Homogeneous Differential Equation

Q.1. Solve the differential equation : $(x^2 + y^2)\,dx - 2xydy = 0$, given that $y = 0$, when $x = 1$.

Solution : 1

We have $(x^2 + y^2)\,dx - 2xy\,dy = 0$ & $y = 0$ when $x = 1$.

The equation may be written as –

$dy/dx = (x^2 + y^2)/2xy$.

This is homogeneous equation . Put $y = vx$

Then $dy/dx = v + x\,dv/dx$

The equation reduces to ,

$v + x\,dv/dx = (x^2 + v^2 x^2)/2x^2v = (1 + v^2)/2v$

Or, $x\,dv/dx = (1 + v^2)/2v - v = (1 - v^2)/2v$

Or, $2v/(1 - v^2)\,dv = dx/x$

Integrating both sides , we get

$\int[2v/(1 - v^2)]\,dv = \int dx/x$

Or, $-\log[1 - v^2] = \log x + c$

Or, $-\log[1 - (y/x)^2] = \log x + c$

Putting $x = 1$ and $y = 0$, we get $c = 0$.

Then $1/[1 - y^2/x2] = x$

Or, $x = x^2 - y^2$.

Q.2. Solve the differential equation : $x\,dy/dx - y = \sqrt{(x^2 + y^2)}$.

Solution : 2

We have x dy/dx – y = $\sqrt{(x^2 + y^2)}$

Or, dy/dx – y/x = $\sqrt{\{(x^2 + y^2)/x^2\}}$ ------------- (1)

Let y = vx => dy/dx = v + dv/dx

Putting in (1) we get v + xdv/dx – v = $\sqrt{(x^2 + v^2x^2)}/x$

Or, xdv/dx = $\sqrt{(1 + v^2)}$

Integrating we get , $\int dv/\sqrt{(1 + v^2)} = \int dx/x$ + const.

Or, log [v + $\sqrt{(1 + v^2)}$ = log x + log c

Or, log[y/x + $\sqrt{(x^2 + y^2)}/x$] = log x + log c

Or, log[{y + $\sqrt{(x^2 + y^2)}$}/x^2] = log c

Or, [y + $\sqrt{(x^2 + y^2)}$] = cx^2 .

Q.3. Solve the differential equation : 2dy/dx = y/x + y^2/x^2 by substituting y = vx.

Solution : 3

We have, 2 dy/dx = y/x + y^2/x^2 --------------- (1)

Putting y = vx, we get dy/dx = v + dv/dx

Putting in (1), we get

2(v + dv/dx) = v + v^2

Or, 2dv/dx = v + v^2 – 2v = v^2 – v

Integrating both sides, we get

$\int 1/(v2 - v)dv = 1/2 \int dx/x$

Or, $\int dv/(v^2 - v + 1/4 - 1/4) = 1/2 \int dx/x$

Or, $\int dv/[(v - 1/2)^2 - (1/2)^2] = 1/2 \int dx/x$

Or, 1/(2×1/2)log[$|v - 1/2 - 1/2| / |v - 1/2 + 1/2|$] = 1/2 log x

Or, $\log [(v - 1)/v] = 1/2 \log x$

Or, $\log [(y - x)/y] = 1/2 \log x + c$.

Q.4. Solve : $x\, dy/dx - y = \sqrt{(x^2 + y^2)}$.

Solution : 4

We are given $x\, dy/dx - y = \sqrt{(x^2 + y^2)}$

Or, $dy/dx = y/x + \{\sqrt{(x^2 + y^2)}\}/x$

Or, $dy/dx = y/x + \sqrt{\{(x^2 + y^2/x^2)\}}$

Or, $dy/dx = y/x + \sqrt{\{1 + (y/x)^2\}}$

Putting $y = vx$ we get

$dy/dx = v + x\, dv/dx$

and our equation reduces to

$v + x\, dv/dx = v + \sqrt{(1 + v^2)}$

Or, $x\, dv/dx = \sqrt{(1 + v^2)}$

Or, $dv/\sqrt{(1 + v^2)} = dx/x$

Integrating both sides we get

$\int dv/\sqrt{(1 + v^2)} = \int dx/x$

Or, $\log [v + \sqrt{(1 + v^2)}] = \log x + \log c$

Or, $\log[y/x + \sqrt{\{1 + (y/x)^2\}}] = \log x + \log c$

Or, $\log[y + \sqrt{(x^2 + y^2)}] - \log x = \log x + \log c$

Or, $\log [y + \sqrt{(x^2 + y^2)}] = 2 \log x + \log c$

Or, $\log [y + \sqrt{(x^2 + y^2)}] = \log x^2 + \log c$

Or, $\log [y + \sqrt{(x^2 + y^2)}] = \log cx^2$

Therefore, $y + \sqrt{(x^2 + y^2)} = cx^2$.

Q.5. Solve : $x(x - y)\, dy + y^2 dx = 0$.

Solution : 5

We are given,

$x(x - y)\, dy + y^2 dx = 0$

Or, $dy/dx = - y^2/(x^2 - xy) = y^2/(xy - x^2)$

Putting $y = v\,x$,

we get $dy/dx = v + x dv/dx$

Or, $v + x dv/dx = v^2 x^2/(x.vx - x^2) = v2x2/(vx^2 - x^2)$

Or, $v + x dv/dx = v^2/(v - 1)$

Or, $x\, dv/dx = v^2/(v - 1) - v = (v^2 - v^2 + v)(v - 1\ )$

Or, $x\, dv/dx = v/(v - 1)$

Or, $[(v - 1)/v]dv = (1/x)\, dx$

Integrating, we get

$\int(1 - 1/v)\, dv = \int(1/x)\, dx + c$

Or, $v - \log v = \log x + c$

Or, $y/x - \log (y/x) = \log x + c$

Or, $y/x - [\log y/x + \log x] = c$

Or, $y/x - [\log y - \log x + \log x] = c$

Or, $y/x - \log y = c$

Or, $y = x \log y = c\, x$.