09

We know how to find derivative of a given function f with respect to an independent variable. Also, we have discussed various methods of integration to find a function f whose derivative is the function g, we may formulate this as follows

For a given function g, find a function f such that $\frac{df}{dx} = g(x)$ or we can write it as $\frac{dy}{dx} = g(x)$, where $y = f(x)$. Such type of equation is known as differential equation.

DIFFERENTIAL EQUATIONS

In this chapter, we will study some basic concepts related to differential equation, general and particular solutions of a differential equation, formation of differential equations, some methods to solve differential equations and applications of differential equations.

CHAPTER CHECKLIST

- Order, Degree and Solution of Differential Equation
- Methods of Solving First Order and First Degree Differential Equations
- Linear Differential Equation

|TOPIC 1|

Order, Degree and Solution of Differential Equation

DIFFERENTIAL EQUATION

An equation involving independent variable (variables), dependent variable and derivative or derivatives of dependent variable with respect to independent variable (variables) is called a differential equation. e.g. $2x\frac{dy}{dx} - 3y = 5$ and $4\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$ are differential equations, but $2x - 3y = 0$ is not a differential equation as derivatives of dependent variable (y) with respect to independent variable (x) is not present.

Ordinary Differential Equation

A differential equation involving derivative or derivatives of the dependent variable with respect to only one independent variable is called an ordinary differential equation. e.g. $2\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 = 0$ is an ordinary differential equation, because here dependent variable y have derivatives with respect to only one independent variable, i.e. x.

There are differential equation involving derivatives with respect to more than one independent variables, such equation is called partial differential equation. But here, we shall only deal with ordinary differential equation.

The derivatives are denoted by the symbols

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n} \text{ or } y', y'', \ldots, y^{(n)} \text{ or } y_1, y_2, \ldots, y_n.$$

If the derivative is of higher order say n, then it is convenient to use y_n to represent nth order derivative.

Order of a Differential Equation

The order of the highest order derivative of dependent variable with respect to independent variable involved in a differential equation, is called order of differential equation.

e.g. (i) $\frac{dy}{dx} - \sin x = 0$ (ii) $\frac{d^3y}{dx^3} + x^2\left(\frac{d^2y}{dx^2}\right) = 0$

Here, in e.g. (i), equation has the highest derivative of first order and in e.g. (ii), equation has the highest derivative of third order. So, orders of the differential equations in e.g. (i) and (ii) are 1 and 3, respectively.

Degree of a Differential Equation

The highest power (positive integral index) of the highest order derivative involved in a differential equation, when it is written as a polynomial in derivatives, is called the degree of a differential equation.

e.g. $$\left(\frac{d^3y}{dx^3}\right)^2 + x\left(\frac{d^3y}{dx^3}\right) + 3y\left(\frac{dy}{dx}\right) = 0$$

In this differential equation, highest order derivative is $\left(\frac{d^3y}{dx^3}\right)$, whose highest power is 2. So, degree of differential equation is 2.

Note

(i) The order and degree (if defined) of a differential equation are always positive integers.

(ii) If the given differential equation is not a polynomial equation in its derivatives, then its degree is not defined.

e.g. $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0$ is not a polynomial equation in derivatives because by using expansion of $\sin x$, the differential equation becomes $\frac{dy}{dx} + \left\{\frac{dy}{dx} - \frac{\left(\frac{dy}{dx}\right)^3}{3!} + \frac{\left(\frac{dy}{dx}\right)^5}{5!} - \ldots\right\} = 0$, which is not a polynomial in derivatives, as it is not finite, so its degree is not defined.

(iii) If the differential equation have radicals (like $\sqrt{\ }$, $\sqrt[3]{\ }$, etc.) and fractions $\left(\text{like } \frac{1}{\left(\frac{dy}{dx}\right)}, \frac{1}{\sqrt{1+\frac{d^2y}{dx^2}}}, \text{etc.}\right)$, then to find degree, first made it free from radicals and fractions by simplifying it.

EXAMPLE [1] Find the order and degree (if defined) of each of the following differential equation.

(i) $\frac{dy}{dx} - \sec x = 0$ (ii) $y''' + y^2 + e^{y'} = 0$

(iii) $y = x\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

Sol. (i) Given differential equation is $\frac{dy}{dx} - \sec x = 0$.

The highest order derivative occurring in the differential equation is $\frac{dy}{dx}$, so its order is 1.

It is a polynomial in $\frac{dy}{dx}$ and highest power raised to $\frac{dy}{dx}$ is 1.

So, its degree is 1.

(ii) Given differential equation is $y''' + y^2 + e^{y'} = 0$

The highest order derivative occurring in the differential equation is y''', so its order is 3.

The given differential equation is not a polynomial equation in derivatives of y, so its degree is not defined.

(iii) Given differential equation is $y = x\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

It has radical sign in power, i.e. not have positive integer powers, so we first convert it into differential equation having positive integer powers.

We can rewrite the given differential equations as

$$y - x\frac{dy}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} = 1 + \left(\frac{dy}{dx}\right)^2$$

[squaring both sides]

$$\Rightarrow (x^2 - 1)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + y^2 - 1 = 0$$

which represents a quadratic polynomial in $\frac{dy}{dx}$.

Since, highest order derivative is $\frac{dy}{dx}$.

So, order of the differential equation is 1 and degree of the differential equation is 2.

EXAMPLE |2| Find the order and degree of differential equation $y = px + \sqrt{a^2p^2 + b^2}$, where $p = dy/dx$.

Sol. Given differential equation is

$$y = px + \sqrt{a^2p^2 + b^2}$$

where, $p = \frac{dy}{dx}$.

$$\Rightarrow \quad y - px = \sqrt{a^2p^2 + b^2}$$

$$\Rightarrow \quad (y - px)^2 = a^2p^2 + b^2$$

$$\Rightarrow \quad y^2 + p^2x^2 - 2xpy = a^2p^2 + b^2$$

$$\Rightarrow \quad (x^2 - a^2)p^2 - 2xyp + (y^2 - b^2) = 0$$

$$\Rightarrow \quad (x^2 - a^2)\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) + (y^2 - b^2) = 0$$

Clearly, order = 1 and degree = 2.

EXAMPLE |3| Write the sum of the order and degree of the differential equation $\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0$. **[All India 2015]**

Sol. Given equation is $\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0$

$$\Rightarrow \quad 3\left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dx^2} = 0$$

Clearly, the highest order derivative occurring in the differential equation is d^2y/dx^2.

So, its order is 2. Also, it is a polynomial equation in derivatives and highest power raised to d^2y/dx^2 is 1, so its degree is 1.

Hence, the sum of the order and degree of the above differential equation is $2 + 1 = 3$.

SOLUTION OF A DIFFERENTIAL EQUATION

We know that the solutions of a polynomial equation are numbers (real or complex), which satisfies the given equation. Similarly, a function of independent variable will be a solution of differential equation, if this function satisfies the given differential equation.

In other words, suppose a differential equation is given to us, in which y is dependent variable and x is independent variable. Then, the function $\phi(x)$ will be its solution, if it satisfy the given differential equation, i.e. when the function ϕ is substituted for the unknown y (dependent variable) in the given differential equation, LHS becomes equal to RHS. The solution of a differential equation is of two types, which are given below

General Solution of a Differential Equation

If the solution of a differential equation of order n, contains n arbitrary constants, then it is called a general solution.

e.g. The general solution of $\frac{d^2y}{dx^2} + y = 0$ is

$y = A\cos x + B\sin x$.

But $y = A\cos x + \sin x$ and $y = \cos x + B\sin x$ is not the general solution of given differential equation, as it contains only one arbitrary constant.

METHOD TO VERIFY THAT GIVEN FUNCTION IS A SOLUTION OF DIFFERENTIAL EQUATION

Suppose, a differential equation is given to us in which y is dependent variable and x is independent variable. Then, to verify whether the function $y = \phi(x)$ is a solution of given differential equation or not, we differentiate $y = \phi(x)$ with respect to x as many times as the order of given differential equation. Then, put $y = \phi(x)$ and values of its derivatives into LHS of given differential equation and check that LHS = RHS or not. If LHS = RHS, then function $y = \phi(x)$ is a solution of given differential equation, otherwise not.

(i) If it is possible to obtain the given differential equation by using given function $y = \phi(x)$ and its derivatives, then the given function $y = \phi(x)$ is a solution of given differential equation.

(ii) Sometimes, a relation between dependent variable (y) and independent variable (x) is given to us, where y is not directly expressed as function of x (e.g. $y - \cos y = x$) and we have to check whether the given relation is a solution of given differential equation or not. In such case, we differentiate the given relation as many times as the order of differential equation to get the given differential equation. If we obtained the given differential equation. Then, the given relation is a solution, otherwise not.

EXAMPLE |4| Verify that the function $y = e^{-3x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$.

Sol. Given differential equation is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0 \quad \text{...(i)}$$

and given function is $y = e^{-3x}$. ...(ii)

Here, order of the given differential equation is 2.

So, we differentiate Eq. (ii) two times.

On differentiating Eq. (ii), w.r.t. x two times, we get

$$\frac{dy}{dx} = -3e^{-3x} \quad \text{and} \quad \frac{d^2y}{dx^2} = 9e^{-3x}$$

On putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in LHS of Eq. (i), we get

$$\begin{aligned} \text{LHS} &= 9e^{-3x} + (-3e^{-3x}) - 6e^{-3x} \\ &= 9e^{-3x} - 3e^{-3x} - 6e^{-3x} = 0 = \text{RHS} \end{aligned}$$

Hence, the given function is a solution of given differential equation.

EXAMPLE |5| Verify that $y = A\cos x - B\sin x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Sol. Given, $\quad y = A\cos x - B\sin x \quad$ ...(i)

Here, order of the given differential equation is 2. So, we differentiate the Eq. (i) two times.

On differentiating both sides of Eq. (i) w.r.t. x two times, we get

$$\frac{dy}{dx} = -A\sin x - B\cos x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -A\cos x + B\sin x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -(A\cos x - B\sin x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -y \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + y = 0,$$

which is the given differential equation.

Hence, $y = A\cos x - B\sin x$ is a solution of given equation.

EXAMPLE |6| Verify that the function $x + y = \tan^{-1} y$ is a solution of the differential equation $y^2y' + y^2 + 1 = 0$.

Sol. Given function is $x + y = \tan^{-1} y$. ...(i)

Here, order of differential equation is 1. So, we differentiate Eq. (i) one time.

On differentiating Eq. (i), w.r.t. x one time, we get

$$1 + y' = \frac{1}{1+y^2} \cdot y'$$

$$\Rightarrow \quad (1 + y')(1 + y^2) = y'$$

$$\Rightarrow \quad 1 + y' + y^2(1 + y') = y'$$

$$\Rightarrow \quad 1 + y^2 + y^2y' = 0$$

which is the given differential equation.

Hence, $x + y = \tan^{-1} y$ is a solution of the given differential equation.

Particular Solution of a Differential Equation

The solution of a differential equation obtained by giving particular values to the arbitrary constants in the general solution, is called the particular solution. In other words, the solution free from arbitrary constant is called particular solution.

e.g. The general solution of $\frac{d^2y}{dx^2} + y = 0$ is

$$y = A\cos x + B\sin x.$$

If $A = B = 1$, then $y = \cos x + \sin x$ is a particular solution of the given differential equation.

INITIAL VALUE PROBLEMS

An initial value problem is a differential equation together with specified value of the unknown function and its derivatives at a point of the domain of definition of independent variable.

These specified values are called initial conditions and given for specify a particular solution of the differential equation. Often, initial conditions are of the form $y(x_0) = y_0$ and / or $y'(x_0) = y_{01}$, $y''(x_0) = y_{02}$ and so on.

e.g. (i) $x\frac{dy}{dx} = 1$, $y(1) = 0$

(ii) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$, $y(0) = 2$, $y'(0) = 3$ are initial value problems.

(i) If the differential equation in the initial value problem is of order 1, then initial value problem involves only specified value of the unknown function, i.e. $y(x_0) = y_0$.

(ii) A function is said to be a solution of initial value problem, if it satisfy the given differential equation and its initial conditions.

EXAMPLE |7| Show that $y = x^2 + 2x + 1$ is the solution of the initial value problem

$$\frac{d^3y}{dx^3} = 0,\ y(0) = 1,$$
$$y'(0) = 2,\ y''(0) = 2.$$

First, show that $y = \phi(x)$ satisfy the given differential equation and then show that it also satisfy given initial conditions.

Sol. Given function is

$$y = x^2 + 2x + 1 \qquad \text{...(i)}$$

Here, order of the given differential equation is 3. So, we differentiate Eq. (i) three times.

On differentiating Eq. (i) w.r.t. x three times, we get

$$\frac{dy}{dx} = 2x + 2 \quad \text{...(ii)}$$

$$\frac{d^2y}{dx^2} = 2 \quad \text{...(iii)}$$

$\frac{d^3y}{dx^3} = 0$, which is the given differential equation.

So, $y = x^2 + 2x + 1$ is the solution of given differential equation.

Now, $y(0) = (0)^2 + 2(0) + 1 = 1$ [using Eq. (i)]

$y'(0) = \left(\frac{dy}{dx}\right)_{\text{at } x=0} = 2(0) + 2 = 2$ [using Eq. (ii)]

and $\left(\frac{d^2y}{dx^2}\right)_{\text{at } x=0} = 2$ [using Eq. (iii)]

Hence, $y = x^2 + 2x + 1$ is the solution of initial value problem.

TOPIC PRACTICE 1

OBJECTIVE TYPE QUESTIONS

1 The order and degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{1/4} + x^{1/5} = 0$ respectively, are

(a) 2 and 4 (b) 2 and 2
(c) 2 and 3 (d) 3 and 3

2 The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{dy}{dx}\right)$ is

(a) 1 (b) 2
(c) 3 (d) not defined

3 The degree of the differential equation $1 + \left(\frac{dy}{dx}\right)^2 = x$ is

(a) 1 (b) 0
(c) 2 (d) None of these

4 The sum of the order and degree of the following differential equation $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 5$ is

(a) 2 (b) 3
(c) 4 (d) 5

5 The order and degree of the differential equation $\left(1 + \frac{3dy}{dx}\right)^{2/3} = 4\frac{d^3y}{dx^3}$ is

(a) $\left(1, \frac{2}{3}\right)$ (b) (3, 1)
(c) (3, 3) (d) (1, 2)

VERY SHORT ANSWER Type Questions

Directions (Q. Nos. 6-11) *Find the order and degree, each of the following differential equations, if defined.*

6 $x\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{dy}{dx}\right)^4 + y^2 = e^{-x}$

7 $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$ **[NCERT]**

8 $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

9 $y + \frac{dy}{dx} = \frac{1}{4}\int y\,dx$

10 $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$ **[NCERT]**

11 $\left(\frac{d^2y}{dx^2}\right) + \cos\left(\frac{dy}{dx}\right) = 0$ **[NCERT]**

12 Find the order and degree of the differential equation $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2}$.

13 Write the sum of order and degree of differential equation $1 + \left(\frac{dy}{dx}\right)^4 = 7\left(\frac{d^2y}{dx^2}\right)^3$. **[Delhi 2015C]**

SHORT ANSWER Type I Questions

14 Find the order and degree of the differential equation $x - \sin\left(\frac{dy}{dx}\right) = 0$.

15 Find the order and degree (if defined) of the differential equation

$$\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = 2x^2\log\left(\frac{d^2y}{dx^2}\right).$$ **[Delhi 2020]**

16 If $\log_e\left(1 + \frac{d^2y}{dx^2}\right) = x$, then find the order and degree of the given differential equation.

17 Verify that the function $y = \sqrt{a^2 - x^2}$, $x \in (-a, a)$ is a solution of differential equation
$x + y\dfrac{dy}{dx} = 0\ (y \neq 0)$. **[NCERT]**

18 Verify that $y = e^{-x} + Ax + B$ is a solution of the differential equation $e^x\left(\dfrac{d^2y}{dx^2}\right) = 1$. **[NCERT]**

19 Verify that the function defined by $y = \sin x - \cos x$, $x \in R$ is a solution of the initial value problem $\dfrac{dy}{dx} = \sin x + \cos x$, $y(0) = -1$. **[NCERT]**

SHORT ANSWER Type II Questions

20 Verify that the function $y - \cos y = x$ is a solution of the differential equation
$(y \sin y + \cos y + x)\, y' = y$. **[NCERT]**

21 Verify that $y = x \sin x$ is a solution of differential equation $xy' = y + x\sqrt{x^2 - y^2}$,
$(x \neq 0$ and $x > y$ or $x < -y)$. **[NCERT]**

22 Verify that the function $x^2 = 2y^2 \log y$ is a solution of the differential equation

$$(x^2 + y^2)\frac{dy}{dx} - xy = 0.$$ **[NCERT]**

23 Show that the function $xy = \log y + C$ is a solution of the differential equation

$$y' = \frac{y^2}{1 - xy},\ (xy \neq 1).$$ **[NCERT]**

24 Show that $y = Ae^{mx} + Be^{nx}$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} - (m+n)\dfrac{dy}{dx} + mny = 0$.

25 Verify that $y = e^{m \sin^{-1} x}$ is a solution of the differential equation $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - m^2y = 0$.

HINTS & SOLUTIONS

1. (a) Given that $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^{1/4} = -x^{1/5}$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{1/4} = -x^{1/5}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^{1/4} = -\left(x^{1/5} + \frac{d^2y}{dx^2}\right)$$

On squaring both sides, we get

$$\left(\frac{dy}{dx}\right)^{1/2} = \left(x^{1/5} + \frac{d^2y}{dx^2}\right)^2$$

Again, on squaring both sides, we have

$$\frac{dy}{dx} = \left(x^{1/5} + \frac{d^2y}{dx^2}\right)^4$$

Order = 2, degree = 4

2. (d) The degree of given differential equation is not defined because when we expand $\sin\left(\dfrac{dy}{dx}\right)$, we get an infinite series in the increasing powers of $\dfrac{dy}{dx}$. Therefore its degree is not defined.

3. (c) We have, $1 + \left(\dfrac{dy}{dx}\right)^2 = x$

Clearly, degree = 2

4. (b) We have, $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = 5$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 5$$

$\therefore$ Order = 2, degree = 1
Hence, sum = 2+1 = 3

5. (c) We have, $\left(1 + 3\dfrac{dy}{dx}\right)^{2/3} = 4\dfrac{d^3y}{dx^3}$

Cubing both sides, we get

$$\left(1 + 3\frac{dy}{dx}\right)^2 = \left(4\frac{d^3y}{dx^3}\right)^3$$

$\therefore$ Order = 3, degree = 3

6. Given differential equation is $x\left(\dfrac{d^3y}{dx^3}\right)^2 + \left(\dfrac{dy}{dx}\right)^4 + y^2 = e^{-x}$.

Since, the highest order derivative is $\dfrac{d^3y}{dx^3}$. So, its order is 3.

It is a polynomial equation in $\dfrac{dy}{dx}, \dfrac{d^3y}{dx^3}$ and the highest power of $\dfrac{d^3y}{dx^3}$ is 2. So, its degree is 2.

7. Solve as Question 6. [**Ans.** Order = 2, Degree = 1]

8. The given equation can be rewritten as

$$x\sqrt{1-y^2}\,dx = -y\sqrt{1-x^2}\,dy \Rightarrow \frac{dy}{dx} = \frac{-x\sqrt{1-y^2}}{y\sqrt{1-x^2}}$$

Since, the highest order derivative is $\dfrac{dy}{dx}$. So, its order is 1.

It is a polynomial equation in $\dfrac{dy}{dx}$ and highest power of $\dfrac{dy}{dx}$ is 1. So, its degree is 1.

9. Given differential equation is

$$y + \frac{dy}{dx} = \frac{1}{4}\int y\, dx \qquad \text{...(i)}$$

Let us first remove the integral sign by differentiating it w.r.t. x.

On differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} + \frac{d^2y}{dx^2} = \frac{1}{4}y$$

Since, the highest order derivative is $\frac{d^2y}{dx^2}$. So, its order is 2.

Also, it is a polynomial equation in $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$, where highest power of $\frac{d^2y}{dx^2}$ is 1. So, its degree is 1.

10. Given differential equation is

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$$

Since, the highest order derivative is $\frac{d^2y}{dx^2}$. So, its order is 2.

Degree is not defined as it is not a polynomial in derivatives.

11. Solve as Question 10.
[**Ans.** Order = 2, Degree = not defined]

12. Given differential equation is $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2}$.

On squaring both sides, we get

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$$

Here, highest order derivative is $\frac{d^2y}{dx^2}$, whose highest power is 2.
So, order of differential equation is 2 and degree is also 2.

13. Given differential equation is $1 + \left(\frac{dy}{dx}\right)^4 = 7\left(\frac{d^2y}{dx^2}\right)^3$.

Here, highest order derivative is $\frac{d^2y}{dx^2}$, whose highest power is 3.
So, order = 2 and degree = 3
Sum of order and degree = 2 + 3 = 5

14. Given differential equation is $x - \sin\left(\frac{dy}{dx}\right) = 0$.

Since, the highest order derivative is $\frac{dy}{dx}$.

So, its order is 1.
Now, if possible, then convert the given differential equation into polynomial equation of derivatives.

Consider, $x - \sin\left(\frac{dy}{dx}\right) = 0$

$$\Rightarrow \quad \sin\left(\frac{dy}{dx}\right) = x$$

$\Rightarrow \frac{dy}{dx} = \sin^{-1} x$, which is a polynomial in $\frac{dy}{dx}$.
Also, highest power of $\frac{dy}{dx}$ is 1. So, its degree is 1.

15. Since, highest order derivative occurring in the differential equation is $\frac{d^2y}{dx^2}$, therefore order is 2 and as the differential equation is not a polynomial in derivatives, therefore its degree is not defined.

16. Given differential equation is
$\log_e\left(1 + \frac{d^2y}{dx^2}\right) = x$, which can be rewritten as $1 + \frac{d^2y}{dx^2} = e^x$

Clearly, the order of differential equation is 2 and its degree is 1.

17. Hint $\frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}} \Rightarrow \frac{dy}{dx} = \frac{-x}{y} \Rightarrow x + y\frac{dy}{dx} = 0$

18. Hint $\frac{dy}{dx} = e^{-x}(-1) + A;\ \frac{d^2y}{dx^2} = e^{-x}$

19. Solve as Example 7.

20. Given differential equation is

$$(y\sin y + \cos y + x)\frac{dy}{dx} = y \qquad \text{...(i)}$$

and given function is $y - \cos y = x$ ...(ii)

Since, the order of differential equation is 1, therefore we differentiate Eq. (ii) w.r.t. x only once.

Now, differentiating Eq. (ii) w.r.t. x, we get

$$\frac{dy}{dx} + \sin y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx}(1 + \sin y) = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{1 + \sin y}$$

Now, consider

$$\text{LHS} = (y\sin y + \cos y + x)\, y' = \frac{y\sin y + \cos y + x}{1 + \sin y}$$

$$= \frac{y\sin y + \cos y + y - \cos y}{1 + \sin y} \qquad \text{[using Eq. (ii)]}$$

$$= \frac{y\sin y + y}{1 + \sin y} = \frac{y(\sin y + 1)}{1 + \sin y} = y = \text{RHS}$$

Hence, the given function is a solution of the differential equation.

21. Hint $\frac{dy}{dx} = \sin x + x\cos x$

$$\text{LHS} = x\sin x + x^2\cos x$$

$$= y + x^2\sqrt{1 - \sin^2 x} = y + x^2\sqrt{1 - \left(\frac{y}{x}\right)^2} = \text{RHS}$$

22. **Hint** $\frac{dy}{dx} = \frac{x}{y(1 + 2\log y)}$

Now, substitute the value of x^2 from the given equation and $\frac{dy}{dx}$ in LHS.

23. Given, $xy = \log y + C$

On differentiating both sides w.r.t. x, we get

$xy' + y \cdot 1 = \frac{1}{y} \cdot y' + 0$ [using product rule of derivative]

$$\Rightarrow xy' + y = \frac{y'}{y} \Rightarrow xy \cdot y' + y^2 = y'$$

$$\Rightarrow y^2 = y' - xy \cdot y'$$

$$\Rightarrow y^2 = y'(1 - xy)$$

$$\Rightarrow y' = \frac{y^2}{1 - xy},\ xy \neq 1$$

Hence proved.

24. Given, $y = Ae^{mx} + Be^{nx}$ ...(i)

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = A \cdot e^{mx} \cdot m + Be^{nx} \cdot n \quad ...(ii)$$

Again, on differentiating both sides of Eq. (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = m \cdot Ae^{mx} \cdot m + n \cdot Be^{nx} \cdot n$$

$$\Rightarrow \frac{d^2y}{dx^2} = m^2 Ae^{mx} + n^2 Be^{nx}$$

Now, consider

$$\text{LHS} = \frac{d^2y}{dx^2} - (m + n)\frac{dy}{dx} + mny$$

$$= m^2Ae^{mx} + n^2Be^{nx} - (m + n)(mAe^{mx} + nBe^{nx}) + mn(Ae^{mx} + Be^{nx})$$

$$= m^2Ae^{mx} + n^2Be^{nx} - m^2Ae^{mx} - mnBe^{nx} - nmAe^{mx} - n^2Be^{nx} + mnAe^{mx} + mnBe^{nx}$$

$$= 0 = \text{RHS}$$

Thus, LHS = RHS

Hence, $y = Ae^{mx} + Be^{nx}$ is a solution of given equation.

Hence proved.

25. **Hint** $\frac{dy}{dx} = \frac{my}{\sqrt{1 - x^2}}$

$$\Rightarrow (1 - x^2)\left(\frac{dy}{dx}\right)^2 = m^2y^2$$

$$\Rightarrow (1 - x^2)2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2(-2x) = m^2 2y\frac{dy}{dx}$$

On dividing both sides by $2\left(\frac{dy}{dx}\right)$ to verify required result.

|TOPIC 2|

Methods of Solving First Order and First Degree Differential Equations

DIFFERENTIAL EQUATION WITH VARIABLES SEPARABLE

A first order and first degree differential equation $\frac{dy}{dx} = F(x, y)$ is in the form of **variable separable**, if the function F can be expressed as the product of the functions of x and the functions of y. Suppose a first order and first degree differential equation is given to us, i.e.

$$\frac{dy}{dx} = F(x, y) \quad ...(i)$$

Now, expressed it as $\frac{dy}{dx} = h(y) \cdot g(x)$ ...(ii)

If $h(y) \neq 0$, then separate the variables, i.e. write Eq. (ii) as $\frac{1}{h(y)}dy = g(x)dx$

On integrating both sides, we get

$$\int \frac{1}{h(y)}dy = \int g(x)dx \quad \text{or} \quad H(y) = G(x) + C$$

which is the required solution of given differential equation, where $H(y)$, $G(x)$ are the anti-derivatives of $\frac{1}{h(y)}$, $g(x)$ respectively and C is the arbitrary constant.

EXAMPLE |1| Solve the differential equation $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$. **[NCERT]**

Sol. Given differential equation is $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

On separating the variables, we get

$$\left(\frac{1}{1 + y^2}\right)dy = (1 + x^2)dx$$

On integrating both sides, we get

$$\int \frac{1}{1+y^2}\,dy = \int (1+x^2)\,dx$$

$$\Rightarrow \quad \tan^{-1} y = x + \frac{x^3}{3} + C$$

where, C is a constant of integration.

Note Here, is no need to introduce arbitrary constants of integration on both sides because both constant may be combined together and give just one arbitrary constant.

EXAMPLE |2| Find the general solution of the differential equation $(x+2)\frac{dy}{dx} = x^2 + 5x - 3,\ x \neq -2$.

Sol. Given differential equation is $(x+2)\frac{dy}{dx} = x^2 + 5x - 3$

$$\Rightarrow \quad dy = \frac{x^2+5x-3}{(x+2)}\,dx \qquad \text{[separating the variables]}$$

On integrating both sides, we get

$$\int dy = \int \frac{x^2+5x-3}{(x+2)}dx \Rightarrow y = \int \left\{(x+3) - \frac{9}{(x+2)}\right\}dx$$

[dividing (x^2+5x-3) by $(x+2)$]

$$\Rightarrow \quad y = \frac{x^2}{2} + 3x - 9\log|x+2| + C$$

which is the required general solution.

EXAMPLE |3| Solve the differential equation $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$.

To make it in a variable separable form, take e^{-y} as common. Solve the resulting differential equation by applying the method of variable separable.

Sol. Given differential equation is

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{-y}(e^x + x^2)$$

$$\Rightarrow \quad \frac{1}{e^{-y}}dy = (e^x + x^2)\,dx \Rightarrow e^y dy = (e^x + x^2)dx$$

[separating the variables]

On integrating both sides, we get

$$\int e^y dy = \int (e^x + x^2)\,dx$$

$$\Rightarrow \quad e^y = e^x + \frac{x^3}{3} + C$$

$$\left[\because \int e^x\,dx = e^x \text{ and } \int x^n\,dx = \frac{x^{n+1}}{n+1}, \text{ where } n \neq -1\right]$$

which is the required solution.

EXAMPLE |4| Find the particular solution of differential equation $\frac{dy}{dx} = -4xy^2$, given that $y = 1$, when $x = 0$.

First, separate the variables x and y and then integrate to find a general solution. Further, put the given condition in the general solution to find the value of arbitrary constant and then put the value of arbitrary constant in the general solution to find the particular solution.

Sol. Given differential equation is

$$\frac{dy}{dx} = -4xy^2 \qquad \text{...(i)}$$

$$\Rightarrow \quad \frac{dy}{y^2} = -4x\,dx \qquad \text{[separating the variables]}$$

On integrating both sides, we get

$$\int \frac{dy}{y^2} = -\int 4x\,dx \quad \Rightarrow \quad -\frac{1}{y} = -2x^2 + C$$

$$\Rightarrow \quad y = \frac{1}{2x^2 - C} \qquad \text{...(ii)}$$

which is a general solution of the given differential equation.

Also, given $y = 1$, when $x = 0$

On putting $x = 0$ and $y = 1$ in Eq. (ii), we get

$$1 = \frac{1}{2(0)^2 - C} \Rightarrow C = -1$$

On putting the value of C in Eq. (ii), we get

$$y = \frac{1}{2x^2 - (-1)} \Rightarrow y = \frac{1}{2x^2 + 1}$$

which is a particular solution of given differential equation.

HOMOGENEOUS DIFFERENTIAL EQUATIONS

A function $F(x, y)$ is said to be homogeneous function of degree n, if $F(x, y) = x^n g\left(\frac{y}{x}\right)$ or $y^n h\left(\frac{x}{y}\right)$

e.g. $F(x, y) = \frac{x+2y}{x-y}$ is a homogeneous function of degree 0, as $F(x, y) = x^0\left[\frac{1+2\left(\frac{y}{x}\right)}{1-\frac{y}{x}}\right]$

A differential equation of the form $\frac{dy}{dx} = F(x, y)$ is called a homogeneous differential equation, if $F(x, y)$ is a homogeneous function of degree zero.

Thus, if $F(x, y) = g\left(\frac{y}{x}\right)$ or $h\left(\frac{x}{y}\right)$, then the differential equation is homogeneous.

e.g. $\frac{dy}{dx} = \frac{x+2y}{x-y}$ is a homogeneous differential equation.

METHOD TO CHECK WHETHER GIVEN DIFFERENTIAL EQUATION IS HOMOGENEOUS OR NOT

Suppose, a differential equation is given to us, then to check whether it is homogeneous or not, we use the following steps

I. First, write the given differential equation as

$$\frac{dy}{dx} = F(x, y) \text{ or } \frac{dx}{dy} = F(x, y)$$

II. Replace y by λy and x by λx in $F(x, y)$.

III. If it is possible to write $F(\lambda x, \lambda y) = \lambda^0 F(x, y)$, then the given differential equation is homogeneous, otherwise the given differential equation is not homogeneous.

Note Generally, we use this method, only when we have to show that the differential equation is homogeneous.

EXAMPLE |5| Show that the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$ is homogeneous.

Sol. Given differential equation is,

$$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\cos(y/x) + x}{x\cos(y/x)}$$

Let $\quad F(x, y) = \frac{y\cos(y/x) + x}{x\cos(y/x)} \quad$ ...(i)

On replacing y by λy and x by λx in Eq. (i), we get

$$F(\lambda x, \lambda y) = \frac{\lambda y\cos\left(\frac{\lambda y}{\lambda x}\right) + \lambda x}{\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right)}$$

$$= \frac{\lambda[y\cos(y/x) + x]}{\lambda x\cos(y/x)}$$

$$= \lambda^0 F(x, y) \quad \text{[from Eq. (i)]}$$

Since, power of λ is 0 (zero), so $F(x, y)$ is a homogeneous function of degree 0. Hence, given differential equation is a homogeneous differential equation.

Note In short, to check whether the given differential equation is homogeneous or not, we check degree of each term of numerator and denominator of $F(x,y)$. If degree is same, then differential equation is homogeneous, otherwise not.

Solution of Homogeneous Differential Equation

Suppose given differential equation is of the form $f(x, y)dy = g(x, y)dx$, then to find its solution, we use the following steps

I. Write the given differential equation in the form

$$\frac{dy}{dx} = F(x, y), \text{ where } F(x, y) = \frac{g(x, y)}{f(x, y)} \quad \ldots(i)$$

or

$$\frac{dx}{dy} = F(x, y), \text{ where } F(x, y) = \frac{f(x, y)}{g(x, y)} \quad \ldots(i)$$

and check whether it is homogeneous or not. If Eq. (i) is homogeneous differential equation, then go to next step.

II. ***Case*** I If $F(x, y) = h\left(\frac{y}{x}\right)$, then put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i).

Case II If $F(x, y) = h\left(\frac{x}{y}\right)$, then put $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$ in Eq. (i).

III. Write the equation obtained in step II in variable separable form and then integrate.

IV. If case I occurred, then replace v by $\frac{y}{x}$ otherwise for case II, replace v by $\frac{x}{y}$ in the equation obtained in step III, to get the required solution.

EXAMPLE |6| Show that given differential equation $(x - y)\,dy = (x + y)\,dx$ is homogeneous and solve it. **[NCERT]**

Sol. Given differential equation is $(x - y)\,dy = (x + y)\,dx$

or $\quad \frac{dy}{dx} = \frac{x + y}{x - y} \quad$ ...(i)

Let $\quad F(x, y) = \frac{x + y}{x - y} \quad$ ...(ii)

On replacing x by λx and y by λy in Eq. (ii), we get

$$F(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x - \lambda y} = \frac{\lambda(x + y)}{\lambda(x - y)} = \lambda^0 F(x, y)$$

Thus, $F(x, y)$ is a homogeneous function of degree zero.

So, put $y = vx$.

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

On putting the values of y and $\frac{dy}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{x + vx}{x - vx} \Rightarrow v + x\frac{dv}{dx} = \frac{1 + v}{1 - v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 + v}{1 - v} - v \Rightarrow \frac{x\,dv}{dx} = \frac{1 + v - v + v^2}{1 - v}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{1+v^2}{1-v}$$

On separating the variables, we get

$$\frac{1-v}{1+v^2}\,dv=\frac{1}{x}\,dx$$

On integrating both sides, we get $\int\frac{1-v}{1+v^2}\,dv=\int\frac{1}{x}\,dx$

$$\Rightarrow \quad \int\frac{1}{1+v^2}\,dv-\frac{1}{2}\int\frac{2v}{1+v^2}\,dv=\int\frac{1}{x}\,dx$$

$$\Rightarrow \quad \tan^{-1}v-\frac{1}{2}\log|(1+v^2)|=\log|x|+C$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\log\left|\left(1+\frac{y^2}{x^2}\right)\right|=\log|x|+C \qquad \left[\text{put } v=\frac{y}{x}\right]$$

$$\Rightarrow \quad 2\tan^{-1}\left(\frac{y}{x}\right)-\log\left|\left(1+\frac{y^2}{x^2}\right)\right|=2(\log|x|+C)$$

$$\Rightarrow \quad 2\tan^{-1}\left(\frac{y}{x}\right)=\log\left|\frac{(x^2+y^2)}{x^2}\right|+\log|x|^2+2C$$

$$\Rightarrow \quad 2\tan^{-1}\left(\frac{y}{x}\right)=\log\left[\left|\frac{x^2+y^2}{x^2}\right||x|^2\right]+2C$$

$$[\because \log m+\log n=\log mn]$$

$$\Rightarrow \quad 2\tan^{-1}\left(\frac{y}{x}\right)-\log|x^2+y^2|=2C$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\log|x^2+y^2|=C,$$

which is the required solution.

EXAMPLE [7] Show that the differential equation $(x^2-y^2)\,dx+2xy\,dy=0$ is homogeneous and solve it. **[NCERT]**

Sol. Given differential equation is $(x^2-y^2)\,dx+2xy\,dy=0$.

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y^2-x^2}{2xy} \qquad \text{...(i)}$$

$$\text{Let } F(x,y)=\frac{y^2-x^2}{2xy} \qquad \text{...(ii)}$$

On replacing x by λx and y by λy in Eq. (ii), we get

$$F(\lambda x,\lambda y)=\frac{(\lambda y)^2+(\lambda x)^2}{2\lambda x\,\lambda y}=\frac{\lambda^2(y^2+x^2)}{\lambda^2\,2xy}=\lambda^0\,F(x,y)$$

Thus, $F(x,y)$ is a homogeneous function of degree zero.

On putting $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in Eq. (i), we get

$$v+x\frac{dv}{dx}=\frac{v^2x^2-x^2}{2vx^2}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{v^2-1}{2v}-v=\frac{-v^2-1}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{-v^2-1}{2v} \qquad \text{...(iii)}$$

From Eq. (iii), we get

$$\frac{2v}{1+v^2}\,dv=-\frac{1}{x}\,dx$$

On integrating both sides, we get

$$\int\frac{2v}{1+v^2}\,dv=-\int\frac{1}{x}\,dx$$

$$\Rightarrow \quad \log|(1+v^2)|=-\log|x|+\log|C|$$

$$\left[\begin{array}{l}\text{put } 1+v^2=t\Rightarrow 2v\,dv=dt\\ \therefore \int\frac{2v}{1+v^2}dv=\int\frac{1}{t}dt=\log|t|=\log|1+v^2|\end{array}\right]$$

$$\Rightarrow \quad \log|(1+v^2)|+\log|x|=\log|C|$$

$$\Rightarrow \quad \log|(1+v^2)\,x|=\log|C|$$

$$[\because \log m+\log n=\log mn]$$

$$\Rightarrow \quad x(1+v^2)=C$$

$$\Rightarrow \quad x\left(1+\frac{y^2}{x^2}\right)=C \qquad \left[\because v=\frac{y}{x}\right]$$

$$\Rightarrow \quad x\left(\frac{x^2+y^2}{x^2}\right)=C$$

$$\Rightarrow \quad x^2+y^2=Cx,$$

which is the required solution.

EXAMPLE [8] Find the general solution of $y^2dx+(x^2-xy+y^2)\,dy=0$. **[NCERT Exemplar]**

Sol. Given differential equation is

$$y^2dx+(x^2-xy+y^2)dy=0$$

$$\Rightarrow \quad y^2dx=-(x^2-xy+y^2)dy$$

$$\Rightarrow \quad y^2\frac{dx}{dy}=-(x^2-xy+y^2)$$

$$\Rightarrow \quad \frac{dx}{dy}=-\left(\frac{x^2}{y^2}-\frac{x}{y}+1\right) \qquad \text{...(i)}$$

which is a homogeneous differential equation as

$$\frac{dx}{dy}=F\left(\frac{x}{y}\right)$$

$\therefore$ Put $\frac{x}{y}=v$ or $x=vy$

$$\Rightarrow \quad \frac{dx}{dy}=v+y\frac{dv}{dy}$$

On substituting these values in Eq. (i), we get

$$v+y\frac{dv}{dy}=-(v^2-v+1)$$

$$\Rightarrow \quad y\frac{dv}{dy} = -v^2 + v - 1 - v$$

$$\Rightarrow \quad y\frac{dv}{dy} = -v^2 - 1$$

$$\Rightarrow \quad \frac{dv}{v^2+1} = -\frac{dy}{y}$$

On integrating both sides, we get

$$\int \frac{dv}{v^2+1} = \int -\frac{dy}{y}$$

$$\Rightarrow \quad \tan^{-1}(v) = -\log|y| + C$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x}{y}\right) + \log|y| = C \quad \left[\text{put } v = \frac{x}{y}\right]$$

which is the required general solution.

TOPIC PRACTICE 2

OBJECTIVE TYPE QUESTIONS

1 The general solution of $e^x \cos y dx - e^x \sin y dy = 0$ is

(a) $e^x \cos y = k$ (b) $e^x \sin y = k$

(c) $e^x = k \cos y$ (d) $e^x = k \sin y$

2 The general solution of $\frac{dy}{dx} = 2x e^{x^2 - y}$ is

(a) $e^{x^2 - y} = C$ (b) $e^{-y} + e^{x^2} = C$

(c) $e^y = e^{x^2} + C$ (d) $e^{x^2 + y} = C$

3 The number of solutions of $\frac{dy}{dx} = \frac{y+1}{x-1}$, when $y(1) = 2$ is

(a) none (b) one

(c) two (d) infinite

4 Which of the following is not a homogeneous function of x and y?

(a) $x^2 + 2xy$ b) $2x - y$

(c) $\cos^2\left(\frac{y}{x}\right) + \frac{y}{x}$ (d) $\sin x - \cos y$

5 The curve for which the slope of the tangent at any point is equal to the ratio of the abscissa to the ordinate of the point is

(a) an ellipse (b) parabola

(c) circle (d) rectangular hyperbola

VERY SHORT ANSWER Type Questions

Directions (Q. Nos. 6-9) *Solve the following differential equations.*

6 $\frac{dy}{dx} - \frac{x}{x^2+1} = 0$

7 $\frac{dy}{dx} = x^3 e^{-2y}$ **[All India 2015C]**

8 $\frac{dy}{dx} = \sqrt{4 - y^2}$ $(-2 < y < 2)$ **[NCERT]**

9 $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$ **[NCERT]**

SHORT ANSWER Type I Questions

Directions (Q. Nos. 10-23) *Find the general solution of the following differential equations.*

10 $(x+1)\frac{dy}{dx} = 2xy$

11 $\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$ **[NCERT]**

12 $\frac{dy}{dx} = \sin^{-1} x$ **[NCERT]**

13 $\frac{dy}{dx} = \frac{xy + y}{xy + x}$

14 $xy\frac{dy}{dx} = (x+2)(y+2)$ **[All India 2017C]**

15 $\frac{dy}{dx} = 1 - x + y - xy$

16 $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

17 $y \log y\, dx - x\, dy = 0$ **[NCERT]**

18 $\sin^3 x \frac{dx}{dy} = \sin y$

19 $(1+y^2)\tan^{-1} x\, dx + 2y(1+x^2)\, dy = 0$ **[NCERT Exemplar]**

20 $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

21 $\cos x(1 + \cos y)\, dx - \sin y(1 + \sin x)\, dy = 0$

22 $\sin\left(\frac{dy}{dx}\right) = a$

23 $\log\left(\frac{dy}{dx}\right) = 3x + 4y$ **[All India 2017C]**

24 Find the particular solution of the differential equation $e^x \tan y\, dx + (2 - e^x) \sec^2 y\, dy = 0$, given that $y = \frac{\pi}{4}$ when $x = 0$. **[CBSE 2018]**

25 Given that $\frac{dy}{dx} = e^{-2y}$ and $y = 0$, when $x = 5$. Find the value of x, when $y = 3$. **[NCERT Exemplar]**

SHORT ANSWER Type II Questions

26 Solve $(x - 1)\frac{dy}{dx} = 2x^3 y$.

27 Solve the following differential equation.
$e^x \tan y dx + (1 - e^x) \sec^2 y dy = 0$ **[Delhi 2011]**

28 Find the particular solution of the differential equation $e^x \sqrt{1 - y^2} dx + \frac{y}{x} dy = 0$, given that $y = 1$, when $x = 0$. **[Delhi 2014]**

29 Solve the initial value problem $e^{\frac{dy}{dx}} = x + 1$, $y(0) = 5$.

30 Find the particular solution of the differential equation $\frac{dy}{dx} = 1 + x + y + xy$, given that $y = 0$, when $x = 1$. **[All India 2014]**

31 Solve the differential equation $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2 y^2$, given that $y = 1$, when $x = 0$. **[Delhi 2012]**

32 Find the particular solution of the differential equation $x(1 + y^2)\, dx - y(1 + x^2)\, dy = 0$, given that $y = 1$, when $x = 0$.

33 Find the particular solution of the differential equation $(1 - y^2)(1 + \log|x|)\, dx + 2xy\, dy = 0$, given that $y = 0$ when $x = 1$. **[Delhi 2016]**

34 Find a particular solution of the differential equation $(x - y)(dx + dy) = dx - dy$, given that $y = -1$, when $x = 0$. **[NCERT]**

35 If $y(x)$ is a solution of $\left(\frac{2 + \sin x}{1 + y}\right)\frac{dy}{dx} = -\cos x$ and $y(0) = 1$, then find the value of $y\left(\frac{\pi}{2}\right)$. **[NCERT Exemplar; Delhi 2014]**

36 Find the equation of the curve passing through the point $\left(0, \frac{\pi}{4}\right)$, whose differential equation is $\sin x \cos y\, dx + \cos x \sin y\, dy = 0$. **[NCERT]**

37 Find the equation of a curve passing through the point $(0, -2)$, given that at any point (x, y) on the curve, the product of the slope of its tangent and y-coordinate of the point is equal to the x-coordinate of the point. **[NCERT]**

38 At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point $(-4, -3)$. Find the equation of the curve given that it passes through $(-2, 1)$. **[NCERT]**

39 Find the equation of a curve whose tangent at any point on it, different from origin, has slope $y + \frac{y}{x}$. **[NCERT Exemplar]**

40 Find the equation of a curve passing through the point $(1, 1)$, if the tangent drawn at any point $P(x, y)$ on the curve meets the coordinate axes at A and B such that P is the mid-point of AB. **[NCERT Exemplar]**

41 Find the equation of the curve passes through the point $(1, 0)$, if the slope of the tangent to the curve at any point (x, y) is $\frac{y - 1}{x^2 + x}$. **[NCERT Exemplar]**

42 Find the equation of a curve passing through the point $(0, 0)$ and whose differential equation is $\frac{dy}{dx} = e^x \sin x$. **[NCERT]**

43 Find the particular solution of the differential equation $\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0$, $y = 0$ when $x = 1$. **[All India 2014C, 2011]**

44 Show that the given differential equation is homogeneous and solve it

$x\frac{dy}{dx}-y+x\sin\left(\frac{y}{x}\right)=0.$ **[NCERT]**

45 Solve the following differential equation.

$$xy\log\left(\frac{y}{x}\right)dx+\left[y^2-x^2\log\left(\frac{y}{x}\right)\right]dy=0$$

[Delhi 2010C]

46 Find the general solution of the following differential equation.

$$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx}=y\cos\left(\frac{y}{x}\right)+x,\ x\neq 0$$

[Delhi 2017C; All India 2014C]

47 Solve $x+y\frac{dy}{dx}=2y$.

48 Solve $(x^3+y^3)\,dy-x^2y\,dx=0$.

49 Solve the differential equation $(1+x)(1+y^2)\,dx+(1+y)(1+x^2)\,dy=0$.

50 Solve the differential equation

$$x^2\frac{dy}{dx}=x^2-2y^2+xy.$$

51 Solve the differential equation

$$y+x\frac{dy}{dx}=x-y\frac{dy}{dx}.$$

[All India 2016]

52 Find the solution of the differential equation $(x\,dy-y\,dx)\,y\sin\left(\frac{y}{x}\right)=(y\,dx+x\,dy)\,x\cos\left(\frac{y}{x}\right).$

[NCERT, All India 2016C]

53 Solve the differential equation $ye^{x/y}\,dx=(xe^{x/y}+y^2)\,dy\,(y\neq 0).$

[NCERT; All India 2017C]

54 Show that the differential equation $2y\cdot e^{x/y}\,dx+(y-2xe^{x/y})\,dy=0$ is homogeneous and find its particular solution, given that $x=0$, when $y=1$. **[NCERT; All India 2016; Delhi 2013]**

LONG ANSWER Type Questions

55 Solve the following differential equation.

$x(x^2-1)\frac{dy}{dx}=1,\ y=0$, when $x=2$

[NCERT; All India 2012]

56 Find the particular solution of the differential equation $\frac{dy}{dx}=\frac{x(2\log x+1)}{\sin y+y\cos y}$, given that $y=\frac{\pi}{2}$, when $x=1$. **[Delhi 2014]**

57 Find the general solution of the following differential equation.

$$\left(y-x\frac{dy}{dx}\right)=a\left(y^2+\frac{dy}{dx}\right)$$

[All India 2016C]

58 Show that the general solution of the differential equation $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$ is given by $(x+y+1)=A(1-x-y-2xy)$, where A is a parameter. **[NCERT]**

59 Show that the differential equation $(x-y)\frac{dy}{dx}=x+2y$ is a homogeneous and solve it.

[NCERT; Delhi 2014]

60 Find the particular solution of the differential equation $(x-y)\frac{dy}{dx}=(x+2y)$, given that $y=0$ when $x=1$. **[All India 2017]**

61 Solve the differential equation

$\left[x\sin^2\left(\frac{y}{x}\right)-y\right]dx+x\,dy=0$, given $y=\frac{\pi}{4}$, when $x=1$.

[All India 2015C, 2014C, 2013; Delhi 2011C]

62 Show that the differential equation $x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)+x-y\sin\left(\frac{y}{x}\right)=0$ is homogeneous. Find the particular solution of this differential equation, given that $x=1$, when $y=\frac{\pi}{2}$.

[All India 2020; Delhi 2013]

63 Show that the following differential equation is homogeneous and then solve it.

$y\,dx+x\log\left|\frac{y}{x}\right|dy-2x\,dy=0$

[NCERT; All India 2011C]

64 Find the particular solution of the differential equation $\frac{dy}{dx}=\frac{xy}{x^2+y^2}$ given that $y=1$, when $x=0$. **[Delhi 2015]**

65 Solve the differential equation $x^2dy + (xy + y^2)\,dx = 0$ given $y = 1$, when $x = 1$. **[Delhi 2015]**

66 Show that the differential equation $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$ is homogeneous and also solve it. **[All India 2015]**

67 Find the particular solution of the differential equation $x^2dy = (2xy + y^2)\,dx$, given that $y = 1$, when $x = 1$. **[All India 2015]**

68 Find the particular solution $(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x$, $y = 1$, when $x = 0$.

69 Find the general solution of differential equation $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$.

70 Find the particular solution of the differential equation $(3xy + y^2)dx + (x^2 + xy)\,dy = 0$, for $x = 1$ and $y = 1$. **[Delhi 2013C]**

71 Show that the differential equation $(x\sqrt{x^2 + y^2} - y^2)\,dx + xy\,dy = 0$ is homogeneous and solve it.

72 Find the solution of the differential equation

$$\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\} y\,dx = \left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\} x\,dy.$$

[NCERT; All India 2010C]

73 A population grows at the rate of 5% per year. How long does it take for the population to double?

74 The population of a village increases at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20000 in year 1999 and 25000 in the year 2004, then what will be the population of the village in 2009?

75 In a bank, principal increases continuously at the rate of 5% per year. An amount of ₹ 1000 is deposited with this bank, how much will it be worth after 10 yr? ($e^{0.5} = 1.648$) **[NCERT]**

76 In a bank, principal increases at the rate of r% per year. Find the value of r, if ₹ 100 double itself in 10 yr. ($\log_e 2 = 0.6931$) **[NCERT]**

77 The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 s, it is 6 units. Find the radius of balloon after t s. **[NCERT]**

HINTS & SOLUTIONS

1. (a) Given that

$$e^x\cos y\,dx - e^x\sin y\,dy = 0$$

$\Rightarrow \quad e^x\cos y\,dx = e^x\sin y\,dy$

$\Rightarrow \quad dx = \tan y\,dy$

On integrating both sides, we get

$$x = \log\sec y + C$$

$\Rightarrow \quad x - C = \log\sec y$

$\Rightarrow \quad \sec y = e^{x-C}$

$\Rightarrow \quad \frac{1}{\cos y} = \frac{e^x}{e^C}$

$\Rightarrow \quad e^x\cos y = k$ [where, $k = e^C$]

2. (c) Given that $\frac{dy}{dx} = 2x\,e^{x^2 - y} = 2x\,e^{x^2}\cdot e^{-y}$

$\Rightarrow \quad e^y\,dy = 2x\,e^{x^2}\,dx$

On integrating both sides, we get

$$\int e^y\,dy = \int 2xe^{x^2}\,dx$$

Put $x^2 = t$ in RHS integral, we get

$$2x\,dx = dt$$

$$\int e^y\,dy = \int e^t\,dt$$

$\Rightarrow \quad e^y = e^t + C$

$\Rightarrow \quad e^y = e^{x^2} + C$

3. (b) Given that $\frac{dy}{dx} = \frac{y+1}{x-1}$

$\Rightarrow \quad \frac{dy}{y+1} = \frac{dx}{x-1}$

On integrating both sides, we get

$$\log(y+1) = \log(x-1) - \log C$$

$$C(y+1) = (x-1)$$

$\Rightarrow \quad C = \frac{x-1}{y+1}$

When $x = 1$ and $y = 2$, then $C = 0$

So, the required solution is $x - 1 = 0$.

Hence, only one solution exist.

4. (d) Since $\sin x - \cos y$ can't expressed in the form $x^n g\left(\frac{y}{x}\right)$ or $y^n h\left(\frac{x}{y}\right)$, therefore it is not a homogeneous function.

5. (d) It is given that

$$\frac{dy}{dx}=\frac{x}{y} \quad \text{...(i)}$$

$$ydy = xdx$$

$$\Rightarrow \frac{y^2}{2}=\frac{x^2}{x}+C_1$$

$$\Rightarrow x^2-y^2=C$$

where, $C=-2C_1$

This equation is representing a rectangular hyperbola.

6. Given differential equation is

$$\frac{dy}{dx}-\frac{x}{x^2+1}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{x}{x^2+1}$$

On separating the variables, we get

$$dy=\frac{x}{x^2+1}dx$$

On integrating both sides, we get

$$\int dy=\int\frac{x}{x^2+1}dx \Rightarrow \int dy=\frac{1}{2}\int\frac{2x}{x^2+1}dx$$

$$\left[\begin{array}{l}\text{put } x^2+1=t \Rightarrow 2x\,dx=dt\\ \therefore \int\frac{2x}{x^2+1}dx=\int\frac{1}{t}dt=\log|t|+C=\log|x^2+1|+C\end{array}\right]$$

$$\therefore \quad y=\frac{1}{2}\log|x^2+1|+C$$

which is the required solution.

7. **Hint** Write the given equation as $e^{2y}dy=x^3dx$ and solve it.

$$\left[\text{Ans. } \frac{e^{2y}}{2}=\frac{x^4}{4}+C\right]$$

8. Given differential equation is $\frac{dy}{dx}=\sqrt{4-y^2}$

On separating the variables, we get $\frac{dy}{\sqrt{4-y^2}}=dx$

On integrating both sides, we get

$$\int\frac{dy}{\sqrt{4-y^2}}=\int dx$$

$$\Rightarrow \sin^{-1}\left(\frac{y}{2}\right)=x+C \quad \left[\because \int\frac{dy}{\sqrt{a^2-y^2}}=\sin^{-1}\left(\frac{y}{a}\right)\right]$$

$$\Rightarrow \frac{y}{2}=\sin(x+C)$$

$$\Rightarrow y=2\sin(x+C),$$

which is the required solution.

9. Given differential equation is

$$\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

On separating the variables, we get

$$\frac{1}{\sqrt{1-y^2}}dy=-\frac{1}{\sqrt{1-x^2}}dx$$

On integrating both sides, we get

$$\int\frac{1}{\sqrt{1-y^2}}dy=-\int\frac{1}{\sqrt{1-x^2}}dx$$

$$\Rightarrow \sin^{-1}y=-\sin^{-1}x+C$$

$$\Rightarrow \sin^{-1}x+\sin^{-1}y=C,$$

which is the required general solution.

10. Given differential equation is

$$(x+1)\frac{dy}{dx}=2xy$$

On separating the variables, we get

$$\frac{dy}{y}=\frac{2x}{x+1}dx$$

On integrating both sides, we get

$$\int\frac{dy}{y}=2\int\frac{x}{x+1}dx$$

$$\Rightarrow \int\frac{dy}{y}=2\int\frac{(x+1)-1}{(x+1)}dx$$

$$\Rightarrow \int\frac{dy}{y}=2\left[\int dx-\int\frac{1}{(x+1)}dx\right]$$

$$\Rightarrow \log|y|=2\int dx-2\int\frac{1}{x+1}dx$$

$$\Rightarrow \log|y|=2x-2\log|x+1|+C$$

$$\Rightarrow \log|y|=2(x-\log|x+1|)+C,$$

which is the required solution.

11. **Hint**

(i) Use the following formulae

(a) $1-\cos x=2\sin^2\frac{x}{2}$

(b) $1+\cos x=2\cos^2\frac{x}{2}$

(ii) Write the given equation as $dy=\tan^2\frac{x}{2}dx$ and solve it.

$$\left[\text{Ans. } y=2\tan\frac{x}{2}-x+C\right]$$

12. **Hint** Evaluate $\int\sin^{-1}x\,dx$ by using integration by parts.

[**Ans.** $y=x(\sin^{-1}x)+\sqrt{1-x^2}+C$]

13. **Hint** Write the given equation as $\left(\frac{y+1}{y}\right)dy=\left(\frac{x+1}{x}\right)dx$ and solve it. [**Ans.** $y=C\,x\,e^{(x-y)}$]

14. Given differential equation is

$$xy\frac{dy}{dx} = (x+2)(y+2) \quad ...(i)$$

On separating the variables, we get

$$\frac{y}{y+2}dy = \frac{x+2}{x}dx$$

On integrating both sides, we get

$$\int \frac{y}{y+2}dy = \int \frac{x+2}{2}dx$$

$$\Rightarrow \int \left(\frac{y+2-2}{y+2}\right)dy = \int \frac{x+2}{x}dx$$

$$\Rightarrow \int \frac{y+2}{y+2}dy - \int \frac{2}{y+2}dy = \int \frac{x}{x}dx + \int \frac{2}{x}dx$$

$$\Rightarrow \int dy - 2\int \frac{1}{y+2}dy = \int dx + 2\int \frac{1}{x}dx$$

$$\Rightarrow y - 2\log|y+2| = x + 2\log|x| + C$$

15. **Hint** Write the given equation as $\frac{dy}{dx} = (1-x)(1+y)$ and solve it.

$\left[\textbf{Ans. } \log|1+y| = x - \frac{x^2}{2} + C\right]$

16. **Hint** Write the given equation as $\frac{x}{\sqrt{1-x^2}}dx = \frac{-y}{\sqrt{1-y^2}}dy$ and solve it.

[**Ans.** $\sqrt{1-x^2} + \sqrt{1-y^2} = C$]

17. **Hint** Write the given equation as $\frac{dy}{y\log y} = \frac{1}{x}dx$ and solve it. [**Ans.** $y = e^{cx}$]

18. **Hint** Use the formula, $\sin 3x = 3\sin x - 4\sin^3 x$

$\left[\textbf{Ans. } \cos y - \frac{3}{4}\cos x + \frac{1}{12}\cos 3x = C\right]$

19. **Hint** Write the given equation as $\frac{\tan^{-1}x}{1+x^2}dx = \frac{-2y}{1+y^2}dy$ and solve it.

$\left[\textbf{Ans. } \frac{1}{2}(\tan^{-1}x)^2 + \log|1+y^2| = C\right]$

20. **Hint** Write the given equation as $\frac{\sec^2 x}{\tan x}dx + \frac{\sec^2 y}{\tan y}dy = 0$ and solve it. [**Ans.** $\tan x \cdot \tan y = C$]

21. **Hint**

(i) Write the given equation as $\frac{\cos x}{1+\sin x}dx = \frac{\sin y}{1+\cos y}dy$

(ii) Put $1+\sin x = u$ in LHS and $1+\cos y = v$ in RHS to solve it.

[**Ans.** $(1+\sin x)(1+\cos y) = C$]

22. Given differential equation is $\sin\left(\frac{dy}{dx}\right) = a$.

$$\Rightarrow \frac{dy}{dx} = \sin^{-1}a$$

On separating the variables, we get

$$dy = \sin^{-1}a\,dx$$

On integrating both sides, we get

$$\int dy = \int \sin^{-1}a\,dx \Rightarrow y = x\sin^{-1}a + C$$

which is the required solution.

23. Given differential equation is $\log\left(\frac{dy}{dx}\right) = 3x + 4y$

$$\Rightarrow \frac{dy}{dx} = e^{3x+4y} \Rightarrow \frac{dy}{dx} = e^{3x}e^{4y}$$

On separating the variables, we get

$$\frac{1}{e^{4y}}dy = e^{3x}dx$$

On integrating both sides, we get

$$\int e^{-4y}dy = \int e^{3x}dx \Rightarrow \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C \quad ...(i)$$

which is the required general solution of given differential equation.

24. Given differential equation is

$$e^x \tan y\,dx + (2-e^x)\sec^2 y\,dy = 0$$

which can be rewritten as

$$e^x \tan y\,dx = (e^x - 2)\sec^2 y\,dy$$

$$\Rightarrow \frac{\sec^2 y}{\tan y}dy = \frac{e^x}{e^x-2}dx$$

$$\Rightarrow \int \frac{\sec^2 y}{\tan y}dy = \int \frac{e^x}{e^x-2}dx$$

$$\Rightarrow \log|\tan y| = \log|e^x - 2| + C$$

$$\left[\because \int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C\right]$$

$$\Rightarrow \log|\tan y| - \log|e^x - 2| = C$$

$$\Rightarrow \log\left|\frac{\tan y}{e^x-2}\right| = C \quad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$\Rightarrow \frac{\tan y}{e^x-2} = e^C \quad [\because \log m = n \Rightarrow m = e^n]$$

$$\Rightarrow \tan y = e^C(e^x - 2)$$

Now, it is given that $y = \frac{\pi}{4}$ when $x = 0$

$$\therefore \tan\frac{\pi}{4} = e^C(e^0 - 2) \Rightarrow 1 = e^C(1-2) \Rightarrow e^C = -1$$

Thus, the particular solution of the given differential equation is $\tan y = 2 - e^x$.

25. **Hint** (i) $\int e^{2y}dy = \int 1dx$

(ii) Put $y = 3$, to get the required value of x. $\left[\textbf{Ans. } \frac{e^6+9}{2}\right]$

26. Given differential equation is $(x-1)\dfrac{dy}{dx}=2x^3y$.

On separating the variables, we get

$$\frac{dy}{y}=\frac{2x^3}{(x-1)}\,dx$$

$$\Rightarrow \quad \frac{dy}{y}=2\left(\frac{x^3}{x-1}\right)dx \Rightarrow \frac{dy}{y}=2\left[\frac{(x^3-1)+1}{x-1}\right]dx$$

$$=2\left[\frac{(x-1)(x^2+x+1)}{(x-1)}+\frac{1}{(x-1)}\right]dx$$

$$[\because a^3-b^3=(a-b)(a^2+ab+b^2)]$$

$$\Rightarrow \quad \frac{dy}{y}=2\left(x^2+x+1+\frac{1}{x-1}\right)dx$$

On integrating both sides, we get

$$\int\frac{dy}{y}=2\int\left(x^2+x+1+\frac{1}{x-1}\right)dx$$

$$\Rightarrow \log|y|=2\left(\frac{x^3}{3}+\frac{x^2}{2}+x+\log|x-1|\right)+C,$$

which is the required solution.

27. **Hint** Differential equation reduces to

$\displaystyle\int\frac{e^x}{(e^x-1)}dx=\int\frac{\sec^2 y}{\tan y}dy$ and for simplifying this put

$e^x-1=t$ and $\tan y=z$. $\left[\text{Ans. } y=\tan^{-1}\left(\dfrac{e^x-1}{C}\right)\right]$

28. Given differential equation is

$$e^x\sqrt{1-y^2}\,dx+\frac{y}{x}dy=0$$

$$\Rightarrow \quad e^x\sqrt{1-y^2}\,dx=\frac{-y}{x}dy$$

On separating the variables, we get

$$\frac{-y}{\sqrt{1-y^2}}dy=xe^x dx$$

On integrating both sides, we get

$$\int\frac{-y}{\sqrt{1-y^2}}dy=\int xe^x dx$$

On putting $1-y^2=t$, then $-y\,dy=\dfrac{dt}{2}$ in LHS, we get

$$\int\frac{1}{2\sqrt{t}}dt=\int xe^x dx$$

$$\Rightarrow \quad \frac{1}{2}[2\sqrt{t}]=x\int e^x dx-\int\left[\frac{d}{dx}(x)\int e^x dx\right]dx$$

$$\Rightarrow \quad \sqrt{1-y^2}=xe^x-\int e^x dx \qquad [\text{putting } t=1-y^2]$$

$$\Rightarrow \quad \sqrt{1-y^2}=xe^x-e^x+C \qquad \text{...(i)}$$

On putting $y=1$ and $x=0$ in Eq. (i), we get

$$\sqrt{1-1}=0-e^0+C \Rightarrow C=1 \qquad [\because e^0=1]$$

On substituting the value of C in Eq. (i), we get

$$\sqrt{1-y^2}=xe^x-e^x+1$$

which is the required particular solution of given differential equation.

29. Given differential equation is

$$e^{\frac{dy}{dx}}=x+1$$

Now, taking log on both sides, we get

$$\frac{dy}{dx}=\log|x+1|$$

On separating the variables, we get

$$dy=\log|x+1|\,dx$$

On integrating both sides, we get

$$\int dy=\int\log|x+1|\,dx$$

$$\Rightarrow \quad y=\int \underset{\text{II}}{1}\cdot\log\underset{\text{I}}{|x+1|}\,dx$$

$$\Rightarrow \quad y=\log|x+1|\cdot x-\int\frac{1}{x+1}\cdot x\,dx$$

$$\Rightarrow \quad y=x\log|x+1|-\int\frac{(x+1-1)}{(x+1)}dx$$

$$\Rightarrow \quad y=x\log|x+1|-\left\{\int\frac{(x+1)}{(x+1)}dx-\int\frac{1}{x+1}dx\right\}$$

$$\Rightarrow \quad y=x\log|x+1|-\int dx+\int\frac{dx}{x+1}$$

$$\Rightarrow \quad y=x\log|x+1|-x+\log|x+1|+C \qquad \text{...(i)}$$

$\because$ It is given that $y=5$, when $x=0$.

$$\therefore \quad 5=0\cdot\log(0+1)-0+\log|0+1|+C$$

$$\Rightarrow \quad C=5$$

Now, substituting the value of C in Eq. (i), we get

$$y=x\log(x+1)-x+\log|x+1|+5,$$

which is the required solution.

30. Given, $\quad \dfrac{dy}{dx}=1+x+y+xy$

$$\Rightarrow \quad \frac{dy}{dx}=1(1+x)+y(1+x)$$

$$\Rightarrow \quad \frac{dy}{dx}=(1+x)(1+y) \qquad \text{...(i)}$$

On separating the variables, we get

$$\frac{1}{(1+y)}dy=(1+x)dx \qquad \text{...(ii)}$$

On integrating both sides of Eq. (ii), we get

$$\int\frac{1}{1+y}dy=\int(1+x)dx$$

$$\Rightarrow \quad \log|1+y|=x+\frac{x^2}{2}+C \qquad \text{...(iii)}$$

On substituting $x=1$, $y=0$ in Eq. (iii), we get

$$\log|1+0|=1+\frac{1}{2}+C$$

$$\Rightarrow \quad C=-\frac{3}{2} \qquad [\because \log 1=0]$$

Now, substituting the value of C in Eq. (iii), we get

$$\log|1+y|=x+\frac{x^2}{2}-\frac{3}{2}$$

which is the required particular solution of differential equation.

31. **Hint** (i) Write given equation as, $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$.

Further similar as Example 1.

(ii) Given that, $y = 1$, when $x = 0$

$\therefore$ By using these values in general solution, we get

$\tan^{-1}(1) = 0 + 0 + C \Rightarrow C = \frac{\pi}{4}$

$\left[\text{Ans. } \tan^{-1} y = x + \frac{x^3}{3} + \frac{\pi}{4}\right]$

32. **Hint** (i) Write the given equation as $\frac{x}{1+x^2}dx = \frac{y}{1+y^2}dy$

and solve it.

(ii) Use $x = 0$ and $y = 1$ to find the value of C.

[**Ans.** $y^2 - 2x^2 - 1 = 0$]

33. Given differential equation is

$$(1 - y^2)(1 + \log|x|)\,dx + 2xy\,dy = 0$$

On separating the variables, we get

$$\frac{(1 + \log|x|)}{x}dx + \frac{2y}{1 - y^2}dy = 0$$

On integrating, we get

$$\int\left(\frac{1}{x} + \frac{\log|x|}{x}\right)dx + \int\frac{2y}{1 - y^2}dy = 0$$

$$\Rightarrow \log|x| + \frac{(\log|x|)^2}{2} - \log|1 - y^2| = \log C \quad \text{...(i)}$$

Also, given $y = 0$ and $x = 1$

$$\therefore \log 1 + \frac{(\log 1)^2}{2} - \log|1 - 0| = \log C$$

$$\Rightarrow 0 + 0 - 0 = \log C$$

$$\Rightarrow \log C = 0$$

Putting $\log C = 0$ in Eq. (i), we get

$$\log|x| + \frac{(\log|x|)^2}{2} - \log|1 - y^2| = 0$$

34. Given differential equation is

$$(x - y)(dx + dy) = dx - dy$$

$$\Rightarrow dx + dy = \frac{dx - dy}{x - y}$$

On integrating both sides, we get

$$\int(dx + dy) = \int\frac{dx - dy}{x - y} \quad \text{...(i)}$$

Put $x - y = t$, then $dx - dy = dt$

From Eq. (i), we get

$$x + y = \int\frac{dt}{t} = \log|t| + C$$

$$\Rightarrow x + y = \log|x - y| + C \quad [\text{putting } t = x - y] \text{...(ii)}$$

It is given that when $x = 0$, then $y = -1$

$$\therefore 0 + (-1) = \log|0 + 1| + C$$

$$\Rightarrow C = -1 \quad [\because \log 1 = 0]$$

On substituting the value of C in Eq. (ii), we get the required particular solution

$$x + y = \log|x - y| - 1$$

$$\Rightarrow \log|x - y| = x + y + 1$$

35. Given that $\left(\frac{2 + \sin x}{1 + y}\right)\frac{dy}{dx} = -\cos x$

$$\Rightarrow \frac{dy}{1 + y} = -\frac{\cos x}{2 + \sin x}dx$$

On integrating both sides, we get

$$\int\frac{1}{1 + y}dy = -\int\frac{\cos x}{2 + \sin x}dx$$

$$\Rightarrow \log|1 + y| = -\log|2 + \sin x| + \log|C|$$

$$\Rightarrow \log|1 + y| + \log|2 + \sin x| = \log|C|$$

$$\Rightarrow \log|(1 + y)(2 + \sin x)| = \log|C|$$

$$\Rightarrow (1 + y)(2 + \sin x) = C$$

$$\Rightarrow 1 + y = \frac{C}{2 + \sin x}$$

$$\Rightarrow y = \frac{C}{2 + \sin x} - 1 \quad \text{...(i)}$$

It is given that when $x = 0$, then $y = 1$.

$$\therefore 1 = \frac{C}{2} - 1 \Rightarrow C = 4$$

On putting $C = 4$ in Eq. (i), we get

$$y = \frac{4}{2 + \sin x} - 1$$

Now, $y\left(\frac{\pi}{2}\right) = \frac{4}{2 + \sin\frac{\pi}{2}} - 1 = \frac{4}{2 + 1} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$

36. The differential equation of the given curve is

$$\sin x \cos y\,dx + \cos x \sin y\,dy = 0$$

$$\Rightarrow \frac{\sin x}{\cos x}dx + \frac{\sin y}{\cos y}dy = 0$$

$$\Rightarrow \tan x\,dx + \tan y\,dy = 0$$

On integrating both sides, we get

$$\int\tan x\,dx + \int\tan y\,dy = 0$$

$$\Rightarrow \log|\sec x| + \log|\sec y| = \log|C|$$

$$\Rightarrow \log|\sec x \cdot \sec y| = \log|C|$$

$$\Rightarrow \sec x \cdot \sec y = C \quad \text{...(i)}$$

Since, the curve passes through the point $\left(0, \frac{\pi}{4}\right)$.

Therefore, put $x = 0$ and $y = \frac{\pi}{4}$ in Eq. (i), we get

$$\sec 0 \sec\frac{\pi}{4} = C \Rightarrow C = \sqrt{2}$$

On putting the value of C in Eq. (i), we get

$$\sec x \cdot \sec y = \sqrt{2}$$

$$\Rightarrow \sec x \cdot \frac{1}{\cos y} = \sqrt{2} \Rightarrow \cos y = \frac{\sec x}{\sqrt{2}}$$

Hence, required equation of the curve is $\cos y = \frac{\sec x}{\sqrt{2}}$.

37. Let x and y be the x and y-coordinates of the point on the curve, respectively.

We know that the slope of a tangent to the curve is given by $\frac{dy}{dx}$.

According to the question, product of the slope of tangent with y-coordinate = x-coordinate

i.e. $$y \cdot \frac{dy}{dx} = x$$

Now, separating the variables, we get

$$y\,dy = x\,dx$$

On integrating both sides, we get

$$\int y\,dy = \int x\,dx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C \quad \text{...(i)}$$

Since, the curve passes through the point $(0, -2)$, therefore we have $\frac{(-2)^2}{2} = 0 + C \Rightarrow C = \frac{4}{2} = 2$

On putting the value of C in Eq. (i), we get

$$\frac{y^2}{2} = \frac{x^2}{2} + 2$$

$$\Rightarrow \quad x^2 - y^2 + 4 = 0$$

which is the required equation of the curve.

38. It is given that (x, y) is the point of contact of tangent to the curve.

The slope of the line segment joining the points $(x_2, y_2) = (x, y)$ and $(x_1, y_1) = (-4, -3)$

$$= \frac{y-(-3)}{x-(-4)} = \frac{y+3}{x+4} \quad \left[\because \text{slope of tangent} = \frac{y_2 - y_1}{x_2 - x_1}\right]$$

According to the question, slope of tangent is twice the slope of the line segment.

$$\therefore \quad \frac{dy}{dx} = 2\left(\frac{y+3}{x+4}\right)$$

Now, separating the variables, we get

$$\frac{dy}{y+3} = \left(\frac{2}{x+4}\right)dx$$

On integrating both sides, we get

$$\int \frac{dy}{y+3} = \int\left(\frac{2}{x+4}\right)dx$$

$$\Rightarrow \quad \log|y+3| = 2\log|x+4| + \log|C|$$

$$\Rightarrow \quad \log|y+3| = \log|x+4|^2 + \log|C|$$

$$\Rightarrow \quad \log\frac{|y+3|}{|x+4|^2} = \log|C| \quad \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

$$\Rightarrow \quad \frac{(y+3)}{(x+4)^2} = C \quad \text{...(i)}$$

Since, the curve passes through the point $(-2, 1)$, therefore we have

$$\frac{(1+3)}{(-2+4)^2} = C \Rightarrow C = 1$$

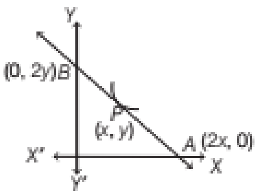
On putting $C = 1$ in Eq. (i), we get

$$\frac{(y+3)}{(x+4)^2} = 1 \Rightarrow y + 3 = (x+4)^2$$

which is the required equation of curve.

39. **Hint** $\frac{dy}{dx} = y + \frac{y}{x}$ **[Ans.** $y = Kxe^x$**]**

40. The figure obtained by the given information is given below



Let the coordinate of the point P be (x, y). It is given that P is mid-point of AB. So, the coordinates of points A and B are $(2x, 0)$ and $(0, 2y)$, respectively.

Now, slope of $AB = \frac{0 - 2y}{2x - 0} = -\frac{y}{x}$

Since, the segment AB is a tangent to the curve at P.

$$\therefore \quad \frac{dy}{dx} = -\frac{y}{x} \Rightarrow \frac{dy}{y} = -\frac{dx}{x}$$

On integrating both sides, we get

$$\log|y| = -\log|x| + \log|C|$$

$$\Rightarrow \quad \log|yx| = \log|C|$$

$$\Rightarrow \quad yx = C \quad \text{...(i)}$$

Since, the given curve passes through the point (1, 1).

$$\therefore \quad 1 \cdot 1 = C \Rightarrow C = 1$$

On putting $C = 1$ in Eq. (i), we get

$$xy = 1$$

41. It is given that slope of tangent to the curve at any point (x, y) is $\frac{y-1}{x^2+x}$.

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(x,y)} = \frac{y-1}{x^2+x} \Rightarrow \frac{dy}{dx} = \frac{y-1}{x^2+x}$$

$$\Rightarrow \quad \frac{dy}{y-1} = \frac{dx}{x^2+x}$$

On integrating both sides, we get

$$\int \frac{dy}{y-1} = \int \frac{dx}{x^2+x}$$

$$\Rightarrow \quad \int \frac{dy}{y-1} = \int \frac{dx}{x(x+1)}$$

$$\Rightarrow \quad \int \frac{dy}{y-1} = \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$\Rightarrow \quad \log|y-1| = \log|x| - \log|x+1| + \log|C|$$

$$\Rightarrow \quad \log|y-1| = \log\left|\frac{xC}{x+1}\right|$$

$$\Rightarrow \quad y - 1 = \frac{xC}{(x+1)}$$

Since, the given curve passes through the point (1, 0).

$$\therefore \quad 0-1=\frac{1\cdot C}{1+1} \Rightarrow C=-2$$

The particular solution is $y-1=\frac{-2x}{x+1}$

$$\Rightarrow \quad (y-1)(x+1)=-2x$$

$$\Rightarrow \quad (y-1)(x+1)+2x=0$$

42. Hint $\int dy=\int e^x \sin x\,dx$

Let $I=\int \underset{\text{II}}{e^x} \underset{\text{I}}{\sin x}\,dx \Rightarrow I=\sin x\int e^x dx-\int \underset{\text{I}}{\cos x}\, \underset{\text{II}}{e^x}\,dx$

$\Rightarrow I=e^x\sin x-e^x\cos x-I$

$$\therefore y=\frac{e^x(\sin x-\cos x)}{2}+C$$

$$\left[\text{Ans. } y=\frac{e^x(\sin x-\cos x)}{2}+\frac{1}{2}\right]$$

43. Given differential equation is

$$\frac{dy}{dx}-\frac{y}{x}+\operatorname{cosec}\left(\frac{y}{x}\right)=0\Rightarrow\frac{dy}{dx}=\frac{y}{x}-\operatorname{cosec}\left(\frac{y}{x}\right) \quad \text{...(i)}$$

which is a homogeneous differential equation as $\frac{dy}{dx}=f\left(\frac{y}{x}\right)$.

On putting $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in Eq. (i), we get

$$v+x\frac{dv}{dx}=v-\operatorname{cosec} v$$

$$\Rightarrow \quad x\frac{dv}{dx}=-\operatorname{cosec} v \Rightarrow \sin v\,dv=-\frac{dx}{x}$$

On integrating both sides, we get

$$\int \sin v\,dv=-\int\frac{dx}{x}$$

$$\Rightarrow \quad -\cos v=-\log|x|+C$$

$$\Rightarrow \quad \cos v=\log|x|-C$$

$$\Rightarrow \quad \cos\left(\frac{y}{x}\right)=\log|x|-C \quad \left[\text{putting } v=\frac{y}{x}\right] \text{...(ii)}$$

Also, given $y=0$, when $x=1$.

Then, $\cos 0=\log 1-C\Rightarrow 1=0-C\Rightarrow C=-1$

So, Eq. (ii) becomes $\cos\left(\frac{y}{x}\right)=\log|x|+1$

which is the required equation.

44. Hint

(i) Show that the given differential equation is homogeneous, i.e. $F(\lambda x, \lambda y)=\lambda^{\circ} F(x, y)$

(ii) Write the given equation as $\frac{dy}{dx}=\left(\frac{y}{x}\right)-\sin\left(\frac{y}{x}\right)$.

(iii) Solve as Question 43.

$$\left[\text{Ans. } x\left\{1-\cos\left(\frac{y}{x}\right)\right\}=C\sin\left(\frac{y}{x}\right)\right]$$

45. Hint Write the given equation as $\frac{dy}{dx}=\frac{xy\log\left(\frac{y}{x}\right)}{x^2\log\left(\frac{y}{x}\right)-y^2}$.

Put $y=vx$, then $v+x\frac{dv}{dx}=\frac{v\log v}{\log v-v^2}$

$$\Rightarrow \quad \int \underset{\text{II}}{v^{-3}}\,\underset{\text{I}}{\log v}\,dv-\int\frac{1}{v}dv=\int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{v^{-2}}{-2}\log v-\int\frac{1}{-2v}(v^{-2})dv=\log|v|+\log|x|+C$$

$$\Rightarrow \quad -\frac{1}{2v^2}\log v-\frac{1}{4v^2}=\log|vx|+C$$

$$\left[\text{Ans. } x^2\left[2\log\left(\frac{y}{x}\right)+1\right]+4y^2\log|y|=4y^2C\right]$$

46. Hint

(i) Write the given equation as $\frac{dy}{dx}=\frac{\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right)+1}{\cos\left(\frac{y}{x}\right)}$.

(ii) Substitute $y=vx$ to solve it. $\left[\text{Ans. } \sin\left(\frac{y}{x}\right)=\log|Cx|\right]$

47. Hint

(i) Write the given equation as $\frac{dy}{dx}=2-\left(\frac{x}{y}\right)$.

(ii) Substitute $y=vx$ to solve it.

$$\left[\text{Ans. } \log|y-x|=C+\frac{x}{(y-x)}\right]$$

48. Hint (i) Write the given equation as $\frac{dy}{dx}=\frac{\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^3}$.

(ii) Substitute $y=vx$ to solve it $\left[\text{Ans. } \frac{-x^3}{3y^3}+\log|y|=C\right]$

49. Given differential equation is

$$(1+x)(1+y^2)\,dx+(1+y)(1+x^2)\,dy=0$$

$$\Rightarrow \quad \frac{(1+x)}{(1+x^2)}dx+\frac{(1+y)}{(1+y^2)}dy=0$$

[dividing both sides by $(1+x^2)(1+y^2)$]

Now, integrating both sides, we get

$$\int\frac{(1+x)}{(1+x^2)}dx+\int\frac{(1+y)}{(1+y^2)}dy=0$$

$$\Rightarrow \int\left\{\frac{1}{1+x^2}+\frac{x}{1+x^2}\right\}dx+\int\left\{\frac{1}{1+y^2}+\frac{y}{1+y^2}\right\}dy=0$$

$$\Rightarrow \int\frac{dx}{1+x^2}+\frac{1}{2}\int\frac{2x}{1+x^2}dx+\int\frac{dy}{1+y^2}+\frac{1}{2}\int\frac{2y}{1+y^2}dy=0$$

$$\Rightarrow \tan^{-1}x+\frac{1}{2}\log|1+x^2|+\tan^{-1}y+\frac{1}{2}\log|1+y^2|=C$$

$$\Rightarrow \tan^{-1}x+\tan^{-1}y+\frac{1}{2}\{\log|1+x^2|+\log|1+y^2|\}=C$$

which is the required solution.

50. Hint

(i) Write the given equation as $\frac{dy}{dx} = 1 - 2\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)$.

(ii) Substitute $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ to solve it.

$$\left[\text{Ans. } \frac{x+\sqrt{2}y}{x-\sqrt{2}y} = (Cx)^{2\sqrt{2}}\right]$$

51. Similar as Example 6.

Hint $y + x\frac{dy}{dx} = x - y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x-y}{x+y}$

[Ans. $y^2 + 2xy - x^2 = C$**]**

52. Given differential equation can be rewritten as

$$xy\sin\left(\frac{y}{x}\right)dy - y^2\sin\left(\frac{y}{x}\right)dx = xy\cos\left(\frac{y}{x}\right)dx + x^2\cos\left(\frac{y}{x}\right)dy$$

$$\Rightarrow \left[xy\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{y}{x}\right)\right]dy = \left[xy\cos\left(\frac{y}{x}\right) + y^2\sin\left(\frac{y}{x}\right)\right]dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{xy\cos\left(\frac{y}{x}\right) + y^2\sin\left(\frac{y}{x}\right)}{xy\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{y}{x}\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x}\cos\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2\sin\left(\frac{y}{x}\right)}{\frac{y}{x}\sin\left(\frac{y}{x}\right) - \cos\left(\frac{y}{x}\right)} \quad \text{...(i)}$$

[dividing numerator and denominator by x^2]

Clearly, the given differential equation is homogeneous as $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

Now, put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ...(ii)

From Eqs. (i) and (ii), we get

$$v + x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} \quad \left[\because v = \frac{y}{x}\right]$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

On separating the variables, we get

$$\frac{v\sin v - \cos v}{v\cos v}dv = 2\frac{dx}{x}$$

On integrating both sides, we get

$$\int\left(\tan v - \frac{1}{v}\right)dv = 2\int\frac{dx}{x}$$

$$\Rightarrow \log|\sec v| - \log|v| = 2\log|x| + \log|C_1|$$

$$\Rightarrow \log\left|\frac{\sec v}{v}\right| - \log|x|^2 = \log|C_1|$$

$$\Rightarrow \log\left|\frac{\sec v}{vx^2}\right| = \log|C_1|$$

$$\Rightarrow \frac{\sec v}{vx^2} = C_1 \Rightarrow \frac{\sec\left(\frac{y}{x}\right)}{\frac{y}{x}(x^2)} = C_1 \quad \left[\because v = \frac{y}{x}\right]$$

$$\Rightarrow \frac{\sec\left(\frac{y}{x}\right)}{xy} = C_1 \Rightarrow \sec\left(\frac{y}{x}\right) = C_1(xy)$$

which is the required solution.

53. Given differential equation is

$$ye^{x/y}dx = (xe^{x/y} + y^2)dy, \; y \neq 0$$

and it can be written as, $\frac{dx}{dy} = \frac{\left(\frac{x}{y}e^{x/y} + y\right)}{e^{x/y}}$ ...(i)

Now, substitute $x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$

$\therefore$ From Eq. (i), we get $v + y\frac{dv}{dy} = \frac{ve^v + y}{e^v}$

$$\Rightarrow y\frac{dv}{dy} = \frac{ve^v + y}{e^v} - v$$

$$\Rightarrow y\frac{dv}{dy} = \frac{ve^v + y - ve^v}{e^v}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{y}{e^v}$$

$$\Rightarrow e^v dv = dy$$

On integrating both sides, we get

$$e^v = y + C \Rightarrow e^{x/y} = y + C \quad \left[\text{putting } v = \frac{x}{y}\right]$$

which is the required solution.

Note Given equation is not homogeneous as it cannot be written as $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ or $g\left(\frac{x}{y}\right)$. But here we have expression of the type $\frac{x}{y}$ in RHS. So, to simplify it put $x = vy$.

54. Given differential equation is

$$2y \cdot e^{x/y}dx + (y - 2x \cdot e^{x/y})\,dy = 0$$

$$\Rightarrow 2y \cdot e^{x/y}\frac{dx}{dy} + (y - 2x \cdot e^{x/y}) = 0$$

$$\Rightarrow 2y \cdot e^{x/y}\frac{dx}{dy} = 2x \cdot e^{x/y} - y$$

$$\Rightarrow \frac{dx}{dy} = \frac{2x \cdot e^{x/y} - y}{2y \cdot e^{x/y}} \quad \text{...(i)}$$

Let $F(x, y)=\dfrac{2x\cdot e^{x/y}-y}{2y\cdot e^{x/y}}$

Then, $F(\lambda x, \lambda y)=\dfrac{2(\lambda x)e^{(\lambda x/\lambda y)}-\lambda y}{2(\lambda y)e^{(\lambda x/\lambda y)}}$

$$=\frac{\lambda(2x\cdot e^{x/y}-y)}{\lambda(2y\cdot e^{x/y})}=\lambda^0[F(x, y)]$$

So, $F(x, y)$ is a homogeneous function of degree zero.

Therefore, the given differential equation is a homogeneous differential equation.

Also, $$\frac{dx}{dy}=\frac{2\left(\frac{x}{y}\right)e^{x/y}-1}{2e^{x/y}}=f\left(\frac{x}{y}\right) \quad \text{...(ii)}$$

On putting $x=vy$ and $\dfrac{dx}{dy}=v+y\dfrac{dv}{dy}$ in Eq. (ii), we get

$$v+y\frac{dv}{dy}=\frac{2v\,e^v-1}{2e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy}=\frac{2v\,e^v-1}{2e^v}-v$$

$$\Rightarrow \quad y\frac{dv}{dy}=\frac{2v\,e^v-1-2v\,e^v}{2e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy}=-\frac{1}{2e^v}$$

On separating the variables, we get

$$2e^v\,dv=-\frac{dy}{y}$$

On integrating both sides, we get

$$\int 2e^v\,dv=-\int\frac{dy}{y}$$

$$\Rightarrow \quad 2e^v=-\log|y|+C$$

$$\Rightarrow \quad 2e^{x/y}=-\log|y|+C \quad \left[\text{putting } v=\frac{x}{y}\right] \text{...(iii)}$$

Also, given $x=0$, when $y=1$.

Now, put $x=0$ and $y=1$ in Eq. (iii), we get

$$2\,e^{0/1}=-\log|1|+C$$

$$\Rightarrow \quad 2=0+C \Rightarrow C=2$$

On putting $C=2$ in Eq. (iii), we get

$$2e^{x/y}=-\log|y|+2$$

$$\Rightarrow \quad 2e^{x/y}+\log|y|=2$$

which is the particular solution.

55. Given differential equation is

$$x(x^2-1)\frac{dy}{dx}=1 \Rightarrow \frac{dy}{dx}=\frac{1}{x(x^2-1)}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x(x-1)(x+1)} \quad [\because a^2-b^2=(a-b)(a+b)]$$

$$\Rightarrow \quad dy=\frac{dx}{x(x-1)(x+1)}$$

On integrating both sides, we get

$$\int dy=\int\frac{dx}{x(x-1)(x+1)}+C$$

$$\Rightarrow \quad y=I+C \quad \text{...(i)}$$

where, $I=\displaystyle\int\frac{dx}{x(x-1)(x+1)}$

Let $\dfrac{1}{x(x-1)(x+1)}=\dfrac{A}{x}+\dfrac{B}{x-1}+\dfrac{C}{x+1}$

$\Rightarrow 1=A(x-1)(x+1)+Bx(x+1)+Cx(x-1)$

On comparing coefficients of x^2, x and constant terms from both sides, we get

$$A+B+C=0 \quad \text{...(ii)}$$

$$B-C=0 \quad \text{...(iii)}$$

and $$-A=1$$

$$\Rightarrow \quad A=-1$$

On putting $A=-1$ in Eq. (ii), we get

$$B+C=1 \quad \text{...(iv)}$$

Now, adding Eqs. (iii) and (iv), we get

$$2B=1 \Rightarrow B=\frac{1}{2}$$

On putting $B=\dfrac{1}{2}$ in Eq. (iii), we get

$$\frac{1}{2}-C=0 \Rightarrow C=\frac{1}{2}$$

$$\therefore \quad A=-1,\ B=\frac{1}{2} \text{ and } C=\frac{1}{2}$$

Thus, $\dfrac{1}{x(x-1)(x+1)}=\dfrac{-1}{x}+\dfrac{1/2}{x-1}+\dfrac{1/2}{x+1}$

On integrating both sides w.r.t. x, we get

$$I=\int\frac{1}{x(x-1)(x+1)}dx$$

$$=\int\frac{-1}{x}dx+\frac{1}{2}\int\frac{dx}{x-1}+\frac{1}{2}\int\frac{dx}{x+1}$$

$$\Rightarrow I=-\log|x|+\frac{1}{2}\log|x-1|+\frac{1}{2}\log|x+1|$$

On putting the value of I in Eq. (i), we get

$$y=-\log|x|+\frac{1}{2}\log|x-1|+\frac{1}{2}\log|x+1|+C \quad \text{...(v)}$$

Also, given that $y=0$, when $x=2$

On putting $y=0$ and $x=2$ in Eq. (v), we get

$$0=-\log 2+\frac{1}{2}\log 1+\frac{1}{2}\log 3+C$$

$$\Rightarrow C=\log 2-\frac{1}{2}\log 1-\frac{1}{2}\log 3$$

$$\Rightarrow C=\log 2-\log\sqrt{3} \quad [\because \log 1=0]$$

$$\Rightarrow C=\log\frac{2}{\sqrt{3}}$$

On putting the value of C in Eq. (v), we get

$$y=-\log|x|+\frac{1}{2}\log|x-1|+\frac{1}{2}\log|x+1|+\log\frac{2}{\sqrt{3}}$$

which is the required solution.

56. Given differential equation is $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

On separating the variables, we get

$$(\sin y + y\cos y)\,dy = x(2\log x + 1)\,dx$$

$$\Rightarrow \quad \sin y\,dy + y\cos y\,dy = 2x\log x\,dx + x\,dx$$

On integrating both sides, we get

$$\int \sin y\,dy + \int \underset{\text{I}}{y}\,\underset{\text{II}}{\cos y}\,dy = 2\int \underset{\text{II}}{x}\,\underset{\text{I}}{\log x}\,dx + \int x\,dx$$

$$\Rightarrow -\cos y + \left[y\int \cos y\,dy - \int\left\{\frac{d}{dy}(y)\int \cos y\,dy\right\}dy\right]$$

$$= 2\left[\log x\int x\,dx - \int\left\{\frac{d}{dx}(\log x)\int x\,dx\right\}dx\right] + \frac{x^2}{2}$$

$$\Rightarrow \quad -\cos y + y\sin y - \int \sin y\,dy$$

$$= 2\left[\frac{x^2}{2}\log x - \int\left\{\frac{1}{x}\times\frac{x^2}{2}\right\}dx\right] + \frac{x^2}{2}$$

$$\Rightarrow \quad -\cos y + y\sin y + \cos y$$

$$= x^2\log x - \int x\,dx + \frac{x^2}{2}$$

$$\Rightarrow y\sin y = x^2\log x - \frac{x^2}{2} + \frac{x^2}{2} + C$$

$$\Rightarrow y\sin y = x^2\log x + C \quad \text{...(i)}$$

On putting $y = \frac{\pi}{2}$ and $x = 1$ in Eq. (i), we get

$$\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = (1)^2\log(1) + C$$

$$\Rightarrow \quad C = \frac{\pi}{2} \quad \left[\because \sin\frac{\pi}{2} = 1, \log 1 = 0\right]$$

On substituting the value of C in Eq. (i), we get

$$y\sin y = x^2\log x + \frac{\pi}{2}$$

which is the required particular solution.

57. **Hint** Given differential equation can be rewritten as

$$\frac{dy}{y(1-ay)} = \frac{dx}{(x+a)} \Rightarrow \int\left[\frac{1}{y} + \frac{a}{(1-ay)}\right]dy = \int\frac{dx}{x+a}$$

[Ans. $y = C(x+a)(1-ay)$]

58. Given differential equation is $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0.$

On separating the variables, we get

$$\frac{dy}{y^2+y+1} + \frac{dx}{x^2+x+1} = 0$$

On integrating both sides, we get

$$\int\frac{dy}{y^2+y+1} + \int\frac{dx}{x^2+x+1} = C$$

$$\Rightarrow \int\frac{dy}{y^2+y+1+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}$$

$$+ \int\frac{dx}{x^2+x+1+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2} = C$$

$$\Rightarrow \int\frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(1-\frac{1}{4}\right)} + \int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(1-\frac{1}{4}\right)} = C$$

$$\Rightarrow \int\frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = C$$

$$\Rightarrow \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = C$$

$$\left[\because \int\frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) = \frac{\sqrt{3}\,C}{2} = k \text{ (say)}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2y+1}{\sqrt{3}} + \frac{2x+1}{\sqrt{3}}}{1-\left(\frac{2y+1}{\sqrt{3}}\right)\left(\frac{2x+1}{\sqrt{3}}\right)}\right] = k$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2y+1+2x+1}{\sqrt{3}}}{1-\left(\frac{4xy+2x+2y+1}{3}\right)}\right] = k$$

$$\Rightarrow \frac{2\sqrt{3}(x+y+1)}{3-(4xy+2x+2y+1)} = \tan k$$

$$\Rightarrow \frac{2\sqrt{3}(x+y+1)}{2(1-x-y-2xy)} = \tan k$$

$$\Rightarrow \quad x+y+1 = \frac{1}{\sqrt{3}}\tan k\,(1-x-y-2xy)$$

$$\Rightarrow \quad x+y+1 = A(1-x-y-2xy)$$

where, $A = \frac{1}{\sqrt{3}}\tan k$ is an arbitrary constant.

59. Given differential equation is

$$(x-y)\frac{dy}{dx} = x+2y$$

or $$\frac{dy}{dx} = \frac{x+2y}{x-y} \quad \text{...(i)}$$

Let $F(x,y) = \frac{x+2y}{x-y}$, then

$$F(\lambda x, \lambda y) = \frac{\lambda x + 2\lambda y}{\lambda x - \lambda y} = \frac{\lambda(x+2y)}{\lambda(x-y)} = \frac{\lambda^0(x+2y)}{x-y}$$

$$= \lambda^0 F(x,y)$$

So, $F(x,y)$ is a homogeneous function of degree zero. Therefore, given differential equation is a homogeneous differential equation.

Also, the given differential equation can be written as

$$\frac{dy}{dx}=\frac{1+2(y/x)}{1-(x/y)}=f\left(\frac{y}{x}\right) \quad \text{...(ii)}$$

On putting $y = vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in Eq. (i), we get

$$v+x\frac{dv}{dx}=\frac{x+2vx}{x-vx}$$

$$\Rightarrow \quad v+x\frac{dv}{dx}=\frac{1+2v}{1-v}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{1+2v}{1-v}-v$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{1+2v-v+v^2}{1-v}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{1+v+v^2}{-(v-1)}$$

On separating the variables, we get

$$\frac{(v-1)}{v^2+v+1}\,dv=-\frac{dx}{x}$$

On integrating both sides, we get

$$\int\frac{v-1}{v^2+v+1}\,dv=-\int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\int\frac{2(v-1)}{v^2+v+1}\,dv=-\int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\int\frac{2v-2}{v^2+v+1}\,dv=-\int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\int\frac{(2v+1)-3}{v^2+v+1}\,dv=-\int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\int\left(\frac{2v+1}{v^2+v+1}\right)dv-\frac{3}{2}\int\frac{dv}{v^2+v+1}=-\int\frac{dx}{x}$$

$$\therefore \frac{1}{2}\log|v^2+v+1|-\frac{3}{2}\int\frac{dv}{\left(v+\frac{1}{2}\right)^2+1-\frac{1}{4}}=-\log|x|+C_1$$

$$\left[\begin{array}{l}\because \text{put } v^2+v+1=t \Rightarrow (2v+1)\,dv=dt\\ \therefore \int\frac{(2v+1)}{v^2+v+1}\,dv=\int\frac{dt}{t}=\log|t|=\log|v^2+v+1|\end{array}\right]$$

$$\Rightarrow \quad \frac{1}{2}\log|v^2+v+1|-\frac{3}{2}\int\frac{dv}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=-\log|x|+C_1$$

$$\Rightarrow \quad \frac{1}{2}\log|v^2+v+1|-\frac{3}{2}\cdot\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left[\frac{\left(v+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}\right]=-\log|x|+C_1 \quad \left[\because \int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$\Rightarrow \quad \frac{1}{2}\log|v^2+v+1|-\frac{3}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)=-\log|x|+C_1$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right|-\frac{3}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\frac{2y}{x}+1}{\sqrt{3}}\right)=-\log|x|+C_1 \quad \left[\text{putting } v=\frac{y}{x}\right]$$

$$\Rightarrow \frac{1}{2}\log\left|\frac{y^2+xy+x^2}{x^2}\right|-\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)=-\log|x|+C_1$$

$$\Rightarrow \frac{1}{2}\log|y^2+xy+x^2|-\frac{1}{2}\log|x^2|-\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)=-\log|x|+C_1$$

$$\left[\because \log\frac{m}{n}=\log m-\log n\right]$$

$$\Rightarrow \frac{1}{2}\log|y^2+xy+x^2|-\log|x|-\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)=-\log|x|+C_1$$

$$\Rightarrow \quad \frac{1}{2}\log|y^2+xy+x^2|=\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+C_1$$

$$\Rightarrow \quad \log|y^2+xy+x^2|=2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+2C_1$$

$$\Rightarrow \quad \log|y^2+xy+x^2|=2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+C \quad \text{...(iii)}$$

[putting $C = 2C_1$]

which is the required general solution.

60. Solve as Question 59.

Ans. $\log|y^2+xy+x^2|=2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)-\frac{\pi}{3}$,

61. Given differential equation is

$$\left[x\sin^2\left(\frac{y}{x}\right)-y\right]dx+x\,dy=0$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y-x\sin^2\left(\frac{y}{x}\right)}{x} \quad \text{...(i)}$$

Let $\quad F(x,y)=\dfrac{y-x\sin^2\left(\frac{y}{x}\right)}{x}$

$$\text{Then,}\quad F(\lambda x, \lambda y) = \frac{\lambda y - \lambda x \sin^2\left(\frac{\lambda y}{\lambda x}\right)}{\lambda x}$$

$$= \frac{\lambda\left[y - x\sin^2\left(\frac{y}{x}\right)\right]}{\lambda x} = \lambda^0 F(x, y)$$

So, $\frac{dy}{dx} = F(x, y)$ is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{vx - x\sin^2\left(\frac{vx}{x}\right)}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v - \sin^2(v) \Rightarrow x\frac{dv}{dx} = -\sin^2(v)$$

$$\Rightarrow \quad \text{cosec}^2\, v\, dv = -\frac{dx}{x}$$

On integrating both sides, we get

$$\int \text{cosec}^2 v\, dv + \int \frac{dx}{x} = 0$$

$$\Rightarrow \quad -\cot v + \log|x| = C$$

$$\Rightarrow \quad -\cot\left(\frac{y}{x}\right) + \log|x| = C \qquad \left[\text{putting } v = \frac{y}{x}\right] \quad \text{...(ii)}$$

Also, given that $y = \frac{\pi}{4}$, when $x = 1$.

$$\therefore \quad -\cot\frac{\pi}{4} + \log|1| = C$$

$$\Rightarrow \quad C = -1 + 0 \Rightarrow C = -1$$

So, the required particular solution is

$$-\cot\left(\frac{y}{x}\right) + \log|x| = -1$$

$$\Rightarrow \quad 1 + \log|x| - \cot\left(\frac{y}{x}\right) = 0$$

62. Hint Given equation can be rewritten as

$$\frac{dy}{dx} = \frac{y}{x} - \frac{1}{\sin\frac{y}{x}} \qquad \text{...(i)}$$

(i) Let $F(x, y) = \frac{y}{x} - \frac{1}{\sin\frac{y}{x}}$

$$F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} - \frac{1}{\sin\frac{\lambda y}{\lambda x}} = \lambda^0\left(\frac{y}{x} - \frac{1}{\sin\frac{y}{x}}\right)$$

$$= \lambda^0 F(x, y)$$

So, the given differential equation is homogeneous.

On putting $y = vx$, then the given equation becomes

$$\sin v\, dv = -\frac{dx}{x} \qquad \left[\textbf{Ans. } \cos\frac{y}{x} = \log|x|\right]$$

63. Hint Given differential equation can be rewritten as

$$F(x, y) = \frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

Verify $F(\lambda x, \lambda y) = F(x, y)$

On putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, then given equation becomes

$$v + x\frac{dv}{dx} = \frac{vx}{2x - x\log\left(\frac{vx}{x}\right)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{2 - \log v} - v$$

$$\Rightarrow \quad \int \frac{2 - \log v}{v(\log v - 1)}\, dv = \int \frac{dx}{x}$$

On putting $\log v = t$ and $\frac{1}{v}dv = dt$, we get

$$\int \frac{2 - t}{t - 1}\, dt = \log|x| + C$$

$$\Rightarrow \quad \int\left(\frac{1}{t - 1} - 1\right)dt = \log|x| + C$$

$$\left[\textbf{Ans. } \log\left|\frac{\log\frac{y}{x} - 1}{y}\right| = C\right]$$

64. Given differential equation is

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2} = F(x, y) \qquad \text{...(i)}$$

On replacing x by λx and y by λy in Eq. (i), we get

$$F(\lambda x, \lambda y) = \frac{\lambda^2 xy}{\lambda^2(x^2 + y^2)} = \lambda^0 F(x, y)$$

Thus, $F(x, y)$ is a homogeneous function of degree zero. Hence, the given differential equation is a homogeneous differential equation.

Now, put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{v}{1 + v^2} \Rightarrow x\frac{dv}{dx} = \frac{v}{1 + v^2} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v - v - v^3}{1 - v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{v^3}{1 + v^2}$$

$$\Rightarrow \quad \frac{1 + v^2}{v^3}\, dv = -\frac{dx}{x}$$

On integrating both sides, we get

$$\int\left(\frac{1}{v^3} + \frac{1}{v}\right)dv = -\int\frac{dx}{x}$$

$$\Rightarrow \quad -\frac{1}{2v^2} + \log|v| = -\log|x| + C$$

$$\Rightarrow -\frac{x^2}{2y^2}+\log\left|\frac{y}{x}\right| = -\log|x| + C \qquad \left[\text{putting } v=\frac{y}{x}\right]$$

$$\Rightarrow -\frac{x^2}{2y^2}+\log|y| - \log|x| = -\log|x| + C$$

$$\Rightarrow -\frac{x^2}{2y^2}+\log|y| = C \qquad \text{...(ii)}$$

It is given that $y=1$, when $x=0$.

From Eq. (ii), we get

$$\log|1| = C$$

$$\Rightarrow \quad C=0 \Rightarrow -\frac{x^2}{2y^2}+\log|y| = 0$$

$$\Rightarrow \quad \log|y| = \frac{x^2}{2y^2} \Rightarrow y = e^{\frac{x^2}{2y^2}}$$

which is the required solution.

65. Given differential equation is

$x^2dy + (xy+y^2)\,dx = 0$

$$\Rightarrow \quad x^2dy = -(xy+y^2)\,dx$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{yx+y^2}{x^2}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{y}{x}\right)-\left(\frac{y}{x}\right)^2 \qquad \text{...(i)}$$

which is a homogeneous, as $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

On putting $y=vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} = -v - v^2$$

$$\Rightarrow \quad x\frac{dv}{dx} = -2v - v^2$$

$$\Rightarrow \quad \frac{1}{2v+v^2}\,dv = -\frac{1}{x}\,dx$$

$$\Rightarrow \quad \frac{1}{v(2+v)}\,dv = -\frac{1}{x}\,dx$$

$$\Rightarrow \quad \frac{2}{2v(2+v)}\,dv = -\frac{1}{x}\,dx$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{1}{v}-\frac{1}{v+2}\right)dv = -\frac{1}{x}\,dx$$

$$\Rightarrow \quad \frac{1}{2}\int\frac{1}{v}\,dv - \frac{1}{2}\int\frac{1}{v+2}\,dv = -\int\frac{1}{x}\,dx + \log C$$

$$\Rightarrow \frac{1}{2}\log|v| - \frac{1}{2}\log|v+2| = -\log|x| + \log C$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{v}{v+2}\right| = \log\left|\frac{C}{x}\right|$$

$$\Rightarrow \quad \log\left|\frac{v}{v+2}\right| = 2\log\left|\frac{C}{x}\right|$$

$$\Rightarrow \quad \frac{v}{v+2} = \left(\frac{C}{x}\right)^2$$

$$\Rightarrow \quad \frac{\frac{y}{x}}{\frac{y}{x}+2} = \left(\frac{C}{x}\right)^2 \qquad \left[\text{putting } v=\frac{y}{x}\right]$$

$$\Rightarrow \quad \frac{y}{y+2x} = \left(\frac{C}{x}\right)^2$$

It is given that $y=1$, when $x=1$.

$$\therefore \quad \frac{1}{1+2} = \left(\frac{C}{1}\right)^2 \Rightarrow \frac{1}{3} = C^2$$

Hence, the particular solution of given differential equation is

$$\frac{y}{y+2x} = \frac{\frac{1}{3}}{x^2}$$

$$\Rightarrow \quad \frac{y}{y+2x} = \frac{1}{3x^2}$$

$\Rightarrow \; 3x^2y = y+2x$, which is the required solution.

66. Hint (i) Show that the given differential equation is homogeneous.

(ii) Write the given equation as $\frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)-1}$.

(iii) Substitute $y=vx$ to solve it. $\left[\textbf{Ans. } \frac{y}{x} - \log|y| = C\right]$

67. Given, $x^2dy = (2xy+y^2)\,dx$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy+y^2}{x^2} = \frac{2y}{x}+\left(\frac{y}{x}\right)^2 \qquad \text{...(i)}$$

[the given differential equation is homogeneous]

On putting $y=vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{2vx}{x}+\frac{v^2x^2}{x^2} = 2v+v^2$$

$$\Rightarrow \quad x\frac{dv}{dx} = v+v^2$$

$$\Rightarrow \quad \frac{1}{v(v+1)}\,dv = \frac{1}{x}\,dx$$

$$\Rightarrow \left(\frac{1}{v}-\frac{1}{v+1}\right)dv = \frac{1}{x}\,dx$$

On integrating both sides, we get

$$\int\frac{1}{v}\,dv - \int\frac{1}{v+1}\,dv = \int\frac{1}{x}\,dx$$

$$\Rightarrow \quad \log v - \log|v+1| = \log|x| + \log|C|$$

$$\Rightarrow \quad \log\left|\frac{v}{v+1}\right| = \log|Cx|$$

$$\Rightarrow \quad \frac{y}{y+x} = Cx \qquad \text{...(ii)}$$

When $y=1$ and $x=1$, then

$$\frac{1}{1+1} = C \Rightarrow C = \frac{1}{2}$$

On putting the value of C in Eq. (ii), we get

$$\frac{y}{y+x}=\frac{1}{2}x \Rightarrow 2y=xy+x^2$$

$$\Rightarrow \quad x^2+xy-2y=0$$

68. Given equation can be written as

$$dy=\frac{2x^2+x}{(x^3+x^2+x+1)}dx$$

On integrating both sides, we get

$$\int dy=\int\frac{2x^2+x}{(x^3+x^2+x+1)}dx$$

$$\Rightarrow \quad y=\int\frac{2x^2+x}{(x+1)(x^2+1)}dx$$

Let $\frac{2x^2+x}{(x+1)(x^2+1)}=\frac{A}{(x+1)}+\frac{Bx+C}{(x^2+1)}$

Then, $\frac{2x^2+x}{(x+1)(x^2+1)}=\frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)}$

$$\Rightarrow \quad 2x^2+x=x^2(A+B)+x(B+C)+(A+C)$$

$$\Rightarrow \quad A+B=2,\ B+C=1 \text{ and } A+C=0$$

$$\Rightarrow \quad A=\frac{1}{2},\ B=\frac{3}{2} \text{ and } C=-\frac{1}{2}$$

$$\Rightarrow \quad y=\frac{1}{2}\int\frac{1}{(x+1)}dx+\int\frac{\frac{3}{2}x-\frac{1}{2}}{(x^2+1)}dx$$

$$\Rightarrow \quad y=\frac{1}{2}\log|x+1|+\frac{3}{2}\int\frac{x}{(x^2+1)}dx-\frac{1}{2}\int\frac{1}{(x^2+1)}dx$$

Put $x^2+1=t$, then $2x\,dx=dt$

$$\therefore \quad y=\frac{1}{2}\log|x+1|+\frac{3}{4}\log|t|-\frac{1}{2}\tan^{-1}x+C$$

$$\Rightarrow \quad y=\frac{1}{4}[\log\{(x+1)^2(x^2+1)^3\}]-\frac{1}{2}\tan^{-1}x+C$$

When $y=1$ and $x=0$, then

$1=C$

$\therefore$ The required particular solution is

$$y=\frac{1}{4}[\log\{(x+1)^2(x^2+1)^3\}]-\frac{1}{2}\tan^{-1}x+1$$

69. Given differential equation can be rewritten as

$$\frac{dy}{dx}=\frac{x^3-3xy^2}{y^3-3x^2y}=\frac{1-3\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^3-3\left(\frac{y}{x}\right)} \quad \text{...(i)}$$

which is a homogeneous differential equation, as

$$\frac{dy}{dx}=f\left(\frac{y}{x}\right)$$

On putting $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in Eq. (i), we get

$$\Rightarrow \quad v+x\frac{dv}{dx}=\frac{1-3v^2}{v^3-3v}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{1-3v^2}{v^3-3v}-v$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{1-3v^2-v^4+3v^2}{v^3-3v}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{1-v^4}{v^3-3v}$$

$$\Rightarrow \quad \left(\frac{v^3-3v}{1-v^4}\right)dv=\frac{dx}{x}$$

On integrating both sides, we get

$$\int\left(\frac{v^3-3v}{1-v^4}\right)dv=\int\frac{dx}{x}$$

$$\Rightarrow \quad \int\left(\frac{v^3-3v}{1-v^4}\right)dv=\log|x|+\log|C_1| \quad \text{...(ii)}$$

$$\Rightarrow \quad \int\frac{v^3}{1-v^4}dv-3\int\frac{v}{1-v^4}dv=\log|C_1x|$$

$$\Rightarrow \quad I_1-3I_2=\log|C_1x| \quad \text{...(iii)}$$

where, $I_1=\int\frac{v^3}{1-v^4}dv$ and $I_2=\int\frac{v}{1-v^4}dv$

Put $1-v^4=t$, then $-4v^3dv=dt \Rightarrow v^3\,dv=-\frac{dt}{4}$

$$\therefore \quad I_1=\int\frac{-dt}{4t}=-\frac{1}{4}\log|t|=-\frac{1}{4}\log|1-v^4| \quad \text{[putting } t=1-v^4\text{]}$$

and $I_2=\int\frac{v\,dv}{1-v^4}=\int\frac{v\,dv}{1-(v^2)^2}$

Put $v^2=z$, then $v\,dv=\frac{dz}{2}$

$$\therefore \quad I_2=\frac{1}{2}\int\frac{dz}{1-z^2}=\frac{1}{2\times2}\log\left|\frac{1+z}{1-z}\right| \quad \left[\because \int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|\right]$$

$$=\frac{1}{4}\log\left|\frac{1+v^2}{1-v^2}\right| \quad \text{[putting } z=v^2\text{]}$$

On substituting the values of I_1 and I_2 in Eq. (iii), we get

$$-\frac{1}{4}\log|1-v^4|-\frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right|=\log|C_1x|$$

$$\Rightarrow \quad -\frac{1}{4}\log\left|(1-v^4)\left(\frac{1+v^2}{1-v^2}\right)^3\right|=\log|C_1x|$$

$$\Rightarrow \quad -\frac{1}{4}\log\left|(1-v^2)(1+v^2)\times\frac{(1+v^2)^3}{(1-v^2)^3}\right|=\log|C_1x|$$

$$\Rightarrow \quad \log\left|\frac{(1+v^2)^4}{(1-v^2)^2}\right|^{-1/4}=\log|C_1x|$$

$$\Rightarrow \quad \left[\frac{(1+v^2)^4}{(1-v^2)^2}\right]^{-1/4}=C_1x$$

$$\Rightarrow \quad \frac{(1+v^2)^4}{(1-v^2)^2} = (C_1 x)^{-4}$$

[raising power (– 4) on both sides]

$$\Rightarrow \quad \frac{\left(1+\frac{y^2}{x^2}\right)^4}{\left(1-\frac{y^2}{x^2}\right)^2} = \frac{1}{C_1^4 x^4} \qquad \left[\text{putting } v = \frac{y}{x}\right]$$

$$\Rightarrow \quad \frac{(x^2+y^2)^4}{x^4(x^2-y^2)^2} = \frac{1}{C_1^4 x^4}$$

$$\Rightarrow \quad (x^2-y^2)^2 = C_1^4(x^2+y^2)^4$$

$$\Rightarrow \quad (x^2-y^2) = C_1^2(x^2+y^2)^2 \qquad \text{[taking square root]}$$

$$\Rightarrow \quad x^2-y^2 = C(x^2+y^2)^2 \qquad [\text{putting } C_1^2 = C]$$

which is the required general solution.

70. **Hint** (i) Write the given differential equation as

$$\frac{dy}{dx} = -\frac{\left[3\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2\right]}{1+\left(\frac{y}{x}\right)}.$$

(ii) Substitute $y = vx$ to solve it.

(iii) Use $x = 1$ and $y = 1$ to find the value of C.

[Ans. $y^2x^2 + 2yx^3 = 3$]

71. **Hint** Given equation can be rewritten as

$$F(x, y) = \frac{dy}{dx} = \frac{y^2 - x\sqrt{x^2+y^2}}{xy} \qquad \text{... (i)}$$

Verify $F(\lambda x, \lambda y) = \lambda^0 F(x, y)$

On putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, then given equation becomes

$$v + x\frac{dv}{dx} = \frac{v^2 - \sqrt{1+v^2}}{v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-\sqrt{1+v^2}}{v}$$

[Ans. $\sqrt{x^2+y^2} + x\log|x| = Cx$]

72. Given, $\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\} y\,dx$

$$= \left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\} x\,dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\left\{x\cos\frac{y}{x} + y\sin\frac{y}{x}\right\}}{x\left\{y\sin\frac{y}{x} - x\cos\frac{y}{x}\right\}} \qquad \text{...(i)}$$

Clearly, the given differential equation is homogeneous.

On putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} = \frac{vx(x\cos v + vx\sin v)}{x(vx\sin v - x\cos v)}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v(\cos v + v\sin v)}{v\sin v - \cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \quad \left(\frac{v\sin v - \cos v}{v\cos v}\right)dv = \frac{2}{x}dx$$

$$\Rightarrow \quad \left(\tan v - \frac{1}{v}\right)dv = \frac{2}{x}dx$$

On integrating both sides, we get

$$\int\left(\tan v - \frac{1}{v}\right)dv = \int\frac{2\,dx}{x}$$

$$\Rightarrow \quad \int\tan v\,dv - \int\frac{1}{v}dv = 2\int\frac{1}{x}dx$$

$$\Rightarrow \quad -\log|\cos v| - \log|v| = 2\log|x| + C$$

$$\Rightarrow \quad \log|v\cos v| + 2\log|x| = -C$$

$$[\because \log m + \log n = \log mn]$$

$$\Rightarrow \quad \log[(v\cos v)x^2] = -C$$

$$\Rightarrow \quad (v\cos v)x^2 = e^{-C} \qquad [\because \log_e x = m \Rightarrow x = e^m]$$

$$\Rightarrow \quad x^2 v\cos v = A \qquad [\text{consider } A = e^{-C}]$$

$$\Rightarrow \quad x^2\frac{y}{x}\cos\frac{y}{x} = A \qquad \left[\text{putting } v = \frac{y}{x}\right]$$

$$\Rightarrow \quad xy\cos\frac{y}{x} = A, \text{ which is the required solution.}$$

73. Let P_0 be the initial population and the population after t years be P.

According to given problem, we get

$$\frac{dP}{dt} = \left(\frac{5}{100}\right)\times P$$

$$\Rightarrow \quad \frac{dP}{dt} = \frac{P}{20}$$

Now, separating the variables, we get

$$\frac{dP}{P} = \frac{dt}{20}$$

On integrating both sides, we get

$$\log P = \frac{t}{20} + C_1 \qquad [\because P > 0]$$

$\Rightarrow P = e^{t/20 + C_1} = e^{t/20}\cdot e^{C_1} = Ce^{t/20}$, where $e^{C_1} = C$

$\because$ At $t = 0$, $P = P_0$, therefore we have $P_0 = C\cdot e^0$

$$\Rightarrow \quad C = P_0$$

Thus, $\quad P = P_0 e^{t/20} \qquad \text{...(i)}$

Now, let t yr be the time required to double the population. Then, we have

$$2P_0 = P_0 e^{t/20}$$

$$\Rightarrow \quad 2 = e^{t/20} \Rightarrow \log 2 = \frac{t}{20}\log e$$

$$\Rightarrow \quad t = 20 \log 2 \qquad [\because \log e = 1]$$

74. Let y be the population at time t, then $\frac{dy}{dt} \propto y$.

$$\Rightarrow \quad \frac{dy}{dt} = ky, \text{ where } k \text{ is a constant.}$$

$$\Rightarrow \quad \frac{dy}{y} = k\,dt$$

On integrating both sides, we get

$$\log|y| = kt + C \qquad [\because y > 0] \quad \text{...(i)}$$

In the year 1999, $t = 0$, $y = 20000$

From Eq. (i), we get

$$\log 20000 = k(0) + C$$

$$\Rightarrow \quad \log 20000 = C \qquad \text{...(ii)}$$

In the year 2004, $t = 5$, $y = 25000$

From Eq. (i), we get

$$\log 25000 = k5 + C$$

$$\Rightarrow \quad \log 25000 = 5k + \log 20000 \qquad [\text{using Eq. (ii)}]$$

$$\Rightarrow \quad 5k = \log\left(\frac{25000}{20000}\right) = \log\left(\frac{5}{4}\right)$$

$$\Rightarrow \quad k = \frac{1}{5}\log\frac{5}{4}$$

For year 2009, $t = 10$ yr

Now, putting the values of t, k and C in Eq. (i), we get

$$\log y = 10 \times \frac{1}{5}\log\left(\frac{5}{4}\right) + \log(20000)$$

$$\Rightarrow \quad \log y = \log\left[20000 \times \left(\frac{5}{4}\right)^2\right]$$

$$[\because \log m + \log n = \log mn]$$

$$\Rightarrow \quad y = 20000 \times \frac{5}{4} \times \frac{5}{4}$$

$$\Rightarrow \quad y = 31250$$

Hence, the population of the village in 2009 will be 31250.

75. Let P and t be the principal and time, respectively. It is given that the principal increases continuously at the rate of 5% per year.

given that the principal increases continuously at the rate of 5% per year.

$$\therefore \quad \frac{dP}{dt} = 5\% \text{ of } P \Rightarrow \frac{dP}{dt} = \frac{5}{100}P$$

On separating the variables, we get

$$\frac{dP}{P} = \frac{1}{20}dt$$

On integrating both sides, we get

$$\int\frac{dP}{P} = \int\frac{1}{20}dt$$

$$\Rightarrow \quad \log(P) = \frac{1}{20}t + C \qquad [\because P > 0] \quad \text{...(i)}$$

Now, initially $t = 0$ and $P = 1000$, we get

$$\log 1000 = C \qquad \text{...(ii)}$$

On putting the value of C in Eq. (i), we get

$$\log(P) = \frac{1}{20}t + \log 1000$$

$$\Rightarrow \log(P) - \log 1000 = \frac{t}{20}$$

$$\Rightarrow \quad \log\left[\frac{P}{1000}\right] = \frac{1}{20}t \qquad \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

When $t = 10$, then $\log\left[\frac{P}{1000}\right] = \frac{1}{20} \times 10$

$$\Rightarrow \quad \frac{P}{1000} = e^{1/2} \qquad [\because \log_e x = m \Rightarrow x = e^m]$$

$$\Rightarrow \quad \frac{P}{1000} = e^{0.5} = 1.648$$

$$\Rightarrow \quad P = 1000 \times 1.648$$

$$\Rightarrow \quad P = 1648$$

Hence, after 10 yr, the amount will worth ₹ 1648.

76. Similar as Question 75. **[Ans.** 6.93%]

77. Let the rate of change of the volume of the balloon be k, where k is constant.

Then, $\frac{d}{dt}(\text{Volume}) = \text{Constant}$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k \qquad \left[\because \text{volume of sphere} = \frac{4}{3}\pi r^3\right]$$

$$\Rightarrow \quad \left(\frac{4}{3}\pi\right)\left(3r^2\frac{dr}{dt}\right) = k$$

On separating the variables, we get

$$4\pi r^2\,dr = k\,dt \qquad \text{...(i)}$$

On integrating both sides, we get

$$4\pi\int r^2\,dr = k\int dt$$

$$\Rightarrow \quad 4\pi\frac{r^3}{3} = kt + C$$

$$\Rightarrow \quad 4\pi r^3 = 3(kt + C) \qquad \text{...(ii)}$$

Now, initially $t = 0$ and $r = 3$, we get

$$4\pi(3)^3 = 3(k \times 0 + C)$$

$$\Rightarrow \quad 108\pi = 3C$$

$$\Rightarrow \quad C = 36\pi$$

Also, when $t = 3$, then $r = 6$.

$\Rightarrow \quad C = 36\pi$

Also, when $t = 3$, then $r = 6$.

Therefore, from Eq. (ii), we get

$$4\pi(6)^3 = 3(k \times 3 + C)$$

$$\Rightarrow \quad 864\pi = 3(3k + 36\pi)$$

$$\Rightarrow \quad 3k = 288\pi - 36\pi = 252\pi$$

$$\Rightarrow \quad k = 84\pi$$

On putting the values of k and C in Eq. (ii), we get

$$4\pi r^3 = 3(84\pi t + 36\pi)$$

$$\Rightarrow \quad 4\pi r^3 = 4\pi(63t + 27)$$

$$\Rightarrow \quad r^3 = 63t + 27$$

$$\Rightarrow \quad r = (63t + 27)^{1/3}$$

which is required radius of the balloon at time t.

|TOPIC 3|

Linear Differential Equation

A differential equation of the form

$$A_0 \frac{d^n y}{dx^n} + A_1 \frac{d^{n-1} y}{dx^{n-1}} + A_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + A_{n-1} \frac{dy}{dx} + A_n y = Q$$

where, $A_0, A_1, A_2, \dots, A_{n-1}, A_n$ are either constants or functions of independent variable x, is called a linear differential equation.

e.g. $\frac{d^3 y}{dx^3} + x^2 \frac{d^2 y}{dx^2} + 2xy = x^2$ is a linear differential equation, but $xy \frac{d^2 y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + ay = x$ is a non-linear differential equation.

Note Linear differential equation is always of 1 degree.

Linear Differential Equation of First Order

A first order differential equation in which the degree of dependent variable and its derivative is one and they do not get multiplied together, is called a linear differential equation of first order.

Here, we will study two types of linear differential equation of first order, as given below

***Type* I** A differential equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are constants or functions of x only.

e.g. $\frac{dy}{dx} + 2y = \sin x$. Here, $P = 2$ and $Q = \sin x$.

***Type* II** A differential equation of the form $\frac{dx}{dy} + Px = Q$, where P and Q are constants or functions of y only.

e.g. $\frac{dx}{dy} - \frac{x}{y} = y^2$.

Here, $P = -\frac{1}{y}$ and $Q = y^2$.

In type II, x is the dependent variable and y is the independent variable.

Integrating Factor (IF)

Linear differential equations are solved when they are multiplied by a factor, which is called the integrating factor, because by multiplying such factor the left hand side of the differential equation become exact differential of same function.

For type I Differential equation, IF is $e^{\int P dx}$.

For type II Differential equation, IF is $e^{\int P dy}$.

EXAMPLE |1| What will be the integrating factor of given differential equation $y\,dx + (x - y^2)\,dy = 0$?

Sol. Given differential equation is

$$y\,dx + (x - y^2)\,dy = 0$$

$$\Rightarrow \quad y\frac{dx}{dy} + (x - y^2) = 0$$

$$\Rightarrow \quad y\frac{dx}{dy} + x = y^2$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{y}x = y \quad \text{[dividing both sides by } y\text{]}$$

which is of the form $\frac{dx}{dy} + Px = Q$.

Here, $P = \frac{1}{y}$ and $Q = y$

$\therefore$ Integrating factor, $\text{IF} = e^{\int P dy} = e^{\int \frac{1}{y} dy}$

$= e^{\log y} = y \quad [\because e^{\log f(x)} = f(x)]$

METHOD TO SOLVE FIRST ORDER LINEAR DIFFERENTIAL EQUATION

To solve a first order linear differential equation, we use the following steps

I. Express the given differential either in the form

$$\frac{dy}{dx} + Py = Q \quad \text{or} \quad \frac{dx}{dy} + Px = Q.$$

II. If the given differential equation is of the form $\frac{dy}{dx} + Py = Q$, then identify P and Q, which are constants or functions of x only.

III. Find the integrating factor IF by using the formula $e^{\int P dx}$.

IV. Now, the solution of the given differential equation is given by

$$y \cdot \text{IF} = \int (Q \times \text{IF})\,dx + C$$

In case, the first order linear differential equation is of the form $\frac{dx}{dy} + Px = Q$, where P and Q are constants or functions of y only.

Then, $\text{IF} = e^{\int P dy}$ and the solution of the differential equation is given by

$$x \cdot \text{IF} = \int (Q \times \text{IF})\,dy + C$$

EXAMPLE |2| Find the general solution of the differential equation $\frac{dy}{dx}+(\sec x)y=\tan x, \left(0 \le x < \frac{\pi}{2}\right)$.

[NCERT; All India 2012]

Sol. Given differential equation is

$$\frac{dy}{dx}+(\sec x)y=\tan x$$

This is of the form $\frac{dy}{dx}+Py=Q$.

Here, $P=\sec x$ and $Q=\tan x$.

Now, Integrating Factor, $\text{IF}=e^{\int P dx}=e^{\int \sec x\,dx}=e^{\log|\sec x+\tan x|}$

$$=\sec x+\tan x$$

and the solution of differential equation is given by

$$y\times \text{IF}=\int (Q\times \text{IF})dx+C$$

$$\Rightarrow y\cdot(\sec x+\tan x)=\int \tan x(\sec x+\tan x)dx+C$$

$$=\int \tan x \sec x\,dx+\int \tan^2 x\,dx+C$$

$$=\sec x+\int (\sec^2 x-1)dx+C$$

$$[\because \sec^2 x-\tan^2 x=1]$$

$\Rightarrow y\cdot(\sec x+\tan x)=\sec x+\tan x-x+C,$
which is the required solution.

EXAMPLE |3| Find the general solution of the differential equation $e^{2x}\frac{dy}{dx}+3e^{2x}y=1$.

Sol. Given differential equation is $e^{2x}\frac{dy}{dx}+3e^{2x}y=1$

$$\Rightarrow \quad e^{2x}\left(\frac{dy}{dx}+3y\right)=1$$

$$\Rightarrow \quad \frac{dy}{dx}+3y=e^{-2x}$$

This is the form of $\frac{dy}{dx}+Py=Q$.

Here, $P=3$ and $Q=e^{-2x}$

Now, Integrating Factor, $\text{IF}=e^{\int 3dx}=e^{3x}$

and the solution of differential equation is given by

$$y\times \text{IF}=\int (Q\times \text{IF})\,dx+C$$

$$\Rightarrow \quad y\times e^{3x}=\int (e^{-2x}\times e^{3x})\,dx+C=\int e^{x}dx+C$$

$$\Rightarrow \quad ye^{3x}=e^{x}+C$$

which is the required solution.

EXAMPLE |4| Find the general solution of the differential equation $(3y^2-x)\,dy=y\,dx$.

Sol. Given differential equation is

$$(3y^2-x)\,dy=y\,dx$$

$$\Rightarrow \quad 3y^2-x=y\frac{dx}{dy}$$

$$\Rightarrow \quad y\frac{dx}{dy}+x=3y^2$$

$$\Rightarrow \quad \frac{dx}{dy}+x\times\frac{1}{y}=3y \quad \text{[dividing both sides by } y]$$

This is of the form $\frac{dx}{dy}+Px=Q$.

Here, $P=\frac{1}{y}$ and $Q=3y$

Now, Integrating Factor,

$$\text{IF}=e^{\int P dy}=e^{\int \frac{1}{y}dy}=e^{\log y}=y \quad [\because e^{\log f(x)}=f(x)]$$

and the solution of differential equation is given by

$$x\times \text{IF}=\int (Q\times \text{IF})\,dy+C$$

$$\Rightarrow \quad x\times y=\int 3y\times y\,dy+C=3\int y^2 dy+C$$

$$\Rightarrow \quad xy=\frac{3y^3}{3}+C \Rightarrow xy=y^3+C$$

which is the required solution.

EXAMPLE |5| Solve the differential equation $(x^2-1)\frac{dy}{dx}+2xy=\frac{2}{x^2-1}$.

[Delhi 2014; All India 2014C]

Sol. Given differential equation is $(x^2-1)\frac{dy}{dx}+2xy=\frac{2}{x^2-1}$

On dividing both sides by (x^2-1), we get

$$\frac{dy}{dx}+\frac{2x}{x^2-1}\cdot y=\frac{2}{(x^2-1)^2} \quad \text{...(i)}$$

which is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q$$

Here, $P=\frac{2x}{x^2-1}$ and $Q=\frac{2}{(x^2-1)^2}$

Now, $\text{IF}=e^{\int P dx}=e^{\int \frac{2x}{x^2-1}dx}=e^{\log(x^2-1)}=x^2-1$

$$\left[\begin{array}{l}\because \text{put } x^2-1=t \Rightarrow 2x\,dx=dt \\ \therefore \int \frac{2x}{x^2-1}dx=\int \frac{1}{t}dt=\log|t|=\log|x^2-1|\end{array}\right]$$

and the solution of differential equation is given by

$$y\times \text{IF}=\int (Q\times \text{IF})\,dx+C$$

$$\Rightarrow \quad y(x^2-1)=\int \frac{2}{(x^2-1)^2}\cdot(x^2-1)\,dx+C$$

$$\Rightarrow \quad y(x^2-1)=\int \frac{2}{(x^2-1)}dx+C$$

$$\Rightarrow \quad y(x^2-1)=\frac{2}{2\times 1}\log\left|\frac{x-1}{x+1}\right|+C$$

$$\because \quad \left[\int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C\right]$$

which is the required solution.

TOPIC PRACTICE 3

OBJECTIVE TYPE QUESTIONS

1 The integrating factor of differential equation $\cos x\frac{dy}{dx}+y\sin x=1$ is

(a) $\cos x$ (b) $\tan x$

(c) $\sec x$ (d) $\sin x$

2 The integrating factor of differential equation $(1-x^2)\frac{dy}{dx}-xy=1$ is

(a) $-x$ (b) $\frac{x}{1+x^2}$

(c) $\sqrt{1-x^2}$ (d) $\frac{1}{2}\log(1-x^2)$

3 The solution of $x\frac{dy}{dx}+y=e^x$ is

(a) $y=\frac{e^x}{x}+\frac{k}{x}$ (b) $y=xe^x+cx$

(c) $y=xe^x+k$ (d) $x=\frac{e^y}{y}+\frac{k}{y}$

4 The general solution of differential equation $\frac{dy}{dx}=e^{\frac{x^2}{2}}+xy$ is

(a) $y=Ce^{-x^2/2}$ (b) $y=Ce^{x^2/2}$

(c) $y=(x+C)e^{x^2/2}$ (d) $y=(C-x)e^{x^2/2}$

5 The solution of differential equation $\frac{dy}{dx}+\frac{y}{x}=\sin x$ is

(a) $x(y+\cos x)=\sin x+C$

(b) $x(y-\cos x)=\sin x+C$

(c) $xy\cos x=\sin x+C$

(d) $x(y+\cos x)=\cos x+C$

VERY SHORT ANSWER Type Questions

6 Find the integrating factor of the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1$. **[Delhi 2015]**

7 Find the integrating factor of $x\frac{dy}{dx}+2y=x\cos x$.

8 Find the integrating factor of $y\frac{dx}{dy}+x=y^3$.

9 Write the integrating factor of the differential equation

$$(1+y^2)+(2xy-\cot y)\frac{dy}{dx}=0.$$ **[All India 2015]**

SHORT ANSWER Type I Questions

10 Find the general solution of the differential equation $\frac{dy}{dx}+\frac{2y}{x}=x$. **[Delhi 2017C]**

11 Solve $\frac{dy}{dx}+y=\cos x-\sin x$.

12 Solve $\frac{dy}{dx}+2xy=y$.

13 Solve $\frac{dy}{dx}-2y=\cos 3x$.

14 Solve the differential equation $x\frac{dy}{dx}+y=x^3$, given that $y=1$, when $x=2$.

15 Solve $y\,dx+(x-y^3)\,dy=0$.

SHORT ANSWER I Type II Questions

16 Solve $x\frac{dy}{dx}-y=\log x$.

17 Solve the differential equation

$$\cos x\cdot\frac{dy}{dx}+2\sin x\cdot y=\sin x\cdot\cos x.$$

18 Solve $\cos^2 x\frac{dy}{dx}+y=\tan x$. **[NCERT]**

19 Solve the differential equation

$$(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}.$$ **[All India 2014]**

20 Find the general solution of the differential equation $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$. **[Delhi 2016; NCERT Exemplar]**

21 Find the general solution of the differential equation $(x+2y^3)\frac{dy}{dx}=y$.

22 Solve $(1+x^2)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}$, where $y=0$ and $x=1$. **[NCERT; Foreign 2011]**

23 Solve $(x^2+1)\frac{dy}{dx}-2xy=(x^2+2)(x^2+1)$. **[Delhi 2019]**

24 Solve the differential equation

$$(1+x^2)dy+2xy\,dx=\sec^2 x\,dx.$$

25 Find the general solution of the differential equation $(x\log x)\frac{dy}{dx}+y=\frac{2}{x}\log x$.

[Foreign 2014; Delhi 2010]

26 Solve the differential equation $(y+3x^2)\frac{dx}{dy}=x$.

[All India 2011]

27 Find the general solution of $(x+y)\frac{dy}{dx}=1$. **[NCERT]**

28 Solve the differential equation

$$(x^2-1)\frac{dy}{dx}+2xy=\frac{1}{x^2-1}.$$

29 Solve the differential equation $(\tan^{-1}x-y)\,dx=(1+x^2)\,dy$. **[All India 2017]**

30 If $y(t)$ is a solution of $(1+t)\frac{dy}{dt}-ty=1$ and $y(0)=-1$, then show that $y(1)=\frac{-1}{2}$.

[NCERT Exemplar]

31 Find the particular solution of differential equation $\frac{dy}{dx}=-\frac{x+y\cos x}{1+\sin x}$, given that $y=1$, when $x=0$. **[All India 2019, 2016]**

32 Find the equation of a curve passing through the point (0, 2), given that the sum of the coordinate of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at the point by 5. **[NCERT]**

33 Find the particular solution of the differential equation $\frac{dy}{dx}+2y\tan x=\sin x$, given that $y=0$ when $x=\frac{\pi}{3}$.

LONG ANSWER Type Questions

34 Solve the differential equation $dy=\cos x(2-y\,\text{cosec}\,x)dx$ given that $y=2$, when $x=\pi/2$.

35 Find the particular solution of the differential equation $x\frac{dy}{dx}+y-x+xy\cot x=0, x\neq 0$, given that when $x=\frac{\pi}{2}, y=0$. **[Delhi 2015C, 2011C]**

36 Solve $\frac{dy}{dx}-3y\cot x=\sin 2x$, where $y=2$ and $x=\frac{\pi}{2}$. **[NCERT; All India 2015C]**

37 Solve the following differential equation $\frac{dy}{dx}+y\cot x=4x\,\text{cosec}\,x$, given that $y=0$, when $x=\pi/2$. **[Delhi 2012C; Foreign 2011]**

38 Solve the differential equation $(x^2+1)\frac{dy}{dx}+2xy=\sqrt{x^2+4}$. **[Delhi 2010]**

39 Solve the differential equation $x\frac{dy}{dx}+y=x\cos x+\sin x$, given that $y=1$ when $x=\frac{\pi}{2}$. **[Delhi 2017]**

40 Solve the following initial value problem $(x^2+1)\,y'-2xy=(x^4+2x^2+1)\cos x,\ y(0)=0$.

41 Find the particular solution of the differential equation $(\tan^{-1}y-x)dy=(1+y^2)dx$, given that $x=1$, when $y=0$. **[Delhi 2015; All India 2015]**

42 Solve the initial value problem $ye^y\,dx=(y^3+2xe^y)dy,\ y(0)=1$.

43 Solve the following differential equation.

$$(1+x^2)\,dy+2xy\,dx=\cot x\,dx,\ x\neq 0$$

[All India 2012C, 2011]

44 Solve the initial value problem $(x-\sin y)\,dy+(\tan y)\,dx=0,\ y(0)=0$.

45 An equation relating to the stability of an aeroplane is given by $\frac{dv}{dt}=g\cos\alpha-kv$, where v is the velocity and g, α, k are constants. Find an expression for the velocity, if $v=0$ at $t=0$.

HINTS & SOLUTIONS

1. (c) Given that $\cos x\frac{dy}{dx}+y\sin x=1$

$$\Rightarrow \quad \frac{dy}{dx}+y\tan x=\sec x$$

Here, $P=\tan x$ and $Q=\sec x$

$$\therefore \quad \text{IF}=e^{\int P\,dx}=e^{\int \tan x\,dx}=e^{\log\sec x}=\sec x$$

2. (c) Given that $(1-x^2)\frac{dy}{dx}-xy=1$

$$\Rightarrow \quad \frac{dy}{dx}-\frac{x}{1-x^2}y=\frac{1}{1-x^2}$$

which is a linear differential equation.

$\therefore \quad IF = e^{-\int \frac{x}{1-x^2}dx}$

Put $\quad 1 - x^2 = t \Rightarrow -2xdx = dt$

$\Rightarrow \quad xdx = -\frac{dt}{2}$

Now, $IF = e^{\frac{1}{2}\int \frac{dt}{t}} = e^{\frac{1}{2}\log t} = e^{\frac{1}{2}\log(1-x^2)} = \sqrt{1-x^2}$

3. (a) Given that $\quad x\frac{dy}{dx} + y = e^x$

$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = \frac{e^x}{x}$

which is a linear differential equation.

$\therefore \quad IF = e^{\int \frac{1}{x}dx} = e^{(\log x)} = x$

The general solution is $\quad y \cdot x = \int \left(\frac{e^x}{x} \cdot x\right) dx$

$\Rightarrow \quad y \cdot x = \int e^x dx$

$\Rightarrow \quad y \cdot x = e^x + k$

$\Rightarrow \quad y = \frac{e^x}{x} + \frac{k}{x}$

4. (c) Given that $\quad \frac{dy}{dx} = e^{x^2/2} + xy$

$\Rightarrow \quad \frac{dy}{dx} - xy = e^{x^2/2}$

Here, $\quad P = -x,\ Q = e^{x^2/2}$

$\therefore \quad IF = e^{\int -x\,dx} = e^{-x^2/2}$

The general solution is

$$y \cdot e^{-x^2/2} = \int e^{-x^2/2} e^{x^2/2}\, dx + C$$

$\Rightarrow \quad y\, e^{-x^2/2} = \int 1\, dx + C$

$\Rightarrow \quad y \cdot e^{-x^2/2} = x + C$

$\Rightarrow \quad y = (x + C)\, e^{x^2/2}$

5. (a) Given differential equation is

$$\frac{dy}{dx} + y\frac{1}{x} = \sin x$$

which is linear differential equation.

Here, $\quad P = \frac{1}{x}$ and $Q = \sin x$

$\therefore \quad IF = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

The general solution is

$$y \cdot x = \int x \cdot \sin x\, dx + C \qquad \text{...(i)}$$

$\Rightarrow \quad xy = -x\cos x + \sin x + C$ [integrating by parts]

$\Rightarrow \quad x(y + \cos x) = \sin x + C$

6. The given differential equation can be rewritten as

$$\frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$$

$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \frac{1}{\sqrt{x}}$ and $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$.

Now, $IF = e^{\int P dx} = e^{\int \frac{1}{\sqrt{x}}dx} = e^{\int \frac{1}{\sqrt{x}}dx} = e^{2\sqrt{x}}$

7. Solve as Question 6. [**Ans.** x^2]

8. Similar as Example 1. [**Ans.** y]

9. Given equation can be rewritten as

$$(\cot y - 2xy)\frac{dy}{dx} = 1 + y^2$$

$\Rightarrow \quad \frac{\cot y - 2xy}{(1 + y^2)} = \frac{dx}{dy}$

$\Rightarrow \quad \frac{dx}{dy} = \frac{\cot y}{1 + y^2} - \frac{2xy}{1 + y^2}$

$\Rightarrow \quad \frac{dx}{dy} + \frac{2y}{1 + y^2} \cdot x = \frac{\cot y}{1 + y^2}$

which is a linear differential equation of the form

$$\frac{dx}{dy} + Px = Q$$

Here, $P = \frac{2y}{1 + y^2}$ and $Q = \frac{\cot y}{1 + y^2}$.

Now, $IF = e^{\int P dy} = e^{\int \frac{2y}{1+y^2}dy}$

Put $1 + y^2 = t \Rightarrow 2y\, dy = dt$

$\therefore \quad IF = e^{\int \frac{dt}{t}} = e^{\log(t)} = t = 1 + y^2$

10. Hint (i) $IF = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$

(ii) Solution is given by $y \cdot x^2 = \int x \cdot x^2 dx + C$

$\left[\textbf{Ans. } y = \frac{x^2}{4} + Cx^{-2}\right]$

11. Hint

(i) $IF = e^x$

(ii) Use the formula, $\int e^x[f(x) + f'(x)]dx = e^x f(x) + C$ to simplify the solution. [**Ans.** $ye^x = e^x \cos x + C$]

12. Given that $\frac{dy}{dx} + 2xy = y$

$\Rightarrow \quad \frac{dy}{dx} + 2xy - y = 0 \Rightarrow \frac{dy}{dx} + (2x - 1)y = 0$

which is a linear differential equation.

On comparing it with $\frac{dy}{dx} + Py = Q$, we get

$$P = (2x - 1) \text{ and } Q = 0$$

Now, $\quad \text{IF} = e^{\int P dx} = e^{\int (2x-1)dx} = e^{\left(\frac{2x^2}{2}-x\right)} = e^{x^2-x}$

and the complete solution is given by

$$y \cdot e^{x^2-x} = \int (Q \cdot e^{x^2-x}) dx + C$$

$$\Rightarrow \quad y \cdot e^{x^2-x} = 0 + C \Rightarrow y = C\, e^{x-x^2}$$

13. Hint (i) $\text{IF} = e^{\int -2dx} = e^{-2x}$

(ii) For simplifying the solution.

Let $I = \int \underset{\text{I}}{e^{-2x}} \underset{\text{II}}{\cos} 3x\, dx$

$$= e^{-2x} \cdot \frac{\sin 3x}{3} - \int e^{-2x} (-2) \frac{\sin 3x}{3} dx$$

$$= \frac{e^{-2x} \sin 3x}{3} + \frac{2}{3} \int \underset{\text{I}}{e^{-2x}} \underset{\text{II}}{\sin 3x}\, dx$$

$$= \frac{e^{-2x} \sin 3x}{3} + \frac{2}{3}$$

$$\left[\int e^{-2x}\left(\frac{-\cos 3x}{3}\right) - \int e^{-2x}(-2)\left(\frac{-\cos 3x}{3}\right) dx\right]$$

$$= \frac{e^{-2x} \sin 3x}{3} - \frac{2}{9} e^{-2x} \cos 3x - \frac{4}{9} I$$

$$\left[\textbf{Ans. } y = \frac{1}{13}(3\sin 3x - 2\cos 3x) + Ce^{2x}\right]$$

14. Similar as Question 10.

$$\left[\textbf{Ans. } xy = \frac{x^4}{4} - 2\right]$$

15. Hint (i) Write the given differential equation as

$$\frac{dx}{dy} + \frac{x}{y} = y^2.$$

(ii) Similar as Example 4. $\left[\textbf{Ans. } xy = \frac{y^4}{4} + C\right]$

16. Given differential equation is

$$x\frac{dy}{dx} - y = \log x$$

and it can be rewritten as $\frac{dy}{dx} - \frac{y}{x} = \frac{\log x}{x}$,

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $\quad P = -\frac{1}{x}$ and $Q = \frac{\log x}{x}$.

Now, $\text{IF} = e^{\int \frac{-1}{x} dx} = e^{-\log x} = x^{-1}$

and the solution of differential equation is given by

$$y \cdot x^{-1} = \int \left(\frac{\log x}{x} \cdot x^{-1}\right) dx + C$$

$$\Rightarrow \quad \frac{y}{x} = \int \frac{\log x}{x^2} dx + C \quad \text{...(i)}$$

Now, put $\log x = t$

$$\Rightarrow \quad \frac{1}{x} dx = dt$$

$$\Rightarrow \quad x = e^t$$

$\therefore$ From Eq. (i), we get

$$\frac{y}{x} = \int \frac{t}{e^t} dt + C \Rightarrow \frac{y}{x} = \int (\underset{\text{I}}{t} \cdot \underset{\text{II}}{e^{-t}}) dt + C$$

$$\Rightarrow \quad \frac{y}{x} = t \cdot \frac{e^{-t}}{(-1)} - \int 1 \cdot \frac{e^{-t}}{(-1)} dt + C$$

[using integrating by parts]

$$\Rightarrow \quad \frac{y}{x} = -t\, e^{-t} + \int e^{-t} dt + C$$

$$\Rightarrow \quad \frac{y}{x} = -t\, e^{-t} - e^{-t} + C$$

$$\Rightarrow \quad \frac{y}{x} = -\log x\left(\frac{1}{x}\right) - \frac{1}{x} + C$$

$$[\because e^{-t} = e^{-\log x} = e^{\log x^{-1}} = x^{-1}]$$

$$\Rightarrow \quad y = -\log x - 1 + Cx$$

$$\Rightarrow \quad y = Cx - (\log x + 1),$$

which is the required solution.

17. Given differential equation is

$$\cos x \frac{dy}{dx} + 2 \sin x \cdot y = \sin x \cdot \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2 \sin x}{\cos x} \cdot y = \frac{\sin x \cdot \cos x}{\cos x}$$

$$\therefore \quad \frac{dy}{dx} + 2 \tan x \cdot y = \sin x \quad \text{...(i)}$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = 2 \tan x$ and $Q = \sin x$.

Now, $\text{IF} = e^{\int P dx} = e^{\int 2 \tan x\, dx} = e^{2 \log \sec x}$

$$[\because \int \tan x\, dx = \log \sec x]$$

$$= e^{\log \sec^2 x} = \sec^2 x$$

and the required solution is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\, dx + C$$

$$\therefore \quad y \cdot \sec^2 x = \int \sin x \cdot \frac{1}{\cos^2 x} dx + C$$

$$\Rightarrow \quad y \cdot \sec^2 x = \int \sec x \cdot \tan x\, dx + C$$

$$\Rightarrow \quad y \cdot \sec^2 x = \sec x + C \quad [\because \int \sec x \tan x\, dx = \sec x]$$

$$\Rightarrow \quad y = \cos x + C \cos^2 x \quad [\text{dividing by } \sec^2 x]$$

which is the required solution.

18. Hint

(i) Given differential equation can be rewritten as

$$\frac{dy}{dx} + \sec^2 x\, y = \sec^2 x \tan x$$

Here, $P = \sec^2 x$ and $Q = \sec^2 x \cdot \tan x$.

(ii) $\text{IF} = e^{\tan x}$ $[\textbf{Ans. } y \cdot e^{\tan x} = e^{\tan x}(\tan x - 1) + C]$

19. Hint Write the given differential equation as

$$\frac{dy}{dx}+\frac{y}{1+x^2}=\frac{e^{\tan^{-1}x}}{(1+x^2)}$$

$\therefore$ IF $= e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$ and required solution is given by

$y\,e^{\tan^{-1}x}=\int \frac{e^{2\tan^{-1}x}}{(1+x^2)}\,dx.$ $\left[\textbf{Ans. } y e^{\tan^{-1}x}=\frac{e^{2\tan^{-1}x}}{2}+C\right]$

20. Given differential equation is $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$

$$\Rightarrow \quad (1+y^2)=-(x-e^{\tan^{-1}y})\frac{dy}{dx}$$

$$\Rightarrow \quad (1+y^2)\frac{dx}{dy}=-x+e^{\tan^{-1}y}$$

$$\Rightarrow \quad (1+y^2)\frac{dx}{dy}+x=e^{\tan^{-1}y}$$

$$\Rightarrow \quad \frac{dx}{dy}+\frac{x}{1+y^2}=\frac{e^{\tan^{-1}y}}{1+y^2}$$

[dividing both sides by $(1+y^2)$]

which is a linear differential equation of the form.

$$\frac{dx}{dy}+Px=Q$$

Here, $P=\frac{1}{1+y^2}$ and $Q=\frac{e^{\tan^{-1}y}}{1+y^2}$.

Now, IF $= e^{\int P dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

and the required solution is given by

$$x\cdot \text{IF}=\int Q\cdot \text{IF}\,dy+C_1$$

$$\Rightarrow \quad x\cdot e^{\tan^{-1}y}=\int \frac{e^{\tan^{-1}y}}{1+y^2}\cdot e^{\tan^{-1}y}dy+C_1$$

$$\Rightarrow \quad x\cdot e^{\tan^{-1}y}=\int \frac{(e^{\tan^{-1}y})^2}{1+y^2}dy+C_1$$

Put $\tan^{-1}y=t$, then $\frac{1}{1+y^2}dy=dt$

$$\therefore \quad x\cdot e^{\tan^{-1}y}=\int e^{2t}dt+C_1$$

$$\Rightarrow \quad x\cdot e^{\tan^{-1}y}=\frac{1}{2}e^{2\tan^{-1}y}+C_1$$

$$\Rightarrow \quad 2x\,e^{\tan^{-1}y}=e^{2\tan^{-1}y}+2C_1$$

$$\Rightarrow \quad 2x\,e^{\tan^{-1}y}=e^{2\tan^{-1}y}+C \qquad [\because C=2C_1]$$

21. Similar as Example 4. [**Ans.** $x=y^3+Cy$]

22. Hint The solution of given differential equation is

$$y(1+x^2)=\tan^{-1}x+C \qquad \text{...(i)}$$

Also, given $y=0$ and $x=1$, then from Eq. (i), we get

$$0(1+1)=\tan^{-1}(1)+C$$

$$\Rightarrow \quad 0=\frac{\pi}{4}+C$$

$$\Rightarrow \quad C=-\frac{\pi}{4}$$

$\left[\textbf{Ans. } (1+x^2)\,y=\tan^{-1}x-\frac{\pi}{4}\right]$

23. Hint (i) Here, $P=\frac{-2x}{x^2+1}$ and $Q=x^2+2$

(ii) IF $=\frac{1}{x^2+1}$

(iii) The required solution is given by

$$\frac{y}{x^2+1}=\int\left(1+\frac{1}{x^2+1}\right)dx+C$$

[**Ans.** $y=(1+x^2)(x+\tan^{-1}x+C)$]

24. Hint (i) Given differential equation can be rewritten as

$$\frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{\sec^2 x}{1+x^2}$$

Here, $P=\frac{2x}{1+x^2}$ and $Q=\frac{\sec^2 x}{1+x^2}$.

(ii) IF $=1+x^2$

(iii) The required solution is given by

$$y\cdot(1+x^2)=\int \sec^2 x\,dx+C$$

$\left[\textbf{Ans. } y=\frac{\tan x}{1+x^2}+\frac{C}{1+x^2}\right]$

25. Hint Given differential equation can be rewritten as

$$\frac{dy}{dx}+\frac{y}{x\log x}=\frac{2}{x^2}$$

IF $= e^{\int \frac{1}{x\log x}dx}=e^{\log\log x}=\log x$

So, required solution is

$$y\log x=\int \underset{\text{II}}{\frac{2}{x^2}}\underset{\text{I}}{\log x}\,dx+C$$

$$=\log x\times 2\left(-\frac{1}{x}\right)-\int \frac{2}{x}\left(-\frac{1}{x}\right)dx+C$$

$\left[\textbf{Ans. } y\log x=-\frac{2}{x}\log x-\frac{2}{x}+C\right]$

26. Hint Write given equation as $\frac{dy}{dx}-\frac{y}{x}=3x$ and solve it.

[**Ans.** $y=3x^2+Cx$]

27. Hint Write the given equation as $\frac{dx}{dy}-x=y$ and solve it.

[**Ans.** $x+y+1=Ce^y$]

28. Similar as Example 5.

$\left[\textbf{Ans. } (x^2-1)y=\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+C\right]$

29. Given, $(\tan^{-1}x - y)\,dx = (1+x^2)\,dy$

$$\Rightarrow \frac{dy}{dx} = \frac{\tan^{-1}x - y}{(1+x^2)} \Rightarrow \frac{dy}{dx} = \frac{\tan^{-1}x}{1+x^2} - \frac{1}{1+x^2}y$$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{1+x^2}y = \frac{\tan^{-1}x}{1+x^2} \quad \text{...(i)}$$

The above differential equation is linear differential equation.

On comparing Eq. (i) with $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{1}{1+x^2} \text{ and } Q = \frac{\tan^{-1}x}{1+x^2}$$

Now, $\quad \text{IF} = e^{\int P\,dx} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$

$\therefore$ Solution is given by

$$y \cdot \text{IF} = \int Q \cdot \text{IF}\,dx + C$$

$$\Rightarrow y \cdot e^{\tan^{-1}x} = \int \frac{\tan^{-1}x}{1+x^2} \cdot e^{\tan^{-1}x}dx + C$$

Put $t = \tan^{-1}x \Rightarrow dt = \frac{1}{1+x^2}dx$

$$\therefore \quad ye^{\tan^{-1}x} = \int t \cdot e^t dt + C = t \cdot e^t - \int 1 \cdot e^t\,dt + C$$

[integration by parts]

$$\Rightarrow ye^{\tan^{-1}x} = t \cdot e^t - e^t + C$$

$$\Rightarrow ye^{\tan^{-1}}x = \tan^{-1}x \cdot e^{\tan^{-1}x} - e^{\tan^{-1}x} + C \quad [\because t = e^{\tan^{-1}x}]$$

$$\Rightarrow ye^{\tan^{-1}x} = (\tan^{-1}x - 1)e^{\tan^{-1}x} + C$$

30. Given differential equation is $(1+t)\frac{dy}{dx} - ty = 1$

and it can be rewritten as $\frac{dy}{dt} - \frac{t}{(1+t)}y = \frac{1}{1+t}$

Here, $P = \frac{-t}{1+t}$ and $Q = \frac{1}{1+t}$

Now, $\text{IF} = e^{\int \frac{-t}{1+t}dt} = e^{-\int \left(1 - \frac{1}{1+t}\right)dt} = e^{-[t - \log(1+t)]}$

$$= e^{-t} \cdot e^{\log(1+t)} = (1+t)e^{-t}$$

and the solution of differential equation is given by

$$y(1+t)e^{-t} = \int (1+t) \cdot e^{-t} \cdot \frac{1}{(1+t)}dt + C$$

$$\Rightarrow y \cdot (1+t)e^{-t} = \int e^{-t}dt + C$$

$$\Rightarrow y \cdot (1+t)\,e^{-t} = -e^{-t} + C \quad \text{...(i)}$$

Also, it is given that $y = -1$, when $t = 0$.

$\therefore$ From Eq. (i), we get

$$(-1)(1+0)\,e^{-0} = -e^{-0} + C$$

$$\Rightarrow \quad -1 = -1 + C \Rightarrow C = 0$$

Now, from Eq. (i), we get $y = \frac{-1}{(1+t)}$

$$\therefore \quad y(1) = \frac{-1}{(1+1)} = \frac{-1}{2}$$

Hence proved.

31. Given differential equation is

$$\frac{dy}{dx} = -\frac{x + y\cos x}{1 + \sin x}$$

It can be rewritten as

$$\frac{dy}{dx} + \frac{\cos x}{1+\sin x}y = -\frac{x}{1+\sin x}$$

This is linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = \frac{\cos x}{1+\sin x}$ and $Q = -\frac{x}{1+\sin x}$.

Now, integrating factor $\text{IF} = e^{\int P\,dx} = e^{\int \frac{\cos x}{1+\sin x}dx}$

Put $1 + \sin x = t \Rightarrow \cos x\,dx = dt$

$$\therefore \quad \text{IF} = e^{\int \frac{dt}{t}} = e^{\log|t|}$$

$$= |t| = 1 + \sin x \quad [\text{putting } t = 1 + \sin x]$$

The general solution is given by

$$y \cdot \text{IF} = \int Q \cdot \text{IF}\,dx + C$$

$$\therefore \quad y \times (1+\sin x) = \int -\frac{x}{1+\sin x} \cdot (1+\sin x) + C$$

$$\Rightarrow y(1+\sin x) = \int -x\,dx + C$$

$$\Rightarrow y(1+\sin x) = -\frac{x^2}{2} + C \quad \text{...(i)}$$

Also, given $y = 1$, when $x = 0$

$$\therefore \quad 1(1+\sin 0) = -\frac{0}{2} + C$$

$$\Rightarrow \quad C = 1 + 0 = 1$$

On putting $C = 1$ in Eq. (i), we get

$$y(1+\sin x) = -\frac{x^2}{2} + 1$$

Hence, particular solution of the given differential equation is $y(1+\sin x) = -\frac{x^2}{2} + 1$.

32. According to the question, the differential equation is

$$x + y = \frac{dy}{dx} + 5$$

$$\Rightarrow \frac{dy}{dx} + (-1)\,y = x - 5 \quad \text{...(i)}$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = -1$ and $Q = x - 5$

Now, $\text{IF} = e^{\int P\,dx} = e^{\int (-1)\,dx} = e^{-x}$

and the required solution is given by

$$y \times \text{IF} = \int Q \times \text{IF}\,dx + C$$

$$\Rightarrow y \cdot e^{-x} = \int \underset{\text{I}}{(x-5)}\underset{\text{II}}{e^{-x}}\,dx + C$$

$$\Rightarrow y \cdot e^{-x} = (x-5)\int e^{-x}dx - \int \left[\frac{d}{dx}(x-5) \cdot \int e^{-x}\,dx\right]dx + C$$

[integration by parts]

$\Rightarrow \quad y \cdot e^{-x} = (x-5)(-e^{-x}) - \int (-e^{-x})\, dx + C$

$\Rightarrow \quad y \cdot e^{-x} = (5-x)\, e^{-x} - e^{-x} + C$ ...(i)

Since, the curve passes through the point (0, 2), therefore we have

$$2 \cdot e^{-0} = (5-0)\, e^{-0} - e^{-0} + C$$

$\Rightarrow \quad 2 = 5 - 1 + C$

$\Rightarrow \quad C = 2 - 4 = -2$

On putting the value of C in Eq. (i), we get

$$e^{-x}\; y = (5-x)\, e^{-x} - e^{-x} - 2$$

$\Rightarrow \quad y = 4 - x - 2e^{x}$ [multiplying both sides by e^{x}]

which is the required equation of the curve.

33. Given differential equation is

$$\frac{dy}{dx} + 2y \tan x = \sin x$$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$.

Here, $P = 2\tan x$ and $Q = \sin x$

Now, $\quad \text{IF} = e^{\int P dx} = e^{\int 2 \tan x\, dx}$

$= e^{2\int \tan x\, dx} = e^{2 \log|\sec x|}$

$= e^{\log \sec^2 x} = \sec^2 x \qquad [\because e^{\log f(x)} = f(x)]$

Then, the solution of differential equation is given by

$$y \cdot (\text{IF}) = \int (\text{IF}) Q\, dx + C$$

$\Rightarrow \quad y \cdot \sec^2 x = \int \sec^2 x \cdot \sin x\, dx + C$

$\Rightarrow \quad y \cdot \sec^2 x = \int \frac{\sin x}{\cos^2 x}\, dx + C$

Put $\cos x = t$, then $-\sin x\, dx = dt$

$\therefore \quad y \cdot \sec^2 x = -\int \frac{dt}{t^2} + C$

$\Rightarrow \quad y \cdot \sec^2 x = -\int t^{-2} dt + C$

$\Rightarrow \quad y \cdot \sec^2 x = -1 \frac{t^{-1}}{(-1)} + C$

$\Rightarrow \quad y \cdot \sec^2 x = \frac{1}{t} + C$

$\Rightarrow \quad y \cdot \sec^2 x = \frac{1}{\cos x} + C \qquad [\because t = \cos x]$

$\Rightarrow \quad y \cdot \sec^2 x = \sec x + C$

$\Rightarrow \quad y = \frac{1}{\sec x} + \frac{C}{\sec^2 x}$

[dividing each term by $\sec^2 x$]

$\Rightarrow \quad y = \cos x + C \cdot \cos^2 x$

Now, it given that $y = 0$ when $x = \frac{\pi}{3}$

$\therefore \quad 0 = \cos \frac{\pi}{3} + C \cdot \cos^2 \frac{\pi}{3}$

$\Rightarrow \quad 0 = \frac{1}{2} + C.\frac{1}{4} \qquad \left[\because \cos \frac{\pi}{3} = \frac{1}{2}\right]$

$\Rightarrow \quad \frac{-1}{2} = \frac{C}{4} \Rightarrow C = -2$

$\therefore$ The required particular solution is

$$y = \cos x - 2\cos^2 x$$

34. Given differential equation is

$$dy = \cos x(2 - y \operatorname{cosec} x) dx$$

$\Rightarrow \quad \frac{dy}{dx} = \cos x(2 - y \operatorname{cosec} x) = 2\cos x - y \operatorname{cosec} x \cdot \cos x$

$\Rightarrow \quad \frac{dy}{dx} = 2\cos x - y \cot x$

$\Rightarrow \quad \frac{dy}{dx} + y \cot x = 2\cos x$

which is a linear differential equation of the form of

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \cot x$ and $Q = 2\cos x$.

Now, $\text{IF} = e^{\int P dx} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$

and the required solution is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) dx + C$$

$$y \cdot \sin x = \int 2\cos x \cdot \sin x\, dx + C$$

$\Rightarrow \quad y \cdot \sin x = \int \sin 2x\, dx + C \qquad [\because \sin 2x = 2 \sin x \cos x]$

$\Rightarrow \quad y \cdot \sin x = -\frac{\cos 2x}{2} + C$ ...(i)

Also, given $y = 2$, when $x = \frac{\pi}{2}$, therefore we have

$$2 \cdot \sin \frac{\pi}{2} = -\frac{\cos\left(2 \times \frac{\pi}{2}\right)}{2} + C$$

$\Rightarrow \quad 2 \cdot 1 = +\frac{1}{2} + C$

$\Rightarrow \quad 2 - \frac{1}{2} = C$

$\Rightarrow \quad \frac{4-1}{2} = C \Rightarrow C = \frac{3}{2}$

On putting the value of C in Eq. (i), we get

$$y \sin x = -\frac{1}{2} \cos 2x + \frac{3}{2}$$

35. Given, $x\frac{dy}{dx} + y - x + xy \cot x = 0$

$\Rightarrow \quad x\frac{dy}{dx} + y(1 + x \cot x) = x$

On dividing both sides by x, we get

$$\frac{dy}{dx} + \frac{y(1 + x\cot x)}{x} = 1$$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$.

Here, $\quad P = \frac{1 + x \cot x}{x}$ and $Q = 1$.

Now, $\text{IF} = e^{\int P\,dx} = e^{\int\left(\frac{1}{x}+\cot x\right)dx}$

$= e^{\log|x| + \log|\sin x|} = e^{\log|x \sin x|}$

$= x \sin x \qquad [\because e^{\log f(x)} = f(x)]$

and the required solution is given by

$$y \cdot \text{IF} = \int (Q \cdot \text{IF})\,dx + C$$

$$\Rightarrow \quad y(x \sin x) = \int \underset{\text{I}}{x} \underset{\text{II}}{\sin x}\,dx + C$$

$$\Rightarrow \quad y(x\sin x) = x\int \sin x\,dx - \int\left[\frac{d}{dx}(x)\int \sin x\,dx\right]dx$$

[integration by parts]

$$\Rightarrow \quad y(x\sin x) = -x\cos x + \int \cos x\,dx$$

$$\Rightarrow \quad y(x\sin x) = -x\cos x + \sin x + C \qquad \text{...(i)}$$

Also, it is given that when $x = \frac{\pi}{2}, y = 0$, therefore from Eq. (i), we get

$$0 = -\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + C$$

$$0 = -\frac{\pi}{2}(0) + 1 + C \Rightarrow C = -1$$

Hence, the required particular solution is

$xy \sin x = -x\cos x + \sin x - 1.$

36. Given, $\frac{dy}{dx} - (3\cot x)y = \sin 2x$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = -3\cot x$ and $Q = \sin 2x$.

Now, $\text{IF} = e^{\int P\,dx} = e^{-3\int \cot x\,dx} = e^{-3\log(\sin x)} = e^{\log(\sin x)^{-3}}$

$$= \frac{1}{\sin^3 x}$$

and the required solution is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\,dx + C$$

$$\Rightarrow \quad y \times \frac{1}{\sin^3 x} = \int \frac{1}{\sin^3 x}\sin 2x\,dx + C$$

$$\Rightarrow \quad y \times \frac{1}{\sin^3 x} = 2\int \frac{\sin x \cos x}{\sin^3 x}\,dx + C$$

$[\because \sin 2x = 2\sin x \cos x]$

$$\Rightarrow \quad \frac{1}{\sin^3 x} \times y = 2\int \frac{\cos x}{\sin^2 x}\,dx + C$$

$$\Rightarrow \quad \frac{y}{\sin^3 x} = 2\int \cot x \operatorname{cosec} x\,dx + C$$

$$\Rightarrow \quad \frac{y}{\sin^3 x} = -2\operatorname{cosec} x + C$$

$$\Rightarrow \quad y = -2\left(\frac{1}{\sin x} \times \sin^3 x\right) + C\sin^3 x$$

$$\Rightarrow \quad y = -2\sin^2 x + C\sin^3 x \qquad \text{...(i)}$$

Also, given $y = 2$ and $x = \frac{\pi}{2}$, therefore from Eq. (i), we get

$$2 = -2\sin^2\left(\frac{\pi}{2}\right) + C\sin^3\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad 2 = -2 + C$$

$$\Rightarrow \quad C = 4$$

On putting the value of C in Eq. (i), we get

$y = -2\sin^2 x + 4\sin^3 x \Rightarrow y = 4\sin^3 x - 2\sin^2 x$

which is the required solution.

37. Similar as Question 36.

$\left[\textbf{Ans. } y = 2x^2 \operatorname{cosec} x - \frac{\pi^2}{2}\operatorname{cosec} x\right]$

38. Given differential equation is

$$(x^2+1)\frac{dy}{dx} + 2xy = \sqrt{x^2+4}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2xy}{(1+x^2)} = \frac{\sqrt{x^2+4}}{(1+x^2)}$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \frac{2x}{1+x^2}$ and $Q = \frac{\sqrt{x^2+4}}{(1+x^2)}$.

Now, $\text{IF} = e^{\int P\,dx} = e^{\int \frac{2x}{(1+x^2)}dx}$

$= e^{\log|1+x^2|} = (1+x^2)$

and the required solution is given by

$$y \cdot \text{IF} = \int (Q \cdot \text{IF})\,dx + C$$

$$\Rightarrow \quad y \cdot (1+x^2) = \int \frac{\sqrt{x^2+4}}{(1+x^2)} \cdot (1+x^2)\,dx + C$$

$$\Rightarrow \quad y \cdot (1+x^2) = \int \sqrt{x^2+(2)^2}\,dx + C$$

$$\Rightarrow \quad y \cdot (1+x^2) = \frac{x}{2}\sqrt{x^2+4} + \frac{4}{2}\log\left|x + \sqrt{x^2+4}\right| + C$$

$$\left[\because \int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x + \sqrt{x^2+a^2}|\right]$$

$$\therefore \quad y(1+x^2) = \frac{x}{2}\sqrt{x^2+4} + 2\log|x + \sqrt{x^2+4}| + C$$

which is the required solution.

39. Given differential equation is

$$x\frac{dy}{dx} + y = x\cos x + \sin x \qquad \text{...(i)}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = \cos x + \frac{\sin x}{x}$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q,$$

where $P = \frac{1}{x}$ and $Q = \cos x + \frac{\sin x}{x}$

$$\text{IF} = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

and the solution of differential equation is given by

$$y \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dx + C$$

$$\Rightarrow \quad yx = \int (x \cos x + \sin x)\, dx + C$$

$$\Rightarrow \quad xy = \int x \cos x\, dx + \int \sin dx + C$$

$$\Rightarrow \quad xy = x \sin x - \int \sin dx + \int \sin x\, dx + C$$

[integrating Ist integral by parts]

$$\Rightarrow \quad xy = x \sin x + C$$

Put $x = \frac{\pi}{2}$ and $y = 1$, we get

$$\frac{\pi}{2} \times 1 = \frac{\pi}{2} \sin \frac{\pi}{2} + C$$

$$\Rightarrow \quad \frac{\pi}{2} = \frac{\pi}{2} + C$$

$$\Rightarrow \quad C = 0$$

Now, $xy = x \sin x + 0 \Rightarrow y = \sin x$

40. **Hint** Write the given differential equation as

$$\frac{dy}{dx} - \frac{2xy}{(x^2+1)} = \frac{(x^2+1)^2}{(x^2+1)} \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{2xy}{x^2+1} = (x^2+1) \cos x$$

and $\quad \text{IF} = e^{\int \frac{-2x}{1+x^2} dx} = e^{-\log|x^2+1|} = (x^2+1)^{-1}$

[**Ans.** $y = (x^2+1) \sin x$]

41. Given differential equation is

$$(\tan^{-1} y - x) dy = (1+y^2) dx$$

$$\Rightarrow \quad \frac{\tan^{-1} y - x}{1+y^2} = \frac{dx}{dy}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{-x}{1+y^2} + \frac{\tan^{-1} y}{1+y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{1+y^2} \cdot x = \frac{\tan^{-1} y}{1+y^2}$$

which is a linear differential equation of first order.

On comparing with $\frac{dx}{dy} + Px = Q$, we get

$$P = \frac{1}{1+y^2} \text{ and } Q = \frac{\tan^{-1} y}{1+y^2}$$

Now, $\text{IF} = e^{\int P dy} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y}$

and the required solution is given by

$$x \cdot \text{IF} = \int Q \cdot \text{IF}\, dy + C$$

$$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} \times e^{\tan^{-1} y}\, dy + C$$

Put $t = \tan^{-1} y$, then $dt = \frac{1}{1+y^2} dy$

$$\therefore \quad x \cdot e^{\tan^{-1} y} = \int t \cdot e^t\, dt + C$$

$$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = t \cdot e^t - \int 1 \cdot e^t dt + C \quad \text{[integration by parts]}$$

$$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = t \cdot e^t - e^t + C$$

$$\Rightarrow x \cdot e^{\tan^{-1} y} = (\tan^{-1} y - 1) e^{\tan^{-1} y} + C$$

[putting $t = \tan^{-1} y$] ...(i)

$\because$ It is given that $x = 1$, when $y = 0$.

Therefore, we have

$$1 \cdot e^0 = (0 - 1) + C$$

$$\Rightarrow \quad 1 = -1 + C$$

$$\Rightarrow \quad C = 2$$

Hence, $\quad x \cdot e^{\tan^{-1} y} = e^{\tan^{-1} y} (\tan^{-1} y - 1) + 2$

which is the required particular solution of the differential equation.

42. **Hint** Write the given differential equation as

$$\frac{dx}{dy} = y^2 e^{-y} + \frac{2x}{y}$$

$$\Rightarrow \quad \frac{dx}{dy} - \frac{2}{y} x = y^2 e^{-y}$$

[**Ans.** $x = y^2 (e^{-1} - e^{-y})$]

43. Given differential equation is

$$(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx \qquad [\because x \neq 0]$$

Above equation can be rewritten as,

$$(1 + x^2)\, dy + (2xy - \cot x)\, dx = 0$$

$$\Rightarrow (1 + x^2)\, dy = (\cot x - 2xy)\, dx$$

On dividing both sides by $1 + x^2$, we get

$$dy = \frac{\cot x - 2xy}{1 + x^2} dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cot x}{1 + x^2} - \frac{2xy}{1 + x^2}$$

$$\Rightarrow \frac{dy}{dx} + \frac{2x}{1 + x^2} y = \frac{\cot x}{1 + x^2}$$

which is a linear differential equation of the form of

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \frac{2x}{1+x^2}$ and $Q = \frac{\cot x}{1+x^2}$

Now, $\text{IF} = e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log|1+x^2|} = 1 + x^2$

$$\left[\because I_1 = \int \frac{2x}{1+x^2} dx, \text{ put } 1 + x^2 = t \Rightarrow 2x\, dx = dt \right.$$

$$\left. \Rightarrow I_1 = \int \frac{dt}{t} = \log|t| = \log|1 + x^2| \right]$$

and the solution of linear differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF})\, dx + C$$

$$\Rightarrow \quad y(1 + x^2) = \int \frac{\cot x}{1+x^2} \times (1 + x^2)\, dx + C$$

$$\Rightarrow \quad y(1 + x^2) = \int \cot x\, dx + C$$

$$\Rightarrow \quad y(1 + x^2) = \log|\sin x| + C$$

$$\Rightarrow \quad y = \frac{\log|\sin x|}{1+x^2} + \frac{C}{1+x^2}$$

which is the required solution.

44. **Hint** Write the given differential equation as

$$\frac{dx}{dy} = -\left(\frac{x - \sin y}{\tan y}\right)$$

$$\Rightarrow \quad \frac{dx}{dy} + (\cot y)\, x = \cos y \qquad [\textbf{Ans. } y = \sin^{-1} 2x]$$

45. Given, $\frac{dv}{dt} = g \cos \alpha - kv$

$$\Rightarrow \quad \frac{dv}{dt} + kv = g \cos \alpha$$

which is a linear differential equation of the form

$$\frac{dv}{dt} + Pv = Q$$

Here, $P = k$ and $Q = g \cos \alpha$.

Now, $\text{IF} = e^{\int P dt} = e^{\int k dt} = e^{kt}$

and the solution of the differential equation is given by

$$v \cdot e^{kt} = \int e^{kt} \cdot g \cos \alpha\, dt + C$$

$$\Rightarrow \quad v \cdot e^{kt} = g \cos \alpha \int e^{kt} dt + C$$

$$\Rightarrow \quad v \cdot e^{kt} = \frac{g \cos \alpha\, e^{kt}}{k} + C \qquad \text{...(i)}$$

It is given that $v = 0$, when $t = 0$.

$\therefore$ From Eq. (i), we get

$$0 = \frac{g \cos \alpha}{k} + C$$

$$\Rightarrow \quad C = \frac{-g \cos \alpha}{k}$$

On putting the value of C in Eq. (i), we get

$$v \cdot e^{kt} = \frac{g \cos \alpha\, e^{kt}}{k} - \frac{g \cos \alpha}{k}$$

$$\Rightarrow \quad v \cdot e^{kt} = \frac{g \cos \alpha}{k} (e^{kt} - 1)$$

$$\Rightarrow \quad v = \frac{g \cos \alpha}{k} (1 - e^{-kt}),$$

which is the required expression.

SUMMARY

- **Differential Equation** An equation involving independent variable (variables), dependent variable and derivative (derivatives) of dependent variable with respect to independent variable (variables) is called a **differential equation.**
- **Ordinary Differential Equation** A differential equation involving derivative or derivatives of the dependent variable with respect to only one independent variable is called an **ordinary differential equation.**
- **Order of a Differential Equation** The order of the highest order derivative involved in a differential equation, is called the **order** of a differential equation.
- **Degree of a Differential Equation** The highest power (positive integral index) of the highest order derivative involved in a differential equation, when it is written as a polynomial in derivatives, is called the degree of a differential equation.
- **Solution of a Differential Equation** A function of independent variable will be a solution of differential equation, if this function satisfies the given differential equation.
- **General Solution of a Differential Equation** If the solution of a differential equation of order n, contains n arbitrary constants, then it is called the **general solution** of the differential equation.
- **Particular Solution of a Differential Equation** The solution of a differential equation obtained by giving particular values to the arbitrary constants in the general solution, is called the **particular solution**. In other words, the solution free from arbitrary constants is called **particular solution.**
- **Differential Equation in Variable Separable Form** A differential equation is said to be in the **variable separable form** if it is expressible in the form $\frac{1}{h(y)}dy = g(x)dx$, i.e. terms containing y should remain with dy and terms containing x should remain with dx.
  The solution of this equation is given by
  $\int \frac{1}{h(y)}dy = \int g(x)dx + C$, where C is a constant of integration and $h(y) \neq 0$.
- **Homogeneous Function** A function $F(x, y)$ is said to be **homogeneous function** of degree n, if $F(x, y) = x^n g\left(\frac{y}{x}\right)$
  or $y^n h\left(\frac{x}{y}\right)$
- **Homogeneous Differential Equations** A differential equation of the form $\frac{dy}{dx} = F(x, y)$ or $\frac{dx}{dy} = F(x, y)$ is called a **homogeneous differential equation**, if $F(x, y)$ is a homogeneous function of degree zero.
- **Solution of Homogeneous Differential Equation** To solve homogeneous differential equation of the form $\frac{dy}{dx} = F(x, y) = f\left(\frac{y}{x}\right)$ put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in given differential equation and then use variable separable method to get required solution.
  To solve homogeneous differential equation of the form $\frac{dx}{dy} = G(x, y) = g\left(\frac{x}{y}\right)$, put $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$ in the given differential equation and then use variable separable method.
- **Linear Differential Equation** A first order differential equation in which the degree of dependent variable and its derivative is one and they do not get multiplied together, is called a **linear differential equation**.
  ***Type* I** A differential equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are constants or functions of x only, is called linear differential equation, whose solution is given by
  $y \cdot (\text{IF}) = \int Q \cdot (\text{IF})dx + C$, where $\text{IF} = e^{\int P dx}$
  ***Type* II** A differential equation of the form $\frac{dx}{dy} + Px = Q$, where P and Q are constants or functions of y only, is called linear differential equation, whose solution is given by
  $x \cdot (\text{IF}) = \int Q \cdot (\text{IF})dy + C$, where $\text{IF} = e^{\int P dy}$.

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

1 If m and n are the order and degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^5 + 4\frac{\left(\frac{d^2y}{dx^2}\right)^3}{\frac{d^3y}{dx^3}} + \frac{d^3y}{dx^3} = x^2 - 1, \text{ then}$$

(a) $m=3, n=3$ (b) $m=3, n=2$
(c) $m=3, n=5$ (d) $m=3, n=1$

2 The order of the differential equation satisfying $\sqrt{1-x^2}+\sqrt{1-y^2} = a(x-y)$ is

(a) 1 (b) 2
(c) 3 (d) None of these

3 If $y = e^{-x}(A\cos x + B\sin x)$, then y is a solution of

(a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$ (b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$
(c) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$ (d) $\frac{d^2y}{dx^2} + 2y = 0$

4 The solution of differential equation $xdy - ydx = 0$ represents

(a) a rectangular hyperbola
(b) parabola whose vertex is at origin
(c) straight line passing through origin
(d) a circle whose centre is at origin

5 The solution of $\frac{dy}{dx} - y = 1$, $y(0) = 1$ is given by

(a) $xy = -e^x$ (b) $xy = -e^{-x}$
(c) $xy = -1$ (d) $y = 2e^x - 1$

6 The differential equation $y\frac{dy}{dx} + x = C$ represents

(a) family of hyperbolas (b) family of parabolas
(c) family of ellipses (d) family of circles

7 The integrating factor of differential equation $\frac{dy}{dx} + y\tan x - \sec x = 0$ is

(a) $\cos x$ (b) $\sec x$ (c) $e^{\cos x}$ (d) $e^{\sec x}$

8 If $\frac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$, then $y(\ln 2)$ is equal to

(a) 5 (b) 13 (c) -2 (d) 7

9 The integrating factor of differential equation $\frac{dy}{dx} + y = \frac{1+y}{x}$ is

(a) $\frac{x}{e^x}$ (b) $\frac{e^x}{x}$ (c) xe^x (d) e^x

10 The solution of $\frac{dy}{dx} + y = e^{-x}$, $y(0) = 0$ is

(a) $y = e^x(x-1)$ (b) $y = xe^{-x}$
(c) $y = xe^{-x} + 1$ (d) $y = (x+1)e^{-x}$

VERY SHORT ANSWER Type Questions

Directions (Q. Nos. 11-14) *Find the order and degree of the following differential equations.*

11 $\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0$ **[NCERT]**

12 $\frac{d^2y}{dx^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

13 $x\frac{dy}{dx} + \frac{2}{(dy/dx)} = y^2$

14 $x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$ **[Delhi 2019]**

15 Find the order and the degree of the differential equation

$$x^2\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^4$$

[Delhi 2019]

Directions (Q. Nos. 16-19) *Solve the following differential equations.*

16 $\frac{dy}{dx} + y = 1$, $y \neq 1$

17 $\frac{dy}{dx} - \frac{y(x+1)}{x} = 0$

18 $\frac{dy}{dx} = 2^{y-x}$ **[NCERT Exemplar]**

19 $\sqrt{a+x}\dfrac{dy}{dx}+x=0$

20 Write the general solution of differential equation $\dfrac{dy}{dx}=\dfrac{y}{x}$.

21 Find the integrating factor of the differential equation $x\dfrac{dy}{dx}-y=2x^2$.

22 Find the integrating factor of the differential equation $(1-y^2)\dfrac{dx}{dy}+yx=ay, -1<y<1$.

23 Write the integrating factor of $\dfrac{dy}{dx}-\dfrac{1}{(1+x)}y=(1+x)e^x$.

24 Write the integrating factor of the differential equation $\sqrt{x}\dfrac{dy}{dx}+y=e^{-2\sqrt{x}}$. **[All India 2015C]**

SHORT ANSWER Type I Questions

25 State whether $y=e^{-x}(x+a)$ is a solution of differential equation $\dfrac{dy}{dx}+y=e^{-x}$.

26 Show that the function $y=(A+Bx)e^{3x}$ is a solution of the equation $\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}+9y=0$.

27 Show that the function ϕ, defined by $\phi(x)=\cos x, (x\in R)$, satisfies the initial value problem $\dfrac{d^2y}{dx^2}+y=0, y(0)=1, y'(0)=0$.

28 Show that $y=Cx+\dfrac{a}{C}$ is a solution of differential equation $y=x\dfrac{dy}{dx}+\dfrac{a}{\left(\dfrac{dy}{dx}\right)}$, where C is a parameter.

29 Prove that $xy=ae^x+be^{-x}+x^2$ is the general solution of the differential equation

$$x\frac{d^2y}{dx^2}+2\frac{dy}{dx}-xy+x^2-2=0.$$

30 Write the solution of differential equation $(e^x+e^{-x})\,dy=(e^x-e^{-x})\,dx$.

31 If $\dfrac{dy}{dx}=ye^x$ and $x=0, y=e$, then find the value of y, when $x=1$. **[NCERT Exemplar]**

32 Solve $2(y+3)-xy\dfrac{dy}{dx}=0$, given that $y(1)=-2$. **[NCERT Exemplar]**

33 Solve the differential equation $\dfrac{dy}{dx}=y\sin 2x$, given that $y(0)=1$.

34 Solve the initial value problem

$$dy=e^{2x+y}\,dx, y(0)=0.$$

35 Show that the given differential equation is homogeneous and solve it $y'=\dfrac{x+y}{x}$. **[NCERT]**

36 Find the integrating factor of the differential equation $\dfrac{dy}{dx}+y=\dfrac{1+y}{x}$. **[All India 2017C]**

37 Find the general solution of the differential equation $\dfrac{dy}{dx}+2y=e^{3x}$. **[Delhi 2017C]**

38 Find the equation of that curve whose tangent at any point on it has slope equal to $y+2x$.

SHORT ANSWER Type II Questions

Directions (Q. Nos. 39-51) *Solve each of the following differential equations.*

39 $x\cos y\,dy=(xe^x\log x+e^x)\,dx$

40 $(1+y^2)(1+\log x)\,dx+x\,dy=0$ **[Delhi 2011]**

41 $\dfrac{dy}{dx}+\dfrac{1+\cos 2y}{1-\cos 2x}=0$

42 $(1+e^{2x})\,dy+(1+y^2)e^x\,dx=0$, given $y=1$, when $x=0$. **[NCERT; Foreign 2011; Delhi 2010]**

43 $(x+1)\dfrac{dy}{dx}=2e^{-y}+1, y=0$, when $x=0$ **[Delhi 2020]**

44 $2xy\,dx+(x^2+2y^2)\,dy=0$ [**Hint** Put $y=vx$]

45 $(x^2-yx^2)\,dy+(y^2+x^2y^2)\,dx=0$ **[Delhi 2010]**

46 $x\,dy-y\,dx=\sqrt{x^2+y^2}$ **[NCERT; All India 2011; Delhi 2019, 2016C]**

47 $x\,dy=(2y+2x^4+x^2)dx$

48 $\dfrac{dy}{dx}-y=\sin x$ **[All India 2017]**

49 $(1+x^2)\dfrac{dy}{dx}-x=2\tan^{-1}x$

50 $x\,dy-(y+2x^2)\,dx=0$ **[All India 2011]**

51 $y\,dx - (x + 2y^2)dy = 0$ **[All India 2017]**

52 Show that the family of curves for which $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$, is given by $x^2 - y^2 = cx$. **[Delhi 2017]**

Or

Find the general solution of the differential equation $2xy\frac{dy}{dx} = x^2 + y^2$. **[All India 2017C]**

53 Solve $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$ subject to the initial condition $y(0) = 0$. **[Delhi 2019, 2016C]**

54 Solve the differential equation

$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$. **[All India 2019]**

55 Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$, given that $y(0) = \sqrt{3}$. **[All India 2017C]**

56 Find the particular solution of the differential equation $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$, $y = 2$, when $x = 1$. **[NCERT; All India 2017C]**

LONG ANSWER Type Questions

57 Solve the differential equation

$$\frac{dy}{dx} = \frac{e^x(\sin^2 x + \sin 2x)}{y(2\log y + 1)}.$$

58 Solve the differential equation

$\frac{dy}{dx} = 1 + x + y^2 + xy^2$, when $y = 0, x = 0$. **[NCERT Exemplar]**

59 Solve $x^2\frac{dy}{dx} = x^2 + xy + y^2$. **[NCERT Exemplar]**

60 Solve $x^2\frac{dy}{dx} - xy = 1 + \cos\left(\frac{y}{x}\right)$, $x \neq 0$ and $x = 1$, $y = \frac{\pi}{2}$. **[NCERT Exemplar]**

61 Find the equation of a curve passing through the point $\left(1, \frac{\pi}{4}\right)$, if the slope of the tangent to the curve at any point $P(x, y)$ is $\frac{y}{x} - \cos^2\frac{y}{x}$. **[NCERT Exemplar]**

62 Solve the differential equation $x\frac{dy}{dx} - ay = x + 1$.

63 Find the particular solution of the differential equation $(1+x^2)\frac{dy}{dx} = e^{m\tan^{-1}x} - y$, given that $y = 1$, when $x = 0$. **[All India 2015]**

64 Solve the initial value problem

$$y - x\frac{dy}{dx} = 2\left(1 + x^2\frac{dy}{dx}\right), \ y(1) = 1$$

[**Hint** $y - x\frac{dy}{dx} = 2\left(1 + x^2\frac{dy}{dx}\right)$

$\Rightarrow \quad \frac{1}{x(2x+1)}dx = \frac{1}{(y-2)}dy$

$\Rightarrow \int\left(\frac{1}{x} - \frac{2}{2x+1}\right)dx = \int\frac{1}{(y-2)}dy$]

65 Show that the differential equation $(x^2 + xy)\,dy = (x^2 + y^2)\,dx$ is homogeneous and solve it. **[NCERT]**

66 Show that the differential equation $(y^2 - x^2)\,dy = 3xy\,dx$ is homogeneous and solve it.

67 Solve the differential equation

$(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$. [**Hint** Put $x = vy$] **[NCERT]**

68 Solve the differential equation

$x^2dy + y(x + y)\,dx = 0$, given that $y = 1$, when $x = 1$

[**Hint** $\frac{dy}{dx} = -\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2$, put $y = vx$, then

$x\frac{dv}{dx} = -2v - v^2$

$\Rightarrow \int\frac{dv}{(v+1)^2 - 1} + \int\frac{dx}{x} = C$]

69 Solve the initial value problem

$(xe^{y/x} + y)dx = x\,dy$, $y(1) = 1$.

70 Solve $y + \frac{d}{dx}(xy) = x(\sin x + \log x)$. **[NCERT Exemplar]**

[**Hint** The given equation can be written as

$$y + x\frac{dy}{dx} + y = x(\sin x + \log x)$$

$\Rightarrow \quad \frac{dy}{dx} + \frac{2}{x}y = \sin x + \log x$]

71 Find the general solution of
$(1+\tan y)(dx-dy)+2x\,dy=0.$

[NCERT Exemplar]

[**Hint** The given equation can be written as

$$\frac{dx}{dy}+\frac{2x}{1+\tan y}=1;\ \text{IF}=e^{\int\frac{2}{1+\tan y}dy}$$

$$\Rightarrow \text{IF}=e^{\int\frac{2\cos y}{\cos y+\sin y}dy}=e^{\int\left(\frac{\cos y+\sin y+\cos y-\sin y}{\cos y+\sin y}\right)dy}$$

$$=e^{\int\left(1+\frac{\cos y-\sin y}{\cos y+\sin y}\right)dy}$$

$$=e^{y+\log|\cos y+\sin y|}=e^{y}\cdot(\cos y+\sin y)]$$

72 Find the particular solution of the differential equation $(1-x^2)\dfrac{dy}{dx}-xy=x^2$, given that $y=2$, when $x=0$.

73 Find the particular solution of the differential equation $x\dfrac{dy}{dx}-y=(x+1)e^{-x}$, given that $y=0$, when $x=1$.

CASE BASED Questions

74. A Veterinary doctor was examining a sick cat brought by a pet lover. When it was brought to the hospital, it was already dead. The pet lover wanted to find its time of death. He took the temperature of the Cat at 11.30 pm which was 94.6°F. He took the temperature again after 1 h; the temperature was lower than the first observation. It was 93.4°F. The room in which the Cat was put is always at 70°F. The normal temperature of the cat is taken as 98.6°F when it was alive.

The doctor estimated the time of death using Newton law of cooling which is governed by the differential equation: $\dfrac{dT}{dt}\propto(T-70)$, where 70°F is the room temperature and T is the temperature of the object at time t.

Substituting the two different observations of T and t made, in the solution of the differential equation $\dfrac{dT}{dt}=k(T-70)$ where k is a constant of proportion, time of death is calculated.

[CBSE Question Bank]

Answer the following questions using the above information.

(i) State the degree of the above given differential equation.

(ii) Which method of solving a differential equation helped in calculation of the time of death?

(a) Variable separable method
(b) Solving homogeneous differential equation
(c) Solving linear differential equation
(d) All of the above

(iii) If the temperature was measured 2 h after 11.30 pm, will the time of death change? (Yes/No)

(iv) The solution of the differential equation $\dfrac{dT}{dt}=k(T-70)$ is given by,

(a) $\log|T-70|=kt+C$
(b) $\log|T-70|=\log|kt|+C$
(c) $T-70=kt+C$
(d) $T-70=kt+C$

(v) If $t=0$ when T is 72, then the value of C is

(a) -2 (b) 0 (c) 2 (d) $\log 2$

75. Polio drops are delivered to 50K children in a district. The rate at which polio drops are given is directly proportional to the number of children who have not been administered the drops.

By the end of 2nd week half the children have been given the polio drops. How many will have been given the drops by the end of 3rd week can be estimated using the solution to the differential equation $\dfrac{dy}{dx}=k(50-y)$ where x denotes the number of weeks and y the number of children who have been given the drops.

[CBSE Question Bank]

Answer the following questions using the above information.

(i) State the order of the above given differential equation.

(ii) Which method of solving a differential equation can be used to solve $\dfrac{dy}{dx}=k(50-y)$?

(a) Variable separable method
(b) Solving homogeneous differential equation
(c) Solving linear differential equation
(d) All of the above

(iii) The solution of the differential equation $\frac{dy}{dx} = k(50-y)$ is given by

(a) $\log|50-y| = kx + C$

(b) $-\log|50-y| = kx + C$

(c) $\log|50-y| = \log|kx| + C$

(d) $50 - y = kx + C$

(iv) The value of c in the particular solution given that $y(0)=0$ and $k = 0.049$ is

(a) $\log 50$ (b) $\log\frac{1}{50}$

(c) 50 (d) – 50

(v) Which of the following solutions may be used to find the number of children who have been given the polio drops?

(a) $y = 50 - e^{kx}$ (b) $y = 50 - e^{-kx}$

(c) $y = 50(1 - e^{-kx})$ (d) $y = 50(e^{kx} - 1)$

ANSWERS

1. (b) 2. (a) 3. (c) 4. (d) 5. (d)
6. (b) 7. (d) 8. (d) 9. (b) 10. (b)
11. Order = 4, degree = not defined 12. Order = 2, degree = 2 13. Order = 1, degree = 2
14. Order = 2, degree = 2 15. Order = 2, degree = 1 16. $x + \log|1-y| = C$ 17. $y = xe^x + C$ 18. $2^{-x} - 2^{-y} = k$
19. $y = -\frac{2}{3}(a+x)^{3/2} + 2a\sqrt{a+x} + C$ 20. $y = Cx$ 21. IF $= \frac{1}{x}$ 22. IF $= \frac{1}{\sqrt{1-y^2}}$
23. IF $= \frac{1}{1+x}$ 24. IF $= e^{2\sqrt{x}}$ 25. Yes 30. $y = \log|e^x + e^{-x}| + C$
31. $y = e^e$ 32. $x^2(y+3)^3 = e^{y+2}$ 33. $y = e^{\sin^2 x}$ 34. $y = \log\left|\frac{2}{3-e^{2x}}\right|$ 35. $y = x\log|x| + Cx$
36. IF $= \frac{e^x}{x}$ 37. $y = \frac{e^{3x}}{5} + Ce^{-2x}$ 38. $y = -2x - 2 + Ce^x$ 39. $\sin y = e^x \log x + C$
40. $\tan^{-1} y + \log|x| + \frac{(\log|x|)^2}{2} = C$ 41. $\tan y = \cot x + C$ 42. $\tan^{-1} y + \tan^{-1} e^x = \frac{\pi}{2}$
43. $y = \log(3x+1)$ 44. $2y^3 + 3x^2y = C$ 45. $\frac{1}{y} + \log y = -\frac{1}{x} + x + C$ 46. $y + \sqrt{x^2+y^2} = Cx^2$
47. $y = x^4 + x^2\log x + Cx^2$ 48. $y = -\frac{1}{2}(\sin x + \cos x + ce^x)$ 49. $y = (\tan^{-1} x)^2 + \frac{1}{2}\log|1+x^2| + C$
50. $y = 2x^2 + Cx$ 51. $x = 2y^2 + Cy$ 53. $y = \frac{4x^3}{3(1+x^2)}$ 54. $x\sin\left(\frac{y}{x}\right) = C$
55. $\tan^{-1} y = \tan^{-1} x + \frac{\pi}{3}$ 56. $y = \frac{2x}{1-\log|x|}\ (x \neq 0,\ x \neq e)$ 57. $y^2(\log y) = e^x \sin^2 x + C$
58. $y = \tan\left(x + \frac{x^2}{2}\right)$ 59. $\tan^{-1}\left(\frac{y}{x}\right) = \log|x| + C$ 60. $\tan\left(\frac{y}{2x}\right) = \frac{-1}{2x^2} + \frac{3}{2}$ 61. $\tan\left(\frac{y}{x}\right) + \log x = 1$
62. $y = \frac{x}{1-a} - \frac{1}{a} + Cx^a$ 63. $y\,(m+1)\,e^{\tan^{-1} x} = e^{(m+1)\tan^{-1} x} + m$ 64. $y = 2 - \frac{3x}{2x+1}$, where $x \neq -\frac{1}{2}$
65. $(x-y)^2 = Cxe^{-y/x}$ 66. $y^2(4x^2 - y^2)^3 = C$ 67. $x + ye^{x/y} = C$ 68. $y + 2x - 3x^2y = 0$
69. $y = x - x\log(1 - e\log|x|)$ 70. $y = -\cos x + \frac{2\sin x}{x} + \frac{2\cos x}{x^2} + \frac{x}{3}\log x - \frac{x}{9} + Cx^{-2}$
71. $x(\sin y + \cos y) = \sin y + Ce^{-y}$ 72. $y = -\frac{x}{2} + \frac{\sin^{-1} x}{2\sqrt{1-x^2}} + \frac{2}{\sqrt{1-x^2}}$ 73. $y = -e^{-x} + xe^{-1}$
74. (i) → (1), (ii) → (a), (iii) → (No), (iv) → (a), (v) → (d) 75. (i) → (1), (ii) → (a), (iii) → (b), (iv) → (b), (v) → (c)