# Chapter 6

# Magnetic Fields in Matter

## 6.1 Magnetization

### 6.1.1 Diamagnets, Paramagnets, Ferromagnets

If you ask the average person what "magnetism" is, you will probably be told about horseshoe magnets, compass needles, and the North Pole—none of which has any obvious connection with moving charges or current-carrying wires. Yet all magnetic phenomena are due to electric charges in motion, and in fact, if you could examine a piece of magnetic material on an atomic scale you *would* find tiny currents: electrons orbiting around nuclei and electrons spinning about their axes. For macroscopic purposes, these current loops are so small that we may treat them as magnetic dipoles. Ordinarily, they cancel each other out because of the random orientation of the atoms. But when a magnetic field is applied, a net alignment of these magnetic dipoles occurs, and the medium becomes magnetically polarized, or **magnetized**.

Unlike electric polarization, which is almost always in the same direction as **E**, some materials acquire a magnetization *parallel* to **B** (*para*magnets) and some *opposite* to **B** (*dia*magnets). A few substances (called *ferro*magnets, in deference to the most common example, iron) retain their magnetization even after the external field has been removed—for these the magnetization is not determined by the *present* field but by the whole magnetic "history" of the object. Permanent magnets made of iron are the most familiar examples of magnetism, though from a theoretical point of view they are the most complicated; I'll save ferromagnetism for the end of the chapter, and begin with qualitative models of paramagnetism and diamagnetism.

### 6.1.2 Torques and Forces on Magnetic Dipoles

A magnetic dipole experiences a torque in a magnetic field, just as an electric dipole does in an electric field. Let's calculate the torque on a rectangular current loop in a uniform field **B**. (Since any current loop could be built up from infinitesimal rectangles, with all

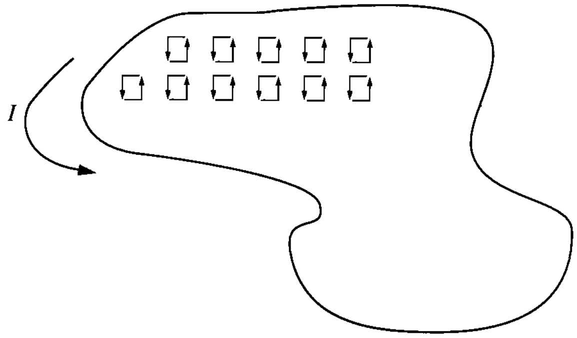



Figure 6.1

the "internal" sides canceling, as indicated in Fig. 6.1, there is no actual loss of generality in using this shape; but if you prefer to start from scratch with an arbitrary shape, see Prob. 6.2.) Center the loop at the origin, and tilt it an angle $\theta$ from the $z$ axis towards the $y$ axis (Fig. 6.2). Let **B** point in the $z$ direction. The forces on the two sloping sides cancel (they tend to *stretch* the loop, but they don't *rotate* it). The forces on the "horizontal" sides are likewise equal and opposite (so the net *force* on the loop is zero), but they do generate a torque:

$$\mathbf{N} = aF\sin\theta\,\hat{\mathbf{x}}.$$

The magnitude of the force on each of these segments is

$$F = IbB,$$

and therefore

$$\mathbf{N} = IabB\sin\theta\,\hat{\mathbf{x}} = mB\sin\theta\,\hat{\mathbf{x}},$$

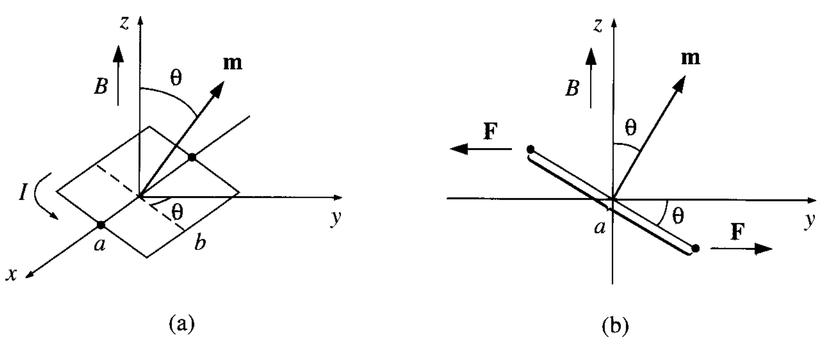



Figure 6.2

or

$$\boxed{\mathbf{N} = \mathbf{m} \times \mathbf{B},} \tag{6.1}$$

where $m = Iab$ is the magnetic dipole moment of the loop. Equation 6.1 gives the exact torque on *any* localized current distribution, in the presence of a *uniform* field; in a *nonuniform* field it is the exact torque (about the center) for a *perfect* dipole of infinitesimal size.

Notice that Eq. 6.1 is identical in form to the electrical analog, Eq. 4.4: $\mathbf{N} = \mathbf{p} \times \mathbf{E}$. In particular, the torque is again in such a direction as to line the dipole up *parallel* to the field. It is this torque that accounts for **paramagnetism**. Since every electron constitutes a magnetic dipole (picture it, if you wish, as a tiny spinning sphere of charge), you might expect paramagnetism to be a universal phenomenon. Actually, the laws of quantum mechanics (specifically, the Pauli exclusion principle) dictate that the electrons within a given atom lock together in pairs with opposing spins, and this effectively neutralizes the torque on the combination. As a result, paramagnetism normally occurs in atoms or molecules with an odd number of electrons, where the "extra" unpaired member is subject to the magnetic torque. Even here the alignment is far from complete, since random thermal collisions tend to destroy the order.

In a *uniform* field, the net *force* on a current loop is zero:

$$\mathbf{F} = I \oint (d\mathbf{l} \times \mathbf{B}) = I \left( \oint d\mathbf{l} \right) \times \mathbf{B} = 0;$$

the constant $\mathbf{B}$ comes outside the integral, and the net displacement $\oint d\mathbf{l}$ around a closed loop vanishes. In a *nonuniform* field this is no longer the case. For example, suppose a circular wire of radius $R$, carrying a current $I$, is suspended above a short solenoid in the "fringing" region (Fig. 6.3). Here $\mathbf{B}$ has a radial component, and there is a net downward force on the loop (Fig. 6.4):

$$F = 2\pi IRB\cos\theta. \tag{6.2}$$

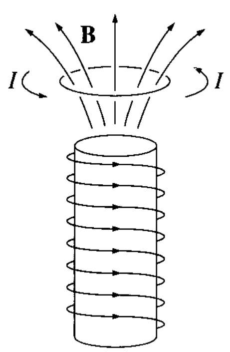



Figure 6.3

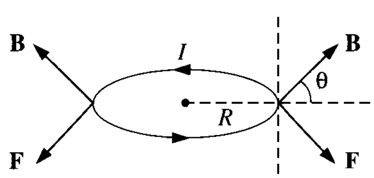


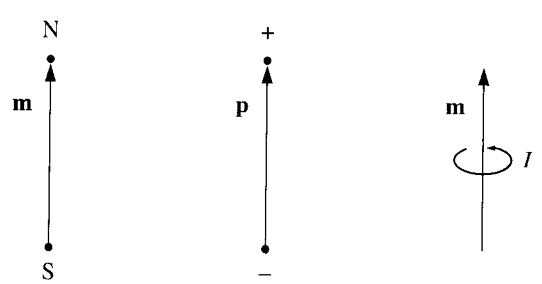

Figure 6.4

For an *infinitesimal* loop, with dipole moment **m**, in a field **B**, the force is

$$\boxed{\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})} \tag{6.3}$$

(see Prob. 6.4). Once again the magnetic formula is identical to its electrical "twin," provided we agree to write the latter in the form $\mathbf{F} = \nabla(\mathbf{p} \cdot \mathbf{E})$.

If you're starting to get a sense of *déjà vu*, perhaps you will have more respect for those early physicists who thought magnetic dipoles consisted of positive and negative magnetic "charges" (north and south "poles," they called them), separated by a small distance, just like electric dipoles (Fig. 6.5(a)). They wrote down a "Coulomb's law" for the attraction and repulsion of these poles, and developed the whole of magnetostatics in exact analogy to electrostatics. It's not a bad model, for many purposes—it gives the correct field of a dipole (at least, away from the origin), the right torque on a dipole (at least, on a *stationary* dipole), and the proper force on a dipole (at least, in the absence of external currents). But it's bad physics, because *there's no such thing* as a single magnetic north pole or south pole. If you break a bar magnet in half, you don't get a north pole in one hand and a south pole in the other; you get two complete magnets. Magnetism is *not* due to magnetic monopoles, but rather to *moving electric charges*; magnetic dipoles are tiny current loops (Fig. 6.5(c)), and it's an extraordinary thing, really, that the formulas involving **m** bear any resemblance at all to the corresponding formulas for **p**. Sometimes it is easier to think in terms of the "Gilbert" model of a magnetic dipole (separated monopoles) instead of the physically correct "Ampère" model (current loop). Indeed, this picture occasionally offers a quick and clever solution to an otherwise cumbersome problem (you just copy the corresponding result from electrostatics, changing **p** to **m**, $1/\epsilon_0$ to $\mu_0$, and **E** to **B**). But whenever the *close-up* features of the dipole come into play, the two models can yield strikingly different answers. My advice is to use the Gilbert model, if you like, to get an intuitive "feel" for a problem, but never rely on it for quantitative results.




(a) Magnetic dipole (Gilbert model) (b) Electric dipole (a) Magnetic dipole (Ampère model)

Figure 6.5

**Problem 6.1** Calculate the torque exerted on the square loop shown in Fig. 6.6, due to the circular loop (assume $r$ is much larger than $a$ or $b$). If the square loop is free to rotate, what will its equilibrium orientation be?

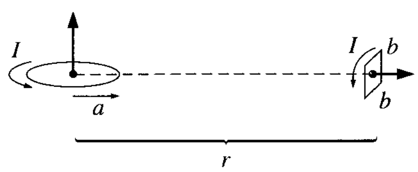



Figure 6.6

**Problem 6.2** Starting from the Lorentz force law, in the form of Eq. 5.16, show that the torque on *any* steady current distribution (not just a square loop) in a uniform field $\mathbf{B}$ is $\mathbf{m} \times \mathbf{B}$.

**Problem 6.3** Find the force of attraction between two magnetic dipoles, $\mathbf{m}_1$ and $\mathbf{m}_2$, oriented as shown in Fig. 6.7, a distance $r$ apart, (a) using Eq. 6.2, and (b) using Eq. 6.3.

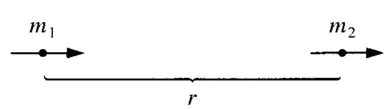



Figure 6.7

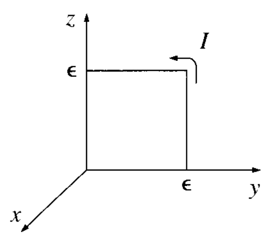



Figure 6.8

**Problem 6.4** Derive Eq. 6.3. [Here's one way to do it: Assume the dipole is an infinitesimal square, of side $\epsilon$ (if it's not, chop it up into squares, and apply the argument to each one). Choose axes as shown in Fig. 6.8, and calculate $\mathbf{F} = I\int(d\mathbf{l} \times \mathbf{B})$ along each of the four sides. Expand $\mathbf{B}$ in a Taylor series—on the right side, for instance,

$$\mathbf{B} = \mathbf{B}(0, \epsilon, z) \cong \mathbf{B}(0, 0, z) + \epsilon \left.\frac{\partial \mathbf{B}}{\partial y}\right|_{(0,0,z)}.$$

For a more sophisticated method, see Prob. 6.22.]

**Problem 6.5** A uniform current density $\mathbf{J} = J_0\,\hat{\mathbf{z}}$ fills a slab straddling the $yz$ plane, from $x = -a$ to $x = +a$. A magnetic dipole $\mathbf{m} = m_0\,\hat{\mathbf{x}}$ is situated at the origin.

(a) Find the force on the dipole, using Eq. 6.3.

(b) Do the same for a dipole pointing in the $y$ direction: $\mathbf{m} = m_0\hat{\mathbf{y}}$.

(c) In the *electrostatic* case the expressions $\mathbf{F} = \nabla(\mathbf{p}\cdot\mathbf{E})$ and $\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}$ are equivalent (prove it), but this is *not* the case for the magnetic analogs (explain why). As an example, calculate $(\mathbf{m}\cdot\nabla)\mathbf{B}$ for the configurations in (a) and (b).

### 6.1.3 Effect of a Magnetic Field on Atomic Orbits

Electrons not only *spin*; they also *revolve* around the nucleus—for simplicity, let's assume the orbit is a circle of radius $R$ (Fig. 6.9). Although technically this orbital motion does not constitute a steady current, in practice the period $T = 2\pi R/v$ is so short that unless you blink awfully fast, it's going to *look* like a steady current:

$$I = \frac{e}{T} = \frac{ev}{2\pi R}.$$

Accordingly, the orbital dipole moment $(I\pi R^2)$ is

$$\mathbf{m} = -\frac{1}{2}evR\,\hat{\mathbf{z}}. \tag{6.4}$$

(The minus sign accounts for the negative charge of the electron.) Like any other magnetic dipole, this one is subject to a torque $(\mathbf{m}\times\mathbf{B})$ when the atom is placed in a magnetic field. But it's a lot harder to tilt the entire orbit than it is the spin, so the orbital contribution to paramagnetism is small. There is, however, a more significant effect on the orbital motion:

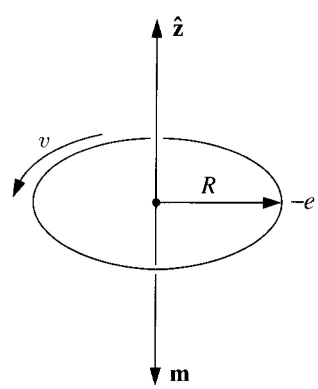



Figure 6.9

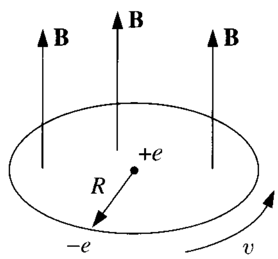



Figure 6.10

The electron *speeds up* or *slows down*, depending on the orientation of **B**. For whereas the centripetal acceleration $v^2/R$ is ordinarily sustained by electrical forces alone,[1]

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} = m_e\frac{v^2}{R}, \tag{6.5}$$

in the presence of a magnetic field there is an additional force, $-e(\mathbf{v}\times\mathbf{B})$. For the sake of argument, let's say that **B** is perpendicular to the plane of the orbit, as shown in Fig. 6.10; then

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + e\bar{v}B = m_e\frac{\bar{v}^2}{R}. \tag{6.6}$$

Under these conditions, the new speed $\bar{v}$ is *greater* than $v$:

$$e\bar{v}B = \frac{m_e}{R}(\bar{v}^2 - v^2) = \frac{m_e}{R}(\bar{v}+v)(\bar{v}-v),$$

or, assuming the change $\Delta v = \bar{v} - v$ is small,

$$\Delta v = \frac{eRB}{2m_e}. \tag{6.7}$$

When **B** is turned on, then, the electron speeds up.[2]

A change in orbital speed means a change in the dipole moment (6.4):

$$\Delta\mathbf{m} = -\frac{1}{2}e(\Delta v)R\,\hat{\mathbf{z}} = -\frac{e^2R^2}{4m_e}\mathbf{B}. \tag{6.8}$$

Notice that *the change in* **m** *is opposite to the direction of* **B**. (An electron circling the other way would have a dipole moment pointing *upward*, but such an orbit would be *slowed*

[1]To avoid confusion with the magnetic dipole moment $m$, I'll write the electron mass with subscript: $m_e$.

[2]I said earlier (Eq. 5.11) that magnetic fields do no work, and are incapable of speeding a particle up. I stand by that. However, as we shall see in Chapter 7, a *changing* magnetic field induces an *electric* field, and it is the latter that accelerates the electrons in this instance.

*down* by the field, so the *change* is *still* opposite to **B**.) Ordinarily, the electron orbits are randomly oriented, and the orbital dipole moments cancel out. But in the presence of a magnetic field, each atom picks up a little "extra" dipole moment, and these increments are all *antiparallel* to the field. This is the mechanism responsible for **diamagnetism**. It is a universal phenomenon, affecting all atoms. However, it is typically much weaker than paramagnetism, and is therefore observed mainly in atoms with *even* numbers of electrons, where paramagnetism is usually absent.

In deriving Eq. 6.8 I assumed that the orbit remains circular, with its original radius $R$. I cannot offer a justification for this at the present stage. If the atom is stationary while the field is turned on, then my assumption can be proved—this is not magneto*statics*, however, and the details will have to await Chapter 7 (see Prob. 7.49). If the atom is moved into the field, the situation is enormously more complicated. But never mind—I'm only trying to give you a qualitative account of diamagnetism. Assume, if you prefer, that the velocity remains the same while the *radius* changes—the formula (6.8) is altered (by a factor of 2), but the *conclusion* is unaffected. The truth is that this classical model is fundamentally flawed (diamagnetism is really a *quantum* phenomenon), so there's not much point in refining the details.[3] What *is* important is the *empirical* fact that in diamagnetic materials the induced dipole moments point *opposite* to the magnetic field.

### 6.1.4 Magnetization

In the presence of a magnetic field, matter becomes *magnetized*; that is, upon microscopic examination it will be found to contain many tiny dipoles, with a net alignment along some direction. We have discussed two mechanisms that account for this magnetic polarization: (1) paramagnetism (the dipoles associated with the spins of unpaired electrons experience a torque tending to line them up parallel to the field) and (2) diamagnetism (the orbital speed of the electrons is altered in such a way as to change the orbital dipole moment in a direction opposite to the field). Whatever the *cause*, we describe the state of magnetic polarization by the vector quantity

$$\mathbf{M} \equiv \textit{magnetic dipole moment per unit volume}. \tag{6.9}$$

**M** is called the **magnetization**; it plays a role analogous to the polarization **P** in electrostatics. In the following section, we will not worry about how the magnetization *got* there—it could be paramagnetism, diamagnetism, or even ferromagnetism—we shall take **M** as *given*, and calculate the field this magnetization itself produces.

Incidentally, it may have surprised you to learn that materials other than the famous ferromagnetic trio (iron, nickel, and cobalt) are affected by a magnetic field *at all*. You cannot, of course, pick up a piece of wood or aluminum with a magnet. The reason is that diamagnetism and paramagnetism are extremely weak: It takes a delicate experiment and a powerful magnet to detect them at all. If you were to suspend a piece of paramagnetic

[3] S. L. O'Dell and R. K. P. Zia, *Am. J. Phys.* **54**, 32, (1986); R. Peierls, *Surprises in Theoretical Physics*, Section 4.3 (Princeton, N.J.: Princeton University Press, 1979); R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Vol. 2, Sec. 34–36 (New York: Addison-Wesley, 1966).

material above a solenoid, as in Fig. 6.3, the induced magnetization would be upward, and hence the force downward. By contrast, the magnetization of a diamagnetic object would be downward and the force upward. In general, when a sample is placed in a region of nonuniform field, the *paramagnet is attracted into the field*, whereas the *diamagnet is repelled away*. But the actual forces are pitifully weak—in a typical experimental arrangement the force on a comparable sample of iron would be $10^4$ or $10^5$ times as great. That's why it was reasonable for us to calculate the field inside a piece of copper wire, say, in Chapter 5, without worrying about the effects of magnetization.

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**Problem 6.6** Of the following materials, which would you expect to be paramagnetic and which diamagnetic? Aluminum, copper, copper chloride ($CuCl_2$), carbon, lead, nitrogen ($N_2$), salt (NaCl), sodium, sulfur, water. (Actually, copper is slightly *dia*magnetic; otherwise they're all what you'd expect.)

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# 6.2 The Field of a Magnetized Object

## 6.2.1 Bound Currents

Suppose we have a piece of magnetized material; the magnetic dipole moment per unit volume, **M**, is given. What field does this object produce? Well, the vector potential of a single dipole **m** is given by Eq. 5.83:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \hat{\boldsymbol{\imath}}}{\imath^2}. \tag{6.10}$$

In the magnetized object, each volume element $d\tau'$ carries a dipole moment $\mathbf{M}\, d\tau'$, so the total vector potential is (Fig. 6.11)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{M}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2}\, d\tau'. \tag{6.11}$$

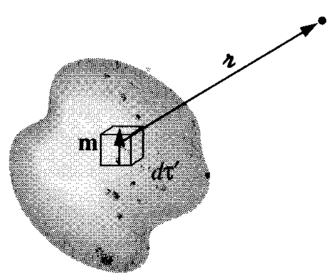


Figure 6.11

That *does* it, in principle. But as in the electrical case (Sect. 4.2.1), the integral can be cast in a more illuminating form by exploiting the identity

$$\nabla' \frac{1}{\mathscr{r}} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}.$$

With this,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \left[ \mathbf{M}(\mathbf{r}') \times \left( \nabla' \frac{1}{\mathscr{r}} \right) \right] d\tau'.$$

Integrating by parts, using product rule 7, gives

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left\{ \int \frac{1}{\mathscr{r}} [\nabla' \times \mathbf{M}(\mathbf{r}')] \, d\tau' - \int \nabla' \times \left[ \frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}} \right] d\tau' \right\}.$$

Problem 1.60(b) invites us to express the latter as a surface integral,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{1}{\mathscr{r}} [\nabla' \times \mathbf{M}(\mathbf{r}')] \, d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{\mathscr{r}} [\mathbf{M}(\mathbf{r}') \times d\mathbf{a}']. \tag{6.12}$$

The first term looks just like the potential of a *volume* current,

$$\boxed{\mathbf{J}_b = \nabla \times \mathbf{M},} \tag{6.13}$$

while the second looks like the potential of a surface current,

$$\boxed{\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}},} \tag{6.14}$$

where $\hat{\mathbf{n}}$ is the normal unit vector. With these definitions,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}} \, d\tau' + \frac{\mu_0}{4\pi} \oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}} \, da'. \tag{6.15}$$

What this means is that the potential (and hence also the field) of a magnetized object is the same as would be produced by a volume current $\mathbf{J}_b = \nabla \times \mathbf{M}$ throughout the material. plus a surface current $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$, on the boundary. Instead of integrating the contributions of all the infinitesimal dipoles, as in Eq. 6.11, we first determine these **bound currents**, and then find the field *they* produce, in the same way we would calculate the field of any other volume and surface currents. Notice the striking parallel with the electrical case: there the field of a polarized object was the same as that of a bound volume charge $\rho_b = -\nabla \cdot \mathbf{P}$ plus a bound surface charge $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$.

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**Example 6.1**

Find the magnetic field of a uniformly magnetized sphere.

**Solution**: Choosing the $z$ axis along the direction of $\mathbf{M}$ (Fig. 6.12), we have

$$\mathbf{J}_b = \nabla \times \mathbf{M} = 0, \quad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = M \sin\theta \, \hat{\boldsymbol{\phi}}.$$

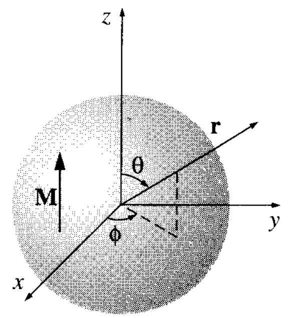


Figure 6.12

Now, a rotating spherical shell, of uniform surface charge $\sigma$, corresponds to a surface current density

$$\mathbf{K} = \sigma \mathbf{v} = \sigma \omega R \sin\theta\, \hat{\boldsymbol{\phi}}.$$

It follows, therefore, that the field of a uniformly magnetized sphere is identical to the field of a spinning spherical shell, with the identification $\sigma R\omega \to \mathbf{M}$. Referring back to Ex. 5.11, I conclude that

$$\mathbf{B} = \frac{2}{3}\mu_0 \mathbf{M}, \tag{6.16}$$

inside the sphere, whereas the field outside is the same as that of a pure dipole,

$$\mathbf{m} = \frac{4}{3}\pi R^3 \mathbf{M}.$$

Notice that the internal field is *uniform*, like the *electric* field inside a uniformly *polarized* sphere (Eq. 4.14), although the actual *formulas* for the two cases are curiously different ($\frac{2}{3}$ in place of $-\frac{1}{3}$). The external fields are also analogous: pure dipole in both instances.

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**Problem 6.7** An infinitely long circular cylinder carries a uniform magnetization $\mathbf{M}$ parallel to its axis. Find the magnetic field (due to $\mathbf{M}$) inside and outside the cylinder.

**Problem 6.8** A long circular cylinder of radius $R$ carries a magnetization $\mathbf{M} = ks^2\,\hat{\boldsymbol{\phi}}$, where $k$ is a constant, $s$ is the distance from the axis, and $\hat{\boldsymbol{\phi}}$ is the usual azimuthal unit vector (Fig. 6.13). Find the magnetic field due to $\mathbf{M}$, for points inside and outside the cylinder.

**Problem 6.9** A short circular cylinder of radius $a$ and length $L$ carries a "frozen-in" uniform magnetization $\mathbf{M}$ parallel to its axis. Find the bound current, and sketch the magnetic field of the cylinder. (Make three sketches: one for $L \gg a$, one for $L \ll a$, and one for $L \approx a$.) Compare this **bar magnet** with the bar electret of Prob. 4.11.

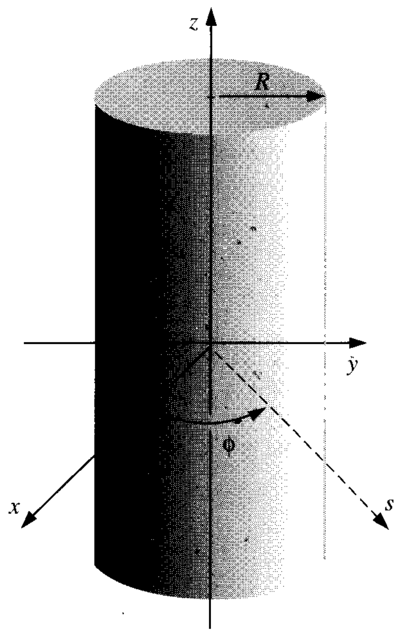


Figure 6.13

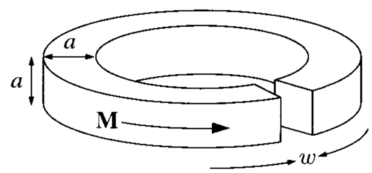


Figure 6.14

**Problem 6.10** An iron rod of length $L$ and square cross section (side $a$), is given a uniform longitudinal magnetization $\mathbf{M}$, and then bent around into a circle with a narrow gap (width $w$), as shown in Fig. 6.14. Find the magnetic field at the center of the gap, assuming $w \ll a \ll L$. [*Hint:* treat it as the superposition of a *complete* torus plus a square loop with reversed current.]

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### 6.2.2 Physical Interpretation of Bound Currents

In the last section we found that the field of a magnetized object is identical to the field that would be produced by a certain distribution of "bound" currents, $\mathbf{J}_b$ and $\mathbf{K}_b$. I want to show you how these bound currents arise physically. This will be a *heuristic* argument—the *rigorous* derivation has already been given. Figure 6.15 depicts a thin slab of uniformly magnetized material, with the dipoles represented by tiny current loops. Notice that all the "internal" currents cancel: every time there is one going to the right, a contiguous one is going to the left. However, at the edge there is *no adjacent loop to do the canceling*. The whole thing, then, is equivalent to a single ribbon of current $I$ flowing around the boundary (Fig. 6.16).

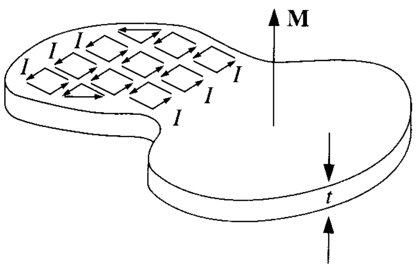



Figure 6.15

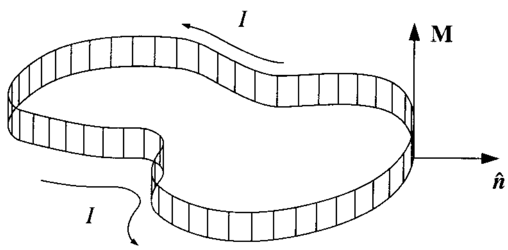



Figure 6.16

What *is* this current, in terms of $\mathbf{M}$? Say that each of the tiny loops has area $a$ and thickness $t$ (Fig. 6.17). In terms of the magnetization $M$, its dipole moment is $m = Mat$. In terms of the circulating current $I$, however, $m = Ia$. Therefore $I = Mt$, so the surface current is $K_b = I/t = M$. Using the outward-drawn unit vector $\hat{\mathbf{n}}$ (Fig. 6.16), the direction of $\mathbf{K}_b$ is conveniently indicated by the cross product:

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}.$$

(This expression also records the fact that there is *no* current on the top or bottom surface of the slab; here $\mathbf{M}$ is parallel to $\hat{\mathbf{n}}$, so the cross product vanishes.)

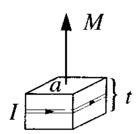



Figure 6.17

This bound surface current is exactly what we obtained in Sect. 6.2.1. It is a peculiar *kind* of current, in the sense that no single charge makes the whole trip—on the contrary, each charge moves only in a tiny little loop within a single atom. Nevertheless, the net effect is a macroscopic current flowing over the surface of the magnetized object. We call it a "bound" current to remind ourselves that every charge is attached to a particular atom, but it's a perfectly genuine current, and it produces a magnetic field in the same way any other current does.

When the magnetization is *non*uniform, the internal currents no longer cancel. Figure 6.18a shows two adjacent chunks of magnetized material, with a larger arrow on the one to the right suggesting greater magnetization at that point. On the surface where they join there is a net current in the $x$-direction, given by

$$I_x = [M_z(y + dy) - M_z(y)]\, dz = \frac{\partial M_z}{\partial y}\, dy\, dz.$$

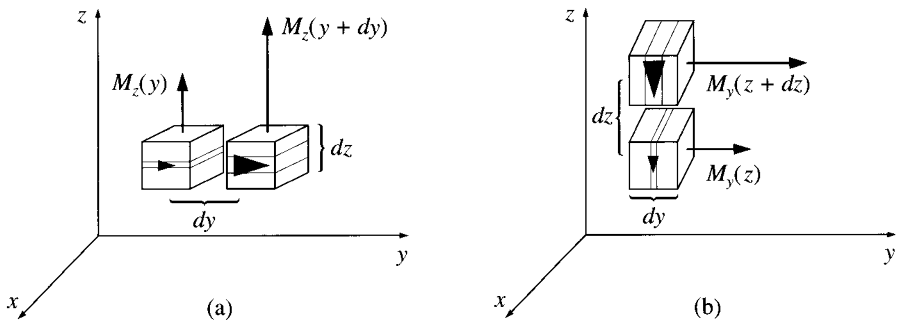



Figure 6.18

The corresponding volume current density is therefore

$$(J_b)_x = \frac{\partial M_z}{\partial y}.$$

By the same token, a nonuniform magnetization in the $y$-direction would contribute an amount $-\partial M_y/\partial z$ (Fig. 6.18b), so

$$(J_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}.$$

In general, then,

$$\mathbf{J}_b = \nabla \times \mathbf{M},$$

consistent, again, with the result of Sect. 6.2.1. Incidentally, like any other steady current, $\mathbf{J}_b$ should obey the conservation law 5.31:

$$\nabla \cdot \mathbf{J}_b = 0.$$

Does it? *Yes*, for the divergence of a curl is *always* zero.

### 6.2.3 The Magnetic Field Inside Matter

Like the electric field, the actual *microscopic* magnetic field inside matter fluctuates wildly from point to point and instant to instant. When we speak of "the" magnetic field in matter, we mean the *macroscopic* field: the average over regions large enough to contain many atoms. (The magnetization $\mathbf{M}$ is "smoothed out" in the same sense.) It is this macroscopic field one obtains when the methods of Sect. 6.2.1 are applied to points inside magnetized material, as you can prove for yourself in the following problem.

**Problem 6.11** In Sect, 6.2.1, we began with the potential of a *perfect* dipole (Eq. 6.10), whereas in *fact* we are dealing with *physical* dipoles. Show, by the method of Sect. 4.2.3, that we nonetheless get the correct macroscopic field.

# 6.3 The Auxiliary Field H

## 6.3.1 Ampère's law in Magnetized Materials

In Sect. 6.2 we found that the effect of magnetization is to establish bound currents $\mathbf{J}_b = \nabla \times \mathbf{M}$ within the material and $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$ on the surface. The field due to magnetization of the medium is just the field produced by these bound currents. We are now ready to put everything together: the field attributable to bound currents, plus the field due to everything else—which I shall call the **free current**. The free current might flow through wires imbedded in the magnetized substance or, if the latter is a conductor, through the material itself. In any event, the total current can be written as

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f. \tag{6.17}$$

There is no new physics in Eq. 6.17; it is simply a *convenience* to separate the current into these two parts because they *got* there by quite different means: the free current is there because somebody hooked up a wire to a battery—it involves actual transport of charge; the bound current is there because of magnetization—it results from the conspiracy of many aligned atomic dipoles.

In view of Eqs. 6.13 and 6.17, Ampère's law can be written

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + (\nabla \times \mathbf{M}),$$

or, collecting together the two curls:

$$\nabla \times \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\right) = \mathbf{J}_f.$$

The quantity in parentheses is designated by the letter **H**:

$$\boxed{\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.} \tag{6.18}$$

In terms of **H**, then, Ampère's law reads

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}_f,} \tag{6.19}$$

or, in integral form,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}}, \tag{6.20}$$

where $I_{f_{\text{enc}}}$ is the total *free* current passing through the Amperian loop.

**H** plays a role in magnetostatics analogous to **D** in electrostatics: Just as **D** allowed us to write *Gauss's* law in terms of the free *charge* alone, **H** permits us to express *Ampère's* law in terms of the free *current* alone—and free current is what we control directly. Bound current, like bound charge, comes along for the ride—the material gets magnetized, and this results in bound currents; we cannot turn them on or off independently, as we can free currents. In applying Eq. 6.20 all we need to worry about is the *free* current, which we know about because we *put* it there. In particular, when symmetry permits, we can calculate $\mathbf{H}$ immediately from Eq. 6.20 by the usual Ampère's law methods. (For example, Probs. 6.7 and 6.8 can be done in one line by noting that $\mathbf{H} = 0$.)

### Example 6.2

A long copper rod of radius $R$ carries a uniformly distributed (free) current $I$ (Fig. 6.19). Find $H$ inside and outside the rod.

**Solution**: Copper is weakly diamagnetic, so the dipoles will line up *opposite* to the field. This results in a bound current running *antiparallel* to $I$ within the wire and *parallel* to $I$ along the surface (see Fig. 6.20). Just how *great* these bound currents will be we are not yet in a position

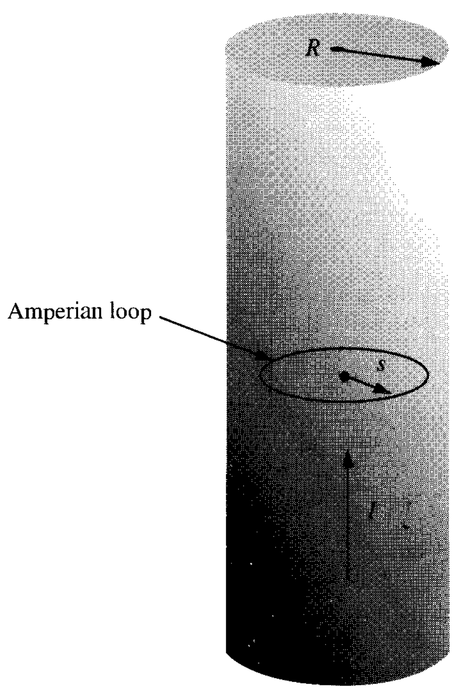



Figure 6.19

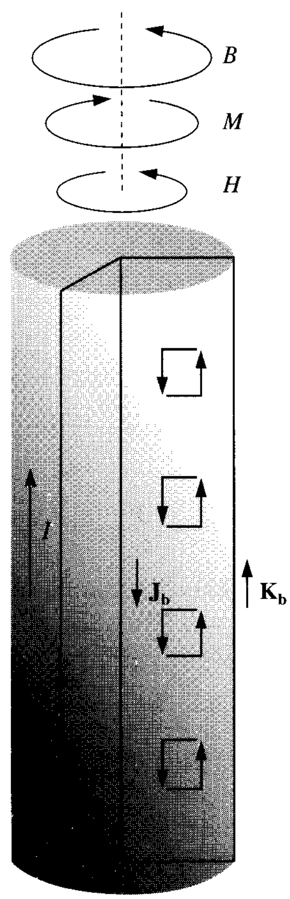



Figure 6.20

to say—but in order to calculate **H** it is sufficient to realize that all the currents are longitudinal, so **B**, **M**, and therefore also **H**, are circumferential. Applying Eq. 6.20 to an Amperian loop of radius $s < R$,

$$H(2\pi s) = I_{f_{\text{enc}}} = I\frac{\pi s^2}{\pi R^2},$$

so

$$\mathbf{H} = \frac{I}{2\pi R^2} s\,\hat{\boldsymbol{\phi}} \qquad (s \leq R), \tag{6.21}$$

within the wire. Meanwhile, outside the wire

$$\mathbf{H} = \frac{I}{2\pi s}\hat{\boldsymbol{\phi}} \qquad (s \geq R). \tag{6.22}$$

In the latter region (as always, in empty space) $\mathbf{M} = 0$, so

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}} \qquad (s \geq R),$$

the same as for a *non*magnetized wire (Ex. 5.7). *Inside* the wire **B** cannot be determined at this stage, since we have no way of knowing **M** (though in practice the magnetization in copper is so slight that for most purposes we can ignore it altogether).

---

As it turns out, **H** is a more useful quantity than **D**. In the laboratory you will frequently hear people talking about **H** (more often even than **B**), but you will never hear anyone speak of **D** (only **E**). The reason is this: To build an electromagnet you run a certain (free) current through a coil. The *current* is the thing you read on the dial, and this determines **H** (or at any rate, the line integral of **H**); **B** depends on the specific materials you used and even, if iron is present, on the history of your magnet. On the other hand, if you want to set up an *electric* field, you do *not* plaster a known free charge on the plates of a parallel plate capacitor; rather, you connect them to a battery of known *voltage*. It's the *potential difference* you read on your dial, and that determines **E** (or at any rate, the line integral of **E**); **D** depends on the details of the dielectric you're using. If it were easy to measure charge, and hard to measure potential, then you'd find experimentalists talking about **D** instead of **E**. So the relative familiarity of **H**, as contrasted with **D**, derives from purely practical considerations; theoretically, they're all on equal footing.

Many authors call **H**, not **B**, the "magnetic field." Then they have to invent a new word for **B**: the "flux density," or magnetic "induction" (an absurd choice, since that term already has at least two other meanings in electrodynamics). Anyway, **B** is indisputably the fundamental quantity, so I shall continue to call it the "magnetic field," as everyone does in the spoken language. **H** has no sensible name: just call it "**H**".[4]

---

[4]For those who disagree, I quote A. Sommerfeld's *Electrodynamics* (New York: Academic Press, 1952), p. 45: "The unhappy term 'magnetic field' for **H** should be avoided as far as possible. It seems to us that this term has led into error none less than Maxwell himself . . ."

**Problem 6.12** An infinitely long cylinder, of radius $R$, carries a "frozen-in" magnetization, parallel to the axis,

$$\mathbf{M} = ks\,\hat{\mathbf{z}},$$

where $k$ is a constant and $s$ is the distance from the axis; there is no free current anywhere. Find the magnetic field inside and outside the cylinder by two different methods:

(a) As in Sect. 6.2, locate all the bound currents, and calculate the field they produce.

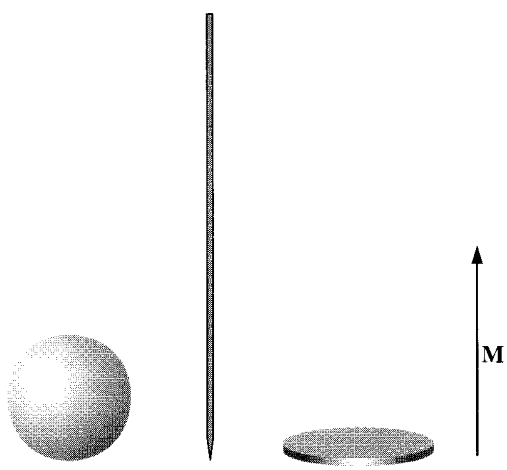
(b) Use Ampère's law (in the form of Eq. 6.20) to find $\mathbf{H}$, and then get $\mathbf{B}$ from Eq. 6.18. (Notice that the second method is much faster, and avoids any explicit reference to the bound currents.)

**Problem 6.13** Suppose the field inside a large piece of magnetic material is $\mathbf{B}_0$, so that $\mathbf{H}_0 = (1/\mu_0)\mathbf{B}_0 - \mathbf{M}$.

(a) Now a small spherical cavity is hollowed out of the material (Fig. 6.21). Find the field at the center of the cavity, in terms of $\mathbf{B}_0$ and $\mathbf{M}$. Also find $\mathbf{H}$ at the center of the cavity, in terms of $\mathbf{H}_0$ and $\mathbf{M}$.

(b) Do the same for a long needle-shaped cavity running parallel to $\mathbf{M}$.

(c) Do the same for a thin wafer-shaped cavity perpendicular to $\mathbf{M}$.




(a) Sphere (b) Needle (c) Wafer

Figure 6.21

Assume the cavities are small enough so that $\mathbf{M}$, $\mathbf{B}_0$, and $\mathbf{H}_0$ are essentially constant. Compare Prob. 4.16. [*Hint:* Carving out a cavity is the same as superimposing an object of the same shape but opposite magnetization.]

### 6.3.2 A Deceptive Parallel

Equation 6.19 looks just like Ampère's original law (5.54), only the *total* current is replaced by the *free* current, and $\mathbf{B}$ is replaced by $\mu_0\mathbf{H}$. As in the case of $\mathbf{D}$, however, I must warn you against reading too much into this correspondence. It does *not* say that $\mu_0\mathbf{H}$ is "just like $\mathbf{B}$, only its source is $\mathbf{J}_f$ instead of $\mathbf{J}$." For the curl alone does not determine a vector field—you must know the divergence as well. And whereas $\nabla \cdot \mathbf{B} = 0$, the divergence of $\mathbf{H}$ is *not*, in general, zero. In fact, from Eq. 6.18

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}. \tag{6.23}$$

Only when the divergence of $\mathbf{M}$ vanishes is the parallel between $\mathbf{B}$ and $\mu_0\mathbf{H}$ faithful.

If you think I'm being pedantic, consider the example of the bar magnet—a short cylinder of iron that carries a permanent uniform magnetization $\mathbf{M}$ parallel to its axis. (See Probs. 6.9 and 6.14.) In this case there is no free current anywhere, and a naïve application of Eq. 6.20 might lead you to suppose that $\mathbf{H} = 0$, and hence that $\mathbf{B} = \mu_0\mathbf{M}$ inside the magnet and $\mathbf{B} = 0$ outside, which is nonsense. It is quite true that the *curl* of $\mathbf{H}$ vanishes everywhere, but the divergence does not. (Can you see where $\nabla \cdot \mathbf{M} = 0$?) *Advice*: When you are asked to find $\mathbf{B}$ or $\mathbf{H}$ in a problem involving magnetic materials, first look for symmetry. If the problem exhibits cylindrical, plane, solenoidal, or toroidal symmetry, then you can get $\mathbf{H}$ directly from Eq. 6.20 by the usual Ampère's law methods. (Evidently, in such cases $\nabla \cdot \mathbf{M}$ is automatically zero, since the free current alone determines the answer.) If the requisite symmetry is absent, you'll have to think of another approach, and in particular you must *not* assume that $\mathbf{H}$ is zero just because you see no free current.

### 6.3.3 Boundary Conditions

The magnetostatic boundary conditions of Sect. 5.4.2 can be rewritten in terms of $\mathbf{H}$ and the *free* current. From Eq. 6.23 it follows that

$$H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}}), \tag{6.24}$$

while Eq. 6.19 says

$$\mathbf{H}^{\parallel}_{\text{above}} - \mathbf{H}^{\parallel}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}}. \tag{6.25}$$

In the presence of materials these are sometimes more useful than the corresponding boundary conditions on **B** (Eqs. 5.72 and 5.73):

$$B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}} = 0, \tag{6.26}$$

and

$$\mathbf{B}^{\parallel}_{\text{above}} - \mathbf{B}^{\parallel}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}). \tag{6.27}$$

You might want to check them, for Ex. 6.2 or Prob. 6.14.

---

**Problem 6.14** For the bar magnet of Prob. 6.9, make careful sketches of **M**, **B**, and **H**, assuming $L$ is about $2a$. Compare Prob. 4.17.

**Problem 6.15** If $\mathbf{J}_f = 0$ everywhere, the curl of **H** vanishes (Eq. 6.19), and we can express **H** as the gradient of a scalar potential $W$:

$$\mathbf{H} = -\nabla W.$$

According to Eq. 6.23, then,

$$\nabla^2 W = (\nabla \cdot \mathbf{M}),$$

so $W$ obeys Poisson's equation, with $\nabla \cdot \mathbf{M}$ as the "source." This opens up all the machinery of Chapter 3. As an example, find the field inside a uniformly magnetized sphere (Ex. 6.1) by separation of variables. [*Hint:* $\nabla \cdot \mathbf{M} = 0$ everywhere except at the surface ($r = R$), so $W$ satisfies Laplace's equation in the regions $r < R$ and $r > R$; use Eq. 3.65, and from Eq. 6.24 figure out the appropriate boundary condition on $W$.]

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## 6.4 Linear and Nonlinear Media

### 6.4.1 Magnetic Susceptibility and Permeability

In paramagnetic and diamagnetic materials, the magnetization is sustained by the field; when **B** is removed, **M** disappears. In fact, for most substances the magnetization is *proportional* to the field, provided the field is not too strong. For notational consistency with the electrical case (Eq. 4.30), I *should* express the proportionality thus:

$$\mathbf{M} = \frac{1}{\mu_0}\chi_m \mathbf{B} \quad \text{(incorrect!)}. \tag{6.28}$$

But custom dictates that it be written in terms of **H**, instead of **B**:

$$\boxed{\mathbf{M} = \chi_m \mathbf{H}.} \tag{6.29}$$

The constant of proportionality $\chi_m$ is called the **magnetic susceptibility**; it is a dimensionless quantity that varies from one substance to another—positive for paramagnets and negative for diamagnets. Typical values are around $10^{-5}$ (see Table 6.1).

| Material | Susceptibility | Material | Susceptibility |
|---|---|---|---|
| *Diamagnetic:* | | *Paramagnetic:* | |
| Bismuth | $-1.6 \times 10^{-4}$ | Oxygen | $1.9 \times 10^{-6}$ |
| Gold | $-3.4 \times 10^{-5}$ | Sodium | $8.5 \times 10^{-6}$ |
| Silver | $-2.4 \times 10^{-5}$ | Aluminum | $2.1 \times 10^{-5}$ |
| Copper | $-9.7 \times 10^{-6}$ | Tungsten | $7.8 \times 10^{-5}$ |
| Water | $-9.0 \times 10^{-6}$ | Platinum | $2.8 \times 10^{-4}$ |
| Carbon Dioxide | $-1.2 \times 10^{-8}$ | Liquid Oxygen ($-200°$ C) | $3.9 \times 10^{-3}$ |
| Hydrogen | $-2.2 \times 10^{-9}$ | Gadolinium | $4.8 \times 10^{-1}$ |

Table 6.1 Magnetic Susceptibilities (unless otherwise specified, values are for 1 atm, 20° C). *Source: Handbook of Chemistry and Physics,* 67th ed. (Boca Raton: CRC Press, Inc., 1986).

Materials that obey Eq. 6.29 are called **linear media.** In view of Eq. 6.18,

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H}, \tag{6.30}$$

for linear media. Thus **B** is *also* proportional to **H**:[5]

$$\mathbf{B} = \mu\mathbf{H}, \tag{6.31}$$

where

$$\mu \equiv \mu_0(1 + \chi_m). \tag{6.32}$$

$\mu$ is called the **permeability** of the material.[6] In a vacuum, where there is no matter to magnetize, the susceptibility $\chi_m$ vanishes, and the permeability is $\mu_0$. That's why $\mu_0$ is called the **permeability of free space**.

**Example 6.3**

An infinite solenoid ($n$ turns per unit length, current $I$) is filled with linear material of susceptibility $\chi_m$. Find the magnetic field inside the solenoid.

**Solution**: Since **B** is due in part to bound currents (which we don't yet know), we cannot compute it directly. However, this is one of those symmetrical cases in which we can get **H** from the free current alone, using Ampère's law in the form of Eq. 6.20:

$$\mathbf{H} = nI\,\hat{\mathbf{z}}$$

[5]Physically, therefore, Eq. 6.28 would say exactly the same as Eq. 6.29, only the constant $\chi_m$ would have a different value. Equation 6.29 is a little more convenient, because experimentalists find it handier to work with **H** than **B**.

[6]If you factor out $\mu_0$, what's left is called the **relative permeability**: $\mu_r \equiv 1 + \chi_m = \mu/\mu_0$. By the way, formulas for **H** in terms of **B** (Eq. 6.31, in the case of linear media) are called **constitutive relations**, just like those for **D** in terms of **E**.

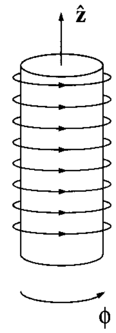



Figure 6.22

(Fig. 6.22). According to Eq. 6.31, then,

$$\mathbf{B} = \mu_0(1 + \chi_m)nI\,\hat{\mathbf{z}}.$$

If the medium is paramagnetic, the field is slightly enhanced; if it's diamagnetic, the field is somewhat reduced. This reflects the fact that the bound surface current

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = \chi_m(\mathbf{H} \times \hat{\mathbf{n}}) = \chi_m nI\,\hat{\boldsymbol{\phi}}$$

is in the same direction as $I$, in the former case ($\chi_m > 0$), and opposite in the latter ($\chi_m < 0$).

---

You might suppose that linear media avoid the defect in the parallel between **B** and **H**: since **M** and **H** are now proportional to **B**, does it not follow that their divergence, like **B**'s, must always vanish? Unfortunately, it does *not*; at the *boundary* between two materials of different permeability the divergence of **M** can actually be infinite. For instance, at the end of a cylinder of linear paramagnetic material, **M** is zero on one side but not on the other. For the "Gaussian pillbox" shown in Fig. 6.23, $\oint \mathbf{M} \cdot d\mathbf{a} \neq 0$, and hence, by the divergence theorem, $\nabla \cdot \mathbf{M}$ cannot vanish everywhere within.

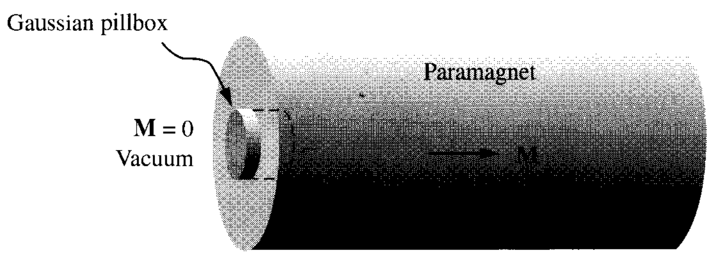



Figure 6.23

Incidentally, the volume bound current density in a homogeneous linear material is proportional to the *free* current density:

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \nabla \times (\chi_m \mathbf{H}) = \chi_m \mathbf{J}_f. \tag{6.33}$$

In particular, unless free current actually flows *through* the material, all bound current will be at the surface.

---

**Problem 6.16** A coaxial cable consists of two very long cylindrical tubes, separated by linear insulating material of magnetic susceptibility $\chi_m$. A current $I$ flows down the inner conductor and returns along the outer one; in each case the current distributes itself uniformly over the surface (Fig. 6.24). Find the magnetic field in the region between the tubes. As a check, calculate the magnetization and the bound currents, and confirm that (together, of course, with the free currents) they generate the correct field.

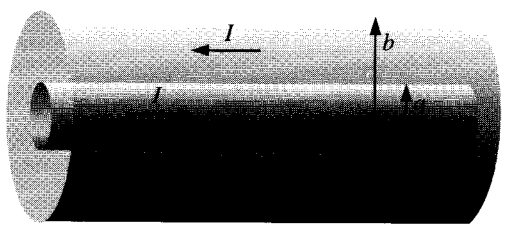



Figure 6.24

**Problem 6.17** A current $I$ flows down a long straight wire of radius $a$. If the wire is made of linear material (copper, say, or aluminum) with susceptibility $\chi_m$, and the current is distributed uniformly, what is the magnetic field a distance $s$ from the axis? Find all the bound currents. What is the *net* bound current flowing down the wire?

! **Problem 6.18** A sphere of linear magnetic material is placed in an otherwise uniform magnetic field $\mathbf{B}_0$. Find the new field inside the sphere. [*Hint:* See Prob. 6.15 or Prob. 4.23.]

**Problem 6.19** On the basis of the naïve model presented in Sect. 6.1.3, estimate the magnetic susceptibility of a diamagnetic metal such as copper. Compare your answer with the empirical value in Table 6.1, and comment on any discrepancy.

---

### 6.4.2 Ferromagnetism

In a linear medium the alignment of atomic dipoles is maintained by a magnetic field imposed from the outside. Ferromagnets—which are emphatically *not* linear[7]—require no external fields to sustain the magnetization; the alignment is "frozen in." Like paramagnetism, ferromagnetism involves the magnetic dipoles associated with the spins of unpaired electrons. The new feature, which makes ferromagnetism so different from paramagnetism, is the interaction between nearby dipoles: In a ferromagnet, *each dipole "likes" to point in the same direction as its neighbors.* The *reason* for this preference is essentially quantum mechanical, and I shall not endeavor to explain it here; it is enough to know that the correlation is so strong as to align virtually 100% of the unpaired electron spins. If you could somehow magnify a piece of iron and "see" the individual dipoles as tiny arrows, it would look something like Fig. 6.25, with all the spins pointing the same way.

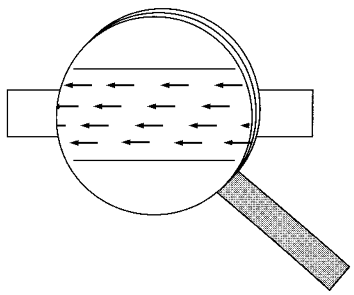
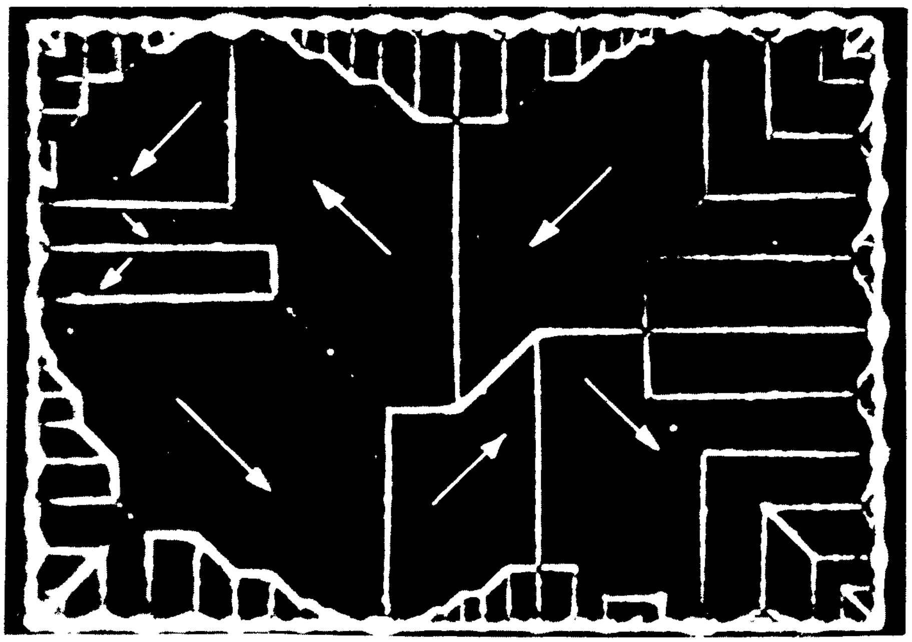


Figure 6.25

But if that is true, why isn't every wrench and nail a powerful magnet? The answer is that the alignment occurs in relatively small patches, called **domains**. Each domain contains billions of dipoles, all lined up (these domains are actually *visible* under a microscope, using suitable etching techniques—see Fig. 6.26), but the domains *themselves* are randomly oriented. The household wrench contains an enormous number of domains, and their magnetic fields cancel, so the wrench as a whole is not magnetized. (Actually, the orientation of domains is not *completely* random; within a given crystal there may be some preferential alignment along the crystal axes. But there will be just as many domains pointing one way as the other, so there is still no large-scale magnetization. Moreover, the crystals themselves are randomly oriented within any sizable chunk of metal.)

How, then, would you produce a **permanent magnet**, such as they sell in toy stores? If you put a piece of iron into a strong magnetic field, the torque $\mathbf{N} = \mathbf{m} \times \mathbf{B}$ tends to align the dipoles parallel to the field. Since they like to stay parallel to their neighbors, most of the dipoles will resist this torque. However, at the *boundary* between two domains, there

[7]In this sense it is misleading to speak of the susceptibility or permeability of a ferromagnet. The terms are used for such materials, but they refer to the proportionality factor between a *differential* increase in **H** and the resulting *differential* change in **M** (or **B**); moreover, they are not *constants*, but functions of **H**.



Ferromagnetic domains. (Photo courtesy of R. W. DeBlois)

Figure 6.26

are *competing* neighbors, and the torque will throw its weight on the side of the domain most nearly parallel to the field; this domain will win over some converts, at the expense of the less favorably oriented one. The net effect of the magnetic field, then, is to *move the domain boundaries*. Domains parallel to the field grow, and the others shrink. If the field is strong enough, one domain takes over entirely, and the iron is said to be "saturated."

It turns out that this process (the shifting of domain boundaries in response to an external field) is not entirely reversible: When the field is switched off, there will be *some* return to randomly oriented domains, but it is far from complete—there remains a preponderance of domains in the original direction. The object is now a permanent magnet.

A simple way to accomplish this, in practice, is to wrap a coil of wire around the object to be magnetized (Fig. 6.27). Run a current $I$ through the coil; this provides the external magnetic field (pointing to the left in the diagram). As you increase the current, the field increases, the domain boundaries move, and the magnetization grows. Eventually, you reach the saturation point, with all the dipoles aligned, and a further increase in current has no effect on $\mathbf{M}$ (Fig. 6.28, point $b$).

Now suppose you *reduce* the current. Instead of retracing the path back to $M = 0$, there is only a *partial* return to randomly oriented domains. $M$ decreases, but even with the current off there is some residual magnetization (point $c$). The wrench is now a permanent

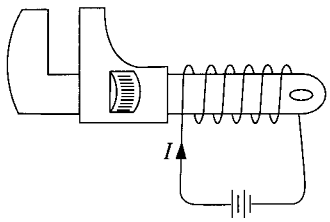



Figure 6.27

magnet. If you want to eliminate the remaining magnetization, you'll have to run a current backwards through the coil (a negative $I$). Now the external field points to the right, and as you increase $I$ (negatively), $M$ drops down to zero (point $d$). If you turn $I$ still higher, you soon reach saturation in the other direction—all the dipoles now pointing to the *right* ($e$). At this stage switching off the current will leave the wrench with a permanent magnetization to the right (point $f$). To complete the story, turn $I$ on again in the positive sense: $M$ returns to zero (point $g$), and eventually to the forward saturation point ($b$).

The path we have traced out is called a **hysteresis loop**. Notice that the magnetization of the wrench depends not only on the applied field (that is, on $I$), but also on its previous magnetic "history."[8] For instance, at three different times in our experiment the current was zero ($a$, $c$, and $f$), yet the magnetization was different for each of them. Actually, it is customary to draw hysteresis loops as plots of $B$ against $H$, rather than $M$ against $I$. (If our coil is approximated by a long solenoid, with $n$ turns per unit length, then $H = nI$, so $H$ and $I$ are proportional. Meanwhile, $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, but in practice $M$ is huge compared to $H$, so to all intents and purposes $\mathbf{B}$ is proportional to $\mathbf{M}$.)

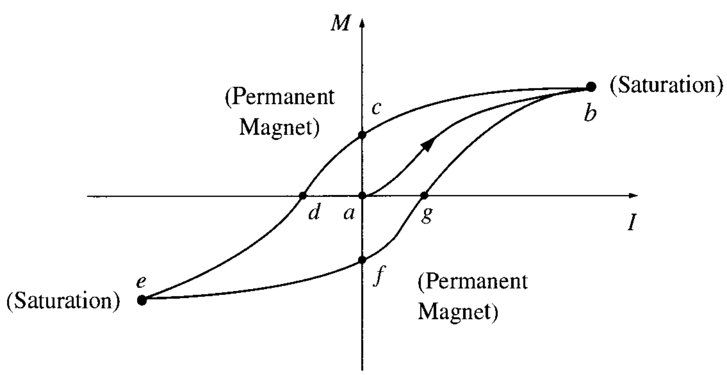



Figure 6.28

[8]Etymologically, the word *hysteresis* has nothing to do with the word *history*—nor with the word *hysteria*. It derives from a Greek verb meaning "to lag behind."

To make the units consistent (teslas), I have plotted $(\mu_0 H)$ horizontally (Fig. 6.29); notice, however, that the vertical scale is $10^4$ times greater than the horizontal one. Roughly speaking, $\mu_0\mathbf{H}$ is the field our coil *would* have produced in the absence of any iron; $\mathbf{B}$ is what we *actually* got, and compared to $\mu_0\mathbf{H}$ it is gigantic. A little current goes a long way when you have ferromagnetic materials around. That's why anyone who wants to make a powerful electromagnet will wrap the coil around an iron core. It doesn't take much of an external field to move the domain boundaries, and as soon as you've done that, you have all the dipoles in the iron working with you.

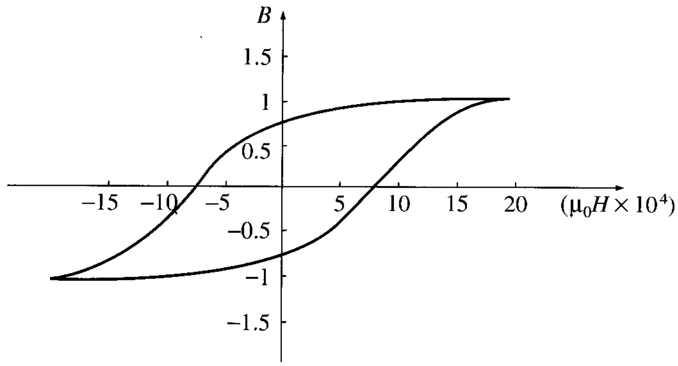



Figure 6.29

One final point concerning ferromagnetism: It all follows, remember, from the fact that the dipoles within a given domain line up parallel to one another. Random thermal motions compete with this ordering, but as long as the temperature doesn't get too high, they cannot budge the dipoles out of line. It's not surprising, though, that *very* high temperatures do destroy the alignment. What *is* surprising is that this occurs at a precise temperature (770° C, for iron). Below this temperature (called the **Curie point**), iron is ferromagnetic; above, it is paramagnetic. The Curie point is rather like the boiling point or the freezing point in that there is no *gradual* transition from ferro- to para-magnetic behavior, any more than there is between water and ice. These abrupt changes in the properties of a substance, occurring at sharply defined temperatures, are known in statistical mechanics as **phase transitions**.

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**Problem 6.20** How would you go about *de*magnetizing a permanent magnet (such as the wrench we have been discussing, at point $c$ in the hysteresis loop)? That is, how could you restore it to its original state, with $M = 0$ at $I = 0$?

**Problem 6.21**

(a) Show that the energy of a magnetic dipole in a magnetic field $\mathbf{B}$ is given by

$$\boxed{U = -\mathbf{m} \cdot \mathbf{B}.} \qquad (6.34)$$

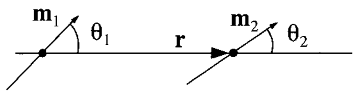


Figure 6.30

[Assume that the *magnitude* of the dipole moment is *fixed*, and all you have to do is move it into place and rotate it into its final orientation. The energy required to keep the current flowing is a different problem, which we will confront in Chapter 7.] Compare Eq. 4.6.

(b) Show that the interaction energy of two magnetic dipoles separated by a displacement **r** is given by

$$U = \frac{\mu_0}{4\pi}\frac{1}{r^3}[\mathbf{m}_1 \cdot \mathbf{m}_2 - 3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}})]. \tag{6.35}$$

Compare Eq. 4.7.

(c) Express your answer to (b) in terms of the angles $\theta_1$ and $\theta_2$ in Fig. 6.30, and use the result to find the stable configuration two dipoles would adopt if held a fixed distance apart, but left free to rotate.

(d) Suppose you had a large collection of compass needles, mounted on pins at regular intervals along a straight line. How would they point (assuming the earth's magnetic field can be neglected)? [A rectangular array of compass needles also aligns itself spontaneously, and this is sometimes used as a demonstration of "ferromagnetic" behavior on a large scale. It's a bit of a fraud, however, since the mechanism here is purely classical, and much weaker than the quantum mechanical **exchange forces** that are actually responsible for ferromagnetism.]

## More Problems on Chapter 6

! **Problem 6.22** In Prob. 6.4 you calculated the force on a dipole by "brute force." Here's a more elegant approach. First write $\mathbf{B}(\mathbf{r})$ as a Taylor expansion about the center of the loop:

$$\mathbf{B}(\mathbf{r}) \cong \mathbf{B}(\mathbf{r}_0) + [(\mathbf{r} - \mathbf{r}_0) \cdot \nabla_0]\mathbf{B}(\mathbf{r}_0),$$

where $\mathbf{r}_0$ is the position of the dipole and $\nabla_0$ denotes differentiation with respect to $\mathbf{r}_0$. Put this into the Lorentz force law (Eq. 5.16) to obtain

$$\mathbf{F} = I \oint d\mathbf{l} \times [(\mathbf{r} \cdot \nabla_0)\mathbf{B}(\mathbf{r}_0)].$$

Or, numbering the Cartesian coordinates from 1 to 3:

$$F_i = I \sum_{j,k,l=1}^{3} \epsilon_{ijk} \left\{ \oint r_l \, dl_j \right\} [\nabla_{0_l} B_k(\mathbf{r}_0)],$$

where $\epsilon_{ijk}$ is the **Levi-Civita symbol** ($+1$ if $ijk = 123$, 231, or 312; $-1$ if $ijk = 132$, 213, or 321; 0 otherwise), in terms of which the cross-product can be written $(\mathbf{A} \times \mathbf{B})_i = \sum_{j,k=1}^{3} \epsilon_{ijk} A_j B_k$. Use Eq. 1.108 to evaluate the integral. Note that

$$\sum_{j=1}^{3} \epsilon_{ijk}\epsilon_{ljm} = \delta_{il}\delta_{km} - \delta_{im}\delta_{kl},$$

where $\delta_{ij}$ is the Kronecker delta (Prob. 3.45).

**Problem 6.23** Notice the following parallel:

$$\begin{cases} \nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{E} = 0, \quad \epsilon_0 \mathbf{E} = \mathbf{D} - \mathbf{P}, \quad \text{(no free charge)}; \\ \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = 0, \quad \mu_0 \mathbf{H} = \mathbf{B} - \mu_0 \mathbf{M}, \quad \text{(no free current)}. \end{cases}$$

Thus, the transcription $\mathbf{D} \to \mathbf{B}$, $\mathbf{E} \to \mathbf{H}$, $\mathbf{P} \to \mu_0 \mathbf{M}$, $\epsilon_0 \to \mu_0$ turns an electrostatic problem into an analogous magnetostatic one. Use this observation, together with your knowledge of the electrostatic results, to rederive

(a) the magnetic field inside a uniformly magnetized sphere (Eq. 6.16);

(b) the magnetic field inside a sphere of linear magnetic material in an otherwise uniform magnetic field (Prob. 6.18);

(c) the average magnetic field over a sphere, due to steady currents within the sphere (Eq. 5.89).

**Problem 6.24** Compare Eqs. 2.15, 4.9, and 6.11. Notice that if $\rho$, $\mathbf{P}$, and $\mathbf{M}$ are *uniform*, the *same integral* is involved in all three:

$$\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'.$$

Therefore, if you happen to know the electric field of a uniformly *charged* object, you can immediately write down the scalar potential of a uniformly *polarized* object, and the vector potential of a uniformly *magnetized* object, of the same shape. Use this observation to obtain $V$ inside and outside a uniformly polarized sphere (Ex. 4.2), and $\mathbf{A}$ inside and outside a uniformly magnetized sphere (Ex. 6.1).

**Problem 6.25** A familiar toy consists of donut-shaped permanent magnets (magnetization parallel to the axis), which slide frictionlessly on a vertical rod (Fig. 6.31). Treat the magnets as dipoles, with mass $m_d$ and dipole moment $\mathbf{m}$.

(a) If you put two back-to-back magnets on the rod, the upper one will "float"—the magnetic force upward balancing the gravitational force downward. At what height ($z$) does it float?

(b) If you now add a *third* magnet (parallel to the bottom one), what is the *ratio* of the two heights? (Determine the actual number, to three significant digits.)
[*Answer:* (a) $[3\mu_0 m^2/2\pi m_d g]^{1/4}$; (b) 0.8501]

**Problem 6.26** At the interface between one linear magnetic material and another the magnetic field lines bend (see Fig. 6.32). Show that $\tan\theta_2/\tan\theta_1 = \mu_2/\mu_1$, assuming there is no free current at the boundary. Compare Eq. 4.68.

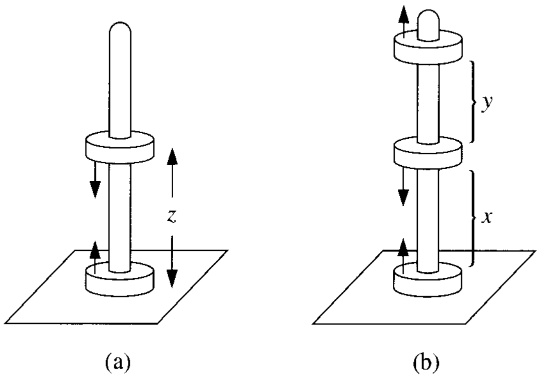


Figure 6.31

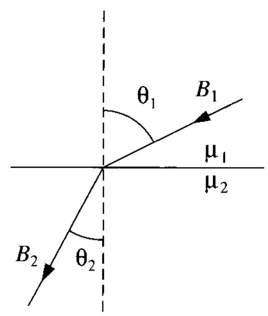


Figure 6.32

! **Problem 6.27** A magnetic dipole $\mathbf{m}$ is imbedded at the center of a sphere (radius $R$) of linear magnetic material (permeability $\mu$). Show that the magnetic field inside the sphere ($0 < r \leq R$) is

$$\frac{\mu}{4\pi}\left\{\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] + \frac{2(\mu_0 - \mu)\mathbf{m}}{(2\mu_0 + \mu)R^3}\right\}.$$

What is the field *outside* the sphere?

**Problem 6.28** You are asked to referee a grant application, which proposes to determine whether the magnetization of iron is due to "Ampère" dipoles (current loops) or "Gilbert" dipoles (separated magnetic monopoles). The experiment will involve a cylinder of iron (radius $R$ and length $L = 10R$), uniformly magnetized along the direction of the axis. If the dipoles are Ampère-type, the magnetization is equivalent to a surface bound current $\mathbf{K}_b = M\hat{\boldsymbol{\phi}}$; if they are Gilbert-type, the magnetization is equivalent to surface monopole densities $\sigma_b = \pm M$ at the two ends. Unfortunately, these two configurations produce identical magnetic fields, at exterior points. However, the *interior* fields are radically different—in the first case $\mathbf{B}$ is in the *same* general direction as $\mathbf{M}$, whereas in the second it is roughly *opposite* to $\mathbf{M}$. The applicant proposes to measure this internal field by carving out a small cavity and finding the torque on a tiny compass needle placed inside.

Assuming that the obvious technical difficulties can be overcome, and that the question itself is worthy of study, would you advise funding this experiment? If so, what shape cavity would you recommend? If not, what is wrong with the proposal? [*Hint:* refer to Probs. 4.11, 4.16, 6.9, and 6.13.]