eta^2 y_0 - y_0) = (\eta^2 - 1)\frac{q_m^2 y_0}{2\,\varepsilon_0 S}$

But $\frac{q_m^2 y_0}{2\,\varepsilon_0 S} = \frac{q_m^2}{2 C_0} = W$ the initial stored energy. Thus.

$$A = (\eta^2 - 1)W.$$

Q.98. In an oscillating circuit shown in Fig. 4.27 the coil inductance is equal to L = 2.5 mH and the capacitor have capacitances C_1 2.0 µF and C_2 = 3.0 µF. The capacitors were charged to a voltage V = 180 V, and then the switch Sw was closed. Find:
(a) the natural oscillation frequency;
(b) the peak value of the current flowing through the coil.

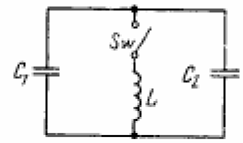


Fig. 4.27.

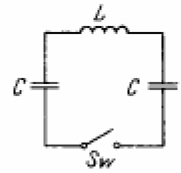


Fig. 4.28.

Ans. The equations of the L - C circuit are

$$L\frac{d}{dt}(I_1+I_)=\frac{C_1V-\int I_1\,dt}{C_1}=\frac{C_2V-\int I_2\,dt}{C_2}$$

Differentiating again $$L\,(I_1+I_2)=-\frac{1}{C_1}I_1=-\frac{1}{C_2}I_2$$

Then

$$I_1=\frac{C_1}{C_1-C_2}I,\ I_2=\frac{C_2}{C_1+C_2}I,$$
$$I=I_1+I_2$$

so $\quad L\,(C_1+C_2)\,I+I=0$

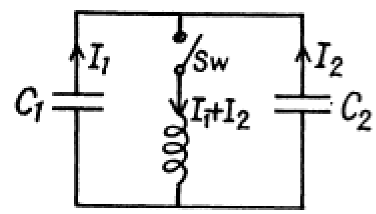
or $\quad I=I_0\sin(\omega_0 t+\alpha)$

where $\quad \omega_0^2=\frac{1}{L\,(C_1+C_2)}$ (Part a)

(Hence $T=\frac{2\pi}{\omega_0}=0.7$ ms)

At $\quad t=0,\ I=0$ so $\alpha=0$

$$I=I_0\sin\omega_0 t$$



The peak value of the current is 70 and it is related to the voltage V by the first equation

$$LI=V-\int I\,dt/(C_1+C_2)$$

or $$+L\,\omega_0 I_0\cos\omega_0 t=V-\frac{1}{C_1+C_2}\int_0^t I_0\sin\omega_0 t\,dt$$

(The P.D. across the inductance is V at t = 0)

$$=V+\frac{1}{C_1+C_2}\cdot\frac{I_0}{\omega_0}(\cos\omega_0 t-1)$$

Hence $$I_0=(C_1+C_2)\,\omega_0 V=V\sqrt{\frac{C_1+C_2}{L}}=8.05\text{ A.}$$

Q.99. An electric circuit shown in Fig. 4.28 has a negligibly small active resistance. The left-hand capacitor was charged to a voltage V_0 and then at the moment t = 0 the switch Sw was closed. Find the time dependence of the voltages in left and right capacitors.

Ans. Initially q_1 = C V 0 and q_2 = 0. After the switch is closed change flows and w e get

$$q_1 + q_2 = C V_0$$

$$\frac{q_1}{C} + L\frac{dI}{dt} - \frac{q_2}{C} = 0$$

Also $I = \dot{q}_1 = -\dot{q}_2$. Thus

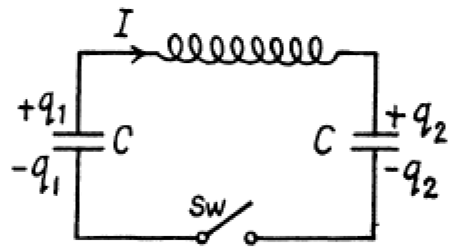
$$L\ddot{I} + \frac{2I}{C} = 0$$

Hence $\ddot{I} + \omega_0^2 I = 0 \quad \omega_0^2 = \frac{2}{LC}$,

The soloution of this equation subject to

$I = 0$ at $t = 0$

is $I = I_0 \sin \omega_0 t$.



Integrating

$$q_1 = A - \frac{I_0}{\omega_0}\cos \omega_0 t$$

$$q_2 = B + \frac{I_0}{\omega_0}\cos \omega_0 t$$

Finally substituting in (1)

$$\frac{A-B}{C} - \frac{2I_0}{\omega_0 C}\cos \omega_0 t + L I_0 \omega_0 \cos \omega_0 t = 0$$

Thus

$$A = B = \frac{C V_0}{2} \text{ and}$$

$$\frac{C V_0}{2} + \frac{I_0}{\omega_0} = 0$$

So

$$q_1 = \frac{C V_0}{2}(1 + \cos \omega_0 t)$$

$$q_2 = \frac{C V_0}{2}(1 - \cos \omega_0 t)$$

Q.100. An oscillating circuit consists of an inductance coil L and a capacitor with capacitance C. The resistance of the coil and the lead wires is negligible. The coil is placed in a permanent magnetic field so that the total flux passing through all the turns of the coil is equal to φ. At the moment t = 0 the magnetic field was switched off. Assuming the switching off time to be negligible compared to the natural oscillation period of the circuit, find the circuit current as a function of time t.

Ans. The flux in the coil is

$$\Phi(t) = \begin{cases} \Phi & t < 0 \\ 0 & t > 0 \end{cases}$$

The equation of the current is $-L\frac{dI}{dt} = \frac{\int_0^t I\,dt}{C}$ (1)

This mean that $LC\frac{d^2I}{dt^2} + I = 0$

or with $\omega_0^2 = \frac{1}{LC}$ $\quad I = I_0 \sin(\omega_0 t + \alpha)$

Putting in (1) $\quad -L I_0 \omega_0 \cos(\omega_0 t + \alpha) = -\frac{I_0}{\omega_0 C}[\cos(\omega_0 t + \alpha) - \cos\alpha]$

This implies $\cos\alpha = 0$ $\quad \therefore \quad I = \pm I_0 \cos\omega_0 t$. From Faraday's law

$$\varepsilon = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$$

or integrating from $t = -\varepsilon$ to $-\varepsilon$ where $\varepsilon \to 0$

$$\Phi = L I_0 \text{ with + sign in } I$$

so, $$I = \frac{\Phi}{L}\cos\omega_0 t.$$

Q.101. The free damped oscillations are maintained in a circuit, such that the voltage across the capacitor varies as $V = V_m e^{-\beta t}\cos\omega t$. Find the moments of time when the modulus of the voltage across the capacitor reaches
(a) peak values;
(b) maximum (extremum) values.

Ans. Given $V = V_m e^{-\beta t}\cos\omega t$

(a) The phrase 'peak values' is not clear. The answer is obtained on taking $|\cos\omega t| = 1$

i.e $t = \frac{\pi n}{\omega}$.

(b) For extrema $\frac{dV}{dt} = 0$

$$- \beta \cos \omega t - \omega \sin \omega t = 0$$

or $$\tan \omega t = - \beta / \omega$$

i.e. $$\omega t = n\pi + \tan^{-1}\left(\frac{-\beta}{\omega}\right).$$

Q.102. A certain oscillating circuit consists of a capacitor with capacitance C, a coil with inductance L and active resistance R, and a switch. When the switch was disconnected, the capacitor was charged; then the switch was closed and oscillations set in. Find the ratio of the voltage across the capacitor to its peak value at the moment immediately after closing the switch.

Ans. The equation of the circuit is

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = 0$$

This has the solution $$Q = Q_m e^{-\beta t} \sin(\omega t + \alpha)$$

where $$\beta = \frac{R}{2L}, \ \omega = \sqrt{\omega_0^2 - \beta^2}, \ \omega_0^2 = \frac{1}{LC}$$

Now $$I = \frac{dQ}{dt} = 0 \quad \text{at } t = 0$$

so, $$Q_m e^{-\beta t} (-\beta \sin(\omega t + \alpha) + \omega \cos(\omega t + \alpha)) = 0 \quad \text{at } t = 0$$

Thus $$\omega \cos \alpha = \beta \sin \alpha \quad \text{or} \quad \alpha = \tan^{-1} \frac{\omega}{\beta}$$

Now $$V_m = \frac{Q_m}{C} \quad \text{and} \quad V_0 = \text{P.D. at } t = 0 = \frac{Q_m}{C} \sin \alpha$$

$$\therefore \quad \frac{V_0}{V_m} = \sin \alpha = \frac{\omega}{\sqrt{\omega^2 + \beta^2}} = \frac{\omega}{\omega_0} = \sqrt{1 - \beta^2/\omega_0^2} = \sqrt{1 - \frac{R^2 C}{4L^2}}$$

Q.103. A circuit with capacitance C and inductance L generates free damped oscillations with current varying with time as I = = $l_m e^{-\beta t}$ sin ωt. Find the voltage across the capacitor as a function of time, and in particular, at the moment t = 0.

Ans. We write

$$-\frac{dQ}{dt} = I = I_m e^{-\beta t} \sin \omega t$$

$$= gm \ I_m \ e^{-\beta t + i\omega t}$$ (gm means imaginary part)

Then

$$Q = gm\, I_m \frac{e^{-\beta t + i\omega t}}{-\beta + i\omega}$$

$$Q = gm\, I_m \frac{e^{-\beta t + i\omega t}}{\beta - i\omega}$$

$$= gm\, I_m \frac{(\beta + i\omega) e^{-\beta t + i\omega t}}{\beta^2 + \omega^2}$$

$$= I_m\, e^{-\beta t} \frac{\beta \sin \omega t + \omega \cos \omega t}{\beta^2 + \omega^2}$$

$$= I_m\, e^{-\beta t} \frac{\sin(\omega t + \delta)}{\sqrt{\beta^2 + \omega^2}}, \quad \tan \delta = \frac{\omega}{\beta}.$$

(An arbitrary constant of integration has been put equal to zero.) Thus

$$V = \frac{Q}{C} = I_m \sqrt{\frac{L}{C}}\, e^{-\beta t} \sin(\omega t + \delta)$$

$$V(0) = I_m \sqrt{\frac{L}{C}} \sin \delta = I_m \sqrt{\frac{L}{C}} \frac{\omega}{\sqrt{\omega^2 + \beta^2}}$$

$$= I_m \sqrt{\frac{L}{C(1 + \beta^2/\omega^2)}}$$

Q.104. An oscillating circuit consists of a capacitor with capacitance C = 4.0 μF and a coil with inductance L = 2.0 mH and active resistance R = 10Ω. Find the ratio of the energy of the coil's magnetic field to that of the capacitor's electric field at the moment when the current has the maximum value.

Ans.

$$I = I_m\, e^{-\beta t} \sin \omega t$$

$$\beta = \frac{R}{2L}, \quad \omega_0 = \sqrt{\frac{1}{LC}}, \quad \omega = \sqrt{\omega_0^2 - \beta^2}$$

$I = -\dot{q}$, $q =$ charge on the capacitor

Then $$q = I_m\, e^{-\beta t} \frac{\sin(\omega t + \delta)}{\sqrt{\omega^2 + \beta^2}}, \quad \tan \delta = \frac{\omega}{\beta}.$$

Thus $$W_M = \frac{1}{2} L\, I_m^2\, e^{-2\beta t} \sin^2 \omega t$$

$$W_E = \frac{I_m^2}{2C} \frac{e^{-2\beta t} \sin^2(\omega t + \delta)}{\omega^2 + \beta^2} = \frac{L I_m^2}{2} e^{-2\beta t} \sin^2(\omega t + \delta)$$

Current is maximum when $\frac{d}{dt} e^{-\beta t} \sin \omega t = 0$

Thus $-\beta \sin \omega t + \omega \cos \omega t = 0$

or $\tan \omega t = \frac{\omega}{\beta} = \tan \delta$

i.e. $\omega t = n\pi + \delta$

and hence
$$\frac{W_M}{W_E} = \frac{\sin^2(\omega t)}{\sin^2(\omega t + \delta)} = \frac{\sin^2 \delta}{\sin^2 2\delta} = \frac{1}{4\cos^2 \delta}$$
$$= \frac{1}{4\beta^2/\omega_0^2} = \frac{\omega_0^2}{4\beta^2} = \frac{1}{LC} \times \frac{L^2}{R^2} = \frac{L}{CR^2} = 5.$$

(W_M is the magnetic energy of the inductance coil and W_E is the electric energy ot t! capacitor.)

Q.105. An oscillating circuit consists of two coils connected in series whose inductances are L_1 and L_2, active resistances are R_1 and R_2, and mutual inductance is negligible. These coils are to be replaced by one, keeping the frequency and the quality factor of the circuit constant. Find the inductance and the active resistance of such a coil.

Ans. Clearly

$$L = L_1 + L_2, \; R = R_1 + R_2$$

Q.106. How soon does the current amplitude in an oscillating circuit with quality factor Q = 5000 decrease η = 2.0 times if the oscillation frequency is v = 2.2 MHz?

Ans.

$$Q = \frac{\pi}{\beta T} \quad \text{or} \quad \beta = \frac{\pi}{QT}$$

Now $\beta t = \ln \eta \quad \text{so} \quad t = \frac{\ln \eta}{\pi} QT$

$$= \frac{Q \ln \eta}{\pi \nu} = 0.5 \text{ ms}$$

Q.107. An oscillating circuit consists of capacitance C = 10μF, inductance L = 25 mH, and active resistance R 1.0 Ω. How many oscillation periods does it take for the current amplitude to decrease e-fold?

Ans. Current decreases e fold in time

$$t = \frac{1}{\beta} = \frac{2L}{R} \text{ sec } = \frac{2L}{RT} \text{ oscillations}$$

$$= \frac{2L}{R}\frac{\omega}{2\pi}$$

$$= \frac{L}{\pi R}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \frac{1}{2\pi}\sqrt{\frac{4L}{R^2 C} - 1} = 15.9 \text{ oscillations}$$

Q.108. How much (in per cent) does the free oscillation frequency ω of a circuit with quality factor Q = 5.0 differ from the natural oscillation frequency ω_0 of that circuit?

Ans.

$$Q = \frac{\pi}{\beta T} = \frac{\omega}{2\beta}$$

$$\therefore \quad \omega = 2\beta Q, \quad \beta = \frac{\omega}{2Q}.$$

Now $$\omega_0 = \omega\sqrt{1 + \frac{1}{4Q^2}} \quad \text{or} \quad \omega = \frac{\omega_0}{\sqrt{1 + \frac{1}{4Q^2}}}$$

so $$\left|\frac{\omega_0 - \omega}{\omega_0}\right| \times 100\,\% = \frac{1}{8Q^2} \times 100\,\% = 0.5\%$$

Q.109. In a circuit shown in Fig. 4.29 the battery emf is equal to �� = 2.0 V, its internal resistance is r = 9.0Ω, the capacitance of the capacitor is C = 10µF, the coil inductance is L = 100 mH, and the resistance

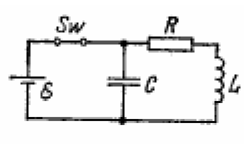
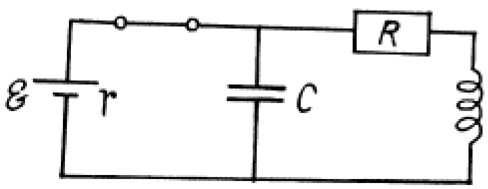


Fig. 4.29.

is R = 1.0 Ω. At a certain moment the switch Sw was disconnected. Find the energy of oscillations in the circuit
(a) immediately after the switch was disconnected;
(b) t = 0.30 s after the switch was disconnected.

Ans.



At t = 0 current through the coil $= \frac{\varepsilon}{R+r}$

P.D. across the condenser $= \frac{\varepsilon}{R+r}$

(a) At t - 0, energy stored = W_0

$$= \frac{1}{2} L \left(\frac{\varepsilon}{R+r} \right)^2 + \frac{1}{2} C \left(\frac{\varepsilon R}{R+r} \right)^2 = \frac{1}{2} \varepsilon^2 \frac{(L + CR^2)}{(R+r)^2} = 2.0 \text{ mJ.}$$

(b) The current and the change stored decrease as $e^{-t R/2L}$ so energy decreases as $e^{-t R/L}$

$$\therefore \quad W = W_0 e^{-tR/L} = 0.10 \text{ mJ.}$$

Q.110. Damped oscillations are induced in a circuit whose quality factor is Q = 50 and natural oscillation frequency is ν_o = 5.5 kHz. How soon will the energy stored in the circuit decrease η = 2.0 times?

Ans.

$$Q = \frac{\pi}{\beta T} = \frac{\pi \nu}{\beta} = \frac{\omega}{2\beta} = \frac{\sqrt{\omega_0^2 - \beta^2}}{2\beta}$$

or $$\frac{\omega_0}{\beta} = \sqrt{1 + 4Q^2} \quad \text{or} \quad \beta = \frac{\omega_0}{\sqrt{1 + Q^2}}$$

Now $$W = W_0 e^{-2\beta t}$$

Thus energy decreases η times in $\frac{\ln \eta}{2\beta}$ sec.

$$= \ln \eta \frac{\sqrt{1 + 4Q^2}}{2\omega_0} = \frac{Q \ln \eta}{2\pi \nu_0} \text{ sec.} = 1.033 \text{ ms.}$$

Electric Oscillations (Part - 2)

Q.111. An oscillating circuit incorporates a leaking capacitor. Its capacitance is equal to C and active resistance to R. The coil inductance is L. The resistance of the coil and the wires is negligible. Find:
(a) the damped oscillation frequency of such a circuit;
(b) its quality factor.

Ans. In a leaky condenser

$$\frac{dq}{dt} = I - I' \quad \text{where } I' = \frac{V}{R} = \text{leak current}$$

Now

$$V = \frac{q}{C} = -L\frac{dI}{dt} = -L\frac{d}{dt}\left(\frac{dq}{dt} + \frac{V}{R}\right)$$

$$= -L\frac{d^2 q}{dt^2} - \frac{L}{RC}\frac{dq}{dt}$$

or

$$\ddot{q} + \frac{1}{RC}\frac{dQ}{dt} + \frac{1}{LC}q = 0$$

Then

$$q = q_m e^{-\beta t}\sin(\omega t + \alpha)$$

(a) $\beta = \frac{1}{2RC}$, $\omega_0^2 = \frac{1}{LC}$, $\omega = \sqrt{\omega_0^2 - \beta^2}$

$$= \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$$

(b) $Q = \frac{\omega}{2\beta} = RC\sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$

$$= \frac{1}{2}\sqrt{\frac{4CR^2}{L} - 1}$$

Q.112. Find the quality factor of a circuit with capacitance C = = 2.0 µF and inductance L = 5.0 mH if the maintenance of undamped oscillations in the circuit with the voltage amplitude across the capacitor being equal to Vm = 1.0 V requires a power (P) 0.10 mW. The damping of oscillations is sufficiently low.

Ans.

Given $V = V_m e^{-\beta t} \sin \omega t$, $\omega = \omega_0$ $\beta T << 1$

$$\text{Power loss} = \frac{\text{Energy loss per cycle}}{T}$$

$$= \frac{1}{2} C V_m^2 \times 2\beta$$

(energy decreases as $W_0 e^{-2\beta t}$ so loss per cycle is $W_0 \times 2\beta T$)

Thus $$\langle P \rangle = \frac{1}{2} C V_m^2 \times \frac{R}{L}$$

or $$R = \frac{2\langle P \rangle}{V_m^2} \frac{L}{C}$$

Hence $$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \sqrt{\frac{C}{L}} \frac{V_m^2}{2\langle P \rangle} = 100 \text{ on putting the vales.}$$

Q.113. What mean power should be fed to an oscillating circuit with active resistance R = 0.45Ω to maintain undamped harmonic oscillations with current amplitude I_m = 30 mA?

Ans. Energy is lost across the resistance and the mean power lass is

$$\langle P \rangle = R \langle I^2 \rangle = \frac{1}{2} R I_m^2 = 0.2 \text{ mW.}$$

This power should be fed to the circuit to maintain undamped oscillations.

Q.114. An oscillating circuit consists of a capacitor with capacitance C = 1.2 nF and a coil with inductance L = 6.0 iLtli and active resistance R = 0.50Ω. What mean power should be fed to the circuit to maintain undamped harmonic oscillations with voltage amplitude across the capacitor being equal to V_m = 10 V?

Ans. $$\langle P \rangle = \frac{R C V_m^2}{2L} \text{ as in (4.112). We get } \langle P \rangle = 5 \text{ mW.}$$

Q.115. Find the damped oscillation frequency of the circuit shown in Fig. 4.30. The capacitance C, inductance L, and active resistance R are supposed to be known. Find how must C, L, and R be interrelated to make oscillations possible.

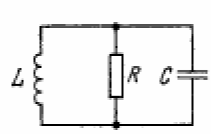


Fig. 4.30.

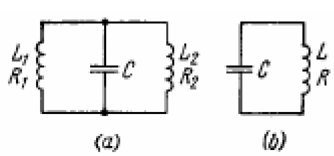


Fig. 4.31.

Ans.

Given $q = q_1 + q_2$

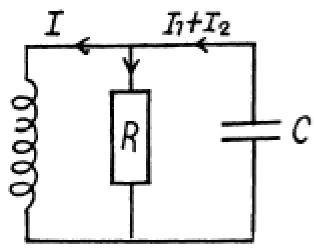
$$I_1 = -\dot{q}_1,\ I_2 = -\dot{q}_2$$

$$L\dot{I}_1 = RI_2 = \frac{q}{C}.$$

Thus $CL\ddot{q}_1 + (q_1 + q_2) = 0$

$RC\dot{q}_2 + q_1 + q_2 = 0$

Putting $q_1 = Ae^{i\omega t}$ $q_2 = Be^{+i\omega t}$



$$(1 - \omega^2 LC)A + B = 0$$

$$A + (1 + i\omega RC)B = 0$$

A solution exists only if

$$(1 - \omega^2 LC)(1 + i\omega RC) = 1$$

or
$$i\omega RC - \omega^2 LC - i\omega^3 LRC^2 = 0$$

or
$$LRC^2\omega^2 - i\omega LC - RC = 0$$

$$\omega^2 - i\omega\frac{1}{RC} - \frac{1}{LC} = 0$$

$$\omega = \frac{i}{2RC} \pm \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}} = i\beta \pm \omega_0$$

Thus $q_1 = (A_1 \cos\omega_0 t + A_2 \sin\omega_0 t)\,e^{-\beta t}$ etc.

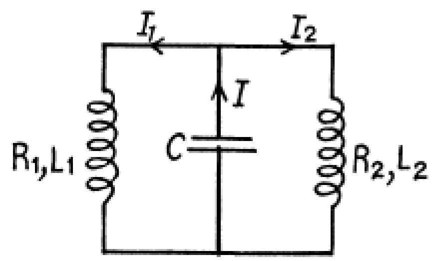
ω_0 is the oscillation frequency. Oscillations are possible only if $\omega_0^2 > 0$

i.e. $\frac{1}{4R^2} < \frac{C}{L}.$

Q.116. There are two oscillating circuits (Fig. 4.31) with capacitors of equal capacitances. How must inductances and active resistances of the coils be interrelated for the frequencies and damping of free oscillations in both circuits to be equal? The mutual inductance of coils in the left circuit is negligible.

Ans. We have

$$L_1 \dot{I}_1 + R_1 I_1 = L_2 \dot{I}_2 + R_2 I_2$$
$$= -\frac{\int I\,dt}{C}$$
$$I = I_1 + I_2$$



Then differentiating we have the equations

$$L_1 C \ddot{I}_1 + R_1 C \dot{I}_1 + (I_1 + I_2) = 0$$
$$L_2 C \ddot{I}_2 + R_2 C \dot{I}_2 + (I_1 + I_2) = 0$$

Look for a solution

$$I_1 = A_1 e^{\alpha t}, \; I_2 = A_2 e^{\alpha t}$$

Then
$$(1 + \alpha^2 L_1 C + \alpha R_1 C) A_1 + A_2 = 0$$
$$A_1 + (1 + \alpha^2 L_2 C + \alpha R_2 C) A_2 = 0$$

This set of simultaneous equations has a nontrivial solution only if

$$(1 + \alpha^2 L_1 C + \alpha R_1 C)(1 + \alpha^2 L_2 C + \alpha R_2 C) = 1$$

or
$$\alpha^3 + \alpha^2 \frac{L_1 R_2 + L_2 R_1}{L_1 L_2} + \alpha \frac{L_1 + L_2 + R_1 R_2 C}{L_1 L_2 C} + \frac{R_1 + R_2}{L_1 L_2 C} = 0$$

This cubic equation has one real root which we ignore and two complex conjugate roots. We require the condition that this pair of complex conjugate roots is identical with the roots of the equation

$$\alpha^2 L C + \alpha R C + 1 = 0$$

The general solution of this problem is not easy. We look for special cases. If $R_1 = R_2 = 0$, their

$R = 0$ and $L = \dfrac{L_1 L_2}{L_1 + L_2}$. If $L_1 = L_2 = 0$, then

$L = 0$ and $R = R_1 R_2 / (R_1 + R_2)$ ∴ These are the quoted solution but they are misleading.

We shall give the solution for small R_1, R_2 . Then we put $\alpha = -\beta + i\omega$ when β is small

We get $(1 - \omega^2 L_1 C - 2i\beta\omega L_1 C - \beta \cancel{R}_1 C + i\omega R_1 C)$

$$(1 - \omega^2 L_2 C - 2i\beta\omega L_2 C - \beta \cancel{R}_2 C + i\omega R_2 C) = 1$$

(we neglect β^2 & βR_1, βR_2). Then

$$(1 - \omega^2 L_1 C)(1 - \omega^2 L_2 C) = 1 \Rightarrow \omega^2 = \frac{L_1 + L_2}{L_1 L_2 C}$$

This is identical with $\omega^2 = \dfrac{1}{LC}$ if $L = \dfrac{L_1 L_2}{L_1 + L_2}$.

also $\quad (2\beta L_1 - R_1)(1 - \omega^2 L_2 C) + (2\beta L_2 - R_2)(1 - \omega^2 L_2 C) = 0$

This gives $\beta = \dfrac{R}{2L} = \dfrac{R_1 L_2^2 + R_2 L_1^2}{2 L_1 L_2 (L_1 + L_2)} \Rightarrow R = \dfrac{R_1 L_2^2 + R_2 L_1^2}{(L_1 + L_2)^2}$.

Q.117. A circuit consists of a capacitor with capacitance C and a coil of inductance L connected in series, as well as a switch and a resistance equal to the critical value for this circuit. With the switch disconnected, the capacitor was charged to a voltage V_o, and at the moment t = 0 the switch was closed. Find the current I in the circuit as a function of time t. What is I_{max} equal to?

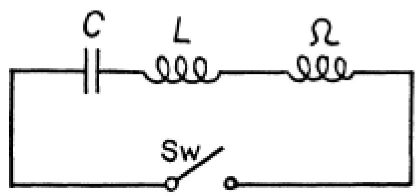
Ans.

$o = \dfrac{q}{C} + L\dfrac{dI}{dt} + RI$, $I = +\dfrac{dq}{dt}$

For the critical case $R = 2\sqrt{\dfrac{L}{C}}$

Thus $LC\ddot{q} + 2\sqrt{LC}\,\dot{q} + q = 0$

Look for a solution with $q\ \alpha\ e^{\alpha t}$



$$\alpha = -\frac{1}{\sqrt{LC}}.$$

An independent solution is $t e^{\alpha t}$. Thus

$$q = (A + Bt)\, e^{-t/\sqrt{LC}},$$

At $t = 0$ $q = CV_0$ thus $A = CV_0$

Also at $t = 0$ $\dot{q} = I = 0$

$$0 = B - A\frac{1}{\sqrt{lC}} \Rightarrow B = V_0\sqrt{\frac{C}{L}}$$

Thus filially

$$I = \frac{dq}{dt} = V_0\sqrt{\frac{C}{L}}\, e^{-t/\sqrt{LC}}$$
$$-\frac{1}{\sqrt{LC}}\left(CV_0 + V_0\sqrt{\frac{C}{L}}\,t\right)e^{-t/\sqrt{LC}}$$
$$= -\frac{V_0}{L}t\, e^{-t/\sqrt{LC}}$$

The current has been defined to increase the charge. Hence the minus sign.
The current is maximum when

$$\frac{dI}{dt} = -\frac{V_0}{L}e^{-t/\sqrt{LC}}\left(1 - \frac{t}{\sqrt{LC}}\right) = 0$$

This gives $t = \sqrt{LC}$ and the magnitude of the maximum current is

$$|I_{max}| = \frac{V_0}{e}\sqrt{\frac{C}{L}}.$$

Q.118. A coil with active resistance R and inductance L was connected at the moment t = 0 to a source of voltage V = V_m cos ωt. Find the current in the coil as a function of time t.

Ans. The equation o f the circuit is (I is th e current)

$$L\frac{dI}{dt}+RI=V_m\cos\omega t$$

From the theory of differential equations

$$I=I_P+I_C$$

Where IP is a particular integral and I_C is the complementary function (Solution of the differential equation with the RHS = 0). Now

$$I_C=I_{CO}\,e^{-tR/L}$$

and for I_P we write $I_P=I_m\cos(\omega t-\varphi)$

Substituting we get

$$I_m=\frac{V_m}{\sqrt{R^2+\omega^2L^2}},\quad \varphi=\tan^{-1}\frac{\omega L}{R}$$

Thus $$I_m=\frac{V_m}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t-\varphi)+I_{CO}\,e^{-tR/L}$$

Now in an inductive circuit I = 0 at t = 0

Because a current cannot change suddenly.

Thus $$I_{CO}=-\frac{V_m}{\sqrt{R^2+\omega^2L^2}}\cos\varphi$$

And so

$$I=\frac{V_m}{\sqrt{R^2+\omega^2L^2}}\left[\cos(\omega t-\varphi)-\cos\varphi\,e^{-tR/L}\right]$$

Q.119. A circuit consisting of a capacitor with capacitance C and a resistance R connected in series was connected at the moment t = 0 to a source of ac voltage V = V_m cos ωt. Find the current in the circuit as a function of time t.

Ans. Here the equation is (Q is charge on the capacitor)

$$\frac{Q}{C} + R\frac{dQ}{dt} = V_m \cos \omega t$$

A solution subject to Q = 0 at t = 0 is of the form (as in the previous problem)

$$Q = Q_m\left[\cos(\omega t - \bar{\varphi}) - \cos\bar{\varphi}\, e^{-t/RC}\right]$$

Substituting back

$$\frac{Q_m}{C}\cos(\omega t - \bar{\varphi}) - \omega R Q_m \sin(\omega t - \bar{\varphi})$$

$$= V_m \cos \omega t$$

$$= V_m\left\{\cos\bar{\varphi}\cos(\omega t - \bar{\varphi}) - \sin\bar{\varphi}\sin(\omega t - \bar{\varphi})\right\}$$

So

$$Q_m = C V_m \cos\bar{\varphi}$$

$$\omega R Q_m = V_m \sin\bar{\varphi}$$

This leads to

$$Q_m = \frac{C V_m}{\sqrt{1 + (\omega R C)^2}}, \ \tan\bar{\varphi} = \omega R C$$

Hence

$$I = \frac{dQ}{dt} = \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}\left[-\sin(\omega t - \bar{\varphi}) + \frac{\cos^2\bar{\varphi}}{\sin\bar{\varphi}}\, e^{-t/RC}\right]$$

The solution given in the book satisfies $I = 0$ at $t = 0$. Then $Q = 0$ at $t = 0$ but this will not satisfy the equation at $t = 0$. Thus $I \neq 0$, (Equation will be satisfied with $I = 0$ only if $Q \neq 0$ at $t = 0$)

With our I, $\quad I(t = 0) = \frac{V_m}{R}$

Q.120. A long one-layer solenoid tightly wound of wire with resistivity p has n turns per unit length. The thickness of the wire insulation is negligible. The cross-sectional radius of the solenoid is equal to a. Find the phase difference between current and alternating voltage fed to the solenoid with frequency v.

Ans. The current lags behind the voltage by the phase angle

$$\varphi = \tan^{-1}\frac{\omega L}{R}$$

Now $L = \mu_0 n^2 \pi a^2 l$, l = length of the solenoid

$$R = \frac{\rho \cdot 2\pi a n \cdot l}{\pi b^2}$$, $2b$ = diameter of the wire

But $\quad 2bn = 1 \quad \therefore \quad b = \frac{1}{2n}$

Then
$$\varphi = \tan^{-1}\frac{\mu_0 n^2 l \pi a^2 \cdot 2\pi\nu}{\rho \cdot 2\pi a n l} \times \pi\frac{1}{4n^2}$$
$$= \tan^{-1}\frac{\mu_0 \pi^2 a \nu}{4\rho n}.$$

Q.121. A circuit consisting of a capacitor and an active resistance R = 110Ω connected in series is fed an alternating voltage with amplitude V_m = 110 V. In this case the amplitude of steady-state current is equal to I_m = 0.50 A. Find the phase difference between the current and the voltage fed.

Ans.

Here $V = V_m \cos \omega t$

$I = I_m \cos(\omega t + \varphi)$

Where

$$I_m = \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}, \quad \tan\varphi = \frac{1}{\omega R C}$$

Now

$$R^2 + \frac{1}{(\omega C)^2} = \left(\frac{V_m}{I_m}\right)^2$$
$$\frac{1}{\omega R C} = \sqrt{\left(\frac{V_m}{R I_m}\right)^2 - 1}$$

Thus the current is ahead of the voltage by

$$\varphi = \tan^{-1}\frac{1}{\omega R C} = \tan^{-1}\sqrt{\left(\frac{V_m}{R I_m}\right)^2 - 1} = 60^\circ$$

Q.122. Fig. 4.32 illustrates the simplest ripple filter. A voltage $V = V_0 \ (1. + \cos \omega t)$ is fed to the left input. Find: (a) the output voltage $V'(t)$; (b) the magnitude of the product RC at which the output amplitude of alternating voltage component is $\eta = 7.0$ times less than the direct voltage component, if $\omega = 314\ s^{-1}$.

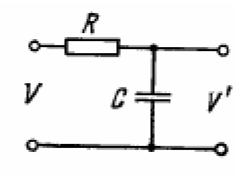


Fig. 4.32.

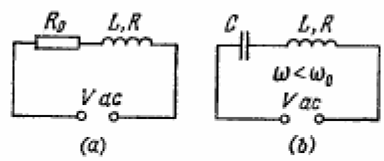
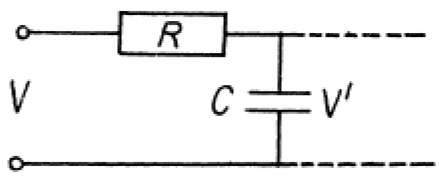


Fig. 4.33.

Ans.

Here $\quad V = IR + \frac{\int_0 I\,dt}{C}$

or $\quad R\dot{I} + \frac{1}{C} I = \dot{V} = -\omega V_0 \sin \omega t$



Ignoring transients, a solution has the form

$I = I_0 \sin(\omega t - \alpha)$

$$\omega R I_0 \cos(\omega t - \alpha) + \frac{I_0}{C} \sin(\omega t - \alpha) = -\omega V_0 \sin \omega t$$

$$= -\omega V_0 \{ \sin(\omega t - \alpha) \cos \alpha + \cos(\omega t - \alpha) \sin \alpha \}$$

so $\qquad R I_0 = -V_0 \sin \alpha$

$$\frac{I_0}{\omega C} = -V_0 \cos \alpha \qquad \alpha = \pi + \tan^{-1}(\omega R C)$$

$$I_0 = \frac{V_0}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}$$

$$I = I_0 \sin(\omega t - \tan^{-1} \omega R C - \pi) = -I_0 \sin(\omega t - \tan^{-1} \omega R C)$$

Then $\quad Q = \int_0^t I\,dt = Q_0 + \frac{I_o}{\omega} \cos(\omega t - \tan^{-1} \omega R C)$

It satisfies $V_0(1+\cos\omega t) = R\frac{dQ}{dt} + \frac{Q}{C}$

if $$V_0(1+\cos\omega t) = -RI_0\sin(\omega t - \tan^{-1}\omega RC) + \frac{Q_0}{C} + \frac{I_0}{\omega C}\cos(\omega t - \tan^{-1}\omega RC)$$

Thus $$Q_0 = CV_0$$

and $$\left.\begin{aligned} \frac{I_0}{\omega C} &= V_0/\sqrt{1+(\omega RC)^2} \\ RI_0 &= \frac{V_0\,\omega RC}{\sqrt{1+(\omega RC)^2}} \end{aligned}\right\} \text{checks}$$

Hence $$V' = \frac{Q}{C} = V_0 + \frac{V_0}{\sqrt{1+(\omega RC)^2}}\cos(\omega t - \alpha)$$

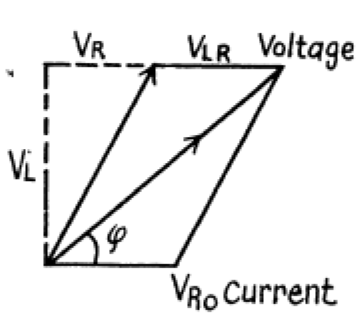
(b) $$\frac{V_0}{\eta} = \frac{V_0}{\sqrt{1+(\omega RC)^2}}$$

or $$\eta^2 - 1 = \omega^2(RC)^2$$

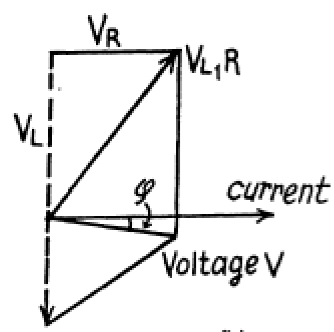
or $$RC = \sqrt{\eta^2-1}/\omega = 22 \text{ ms.}$$

Q.123. Draw the approximate voltage vector diagrams in the electric circuits shown in Fig. 4.33 a, b. The external voltage V is assumed to be alternating harmonically with frequency ω.

Ans.



(a)



(b)

$$\tan\varphi = \frac{\omega L - \frac{1}{\omega C}}{R} = -ve$$

As $\omega^2 < \frac{1}{LC}$

Q.124. A series circuit consisting of a capacitor with capacitance C = 22 μF and a coil with active resistance R = 20Ω and inductance L = 0.35 H is connected to a source of alternating voltage with amplitude V_m = 180 V and frequency ω = 314 s^{-1}. Find:
(a) the current amplitude in the circuit;
(b) the phase difference between the current and the external voltage;
(c) the amplitudes of voltage across the capacitor and the coil.

Ans. (a)

$$I_m = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = 4 \cdot 48 \text{ A}$$

(b) $\tan\varphi = \frac{\omega L - \frac{1}{\omega C}}{R}$, $\varphi = -60°$

Current lags behind the voltage V by φ

(c) $V_C = \frac{I_m}{\omega C} = 0 \cdot 65 \text{ kV}$

$$V_{L/R} = I_m\sqrt{R^2 + \omega^2 L^2} = 0 \cdot 5 \text{ kV}$$

Q.125. A series circuit consisting of a capacitor with capacitance C, a resistance R, and a coil with inductance L and negligible active resistance is connected to an oscillator whose frequency can be varied without changing the voltage amplitude. Find the frequency at which the voltage amplitude is maximum
(a) across the capacitor;
(b) across the coil.

Ans. (a)

$$V_C = \frac{1}{\omega C}\frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$= \frac{V_m}{\sqrt{(\omega R C)^2 + (\omega^2 L C - 1)^2}} = \frac{V_m}{\sqrt{\left(\frac{\omega^2}{\omega_0^2} - 1\right)^2 + 4\beta^2\omega^2/\omega_0^4}}$$

$$= \frac{V_m}{\sqrt{\left(\frac{\omega^2}{\omega_0^2} - 1 + \frac{2\beta^2}{\omega_0^2}\right)^2 + \frac{4\beta^2}{\omega_0^2} - \frac{4\beta^4}{\omega_0^4}}}$$

This is maximum when $\omega^2 = \omega_0^2 - 2\beta^2 = \frac{1}{LC} - \frac{R^2}{2L^2}$

(b)

$$V_L = I_m \omega L = V_m \frac{\omega L}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$= \frac{V_m L}{\sqrt{\frac{R^2}{\omega^2} + \left(L - \frac{1}{\omega^2 C}\right)^2}} = \frac{V_m L}{\sqrt{L^2 - \frac{1}{\omega^2}\left(\frac{2L}{C} - R^2\right) + \frac{1}{\omega^4 C^2}}}$$

$$= \frac{V_m L}{\sqrt{\left(\frac{1}{\omega^2 C} - \left(L - \frac{CR^2}{2}\right)\right)^2 + L^2 - \left(L - \frac{1}{2}CR^2\right)^2}}$$

This is maximum when

$$\frac{1}{\omega^2 C} = L - \frac{1}{2}CR^2$$

Or

$$\omega^2 = \frac{1}{LC - \frac{1}{2}C^2 R^2} = \frac{1}{\frac{1}{\omega_0^2} - \frac{2\beta^2}{\omega_0^4}}$$

$$= \frac{\omega_0^4}{\omega_0^2 - 2\beta^2} \quad \text{or} \quad \omega = \frac{\omega_0^2}{\sqrt{\omega_0^2 - 2\beta^2}}.$$

Q.126. An alternating voltage with frequency $\omega = 314 s^{-1}$ and amplitude $V_m = 180$ V is fed to a series circuit consisting of a capacitor and a coil with active resistance R = 40Ω and inductance L = 0.36 H. At what value of the capacitor's capacitance will the voltage amplitude across the coil be maximum? What is this amplitude equal to? What is the corresponding voltage amplitude across the condenser?

Ans.

$$V_L = I_m \sqrt{R^2 + \omega^2 L^2}$$
$$= \frac{V_m \sqrt{R^2 + \omega^2 L^2}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

For a given ω , L , R , this is maximum when

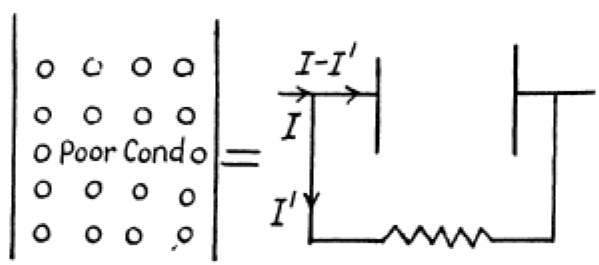
$$\frac{1}{\omega C} = \omega L \quad \text{or} \quad C = \frac{1}{\omega^2 L} = 28.2 \ \mu\text{F}.$$

For that C, $$V_L = \frac{V \sqrt{R^2 + \omega^2 L^2}}{R} = V \sqrt{1 + (\omega L / R)^2} = 0.540 \text{ kV}$$

At this C, $$V_C = \frac{1}{\omega C} \frac{V_m}{R} = \frac{V_m \omega L}{R} = .509 \text{ kV}$$

Q.127. A capacitor with capacitance C whose interelectrode space is filled up with poorly conducting medium with active resistance R is connected to a source of alternating voltage V = V_m cos ωt. Find the time dependence of the steady-state current flowing in lead wires. The resistance of the wires is to be neglected.

Ans.



We use the complex voltage $V = V_m e^{i\omega t}$. Then the voltage across the capacitor is

$$(I - I') \frac{1}{i \omega C}$$

And that across the resistance R I ' and both equal V . Thus

$$I' = \frac{V_m}{R} e^{i\omega t}, \ I - I' = i \omega C V_m e^{i\omega t}$$

Hence

$$I = \frac{V_m}{R}(1 + i\omega RC)\,e^{i\omega t}$$

The actual voltage is obtained by taking the real part Then

$$I = \frac{V_m}{R}\sqrt{1 + (\omega RC)^2}\,\cos(\omega t + \varphi)$$

Where $\tan\varphi = \omega RC$

Note → A condenser with poorly conducting material (dielectric of high resistance) be the plates is equivalent to an ideal condenser with a high resistance joined in p between its plates.

Q.128. An oscillating circuit consists of a capacitor of capacitance C and a solenoid with inductance L_1. The solenoid is inductively connected with a short-circuited coil having an inductance L_2 and a negligible active resistance. Their mutual inductance coefficient is equal to L_{12}. Find the natural frequency of the given oscillating circuit.

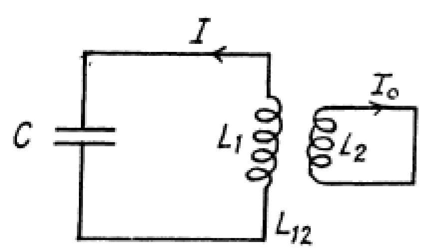
Ans.

$$L_1\frac{dI_1}{dt} + \frac{\int I_1\,dt}{C} = -L_{12}\frac{dI_2}{dt}$$

$$L_2\frac{dI_2}{dt} = -L_{12}\frac{dI_1}{dt}$$

from the second equation

$$L_2 I_2 = -L_{12} I_1$$



Then

$$\left(L_1 - \frac{L_{12}^2}{L_2}\right)\ddot{I}_1 + \frac{I_1}{C} = 0$$

Thus the current oscillates with frequency

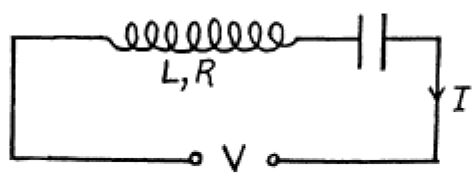
$$\omega = \frac{1}{\sqrt{C\left(L_1 - \frac{L_{12}^2}{L_2}\right)}}$$

Q.129. Find the quality factor of an oscillating circuit connected in series to a source of alternating emf if at resonance the voltage across the capacitor is n times that of the source.

Ans.

Given $V = V_m \cos \omega t$

$I = I_m \cos(\omega t - \varphi)$



Where

$$I_m = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

Then, $V_C = \frac{\int I\,dt}{C} = \frac{I_m \sin(\omega t - \varphi)}{\omega C}$

$$= \frac{V_m}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}} \sin(\omega t - \varphi)$$

As resonance the voltage amplitude across the capacitor

$$= \frac{V_m}{RC\frac{1}{\sqrt{LC}}} = \sqrt{\frac{L}{CR^2}}\, V_m = n V_m$$

So $\frac{L}{CR^2} = n^2$

Now $$Q = \sqrt{\frac{L}{CR^2} - \frac{1}{4}} = \sqrt{n^2 - \frac{1}{4}}$$

Q.130. An oscillating circuit consisting of a coil and a capacitor connected in series is fed an alternating emf, with coil inductance being chosen to provide the maximum current in the circuit. Find the quality factor of the system, provided an n-fold increase of inductance results in an ii-fold decrease of the current in the circuit.

Ans. For maximum current amplitude

$$I_m = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$L = \frac{1}{\omega^2 C} \text{ and then } I_{m0} = \frac{V_m}{R}$$

Now

$$\frac{I_{m0}}{\eta} = \frac{V_m}{\sqrt{R^2 + \frac{(n-1)^2}{\omega^2 C^2}}}$$

So

$$\eta = \sqrt{1 + \frac{(n-1)^2}{(\omega RC)^2}}$$

$$\omega RC = \frac{n-1}{\sqrt{\eta^2 - 1}}$$

Now $$Q = \sqrt{\left(\frac{L}{CR^2}\right)^2 - \frac{1}{4}} = \sqrt{\left(\frac{1}{\omega RC}\right)^2 - \frac{1}{4}} = \sqrt{\frac{\eta^2 - 1}{(n-1)^2} - \frac{1}{4}}$$

Electric Oscillations (Part - 3)

Q.131. A series circuit consisting of a capacitor and a coil with active resistance is connected to a source of harmonic voltage whose frequency can be varied, keeping the voltage amplitude constant. At frequencies ω_1 and ω_2 the current amplitudes are n times less than the resonance amplitude. Find:
(a) the resonance frequency;
(b) the quality factor of the circuit.

Ans. At resonance

$$\omega_0 L = (\omega_0 C)^{-1} \quad \text{or} \quad \omega_0 = \frac{1}{\sqrt{LC}},$$

and

$$(I_m)_{res} = \frac{V_m}{R}.$$

Now $$\frac{V_m}{nR} = \frac{V_m}{\sqrt{R^2 + \left(\omega_1 L - \frac{1}{\omega_1 C}\right)^2}} = \frac{V_m}{\sqrt{R^2 + \left(\omega_2 L - \frac{1}{\omega_2 C}\right)^2}}$$

Then $$\omega_1 L - \frac{1}{\omega_1 C} = \sqrt{n^2 - 1}\, R$$

$$\omega_2 L - \frac{1}{\omega_2 C} = + \sqrt{n^2 - 1}\, R \qquad (\text{assuming } \omega_2 > \omega_1)$$

or $$\omega_1 - \frac{\omega_0^2}{\omega_1} = -\omega_2 + \frac{\omega_0^2}{\omega_2} = -\sqrt{n^2 - 1}\,\frac{R}{L}$$

or $$\omega_1 + \omega_2 = \frac{\omega_0^2}{\omega_1 \omega_2}(\omega_1 + \omega_2) \Rightarrow \omega_0 = \sqrt{\omega_1 \omega_2}$$

and $$\omega_2 - \omega_1 = \sqrt{n^2 - 1}\,\frac{R}{L}$$

$$\beta = \frac{R}{2L} = \frac{\omega_2 - \omega_1}{2\sqrt{n^2 - 1}}$$

And

$$Q = \sqrt{\frac{\omega_0^2}{4\beta^2} - \frac{1}{4}} = \sqrt{\frac{(n^2 - 1)\,\omega_1 \omega_2}{(\omega_2 - \omega_1)^2} - \frac{1}{4}}$$

Q.132. Demonstrate that at low damping the quality factor Q of a circuit maintaining forced oscillations is approximately equal to $\omega_0/\Delta\omega$, where ω_0 is the natural oscillation frequency, $\Delta\omega$ is the width of the resonance curve I (ω) at the "height" which is $\sqrt{2}$ times less than the resonance current amplitude.

Ans.

$$Q = \frac{\omega}{2\beta} \approx \frac{\omega_0}{2\beta} \text{ for low damping.}$$

Now $$\frac{I_m}{\sqrt{2}} = \frac{R I_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}},$$ I_m = current amplitude at resouance.

or $$\omega - \frac{\omega_0^2}{\omega} = \pm \frac{R}{L} = \pm 2\beta$$

Thus $$\omega \approx \omega_0 \pm \beta$$

So $$\Delta\omega = 2\beta \text{ and } Q = \frac{\omega_0}{\Delta\omega}.$$

Q.133. A circuit consisting of a capacitor and a coil connected in series is fed two alternating voltages of equal amplitudes but different frequencies. The frequency of one voltage is equal to the natural oscillation frequency (ω_0) of the circuit, the frequency of the other voltage is η times higher. Find the ratio of the current amplitudes (I_0/I) generated by the two voltages if the quality factor of the system is equal to Q. Calculate this ratio for Q = 10 and 100, if η = 1.10.

Ans. At resonance

$$\omega = \omega_0$$

$$I_m(\omega_0) = \frac{V_m}{R}$$

Then $$I_m(\eta\omega_0) = \frac{V_m}{\sqrt{R^2 + \left(\eta\omega_0 L - \frac{1}{\eta\omega_0 C}\right)^2}}$$

$$= \frac{V_m}{\sqrt{R^2 + \left(\eta - \frac{l}{\eta}\right)^2 \frac{L}{C}}} = \frac{V_m}{\sqrt{1 + \left(Q^2 + \frac{l}{4}\right)\left(\eta - \frac{1}{\eta}\right)^2 \frac{L}{C}}}$$

Q.134. It takes t_0 hours for a direct current I_0 to charge a storage battery. How long will it take to charge such a battery from the mains using a half-wave rectifier, if the effective current value is also equal to I_0?

Ans. The a.c. current must be

$$I = I_0 \sqrt{2}\ \sin \omega t$$

Then D.C. component of the rectified current is

$$< I' > = \frac{1}{T} \int_0^{T/2} I_0 \sqrt{2}\ \sin \omega t\, dt$$

$$= I_0 \sqrt{2}\, \frac{1}{2\pi} \int_0^{\pi} \sin \theta\, d\theta$$

$$= \frac{I_0 \sqrt{2}}{\pi}$$

Since the charge deposited must be the same

$$I_0 t_0 = \frac{I_0 \sqrt{2}}{\pi} t \quad \text{or} \quad t = \frac{\pi t_0}{\sqrt{2}}$$

The answer is incorrect.

Q.135. Find the effective value of current if its mean value is I_0 and its time dependence is
(a) shown in Fig. 4.34;

(b) $I \sim |\sin \omega t|$.

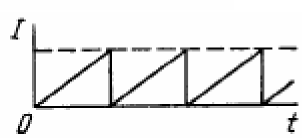


Fig. 4.34.

Ans. (a)

$$I(t) = I_1 \frac{t}{T} \quad 0 \leq t < T$$

$$I(t \pm T) = I(t)$$

Now mean current

$$< I > = \frac{1}{T}\int_0^T I_1 \frac{t}{T}\, dt = I_1 \frac{T^2/2}{T^2} = I_1/2$$

Then $I_1 = 2I_0$ since $< I > = I_0$.

Now mean square current

$$< I^2 >$$

$$= 4 I_0^2 \frac{1}{T}\int_0^T \frac{t^2}{T^2}\, dt = \frac{4 I_0^2}{3}$$

So effective current $= \frac{2 I_0}{\sqrt{3}}$.

(b) In this case $I = I_1 |\sin \omega t|$

and $$I_0 = \frac{1}{T}\int_0^T I_1 |\sin \omega t|\, dt$$

$$= \frac{1}{2\pi} I_1 \int_0^{2\pi} |\sin \theta|\, d\theta = \frac{I_1}{\pi}\int_0^{\pi} \sin\theta\, d\theta = \frac{2 I_1}{\pi}$$

So $I_1 = \frac{\pi I_0}{2}$

Then, mean square current $= < I^2 > = \frac{\pi^2 I_0^2}{4T}\int_0^T \sin^2 \omega t\, dt$

$$= \frac{\pi^2 I_0^2}{4} \times \frac{1}{2\pi}\int_0^{2\pi} \sin^2\theta\, d\theta = \frac{\pi^2 I_0^2}{8}$$

So effective current $= \frac{\pi I_0}{\sqrt{8}}$.

Q.136. A solenoid with inductance L = 7 mH and active resistance R = 44Ω is first connected to a source of direct voltage V_0 and then to a source of sinusoidal voltage with effective value $V = V_0$. At what frequency of the oscillator will the power consumed by the solenoid be η = 5.0 times less than in the former case?

Ans.

$$P_{d.c.} = \frac{V_0^2}{R}$$

$$P_{a.c.} = \frac{V_0^2}{\sqrt{R^2+\omega^2 L^2}} \cdot \frac{R}{\sqrt{R^2+\omega^2 L^2}} = \frac{v_0^2/R}{1+\left(\frac{\omega L}{R}\right)^2} = \frac{P_{d.c.}}{\eta}$$

Thus $\frac{\omega L}{R} = \sqrt{\eta - 1}$

Or $\omega = \frac{R}{L}\sqrt{\eta - 1}$

$\nu = \frac{R}{2\pi L}\sqrt{\eta - 1} = 2$ kH of on putting the values.

Q.137. A coil with inductive resistance $X_L = 30\Omega$ and impedance $Z = 50\Omega$ is connected to the mains with effective voltage value V = 100 V. Find the phase difference between the current and the voltage, as well as the heat power generated in the coil.

Ans.

$Z = \sqrt{R^2 + X_L^2}$ or $R_0 = \sqrt{Z^2 - X_L^2}$

The $\tan\theta = \frac{X_L}{\sqrt{Z^2 - X_L^2}}$

So $\cos\varphi = \frac{\sqrt{Z^2 - X_L^2}}{Z} = \sqrt{1 - \left(\frac{X_L}{Z}\right)^2}$

The current lags by φ behind the voltage.

$$\varphi = \cos^{-1}\sqrt{1 - \left(\frac{X_L}{Z}\right)^2} = 37°.$$

also $P = VI\cos\varphi = \frac{V^2}{Z^2}\sqrt{Z^2 - X_L^2} = .160$ kW.

Q.138. A coil with inductance L =- 0.70 H and active resistance r = 20Ω is connected in series with an inductance-free resistance R. An alternating voltage with effective value V = 220 V and frequency w = 314 s^{-1} i s applied across the terminals of this circuit. At what value of the resistance R will the maximum heat power be generated in the circuit? What is it equal to?

Ans.

$$P = \frac{V^2 (R + r)}{(R + r)^2 + \omega^2 L^2}$$

This is maximum when R + r = ωL for

$$P = \frac{V^2}{R + r + \frac{(\omega L)^2}{R + r}} = \frac{V^2}{\left[\sqrt{R + r} - \frac{\omega L}{\sqrt{R + r}}\right]^2 + 2\,\omega L}$$

Thus R = ωL - r for maximum power and $P_{max} = \frac{V^2}{2\,\omega L}$

Substituting the values, we get $R = 200\ \Omega$ and $P_{max} = .114$ kW.

Q.139. A circuit consisting of a capacitor and a coil in series is connected to the mains. Varying the capacitance of the capacitor, the heat power generated in the coil was increased n = 1.7 times. How much (in per cent) was the value of cos 11) changed in the process?

Ans.

$$P = \frac{V^2 R}{R^2 + (X_L - X_C)^2}$$

Varying the capacitor does not change R so if P increases n times

$Z = \sqrt{R^2 + (X_L - X_C)^2}$ must decreases $\sqrt{n}$ times

Thus

$\cos\varphi = \frac{R}{Z}$ increases $\sqrt{n}$ times

$\therefore$ % increase in $\cos\varphi = \left(\sqrt{n} - 1\right) \times 100\ \% = 30.4\%.$

Q.140. A source of sinusoidal emf with constant voltage is connected in series with an oscillating circuit with quality factor Q = 100. At a certain frequency of the external voltage the heat power generated in the circuit reaches the maximum value. How much (in per cent) should this frequency be shifted to decrease the power generated n = 2.0 times?

Ans.

$$P = \frac{V^2 R}{R^2 + (X_L - X_C)^2}$$

At resonance $X_L = X_C \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$.

Power generated will decrease n times when

$$(X_L - X_C)^2 = \left(\omega L - \frac{1}{\omega C}\right)^2 = (n-1)R^2$$

or
$$\omega - \frac{\omega_0^2}{\omega} = \pm\sqrt{n-1}\,\frac{R}{L} = \pm\sqrt{n-1}\;2\beta.$$

Thus
$$\omega^2 \mp 2\sqrt{n-1}\,\beta\,\omega - \omega_0^2 = 0$$

$$\left(\omega \mp \sqrt{n-1}\,\beta\right)^2 = \omega_0^2 + (n-1)\beta^2$$

or
$$\frac{\omega}{\omega_0} = \sqrt{1 + (n-1)\beta^2/\omega_0^2} \pm \sqrt{n-1}\;\beta/\omega_0$$

(Taking only the positive sign in the first term to ensure positive value for $\frac{\omega}{\omega_0}$)
Now

$$Q = \frac{\omega}{2\beta} = \frac{1}{2}\sqrt{\left(\frac{\omega_0}{\beta}\right)^2 - 1}$$

$$\frac{\omega_0}{\beta} = \sqrt{1 + 4Q^2}$$

Thus

$$\frac{\omega}{\omega_0} = \sqrt{1 + \frac{n-1}{(1+4Q^2)}} \pm \sqrt{n-1}\,/\sqrt{1+4Q^2}$$

For large Q

$$\left|\frac{\omega - \omega_0}{\omega_0}\right| = \frac{\sqrt{en-1}}{2Q} = \frac{\sqrt{en-1}}{2Q} \times 100\,\% = 0.5\%$$

Q.141. A series circuit consisting of an inductance-free resistance R = 0.16 kΩ and a coil with active resistance is connected to the mains with effective voltage V = 220

V, Find the heat power generated in the coil if the effective voltage values across the resistance R and the coil are equal to V_1 = 80 V and V_2 = 180 V respectively.

Ans. We have

$$V_1 = \frac{VR}{\sqrt{(R+R_1)^2 + X_L^2}}, \quad V_2 = \frac{V\sqrt{R_1^2 + X_L^2}}{\sqrt{(R+R_1)^2 + X_L^2}}$$

so $$(R+R_1)^2 + X_L^2 = \left(\frac{VR}{V_1}\right)^2, \quad R_1^2 + X_L^2 = \left(\frac{V_2 R}{V_1}\right)^2$$

Hence $$R^2 + 2RR_1 = \frac{R^2}{V_1^2}(V^2 - V_2^2)$$

or $$R_1 = \frac{R}{2V_1^2}(V^2 - V_2^2 - V_1^2)$$

Heat generated in the coil $$= \frac{V^2 R_1}{(R_1 + R_2)^2 + X_L^2} = \frac{V_1^2}{R^2} \times R_1 = \frac{V_1^2}{R^2} \times \frac{R^2}{2V_1^2}(V^2 - V_1^2 - V_2^2)$$

$$= \frac{V^2 - V_1^2 - V_2^2}{2R} = 30 \text{ W}$$

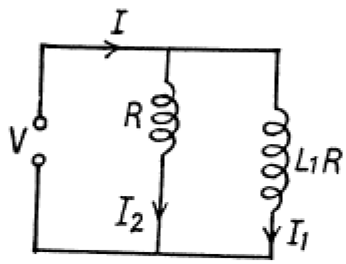
Q.142. A coil and an inductance-free resistance R = 25Ω are connected in parallel to the ac mains. Find the heat power generated in the coil provided a current I = 0.90 A is drawn from the mains. The coil and the resistance R carry currents I_1 = 0.50 A and l_2 = = 0.60 A respectively.

Ans. Here

$I_2 = \frac{V}{R}$, $V =$ Effective voltage

$$I_1 = \frac{V}{\sqrt{R^2 + X_L^2}}$$

and $$I = \frac{V\sqrt{(R+R_1)^2 + X_L^2}}{R\sqrt{R_1^2 + X_L^2}} = \frac{V}{R_{eff}}$$



R_{eff} is the impedance of the coil & the resistance in parallel.

Now
$$\frac{I^2 - I_2^2}{I_2^2} = \frac{R^2 + 2RR_1}{R_1^2 + X_L^2} = \left(\frac{I_1}{I_2}\right)^2 + \frac{2RR_1}{R^2 + X_L^2}$$
$$\frac{I^2 - I_2^2 - I_1^2}{I_2^2} = \frac{2RR_1}{R^2 + X_L^2}$$

Now mean power consumed in the coil

$$= I_1^2 R_1 = \frac{V^2 R_1}{R_2 + X_L^2} = I_2^2 R . \frac{I^2 - I_1^2 - I_2^2}{2 I_2^2} = \frac{1}{2} R (I^2 - I_1^2 - I_2^2) = 2.5 \text{ W.}$$

Q.143. An alternating current of frequency ω = 314 s^{-1} is fed to a circuit consisting of a capacitor of capacitance C = 73 ηF and an active resistance R = 100Ω connected in parallel. Find the impedance of the circuit.

Ans.

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{\frac{1}{i\omega C}} = \frac{1}{R} + i\omega C = \frac{1 + i\omega RC}{R}$$
$$|Z| = \frac{R}{\sqrt{1 + (\omega RC)^2}} = 40\ \Omega$$

Q.144. Draw the approximate vector diagrams of currents in the circuits shown in Fig. 4.35. The voltage applied across the points A and B is assumed to be sinusoidal; the parameters of each circuit are so chosen that the total current I_0 lags in phase behind the external voltage by an angle φ.

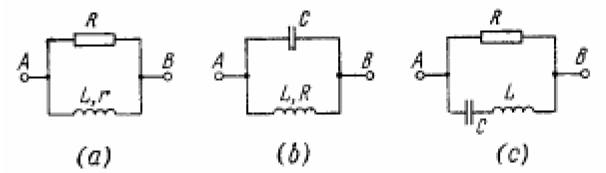
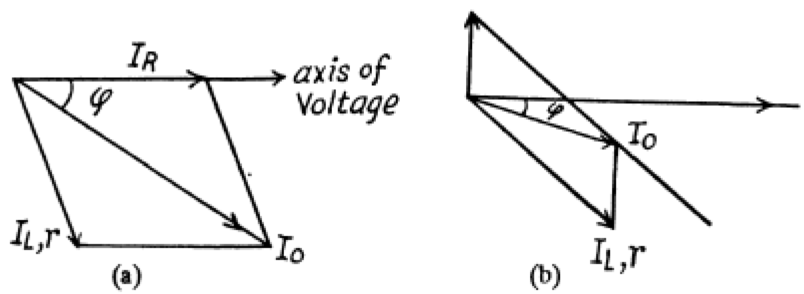


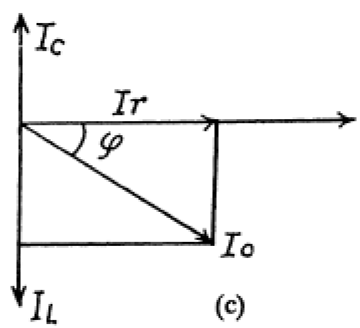
Fig. 4.35.

Ans. (a) For the resistance, the voltage and the current are in phase. For the coil the voltage is ahead of the current by less than 90°. The current is obtained by addition because the elements are in parallel



(b) I_c is ahead of the voltage by 90°.

(c) The coil has no resistance so Ih is 90° behind the voltage.



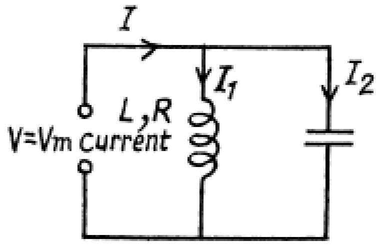
Q.145. A capacitor with capacitance C = 1.0 μF and a coil with active resistance R = 0.10Ω and inductance L = 1.0 mH are connected in parallel to a source of sinusoidal voltage V = 31 V. Find:
(a) the frequency ω at which the resonance sets in;
(b) the effective value of the fed current in resonance, as well as the corresponding currents flowing through the coil and through the capacitor.
Ans. When the coil and the condenser are in parallel, the equation is

$$L\frac{dI_1}{dt} + RI_1 = \frac{\int I_2\,dt}{C} = V_m \cos\omega t$$

$$I = I_1 + I_2$$

Using complex voltages

$$I_1 = \frac{V_m\, e^{i\omega t}}{R + i\omega L}, \quad I_2 = i\omega C V_m\, e^{i\omega t}$$



and

$$I = \left(\frac{1}{R + i\omega L} + i\omega C\right) V_m\, e^{i\omega t} = \left[\frac{R - i\omega L + i\omega C(R^2 + \omega^2 L^2)}{R^2 + \omega^2 L^2}\right] V_m\, e^{i\omega t}$$

Thus, taking real parts $I = \frac{V_m}{|Z(\omega)|}\cos(\omega t - \varphi)$

Where $\frac{1}{|Z(\omega)|} = \frac{[R^2 + \{\omega C(R^2 + \omega^2 L^2) - \omega L\}^2]}{(R^2 + \omega^2 L^2)^{1/2}}$

And $\tan\varphi = \frac{\omega L - \omega C(R^2 + \omega^2 L^2)}{R}$

(a) To get the frequency of resonance we must define what we mean by resonance. One definition requires the extremum (maximum or minimum) of current amplitude. The other definition requires rapid change of phase with φ passing through zero at resonance. For the series circuit.

$$I_m = \frac{V_m}{\left\{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right\}^{1/2}} \quad \text{and} \quad \tan\varphi = \frac{\omega L - \frac{1}{\omega C}}{R}$$

Both definitions give $\omega^2 = \frac{1}{LC}$ at resonance. In the present case the two definitions do not agree (except when R = 0). The definition that has been adopted in the answer given in the book is the vanishing of phase. This requires

$$C(R^2 + \omega^2 L^2) = L$$

or $$\omega^2 = \frac{1}{LC} - \frac{R^2}{L^2} = \omega_{res}^2, \quad \omega_{res} = 31.6 \times 10^3 \text{ rad/s.}$$

Note that for small R , φ rapidly changes from $= -\frac{\pi}{2}$ to $+\frac{\pi}{2}$ as ω passes through

ω_{res} from $< \omega_{res}$ to $> \omega_{res}$.

(b) At resonance

$$I_m = \frac{V_m R}{L/C} = V_m \frac{CR}{L}$$

So l = effective value of total current $= V \frac{CR}{L} = 3.1$ mA.

Similarly $$I_L = \frac{V}{\sqrt{L/C}} = V\sqrt{\frac{C}{L}} = 0.98 \text{ A.}$$

$$I_C = \omega C V = V\sqrt{\frac{C}{L} - \frac{R^2 C^2}{L^2}} = 0.98 \text{ A.}$$

Note:- The vanishing of phase (its passing through zero) is considered a more basic definition of resonance.

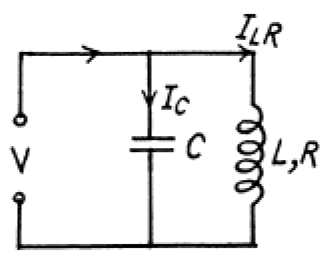
Q.146. A capacitor with capacitance C and a coil with active resistance R and inductance L are connected in parallel to a source of sinusoidal voltage of frequency ω. Find the phase difference between the current fed to the circuit and the source voltage.

Ans. We use the method of complex voltage

$$V = V_0 e^{i\omega t}$$

Then $$I_C = \frac{V_0 e^{i\omega t}}{\frac{1}{i\omega C}} = i\omega C V_0 e^{i\omega t}$$

$$I_{L,R} = \frac{V_0 e^{i\omega t}}{R + i\omega L}$$



$$I = I_C + I_{L,R} = V_0 \frac{R - i\omega L + i\omega C (R^2 + \omega^2 L^2)}{R^2 + \omega^2 L^2} e^{i\omega t}$$

Then taking the real part

$$I = \frac{V_0 \sqrt{R^2 + \{\omega C(R^2 + \omega^2 L^2) - \omega L\}^2}}{R^2 + \omega^2 L^2} \cos(\omega t - \varphi)$$

Where $\tan\varphi = \dfrac{\omega L - \omega C(R^2 + \omega^2 L^2)}{R}$

Q.147. A circuit consists of a capacitor with capacitance C and a coil with active resistance R and inductance L connected in parallel. Find the impedance of the circuit at frequency ω of alternating voltage.

Ans. From the previous problem

$$Z = \frac{R^2 + \omega^2 L^2}{\sqrt{R^2 + \{\omega C(R^2 + \omega^2 L^2) - \omega L\}^2}}$$

$$= \frac{R^2 + \omega^2 L^2}{\sqrt{(R^2 + \omega^2 L^2)(1 - 2\omega^2 LC) + \omega^2 C^2 (R^2 + \omega^2 L^2)^2}}$$

$$= \frac{\sqrt{R^2 + \omega^2 L^2}}{\sqrt{(1 - 2\omega^2 LC) + \omega^2 C^2 (R^2 + \omega^2 L^2)}}$$

$$= \frac{\sqrt{R^2 + \omega^2 L^2}}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}}$$

Q.148. A ring of thin wire with active resistance R and inductance L rotates with constant angular velocity ω in the external uniform magnetic field perpendicular to the rotation axis. In the process, the flux of magnetic induction of external field across the ring varies with time as $\Phi = \Phi_0 \cos \omega t$. Demonstrate that
(a) the inductive current in the ring varies with time as I =
$= I_m \sin(\omega t - \varphi)$, where $I_m = \omega\Phi_0/\sqrt{R^2 + \omega^2 L^2}$ with $\tan\varphi =$
$= \omega L/R$;
(b) The mean mechanical power developed by external forces to maintain rotation is defined by the formula
$\bar{P} = {}^1/_2 \omega^2 \Phi_0^2 R/(R^2 + \omega^2 L^2)$.

Ans. (a) We have

$$\varepsilon = -\frac{d\Phi}{dt} = \omega \Phi_0 \sin \omega t = L\dot{I} + RI$$

Put $\qquad I = I_m \sin(\omega t - \varphi)$. Then

$$\omega \Phi_0 \sin \omega t = \omega \Phi_0 \{\sin(\omega t - \varphi)\cos\varphi + \cos(\omega t - \varphi)\sin\varphi\}$$
$$= L I_m \omega \cos(\omega t - \varphi) + R I_m \sin(\omega t - \varphi)$$

(b) Mean mechanical power required to maintain rotation = energy loss per unit time

so $\qquad R I_m = \omega \Phi_0 \cos\varphi \quad \text{and} \quad L I_m = \Phi_0 \sin\varphi$

or $\qquad I_m = \dfrac{\omega \Phi_0}{\sqrt{R^2 + \omega^2 L^2}} \quad \text{and} \quad \tan\varphi = \dfrac{\omega L}{R}.$

$$= \frac{1}{T}\int_0^T R I^2 dt = \frac{1}{2} R I_m^2 = \frac{1}{2}\frac{\omega^2 \Phi_0^2 R}{R^2 + \omega^2 L^2}$$

Q.149. A wooden core (Fig. 4.36) supports two coils: coil 1 with inductance L_1 a nd short-circuited coil 2 with active resistance R and inductance L_2. The mutual inductance of the coils depends on

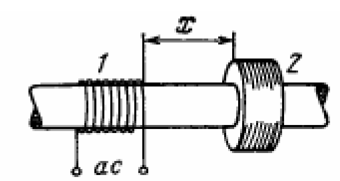


Fig. 4.36.

The distance x between them according to the law $L_{12}(x)$. Find the mean (averaged over time) value of the interaction force between the coils when coil 1 carries an alternating current $I_1 = I_0 \cos \omega t.$

Ans. We consider the force $\vec{F}_{12}$ that a circuit 1 exerts on another closed circuit 2 :-

$$\vec{F}_{12} = \oint I_2 d\vec{l}_2 \times \vec{B}_{12}$$

Here $\vec{B}_{12} =$ magnetic field at the site of the current element $d\vec{l}_2$ due to the current I_1 flowing in 1.

$$= \frac{\mu_0}{4\pi}\int \frac{I_1 d\vec{l}_1 \times \vec{r}_{12}}{r_{12}^3}$$

Where $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1 =$ vector, from current element $d\vec{l}_1$ to the current element $d\vec{l}_2$

Now

$$\vec{F}_{12} = \frac{\mu_0}{4\pi} \iint I_1 I_2 \frac{d\vec{l_2} \times (d\vec{l_1} \times \vec{r_{12}})}{r_{12}^3} = \frac{\mu_0}{4\pi} \iint I_1 I_2 \frac{d\vec{l_1}(d\vec{l_2} \cdot \vec{r_{12}}) - (d\vec{l_1} \cdot d\vec{l_2})\vec{r_{12}}}{r_{12}^3}$$

In the first term, we carry out the integration over $d\vec{l_r}$ first. Then

$$\iint \frac{d\vec{l_1}(d\vec{l_2} \cdot \vec{r_{12}})}{r_{12}^3} = \int d\vec{l_1} \oint \frac{d\vec{l_2} \cdot \vec{r_{12}}}{r_{12}^3} = -\int d\vec{l_1} \oint d\vec{l_2} \cdot \nabla_2 \frac{1}{r_{12}} = 0$$

because $$\oint d\, l_2 \cdot \nabla_2 \frac{1}{r_{12}} = \int d\vec{S_2} \text{ curl } \left(\nabla \frac{1}{r_{12}}\right) = 0$$

Thus $$F_{12} = -\frac{\mu_0}{4\pi} \iint I_1 I_2 \; d\vec{l_1} \cdot d\vec{l_2} \frac{\vec{r_{12}}}{r_{12}^3}$$

The integral involved will depend on the vector $\vec{a}$ that defines the separation of the (suitably chosen)centre of the coils. Let C_1 and C_2 be the centres of the two coil suitably defined.
Write

$$\vec{r_{12}} = \vec{r_2} - \vec{r_1} = \vec{\rho_2} - \vec{\rho_1} + \vec{a}$$

where $\vec{\rho_1}$ ($\vec{\rho_2}$) is the distance of $d\vec{l_1}$ ($d\vec{l_2}$) from C_1 (C_2) and $\vec{a}$ stands for the vector $\overrightarrow{C_1 C_2}$

Then $$\frac{\vec{r_{12}}}{r_{12}^3} = -\vec{\nabla}_{\vec{a}} \frac{1}{r_{12}}$$

and $$\vec{F}_{12} = \vec{\nabla}_a \left[I_1 I_2 \frac{\mu_0}{4\pi} \iint \frac{d\vec{l_1} \cdot d\vec{l_2}}{r_{12}} \right]$$

The bracket defines the mutual inductance L_{12}. Thus noting the definition of x

$$\langle F_x \rangle = \frac{\partial L_{12}}{\partial x} \langle I_1 I_2 \rangle$$

Where < > denotes time average. Now

$$I_1 = I_0 \cos \omega t = \text{Real part of } I_0 e^{i\omega t}$$

The current in the coil 2 satisfies $R I_2 + L_2 \frac{dI_2}{dt} = -L_{12} \frac{dI_1}{dt}$

Or $I_2 = \frac{-i\omega L_{12}}{R + i\omega L_2} I_0 e^{i\omega t}$ (in the complex case)

Taking the real part

$$I_2 = -\frac{\omega L_{12} I_0}{R^2 + \omega^2 L_2^2}\left(\omega L_2 \cos \omega t - R \sin \omega t\right)$$

$$= -\frac{\omega L_{12}}{\sqrt{R^2 + \omega^2 L_\tau^2}} I_0 \cos(\omega t + \varphi)$$

Where tan $\varphi = \frac{R}{\omega L_2}$. Taking time average, we get

$$<F_x> = \frac{\partial L_{12}}{\partial x} I_0 \frac{\omega L_{12} I_0}{\sqrt{R^2 + \omega^2 L_2^2}} \cdot \frac{1}{2}\cos\varphi = \frac{\omega^2 L_2 L_{12} I_0^2}{2\left(R^2 + \omega^2 L_2^2\right)} \frac{\partial L_{12}}{\partial x}$$

The repulsive nature of the force is also consistent with Lenz’s law, assuming, of comes, that L_{12}decreases with x.