Mathematics
Class XII
Sample Paper – 7 Solution

SECTION – A

1. Element at 3rd column and 2nd row

So i =2 and j =3

Substituting in $a_{ij}=\frac{i+2j}{2}$ we get

$$a_{23}=\frac{2+2\times3}{2}=\frac{8}{2}=4$$

2. y + sin y = cos x

differentiating w.r.t. x, we get,

$$\frac{d}{dx}\left(y+\sin y\right)=\frac{d}{dx}\cos x$$
$$\frac{dy}{dx}+\cos y\frac{dy}{dx}=-\sin x$$
$$\frac{dy}{dx}=\frac{-\sin x}{1+\cos y}$$

3.

$$\left(\frac{dy}{dx}\right)^2+\frac{1}{\frac{dy}{dx}}=2$$

rearranging

$$\left(\frac{dy}{dx}\right)^3+1=2\frac{dy}{dx}$$

Order: 1

Degree:3

4. The vector equation of the line passing through the point (5, 2,-4) and parallel to $3\hat{i}+2\hat{j}-8\hat{k}$.

$$\vec{r}=\left(5\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(3\hat{i}+2\hat{j}-8\hat{k}\right)$$

OR

The vector equation of the line passing through the points (-1, 0, 2) and (3, 4, 6) is

$\vec{r}=\left(-\hat{i}+2\hat{k}\right)+\lambda\left(4\hat{i}+4\hat{j}+4\hat{k}\right)$

SECTION – B

5.

A = {1, 2, 3, 4, 5}

R = {(a, b): $|a-b|$ is even}

For R to be an equivalence relation it must be

(i) Reflexive, $|a-a|=0$

$\therefore (a,a)\in R$ for $\forall a\in A$

So R is reflexive.

(ii) Symmetric,

if $(a,b)\in R \Rightarrow |a-b|$ is even

$\Rightarrow |b-a|$ is also even

So R is symmetric.

(iii) Transitive

If (a, b) $\in$R (b, c) $\in$ R then (a, c) $\in$R

(a, b) $\in R \Rightarrow |a-b|$ is even

(b, c) $\in$ R $\Rightarrow |b-c|$ is even

Sum of two even numbers is even

So, $|a-b|+|b-c|$

$=|a-b+b-c|=|a-c|$ is even since, $|a-b|$ and $|b-c|$ are even

So (a, c) $\in$ R

Hence, R is transitive.

Therefore, R is an equivalence relation.

6.

Let $B=\begin{bmatrix}2 & 3\\ -1 & 4\end{bmatrix}$ and $C=\begin{bmatrix}3 & -6\\ -3 & 8\end{bmatrix}$.Then, the given matrix equation is A + B = C

now,

A + B – B = C – B

A = C – B

$$A=\begin{bmatrix}3 & -6\\ -3 & 8\end{bmatrix}-\begin{bmatrix}2 & 3\\ -1 & 4\end{bmatrix}$$

$$A=\begin{bmatrix}3-2 & -6-3\\ -3+1 & 8-4\end{bmatrix}$$

$$A=\begin{bmatrix}1 & -9\\ -2 & 4\end{bmatrix}$$

7.

It is known that, $\sin A\sin B=\frac{1}{2}\left(\cos(A-B)-\cos(A+B)\right)$

$$\therefore\int \sin x\sin 2x\sin 3x\,dx=\int\left[\sin x\times\frac{1}{2}\left(\cos(2x-3x)-\cos(2x+3x)\right)\right]$$

$$=\frac{1}{2}\int\left(\sin x\cos(-x)-\sin x\cos 5x\right)dx$$

$$=\frac{1}{2}\int\left(\sin x\cos x-\sin x\cos 5x\right)dx$$

$$=\frac{1}{2}\int\frac{\sin 2x}{2}dx-\frac{1}{2}\int\sin x\cos 5x$$

$$=\frac{1}{4}\left[\frac{-\cos 2x}{2}\right]-\frac{1}{2}\int\frac{1}{2}\left(\sin(x+5x)+\sin(x-5x)\right)dx$$

$$=\frac{-\cos 2x}{8}-\frac{1}{4}\int\left(\sin 6x+\sin(-4x)\right)dx$$

$$=\frac{-\cos 2x}{8}-\frac{1}{4}\left[\frac{-\cos 6x}{6}+\frac{\cos 4x}{4}\right]+C$$

$$=\frac{-\cos 2x}{8}-\frac{1}{8}\left[\frac{-\cos 6x}{3}+\frac{\cos 4x}{2}\right]+C$$

$$=\frac{-6\cos 2x}{48}-\frac{1}{8}\left[\frac{-2\cos 6x+3\cos 4x}{6}\right]+C$$

$$=\frac{1}{48}\left[2\cos 6x-3\cos 4x-6\cos 2x\right]+C$$

8.

$$\text{Let } \frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2}$$

$$2 = A(1+x^2) + (Bx+C)(1-x)$$

$$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$$

Equating the coefficient of x^2, x, and constant term, we obtain

A − B = 0

B − C = 0

A + C = 2

On solving these equations, we obtain

A = 1, B = 1, and C = 1

$$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

$$\Rightarrow \int \frac{2}{(1-x)(1+x^2)}dx = \int \frac{1}{1-x}dx + \int \frac{x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$$

$$= -\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{2x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$$

$$= -\log|x-1| + \frac{1}{2}\log\left|1+x^2\right| + \tan^{-1}x + C$$

OR

$$I = \int \frac{\sin(x-a)}{\sin(x+a)}dx$$

Let (x + a) = t $\Rightarrow$ dx = dt

$$\therefore I = \int \frac{\sin(t-2a)}{\sin t}dt$$

$$= \int \frac{\sin t\cos 2a - \cos t\sin 2a}{\sin t}dt$$

$$= \int (\cos 2a - \cot t\sin 2a)\,dt$$

$$= (\cos 2a)t - \sin 2a\log|\sin t| + C$$

$$= \cos 2a(x+a) - \sin 2a\log|\sin(x+a)| + C$$

9. $y^2 = a(b - x)(b + x)$

$y^2 = a(b^2 - x^2)$

There are two arbitrary constants so we have to differentiate it two times

Differentiating w.r.t. x

$$2y\frac{dy}{dx} = -2ax$$

$$\frac{y}{x}\frac{dy}{dx} = -a.......(i)$$

$$y\frac{dy}{dx} = -a$$

Differentiating again

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -a$$

putting value of -a

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{y}{x}\frac{dy}{dx} \qquad \text{from}(i)$$

10.

Let the angle between $\vec{a}$ and $\vec{b}$ be θ .

We know that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

Given $\vec{a}\cdot\vec{b} = 60$

$\Rightarrow |\vec{a}||\vec{b}|\cos\theta = 60$

$\Rightarrow 13\times5\times\cos\theta = 60$

$$\Rightarrow \cos\theta = \frac{60}{13\times5} = \frac{12}{13}$$

$$\Rightarrow \sin\theta = \sqrt{1 - \frac{144}{169}} = \frac{5}{13}$$

Also we know that, $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

$$|\vec{a}\times\vec{b}| = 5\times13\times\frac{5}{13} = 25$$

OR

Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar if their scalar product is zero

$$\vec{a}.\left(\vec{b}\times\vec{c}\right)=0$$

$$\vec{a}=2\hat{i}-\hat{j}+\hat{k},\ \vec{b}=\hat{i}+2\hat{j}-3\hat{k}\text{ and }\vec{c}=3\hat{i}-\lambda\hat{j}+5\hat{k}$$

$$\begin{vmatrix}2 & -1 & 1\\ 1 & 2 & -3\\ 3 & -\lambda & 5\end{vmatrix}=0$$

$$2\left(10-3\lambda\right)+1\left(5+9\right)+1\left(-\lambda-6\right)=0$$
$$\Rightarrow 20-6\lambda+14-\lambda-6=0$$
$$\Rightarrow -7\lambda+28=0$$
$$\Rightarrow -7\lambda=-28$$
$$\Rightarrow \lambda=4$$

11. Let X represent the number of Kings drawn and the event here is to successfully draw a King. So X = 0, 1, 2

$$P(0)=\frac{^{48}C_2}{^{52}C_2}=\frac{48\times 47}{52\times 51}=\frac{188}{221}$$
$$P(1)=\frac{^{48}C_1\times {}^{4}C_1}{^{52}C_2}=\frac{4\times 48\times 2}{52\times 51}=\frac{32}{221}$$
$$P(2)=\frac{^{48}C_0\times {}^{4}C_2}{^{52}C_2}=\frac{4\times 3}{52\times 51}=\frac{1}{221}$$

12. Let E_1: First Bag is selected; E_2: Second Bag is selected

and let E: A red ball is drawn

We note that $E_1\cap E_2=\phi$ and $E_1\cup E_2=S$

$\Rightarrow E_1$ and E_2 are mutually exclusive and exhaustive events .

Now $P(E_1)=\frac{1}{2}; P(E_2)=\frac{1}{2}$

$$P(E)=P(E|E_1)P(E_1)+P(E|E_2)P(E_2)$$
$$P(E)=P(E|E_1)\frac{1}{2}+P(E|E_2)\frac{1}{2}=\frac{4}{4+3}\times\frac{1}{2}+\frac{2}{2+4}\times\frac{1}{2}=\frac{2}{7}+\frac{1}{6}=\frac{19}{42}$$

OR

Let E_1: First factory manufactured the machinery;

E_2 : Second factory manufactured the machinery

and let E: selected machinery is of standard quality

We note that $E_1 \cap E_2 = \phi$ and $E_1 \cup E_2 = S$

$\Rightarrow E_1$ and E_2 are mutually exclusive and exhaustive events .

Now $P(E_1) = \frac{7}{10}; P(E_2) = \frac{30}{100} = \frac{3}{10}$

$P(E|E_1) = \frac{80}{100} = \frac{8}{10}; P(E|E_2) = \frac{90}{100} = \frac{9}{10}$

$$P(E_2|E) = \frac{P(E|E_2)P(E_2)}{P(E|E_2)P(E_2) + P(E|E_1)P(E_1)} = \frac{\frac{3}{10} \times \frac{9}{10}}{\frac{3}{10} \times \frac{9}{10} + \frac{7}{10} \times \frac{8}{10}} = \frac{27}{56+27} = \frac{27}{83}$$

SECTION – C

13. (a, b) * (c, d) = (a + c, b + d)

(i) Commutative

(a, b) * (c, d) = (a + c, b + d)

(c, d) * (a, b) = (c + a, d + b)

for all, a, b, c, d $\in$ R

* is commutative on A

(ii) Associative : ______

(a, b), (c, d), (e, f) $\in$A

{ (a, b) * (c, d) } * (e, f)

= (a + c, b + d) * (e, f)

= ((a + c) + e, (b + d) + f)

= (a + (c + e), b + (d + f))

= (a*b) * (c + d, d + f)

= (a*b) {(c, d) * (e, f)}

is associative on A

Let (x, y) be the identity element in A.

then,
(a, b) * (x, y) = (a, b) for all (a, b) $\in$A

(a + x, b + y) = (a, b) for all (a, b) $\in$A

a + x = a, b + y = b for all (a, b) $\in$A

x = 0, y = 0

(0, 0) $\in$A

(0, 0) is the identity element in A.

Let (a, b) be an invertible element of A.

(a, b) * (c, d) = (0, 0) = (c, d) * (a, b)

(a + c, b + d) = (0, 0) = (c + a, d + b)

a + c = 0, b + d = 0

a = - c   b = - d

c = - a   d = - b

(a, b) is an invertible element of A, in such a case the inverse of (a, b) is (-a, -b)

OR

Given that $f(x)=\frac{4x+3}{3x+4}$

For one – one function,

$$f(x_1)=f(x_2)$$

$$\frac{4x_1+3}{3x_1+4}=\frac{4x_2+3}{3x_2+4}$$

$$(4x_1+3)(3x_2+4)=(4x_2+3)(3x_1+4)$$

$$12x_1x_2+16x_1+9x_2+12=12x_1x_2+16x_2+9x_1+12$$

$$7x_1=7x_2$$

$$x_1=x_2$$

Hence, f(x) is one – one.

For onto function,

$$y=\frac{4x+3}{3x+4}$$

$$y(3x+4)=4x+3$$

$$3xy+4y=4x+3$$

$$4y-3=4x-3xy$$

$$4y-3=x(4-3y)$$

$$x=\frac{4y-3}{4-3y}$$

$$f\left(\frac{4y-3}{4-3y}\right)=\frac{4\left(\frac{4y-3}{4-3y}\right)+3}{3\left(\frac{4y-3}{4-3y}\right)+4}$$

$$=\frac{16y-12+12-9y}{12y-9+16-12y}$$

$$=y$$

So, the function is onto.

Hence, function is bijective.

$$f^{-1}(x)=\frac{4x-3}{4-3x}$$

$$f^{-1}(0)=\frac{-3}{4}$$

To find x such that $f^{-1}(x)=2$,

$$\frac{4x-3}{4-3x}=2$$

$$4x-3=2(4-3x)$$

$$4x-3=8-6x$$

$$10x=11$$

$$x=\frac{11}{10}$$

14.

$$\text{To prove}: \sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{16}{65}\right)=\frac{\pi}{2}$$

$$\text{Let } \sin^{-1}\left(\frac{4}{5}\right)=x$$

$$\Rightarrow \sin x=\frac{4}{5}$$

$$\Rightarrow \cos x=\sqrt{1-\sin^2 x}=\frac{3}{5}$$

$$\sin^{-1}\left(\frac{5}{13}\right)=y$$

$$\Rightarrow \sin y=\frac{5}{13}$$

$$\Rightarrow \cos y=\sqrt{1-\sin^2 y}=\frac{12}{13}$$

$$\sin^{-1}\left(\frac{16}{65}\right)=z$$

$$\Rightarrow \sin z=\frac{16}{65}$$

$$\Rightarrow \cos z=\sqrt{1-\sin^2 z}=\frac{63}{65}$$

$$\tan x=\frac{4}{3}, \tan y=\frac{5}{12}, \tan z=\frac{16}{63}$$

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x.\tan y}=\frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{20}{36}}=\frac{63}{16}=\cot z$$

$$\tan(x+y)=\tan\left(\frac{\pi}{2}-z\right)\Rightarrow x+y+z=\frac{\pi}{2}$$

$$\therefore \sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{16}{65}\right)=\frac{\pi}{2}$$

15.

$$\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$

a, b, and c are in A.P.

$b - a = c - b$

$\Rightarrow 2b = a + c \Rightarrow (a+c) - 2b = 0 \qquad$ (i)

Performing the operation : $R_1 \to R_1 + R_3 - 2R_2$

$$\Delta = \begin{vmatrix} 0 & 0 & a+c-2b \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$

But, $(a+c) - 2b = 0$ using this in above determinant

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$

Since a row of the determinant is zero so $\Delta = 0$

16.

Putting $x^2 = \cos 2\theta$, we get

$$y = \tan^{-1}\left\{\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{1+\tan\theta}{1-\tan\theta}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\theta\right)\right\}$$

$$y = \frac{\pi}{4}+\theta$$

$$y = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

$$\frac{dy}{dx} = 0+\frac{-1}{2}\times\frac{1}{\sqrt{1-x^4}}\times 2x$$

$$= \frac{-x}{\sqrt{1-x^4}}$$

OR

Differentiating both sides of the given relation w.r.t. x, we get

$$\Rightarrow \frac{d}{dx}\left\{\log\left(x^2+y^2\right)\right\} = 2\frac{d}{dx}\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$$

$$\Rightarrow \frac{1}{x^2+y^2}\times\frac{d}{dx}\left(x^2+y^2\right) = 2\times\frac{1}{1+\left(\frac{y}{x}\right)^2}\times\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{1}{x^2+y^2}\times\left\{2x+2y\frac{dy}{dx}\right\} = \frac{2}{x^2+y^2}\left\{x\frac{dy}{dx}-y\right\}$$

$$\Rightarrow x+y\frac{dy}{dx} = x\frac{dy}{dx}-y$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$$

17.

we have,

$$\cos^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right)=\tan^{-1}a$$

$$\Rightarrow \frac{x^2-y^2}{x^2+y^2}=\cos\left(\tan^{-1}a\right)=k.....\text{say a constant}$$

by componendo – dividendo

$$\Rightarrow \frac{2x^2}{-2y^2}=\frac{k+1}{k-1}$$

$$\Rightarrow \frac{x^2}{y^2}=\frac{1+k}{1-k}$$

differentiating, w.r.t. x

$$\frac{d}{dx}\left(\frac{x^2}{y^2}\right)=\frac{d}{dx}\left(\frac{1+k}{1-k}\right)$$

$$\frac{y^2\frac{d}{dx}x^2-x^2\frac{d}{dx}y^2}{y^4}=0$$

$$y^2\frac{d}{dx}x^2-x^2\frac{d}{dx}y^2=0$$

$$2xy^2-2x^2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{y}{x}$$

18.

The volume of a sphere(V) with radius (r) is given by,

$$V = \frac{4}{3}\pi r^3$$

$\therefore$ Rate of change of volume (V) w.r.t. (t) is given by,

$$\frac{dV}{dt} = \frac{dV}{dr}.\frac{dr}{dt}$$

$$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right).\frac{dr}{dt}$$

$$= 4\pi r^2.\frac{dr}{dt}$$

It is given that $\frac{dV}{dt} = 900\ cm^3 / s$

$$\therefore 900 = \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{900}{4\pi r^2} = \frac{225}{\pi r^2}$$

Therefore, when radius = 15 cm

$$\frac{dr}{dt} = \frac{225}{\pi(15)^2} = \frac{1}{\pi}$$

Hence, the rate at which the radius of the balloon increases when the radius is 15 cm is $\frac{1}{\pi}$ cm / s.

19.

We need to evaluate the integral

$$\int \frac{x+2}{\sqrt{x^2+5x+6}}dx$$

Let $I=\int \frac{x+2}{\sqrt{x^2+5x+6}}dx$

Consider the integrand as follows:

$$\frac{x+2}{\sqrt{x^2+5x+6}}=\frac{A\frac{d}{dx}\left(x^2+5x+6\right)+B}{\sqrt{x^2+5x+6}}$$

$$\Rightarrow x+2=A\left(2x+5\right)+B$$

$$\Rightarrow x+2=\left(2A\right)x+5A+B$$

Comparing the coefficients, we have

$2A=1;\ 5A+B=2$

Solving the above equations, we have

$A=\frac{1}{2}$ and $B=-\frac{1}{2}$

Thus,

$$I=\int \frac{x+2}{\sqrt{x^2+5x+6}}dx$$

$$=\int \frac{\frac{2x+5}{2}-\frac{1}{2}}{\sqrt{x^2+5x+6}}dx$$

$$=\frac{1}{2}\int \frac{2x+5}{\sqrt{x^2+5x+6}}dx-\frac{1}{2}\int \frac{1}{\sqrt{x^2+5x+6}}dx$$

$$I=\frac{1}{2}I_1-\frac{1}{2}I_2,$$

where $I_1 = \int \frac{2x+5}{\sqrt{x^2+5x+6}}dx$

and $I_2 = \int \frac{1}{\sqrt{x^2+5x+6}}dx$

Now consider I_1 :

$$I_1 = \int \frac{2x+5}{\sqrt{x^2+5x+6}}dx$$

Substitute

$x^2+5x+6 = t; (2x+5)dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}}$$

$$= 2\sqrt{t}$$

$$= 2\sqrt{x^2+5x+6}$$

Now consider I_2 :

$$I_2 = \int \frac{1}{\sqrt{x^2+5x+6}}dx$$

$$= \int \frac{1}{\sqrt{x^2+5x+\left(\frac{5}{2}\right)^2+6-\left(\frac{5}{2}\right)^2}}dx$$

$$= \int \frac{1}{\sqrt{\left(x+\frac{5}{2}\right)^2+6-\frac{25}{4}}}dx$$

$$= \int \frac{1}{\sqrt{\left(x+\frac{5}{2}\right)^2-\frac{1}{4}}}dx$$

$$= \int \frac{1}{\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

$$I_2 = \log\left|x+\frac{5}{2}-\sqrt{x^2+5x+6}\right|+C$$

Thus, $I = \frac{1}{2}I_1 - \frac{1}{2}I_2$

$$I = \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|x+\frac{5}{2}-\sqrt{x^2+5x+6}\right|+C$$

20.

Let $I = \int_0^{\pi} \frac{x}{1+\sin x}\, dx$ ....(i)

[By property of definite integrals]

$$I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}\, dx \qquad \text{using} \quad \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\Rightarrow \qquad I = \int_0^{\pi} \frac{\pi - x}{1+\sin x}\, dx$$

$$= \pi \int_0^{\pi} \frac{1}{1+\sin x} dx - \int_0^{\pi} \frac{x}{1+\sin x}\, dx$$

$$= \pi \int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x} dx - I; \qquad (\text{using (i)})$$

$$\Rightarrow \qquad 2I = \pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x}\, dx$$

$$= \pi \int_0^{\pi} \left(\sec^2 x - \tan x.\sec x\right) dx$$

$$= \pi \left[\tan x - \sec x\right]_0^{\pi}$$

$2I = \pi\ [(\tan\pi - \sec\pi)\text{-}(\tan 0 - \sec 0)]$

$2I = \pi\ [0 - (-1) - (0 - 1)] = 2\pi$

$\therefore\ I = \pi$

21.

$$\sin^{-1}\left(\frac{dy}{dx}\right)=x+y$$

$$\Rightarrow \frac{dy}{dx}=\sin(x+y)$$

put $x+y=v$

$$1+\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow \frac{dv}{dx}-1=\sin(v)$$

$$\Rightarrow \frac{dv}{dx}=1+\sin(v)$$

$$\Rightarrow \frac{1}{1+\sin(v)}dv=dx$$

$$\Rightarrow \int\frac{1}{1+\sin(v)}dv=\int dx$$

$$\Rightarrow \int\frac{1-\sin(v)}{1-\sin^2(v)}dv=\int dx$$

$$\Rightarrow \int\frac{1-\sin(v)}{\cos^2(v)}dv=\int dx$$

$$\Rightarrow \int dx=\int\left(\sec^2 v-\tan v.\sec v\right)dv$$

$$\Rightarrow x=\tan v-\sec v+c$$

$\Rightarrow x=\tan(x+y)-\sec(x+y)+c$......which is required solution

OR

$$\frac{dy}{dx}=\left(4x+y+1\right)^2$$

$$\Rightarrow \text{let } 4x+y+1=v$$

$$\Rightarrow 4+\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow \frac{dv}{dx}-4=v^2$$

$$\Rightarrow \frac{dv}{dx}=v^2+4$$

$$\Rightarrow \int\frac{1}{v^2+4}dv=\int dx$$

$$\Rightarrow \frac{1}{2}\tan^{-1}v=x+c$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\left(4x+y+1\right)=x+c$$

as required

22. Given that

$\vec{b} = 2\hat{i} + 4\hat{j} - 5k$

$\vec{c} = \lambda\hat{i} + 2\hat{j} + 3k$

Now consider the sum of the vectors $\vec{b} + \vec{c}$:

$\vec{b} + \vec{c} = \left(2\hat{i} + 4\hat{j} - 5k\right) + \left(\lambda\hat{i} + 2\hat{j} + 3k\right)$

$\Rightarrow \vec{b} + \vec{c} = (2+\lambda)\hat{i} + 6\hat{j} - 2k$

Let $\hat{n}$ be the unit vector along the sum of vectors $\vec{b} + \vec{c}$:

$$\hat{n} = \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2k}{\sqrt{(2+\lambda)^2 + 6^2 + 2^2}}$$

The scalar product of $\vec{a}$ and n is 1. Thus,

$$\vec{a} \cdot \hat{n} = \left(\hat{i} + \hat{j} + k\right) \cdot \left(\frac{(2+\lambda)\hat{i} + 6\hat{j} - 2k}{\sqrt{(2+\lambda)^2 + 6^2 + 2^2}} \right)$$

$$\Rightarrow 1 = \frac{1(2+\lambda) + 1 \cdot 6 - 1 \cdot 2}{\sqrt{(2+\lambda)^2 + 6^2 + 2^2}}$$

$\Rightarrow \sqrt{(2+\lambda)^2 + 6^2 + 2^2} = 2 + \lambda + 6 - 2$

$\Rightarrow \sqrt{(2+\lambda)^2 + 6^2 + 2^2} = \lambda + 6$

$\Rightarrow (2+\lambda)^2 + 40 = (\lambda+6)^2$

$\Rightarrow \lambda^2 + 4\lambda + 4 + 40 = \lambda^2 + 12\lambda + 36$

$\Rightarrow 4\lambda + 44 = 12\lambda + 36$

$\Rightarrow 8\lambda = 8$

$\Rightarrow \lambda = 1$

Thus, n is:

$$n = \frac{(2+1)\hat{i} + 6\hat{j} - 2k}{\sqrt{(2+1)^2 + 6^2 + 2^2}}$$

$$\Rightarrow n = \frac{3\hat{i} + 6\hat{j} - 2k}{\sqrt{3^2 + 6^2 + 2^2}}$$

$$\Rightarrow \hat{n} = \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{49}}$$

$$\Rightarrow \hat{n} = \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{7} \Rightarrow \hat{n} = \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}$$

23.

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$$

The vector form of this equation is:

$$\vec{r}=\left(3\hat{i}+5\hat{j}+7\hat{k}\right)+\lambda\left(\hat{i}-2\hat{j}+\hat{k}\right)$$

$$\vec{r}=\overrightarrow{a_1}+\lambda\vec{b}_1 \qquad \ldots(1)$$

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$$

The vector form of this equation is:

$$\vec{r}=\left(-\hat{i}-\hat{j}-\hat{k}\right)+\lambda\left(7\hat{i}-6\hat{j}+\hat{k}\right)$$

$$\vec{r}=\vec{a}_2+\lambda\vec{b}_2$$

Therefore, $\vec{a}_1=3\hat{i}+5\hat{j}+7\hat{k}, \vec{b}_1=\hat{i}-2\hat{j}+\hat{k}, \vec{a}_2=-\hat{i}-\hat{j}-\hat{k}$ and $\vec{b}_2=7\hat{i}-6\hat{j}+\hat{k}$

Now, the shortest distance between these two lines is given by:

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right)\cdot\left(\vec{a}_2-\vec{a}_1\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right|$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 1\\ 7 & -6 & 1\end{vmatrix}$$

$$=\hat{i}\left(-2+6\right)-\hat{j}\left(1-7\right)+\hat{k}\left(-6+14\right)$$

$$=4\hat{i}+6\hat{j}+8\hat{k}$$

$$\Rightarrow\left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{4^2+6^2+8^2}=\sqrt{116}$$

$$\vec{a}_2-\vec{a}_1=\left(-\hat{i}-\hat{j}-\hat{k}\right)-\left(3\hat{i}+5\hat{j}+7\hat{k}\right)$$

$$=-4\hat{i}-6\hat{j}-8\hat{k}$$

$$\therefore d=\left|\frac{\left(4\hat{i}+6\hat{j}+8\hat{k}\right)\cdot\left(-4\hat{i}-6\hat{j}-8\hat{k}\right)}{\sqrt{116}}\right|=\left|\frac{-16-36-64}{\sqrt{116}}\right|=\left|\frac{-116}{\sqrt{116}}\right|=\sqrt{116}$$

SECTION – D

24.

We shall prove this result by using mathematical induction on n

Step 1:

When n = 1, we have

$$A^1 = \begin{bmatrix} 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

So, result is true for n = 1

Step 2:

Let the result be true for n = k

$$A^k = \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix}$$

Step 3:

Now we shall prove that the result is true for n = k + 1

$$A^{k+1} = \begin{bmatrix} 3^{k+1-1} & 3^{k+1-1} & 3^{k+1-1} \\ 3^{k+1-1} & 3^{k+1-1} & 3^{k+1-1} \\ 3^{k+1-1} & 3^{k+1-1} & 3^{k+1-1} \end{bmatrix}$$

$$= \begin{bmatrix} 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \end{bmatrix}$$

So it holds true for n = k + 1

Thus in general we can say that

$$A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$$

OR

We have,

$$I+A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & -\tan\left(\frac{x}{2}\right)\\ \tan\left(\frac{x}{2}\right) & 0\end{bmatrix}=\begin{bmatrix}1 & -\tan\left(\frac{x}{2}\right)\\ \tan\left(\frac{x}{2}\right) & 1\end{bmatrix}$$

$$I-A=\begin{bmatrix}1 & \tan\left(\frac{x}{2}\right)\\ -\tan\left(\frac{x}{2}\right) & 1\end{bmatrix}$$

$$(I-A)\begin{bmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{bmatrix}=\begin{bmatrix}1 & \tan\left(\frac{x}{2}\right)\\ -\tan\left(\frac{x}{2}\right) & 1\end{bmatrix}\begin{bmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{bmatrix}$$

$$=\begin{bmatrix}1 & \tan\left(\frac{x}{2}\right)\\ -\tan\left(\frac{x}{2}\right) & 1\end{bmatrix}\begin{bmatrix}\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} & -\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\\ \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} & \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\end{bmatrix}$$

for simplicity take $\tan\left(\frac{x}{2}\right)=t$

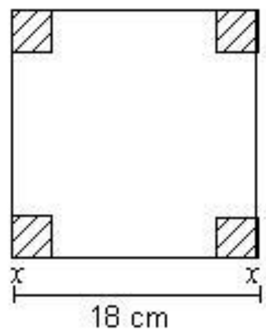
$$=\begin{bmatrix}1 & t\\ -t & 1\end{bmatrix}\begin{bmatrix}\frac{1-t}{1+t^2} & -\frac{2t}{1+t^2}\\ \frac{2t}{1+t^2} & \frac{1-t}{1+t^2}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1+t^2}{1+t^2} & -\frac{t\left(1+t^2\right)}{1+t^2}\\ \frac{t\left(1+t^2\right)}{1+t^2} & \frac{1+t^2}{1+t^2}\end{bmatrix}$$

$$=\begin{bmatrix}1 & -t\\ t & 1\end{bmatrix}=\begin{bmatrix}1 & -\tan\left(\frac{x}{2}\right)\\ \tan\left(\frac{x}{2}\right) & 1\end{bmatrix}=I+A\text{.......hence proved}$$

25.

Let the side of the square piece cut from each corner of the given square plate (side = 18 cm) be x cm. Then the open box has dimensions 18 - 2x, 18 - 2x, x (in cm)



V = Volume of the open box

$=(18-2x)^2 . x$

$=324x-72x^2+4x^3$

$\therefore \frac{dV}{dx}=324-144x+12x^2$

and $\frac{d^2V}{dx^2}=-144+24x.$

For maxima or minima,

$\frac{dV}{dx}=0$

$\Rightarrow 324-144x+12x^2=0$

$\therefore x^2-12x+27=0$

So x = 3, 9

Clearly x = 3

$$\left(\frac{d^2V}{dx^2}\right)_{x=3}=-144+24\times 3=-72<0.$$

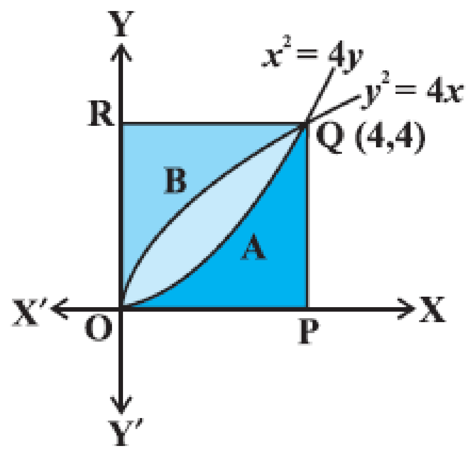
So x = 3 is a point of maxima

∴Volume is maximum, when side of the square cut off is 3 cm.

Maximum volume of box $=\left(18-2\times 3\right)^2\times 3=432\ cm^3$

26.

The point of intersection of the parabolas $y^2 = 4x$ and $x^2 = 4y$ are (0, 0) and (4, 4)



Now, the area of the region OAQBO bounded by curves $y^2 = 4x$ and $x^2 = 4y$

$$\int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx = \left[2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4 = \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{ sq. units} \quad \text{(i)}$$

Again, the area of the region OPQAO bounded by the curves $x^2 = 4y$, $x = 0$, $x = 4$ and the x-axis.

$$\int_0^4 \frac{x^2}{4}dx = \left[\frac{x^3}{12}\right]_0^4 = \left(\frac{64}{12}\right) = \frac{16}{3} \text{ sq units} \quad \text{(ii)}$$

Similarly, the area of the region OBQRO bounded by the curve $y^2 = 4x$ and the y-axis,

$y = 0$ and $y = 4$

$$\int_0^4 \frac{y^2}{4}dy = \left[\frac{y^3}{12}\right]_0^4 = \frac{16}{3} \text{ sq units} \quad \text{(iii)}$$

From (i) (ii), and (iii) it is concluded that

the area of the region OAQBO = area of the region OPQAO = area of the region OBQRO, i.e., area bounded by parabolas $y^2 = 4x$ and $x^2 = 4y$ divides the area of the square in three equal parts.

OR

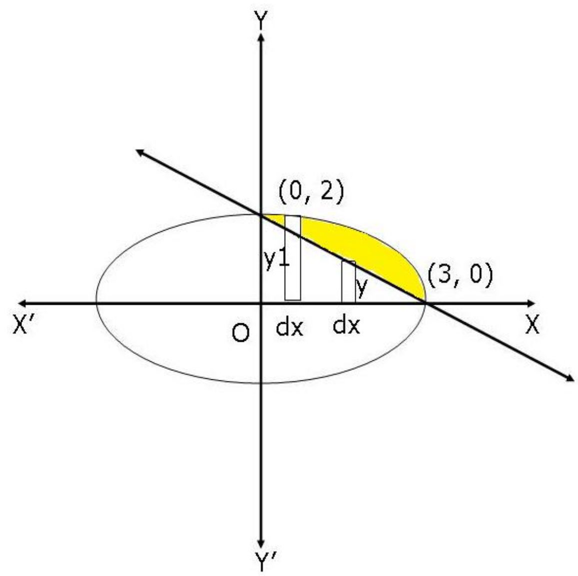
Given ellipse

$$\Rightarrow \frac{x^2}{9}+\frac{y^2}{4}=1$$

$$\Rightarrow y=\frac{2}{3}\sqrt{9-x^2}$$

Given line $\frac{x}{3}+\frac{y}{2}=1$

$$\Rightarrow y=\left(2-\frac{2x}{3}\right)$$



Required Area = $\int_0^3 (y_1 - y_2)\, dx$

$$=\int_0^3 \left[\frac{2}{3}\sqrt{9-x^2}-\left(2-\frac{2x}{3}\right)\right]dx$$

$$=\left[\frac{2}{3}\left(\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\frac{x}{3}\right)-2x+\frac{x^2}{3}\right]_0^3$$

$$=\left[\frac{2}{3}\left(\frac{9}{2}\sin^{-1}1\right)-6+3\right]-0$$

$$=3\times\frac{\pi}{2}-3=\frac{3}{2}(\pi-2)\text{ square units}$$

27. Let $\vec{b}_1$ and $\vec{b}_2$ be the vector parallel to the pair to lines,

$\frac{-x+2}{-2}=\frac{y-1}{7}=\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{2y-8}{4}=\frac{z-5}{4}$, respectively.

Now, $\frac{-x+2}{-2}=\frac{y-1}{7}=\frac{z+3}{-3}$

$\Rightarrow \frac{x-2}{2}=\frac{y-1}{7}=\frac{z+3}{-3}$

$\frac{x+2}{-1}=\frac{2y-8}{4}=\frac{z-5}{4}$

$\Rightarrow \frac{x+2}{-1}=\frac{y-4}{2}=\frac{z-5}{4}$

$\therefore \vec{b}_1 = 2\hat{i} + 7\hat{j} - 3\hat{k}$ and $\vec{b}_2 = -\hat{i} + 2\hat{j} + 4\hat{k}$

$\left|\vec{b}_1\right| = \sqrt{(2)^2+(7)^2+(-3)^2} = \sqrt{62}$

$\left|\vec{b}_2\right| = \sqrt{(-1)^2+(2)^2+(4)^2} = \sqrt{21}$

$\vec{b}_1.\vec{b}_2 = \left(2\hat{i} + 7\hat{j} - 3\hat{k}\right).\left(-\hat{i} + 2\hat{j} + 4\hat{k}\right)$

$= 2(-1) + 7\times 2 + (-3).4$

$= -2 + 14 - 12$

$= 0$

The angle θ between the given pair of lines is given by the relation,

$$\cos\theta = \left|\frac{\vec{b}_1.\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$$

$$\Rightarrow \cos\theta = \frac{0}{\sqrt{62}\times\sqrt{21}} = 0$$

$$\Rightarrow \theta = \cos^{-1}(0) = \frac{\pi}{2}$$

Thus, the given lines are perpendicular to each other and the angle between them is 90°.

OR

Given equation of line is $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$

This can also be written in the standard form as $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-(-4)}{6}$

The vector form of the above equation is,

$$\vec{r}=(\hat{i}+2\hat{j}-4k)+\lambda(2\hat{i}+3\hat{j}+6k)$$
$$\Rightarrow \vec{r}=\overrightarrow{a_1}+\lambda\vec{b} \quad \ldots(1)$$
where, $\overrightarrow{a_1}=\hat{i}+2\hat{j}-4k$ and $\vec{b}=2\hat{i}+3\hat{j}+6k$

The second equation of line is $\frac{x-3}{4}=\frac{y-3}{6}=\frac{z+5}{12}$

The above equation can also be written as $\frac{x-3}{4}=\frac{y-3}{6}=\frac{z-(-5)}{12}$

The vector form of this equation is

$$\vec{r}=(3\hat{i}+3\hat{j}-5k)+\mu(4\hat{i}+6\hat{j}+12k)$$
$$\Rightarrow \vec{r}=(3\hat{i}+3\hat{j}-5k)+2\mu(2\hat{i}+3\hat{j}+6k)$$
$$\Rightarrow \vec{r}=\overrightarrow{a_2}+2\mu\vec{b}\ldots(2)$$
where $\overrightarrow{a_2}=3\hat{i}+3\hat{j}-5k$ and $\vec{b}=2\hat{i}+3\hat{j}+6k$

Since $\vec{b}$ is same in equations (1) and (2), the two lines are parallel.

Distance d, between the two parallel lines is given by the formula,

$$d=\left|\frac{\vec{b}\times(\overrightarrow{a_2}-\overrightarrow{a_1})}{|\vec{b}|}\right|$$

Here, $\vec{b}=(2\hat{i}+3\hat{j}+6k)$ $\quad$ $\overrightarrow{a_2}=(3\hat{i}+3\hat{j}-5k)$ and $\overrightarrow{a_1}=(\hat{i}+2\hat{j}-4k)$

On substitution, we get

$$d = \left| \frac{(2\hat{i}+3\hat{j}+6k)\times((3\hat{i}+3\hat{j}-5k-(\hat{i}+2\hat{j}-4k))}{\sqrt{4+9+36}} \right|$$

$$= \frac{1}{\sqrt{49}}\left|(2\hat{i}+3\hat{j}+6k)\times(2\hat{i}+\hat{j}-k)\right|$$

$$= \frac{1}{7}\begin{vmatrix} \hat{i} & \hat{j} & k \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \frac{1}{7}\left|\hat{i}(-3-6)-\hat{j}(-2-12)+k(2-6)\right|$$

$$= \frac{1}{7}\left|-9\hat{i}+14\hat{j}-4k\right|$$

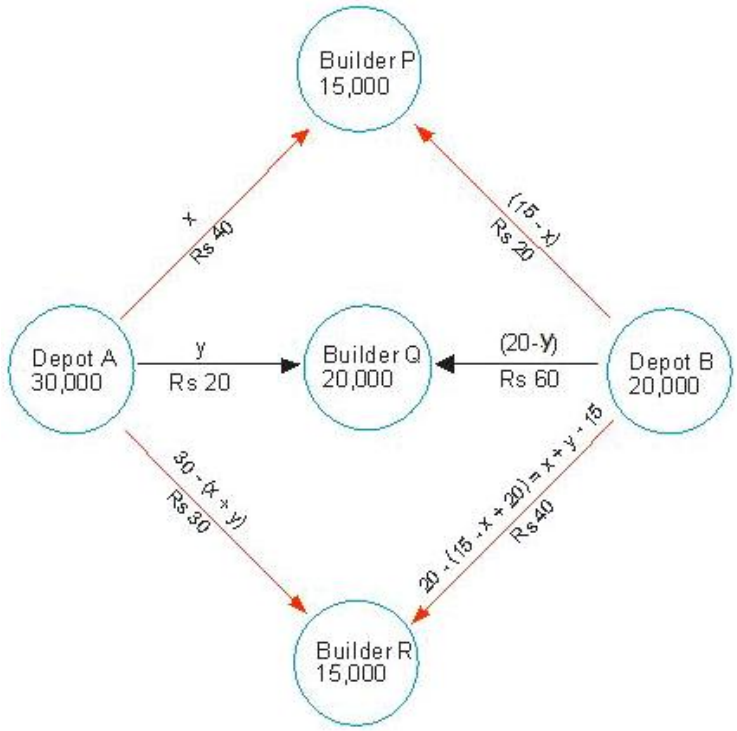
$$= \frac{1}{7}\left|\sqrt{81+196+16}\right|$$

$$= \frac{\sqrt{293}}{7}$$

Thus, the distance between the two given lines is $\frac{\sqrt{293}}{7}$

28.

Let the depot A transport x thousand bricks to builder P and y thousand bricks to builder Q and 30 - (x + y) thousand bricks to builder R. Let the depot B transport (15 – x) thousand bricks to builder P and (20 – y) thousand bricks to builder Q and (x + y) - 15 thousand bricks to builder R



Then, the LPP can be stated mathematically as follows:

Minimise Z = 30x -30y+ 1800

Subject to

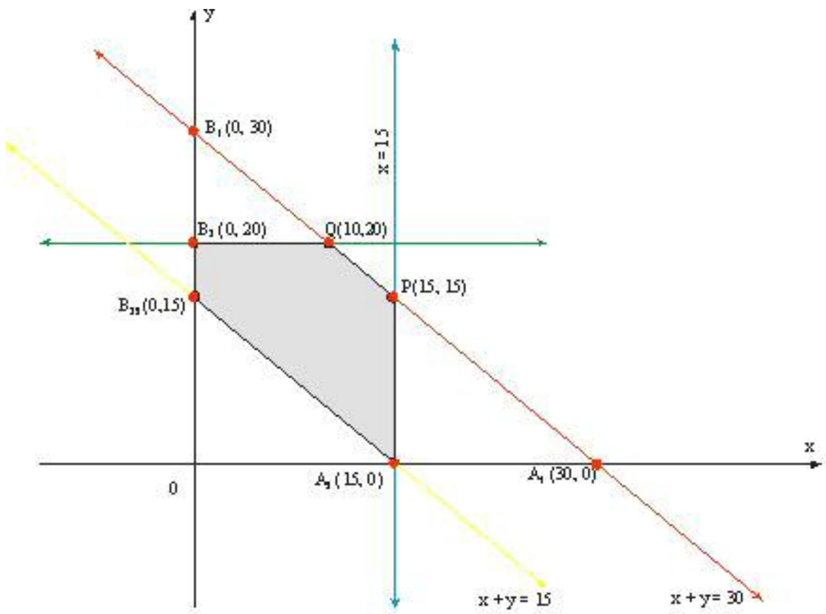
$x + y \leq 30$

$x + y \leq 15$

$x \leq 20$

$y \leq 15$ and $x \geq 0, y \geq 0$

To solve this LPP graphically, we first convert inequations into equations and then draw the corresponding lines. The feasible region of the LPP is shaded in the figure given below. The co-ordinates of the corner points of the feasible region A_2 PQ B_3 B_2 are A_2 (15, 0), P (15, 15), Q (10, 20), B_3 (0, 20) and B_2 (0, 15). These points have been obtained by solving the corresponding intersecting lines simultaneously.



The value of the objective function at the corner points of the feasible region are given in the following table

Point (x, y)	Value of the objective function $X = 30x - 30y + 1800$
A_2 (15, 0)	$Z = 30 \times 15 - 30 \times 0 + 1800 = 2250$
P (15, 15)	$Z = 30 \times 15 - 30 \times 15 + 1800 = 1800$
Q (10, 20)	$Z = 30 \times 10 - 30 \times 20 + 1800 = 1500$
B_3 (0, 20)	$Z = 30 \times 0 - 30 \times 20 + 1800 = 1200$
B_2 (0, 15)	$Z = 30 \times 0 - 30 \times 15 + 1800 = 1350$

Clearly, Z is minimum at x = 0, y = 20 and the minimum value Z is 1200.

Thus, the manufacturer should supply 0, 20 and 10 thousand bricks to builders P, Q and R from depot A and 15, 0 and 5 thousand bricks to builders P, Q and R from dept B respectively. In this case the minimum transportation cost will be Rs. 1200.

29.

S_1: the screw is manufactured by machine X

S_2: the screw is manufactured by machine Y

S_3: the screw is manufactured by machine Z

E: the screw manufactured is defective

Required probability: $P(S_1/E)$

$P(S_1) = 1/6$

$P(S_2) = 1/3$

$P(S_3) = ½$

$$P(E\,|\,S_1) = \frac{1}{100}$$

$$P(E\,|\,S_2) = \frac{3}{200}$$

$$P(E\,|\,S_3) = \frac{2}{100}$$

$$P(S_1|E) = \frac{P(S_1)(P(E|S_1)}{P(S_1)(P(E|S_1)) + P(S_2)(P(E|S_2)) + P(S_3)(P(E|S_3))}$$

$$= \frac{\frac{1}{6} \times \frac{1}{100}}{\frac{1}{6} \times \frac{1}{100} + \frac{1}{3} \times \frac{3}{200} + \frac{1}{2} \times \frac{2}{100}}$$

$$= \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{2} + 1}$$

$$= \frac{1}{1+3+6} = \frac{1}{10}$$