GOVERNMENT OF TAMIL NADU

MATHEMATICS

HIGHER SECONDARY FIRST YEAR

VOLUME - I

Untouchability is Inhuman and a Crime

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NOT FOR SALE

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HOW TO USE THE BOOK

Scope of Mathematics

- Awareness on the scope of higher eductaional opportunities; courses, institutions and required competitive examinations.
- Possible financial assistance to help students climb academic ladder.

Learning Objectives:

- Overview of the unit
- Give clarity on the intended learning outcomes of the unit.

DO YOU KNOW?

- Additional facts related to the topics have been included to arouse interest for searching of more information for deeper and wider learning.

- Visual representation of concepts with illustrations
- Videos, animations, and tutorials.

ICT

- To increase the span of attention of concepts
- To visualize the concepts for strengthening and understanding
- To link concepts related to one unit with other units.
- To utilize the digital skills in classroom learning and providing students experimental learning.

Summary

- Recapitutation of the salient points of each chapter for recalling the concepts learnt.

Evaluation

- Assessing student's understanding of concepts and get them acquainted with solving exercise problems.

Books for Reference

- List of relevant books for further reading.

Scope for Higher Order Thinking

- To motivate students aspiring to take up competitive examinations such as JEE, KVPY, Math olympiad, etc., the concepts and questions based on Higher Order Thinking are incorporated in the content of this book.

Glossary

- Frequently used Mathematical terms have been given with their Tamil equivalents.

Mathematics Learning

The correct way to learn is to understand the concepts throughly. Each chapter opens with an Introduction, Learning Objectives, Various Definitions, Theorems, Results and Illustrations. These in turn are followed by solved examples and exercise problems which have been classified in to various types for quick and effective revision. One can develop the skill of solving mathematical problems only by doing them. So the teacher's role is to teach the basic concepts and problems related to it and to scaffold students to try the other problems on their own. Since the first year of Higher Secondary is considered to be the foundation for learning higher mathematics, the students must be given more attention to each and every concept mentioned in this book.

CONTENTS

MATHEMATICS

2197M2

E-book

21I3NP

Assessment

B162_11_MAT_EM

DIGI links

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- Open the QR code scanner application
- Once the scanner button in the application is clicked, camera opens and then bring it closer to the QR code in the text book.
- Once the camera detects the QR code, a url appears in the screen.Click the url and goto the content page.

Scope for students after completing Higher Secondary

EXAMS

MEDICAL RELATED COURSES:	
JEE (Main & Advanced)	Bio Informatics
NEET, JIPMER, AIIMS	M.B.B.S / B.D.S
ENGINEERING RELATED COURSES:	
JEE (Main & Advanced)	B.E. / B.Tech. / B.Arch in IITs and NITs
B.Tech.	Avionics / Aerospace
B.Tech. & M.Tech	Dual degree programme in IIST Trivandram.
B.E	Agricultural Engineering (TamilNadu Agricultural University Entrance Exam)
CET	Common Entrance Test for Maritime Courses (www.imu.edu.in)
BITS	Birla Institute of Technology and science Admission Test (www.bits – pilani.ac.in)
NATA	National Aptitude Test in Architecture (www.nata.in)
IIIT	Indian Institute of Information Technology (www.iiit.ac.in)
BASIC SCIENCE COURSES:	
TIFR	Tata Institute of Fundamental Research conducts Graduate School admission examination (Univ.tirf.res.in)
NEST, NISER	National Institute of Science Education and Research (www.niser.ac.in)
IISC	Indian Institute of Science (www.iisc.ac.in)
RIE	Regional Institute of Education
CEE	Common Entrance Examination (www.rie ajmer.raj.nic.in)
CUCET	Central Universities Common Entrance Test (http://Ucetexam.in)
GATE	Graduate Aptitude Test in Engineering (gate.iitg.ac.in)
NET	CSIR and UGC - National Eligibility Test (http://cbsenet.nic.in)
JAM	Joint Admission Test for M.Sc (Jam.iitb.ac.in)
ISI	Indian Statistical Institute Admission Test (www.isical.ac.in)
IISER - KVPY / IIT – JEE / NEET	Indian Institute of science Education and Research (www.iiser.ac.in & www.iiseradmission.in)

AFTER COMPLETING +2

UNDER GRADURATE COURSES:	
B.Sc	Mathematics / Applied Mathematics / Statistics
B.Sc	Physics, Chemistry
B.Stat.(Hons.)	Bachelor of Statistics
B.Math (Hons.)	Bachelor of Mathematics
B.S. - M.S Dual Degree	5 years programme
B.Sc. Ed.	4 years programme
POST GRADURATE COURSES:	
M.Sc	Mathematics / Applied Mathematics / Statistics
M.Sc	Astronomy
M.Sc	Operational Research
M.Stat.(Hons.)	Master of Statistics
M.Math (Hons.)	Master of Mathematics
M.Sc. Ed.	6 years programme
M.Sc	Mathematics - 5 years Integrated programme

Job and Scholarship Opportunities

JOB OPPORTUNITIES

- Indian Forest Services
- Scientist Jobs in ISRO, DRDO, CSIR labs
- Union Public Service Commission
- Staff Selection Commission
- Indian Defense Services
- Public Sector Bank
- Tax Assistant
- Statistical Investigator
- Combined Graduate Level Exam
- Tamil Nadu Public Service Commission
- Teaching Profession

FINANCIAL ASSISTANCE

- Scholarship for Graduate and Post – Graduate courses
- NTSE at the end of X (from class XI to Ph.D)
- International Olympiad: for getting stipend for Higher Education in Science and Mathematics
- DST – INSPIRE Scholarships (for UG and PG)
- DST – INSPIRE Fellowships (for Ph.D)
- UGC National Fellowships (for Ph.D)
- Indira Gandhi Fellowship for Single Girl Child (for UG and PG)
- Moulana Azad Fellowship for minorities (for Ph.D)
- In addition various fellowships for SC / ST / PWD / OBC etc are available. (Visit website of University Grants Commission (UGC) and Department of Science and Technology (DST))
- University Fellowships
- Tamil Nadu Collegiate Education fellowship.

Institutes in India to pursue research in Mathematics

RESEARCH INSTITUTIONS IN VARIOUS AREAS OF SCIENCE

NAME OF THE INSTITUTION	WEBSITE
Indian Institute of Science (IISc) Bangalore	www.iisc.ac.in
Chennai Mathematical Institute (CMI) Chennaie	www.cmi.ac.in
Tata Institue of Fundamental Research (TIFR) Mumbai	www.tifr.res.in
Indian Institute of Space Science and Technology (IIST) Trivandrum	www.iist.ac.in
National Institute of Science Education and Research (NISER)	www.niser.ac.in
Birla Institute of Technology and Science, Pilani	www.bits-pilani.ac.in
Indian Institute of Science Education and Research	www.iiseradmission.in
Anna University	https://www.annauniv.edu/
Indian Institute of Technology in various places (IIT's)	www.iitm.ac.in
National Institute of Technology (NITs)	www.nitt.edu
Central Universities	www.cucet.ac.in
State Universities	www.ugc.ac.in
Tamil Nadu agricultural University (tnau.ac.in)	tnau.ac.in
International Institute of Information Technology	www.iiit.ac.in
The Institute of Mathematical Sciences (IMSC) Chennai.	www.imsc.res.in
Hyderabad Central university, Hyderabad.	www.uohyd.ac.in
Delhi University, Delhi	www.du.ac.in
Mumbai University, Mumbai	www.mu.ac.in
Savithiri Bai Phule Pune University, Pune	www.unipune.ac.in
Indian Statistical Institute	www.iisical.ac.in
Regional Institute of Education	www.rieajmer.raj.nic.in

Chapter 1 Sets, Relations and Functions

"a set is many that allows itself to be thought of as a one"

Cantor

1.1 Introduction

The concepts of sets, relations and functions occupy a fundamental place in the mainstream of mathematical thinking. As rightly stated by the Russian mathematician Luzin the concept of functions did not arise suddenly. It underwent profound changes in time. Galileo (1564-1642) explicitly used the dependency of one quantity on another in the study of planetary motions. Descartes (1596-1650) clearly stated that an equation in two variables, geometrically represented by a curve, indicates dependence between variable quantities. Leibnitz (1646-1716) used the word "function", in a 1673 manuscript, to mean any quantity varying from point to point of a curve. Dirichlet (1805-1859), a student of Gauss, was credited with the modern "formal" definition of function with notation $y = f(x)$. In the 20^{th} century, this concept was extended to include all arbitrary correspondence satisfying the uniqueness condition between sets and numerical or non-numerical values.

With the development of set theory, initiated by Cantor (1845-1918), the notion of function continued to evolve. From the notion of correspondence, mathematicians moved to the notion of relation. However even now in the theory of computation, a function is not viewed as a relation but as a computational rule. The modern definition of a function is given in terms of relation so as to suit to develop artificial intelligence.

Cantor
1845 - 1918

In the previous classes, we have studied and are well versed with the real numbers and arithmetic operations on them. We also learnt about sets of real numbers, Venn diagrams, Cartesian product of sets, basic definitions of relations and functions. For better understanding, we recall more about sets and Cartesian products of sets. In this chapter, we see a new facelift to the mathematical notions of "Relations" and "Functions".

Learning Objectives

On completion of this chapter, the students are expected to

- list and work with many properties of sets and Cartesian product;
- know the concepts of constants, variables, intervals and neighbourhoods;
- understand about various types of relations; create relations of any required type;
- represent functions in different ways;
- work with elementary functions, types of functions, operations on functions including inverse of a bijective function;
- identify the graphs of some special functions;
- visualize and sketch the graphs of some relatively complicated functions.

1.2 Sets

In the earlier classes, we have seen that a ***set*** is a collection of well-defined objects. As the theory of sets is the building blocks of modern mathematics, one has to learn the concepts of sets carefully and deeply. Now we look at the term "well-defined" a little more deeply. Consider the two statements:

(i) The collection of all beautiful flowers in Ooty Rose Garden.
(ii) The collection of all old men in Tamilnadu.

The terms "beautiful flowers" and "old men" are not well-defined. We cannot define the term "beautiful flower" in a sharp way as there is no concrete definition for beauty because the concept of beauty varies from person to person, content to content and object to object. We should not consider statements like "the collection of all beautiful flowers in Ooty Rose Garden" as a set. Now, can we say "the collection of all red flowers in Ooty Rose Garden" a set? The answer is "yes".

One may consider a person of age 60 as old and others may not agree. There is no specific and concrete definition for "old men". The second statement can be made more sharply as

"the collection of all men in Tamilnadu of age greater than 70".

Now, the above collection becomes a set because of definiteness in the age. Thus, the description of a set should enable us to concretely decide whether a given particular object (element) is available in the collection or not. So set is a distinguishable collection of objects.

We have also seen and learnt to use symbols like $\in$, $\subset$, $\subseteq$, $\cup$ and $\cap$. Let us start with the question:

"If A and B are two sets, is it meaningful to write $A \in B$?".

At the first sight one may hurry to say that this is always meaningless by telling, "the symbol $\in$ should be used between an element and a set and it should not be used between two sets". The first part of the statement is true whereas the second part is not true. For example, if $A = \{1, 2\}$ and $B = \{1, \{1, 2\}, 3, 4\}$, then $A \in B$. In this section we shall discuss the meaning of such symbols more deeply.

As we learnt in the earlier classes the set containing no elements is called an ***empty set*** or a ***void set***. It is usually denoted by $\emptyset$ or $\{\ \}$. By $A \subseteq B$, we mean every element of the set A is an element of the set B. In this case, we say A is a ***subset*** of B and B is a ***super set*** of A. For any two sets A and B, if $A \subseteq B$ and $B \subseteq A$, then the two sets are ***equal***. For any set A, the empty set $\emptyset$ and the set A are always subsets of A. These two subsets are called ***trivial subsets***. Further, we say A is a ***proper subset*** of B if A is a subset of B and $A \neq B$. That is, B contains all elements of A and at least one element which is not in A. Note that, as every element of A is an element of A, we have $A \subseteq A$. Thus, any set is a subset of itself. This subset is called an ***improper subset***. In other words, for any set A, A is the improper subset of A. It is known that, $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, where $\mathbb{N}$ denotes the set of all natural numbers or positive integers, $\mathbb{W}$ denotes the set of all non-negative integers, $\mathbb{Z}$ denotes the set of all integers, $\mathbb{Q}$ denotes the set of all rational numbers and $\mathbb{R}$ denotes the set of all real numbers. Note that, the set of all irrational numbers is a subset of $\mathbb{R}$ but not a subset of any other set mentioned above.

We learnt that the ***union*** of two sets A and B is denoted by $A \cup B$ and is defined as

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

and the ***intersection*** as

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Two sets A and B are ***disjoint*** if they do not have any common element. That is, A and B are disjoint if $A \cap B = \emptyset$.

Let us see some more notations. We are familiar with notations like $\sum_{i=1}^{n} a_i$. This in fact stands for $a_1 + a_2 + \cdots + a_n$. Similarly we can use the notations $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$ to denote $A_1 \cup A_2 \cup \cdots \cup A_n$ and $A_1 \cap A_2 \cap \cdots \cap A_n$ respectively.

Thus, $\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for some } i\}$ and $\bigcap_{i=1}^{n} A_i = \{x : x \in A_i \text{ for each } i\}$. These notations are useful when we discuss more number of sets.

If A is a set, then the set of all subsets of A is called the ***power set*** of A and is usually denoted as $\mathscr{P}(A)$. That is, $\mathscr{P}(A) = \{B : B \subseteq A\}$. The number of elements in $\mathscr{P}(A)$ is 2^n, where n is the number of elements in A.

Now, to define the complement of a set, it is necessary to know about the concept of universal set. Usually all sets under consideration in a mathematical process are assumed to be subsets of some fixed set. This basic set is called the ***universal set***. For example, depending on the situation, for the set of prime numbers, the universal set can be any one of the sets containing the set of prime numbers. Thus, one of the sets $\mathbb{N}, \mathbb{W}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ may be taken as a universal set for the set of prime numbers, depending on the requirement. Universal set is usually denoted by U.

To define the complement of a set, we have to fix the universal set. Let A be a subset of the universal set U. The ***complement*** of A with respect to U is denoted as A' or A^c and defined as $A' = \{x : x \in U \text{ and } x \notin A\}$.

The ***set difference*** of the set A to the set B is denoted by either $A - B$ or $A \smallsetminus B$ and is defined as

$$A - B = \{a : a \in A \text{ and } a \notin B\}.$$

Note that,

(i) $U - A = A'$ (ii) $A - A = \emptyset$ (iii) $\emptyset - A = \emptyset$ (iv) $A - \emptyset = A$ (v) $A - U = \emptyset$.

The ***symmetric difference*** between two sets A and B is denoted by $A\Delta B$ and is defined as $A\Delta B = (A - B) \cup (B - A)$. Actually the elements of $A\Delta B$ are the elements of $A \cup B$ which are not in $A \cap B$. Thus $A\Delta B = (A \cup B) - (A \cap B)$.

A set X is said to be a ***finite set*** if it has k elements for some $k \in \mathbb{W}$. In this case, we say the finite set X is of ***cardinality*** k and is denoted by $n(X)$. A set is an ***infinite set*** if it is not finite. For an infinite set A, the cardinality is infinity. If $n(A) = 1$, then it is called a ***singleton set***. Note that $n(\emptyset) = 0$ and $n(\{\emptyset\}) = 1$.

1.2.1 Properties of Set Operations

We now list out some of the properties.

Commutative

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$.

Associative

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $(A \cap B) \cap C = A \cap (B \cap C)$.

Distributive

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Identity

(i) $A \cup \emptyset = A$ (ii) $A \cap U = A$.

Idempotent

(i) $A \cup A = A$ (ii) $A \cap A = A$.

Absorption

(i) $A \cup (A \cap B) = A$ (ii) $A \cap (A \cup B) = A$.

De Morgan Laws

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$
(iii) $A - (B \cup C) = (A - B) \cap (A - C)$ (iv) $A - (B \cap C) = (A - B) \cup (A - C)$.

On Symmetric Difference

(i) $A \Delta B = B \Delta A$ (ii) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$
(iii) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$.

On Empty Set and Universal Set

(i) $\emptyset' = U$ (ii) $U' = \emptyset$
(iii) $A \cup A' = U$ (iv) $A \cap A' = \emptyset$
(v) $A \cup U = U$ (vi) $A \cap U = A$.

On Cardinality

(i) For any two finite sets A and B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
(ii) If A and B are disjoint finite sets, then $n(A \cup B) = n(A) + n(B)$.
(iii) For any three finite sets A, B and C,
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$.

1.3 Cartesian Product

We know that the Cartesian product of sets is nothing but a set of ordered elements. In particular, Cartesian product of two sets is a set of ordered pairs, while the Cartesian product of three sets is a set of ordered triplets. Precisely, let A, B and C be three non-empty sets. Then the ***Cartesian product*** of A with B is denoted by $A \times B$. It is defined by

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Similarly, the Cartesian product $A \times B \times C$ is defined by

$$A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}.$$

Thus $A \times A = \{(a, b) : a, b \in A\}$.

$\boxed{\text{Is it correct to say } A \times A = \{(a, a) : a \in A\}?}$

It is important that the elements of the Cartesian product are ordered and hence, for non-empty sets,

$$A \times B \neq B \times A, \text{ unless } A = B.$$

That is, $A \times B = B \times A$ only if $A = B$. We know that $\mathbb{R}$ denotes the set of real numbers and

$$\begin{aligned} \mathbb{R} \times \mathbb{R} &= \{(x, y) : x, y \in \mathbb{R}\}. \\ \mathbb{R} \times \mathbb{R} \times \mathbb{R} &= \{(x, y, z) : x, y, z \in \mathbb{R}\}. \end{aligned}$$

Symbolically, $\mathbb{R} \times \mathbb{R}$ can be represented as $\mathbb{R}^2$ and $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ as $\mathbb{R}^3$. Note that $\mathbb{R} \times \mathbb{R}$ is a set of ordered pairs and $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is a set of ordered triplets.

If $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$ then

$$A \times B = \{(1, 2), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 2), (3, 4), (3, 6)\}.$$

Here $A \times B$ is a subset of $\mathbb{R} \times \mathbb{R}$. The number of elements in $A \times B$ is the product of the number of elements in A and the number of elements in B, that is, $n(A \times B) = n(A)n(B)$, if A and B are finite. Further $n(A \times B \times C) = n(A)n(B)n(C)$, if A, B and C are finite.

It is easy to see that the following are the subsets of $\mathbb{R} \times \mathbb{R}$.

(i) $\{(x, 2x) : x \in \mathbb{R}\}$

(ii) $\{(x, x^2) : x \in \mathbb{R}\}$

(iii) $\{(x, \sqrt{x}) : x$ is a non-negative real number$\}$

(iv) $\{(x^2, x) : x \in \mathbb{R}\}$.

(v) $\{(x, -\sqrt{x}) : x$ is a non-negative real number$\}$

Example 1.1 Find the number of subsets of A if $A = \{x : x = 4n + 1,\ 2 \leq n \leq 5, n \in \mathbb{N}\}$.

Solution:
Clearly $A = \{x : x = 4n + 1,\ n = 2, 3, 4, 5\} = \{9, 13, 17, 21\}$.
Hence $n(A) = 4$. This implies that $n(\mathscr{P}(A)) = 2^4 = 16$.

Example 1.2 In a survey of 5000 persons in a town, it was found that 45% of the persons know Language A, 25% know Language B, 10% know Language C, 5% know Languages A and B, 4% know Languages B and C, and 4% know Languages A and C. If 3% of the persons know all the three Languages, find the number of persons who knows only Language A.

Solution:
This problem can be solved either by property of cardinality or by Venn diagram.

Cardinality: Given that $n(A) = 45\%$ of $5000 = 2250$

Similarly, $n(B) = 1250, n(C) = 500, n(A \cap B) = 250, n(B \cap C) = 200, n(C \cap A) = 200$ and $n(A \cap B \cap C) = 150$.

The number of persons who knows only Language A is

$$\begin{aligned} n(A \cap B' \cap C') &= n\{A \cap (B \cup C)'\} = n(A) - n\{A \cap (B \cup C)\}. \\ &= n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C). \\ &= 2250 - 250 - 200 + 150 = 1950. \end{aligned}$$

Thus the required number of persons is 1950.

Venn diagram: We draw the Venn Diagram using percentage.

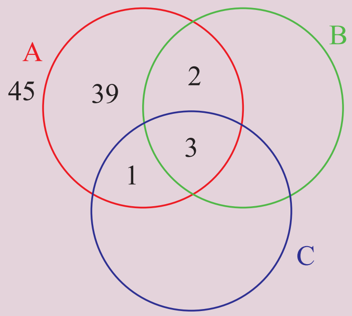


Figure 1.1

From Figure 1.1, the percentage of persons who knows only Language A is 39. Therefore, the required number of persons is $5000 \times \frac{39}{100} = 1950$.

Example 1.3 Prove that

$$((A \cup B' \cup C) \cap (A \cap B' \cap C')) \cup ((A \cup B \cup C') \cap (B' \cap C')) = B' \cap C'.$$

Solution:
We have $A \cap B' \cap C' \subseteq A \subseteq A \cup B' \cup C$ and hence $(A \cup B' \cup C) \cap (A \cap B' \cap C') = A \cap B' \cap C'$.
Also, $B' \cap C' \subseteq C' \subseteq A \cup B \cup C'$ and hence $(A \cup B \cup C') \cap (B' \cap C') = B' \cap C'$.
Now as $A \cap B' \cap C' \subseteq B' \cap C'$, we have

$$((A \cup B' \cup C) \cap (A \cap B' \cap C')) \cup ((A \cup B \cup C') \cap (B' \cap C')) = B' \cap C'.$$

Try to simplify the above expression using Venn diagram.

Example 1.4 If $X = \{1, 2, 3, \ldots 10\}$ and $A = \{1, 2, 3, 4, 5\}$, find the number of sets $B \subseteq X$ such that $A - B = \{4\}$

Solution:
For every subset C of $\{6, 7, 8, 9, 10\}$, let $B = C \cup \{1, 2, 3, 5\}$. Then $A - B = \{4\}$. In other words, for every subset C of $\{6, 7, 8, 9, 10\}$, we have a unique set B so that $A - B = \{4\}$. So number of sets $B \subseteq X$ such that $A - B = \{4\}$ and the number of subsets of $\{6, 7, 8, 9, 10\}$ are the same. So the number of sets $B \subseteq X$ such that $A - B = \{4\}$ is $2^5 = 32$.

Example 1.5 If A and B are two sets so that $n(B - A) = 2n(A - B) = 4n(A \cap B)$ and if $n(A \cup B) = 14$, then find $n(\mathscr{P}(A))$.

Solution:
To find $n(\mathscr{P}(A))$, we need $n(A)$.
Let $n(A \cap B) = k$. Then $n(A - B) = 2k$ and $n(B - A) = 4k$.
Now $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B) = 7k$.
It is given that $n(A \cup B) = 14$. Thus $7k = 14$ and hence $k = 2$.
So $n(A - B) = 4$ and $n(B - A) = 8$. As $n(A) = n(A - B) + n(A \cap B)$, we get $n(A) = 6$ and hence $n(\mathscr{P}(A)) = 2^6 = 64$.

Example 1.6 Two sets have m and k elements. If the total number of subsets of the first set is 112 more than that of the second set, find the values of m and k.

Solution:
Let A and B be the two sets with $n(A) = m$ and $n(B) = k$. Since A contains more elements than B, we have $m > k$. From the given conditions we see that $2^m - 2^k = 112$. Thus we get, $2^k(2^{m-k} - 1) = 2^4 \times 7$.

Then the only possibility is $k = 4$ and $2^{m-k} - 1 = 7$. So $m - k = 3$ and hence $m = 7$.

Example 1.7 If $n(A) = 10$ and $n(A \cap B) = 3$, find $n((A \cap B)' \cap A)$.

Solution:
$(A \cap B)' \cap A = (A' \cup B') \cap A = (A' \cap A) \cup (B' \cap A) = \emptyset \cup (B' \cap A) = (B' \cap A) = A - B$.
So $n((A \cap B)' \cap A) = n(A - B) = n(A) - n(A \cap B) = 7$.

Example 1.8 If $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, find $n((A \cup B) \times (A \cap B) \times (A \Delta B))$.

Solution:
We have $n(A \cup B) = 6, n(A \cap B) = 2$ and $n(A \Delta B) = 4$.
So, $n((A \cup B) \times (A \cap B) \times (A \Delta B)) = n(A \cup B) \times n(A \cap B) \times n(A \Delta B) = 6 \times 2 \times 4 = 48$.

Example 1.9 If $\mathscr{P}(A)$ denotes the power set of A, then find $n(\mathscr{P}(\mathscr{P}(\mathscr{P}(\emptyset))))$.

Solution:
Since $\mathscr{P}(\emptyset)$ contains 1 element, $\mathscr{P}(\mathscr{P}(\emptyset))$ contains 2^1 elements and hence $\mathscr{P}(\mathscr{P}(\mathscr{P}(\emptyset)))$ contains 2^2 elements. That is, 4 elements.

Exercise - 1.1

1. Write the following in roster form.
   (i) $\{x \in \mathbb{N} : x^2 < 121 \text{ and } x \text{ is a prime}\}$.
   (ii) the set of all positive roots of the equation $(x - 1)(x + 1)(x^2 - 1) = 0$.
   (iii) $\{x \in \mathbb{N} : 4x + 9 < 52\}$.
   (iv) $\{x : \frac{x-4}{x+2} = 3, x \in \mathbb{R} - \{-2\}\}$.
2. Write the set $\{-1, 1\}$ in set builder form.
3. State whether the following sets are finite or infinite.
   (i) $\{x \in \mathbb{N} : x \text{ is an even prime number}\}$.
   (ii) $\{x \in \mathbb{N} : x \text{ is an odd prime number}\}$.
   (iii) $\{x \in \mathbb{Z} : x \text{ is even and less than } 10\}$.
   (iv) $\{x \in \mathbb{R} : x \text{ is a rational number}\}$.
   (v) $\{x \in \mathbb{N} : x \text{ is a rational number}\}$.
4. By taking suitable sets A, B, C, verify the following results:
   (i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
   (ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
   (iii) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$.
   (iv) $C - (B - A) = (C \cap A) \cup (C \cap B')$.
   (v) $(B - A) \cap C = (B \cap C) - A = B \cap (C - A)$.
   (vi) $(B - A) \cup C = (B \cup C) - (A - C)$.
5. Justify the trueness of the statement:

   "An element of a set can never be a subset of itself."
6. If $n(\mathscr{P}(A)) = 1024, n(A \cup B) = 15$ and $n(\mathscr{P}(B)) = 32$, then find $n(A \cap B)$.
7. If $n(A \cap B) = 3$ and $n(A \cup B) = 10$, then find $n(\mathscr{P}(A \Delta B))$.
8. For a set $A, A \times A$ contains 16 elements and two of its elements are $(1, 3)$ and $(0, 2)$. Find the elements of A.

9. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If $(x, 1), (y, 2), (z, 1)$ are in $A \times B$, find A and B, where x, y, z are distinct elements.
10. If $A \times A$ has 16 elements, $S = \{(a, b) \in A \times A : a < b\}$; $(-1, 2)$ and $(0, 1)$ are two elements of S, then find the remaining elements of S.

1.4 Constants and Variables, Intervals and Neighbourhoods

To continue our discussion, we need certain prerequisites namely, constants, variables, independent variables, dependent variables, intervals and neighbourhoods.

1.4.1 Constants and Variables

A quantity that remains unaltered throughout a mathematical process is called a ***constant***. A quantity that varies in a mathematical process is called a ***variable***. A variable is an ***independent variable*** when it takes any arbitrary (independent) value not depending on any other variables, whereas if its value depends on other variables, then it is called a ***dependent variable***.

We know the area A of a triangle is given by $A = \frac{1}{2}bh$. Here $\frac{1}{2}$ is a constant and A, b, h are variables. Moreover b and h are independent variables and A is a dependent variable. We ought to note that the terms *dependent* and *independent* are relative terms. For example in the equation $x + y = 1$, x, y are variables and 1 is a constant. Which of x and y is dependent and which one is independent? If we consider x as an independent variable, then y becomes a dependent whereas if we consider y as an independent variable, then x becomes dependent.

Further consider the following examples:

(i) area of a rectangle $A = \ell b$.
(ii) area of a circle $A = \pi r^2$.
(iii) volume of a cuboid $V = \ell bh$.

From the above examples we can directly infer that b, h, ℓ, r are independent variables; A and V are dependent variables and π is a constant.

1.4.2 Intervals and Neighbourhoods

The system $\mathbb{R}$ of real numbers can be represented by the points on a line and a point on the line can be related to a unique real number as in Figure 1.2. By this, we mean that any real number can be identified as a point on the line. With this identification we call the line as the ***real line***.

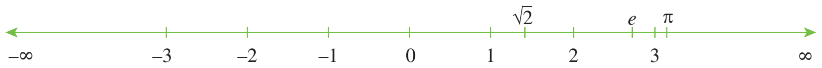


Figure 1.2

The value increases as we go right and decreases as we go left. If x lies to the left of y on the real line then $x < y$. As there is no gap in a line, we have infinitely many real numbers between any two real numbers.

> **Definition 1.1**
>
> A subset I of $\mathbb{R}$ is said to be an ***interval*** if
>
> (i) I contains at least two elements and
> (ii) $a, b \in I$ and $a < c < b$ then $c \in I$.
>
> Geometrically, intervals correspond to rays and line segments on the real line.

Note that the set of all natural numbers, the set of all non-negative integers, set of all odd integers, set of all even integers, set of all prime numbers are not intervals. Further observe that, between any two real numbers there are infinitely many real numbers and hence the above examples are not intervals.

Consider the following sets:

(i) The set of all real numbers greater than 0.
(ii) The set of all real numbers greater than 5 and less than 7.
(iii) The set of all real numbers x such that $1 \leq x \leq 3$.
(iv) The set of all real numbers x such that $1 < x \leq 2$.

The above four sets are intervals. In particular (i) is an infinite interval and (ii), (iii) and (iv) are finite intervals. The term "finite interval" does not mean that the interval contains only finitely many real numbers, however both ends are finite numbers. Both finite and infinite intervals are infinite sets. The intervals correspond to line segments are finite intervals whereas the intervals that correspond to rays and the entire real line are infinite intervals.

A finite interval is said to be ***closed*** if it contains both of its end points and ***open*** if it contains neither of its end points. Symbolically the above four intervals can be written as $(0, \infty), (5, 7), [1, 3], (1, 2]$. Note that for symbolic form we used parentheses and square brackets to denote intervals. ( ) parentheses indicate open interval and [ ] square brackets indicate closed interval. The first two examples are open intervals, third one is a closed interval. Note that fourth example is neither open nor closed, that is, one end open and other end closed.

In particular $[1, 3]$ contains both 1 and 3 and in between real numbers. The interval $(1, 3)$ does not contain 1 and 3 but contains all in between the numbers. The interval $(1, 2]$ does not contain 1 but contains 2 and all in between numbers.

Note that ∞ is not a number. The symbols $-\infty$ and ∞ are used to indicate the ends of real line. Further, the intervals (a, b) and $[a, b]$ are subsets of $\mathbb{R}$.

Type of Intervals

There are many types of intervals. Let $a, b \in \mathbb{R}$ such that $a < b$. The following table describes various types of intervals. It is not possible to draw a line if a point is removed. So we use an unfilled circle "○" to indicate that the point is removed and use a filled circle "●" to indicate that the point is included.

Interval	*Notation*	*Set*	*Diagramatic Representation*
finite	(a, b)	$\{x : a < x < b\}$	 a b
	$[a, b]$	$\{x : a \leq x \leq b\}$	 a b
	$(a, b]$	$\{x : a < x \leq b\}$	 a b
	$[a, b)$	$\{x : a \leq x < b\}$	 a b
infinite	(a, ∞)	$\{x : a < x < \infty\}$	 a
	$[a, \infty)$	$\{x : a \leq x < \infty\}$	 a
	$(-\infty, b)$	$\{x : -\infty < x < b\}$	 b
	$(-\infty, b]$	$\{x : -\infty < x \leq b\}$	 b
	$(-\infty, \infty)$ or $\mathbb{R}$	$\{x : -\infty < x < \infty\}$ or the set of real numbers	

Try to write the following intervals in symbolic form.

(i) $\{x : x \in \mathbb{R}, -2 \leq x \leq 0\}$ (ii) $\{x : x \in \mathbb{R}, 0 < x < 8\}$

(iii) $\{x : x \in \mathbb{R}, -8 < x \leq -2\}$ (iv) $\{x : x \in \mathbb{R}, -5 \leq x \leq 9\}$.

Neighbourhood

Neighbourhood of a point 'a' is any open interval containing 'a'. In particular, if ϵ is a positive number, usually very small, then the ϵ-neighbourhood of 'a' is the open interval $(a-\epsilon, a+\epsilon)$. The set $(a-\epsilon, a+\epsilon) - \{a\}$ is called ***deleted neighbourhood*** of 'a' and it is denoted as $0 < |x-a| < \epsilon$ (See Figure 1.3).

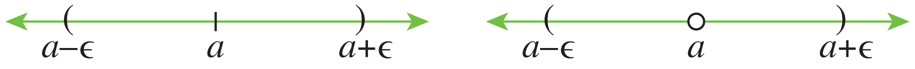


Figure 1.3

1.5 Relations

We approach the concept of relations in different aspects using real life sense, Cryptography and Geometry through Cartesian products of sets.

In our day to day life very often we come across questions like, "*How is he related to you?*". Some probable answers are,

(i) He is my father.
(ii) He is my teacher.
(iii) He is not related to me.

From this we see that the word relation connects a person with another person. Extending this idea, in mathematics we consider relations as one which connects mathematical objects. Examples,

(i) A number m is related to a number n if m divides n in $\mathbb{N}$.
(ii) A real number x is related to a real number y if $x \leq y$.
(iii) A point p is related to a line L if p lies on L.
(iv) A student X is related to a school S if X is a student of S.

Illustration 1.1 (Cryptography) For centuries, people have used ciphers or codes, to keep confidential information secure. Effective ciphers are essential to the military, to financial institutions and to computer programmers. The study of the techniques used in creating coding and decoding these ciphers is called cryptography.

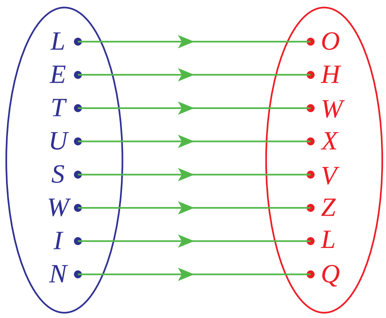


Figure 1.4

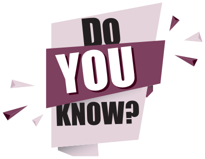
One of the earliest methods of coding a message was a simple substitution. For example, each letter in a message might be replaced by the letter that appears three places later in the alphabet.

Using this coding scheme, "LET US WIN" becomes "OHW XVZ LQ". This scheme was used by Julius Caesar and is called the Caesars cipher. To decode, replace each letter by the letter three places before it. This concept is used often in Mental Ability Tests. The above can be represented as an arrow diagram as given in Figure 1.4.

This can be viewed as the set of ordered pairs

$$\{(L,O),(E,H),(T,W),(U,X),(S,V),(W,Z),(I,L),(N,Q)\}$$

which is a subset of the Cartesian product $C \times D$ where $C = \{L,E,T,U,S,W,I,N\}$ and $D = \{O,H,W,X,V,Z,L,Q\}$.



If "KDUGZRUN" means "HARDWORK", then "DFKLHYHPHQW" becomes "ACHIEVEMENT"

"Is it $f(x) = x - 3$?".

Illustration 1.2 (Geometry) Consider the following three equations

$$\text{(i) } 2x - y = 0 \quad \text{(ii) } x^2 - y = 0 \quad \text{(iii) } x - y^2 = 0$$

(i) $2x - y = 0$

The equation $2x - y = 0$ represents a straight line. Clearly the points, $(1,2),(3,6)$ lie on it whereas $(1,1),(3,5),(4,5)$ are not lying on the straight line. The analytical relation between x and y is given by $y = 2x$. Here the values of y depends on the values of x. To denote this dependence, we write $y = f(x)$. The set of all points that lie on the straight line is given as $\{(x, 2x) : x \in \mathbb{R}\}$. Clearly this is a subset of $\mathbb{R} \times \mathbb{R}$. (See Figure 1.5.)

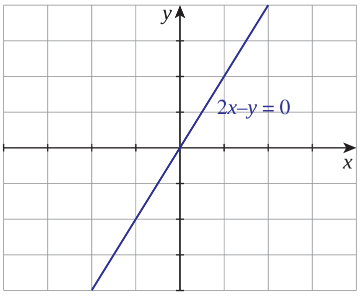


Figure 1.5

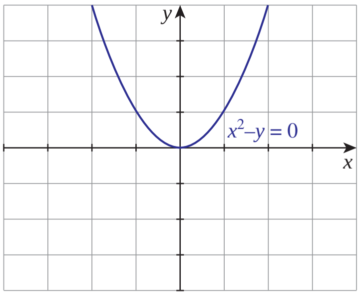


Figure 1.6

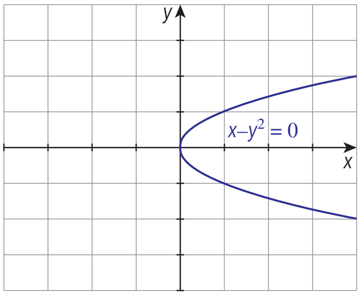


Figure 1.7

(ii) $x^2 - y = 0$.

As we discussed earlier, the relation between x and y is $y = x^2$. The set of all points on the curve is $\{(x, x^2) : x \in \mathbb{R}\}$ (See Figure 1.6). Again this is a subset of the Cartesian product $\mathbb{R} \times \mathbb{R}$.

(iii) $x - y^2 = 0$

As above, the relation between x and y is $y^2 = x$ or $y = \pm\sqrt{x},\ x \geq 0$. The equation can also be re-written as $y = +\sqrt{x}$ and $y = -\sqrt{x}$. The set of all points on the curve is the union of the sets $\{(x, \sqrt{x})\}$ and $\{(x, -\sqrt{x})\}$, where x is a non-negative real number, are the subsets of the Cartesian product $\mathbb{R} \times \mathbb{R}$. (See Figure 1.7).

From the above examples we intuitively understand what a relation is. But in mathematics, we have to give a rigorous definition for each and every technical term we are using. Now let us start defining the term "relation" mathematically.

Definition of Relation

Let $A = \{p, q, r, s, t, u\}$ be a set of students and let $B = \{X, Y, Z, W\}$ be a set of schools. Let us consider the following "relation".

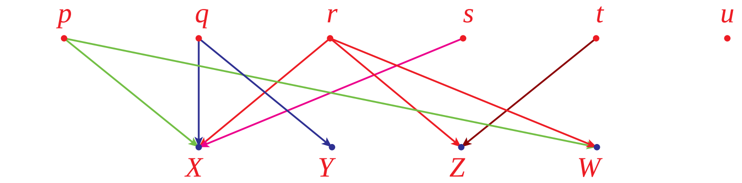
A student $a \in A$ is related to a school $S \in B$ if "a" is studying or studied in the school S.

Let us assume that p studied in X and now studying in W, q studied in X and now studying in Y, r studied in X and W, and now studying in Z, s has been studying in X from the beginning, t studied in Z and now studying in no school, and u never studied in any of these four schools.

Though the relations are given explicitly, it is not possible to give a relation always in this way. So let us try some other representations for expressing the same relation:

(i) $\begin{array}{ccccccccc} p & p & q & q & r & r & r & s & t \\ X & W & X & Y & X & Z & W & X & Z \end{array}$

(ii)



(iii) $\{\ (p, X),\ (p, W),\ (q, X),\ (q, Y),\ (r, X),\ (r, Z),\ (r, W),\ (s, X),\ (t, Z)\ \}$

(iv) $pRX, pRW, qRX, qRY, rRX, rRZ, rRW, sRX, tRZ.$

Among these four representations of the relation, the third one seems to be more convenient and comfortable to deal with a relation in terms of sets.

The set given in the third representation is a subset of the Cartesian product $A \times B$. In Illustrations 1.1 and 1.2 also, we arrived at subsets of a Cartesian product.

> **Definition 1.2**
>
> Let A and B be any two non-empty sets. A ***relation*** R from A to B is defined as a subset of the Cartesian product of A and B. Symbolically $R \subseteq A \times B$.

A relation from A to B is different from a relation from B to A.

The set $\{a \in A : (a, b) \in R \text{ for some } b \in B\}$ is called the ***domain*** of the relation.

The set $\{b \in B : (a, b) \in R \text{ for some } a \in A\}$ is called the ***range*** of the relation.

Thus the domain of the relation R is the set of all first coordinates of the ordered pairs and the range of the relation R is the set of all second coordinates of the ordered pairs.

Illustration 1.3 Consider the diagram in Figure 1.8. Here the alphabets are mapped onto the natural numbers. A simple cipher is to assign a natural number to each alphabet. Here a is represented by 1, b is represented by 2, $\ldots$, z is represented by 26. This correspondence can be written as the set

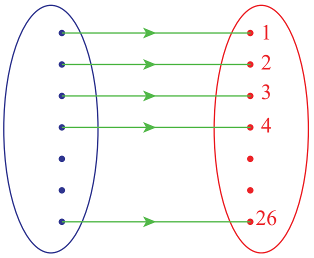


Figure 1.8

of ordered pairs $\{(a,1),(b,2),\ldots,(z,26)\}$. This set of ordered pairs is a relation. The domain of the relation is $\{a,b,\ldots,z\}$ and the range is $\{1,2,\ldots,26\}$.

Now we recall that the relations discussed in Illustrations 1.1 and 1.2 also end up with subsets of the cartesian product of two sets. So the term *relation* used in all discussions we had so far, fits with the mathematical term *relation* defined in Definition 1.2.

The domain of the relation discussed in Illustration 1.1 is the set $\{L,E,T,U,S,W,I,N\}$ and the range is $\{O,H,W,X,V,Z,L,Q\}$. In Illustration 1.2, the domain and range of the relation discussed for the equation $2x-y=0$ are $\mathbb{R}$ and $\mathbb{R}$ (See Figure 1.9); for the equation $x^2-y=0$, the domain is $\mathbb{R}$ and the range is $[0,\infty)$ (See Figure 1.10); and in the case of the third equation $x-y^2=0$, the domain is $[0,\infty)$ and the range is $\mathbb{R}$ (See Figures 1.11 and 1.12).

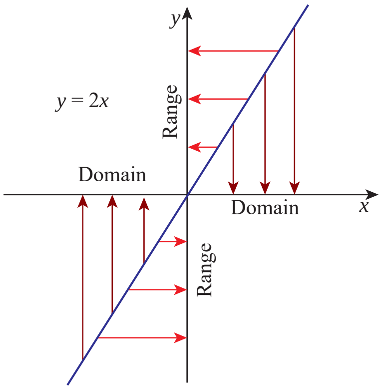


Figure 1.9

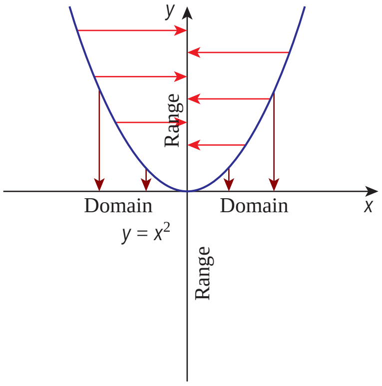


Figure 1.10

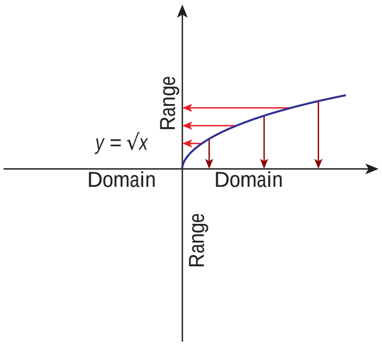


Figure 1.11

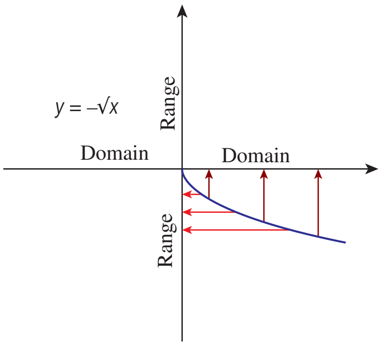


Figure 1.12

Note that, the domain of a relation is a subset of the first set in the Cartesian product and the range is a subset of second set. Usually we call the second set as ***co-domain*** of the relation. Thus, the range of a relation is the collection of all elements in the co-domain which are related to some element in the domain. Let us note that the range of a relation is a subset of the co-domain.

For any set A, $\emptyset$ and $A\times A$ are subsets of $A\times A$. These two relations are called ***extreme relations***. The former relation is an ***empty relation*** and the later is an ***universal relation***.

We will discuss more about domain, co-domain and the range in the next section namely, "Functions".

If R is a relation from A to B and if $(x,y)\in R$, then sometimes we write xRy (read this as "x is related to y") and if $(x,y)\notin R$, then sometimes we write $x\not R y$ (read this as "x is not related to y").

Though the general definition of a relation is defined from one set to another set, relations defined **on** a set are of more interest in mathematical point of view. That is, relations in which the domain and the co-domain are the same are of more interest. So let us concentrate on relations defined on a set.

1.5.1 Type of Relations

Consider the following examples:

(i) Let $S = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 3), (2, 3)\}$ on S.
(ii) Let $S = \{1, 2, 3, \dots 10\}$ and define "m is related to n, if m divides n".
(iii) Let $\mathscr{C}$ be the set of all circles in a plane and define "a circle C is related to a circle C', if the radius of C is equal to the radius of C'".
(iv) In the set S of all people define "a is related to b, if a is a brother of b".
(v) Let S be the set of all people. Define the relation on S by the rule "mother of".

In the second example, as every number divides itself, "a is related a for all $a \in S$"; the same is true in the third relation also. In the first example "a is related a for all $a \in S$" is not true as 2 is not related to 2.

It is easy to see that the property "if a is related to b, then b is related to a" is true in the third but not in the second.

It is easy to see that the property "if a is related to b and b is related to c, then a is related to c" is true in the second and third relations but not in the fifth.

These properties, together with some more properties are very much studied in mathematical structures. Let us define them now.

Definition 1.3

Let S be any non-empty set. Let R be a relation on S. Then

- R is said to be ***reflexive*** if a is related to a for all $a \in S$.
- R is said to be ***symmetric*** if a is related to b implies that b is related to a.
- R is said to be ***transitive*** if "a is related to b and b is related to c" implies that a is related to c.

These three relations are called ***basic relations***.

Let us rewrite the definitions of these basic relations in a different form:

Let S be any non-empty set. Let R be a relation on S. Then R is

- reflexive if "$(a, a) \in R$ for all $a \in S$".
- symmetric if "$(a, b) \in R \Rightarrow (b, a) \in R$".
- transitive if "$(a, b), (b, c) \in R \Rightarrow (a, c) \in R$".

Definition 1.4

Let S be any set. A relation on S is said to be an ***equivalence relation*** if it is reflexive, symmetric and transitive.

Let us consider the following two relations.

(1) In the set S_1 of all people, define a relation R_1 by the rule: "a is related to b, if a is a brother of b".
(2) In the set S_2 of all males, define a relation R_2 by the rule: "a is related to b, if a is a brother of b".

The rules that define the relations on S_1 and S_2 are the same. But the sets are not same. R_1 is not a symmetric relation on S_1 whereas R_2 is a symmetric relation on S_2. This shows that not only the rule defining the relation is important, the set on which the relation is defined, is also important. So whenever one considers a relation, both the relation as well as the set on which the relation is defined have to be given explicitly. Note that the relation $\{(1, 1), (2, 2), (3, 3), (1, 2)\}$ is reflexive if it is defined on the set $\{1, 2, 3\}$; it is not reflexive if it is defined on the set $\{1, 2, 3, 4\}$.

Illustration 1.4

1. Let $X = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (2, 1), (2, 2), (3, 3), (1, 3), (4, 4), (1, 2), (3, 1)\}$. As $(1, 1), (2, 2), (3, 3)$ and $(4, 4)$ are all in R, it is reflexive. Also for each pair $(a, b) \in R$ the pair (b, a) is also in R. So R is symmetric. As $(2, 1), (1, 3) \in R$ and $(2, 3) \notin R$, we see that R is not transitive. Thus R is not an equivalence relation.
2. Let P denote the set of all straight lines in a plane. Let R be the relation defined on P as $\ell R m$ if ℓ is parallel to m.

   This relation is reflexive, symmetric and transitive. Thus it is an equivalence relation.
3. Let A be the set consisting of children and elders of a family. Let R be the relation defined by aRb if a is a sister of b.

   This relation is to be looked into carefully. A woman is not a sister of herself. So it is not reflexive. It is not symmetric also. Clearly it is not transitive. So it is not an equivalence relation. (If we consider the same relation on a set consisting only of females, then it becomes symmetric; even in this case it is not transitive).
4. On the set of natural numbers let R be the relation defined by xRy if $x + 2y = 21$. It is better to write the relation explicitly. The relation R is the set

   $$\{(1, 10), (3, 9), (5, 8), (7, 7), (9, 6), (11, 5), (13, 4), (15, 3), (17, 2), (19, 1)\}.$$

   As $(1, 1) \notin R$ it is not reflexive; as $(1, 10) \in R$ and $(10, 1) \notin R$ it is not symmetric. As $(3, 9) \in R$, $(9, 6) \in R$ but $(3, 6) \notin R$, the relation is not transitive.
5. Let $X = \{1, 2, 3, 4\}$ and $R = \emptyset$, where $\emptyset$ is the empty set.

   As $(1, 1) \notin R$ it is not reflexive. As we cannot find a pair (x, y) in R such that $(y, x) \notin R$, the relation is not 'not symmetric'; so it is symmetric. Similarly it is transitive.
6. The universal relation is always an equivalence relation.
7. An empty relation can be considered as symmetric and transitive.
8. If a relation contains a single element, then the relation is transitive.

Let us discuss some more special relations now.

Example 1.10 Check the relation $R = \{(1, 1), (2, 2), (3, 3), \ldots, (n, n)\}$ defined on the set $S = \{1, 2, 3, \ldots, n\}$ for the three basic relations.

Solution:
As $(a, a) \in R$ for all $a \in S$, R is reflexive.
There is no pair (a, b) in R such that $(b, a) \notin R$. In other words, for every pair $(a, b) \in R$, (b, a) is also in R. Thus R is symmetric.
We cannot find two pairs (a, b) and (b, c) in R, such that $(a, c) \notin R$. Thus the statement "R is not transitive" is not true; therefore, the statement "R is transitive" is true; hence R is transitive.

Since R is reflexive, symmetric and transitive, this relation is an equivalence relation.

From the very beginning we have denoted all the relations by the same letter R. It is not necessary to do so. We may use the Greek letter ρ (Read as rho) to denote relations. Equivalence relations are mostly denoted by "$\sim$".

If a relation is not of required type, then by inserting or deleting some pairs we can make it of the required type. We do this in the following problem.

Example 1.11 Let $S = \{1, 2, 3\}$ and $\rho = \{(1, 1), (1, 2), (2, 2), (1, 3), (3, 1)\}$.

(i) Is ρ reflexive? If not, state the reason and write the minimum set of ordered pairs to be included to ρ so as to make it reflexive.

(ii) Is ρ symmetric? If not, state the reason, write minimum number of ordered pairs to be included to ρ so as to make it symmetric and write minimum number of ordered pairs to be deleted from ρ so as to make it symmetric.
(iii) Is ρ transitive? If not, state the reason, write minimum number of ordered pairs to be included to ρ so as to make it transitive and write minimum number of ordered pairs to be deleted from ρ so as to make it transitive.
(iv) Is ρ an equivalence relation? If not, write the minimum ordered pairs to be included to ρ so as to make it an equivalence relation.

Solution:

(i) ρ is not reflexive because $(3,3)$ is not in ρ. As $(1,1)$ and $(2,2)$ are in ρ, it is enough to include the pair $(3,3)$ to ρ so as to make it reflexive.
(ii) ρ is not symmetric because $(1,2)$ is in ρ, but $(2,1)$ is not in ρ. It is enough to include the pair $(2,1)$ to ρ so as to make it symmetric.
It is enough to remove the pair $(1,2)$ from ρ so as to make it symmetric
(iii) ρ is not transitive because $(3,1)$ and $(1,3)$ are in ρ, but $(3,3)$ is not in ρ. To make it transitive we have to include $(3,3)$ in ρ. Even after including $(3,3)$, the relation is not transitive because $(3,1)$ and $(1,2)$ are in ρ, but $(3,2)$ is not in ρ. To make it transitive we have to include $(3,2)$ also in ρ. Now it becomes transitive. So $(3,3)$ and $(3,2)$ are to be included so as to make ρ transitive.
But if we remove $(3,1)$ from ρ, then it becomes transitive.
(iv) We have seen that
- to make ρ reflexive, we have to include $(3,3)$;
- to make ρ symmetric, we have to include $(2,1)$;
- and to make ρ transitive, we have to include $(3,3)$ and $(3,2)$.

To make ρ as an equivalence relation we have to include all these pairs. So after including the pairs the relation becomes $\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(3,2)\}$

But this relation is not symmetric because $(3,2)$ is in the relation and $(2,3)$ is not in the relation. So we have to include $(2,3)$ also. Now the new relation becomes

$$\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(3,2),(2,3)\}$$

It can be seen that this relation is reflexive, symmetric and transitive, and hence it is an equivalence relation. Thus we have to include $(3,3)$, $(2,1)$, $(3,2)$ and $(2,3)$ to ρ so as to make it an equivalence relation.

Now let us learn how to create relations having certain properties through the following example.

Example 1.12 Let $A = \{0,1,2,3\}$. Construct relations on A of the following types:

(i) not reflexive, not symmetric, not transitive.
(ii) not reflexive, not symmetric, transitive.
(iii) not reflexive, symmetric, not transitive.
(iv) not reflexive, symmetric, transitive.
(v) reflexive, not symmetric, not transitive.
(vi) reflexive, not symmetric, transitive.
(vii) reflexive, symmetric, not transitive.
(viii) reflexive, symmetric, transitive.

Solution:

(i) Let us use the pair $(1, 2)$ to make the relation "not symmetric" and consider the relation $\{(1, 2)\}$. It is transitive. If we include $(2, 3)$ and not include $(1, 3)$, then the relation is not transitive. So the relation $\{(1, 2), (2, 3)\}$ is not reflexive, not symmetric and not transitive. Similarly we can construct more examples.

(ii) Just now we have seen that the relation $\{(1, 2)\}$ is transitive, not reflexive and not symmetric.

(iii) Let us start with the pair $(1, 2)$. Since we need symmetricity, we have to include the pair $(2, 1)$. At this stage as $(1, 1), (2, 2)$ are not here, the relation is not transitive. Thus $\{(1, 2), (2, 1)\}$ is not reflexive; it is symmetric; and it is not transitive.

(iv) If we include the pairs $(1, 1)$ and $(2, 2)$ to the relation discussed in (iii), it will become transitive. Thus $\{(1, 2), (2, 1), (1, 1), (2, 2)\}$ is not reflexive; it is symmetric and it is transitive.

(v) For a relation on $\{0, 1, 2, 3\}$ to be reflexive, it must have the pairs $(0, 0), (1, 1), (2, 2), (3, 3)$. Fortunately, it becomes symmetric and transitive. Therefore, as in (i) if we insert $(1, 2)$ and $(2, 3)$ we get the required one. Thus $\{(0, 0), (1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$ is reflexive; it is not symmetric and it is not transitive.

(vi) Proceeding like this we get the relation $\{(0, 0), (1, 1), (2, 2), (3, 3), (1, 2)\}$ that is reflexive, transitive and not symmetric.

(vii) As above we get the relation $\{(0, 0), (1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (2, 1), (3, 2)\}$ that is reflexive, symmetric and not transitive.

(viii) We have the relation $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$ which is reflexive, symmetric and transitive.

Example 1.13 In the set $\mathbb{Z}$ of integers, define mRn if $m - n$ is a multiple of 12. Prove that R is an equivalence relation.

Solution:

As $m - m = 0$ and $0 = 0 \times 12$, hence mRm proving that R is reflexive.

Let mRn. Then $m - n = 12k$ for some integer k; thus $n - m = 12(-k)$ and hence nRm. This shows that R is symmetric.

Let mRn and nRp; then $m - n = 12k$ and $n - p = 12\ell$ for some integers k and ℓ. So $m - p = 12(k + \ell)$ and hence mRp. This shows that R is transitive.

Thus R is an equivalence relation.

Theorem 1.1: The number of relations from a set containing m elements to a set containing n elements is 2^{mn}. In particular the number of relations on a set containing n elements is 2^{n^2}.

Proof. Let A and B be sets containing m and n elements respectively. Then $A \times B$ contains mn elements and $A \times B$ has 2^{mn} subsets. Since every subset of $A \times B$ is a relation from A to B, there are 2^{mn} relations from a set containing m elements to a set containing n elements.

Taking $A = B$, we see that the number of relations on a set containing n elements is 2^{n^2}. □

(i) The number of reflexive relations on a set containing n elements is 2^{n^2-n}.

(ii) The number of symmetric relations on a set containing n elements is $2^{\frac{(n^2+n)}{2}}$.

Definition 1.5

If R is a relation from A to B, then the relation R^{-1} defined from B to A by

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

is called the ***inverse*** of the relation R.

For example, if $R = \{(1, a), (2, b), (2, c), (3, a)\}$, then

$$R^{-1} = \{(a, 1), (b, 2), (c, 2), (a, 3)\}.$$

It is easy to see that the domain of R becomes the range of R^{-1} and the range of R becomes the domain of R^{-1}.

DO YOU KNOW?

An equivalence relation on a set decomposes it into a disjoint union of its subsets (equivalence classes). Such a decomposition is called a partition. This is explained in the following example.

For $a, b \in \mathbb{Z}$, aRb if and only if $a - b = 3k$, $k \in \mathbb{Z}$ is an equivalence relation on $\mathbb{Z}$.

$$\begin{array}{lclcl} Z_0 & = & \{x \in \mathbb{Z} : xR0\} & = & \{\ldots, -6, -3, 0, 3, 6, \ldots\} \\ Z_1 & = & \{x \in \mathbb{Z} : xR1\} & = & \{\ldots, -5, -2, 1, 4, 7, \ldots\} \\ Z_2 & = & \{x \in \mathbb{Z} : xR0\} & = & \{\ldots, -4, -1, 2, 5, 8, \ldots\} \end{array}$$

Thus $\mathbb{Z} = Z_0 \cup Z_1 \cup Z_2$ and all are disjoint subsets.

For a given partition $S_1 \cup S_2 \cup \cdots \cup S_n$ of a set S into disjoint subsets, one can construct an equivalence relation R on S by xRy if $x, y \in S_i$ for some i.

Equivalence relation is used in almost all branches of higher mathematics.

Exercise - 1.2

1. Discuss the following relations for reflexivity, symmetricity and transitivity:
   (i) The relation R defined on the set of all positive integers by "mRn if m divides n".
   (ii) Let P denote the set of all straight lines in a plane. The relation R defined by "ℓRm if ℓ is perpendicular to m".
   (iii) Let A be the set consisting of all the members of a family. The relation R defined by "aRb if a is not a sister of b".
   (iv) Let A be the set consisting of all the female members of a family. The relation R defined by "aRb if a is not a sister of b".
   (v) On the set of natural numbers the relation R defined by "xRy if $x + 2y = 1$".
2. Let $X = \{a, b, c, d\}$ and $R = \{(a, a), (b, b), (a, c)\}$. Write down the minimum number of ordered pairs to be included to R to make it
   (i) reflexive (ii) symmetric (iii) transitive (iv) equivalence
3. Let $A = \{a, b, c\}$ and $R = \{(a, a), (b, b), (a, c)\}$. Write down the minimum number of ordered pairs to be included to R to make it
   (i) reflexive (ii) symmetric (iii) transitive (iv) equivalence
4. Let P be the set of all triangles in a plane and R be the relation defined on P as aRb if a is similar to b. Prove that R is an equivalence relation.
5. On the set of natural numbers let R be the relation defined by aRb if $2a + 3b = 30$. Write down the relation by listing all the pairs. Check whether it is
   (i) reflexive (ii) symmetric (iii) transitive (iv) equivalence
6. Prove that the relation "friendship" is not an equivalence relation on the set of all people in Chennai.
7. On the set of natural numbers let R be the relation defined by aRb if $a + b \leq 6$. Write down the relation by listing all the pairs. Check whether it is
   (i) reflexive (ii) symmetric (iii) transitive (iv) equivalence

8. Let $A = \{a, b, c\}$. What is the equivalence relation of smallest cardinality on A? What is the equivalence relation of largest cardinality on A?
9. In the set Z of integers, define mRn if $m - n$ is divisible by 7. Prove that R is an equivalence relation.

1.6 Functions

Suppose that a particle is moving in the space. We assume the physical particle as a point. As time varies, the particle changes its position. Mathematically at any time the point occupies a position in the three dimensional space $\mathbb{R}^3$. Let us assume that the time varies from 0 to 1. So the movement or *functioning* of the particle decides the position of the particle at any given time t between 0 and 1. In other words, for each $t \in [0, 1]$, the functioning of the particle gives a point in $\mathbb{R}^3$. Let us denote the position of the particle at time t as $f(t)$.

Let us see another simple example. We know that the equation $2x - y = 0$ describes a straight line. Here whenever x assumes a value, y assumes some value accordingly. The movement or *functioning* of y is decided by that of x. Let us denote y by $f(x)$. We may see many situation like this in nature. In the study of natural phenomena, we find that it is necessary to consider the variation of one quantity depending on the variation of another.

The relation of the time and the position of the particle, the relation of a point in the x-axis to a point in the y-axis and many more such relations are studied for a very long period in the name *function*. Before Cantor, the term function is defined as a rule which associates a variable with another variable. After the development of the concept of sets, a function is defined as a rule that associates for every element in a set A, a unique element in a set B. However the terms *rule* and *associate* are not properly defined mathematical terminologies. In modern mathematics every term we use has to be defined properly. So a definition for function is given using relations.

Suppose that we want to discuss a test written by a set of students. We shall see this as a relation.

Let A be the set of students appeared for an examination and let $B = \{0, 1, 2, 3, \ldots 100\}$ be the set of possible marks. We define a relation R as follows:

A student a is related to a mark b if a got b marks in the test.

We observe the following from this example:

- Every student got a mark. In other words, for every $a \in A$, there is an element $b \in B$ such that $(a, b) \in R$.
- A student cannot get two different marks in any test. In other words, for every $a \in A$, there is definitely only one $b \in B$ such that $(a, b) \in R$. This can be restated in a different way: If $(a, b), (a, c) \in R$ then $b = c$.

Relations having the above two properties form a very important class of relations, called functions. Let us now have a rigorous definition of a function through relations.

Definition 1.6

Let A and B be two sets. A relation f from A to B, a subset of $A \times B$, is called a **function** from A to B if it satisfies the following:

(i) for all $a \in A$, there is an element $b \in B$ such that $(a, b) \in f$.
(ii) if $(a, b) \in f$ and $(a, c) \in f$ then $b = c$.

That is, a function is a relation in which each element in the domain is mapped to exactly one element in the range.

A is called the ***domain*** of f and B is called the ***co-domain*** of f. If (a, b) is in f, then we write $f(a) = b$; the element b is called the ***image*** of a and the element a is called a ***pre-image*** of b and $f(a)$ is known as the value of f at a. The set $\{b : (a, b) \in f \text{ for some } a \in A\}$ is called the ***range*** of the function. If B is a subset of $\mathbb{R}$, then we say that the function is a ***real-valued function***.

Two functions f and g are said to be ***equal functions*** if their domains are same and $f(a) = g(a)$ for all a in the domain.

If f is a function with domain A and co-domain B, we write $f : A \to B$ (Read this as$f$ is from A to B or f be a function from A to B). We also say that f maps A into B. If $f(a) = b$, then we say f maps a to b or a is mapped onto b by f, and so on.

The range of a function is the collection of all elements in the co-domain which have pre-images. Clearly the range of a function is a subset of the co-domain. Further the first condition says that every element in the domain must have an image; this is the reason for defining the domain of a relation R from a set A to a set B as the set of all elements of A having images and not as A. The second condition says that an element in the domain cannot have two or more images.

Naturally one may have the following doubts:

- In the definition, why we use the definite article "the" for image of a and the indefinite article "a" for pre-image of b?
- We have a condition stating that every element in the domain must have an image; is there any condition like "every element in the co-domain must have a pre-image"? If not, why?
- We have a condition stating that an element in the domain cannot have two or more images; is there any condition like "an element in the co-domain cannot have two or more pre-images"? If not, why?

As an element in the domain has exactly one image and an element in the co-domain can have more than one pre-image according to the definition, we use the definite article "the" for image of a and the indefinite article "a" for pre-image of b. There are no conditions as asked in the other two questions; the reason behind it can be understood from the problem of students' mark we considered above.

We observe that every function is a relation but a relation need not be a function.

Let $f = \{(a, 1), (b, 2), (c, 2), (d, 4)\}$.

Is f a function? This is a function from the set $\{a, b, c, d\}$ to $\{1, 2, 4\}$. This is not a function from $\{a, b, c, d, e\}$ to $\{1, 2, 3, 4\}$ because e has no image. This is not a function from $\{a, b, c, d\}$ to $\{1, 2, 3, 5\}$ because the image of d is not in the co-domain; f is not a subset of $\{a, b, c, d\} \times \{1, 2, 3, 5\}$. So whenever we consider a function the domain and the co-domain must be stated explicitly.

The relation discussed in Illustration 1.1 is a function with domain $\{L, E, T, U, S, W, I, N\}$ and co-domain $\{O, H, W, X, V, Z, L, Q\}$. The relation discussed in Illustration 1.3 is again a function with domain $\{a, b, \ldots, z\}$ and the co-domain $\{1, 2, 3, \ldots, 26\}$.

In Illustration 1.2, we discussed three relations, namely

$$\text{(i)}\ \ y = 2x \quad \text{(ii)}\ \ y = x^2 \quad \text{(iii)}\ \ y^2 = x.$$

Clearly (i) and (ii) are functions whereas (iii) is not a function, if the domain and the co-domain are $\mathbb{R}$. In (iii) for the same x, we have two y values which contradict the definition of the function. But if we split into two relations, that is, $y = \sqrt{x}$ and $y = -\sqrt{x}$ then both become functions with same domain non-negative real numbers and the co-domains $[0, \infty)$ and $(-\infty, 0]$ respectively.

1.6.1 Ways of Representing Functions

(a) Tabular Representation of a Function

When the elements of the domain are listed like $x_1, x_2, x_3 \ldots x_n$, we can use this tabular form.Here, the values of the arguments $x_1, x_2, x_3 \ldots x_n$ and the corresponding values of the function $y_1, y_2, y_3 \ldots y_n$ are written out in a definite order.

x	x_1	x_2	$\ldots$	x_n
y	y_1	y_2	$\ldots$	y_n

(b) Graphical Representation of a Function

When the domain and the co-domain are subsets of $\mathbb{R}$, many functions can be represented using a graph with x-axis representing the domain and y-axis representing the co-domain in the (x, y)-plane.

We note that the first and second figures in Illustration 1.2 represent the functions $f(x) = 2x$ and $f(x) = x^2$ respectively. Usually the variable x is treated as independent variable and y as a dependent variable. The variable x is called the ***argument*** and $f(x)$ is called the value.

(c) Analytical Representation of a Function

If the functional relation $y = f(x)$ is such that f denotes an analytical expression, we say that the function y of x is represented or defined analytically. Some examples of analytical expressions are

$$x^3 + 5, \ \frac{\sin x + \cos x}{x^2 + 1}, \ \log x + 5\sqrt{x}.$$

That is, a series of symbols denoting certain mathematical operations that are performed in a definite sequence on numbers, letters which designate constants or variable quantities.

Examples of functions defined analytically are

$$\text{(i)}\ \ y = \frac{x-1}{x+1} \quad \text{(ii)}\ \ y = \sqrt{9 - x^2} \quad (iii)\ \ y = \sin x + \cos x \quad \text{(iv)}\ \ A = \pi r^2.$$

One of the usages of writing functions analytically is finding domains naturally. That is, the set of values of x for which the analytical expressions on the right-hand side has a definite value is the natural domain of definition of a function represented analytically.

Thus, the natural domain of the function,

$$\begin{array}{llllllll} \text{(i)} & y = x^3 + 3 & \text{is} & (-\infty, \infty) & \text{(ii)} & y = x^4 - 2 & \text{is} & (-\infty, \infty) \\ \text{(iii)} & y = \frac{x-1}{x+1} & \text{is} & \mathbb{R} - \{-1\} & \text{(iv)} & y = +\sqrt{4 - x^2} & \text{is} & -2 \le x \le 2. \end{array}$$

Now recall the domain of the functions (i) $y = 2x$, (ii) $y = x^2$, (iii) $y = +\sqrt{x}$, (iv) $y = -\sqrt{x}$ which are analytical in nature described earlier.

Sometimes we may come across piece-wise defined functions. For example, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0 & \text{if} \quad -\infty < x \le -2 \\ 2x & \text{if} \quad -2 < x \le 3 \\ x^2 & \text{if} \quad 3 < x \le \infty \end{cases}$$

Depending upon the value of x, we have to select the formula to be used to find the value of f at any point x. To find the value of f at any real number, first we have to find to which interval x belongs to; then using the corresponding formula we can find the value of f at that point. To find $f(6)$ we know $3 \le 6 \le \infty$ (or $6 \in [3, \infty)$); so we use the formula $f(x) = x^2$ and find $f(6) = 36$. Similarly $f(-1) = -2$, $f(-5) = 0$ and so on.

If the function is defined from $\mathbb{R}$ or a subset of $\mathbb{R}$ then we can draw the graph of the function. For example, if $f : [0, 4] \to \mathbb{R}$ is defined by $f(x) = \frac{x}{2} + 1$, then we can plot the points $(x, \frac{x}{2} + 1)$ for all $x \in [0, 4]$. Then we will get a straight line segment joining $(0, 1)$ and $(4, 3)$. (See Figure 1.13)

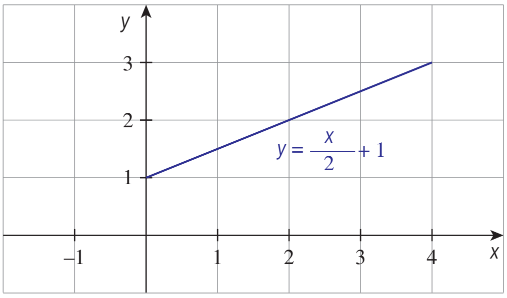


Figure 1.13

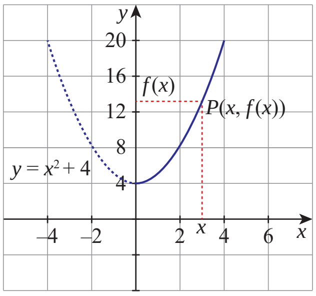


Figure 1.14

Consider another function $f(x) = x^2 + 4, x \geq 0$. The function will be given by its graph. (See Figure 1.14)

Let x be a point in the domain. Let us draw a vertical line through the point x. Let it meet the curve at P. The point at which the horizontal line drawn through P meets the y-axis is $f(x)$. Similarly using horizontal lines through a point y in the co-domain, we can find the pre-images of y.

Can we say that any curve drawn on the plane be considered as a function from a subset of $\mathbb{R}$ to $\mathbb{R}$? No, we cannot. There is a simple test to find this.

Vertical Line Test

As we noted earlier, the vertical line through any point x in the domain meets the curve at some point, then the y-coordinate of the point is $f(x)$. If the vertical line through a point x in the domain meets the curve at more than one point, we will get more than one value for $f(x)$ for one x. This is not allowed in a function. Further, if the vertical line through a point x in the domain does not meet the curve, then there will be no image for x; this is also not possible in a function. So we can say,

> "if the vertical line through a point x in the domain meets the curve at more than one point or does not meet the curve, then the curve will not represent a function".

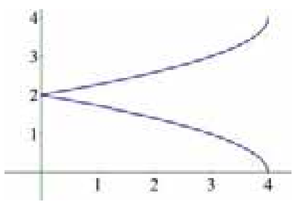


Figure 1.15

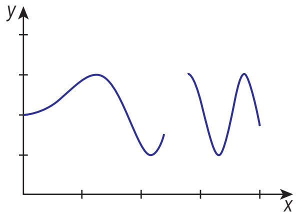


Figure 1.16

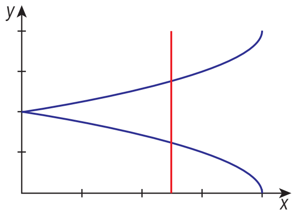


Figure 1.17

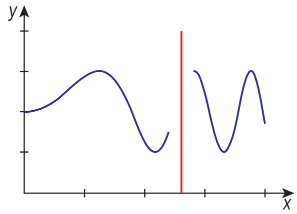


Figure 1.18

The curve indicated in Figure 1.15 does not represent a function from $[0, 4]$ to $\mathbb{R}$ because a vertical line meets the curve at more than one point (See Figure 1.17). The curve indicated in Figure 1.16 does not represent a function from $[0, 4]$ to $\mathbb{R}$ because a vertical line drawn through $x = 2.5$ in $[0, 4]$ does not meet the curve (See Figure 1.18).

Testing whether a given curve represents a function or not by drawing vertical lines is called ***vertical line test*** or simply vertical test.

The third curve $y^2 = x$ in Illustration 1.2 fails in the vertical line test and hence it is not a function from $\mathbb{R}$ to $\mathbb{R}$.

1.6.2 Some Elementary Functions

Some frequently used functions are known by names. Let us list some of them.

(i) Let X be any non-empty set. The function $f : X \to X$ defined by $f(x) = x$ for all $x \in X$ is called the ***identity function*** on X (See Figure 1.19). It is denoted by I_X or I.

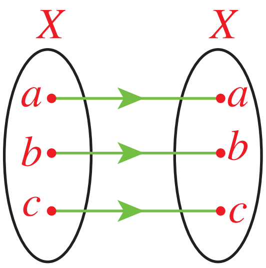


Figure 1.19

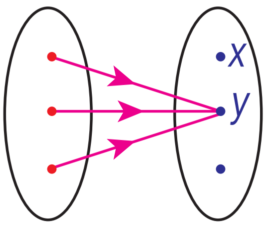


Figure 1.20

(ii) Let X and Y be two sets. Let c be a fixed element of Y. The function $f : X \to Y$ defined by $f(x) = c$ for all $x \in X$ is called a ***constant function*** (See Figure 1.20).

The value of a constant function is same for all values of x throughout the domain.

If X and Y are $\mathbb{R}$, then the graph of the identity function and a constant function are as in Figures 1.21 and 1.22. If X is any set, then the constant function defined by $f(x) = 0$ for all x is called the ***zero function***. So zero function is a particular case of constant function.

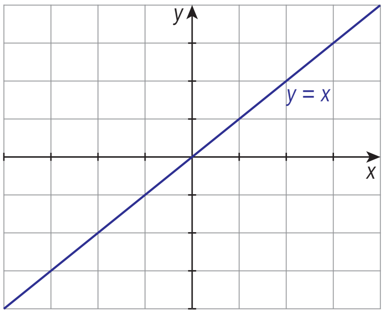


Figure 1.21

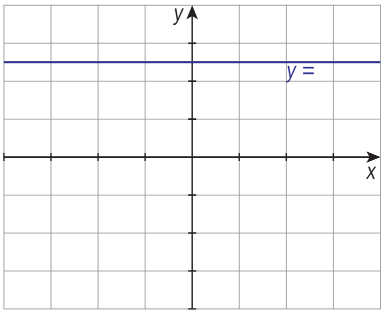


Figure 1.22

(iii) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$, where $|x|$ is the modulus or absolute value of x, is called the ***modulus function*** or ***absolute value function***. (See Figure 1.23.)

Let us recall that $|x|$ is defined as

$$|x| = \begin{cases} -x & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ x & \text{if} \quad x > 0 \end{cases} \quad \text{or } |x| = \begin{cases} -x & \text{if} \quad x \leq 0 \\ x & \text{if} \quad x > 0 \end{cases} \quad \text{or } |x| = \begin{cases} -x & \text{if} \quad x < 0 \\ x & \text{if} \quad x \geq 0 \end{cases}$$

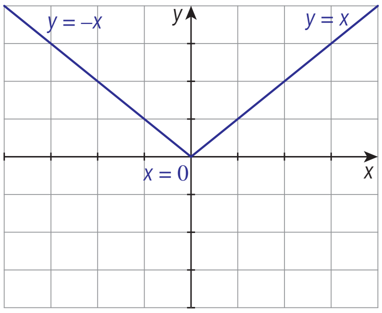


Figure 1.23

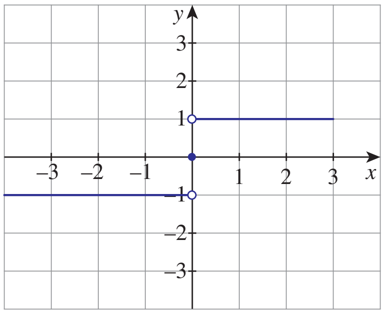


Figure 1.24

(iv) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} \frac{x}{|x|} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$ is called the ***signum function***.

This function is denoted by sgn. (See Figure 1.24)

(v) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x)$ is the greatest integer less than or equal to x is called the integral part function or the ***greatest integer function*** or the ***floor function***. This function is denoted by $\lfloor x \rfloor$. (See Figure 1.25.)

(vi) The function $f : \mathbb{R} \to \mathbb{R}$defined by $f(x)$ is the smallest integer greater than or equal to x is called the ***smallest integer function*** or the ***ceil function*** (See Figure 1.26.). This function is denoted by $\lceil \cdot \rceil$; that is $f(x)$ is denoted by $\lceil x \rceil$.

The functions (v) and (vi) are also called ***step functions***.

Let us note that $\lfloor 1\frac{1}{5} \rfloor = 1$, $\lfloor 7.23 \rfloor = 7$, $\lfloor -2\frac{1}{2} \rfloor = -3$ (not -2), $\lfloor 6 \rfloor = 6$ and $\lfloor -4 \rfloor = -4$.

Let us note that $\lceil 1\frac{1}{5} \rceil = 2$, $\lceil 7.23 \rceil = 8$, $\lceil -2\frac{1}{2} \rceil = -2$ (not -3), $\lceil 6 \rceil = 6$ and $\lceil -4 \rceil = -4$.

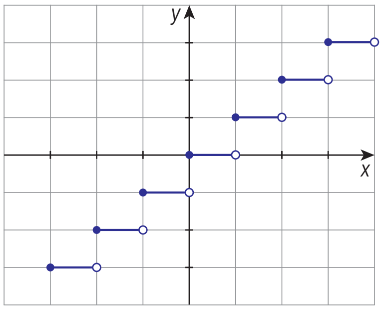


Figure 1.25

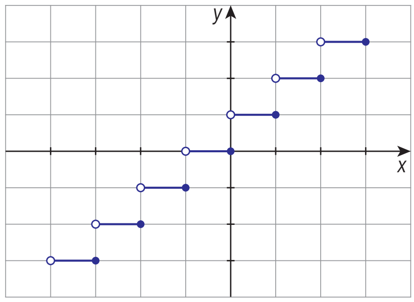


Figure 1.26

One may note the relations among the names of these functions, the symbols denoting the functions and the commonly used words ceiling and floor of a room and their graphs are given in Figures 1.25 and 1.26.

1.6.3 Types of Functions

Though functions can be classified into various types according to the need, we are going to concentrate on two basic types: *one-to-one functions and onto functions*.

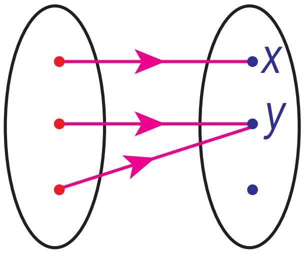


Figure 1.27

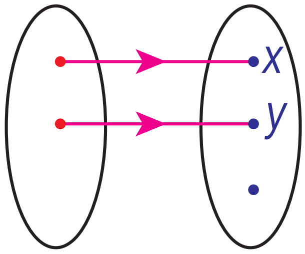


Figure 1.28

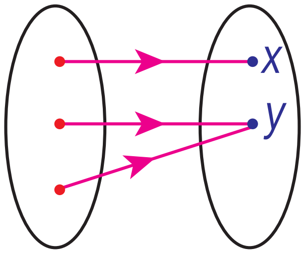


Figure 1.29

Let us look at the two simple functions given in Figure 1.27 and Figure 1.28. In the first function two elements of the domain, b and c, are mapped into the same element y, whereas it is not the case in the Figure 1.28. Functions like the second one are examples of one-to-one functions.

Let us look at the two functions given in Figures 1.28 and 1.29. In Figure 1.28 the element z in the co-domain has no pre-image, whereas it is not the case in Figure 1.29. Functions like in Figure 1.29 are example of onto functions. Now we define one-to-one and onto functions.

Definition 1.7

A function $f : A \to B$ is said to be ***one-to-one*** if $x, y \in A, x \neq y \Rightarrow f(x) \neq f(y)$ [or equivalently $f(x) = f(y) \Rightarrow x = y$]. A function $f : A \to B$ is said to be ***onto***, if for each $b \in B$ there exists at least one element $a \in A$ such that $f(a) = b$. That is, the range of f is B.

Another name for one-to-one function is ***injective function***; onto function is ***surjective function***. A function $f : A \to B$ is said to be ***bijective*** if it is both one-to-one and onto.

To prove a function $f : A \to B$ to be one-to-one, it is enough to prove any one of the following:

if $x \neq y$, then $f(x) \neq f(y)$, or equivalently if $f(x) = f(y)$, then $x = y$.

It is easy to observe that every identity function is one-to-one function as well as onto. A constant function is not onto unless the co-domain contains only one element. The following statements are some important simple results.

Let A and B be two sets with m and n elements.

(i) There is no one-to-one function from A to B if $m > n$.
(ii) If there is an one-to-one function from A to B, then $m \leq n$.
(iii) There is no onto function from A to B if $m < n$.
(iv) If there is an onto function from A to B, then $m \geq n$.
(v) There is a bijection from A to B, if and only if, $m = n$.
(vi) There is no bijection from A to B if and only if, $m \neq n$.

A function which is not onto is called into function. That is, the range of the function is a proper subset of its co-domain. Let us see some illustrations.

(1) $X = \{1, 2, 3, 4\}, Y = \{a, b, c, d, e\}$and $f = \{(1, a), (2, c), (3, e), (4, b)\}$.
This function is one-to-one but not onto.

(2) $X = \{1, 2, 3, 4\}, Y = \{a, b\}$ and $f = \{(1, a), (2, a), (3, a), (4, a)\}$.
This function is not one-to-one; it is not onto.

(3) $X = \{1, 2, 3, 4\}, Y = \{a\}$ and $f = \{(1, a), (2, a), (3, a), (4, a)\}$.
This function is not one-to-one but it is onto. It seems that this function is same as the previous one. The co-domain of the function is very important when deciding whether the function is onto or not.

(4) $X = \{1, 2, 3, 4\}, Y = \{a, b, c, d, e\}$ and $f = \{(1, a), (2, c), (3, b), (4, b)\}$.
This function is neither one-to-one nor onto.

(5) $X = \{1, 2, 3, 4\}, Y = \{a, b, c, d\}$ and $f = \{(1, a), (2, c), (3, d), (4, b)\}$.
This function is both one-to-one and onto.

(6) $X = \{1, 2, 3, 4\}, Y = \{a, b, c, d, e\}$ and $f = \{(1, a), (2, c), (3, e)\}$.
This is not at all a function, only a relation.

(7) Let X be a finite set with k elements. Then, we have a bijection from X to $\{1, 2, \ldots k\}$.

Let us consider functions defined on some known sets through a formula rule.

Example 1.14 Check whether the following functions are one-to-one and onto.

(i) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n + 2$.
(ii) $f : \mathbb{N} \cup \{-1, 0\} \to \mathbb{N}$ defined by $f(n) = n + 2$.

Solution:

(i) If $f(n) = f(m)$, then $n + 2 = m + 2$ and hence $m = n$. Thus f is one-to-one. As 1 has no pre-image, this function is not onto. (See Figure 1.30)
(ii) As above, this function is one-to-one. If m is in the co-domain, then $m - 2$ is in the domain and $f(m - 2) = (m - 2) + 2 = m$; thus m has a pre-image and hence this function is onto. (See Figure 1.31)

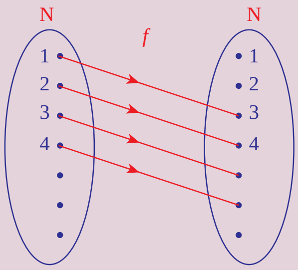


Figure 1.30

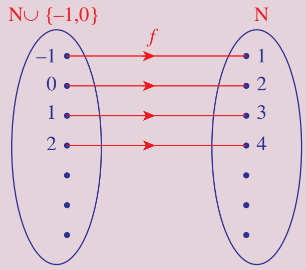


Figure 1.31

It seems that the second function (ii) is same as the first function (i). But the domains are different. From this we see that the domain of the function is also important in deciding whether the function is onto or not. The co-domain has no role in deciding whether the function is one-to-one or not. But it is important to decide whether the function is onto or not.

Example 1.15 Check the following functions for one-to-oneness and ontoness.

(i) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n^2$.
(ii) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(n) = n^2$.

Solution:

(i) $f(m) = f(n) \Rightarrow m^2 = n^2 \Rightarrow m = n$ since $m, n \in \mathbb{N}$. Thus f is one-to-one. But, non-perfect square elements in the co-domain do not have pre-images and hence not onto.
(ii) Two different elements in the domain have same images and hence f is not one-to-one. Clearly the range of f is a proper subset of $\mathbb{R}$. Thus it is not onto.

Now, we recall Illustration 1.1. In this illustration the function $f : C \to D$ is defined by

$$f(L) = O, f(E) = H, f(T) = W, f(U) = X, f(S) = V, f(W) = Z, f(I) = L, f(N) = Q$$

where $C = \{L, E, T, U, S, W, I, N\}$ and $D = \{O, H, W, X, V, Z, L, Q\}$, is an one-to-one and onto function.

In Illustration 1.3, the function $f : A \to \mathbb{N}$ is defined by $f(a) = 1, f(b) = 2, \ldots, f(z) = 26$, where $A = \{a, b, \ldots z\}$. This function is one-to-one. If we take $\mathbb{N}$ as co-domain, the function is not onto. Instead of $\mathbb{N}$ if we take the co-domain as $\{1, 2, 3 \ldots, 26\}$ then it becomes onto.

Example 1.16 Check whether the following for one-to-oneness and ontoness.

(i) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$.
(ii) $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$.

Solution:

(i) This is not at all a function because $f(x)$ is not defined for $x = 0$.
(ii) This function is one-to-one (verify) but not onto because 0 has no pre-image.

If we consider $\mathbb{R} - \{0\}$ as the co-domain for the second, then f will become a bijection.

Example 1.17 If $f : \mathbb{R} - \{-1, 1\} \to \mathbb{R}$ is defined by $f(x) = \frac{x}{x^2-1}$, verify whether f is one-to-one or not.

Solution:
We start with the assumption $f(x) = f(y)$. Then,

$$\begin{aligned} \frac{x}{x^2-1} &= \frac{y}{y^2-1} \\ \Rightarrow \quad x(y^2-1) &= y(x^2-1) \\ \Rightarrow \quad xy^2 - x - yx^2 + y &= 0 \\ \Rightarrow \quad (y-x)(xy+1) &= 0 \end{aligned}$$

This implies that $x = y$ or $xy = -1$. So if we select two numbers x and y so that $xy = -1$, then $f(x) = f(y)$. $(2, -\frac{1}{2}), (7, -\frac{1}{7}), (-2, \frac{1}{2})$ are some among the infinitely many possible pairs. Thus $f(2) = f(\frac{-1}{2}) = \frac{2}{3}$. That is, $f(x) = f(y)$ does not imply $x = y$. Hence it is not one-to-one.

Example 1.18 If $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = 2x^2 - 1$, find the pre-images of $17, 4$ and -2.

Solution:
To find the pre-image of 17, we solve the equation $2x^2 - 1 = 17$. The two solutions of this equation, 3 and -3 are the pre-images of 17 under f. The equation $2x^2 - 1 = 4$ yields $\sqrt{\frac{5}{2}}$ and $-\sqrt{\frac{5}{2}}$ as the two pre-images of 4. To find the pre-image of -2, we solve the equation $2x^2 - 1 = -2$. This shows that $x^2 = -\frac{1}{2}$ which has no solution in $\mathbb{R}$ because square of a number cannot be negative and hence -2 has no pre-image under f.

Example 1.19 If $f : [-2, 2] \to B$ is given by $f(x) = 2x^3$, then find B so that f is onto.

Solution:
The minimum value is $f(-2)$ and its maximum value is $f(2)$ which are equal to -16 and 16 respectively. So B is $[-16, 16]$.

As $f(x) = 2x^3$ is an increasing function on $[-2, 2]$, the minimum value is attained at the left end and the maximum value is attained at the right end. (For more about increasing / decreasing functions one may refer later chapters.)

Example 1.20 Check whether the function $f(x) = x|x|$ defined on $[-2, 2]$ is one-to-one or not. If it is one-to-one, find a suitable co-domain so that the function becomes a bijection.

Solution:
Let $x, y \in [-2, 2]$ such that $f(x) = f(y)$. If $y = 0$, then $x = 0$. So let $y \neq 0$ and hence $x \neq 0$. Now $x|x| = y|y|$ since $f(x) = f(y)$. This implies that $\frac{x}{y} = \frac{|y|}{|x|}$. Since $\frac{|y|}{|x|} > 0$, $\frac{x}{y} > 0$; thus x and y are either both positive or both negative and hence $x^2 = y^2$.

So if $f(x) = f(y)$, we must have $x^2 = y^2$. Also x and y are either both negative or both positive. This is possible only if $x = y$. Thus f is one-to-one. When $x < 0$, $f(x) = -x^2$ and when $x \geq 0$, $f(x) = x^2$. So the range is $[-4, 4]$. So f becomes a bijection from $[-2, 2]$ to $[-4, 4]$.

$f(x) = x|x|$ is also an increasing function.

Horizontal Test

Similar to the vertical line test we have a test called horizontal test to check whether a function is one-to-one, onto or not. Let a function be given as a curve in the plane. If the horizontal line through a point y in the co-domain meets the curve at some points, then the x-coordinate of all the points give pre-images for y.

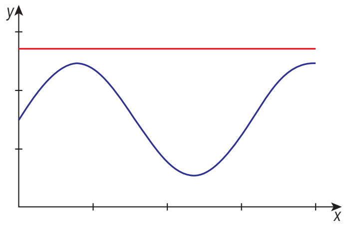


Figure 1.32

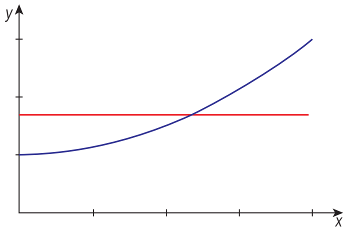


Figure 1.33

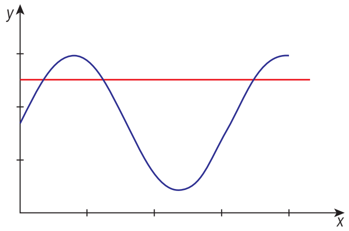


Figure 1.34

(i) If the horizontal line through a point y in the co-domain does not meet the curve, then there will be no pre-image for y and hence the function is not onto.

(ii) If the horizontal line through atleast one point meets the curve at more than one point, then the function is not one-to-one.

(iii) If for all y in the range the horizontal line through y meets the curve at only one point, then the function is one-to-one.

So we may say, the function represented by a curve is one-to-one if and only if for all y in the range, the horizontal line through the point y meets the curve at exactly one point.

The function represented by a curve is onto if and only if for all y in the co-domain, the horizontal line through the point y meets the curve atleast one point.

The curve given in Figure 1.32 represents a function from $[0, 4]$ which is not onto if the co-domain contains $[1, 3]$. The curve given in Figure 1.33 represents a one-to-one function from $[0, 4]$ to $\mathbb{R}$ and the curve given in Figure 1.34 represents a function from $[0, 4]$ to $\mathbb{R}$ which is not one-to-one.

Testing whether a given curve represents a one-to-one function, onto function or not by drawing horizontal lines is called ***horizontal line test*** or simply horizontal test.

Further by seeing the diagrams in Illustration 1.2 and Figures 1.5 to Figure 1.7, the function

(i) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$ is an one-to-one and onto function.

(ii) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is neither one-to-one nor onto.

(iii) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = +\sqrt{x}$ is an one-to-one but not onto function.

(iv) $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = +\sqrt{x}$ is an one-to-one and onto function.

(v) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = -\sqrt{x}$ is one-to-one but not onto function.

(vi) $f : [0, \infty) \to (-\infty, 0]$ defined by $f(x) = -\sqrt{x}$ is one-to-one and onto function.

Example 1.21 Find the largest possible domain for the real valued function f defined by $f(x) = \sqrt{x^2 - 5x + 6}$.

Solution:

As we are finding the square root of $x^2 - 5x + 6$, we must have $x^2 - 5x + 6 \geq 0$ for all x in the domain. For this, follow the given procedure.

Solving $x^2 - 5x + 6 = 0$, we get $x = 2$ and 3. Now draw the number line as in Figure 1.35.

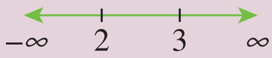


Figure 1.35

Now we have three intervals. $(-\infty, 2), (2, 3)$ and $(3, \infty)$

(i) Take any point in $(-\infty, 2)$, say $x = 1$. Clearly $x^2 - 5x + 6$ is positive.
(ii) Take any point in $(2, 3)$, say $x = 2.5$. Clearly $x^2 - 5x + 6$ is negative.
(iii) Take any point in $(3, \infty)$, say $x = 4$. Clearly $x^2 - 5x + 6$ is positive.

For all x, in the intervals $(-\infty, 2)$ and $(3, \infty)$, $x^2 - 5x + 6$ is positive. At $x = 2, 3$ the value of $x^2 - 5x + 6$ is zero. Thus, $\sqrt{x^2 - 5x + 6}$ is defined for all x in $(-\infty, 2] \cup [3, \infty)$.
Hence the domain of $\sqrt{x^2 - 5x + 6}$ is $(-\infty, 2] \cup [3, \infty)$.

Example 1.22 Find the domain of $f(x) = \frac{1}{1-2\cos x}$.

Solution:
The function is defined for all $x \in \mathbb{R}$ except $1 - 2\cos x = 0$. That is, except $\cos x = \frac{1}{2}$. That is except $x = 2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$. Hence the domain is $\mathbb{R} - \{2n\pi \pm \frac{\pi}{3}\}, n \in \mathbb{Z}$

Example 1.23 Find the range of the function $f(x) = \frac{1}{1-3\cos x}$.

Solution:
Clearly,

$$\begin{array}{rrcrcr} & -1 & \leq & \cos x & \leq & 1 \\ \Rightarrow & 3 & \geq & -3\cos x & \geq & -3 \\ \Rightarrow & -3 & \leq & -3\cos x & \leq & 3 \\ \Rightarrow & 1-3 & \leq & 1-3\cos x & \leq & 1+3 \end{array}$$

Thus we get $-2 \leq 1 - 3\cos x$ and $1 - 3\cos x \leq 4$.
By taking reciprocals, we get $\frac{1}{1-3\cos x} \leq -\frac{1}{2}$ and $\frac{1}{1-3\cos x} \geq \frac{1}{4}$.
Hence the range of f is $(-\infty, -\frac{1}{2}] \cup [\frac{1}{4}, \infty)$.

Example 1.24 Find the largest possible domain for the real valued function given by $f(x) = \frac{\sqrt{9-x^2}}{\sqrt{x^2-1}}$.

Solution:
If $x < -3$ or $x > 3$, then x^2 will be greater than 9 and hence $9 - x^2$ will become negative which has no square root in $\mathbb{R}$. So x must lie on the interval $[-3, 3]$.

Also if $x \geq -1$ and $x \leq 1$, then $x^2 - 1$ will become negative or zero. If it is negative, $x^2 - 1$ has no square root in $\mathbb{R}$. If it is zero, f is not defined. So x must lie outside $[-1, 1]$. That is, x must lie on $(-\infty, -1) \cup (1, \infty)$. Combining these two conditions, the largest possible domain for$f$ is $[-3, 3] \cap ((-\infty, -1) \cup (1, \infty))$. That is, $[-3, -1) \cup (1, 3]$.

Draw the number line and plot the intervals to get the required domain interval.

1.6.4 Operations on Functions

Composition

Let there be two functions f and g as given in the Figure 1.36 and Figure 1.37. Let us note that the co-domain of f and the domain of g are the same. Let us cut off Figure 1.37 of g and paste it on the Figure 1.36 of f so that the domain Y of g is pasted on co-domain Y of f. (See Figure 1.38.)

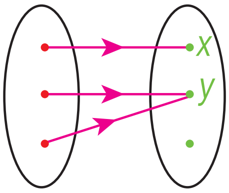


Figure 1.36

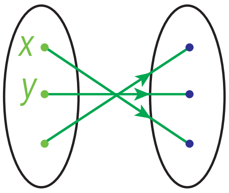


Figure 1.37

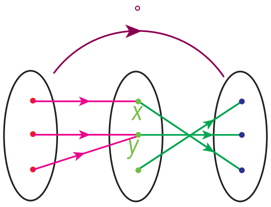


Figure 1.38

Now we can define a function $h : X \to Z$ in a natural way. To find the image of a under h, we first see the image of a under f; it is x; then we see the image of this x under g; this is r. That is, $h(a) = r$. Similarly, we declare $h(b) = q$ and $h(c) = q$. In this way we can define a new function h. This h is called the composition of f with g.

Definition 1.8

Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Then the function $h : X \to Z$ defined as $h(x) = g(f(x))$ for every $x \in X$ is called the ***composition*** of f with g. It is denoted by $g \circ f$ (Read this as f composite with g). (See Figures 1.38 and 1.39.)

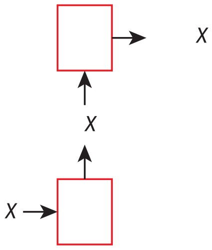


Figure 1.39

We can note that the range of f need not be Y. If $f : X \to Y_1, g : Y_2 \to Z$ and $Y_1 \subseteq Y_2$, then also we can define $g \circ f$; we can take Y_2 as the co-domain of f and use the same definition. So we can define $g \circ f$ if and only if the range of f is contained in the domain of g.

Example 1.25 Let $f = \{(1,2),(3,4),(2,2)\}$ and $g = \{(2,1),(3,1),(4,2)\}$. Find $g \circ f$ and $f \circ g$.

Solution:

To check whether compositions can be defined, let us find the domain and range of these functions.

Domain of $f = \{1,2,3\}$, Range of $f = \{2,4\}$, Domain of $g = \{2,3,4\}$ and Range of $g = \{1,2\}$. Since the range of f is contained in the domain of g we can define $g \circ f$; so as to find the image of 1 under $g \circ f$, we first find the image of 1 under f and then its image under g. The image of 1 under f is 2 and its image under g is 1. So $(g \circ f)(1) = g(f(1)) = g(2) = 1$.

Similarly we find that $(g \circ f)(2) = 1$ and $(g \circ f)(3) = 2$. So $g \circ f = \{(1,1),(2,1),(3,2)\}$. Similarly $f \circ g = \{(2,2),(3,2),(4,2)\}$.

Example 1.26 Let $f = \{(1,4),(2,5),(3,5)\}$ and $g = \{(4,1),(5,2),(6,4)\}$. Find $g \circ f$. Can you find $f \circ g$?

Solution:
Clearly, $g \circ f = \{(1,1),(2,2),(3,2)\}$. But $f \circ g$ is not defined because the range of $g = \{1,2,4\}$ is not contained in the domain of $f = \{1,2,3\}$.

Example 1.27 Let f and g be the two functions from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = 3x - 4$ and $g(x) = x^2 + 3$. Find $g \circ f$ and $f \circ g$.

Solution:
We have,
$(g \circ f)(x) = g(f(x)) = g(3x-4) = (3x-4)^2 + 3 = 9x^2 - 24x + 19.$
$(f \circ g)(x) = f(g(x)) = f(x^2+3) = 3(x^2+3) - 4 = 3x^2 + 5.$

Here we have $f \circ g \neq g \circ f$. Thus the operation "composition of functions" is in general not commutative.

Theorem 1.2: Let $f : A \to B$ and $g : B \to C$ be two functions. If f and g are one-to-one, then $g \circ f$ is one-to-one.

Proof. Let $x \neq y$ in A. Since f is one-to-one, $f(x) \neq f(y)$. Since g is one-to-one, $g(f(x)) \neq g(f(y))$. That is, $x \neq y \Rightarrow (g \circ f)(x) \neq (f \circ g)(y)$. Hence $g \circ f$ is one-to-one. □

Example 1.28 Show that the statement,

"**if f and $g \circ f$ are one-to-one, then g is one-to-one**" is not true.

Solution:
To claim a statement is not true we have to prove by giving one counter example. Consider the diagram given in Figure 1.40.

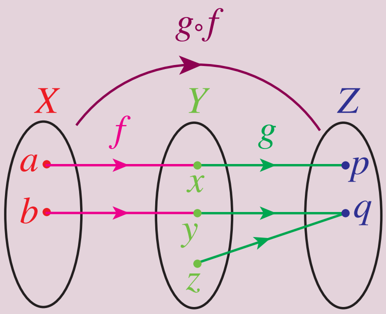


Figure 1.40

Clearly f and $g \circ f$ are one-to-one. But g is not one-to-one. Thus from the above diagram it shows that the statement is not true.

Example 1.29 Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x - |x|$ and $g(x) = 2x + |x|$. Find $f \circ g$.

Solution:
We know

$$|x| = \begin{cases} -x & \text{if} \quad x \leq 0 \\ x & \text{if} \quad x > 0 \end{cases}$$

So

$$f(x) = \begin{cases} 2x - (-x) & \text{if} \quad x \leq 0 \\ 2x - x & \text{if} \quad x > 0 \end{cases}$$

Thus

$$f(x) = \begin{cases} 3x & \text{if} \quad x \leq 0 \\ x & \text{if} \quad x > 0 \end{cases}$$

Also

$$g(x) = \begin{cases} 2x + (-x) & \text{if} \quad x \leq 0 \\ 2x + x & \text{if} \quad x > 0 \end{cases}$$

Thus

$$g(x) = \begin{cases} x & \text{if} \quad x \leq 0 \\ 3x & \text{if} \quad x > 0 \end{cases}$$

Let $x \leq 0$. Then

$$(g \circ f)(x) = g(f(x)) = g(3x) = 3x.$$

The last equality is taken because $3x \leq 0$ whenever $x \leq 0$.

Let $x > 0$. Then

$$(g \circ f)(x) = g(f(x)) = g(x) = 3x.$$

Thus $(g \circ f)(x) = 3x$ for all x.

1.6.5 Inverse of a Function

Let there be a bijection $f : X \to Y$ as given in the Figure 1.41.

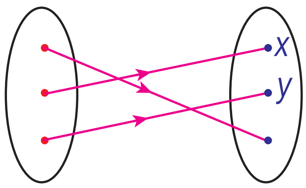


Figure 1.41

If we look this function in a mirror, we get a function from Y to X. Let us call that function as g. Then g is a function from Y to X defined by $g(x) = b, g(y) = c, g(z) = a$.

This function g is an example for the inverse of f. Now we define the inverse of a function.

Definition 1.9

Let $f : X \to Y$ be a bijection. The function $g : Y \to X$ defined by $g(y) = x$ if $f(x) = y$, is called the ***inverse*** of f and is denoted by f^{-1}.

If a function f has an inverse, then we say that f is ***invertible***. There is a nice relationship between composition of functions and inverse.

Let $f : X \to Y$ be a bijection and $g : Y \to X$ be its inverse. Then $g \circ f = I_X$ and $f \circ g = I_Y$ where I_X and I_Y are identity functions on X and Y respectively. Moreover, if $f : X \to Y$ and $g : Y \to X$ are functions such that $g \circ f = I_X$ and $f \circ g = I_Y$, then both f and g are bijections and they are inverses to each other; that is $f^{-1} = g$ and $g^{-1} = f$.

Using the discussions above, the terms invertible and inverse can be defined in some other way as follows:

Definition 1.10

A function $f : X \to Y$ is said to be ***invertible*** if there exists a function $g : Y \to X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$ where I_X and I_Y are identity functions on X and Y respectively. In this case, g is called the inverse of f and g is denoted by f^{-1}.

We may use this concept to prove some functions are bijective.

If f is a bijection, then $f^{-1}(y)$ is nothing but the pre-image of y under f. Let us note that the inverses are defined only for bijections. If f is not one-to-one , then there exists a and b such that $a \neq b$ and $f(a) = f(b)$. Let this value be y. Then we cannot define $f^{-1}(y)$ because both a and b are pre-images of y under f, as f^{-1} cannot assume two different values for y. If f is not onto, then there will be a y in Y without a pre-image. In this case also we cannot assign any value to $f^{-1}(y)$.

For example, if $A = \{1, 2, 3, 4\}$ and $f = \{(1, 2), (2, 4), (3, 1), (4, 3)\}$. Then the range of f is $\{1, 2, 3, 4\}$; the inverse of f is $\{(1, 3), (2, 1), (3, 4), (4, 2)\}$.

Working Rule to Find the Inverse of Functions from $\mathbb{R}$ to $\mathbb{R}$:

Let $f : \mathbb{R} \to \mathbb{R}$ be the given function.

i. write $y = f(x)$;
ii. write x in terms of y;
iii. write $f^{-1}(y) =$ the expression in y.
iv. replace y as x.

Example 1.30 If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 2x - 3$ prove that f is a bijection and find its inverse.

Solution:
Method 1:

One-to-one : Let $f(x) = f(y)$. Then $2x-3 = 2y-3$; this implies that $x = y$. That is, $f(x) = f(y)$ implies that $x = y$. Thus f is one-to-one.

Onto : Let $y \in \mathbb{R}$. Let $x = \frac{y+3}{2}$. Then $f(x) = 2(\frac{y+3}{2}) - 3 = y$. Thus f is onto. This also can be proved by saying the following statement. The range of f is $\mathbb{R}$ (how?) which is equal to the co-domain and hence f is onto.

Inverse Let $y = 2x - 3$. Then $y + 3 = 2x$ and hence $x = \dfrac{y+3}{2}$. Thus $f^{-1}(y) = \dfrac{y+3}{2}$. By replacing y as x, we get $f^{-1}(x) = \dfrac{x+3}{2}$.

Method 2:
Let $y = 2x - 3$. Then $x = \dfrac{y+3}{2}$. Let $g(y) = \dfrac{y+3}{2}$.

Now

$$(g \circ f)(x) = g(f(x)) = g(2x-3) = \frac{(2x-3)+3}{2} = x.$$

$$(f \circ g)(y) = f(g(y)) = f\left(\frac{y+3}{2}\right) = 2\left(\frac{y+3}{2}\right) - 3 = y.$$

Thus, $g \circ f = I_X$ and$f \circ g = I_Y$

This implies that f and g are bijections and inverses to each other. Hence f is a bijection and $f^{-1}(y) = \dfrac{y+3}{2}$. Replacing y by x we get, $f^{-1}(x) = \dfrac{x+3}{2}$.

1.6.6 Algebra of Functions

A function whose co-domain is $\mathbb{R}$ or a subset of $\mathbb{R}$ is called a real valued function. We can discuss many more operations on functions if it is real valued.

Let f and g be two real valued functions. Can we define addition of f and g? Naturally we expect the sum of two functions to be a function. The value of $f+g$ at a point x should be related to the values of f and g at x. So to define $f+g$ at a point x, we must know both $f(x)$ and $g(x)$. In other words x must be in the domain of f as well as in the domain of g. And the natural way of defining $f+g$ at x is $f(x)+g(x)$. So if we impose a condition that the domains of f and g to be the same, then we can define $f+g$. In the same way we can define subtraction, multiplication and many more algebraic operations available on the set $\mathbb{R}$ of the real numbers.

Definition 1.11

Let X be any set. Let f and g be real valued functions defined on X. Define, for all $x \in X$

- $(f+g)(x) = f(x) + g(x)$.
- $(f-g)(x) = f(x) - g(x)$.
- $(fg)(x) = f(x)g(x)$.
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$.
- $(cf)(x) = cf(x)$, where c is a real constant.
- $(-f)(x) = -f(x)$.

Note that the domain may be any set, not necessarily a set of numbers. For example if X is a set of students of a class, f and g functions representing the marks obtained by the students in two tests, then the function $f+g$ represent the total marks of the students in the two tests. It is easy to see that the operations addition, subtraction, multiplication and division defined above satisfy the following properties.

- $(f+g)+h = f+(g+h)$
- $f+g = g+f$
- $\mathbf{0}+f = f+\mathbf{0}$, where $\mathbf{0}$ is the zero function defined by $\mathbf{0}(x) = 0$ for all x.
- $f+(-f) = (-f)+f = \mathbf{0}$
- $f(g+h) = fg+fh$
- $(c_1+c_2)f = c_1f + c_2f$ where c_1 and c_2 are real constants.

We can list many more properties of these operations. The proofs are simple; however let us prove only one to show a way in which these properties can be proved.

Let us prove $f(g+h) = fg+fh$. To prove $f(g+h) = fg+fh$ we have to prove that $(f(g+h))(x) = (fg+fh)(x)$ for all x in the domain.

Theorem 1.3: If f and g are real-valued functions, then $f(g+h) = fg + fh$.

Proof. Let X be any set and f and g be real-valued functions defined on X. Let $x \in X$.

$$\begin{aligned}
(f(g+h))(x) &= f(x)(g+h)(x) && \text{(by the definition of product)}\\
&= f(x)[g(x)+h(x)] && \text{(by the definition of addition)}\\
&= f(x)g(x)+f(x)h(x) && \text{(by the distributivity of reals)}\\
&= (fg)(x)+(fh)(x) && \text{(by the definition of product)}\\
&= (fg+fh)(x) && \text{(by the definition of addition)}
\end{aligned}$$

Thus $(f(g+h))(x) = (fg+fh)(x)$ for all $x \in X$; hence $f(g+h) = fg + fh$. □

1.6.7 Some Special Functions

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Now let us see some special functions.

(i) The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n,$$

where a_i are constants, is called a ***polynomial function***. Since the right hand side of the equality defining the function is a polynomial, this function is called a polynomial function.

(ii) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax + b$ where $a \neq 0$ and b are constants, is called a ***linear function***. A function which is not linear is called a ***non-linear function***.

Clearly a linear function is a polynomial function. The graph of this function is a straight line; a straight line is called a linear curve; so this function is called a linear function. (one may come across different definitions for linear functions in higher study of mathematics.)

(iii) Let a be a non-negative constant. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a^x$. If $a = 0, x \neq 0$ then the function becomes the zero function and if $a = 1$, then function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a^x$ is the constant function $f(x) = 1$. [See, Figures 1.42 and 1.43].

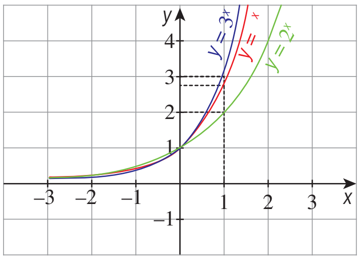


Figure 1.42

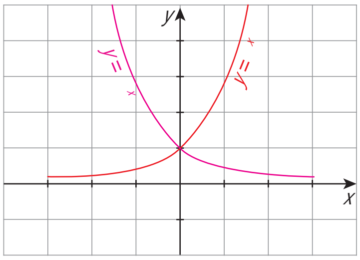


Figure 1.43

When $a > 1$, the function $f(x) = a^x$ is called an ***exponential function***. Moreover, any function having x in the "power" is called as an exponential function.

e is a special irrational number lies between 2 and 3. We will study more about e in the subsequent chapters.

(iv) Let $a > 1$ be a constant. The function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \log_a x$ is called a ***logarithmic function***. In fact, the inverse of an exponential function $f(x) = a^x$ on a suitable domain is called a logarithmic function. [See, Figure 1.44].

(v) The real valued function f defined by $f(x) = \frac{p(x)}{q(x)}$ on a suitable domain, where $p(x)$ and $q(x)$ are polynomials, $q(x) \neq 0$, is called a ***rational function***. In fact, the domain of these function are the sets obtained from $\mathbb{R}$ by removing the real numbers at which $q(x) = 0$.

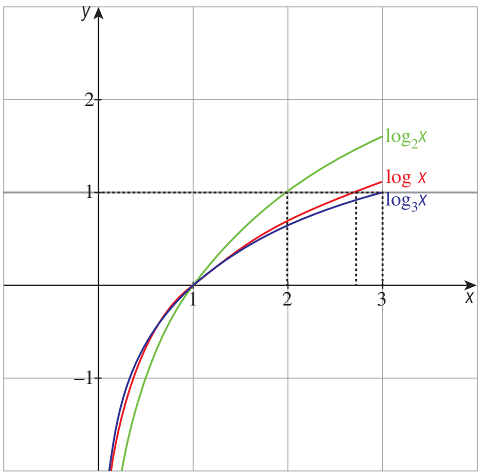


Figure 1.44

(vi) If f is a real valued function such that $f(x) \neq 0$, then the real valued function g defined by $g(x) = \frac{1}{f(x)}$ on a suitable domain is called the ***reciprocal function*** of f. The domain of g is the set obtained from $\mathbb{R}$ by removing the real numbers at which $f(x) = 0$. For example, the largest possible domain of $f(x) = \frac{1}{x-1}$ is $\mathbb{R} - \{1\}$.

Let us see two more categories of functions.

Definition 1.12

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be an ***odd function*** if $f(-x) = -f(x)$ for all $\in \mathbb{R}$. It is said to be an ***even function*** if $f(-x) = f(x)$ for all $x \in \mathbb{R}$. [See, Figures 1.45 and 1.46].

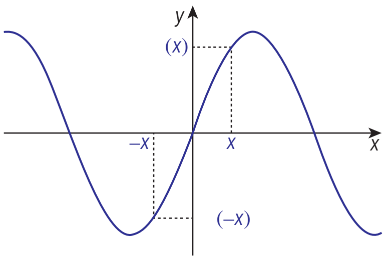


Figure 1.45

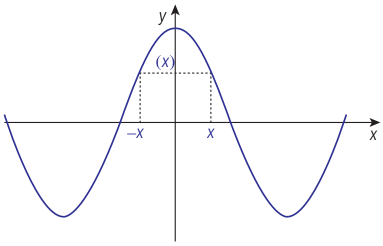


Figure 1.46

The function defined by $f(x) = x$, $f(x) = 2x$ and $f(x) = x^3 + 2x$ are some examples for odd functions. The functions defined by $f(x) = x^2$, $f(x) = 3$, $f(x) = x^4 + x^2$ and $f(x) = |x|$ are some examples for even functions. Note that the function $f(x) = x + x^2$ is neither even nor odd.

We can prove the following results.

(i) The sum of two odd functions is an odd function.
(ii) The sum of two even functions is an even function
(iii) The product of two odd functions is an even function.
(iv) The product of two even functions is an even function.
(v) The product of an odd function and an even function is an odd function.
(vi) The only function which is both odd and even function is the zero function.
(vii) The product of a positive constant and an even function is an even function.
(viii) The product of a negative constant and an even function is also an even function.
(ix) The product of a constant and an odd function is an odd function.
(x) There are functions which are neither odd nor even.

Let us prove one of the above properties. The other properties can be proved similarly.

Theorem 1.4: The product of an odd function and an even function is an odd function.

Proof. Let f be an odd function and g be an even function. Let $h = fg$. Now
$h(-x) = (fg)(-x) = f(-x)g(-x) = -f(x)g(x)$ (as f is odd and g is even)
$= -h(x)$

Thus h is an odd function. This shows that fg is an odd function. $\square$

If one function is not odd then don't think that the function is an even function. There are plenty of functions which are neither even nor odd.

Exercise - 1.3

1. Suppose that 120 students are studying in 4 sections of eleventh standard in a school. Let A denote the set of students and B denote the set of the sections. Define a relation from A to B as "*x related to y if the student x belongs to the section y*". Is this relation a function? What can you say about the inverse relation? Explain your answer.
2. Write the values of f at $-4, 1, -2, 7, 0$ if
$$f(x) = \begin{cases} -x+4 & \text{if } -\infty < x \leq -3 \\ x+4 & \text{if } -3 < x < -2 \\ x^2 - x & \text{if } -2 \leq x < 1 \\ x - x^2 & \text{if } 1 \leq x < 7 \\ 0 & \text{otherwise} \end{cases}$$
3. Write the values of f at $-3, 5, 2, -1, 0$ if
$$f(x) = \begin{cases} x^2 + x - 5 & \text{if } x \in (-\infty, 0) \\ x^2 + 3x - 2 & \text{if } x \in (3, \infty) \\ x^2 & \text{if } x \in (0, 2) \\ x^2 - 3 & \text{otherwise} \end{cases}$$
4. State whether the following relations are functions or not. If it is a function check for one-to-oneness and ontoness. If it is not a function, state why?

   (i) If $A = \{a, b, c\}$ and $f = \{(a, c), (b, c), (c, b)\}; (f : A \to A)$.
   (ii) If $X = \{x, y, z\}$ and $f = \{(x, y), (x, z), (z, x)\}; (f : X \to X)$.
5. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$. Give a function from $A \to B$ for each of the following:

   (i) neither one-to-one nor onto. (ii) not one-to-one but onto.
   (iii) one-to-one but not onto. (iv) one-to-one and onto.
6. Find the domain of $\dfrac{1}{1 - 2\sin x}$.
7. Find the largest possible domain of the real valued function $f(x) = \dfrac{\sqrt{4 - x^2}}{\sqrt{x^2 - 9}}$.
8. Find the range of the function $\dfrac{1}{2\cos x - 1}$.
9. Show that the relation $xy = -2$ is a function for a suitable domain. Find the domain and the range of the function.
10. If $f, g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = |x| + x$ and $g(x) = |x| - x$, find $g \circ f$ and $f \circ g$.

11. If f, g, h are real valued functions defined on $\mathbb{R}$, then prove that $(f+g)\circ h = f\circ h + g\circ h$. What can you say about $f\circ(g+h)$? Justify your answer.
12. If$f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x - 5$, prove that f is a bijection and find its inverse.
13. The weight of the muscles of a man is a function of his body weight x and can be expressed as $W(x) = 0.35x$. Determine the domain of this function.
14. The distance of an object falling is a function of time t and can be expressed as $s(t) = -16t^2$. Graph the function and determine if it is one-to-one.
15. The total cost of airfare on a given route is comprised of the base cost C and the fuel surcharge S in rupee. Both C and S are functions of the mileage m; $C(m) = 0.4m + 50$ and $S(m) = 0.03m$. Determine a function for the total cost of a ticket in terms of the mileage and find the airfare for flying 1600 miles.
16. A salesperson whose annual earnings can be represented by the function $A(x) = 30,000 + 0.04x$, where x is the rupee value of the merchandise he sells. His son is also in sales and his earnings are represented by the function $S(x) = 25,000 + 0.05x$. Find $(A+S)(x)$ and determine the total family income if they each sell Rupees $1,50,00,000$ worth of merchandise.
17. The function for exchanging American dollars for Singapore Dollar on a given day is $f(x) = 1.23x$, where x represents the number of American dollars. On the same day the function for exchanging Singapore Dollar to Indian Rupee is $g(y) = 50.50y$, where y represents the number of Singapore dollars. Write a function which will give the exchange rate of American dollars in terms of Indian rupee.
18. The owner of a small restaurant can prepare a particular meal at a cost of Rupees 100. He estimates that if the menu price of the meal is x rupees, then the number of customers who will order that meal at that price in an evening is given by the function $D(x) = 200 - x$. Express his day revenue, total cost and profit on this meal as functions of x.
19. The formula for converting from Fahrenheit to Celsius temperatures is $y = \dfrac{5x}{9} - \dfrac{160}{9}$. Find the inverse of this function and determine whether the inverse is also a function.
20. A simple cipher takes a number and codes it, using the function $f(x) = 3x - 4$. Find the inverse of this function, determine whether the inverse is also a function and verify the symmetrical property about the line $y = x$ (by drawing the lines).

1.7 Graphing Functions using Transformations

"*A picture is worth a thousand words*" is a well known proverb. To know about a function well, its graph will help us more than its analytical expression. To draw graphs quickly without plotting many points is an invaluable skill. Familiarity with shapes of some basic functions will help to graph other complicated functions. Understanding and usage of symmetry and transformations will then enable to strengthen graphing abilities. This section is not simply a data base of graphs, we learn some methods to graph certain functions.

Suppose that we want to draw or sketch the curve of the function $y = 2\sin(x-1) + 3$. At the very first sight it looks that it is very difficult to draw the curve representing this function. But it will be very easy to draw after understanding the content of this section.

If we know a half of a graph is the mirror image of the other half with respective to a line, or a graph can be obtained just by moving a known graph in some direction, then we can draw the new one using the known one. Moreover if we know that a graph can be obtained by enlarging or shrinking a known one, then also we can draw the new one using the known ones.

The following type of transformations play very important roles in graphing.

(i) Reflection
(ii) Translation
(iii) Dilations.

In the case of reflections and translations, they produce graphs congruent to the original graph; that is, the size and the shape of the graph does not change, but in dilation, it produce graphs with shapes related to those of the original graph.

Reflection

The ***reflection*** of the graph of a function with respect to a line ℓ is the graph that is symmetric to it with respect to ℓ. A reflection is the mirror image of the graph where line ℓ is the mirror of the reflection. (See Figure 1.47.)

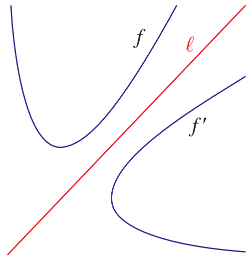


Figure 1.47

Here f' is the mirror image of f with respect to ℓ. Every point of f has a corresponding image in f'. Some useful reflections of $y = f(x)$ are

(i) The graph $y = -f(x)$ is the reflection of the graph of f about the x-axis.
(ii) The graph $y = f(-x)$ is the reflection of the graph of f about the y-axis.
(iii) The graph of $y = f^{-1}(x)$ is the reflection of the graph of f in $y = x$.

Illustration 1.5 Consider the functions:

$$(i)\ \ y = x^2 \quad (ii)\ \ y = -x^2.$$

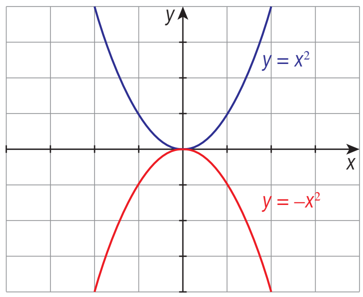


Figure 1.48

For the curve $f(x) = x^2$, $-f(x) = -x^2$. Hence, $y = -x^2$ is the reflection of $y = x^2$ about x-axis. (See Figure 1.48.)

Illustration 1.6 Consider the positive branches of

$$y^2 = x \text{ and } y^2 = -x.$$

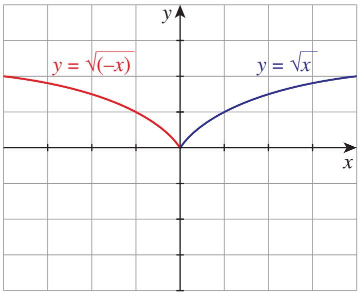


Figure 1.49

For the curve $f(x) = \sqrt{x}$, we have $f(-x) = \sqrt{-x}$ and hence $f(-x) = \sqrt{-x}$ where $x < 0$, is the reflection of $f(x) = \sqrt{x}$ about y-axis. (See Figure 1.49.)

Illustration 1.7 Consider the functions:

$$\text{(i) } y = e^x \quad \text{(ii) } y = \log_e x.$$

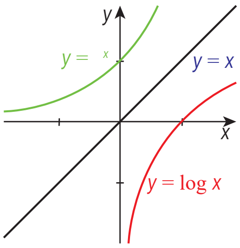


Figure 1.50

We know that, $y = e^x$ is the inverse function of $y = \log_e x$ and hence $y = e^x$ is the reflection of $y = \log_e x$ about $y = x$. (See Figure 1.50.)

Translation

A ***translation*** of a graph is a vertical or horizontal shift of the graph that produces congruent graphs.

The graph of	$y = f(x + c), c > 0$	causes the shift to the left.
	$y = f(x - c), c > 0$	causes the shift to the right.
	$y = f(x) + d, d > 0$	causes the shift to the upward.
	$y = f(x) - d, d > 0$	causes the shift to the downward.

Illustration 1.8 Consider the functions:

$$\text{(i) } f(x) = |x| \text{ (ii) } f(x) = |x - 1| \text{ (iii) } f(x) = |x + 1|$$

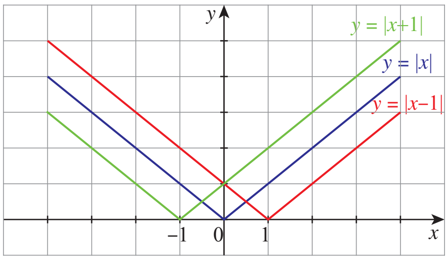


Figure 1.51

$f(x) = |x-1|$ causes the graph of the function $f(x) = |x|$ shifts to the right for one unit.

$f(x) = |x+1|$ causes the graph of the function $f(x) = |x|$ shifts to the left for one unit. (See Figure 1.51.)

Illustration 1.9 Consider the functions:

$$\text{(i) } f(x) = |x| \text{ (ii) } f(x) = |x| - 1 \text{ (iii) } f(x) = |x| + 1$$

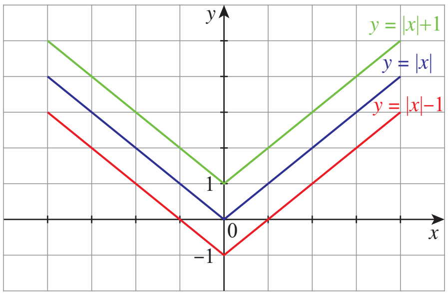


Figure 1.52

$f(x) = |x| - 1$ causes the graph of the function $f(x) = |x|$ shifts to the downward for one unit.

$f(x) = |x| + 1$ causes the graph of the function $f(x) = |x|$ shifts to the upward for one unit. (See Figure 1.52.)

Dilation

Dilation is also a transformation which causes the curve stretches (expands) or compresses (contracts). Multiplying a function by a positive constant vertically stretches or compresses its graph; that is, the graph moves away from x-axis or towards x-axis.

If the positive constant is greater than one, the graph moves away from the x-axis. If the positive constant is less than one, the graph moves towards the x-axis.

Illustration 1.10 Consider the functions:

$$\text{(i) } f(x) = x^2 \text{ (ii) } f(x) = \tfrac{1}{2}x^2 \text{ (iii) } f(x) = 2x^2$$

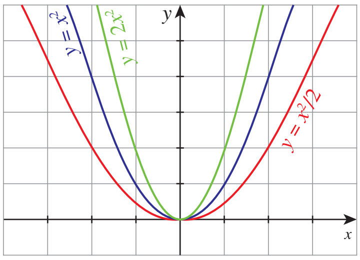


Figure 1.53

$f(x) = \frac{1}{2}x^2$ causes the graph of the function $f(x) = x^2$ stretches towards the x-axis since the multiplying factor is $\frac{1}{2}$ which is less than one.

$f(x) = 2x^2$ causes the graph of the function $f(x) = x^2$ compresses towards the $y-$ axis that is, moves away from the $x-$axis since the multiplying factor is 2 which is greater than one. (See Figure 1.53.)

Illustration 1.11 Consider the functions:

$$\text{(i)}\ \ f(x) = x^2 \quad \text{(ii)}\ \ f(x) = x^2 + 1 \quad \text{(iii)}\ \ f(x) = (x+1)^2.$$

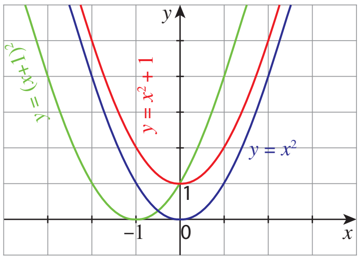


Figure 1.54

$f(x) = x^2 + 1$ causes the graph of the function $f(x) = x^2$ shifts to the upward for one unit.

$f(x) = (x+1)^2$ causes the graph of the function $f(x) = x^2$ shifts to the left for one unit. (See Figure 1.54.)

Illustration 1.12 Compare and contrast the graphs $y = x^2 - 1$, $y = 4(x^2 - 1)$ and $y = (4x)^2 - 1$

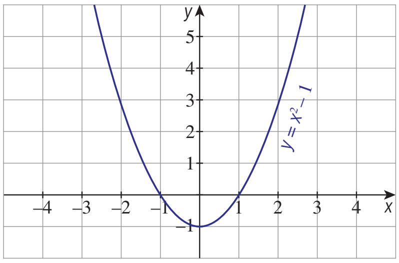


Figure 1.55

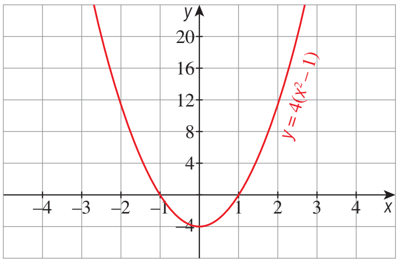


Figure 1.56

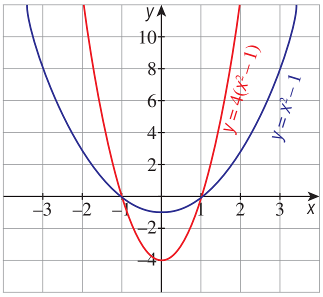


Figure 1.57

The graphs Figures 1.55 and 1.56 look identical until we compare the scales on the y-axis. The scale in Figure 1.56 is four times as large, reflecting the multiplication of the original function by 4 (Figure 1.55). The effect looks different when the functions are plotted on the same scale as in Figure 1.57.

The graph of $y = (4x)^2 - 1$ is shown in Figure 1.58. Can you spot the difference between Figure 1.55 and Figure 1.58? In this case, x-scale has now changed, by the same factor of 4 as in the function (Figure 1.58). To see this, note that substituting $x = \frac{1}{4}$ into $(4x)^2 - 1$ produces $1^2 - 1$, exactly the same as substituting $x = 1$ into the original function (Figure 1.55). When plotted on the same set of axes (as in Figure 1.59) the parabola $y = (4x)^2 - 1$ looks thinner. Here, the x-intercepts are different, but y-intercepts are the same.

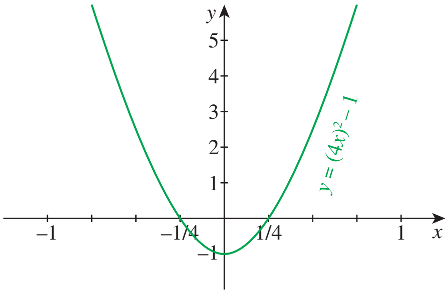


Figure 1.58

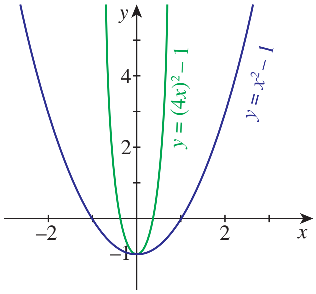


Figure 1.59

Illustration 1.13 By using the same concept applied in Illustration 1.12, graphs of $y = \sin x$ and $y = \sin 2x$, and also their combined graphs are given Figures 1.60, 1.61 and 1.62. The minimum and maximum values of $\sin x$ and $\sin 2x$ are the same. But they have different x-intercepts. The x-intercepts for $y = \sin x$ are $\pm n\pi$ and for $y = \sin 2x$ are $\pm\frac{1}{2}n\pi, n \in \mathbb{Z}$.

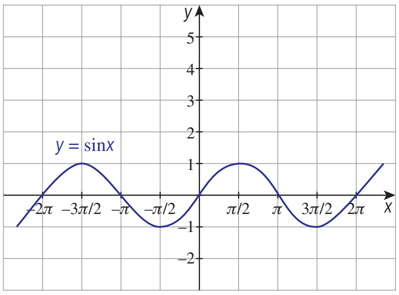


Figure 1.60

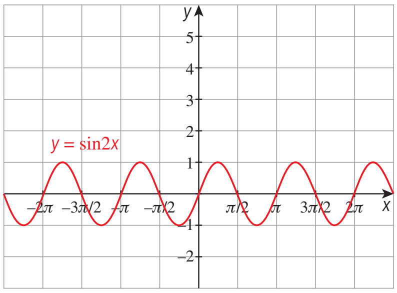


Figure 1.61

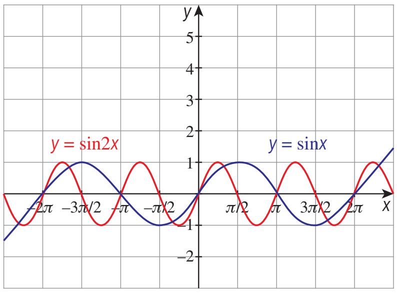


Figure 1.62

In the beginning of the section we talked about drawing the graph of $y = 2\sin(x-1)+3$. Now we are well equipped to draw the curve and even we can draw more complicated curve.

Illustration 1.14 Let us now draw the graph of $y = 2\sin(x-1)+3$.

It is clear that the curve can be obtained from that of $y = \sin x$ using translation and dilation.

So first we draw $y = \sin x$. From that it is easy to draw the curve $y = \sin(x-1)$; then draw $y = 2\sin(x-1)$ and finally $y = 2\sin(x-1)+3$. (See Figures 1.63 to 1.66.)

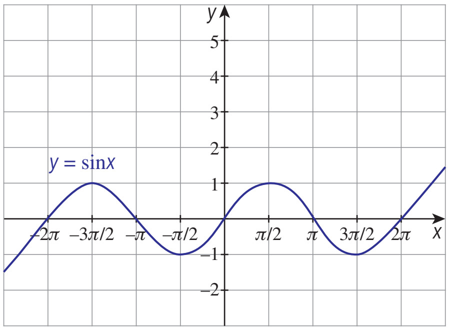


Figure 1.63

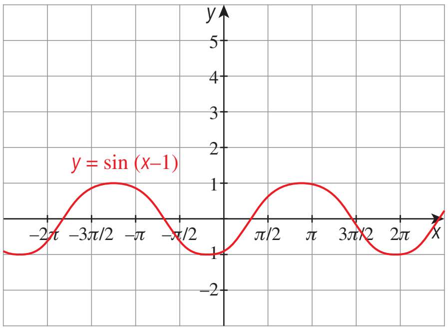


Figure 1.64

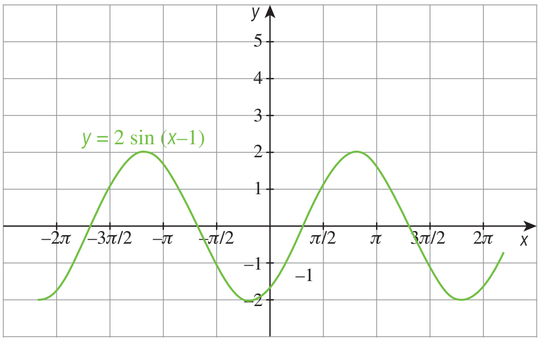


Figure 1.65

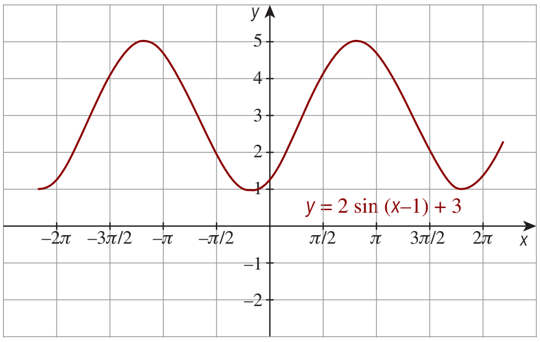


Figure 1.66

Exercise - 1.4

1. For the curve $y = x^3$ given in Figure 1.67, draw

   (i) $y = -x^3$ (ii) $y = x^3 + 1$ (iii) $y = x^3 - 1$ (iv) $y = (x+1)^3$ with the same scale.

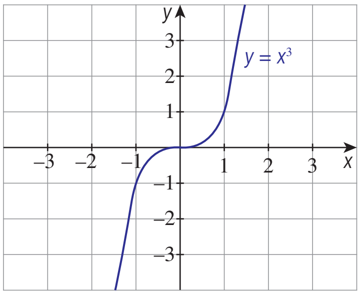


Figure 1.67

2. For the curve $y = x^{\left(\frac{1}{3}\right)}$ given in Figure 1.68, draw

   (i) $y = -x^{\left(\frac{1}{3}\right)}$ (ii) $y = x^{\left(\frac{1}{3}\right)} + 1$ (iii) $y = x^{\left(\frac{1}{3}\right)} - 1$ (iv) $y = (x+1)^{\left(\frac{1}{3}\right)}$

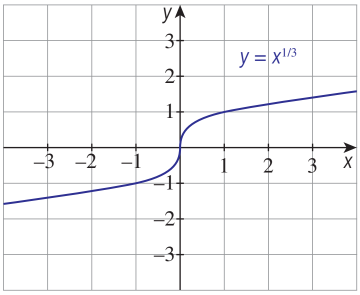


Figure 1.68

3. Graph the functions $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ on the same coordinate plane. Find $f \circ g$ and graph it on the plane as well. Explain your results.
4. Write the steps to obtain the graph of the function $y = 3(x-1)^2 + 5$ from the graph $y = x^2$.
5. From the curve $y = \sin x$, graph the functions
   (i) $y = \sin(-x)$ (ii) $y = -\sin(-x)$ (iii) $y = \sin\left(\frac{\pi}{2} + x\right)$ which is $\cos x$
   (iv) $y = \sin\left(\frac{\pi}{2} - x\right)$ which is also $\cos x$ (refer trigonometry)
6. From the curve $y = x$, draw
   (i) $y = -x$ (ii) $y = 2x$ (iii) $y = x + 1$ (iv) $y = \frac{1}{2}x + 1$ (v) $2x + y + 3 = 0$.
7. From the curve $y = |x|$, draw (i) $y = |x-1| + 1$ (ii) $y = |x+1| - 1$ (iii) $y = |x+2| - 3$.
8. From the curve $y = \sin x$, draw $y = \sin|x|$ (Hint: $\sin(-x) = -\sin x$.)

Activities

Balls and Runs

What a Celebration! What a Relation!! What a Function!!!

Source: Times of India

1. Interpret the above data as,
   (i) a relation
   (ii) a function
   (iii) an onto function
   (iv) can you make the data as an one-to-one function? If not, why?
2. Identify the curves in Figure 1.69 and the corresponding equations for the base curve $y = x^2$ (graph with dotted line) by seeing the scale.

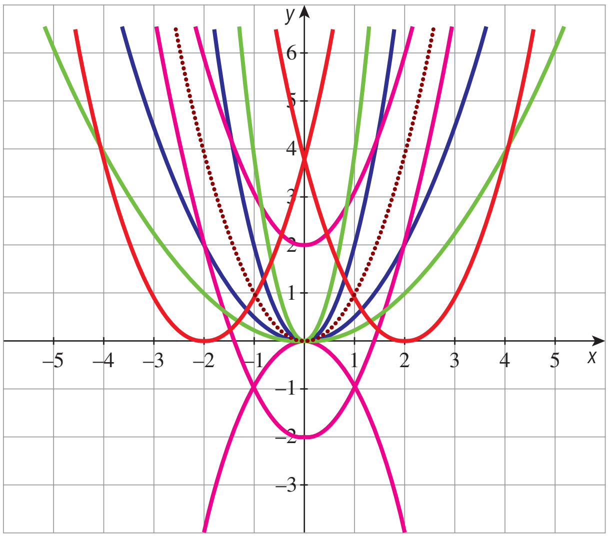


Figure 1.69

22AQTM

Exercise - 1.5

Choose the correct or the most suitable answer.

1. If $A = \{(x, y) : y = e^x, x \in R\}$ and $B = \{(x, y) : y = e^{-x}, x \in R\}$ then $n(A \cap B)$ is

   (1) Infinity (2) 0 (3) 1 (4) 2

2. If $A = \{(x, y) : y = \sin x, x \in R\}$ and $B = \{(x, y) : y = \cos x, x \in R\}$ then $A \cap B$ contains

   (1) no element (2) infinitely many elements
   (3) only one element (4) cannot be determined.

3. The relation R defined on a set $A = \{0, -1, 1, 2\}$ by xRy if $|x^2 + y^2| \le 2$, then which one of the following is true?

   (1) $R = \{(0,0), (0,-1), (0,1), (-1,0), (-1,1), (1,2), (1,0)\}$
   (2) $R^{-1} = \{(0,0), (0,-1), (0,1), (-1,0), (1,0)\}$
   (3) Domain of R is $\{0, -1, 1, 2\}$
   (4) Range of R is $\{0, -1, 1\}$

4. If $f(x) = |x - 2| + |x + 2|, x \in \mathbb{R}$, then

   (1) $f(x) = \begin{cases} -2x & \text{if} \quad x \in (-\infty, -2] \\ 4 & \text{if} \quad x \in (-2, 2] \\ 2x & \text{if} \quad x \in (2, \infty) \end{cases}$

   (2) $f(x) = \begin{cases} 2x & \text{if} \quad x \in (-\infty, -2] \\ 4x & \text{if} \quad x \in (-2, 2] \\ -2x & \text{if} \quad x \in (2, \infty) \end{cases}$

   (3) $f(x) = \begin{cases} -2x & \text{if} \quad x \in (-\infty, -2] \\ -4x & \text{if} \quad x \in (-2, 2] \\ 2x & \text{if} \quad x \in (2, \infty) \end{cases}$

   (4) $f(x) = \begin{cases} -2x & \text{if} \quad x \in (-\infty, -2] \\ 2x & \text{if} \quad x \in (-2, 2] \\ 2x & \text{if} \quad x \in (2, \infty) \end{cases}$

5. Let $\mathbb{R}$ be the set of all real numbers. Consider the following subsets of the plane $\mathbb{R} \times \mathbb{R}$:

   $S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$ and $T = \{(x, y) : x - y \text{ is an integer }\}$

   Then which of the following is true?

   (1) T is an equivalence relation but S is not an equivalence relation.
   (2) Neither S nor T is an equivalence relation
   (3) Both S and T are equivalence relation
   (4) S is an equivalence relation but T is not an equivalence relation.

6. Let A and B be subsets of the universal set $\mathbb{N}$, the set of natural numbers. Then $A' \cup [(A \cap B) \cup B']$ is

   (1) A (2) A' (3) B (4) $\mathbb{N}$

7. The number of students who take both the subjects Mathematics and Chemistry is 70. This represents 10% of the enrollment in Mathematics and 14% of the enrollment in Chemistry. The number of students take at least one of these two subjects, is

   (1) 1120 (2) 1130 (3) 1100 (4) insufficient data

8. If $n((A \times B) \cap (A \times C)) = 8$ and $n(B \cap C) = 2$, then $n(A)$ is

   (1) 6 (2) 4 (3) 8 (4) 16

9. If $n(A) = 2$ and $n(B \cup C) = 3$, then $n[(A \times B) \cup (A \times C)]$ is

   (1) 2^3 (2) 3^2 (3) 6 (4) 5

10. If two sets A and B have 17 elements in common, then the number of elements common to the set $A \times B$ and $B \times A$ is

    (1) 2^{17} (2) 17^2 (3) 34 (4) insufficient data

11. For non-empty sets A and B, if $A \subset B$ then $(A \times B) \cap (B \times A)$ is equal to

    (1) $A \cap B$ (2) $A \times A$ (3) $B \times B$ (4) none of these.

12. The number of relations on a set containing 3 elements is

    (1) 9 (2) 81 (3) 512 (4) 1024

13. Let R be the universal relation on a set X with more than one element. Then R is

    (1) not reflexive (2) not symmetric (3) transitive (4) none of the above

14. Let $X = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3), (2, 1), (3, 1), (1, 4), (4, 1)\}$. Then R is

    (1) reflexive (2) symmetric (3) transitive (4) equivalence

15. The range of the function $\frac{1}{1-2\sin x}$ is

    (1) $(-\infty, -1) \cup (\frac{1}{3}, \infty)$ (2) $(-1, \frac{1}{3})$ (3) $[-1, \frac{1}{3}]$ (4) $(-\infty, -1] \cup [\frac{1}{3}, \infty)$.

16. The range of the function $f(x) = |\lfloor x \rfloor - x|, x \in \mathbb{R}$ is

    (1) $[0, 1]$ (2) $[0, \infty)$ (3) $[0, 1)$ (4) $(0, 1)$

17. The rule $f(x) = x^2$ is a bijection if the domain and the co-domain are given by

    (1) $\mathbb{R}, \mathbb{R}$ (2) $\mathbb{R}, (0, \infty)$ (3) $(0, \infty), \mathbb{R}$ (4) $[0, \infty), [0, \infty)$

18. The number of constant functions from a set containing m elements to a set containing n elements is

    (1) mn (2) m (3) n (4) $m + n$

19. The function $f : [0, 2\pi] \to [-1, 1]$ defined by $f(x) = \sin x$ is

    (1) one-to-one (2) onto (3) bijection (4) cannot be defined

20. If the function $f : [-3, 3] \to S$ defined by $f(x) = x^2$ is onto, then S is

    (1) $[-9, 9]$ (2) $\mathbb{R}$ (3) $[-3, 3]$ (4) $[0, 9]$

21. Let $X = \{1, 2, 3, 4\}, Y = \{a, b, c, d\}$ and $f = \{(1, a), (4, b), (2, c), (3, d), (2, d)\}$. Then f is

    (1) an one-to-one function (2) an onto function
    (3) a function which is not one-to-one (4) not a function

22. The inverse of $f(x) = \begin{cases} x & \text{if} \quad x < 1 \\ x^2 & \text{if} \quad 1 \le x \le 4 \\ 8\sqrt{x} & \text{if} \quad x > 4 \end{cases}$ is

(1) $f^{-1}(x) = \begin{cases} x & \text{if} \quad x < 1 \\ \sqrt{x} & \text{if} \quad 1 \le x \le 16 \\ \frac{x^2}{64} & \text{if} \quad x > 16 \end{cases}$

(2) $f^{-1}(x) = \begin{cases} -x & \text{if} \quad x < 1 \\ \sqrt{x} & \text{if} \quad 1 \le x \le 16 \\ \frac{x^2}{64} & \text{if} \quad x > 16 \end{cases}$

(3) $f^{-1}(x) = \begin{cases} x^2 & \text{if} \quad x < 1 \\ \sqrt{x} & \text{if} \quad 1 \le x \le 16 \\ \frac{x^2}{64} & \text{if} \quad x > 16 \end{cases}$

(4) $f^{-1}(x) = \begin{cases} 2x & \text{if} \quad x < 1 \\ \sqrt{x} & \text{if} \quad 1 \le x \le 16 \\ \frac{x^2}{8} & \text{if} \quad x > 16 \end{cases}$

23. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1 - |x|$. Then the range of f is

(1) $\mathbb{R}$ (2) $(1, \infty)$ (3) $(-1, \infty)$ (4) $(-\infty, 1]$

24. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \sin x + \cos x$ is

(1) an odd function
(2) neither an odd function nor an even function
(3) an even function
(4) both odd function and even function.

25. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \frac{(x^2 + \cos x)(1 + x^4)}{(x - \sin x)(2x - x^3)} + e^{-|x|}$$

is (1) an odd function (2) neither an odd function nor an even function
(3) an even function (4) both odd function and even function.

Summary

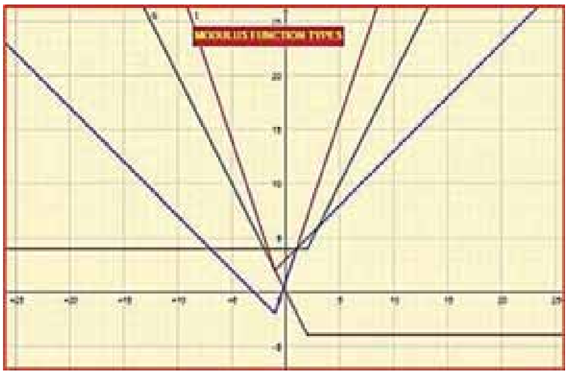
In this chapter we have acquired the knowledge of

- Set
  - Subset, super set, trivial subset, proper subset, improper subset
  - Empty set, power set, universal set, singleton set, finite set, infinite set
  - Cardinality of a set
  - Union, Intersection, Complement, Set Difference, Symmetric Difference
  - Properties and De Morgan Laws
  - Cartesian Product
- Intervals
  - Constants, dependent and independent variables
  - Open, Closed, finite and infinite intervals and neighbourhoods;

- Relations
  - Domain and range of relation
  - Extreme relations (empty and universal)
  - Inverse of a relation
  - Reflexive, Symmetric, Transitive, Equivalence Relations
- Functions
  - Definition, domain, co-domain, range, image, pre-image,
  - Tabular, graphical, analytical and piecewise representations,
  - Identity function, constant function, zero function, modulus function, signum function, greatest integer function, smallest integer function,
  - Injective, surjective and bijective functions,
  - Vertical test and Horizontal test,
  - Composition of functions, inverse of a function,
  - Addition and multiplication of real valued functions,
  - Polynomial function, linear function, exponential function, logarithmic function, rational function, reciprocal function,
  - Odd and Even functions.
- Graphing functions
  - Reflection, translation, dilation
  - Drawing graph of some seems to be complicated functions.

ICT CORNER-1(a)

Expected Outcome ⇒

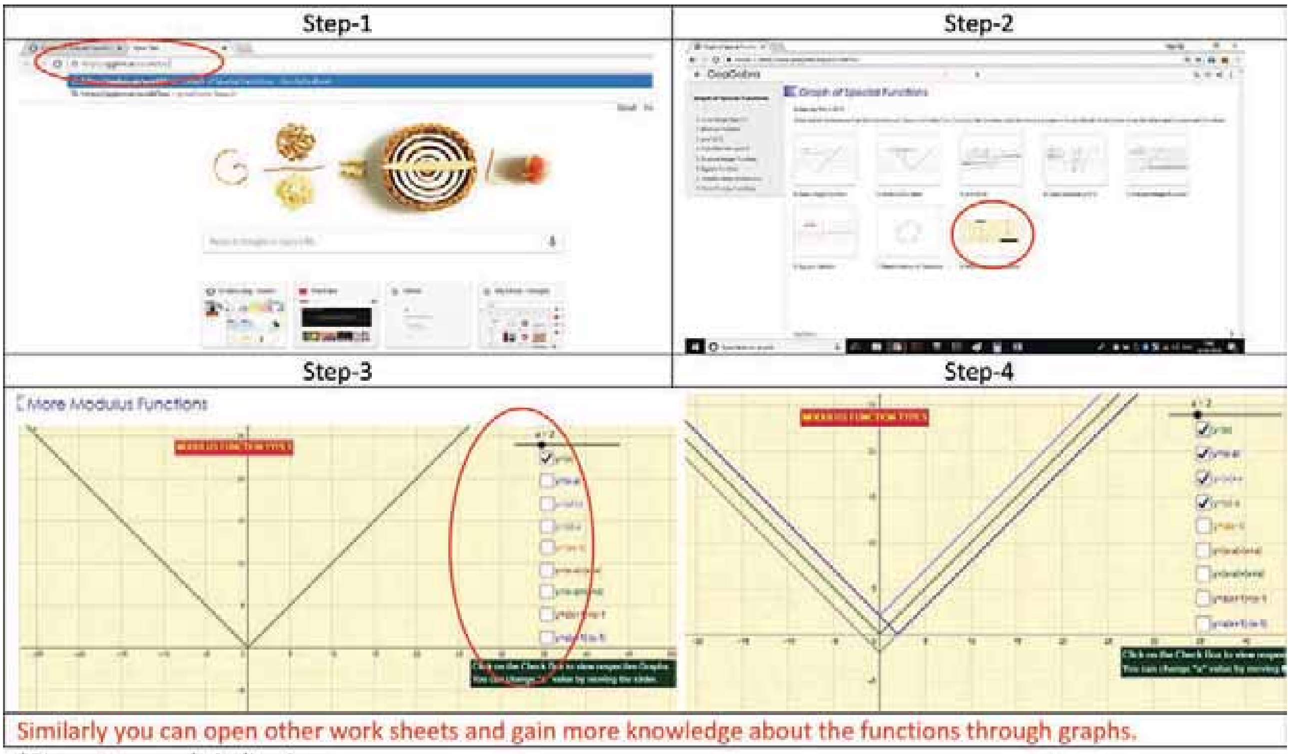


Step–1
Open the Browser and type the URL Link given below (or) Scan the QR Code.

Step–2
A workbook named "Graph of Special Functions" will open. 7 worksheets are given in this workbook related to Functions and Graph. Select a worksheet named "More Modulus Functions"

Step–3
Right side of the work sheet There are check boxes for many modulus functions

Step–4
You can click on any check box to view the Graph. Now Move the Slider a to change the function and observe.

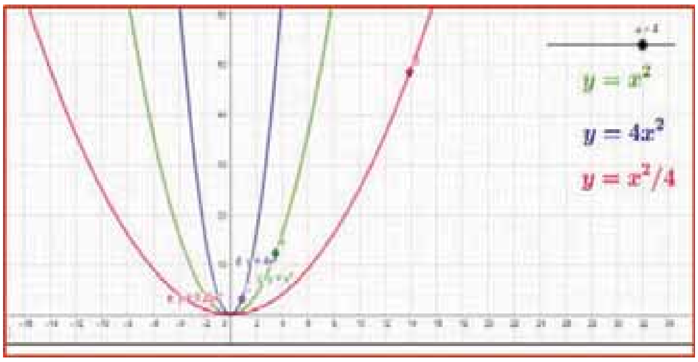


*Pictures are only indicatives.

Browse in the link Graph of Special Functions: https://ggbm.at/ucz465auor Scan the QR Code.

ICT CORNER-1(b)

Expected Outcome ⇒



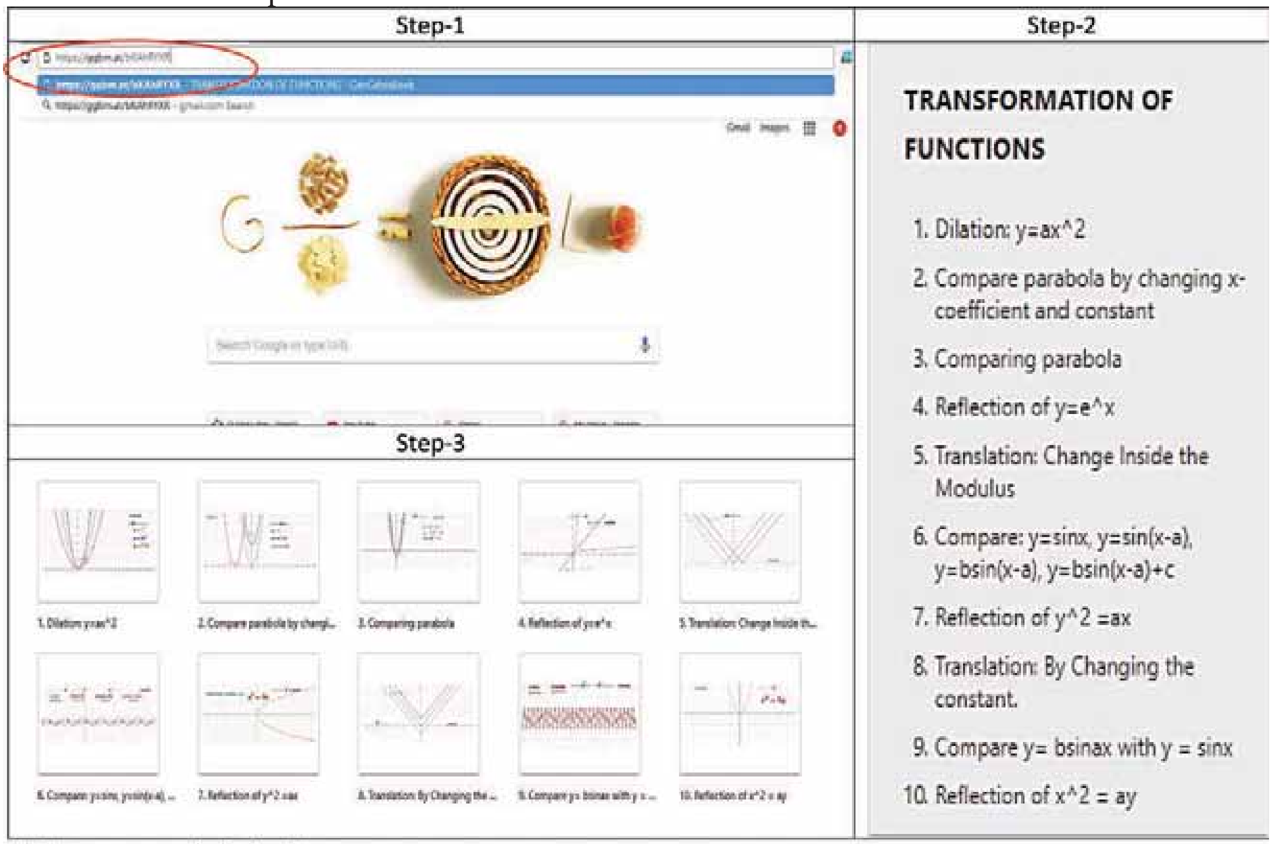
Step–1
Open the Browser and type the URL Link given below (or) Scan the QR Code.

Step–2
GeoGebra Workbook called "**TRANSFORMATION OF FUNCTIONS**" will appear. In that there are 10 worksheets related to your lesson.

Step–3

Select the work sheets one-one by one and analyse the transformations 1. Translation, 2. Reflection and 3. Dilation.

Also, there is a worksheet for comparison of Sine functions. All you have to do is move the sliders in the worksheets and compare.



*Pictures are only indicatives.

Browse in the link Transformation of Functions: https://ggbm.at/bKAhRYXR

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Chapter 2 Basic Algebra

I see it but I don't believe it.

Richard Dedekind

2.1 Introduction

Algebra is a branch of mathematics in which one expresses relations among quantities by using symbols to represent these quantities. The symbols are called the *variables*. In this class we shall allow the variables to represent real numbers only. One can carry out manipulations and computations using variables just as one does with numbers. That is, one may substitute real numbers for the variables in the expression and the resulting value will also be a real number. Once a quantity or a mathematical statement is expressed in terms of variables, it is possible to substitute specific numerical values for those variables. This makes algebra a very powerful tool. For this reason the subject of algebra has very wide application, not only within mathematics, but also in other disciplines and in real life. The notion of real numbers is fundamental to the whole of mathematics. The real number system was well understood only in the nineteenth century. The need for extending the rational numbers arose quite early in the history of mathematics. Pythagoreans knew that $\sqrt{2}$ was not a rational number. Certain constructions involving irrational numbers can be found in Shulbha Sutras, which date back to around 800 BCE. Aryabhata (476-550) had found approximations to π.

Indian mathematicians like Brahmagupta (598-670) and Bhaskaracharya (1114-1185) had made contributions to the understanding of the real numbers system and algebra. In his work Brahmagupta had solved the general quadratic equation for both positive and negative roots. Bhaskaracharya solved quadratic equations with more than one unknown and found negative and irrational solutions. The most important real number **zero** was the contribution by Indians.

Rene Descartes (1596-1650) introduced the term "real" to describe roots of a polynomial equation distinguishing them from imaginary ones. A rigorous construction of real number system was due to Richard Dedekind (1831-1916).

Richard Dedekind
(1831-1916)

Learning Objectives

On completion of this chapter, the students are expected to know

- the concept of real numbers and their properties.
- the absolute value, polynomials, exponents, radicals, logarithms and functions of one variables involving these concepts.
- how to solve equations, inequalities involving above mentioned functions.
- how to solve linear inequalities involving two variables and representing the solutions graphically in the cartesian plane.

2.2 Real Number System

First we shall recall how the real number system was developed. We start with natural numbers $\mathbb{N}$.

2.2.1 Rational Numbers

Note that $\mathbb{N} = \{1, 2, 3, \ldots\}$ is enough for counting objects. In order to deal with loss or debts, we enlarged $\mathbb{N}$ to the set of all integers, $\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, \ldots\}$, which consists of the natural numbers, zero, and the negatives of natural numbers. We call $\{0, 1, 2, 3, \cdots\}$ as the set of whole numbers and denote it by $\mathbb{W}$. Note that it differs from $\mathbb{N}$ by just one element, namely, zero. Now imagine dividing a cake into five equal parts, which is equivalent to finding a solution of $5x = 1$. But this equation cannot be solved within $\mathbb{Z}$. Hence we have enlarged $\mathbb{Z}$ to the set $\mathbb{Q} = \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \right\}$ of *ratios*; so we call each $x \in \mathbb{Q}$ as a *rational* number. Some examples of rational numbers are

$$-5, \ \frac{-7}{3}, \ \ 0, \ \frac{22}{7}, \ 7, \ 12.$$

We have seen in earlier classes that rational numbers are precisely the set of terminating or infinite periodic decimals. For example,

$$-5.0, \ -2.333\cdots, \ \frac{25}{99} = 0.252525\cdots, \ \frac{2}{3} = 0.66666\cdots, 7.145278362312312 31\cdots$$

are rational numbers.

2.2.2 The Number Line

Let us recall "*The Number Line*". It is a horizontal line with the *origin*, to represent 0, and another point marked to the right of 0 to represent 1. The distance from 0 to 1 defines one unit of length. Now put 2 one unit to the right of 1. Similarly we put any positive rational number x to the right of 0 and x units away. Also, we put a negative rational number $-r, r > 0$, to the left of 0 by r units. Note that for any $x, y \in \mathbb{Q}$ if $x < y$, then x is to the left of y; also $x < \dfrac{x+y}{2} < y$ and hence between any two distinct rational numbers there is another rational number between them.

Question:

Have we filled the whole line with rational numbers?

The answer to the above question is "No" as the following consideration demonstrates. Consider a square whose side has length 1 unit. Then by Pythagoras theorem its diagonal has length $\sqrt{2}$ units.

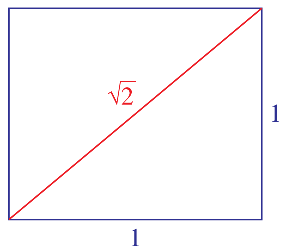


Figure 2.1

2.2.3 Irrational Numbers

Theorem 2.1: $\sqrt{2}$ is not a rational number.

Proof. Suppose that $\sqrt{2}$ is a rational number. Let $\sqrt{2} = \frac{m}{n}$, where m and n are positive integers with no common factors greater than 1. Then, we have $m^2 = 2n^2$, which implies that m^2 is even and hence m is even.

Let $m = 2k$. Then, we have $2n^2 = 4k^2$ which gives $n^2 = 2k^2$.
Thus, n is also even.
It follows, that m and n are even numbers having a common factor 2.
Thus, we arrived at a contradiction.
Hence, $\sqrt{2}$ is an irrational number. $\square$

Remark:

(i) Note that in the above proof we have assumed the contrary of what we wanted to prove and arrived at a contradiction. This method of proof is called '*proof by contradiction*'.
(ii) There are points on the number line that are not represented by rational numbers.
(iii) We call those numbers on the number line that do not correspond to rational numbers as *irrational numbers*. The set of all irrational numbers is denoted by $\mathbb{Q}'$ (For number line see Figure 1.2.)

Every real number is either a rational number or an irrational number, but not both. Thus, $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$ and $\mathbb{Q} \cap \mathbb{Q}' = \emptyset$.

As we already knew that every terminating or infinite periodic decimal is a rational number, we see that the decimal representation of an irrational number will neither be terminating nor infinite periodic. The set $\mathbb{R}$ of real numbers can be visualized as the set of points on the number line such that if $x < y$, then x lies to left of y.

Figure 2.2 demonstrates how the square roots of 2 and 3 can be identified on a number line.

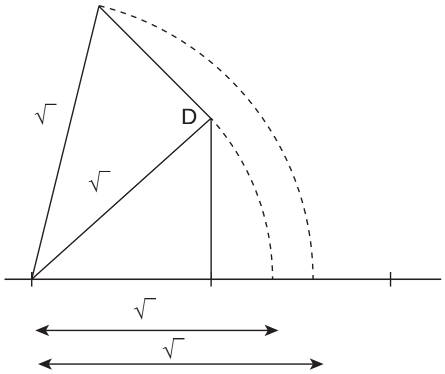


Figure 2.2

We notice that $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

As we have already observed, irrational numbers occur in real life situations. Over 2000 years ago people in the Orient and Egypt observed that the ratio of the circumference to the diameter is the same for any circle. This constant was proved to be an irrational number by Johann Heinrich Lambert in 1767. The value of π rounded off to nine decimal places is equal to 3.141592654. The values $\frac{22}{7}$ and 3.14, used in calculations, such as area of a circle or volume of a sphere, are only approximate values for π.

The number π, which is the ratio of the circumference of a circle to its diameter, is an irrational number.

Now let us recall the properties of the real number system which is the foundation for mathematics.

2.2.4 Properties of Real Numbers

(i) For any $a, b \in \mathbb{R}$, $a + b \in \mathbb{R}$ and $ab \in \mathbb{R}$.
[Sum of two real numbers is again a real number and multiplication of two real numbers is again a real number.]

(ii) For any $a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$ and $a(bc) = (ab)c$.
[While adding (or multiplying) finite number of real numbers, we can add (or multiply) by grouping them in any order.]

(iii) For all $a \in \mathbb{R}$, $a + 0 = a$ and $a(1) = a$.

(iv) For every $a \in \mathbb{R}$, $a + (-a) = 0$ and for every $b \in \mathbb{R} - \{0\}$, $b\left(\frac{1}{b}\right) = 1$.

(v) For any $a, b \in \mathbb{R}$, $a + b = b + a$ and $ab = ba$.

(vi) For $a, b, c \in \mathbb{R}$, $a(b + c) = ab + ac$.

(vii) For $a, b \in \mathbb{R}$, $a < b$ if and only if $b - a > 0$.

(viii) For any $a \in \mathbb{R}$, $a^2 \geq 0$.

(ix) For any $a, b \in \mathbb{R}$, only one of the following holds: $a = b$ or $a < b$ or $a > b$.

(x) If $a, b \in \mathbb{R}$ and $a < b$, then $a + c < b + c$ for all $c \in \mathbb{R}$.

(xi) If $a, b \in \mathbb{R}$ and $a < b$, then $ax < bx$ for all $x > 0$.

(xii) If $a, b \in \mathbb{R}$ and $a < b$, then $ay > by$ for all $y < 0$.

Exercise - 2.1

1. Classify each element of $\{\sqrt{7}, \frac{-1}{4}, 0, 3.14, 4, \frac{22}{7}\}$ as a member of $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R} - \mathbb{Q}$ or $\mathbb{Z}$.
2. Prove that $\sqrt{3}$ is an irrational number.
(Hint: Follow the method that we have used to prove $\sqrt{2} \notin \mathbb{Q}$.)
3. Are there two distinct irrational numbers such that their difference is a rational number? Justify.
4. Find two irrational numbers such that their sum is a rational number. Can you find two irrational numbers whose product is a rational number.
5. Find a positive number smaller than $\dfrac{1}{2^{1000}}$. Justify.

2.3 Absolute Value

2.3.1 Definition and Properties

As we have observed that there is an order preserving one-to-one correspondence between elements of $\mathbb{R}$ and points on the number line. Note that for each $x \in \mathbb{R}$, x and $-x$ are equal distance from the origin. The distance of the number $a \in \mathbb{R}$ from 0 on the number line is called the *absolute value* of that number a and is denoted by $|a|$. Thus, for any $x \in \mathbb{R}$, we have

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

and hence $|\cdot|$ defines a function known as absolute value function, from $\mathbb{R}$ onto $[0, \infty)$ and the graph of this function is discussed in Chapter 1.

(i) For any $x \in \mathbb{R}$, we have $|x| = |-x|$ and thus, $|x| = |y|$ if and only if $x = y$ or $x = -y$.
(ii) $|x - a| = r$ if and only if $r \geq 0$ and $x - a = r$ or $x - a = -r$.

2.3.2 Equations Involving Absolute Value

Note that *a real number a is said to be a* ***solution*** *of an equation or an inequality, if the statement obtained after replacing the variable by a is true.*

Next we shall learn solving equations involving absolute value.

Example 2.1 Solve $|2x - 17| = 3$ for x.

Solution:
$|2x - 17| = 3$. Then, we have $2x - 17 = \pm 3$ which implies $x = 10$ or $x = 7$.

Example 2.2 Solve $3|x - 2| + 7 = 19$ for x.

Solution:
$3|x - 2| + 7 = 19$. So that we have, $|x - 2| = \frac{19-7}{3} = 4$.
Thus, we have either $x - 2 = 4$ or $x - 2 = -4$.
Therefore the solutions are $x = -2$ and $x = 6$.

Example 2.3 Solve $|2x - 3| = |x - 5|$.

Solution:
We know that $|u| = |v|$ if and only if $u = v$ or $u = -v$.
Therefore, $|2x - 3| = |x - 5|$ implies $2x - 3 = x - 5$ or $2x - 3 = 5 - x$.
Solving these two equations, we get $x = -2$ and $x = \dfrac{8}{3}$.
Hence, both $x = -2$ and $x = \dfrac{8}{3}$ are solutions.

2.3.3 Some Results For Absolute Value

(i) If $x, y \in \mathbb{R}$, $|y + x| = |x - y|$, then $xy = 0$.
(ii) For any $x, y \in \mathbb{R}$, $|xy| = |x||y|$.
(iii) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, for all $x, y \in \mathbb{R}$ and $y \neq 0$.
(iv) For any $x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$.

2.3.4 Inequalities Involving Absolute Value

Here we shall learn to solve inequalities involving absolute values. First we analyze very simple inequalities such as (i) $|x| < r$ and (ii) $|x| > r$.

(i) Let us prove that $|x| < r$ if and only if $-r < x < r$. Note that $r > 0$ as $|x| \geq 0$.
There are two possibilities to consider depending on the sign of x.

Case (1). If $x \geq 0$, then $|x| = x$, so $|x| < r$ implies $x < r$.
Case (2). If $x < 0$, then $|x| = -x$, so $|x| < r$ implies $-x < r$ that is, $x > -r$.

Therefore we have, $|x| < r$ if and only if $-r < x < r$, that is $x \in (-r, r)$.

(ii) Let us prove that $|x| > r$ if and only if $x < -r$ or $x > r$.
Consider $|x| > r$. If $r < 0$, then every $x \in \mathbb{R}$ satisfies the inequality.
For $r \geq 0$, there are two possibilities to consider.

Case (1). If $x \geq 0$, then $|x| = x > r$.
Case (2). If $x < 0$, then $|x| = -x > r$, that is, $x < -r$.

So we have $|x| > r$, if and only if $x < -r$ or $x > r$, that is, $x \in (-\infty, -r) \cup (r, \infty)$.

Remark:

(i) For any $a \in \mathbb{R}$, $|x - a| \leq r$ if and only if $-r \leq x - a \leq r$ if and only if $x \in [a - r, a + r]$.
(ii) For any $a \in \mathbb{R}$, $|x - a| \geq r$ is equivalent to $x - a \leq -r$ or $x - a \geq r$ if and only if $x \in (-\infty, a - r] \cup [a + r, \infty)$.

Example 2.4 Solve $|x - 9| < 2$ for x.

Solution:
$|x - 9| < 2$ implies $-2 < x - 9 < 2$. Thus, $7 < x < 11$.

Example 2.5 Solve $\left|\dfrac{2}{x-4}\right| > 1$, $x \neq 4$.

Solution:
From the given inequality, we have that $2 > |x - 4|$.
That is, $-2 < x - 4 < 2$ and $x \neq 4$.
Adding 4 throughout the inequality, we obtain $2 < x < 6$ and $x \neq 4$.
So the solution set is $(2, 4) \cup (4, 6)$.

Exercise - 2.2

1. Solve for x:
   (i) $|3 - x| < 7$. (ii) $|4x - 5| \geq -2$. (iii) $\left|3 - \frac{3}{4}x\right| \leq \frac{1}{4}$.
   (iv) $|x| - 10 < -3$.
2. Solve $\frac{1}{|2x-1|} < 6$ and express the solution using the interval notation.
3. Solve $-3|x| + 5 \leq -2$ and graph the solution set in a number line.
4. Solve $2|x + 1| - 6 \leq 7$ and graph the solution set in a number line.
5. Solve $\frac{1}{5}|10x - 2| < 1$.
6. Solve $|5x - 12| < -2$.

2.4 Linear Inequalities

Recall that a function of the form $f(x) = ax + b$, $a, b \in \mathbb{R}$ are constants, is called a linear function, because its graph is a straight line. Here a is the slope of the line and b is the y-intercept. If $a \neq 0$, then x-intercept $x = \frac{-b}{a}$ is obtained by solving $f(x) = ax + b = 0$.
But there are situations where we need to consider linear inequalities.
For example to describe a statement like "*A tower is not taller than fifty feet.*"
If x denotes the height of the tower in feet, then the above statement can be expressed as $x \leq 50$.

Example 2.6 Our monthly electricity bill contains a basic charge, which does not change with number of units used, and a charge that depends only on how many units we use. Let us say Electricity Board charges Rs.110 as basic charge and charges Rs. 4 for each unit we use. If a person wants to keep his electricity bill below Rs.250, then what should be his electricity usage?

Solution:
Let x denote the number of units used. Note that $x \geq 0$. Then, his electricity bill is Rs. $110 + 4x$. The person wants his bill to be below Rs.250. Let us solve the inequality $110 + 4x < 250$. Thus, $4x < 140$; which gives $0 \leq x < 35$.
The person should keep his usage below 35 units in order to keep his bill below Rs.250.

Example 2.7 Solve $3x - 5 \leq x + 1$ for x.

Solution:
We have $3x - 5 \leq x + 1$; which is equivalent to $2x \leq 6$. Hence we have $x \leq 3$; the solution set is $(-\infty, 3]$.

We can also solve the above inequality graphically.
Let us consider the graphs of $f(x) = 3x - 5$ and $g(x) = x + 1$ (See Figure 2.3). Now, find all the x-values for which the graph of f is below the graph of g.

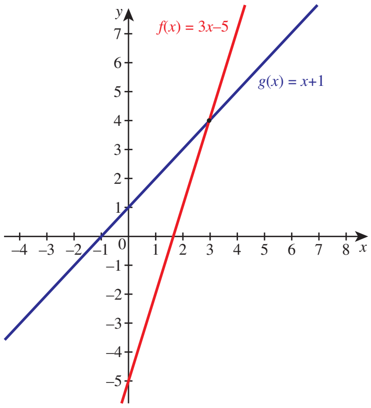


Figure 2.3

Example 2.8 Solve the following system of linear inequalities.
$3x - 9 \geq 0, \quad 4x - 10 \leq 6.$

Solution:
Note that $3x - 9 \geq 0$ implies $3x \geq 9$, by multiplying both sides by $1/3$ we get $x \geq 3$. Similarly, $4x - 10 \leq 6$ implies $4x \leq 16$ and hence $x \leq 4$.

So the solution set of $3x - 9 \geq 0, 4x - 10 \leq 6$ is the intersection of $[3, \infty)$ and $(-\infty, 4]$. Clearly, the intersection of these intervals give $[3, 4]$.

Example 2.9 A girl A is reading a book having 446 pages and she has already finished reading 271 pages. She wants to finish reading this book within a week. What is the minimum number of pages she should read per day to complete reading the book within a week?

Solution:
Let x denote the number of pages the girl should read per day. Then we need our x to satisfy $7x + 271 \geq 446$. Hence $x \geq 25$; which implies that she should read at least 25 pages per day.

In all the above examples observe that each inequality has more than one solution. Inequalities in general give rise to a range of solutions.

Exercise - 2.3

1. Represent the following inequalities in the interval notation:
   (i) $x \geq -1$ and $x < 4$ (ii) $x \leq 5$ and $x \geq -3$
   (iii) $x < -1$ or $x < 3$ (iv) $-2x > 0$ or $3x - 4 < 11$.
2. Solve $23x < 100$ when (i) x is a natural number, (ii) x is an integer.
3. Solve $-2x \geq 9$ when (i) x is a real number, (ii) x is an integer, (iii) x is a natural number.
4. Solve: (i) $\dfrac{3(x-2)}{5} \leq \dfrac{5(2-x)}{3}$. (ii) $\dfrac{5-x}{3} < \dfrac{x}{2} - 4$.
5. To secure A grade one must obtain an average of 90 marks or more in 5 subjects each of maximum 100 marks. If one scored $84, 87, 95, 91$ in first four subjects, what is the minimum mark one scored in the fifth subject to get A grade in the course?
6. A manufacturer has 600 litres of a 12 percent solution of acid. How many litres of a 30 percent acid solution must be added to it so that the acid content in the resulting mixture will be more than 15 percent but less than 18 percent?
7. Find all pairs of consecutive odd natural numbers both of which are larger than 10 and their sum is less than 40.
8. A model rocket is launched from the ground. The height h reached by the rocket after t seconds from lift off is given by $h(t) = -5t^2 + 100t, 0 \leq t \leq 20$. At what time the rocket is 495 feet above the ground?
9. A plumber can be paid according to the following schemes: In the first scheme he will be paid rupees 500 plus rupees 70 per hour, and in the second scheme he will paid rupees 120 per hour. If he works x hours, then for what value of x does the first scheme give better wages?
10. A and B are working on similar jobs but their annual salaries differ by more than Rs 6000. If B earns rupees 27000 per month, then what are the possibilities of A's salary per month?

2.5 Quadratic Functions

22THHX

In earlier classes we have learnt that for any $z \in \mathbb{R}$ and $n \in \mathbb{N}$, $z^n = z \cdot z \cdot z \cdots z$ (n-times).
A function of the form $P(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ are constants and $a \neq 0$, is called a quadratic function. If $P(t) = 0$ for some $t \in \mathbb{R}$, then we say t is a zero of $P(x)$.

2.5.1 Quadratic Formula

Is it possible to write the general quadratic function $P(x) = ax^2+bx+c$ in the form $a(x-k)^2 + d$? The answer is yes. We can do this by the method called "**completing the square.**" We shall rewrite the function $P(x)$ as follows:

$$\begin{aligned} P(x) &= ax^2+bx+c \\ &= a\left(x^2+2x\frac{b}{2a}+\frac{c}{a}\right) \\ &= a\left(x^2+2x\frac{b}{2a}+\left(\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right) \\ &= a\left(x+\frac{b}{2a}\right)^2 - a\frac{b^2}{4a^2}+c \\ &= a\left(x+\frac{b}{2a}\right)^2 + \left(a\left(\frac{b}{2a}\right)^2 - b\frac{b}{2a}+c\right). \end{aligned}$$

$$\text{Thus,}\quad P(x) = a\left(x+\frac{b}{2a}\right)^2 + P\left(\frac{b}{2a}\right). \qquad (1)$$

Now, to find the x- intercepts of the curve described by $P(x)$, let us solve for $P(x) = 0$.

Considering $P(x) = 0$ from (1) it follows that $a\left(x+\dfrac{b}{2a}\right)^2 + P\left(\frac{b}{2a}\right) = 0$.

$$\begin{aligned} a\left(x+\tfrac{b}{2a}\right)^2 &= -P\left(\tfrac{b}{2a}\right) \\ &= -\tfrac{(b^2-4ac)}{4a} \\ \left(x+\tfrac{b}{2a}\right)^2 &= \tfrac{b^2-4ac}{4a^2}. \end{aligned}$$

So $x = \frac{\sqrt{b^2-4ac}}{2a} - \frac{b}{2a}$ or $x = -\frac{\sqrt{b^2-4ac}}{2a} - \dfrac{b}{2a}$.

Hence, $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$; which is called the ***quadratic formula***.

Remark:

(i) Note that $\sqrt{u}$ is defined as a real number only for $u \geq 0$.
(ii) when we write $\sqrt{u}$, we mean only the nonnegative root.

Note that $P(x) = 0$ has two distinct real solutions if $b^2-4ac > 0$, the roots are real and equal if $b^2-4ac = 0$, and no real root if $b^2-4ac < 0$.

Thus the curve intersects x-axis in two places if $b^2-4ac > 0$, touches x-axis at only one point if $b^2-4ac = 0$, and does not intersect x-axis at any point if $b^2-4ac < 0$.

That is why $D = b^2-4ac$ is called the ***discriminant*** of the quadratic function $P(x) = ax^2+bx+c$.

(i) If α and β are roots of $ax^2+bx+c = 0$, then $\alpha+\beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$.
(ii) If the discriminant b^2-4ac is negative, then the quadratic equation $ax^2+bx+c=0$, has no real roots. In this case, we have complex roots given by

$$\alpha = \frac{-b+i\sqrt{4ac-b^2}}{2a}, \quad \beta = \frac{-b-i\sqrt{4ac-b^2}}{2a}, \quad \text{where } i^2 = -1,$$

which we will study in Higher Secondary Second year

(iii) For example, let us look at the graph of $y = x^2-4x+5$. (See Figure 2.4.) Since the graph does not intersect the $x-$axis, $x^2-4x+5 = 0$ has no real roots.
(iv) We have the following table describing the nature of the roots of a quadratic equation and the sign of the discriminant $D = b^2-4ac$.

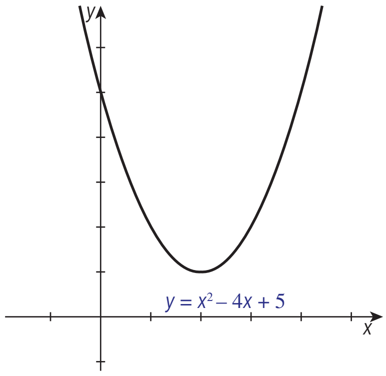


Figure 2.4

Discriminant	Nature of roots	Parabola
Positive	real and distinct	intersects x-axis at two points
Zero	real and equal	touches x-axis at one point
Negative	no real roots	does not meet x-axis

Example 2.10 If a and b are the roots of the equation $x^2 - px + q = 0$, find the value of $\frac{1}{a} + \frac{1}{b}$.

Solution:
Given that a and b are the roots of $x^2 - px + q = 0$. Then, $a + b = p$ and $ab = q$. Thus, $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{p}{q}$.

Example 2.11 Find the complete set of values of a for which the quadratic $x^2 - ax + a + 2 = 0$ has equal roots.

Solution:
The quadratic equation $x^2 - ax + a + 2 = 0$ has equal roots.
So, its discriminant is zero. Thus, $D = b^2 - 4ac = a^2 - 4a - 8 = 0$.
So, $a = \frac{4 \pm \sqrt{48}}{2}$ which gives $a = 2 + \sqrt{12}, \quad 2 - \sqrt{12}$.

Example 2.12 Find the number of solutions of $x^2 + |x - 1| = 1$.

Solution:

Case (1). For $x \geq 1$, $|x - 1| = x - 1$.
Then the given equation reduces to $x^2 + x - 2 = 0$. Factoring we get $(x+2)(x-1) = 0$, which implies $x = -2$ or 1. As $x \geq 1$, we obtain $x = 1$.

Case (2). For $x < 1$, $|x - 1| = 1 - x$
Then the given equation becomes $x^2 + 1 - x = 1$. Thus we have $x(x - 1) = 0$ which implies $x = 0$ or $x = 1$. As $x < 1$, we have to choose $x = 0$.

Thus, the solution set is $\{0, 1\}$. Hence, the equation has two solutions.

Exercise - 2.4

1. Construct a quadratic equation with roots 7 and -3.
2. A quadratic polynomial has one of its zeros $1+\sqrt{5}$ and it satisfies $p(1)=2$. Find the quadratic polynomial.
3. If α and β are the roots of the quadratic equation $x^2+\sqrt{2}x+3=0$, form a quadratic polynomial with zeroes $\frac{1}{\alpha}, \frac{1}{\beta}$.
4. If one root of $k(x-1)^2=5x-7$ is double the other root, show that $k=2$ or -25.
5. If the difference of the roots of the equation $2x^2-(a+1)x+a-1=0$ is equal to their product, then prove that $a=2$.
6. Find the condition that one of the roots of ax^2+bx+c may be (i) negative of the other, (ii) thrice the other, (iii) reciprocal of the other.
7. If the equations $x^2-ax+b=0$ and $x^2-ex+f=0$ have one root in common and if the second equation has equal roots, then prove that $ae=2(b+f)$.
8. Discuss the nature of roots of (i) $-x^2+3x+1=0$, (ii) $4x^2-x-2=0$, (iii) $9x^2+5x=0$.
9. Without sketching the graphs, find whether the graphs of the following functions will intersect the x-axis and if so in how many points.
   (i) $y=x^2+x+2$, (ii) $y=x^2-3x-7$, (iii) $y=x^2+6x+9$.
10. Write $f(x)=x^2+5x+4$ in completed square form.

2.5.2 Quadratic Inequalities

Here we shall learn to solve the quadratic inequalities $ax^2+bx+c<0$ or $ax^2+bx+c>0$.

Steps to Solve Quadratic Inequalities:

(i) First solve $ax^2+bx+c=0$.
(ii) If there are no real solutions, then one of the above inequality holds for all $x \in \mathbb{R}$
(iii) If there are real solutions, which are called *critical points*, then label those points on the number line.
(iv) Note that these critical points divide the number line into disjoint intervals. (It is possible that there may be only one critical point.)
(v) Choose one representative number from each interval.
(vi) Substitute that these representative numbers in the inequality.
(vii) Identify the intervals where the inequality is satisfied.

Example 2.13 Solve $3x^2+5x-2 \leq 0$.

Solution:
On factorizing the quadratic polynomial we get $3(x+2)(x-\frac{1}{3}) \leq 0$. Draw the number line. Mark the critical points -2 and $\frac{1}{3}$ where the factors vanish (See Figure 2.5). On each sub-interval check the sign of $(x+2)(x-\frac{1}{3})$. To do this pick an arbitrary point anywhere in the interval. Whatever sign the resulting value has, the polynomial has the same sign throughout the whole corresponding interval. (Otherwise, there would be another critical point within the interval). This process is easily organized in the following table.

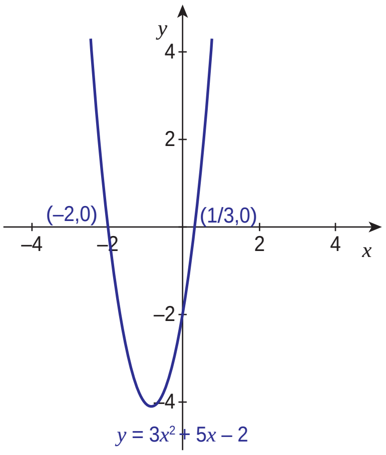


Figure 2.5

Interval	Sign of $(x+2)$	Sign of $(x-1/3)$	Sign of $3x^2+5x-2$
$(-\infty,-2)$	-	-	+
$(-2,1/3)$	+	-	-
$(1/3,\infty)$	+	+	+

You can see the inequality is satisfied in $[-2,1/3]$.

Example 2.14 Solve $\sqrt{x+14} < x+2$.

Solution:
The function $\sqrt{x+14}$ is defined for $x+14 \geq 0$. Therefore $x \geq -14$, $x+2>0$ implies $x \geq -2$.
$(x+14) < (x+2)^2$ gives $x^2+3x-10>0$.
Hence, $(x+5)(x-2)>0$. Dividing the number line with the critical points $x=-5$ and $x=2$.
Substituting a reference point in the sub-interval we get the solution set to be $x<-5$ and $x>2$.
Since $x \geq -2$, we have the solution to be $x>2$.

Example 2.15 Solve the equation $\sqrt{6-4x-x^2} = x+4$.

Solution:
The given equation is equivalent to the system
$(x+4) \geq 0$ and $6-4x-x^2 = (x+4)^2$.
This implies $x \geq -4$ and $x^2+6x+5=0$. Thus, $x=-1,-5$.
But only $x=-1$ satisfies both the conditions. Hence, $x=-1$.

Exercise - 2.5

1. Solve $2x^2+x-15 \leq 0$.
2. Solve $-x^2+3x-2 \geq 0$.

2.6 Polynomial Functions

So far we have understood about linear functions and quadratic functions. Now we shall generalize these ideas. We call an expression of the form $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_i \in \mathbb{R}$, $i = 0, 1, 2, \cdots, n$, is called a ***polynomial*** in the variable x. Here n is a non-negative integer. When $a_n \neq 0$, we say that the polynomial has **degree** n. The numbers $a_0, a_1, \ldots, a_n \in \mathbb{R}$ are called the coefficients of the polynomial. The number a_0 is called the constant term and a_n is called the leading coefficient (when it is non-zero). It is clear that:

(i) $100x^7 - \pi x^5 + 20\sqrt{2}x^2 + 7x + 1.22$ is a polynomial of degree 7.
(ii) $(17x - 3)(x + 3)(2x - \sqrt{\pi})(x + 2.3)$ is a polynomial of degree 4.
(iii) $(x^2 + x + 1)(x^3 + 2x + 2)(x^5 - 5x + \sqrt{3})$ is a polynomial of degree 10.

One may substitute specific values for x, say $x = c$ and obtain $a_nc^n + a_{n-1}c^{n-1} + \cdots + a_1c + a_0$. A function of the form $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ is called a ***polynomial function*** which is defined from $\mathbb{R}$ to $\mathbb{R}$. We shall treat polynomial and polynomial function as one and the same.

A polynomial with degree 1 is called a ***linear polynomial***. A polynomial with degree 2 is called a ***quadratic polynomial***. A **cubic polynomial** is one that has degree three. Likewise, degree 4 and degree 5 polynomials are called ***quartic*** and ***quintic*** polynomials respectively. Note that any constant $a \neq 0$ is a polynomial of degree zero!

Two polynomials $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_n \neq 0$ and $g(x) = b_mx^m + b_{m-1}x^{m-1} + \cdots + b_0$, $b_m \neq 0$ are equal if and only if $f(x) = g(x)$ for all $x \in \mathbb{R}$. It can be proved that $f(x) = g(x)$ if and only if $n = m$ and $a_k = b_k, k = 0, 1, 2, \cdots n$. Given two polynomials, one can form their **sum** and **product**. For example if $P(x) = 2x^3 + 7x^2 - 5$ and $Q(x) = x^4 - 2x^3 + x^2 + x + 1$, then $P(x) + Q(x) = x^4 + 8x^2 + x - 6$ (by adding the corresponding coefficients of the like powers of x) and $P(x)Q(x) = 2x^7 + 3x^6 - 12x^5 + 4x^4 + 19x^3 + 2x^2 - 5x - 5$ by multiplying each term of $P(x)$ by every term of $Q(x)$. It is easy to see that the degree of $P(x)Q(x)$ is the sum of the degrees of $P(x)$ and $Q(x)$, whereas the degree of $P(x) + Q(x)$ is at most the maximum of degrees of $P(x)$ and $Q(x)$. Here is an example of the graph of a cubic polynomial function.

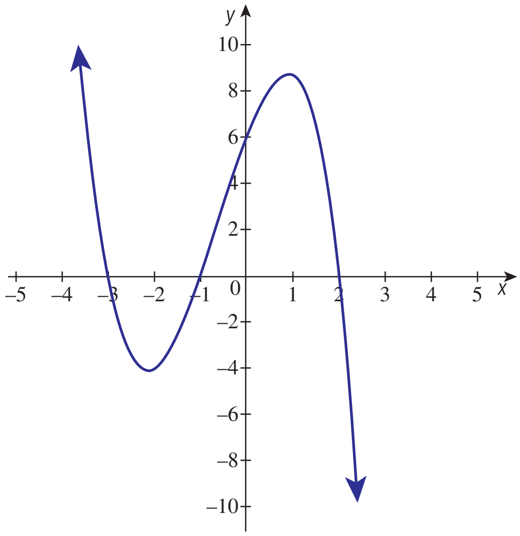


Figure 2.6

Suppose that $f(x)$ and $g(x)$ are polynomials where $g(x)$ is not zero. The quotient $\frac{f(x)}{g(x)}$ is called a **rational function,** which is defined for all $x \in \mathbb{R}$ such that $g(x) \neq 0$. In general, a rational function need not be a polynomial.

2.6.1 Division Algorithm

Given two polynomials $f(x)$ and $g(x)$, where $g(x)$ is not the zero polynomial, there exist two polynomials $q(x)$ and $r(x)$ such that $f(x) = q(x)g(x) + r(x)$ where degree of $r(x) <$ degree of $g(x)$. Here, $q(x)$ is called the quotient polynomial, and $r(x)$ is called the remainder polynomial. If $r(x)$ is the zero polynomial, then $q(x)$, $g(x)$ are factors of $f(x)$ and $f(x) = q(x)g(x)$.

These terminologies are similar to terminologies used in division done with integers.

If $g(x) = x - a$, then the remainder $r(x)$ should have degree zero and hence $r(x)$ is a constant. To determine the constant, write $f(x) = (x - a)q(x) + c$. Substituting $x = a$ we get $c = f(a)$.

Remainder Theorem

If a polynomial $f(x)$ is divided by $x - a$, then the remainder is $f(a)$. Thus the remainder $c = f(a) = 0$ if and only if $x - a$ is a factor for $f(x)$.

Definition 2.1

A real number a is said to be a ***zero of the polynomial*** $f(x)$ if $f(a) = 0$. If $x = a$ is a zero of $f(x)$, then $x - a$ is a ***factor*** for $f(x)$.

In general, if we can express $f(x)$ as $f(x) = (x - a)^k . g(x)$ where $g(a) \neq 0$, then the value of k, which depends on a, cannot exceed the degree of $f(x)$. The value k is called the ***multiplicity*** of the zero a.

(i) A polynomial function of degree n can have at most n distinct real zeros. It is also possible that a polynomial function like $P(x) = x^2 + 1$ has no real zeros at all.

(ii) Suppose that $P(x)$ is a polynomial function having rational coefficients. If $a + b\sqrt{p}$ where $a, b \in \mathbb{Q}$, p a prime, is a zero of $P(x)$, then its ***conjugate*** $a - b\sqrt{p}$ is also a zero.

Two important problems relating to polynomials are

(i) Finding zeros of a given polynomial function; and hence factoring the polynomial into linear factors and

(ii) Constructing polynomials with the given zeros and/or satisfying some additional conditions.

To address the problem of finding zeros of a polynomial function, some well known algebraic identities are useful. What is an identity?

An equation is said to be an ***identity*** if that equation remains valid for all values in its domain. An equation is called ***conditional equation*** if it is true only for some (not all) of values in its domain. Let us recall the following identities.

2.6.2 Important Identities

For all $x, a, b \in \mathbb{R}$ we have

1. $(x+a)^3 = (x+a)^2(x+a) = x^3 + 3x^2a + 3xa^2 + a^3 = x^3 + 3xa(x+a) + a^3$
2. $(x-b)^3 = x^3 - 3x^2b + 3xb^2 - b^3 = x^3 - 3xb(x-b) + b^3$ taking $a = -b$ in (1)
3. $x^3 + a^3 = (x+a)(x^2 - xa + a^2)$
4. $x^3 - b^3 = (x-b)(x^2 + xb + b^2)$ taking $a = -b$ in (3)
5. $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + x^{n-k-1}a^k + \cdots + a^{n-1}),\ n \in \mathbb{N}$
6. $x^n + b^n = (x+b)(x^{n-1} - x^{n-2}b + \cdots + x^{n-k-1}(-b)^k + \cdots + (-b)^{n-1}),\ n \in \mathbb{N}$

Exercise - 2.6

1. Find the zeros of the polynomial function $f(x) = 4x^2 - 25$.
2. If $x = -2$ is one root of $x^3 - x^2 - 17x = 22$, then find the other roots of equation.
3. Find the real roots of $x^4 = 16$.
4. Solve $(2x+1)^2 - (3x+2)^2 = 0$.

Method of Undetermined Coefficients

Now let us focus on constructing polynomials with the given information using the method of undetermined coefficients. That is, we shall determine coefficients of the required polynomial using the given conditions. The main idea here is that two polynomials are equal if and only if the coefficients of same powers of the variables in the two polynomials are equal.

Example 2.16 Find a quadratic polynomial $f(x)$ such that, $f(0) = 1$, $f(-2) = 0$ and $f(1) = 0$.

Solution:
Let $f(x) = ax^2 + bx + c$ be the polynomial satisfying the given conditions.
$f(0) = a0^2 + b0 + c = 1$, implies that $c = 1$. Now the other two conditions $f(-2) = 0$, $f(1) = 0$ give $4a - 2b + c = 0$ and $a + b + c = 0$.
Using $c = 1$, we get $4a - 2b = -1$ and $a + b = -1$. Solving these two equations we get $a = b = -\dfrac{1}{2}$ and thus, we have $f(x) = -\dfrac{1}{2}x^2 - \dfrac{1}{2}x + 1$.

The above problem can also be solved in another way. $x = -2$, $x = 1$ are zeros of $f(x)$. Thus, $f(x) = d(x+2)(x-1)$ for some constant d.
Now using $f(0) = 1$ gives $-2d = 1$, hence $d = -\frac{1}{2}$. So, $f(x) = -\frac{1}{2}(x+2)(x-1) = -\frac{1}{2}x^2 - \frac{1}{2}x + 1$.

Example 2.17 Construct a cubic polynomial function having zeros at $x = \frac{2}{5}, 1 + \sqrt{3}$ such that $f(0) = -8$.

Solution:
Given that $\frac{2}{5}$ and $1 + \sqrt{3}$ are zeros of $f(x)$. Thus, $1 - \sqrt{3}$ is also a zero of $f(x)$.
Let $f(x) = a(x - \frac{2}{5})[(x - (1 + \sqrt{3}))][x - (1 - \sqrt{3})] = a(x - \frac{2}{5})[(x-1)^2 - 3]$.
Using $f(0) = -8$, we have, $\left(-\frac{2}{5}a\right)(-2) = -8$ which give $a = -10$.
Thus the required polynomial is $f(x) = (-10)(x - \frac{2}{5})[x^2 - 2x - 2] = -10x^3 + 24x^2 + 12x - 8$.

Example 2.18 Prove that $ap + q = 0$ if $f(x) = x^3 - 3px + 2q$ is divisible by $g(x) = x^2 + 2ax + a^2$.

Solution:
Note that the degree of $f(x)$ is 3 and the leading coefficient is 1. Since $g(x)$ divides $f(x)$, we have $f(x) = (x+b)g(x)$, for some $b \in \mathbb{R}$. Thus, $x^3 - 3px + 2q = (x+b)(x^2 + 2ax + a^2)$.
Equating like coefficients on both sides, we have $2a + b = 0$, $a^2 + 2ab = -3p$ and $2q = ba^2$. Thus, $b = -2a$, $p = a^2$, and $q = -a^3$.
Now, $q = -a^3 = -a(a^2) = -ap$, which gives $ap + q = 0$.

Example 2.19 Use the method of undetermined coefficients to find the sum of $1+2+3+\cdots+(n-1)+n, n \in \mathbb{N}$

Solution:

$$\begin{aligned}
\text{Let } S(n) &= n+(n-1)+(n-2)+\cdots+2+1 \\
&= n+(n-1)+(n-2)+\cdots+[n-(n-2)]+[n-(n-1)] \\
&= n\left[1+\frac{n-1}{n}+\frac{n-2}{2}+\cdots+\frac{n-(n-2)}{n}+\frac{n-(n-1)}{n}\right] \\
&\leq n[1+1+\cdots+1] \quad \text{since } \frac{n-1}{n}<1,\ \frac{n-2}{n}<1,\cdots \\
\text{Thus, } S(n) &\leq n^2.
\end{aligned}$$

$$\begin{aligned}
\text{Let } S(n) &= a+bn+cn^2, \text{ where } a,b,c,\in \mathbb{R}. \\
\text{Now, } S(n+1)-S(n) &= n+1 \\
a+b(n+1)+c(n+1)^2-[a+bn+cn^2] &= n+1 \\
b+2cn+c &= n+1
\end{aligned}$$

Thus, $b+c=1$ and $2c=1$ (Equating like coefficients) which give $b=\frac{1}{2}$; $c=\frac{1}{2}$
Now, $S(1)=1$ $a+b+c=1$ which gives $a=0$
Hence, $S(n)=\frac{1}{2}n+\frac{1}{2}n^2=\frac{n(n+1)}{2}, n \in \mathbb{N}$.

Example 2.20 Find the roots of the polynomial equation $(x-1)^3(x+1)^2(x+5)=0$ and state their multiplicity.

Solution:
Let $f(x)=(x-1)^3(x+1)^2(x+5)=0$. Clearly, we have $x=1,-1,-5$.
Hence, the roots are 1 with multiplicity 3, -1 with multiplicity 2 and -5 with multiplicity 1.

When the root has multiplicity 1, it is called a simple root.

Example 2.21 Solve $x=\sqrt{x+20}$ for $x \in \mathbb{R}$.

Solution:
Observe that $\sqrt{x+20}$ is defined only if $x+20 \geq 0$.
By definition, $\sqrt{x+20} \geq 0$ is positive. So, x is positive.
Now squaring we get $x^2=x+20$. $x^2-x-20=0$
$(x-5)(x+4)=0$, which gives $x=5, x=-4$
Since, x is positive, the required solution is $x=5$.

Example 2.22 The equations $x^2 - 6x + a = 0$ and $x^2 - bx + 6 = 0$ have one root in common. The other root of the first and the second equations are integers in the ratio $4 : 3$. Find the common root.

Solution:
Let α be the common root.
Let $\alpha, 4\beta$ be the roots of $x^2 - 6x + a = 0$.
Let $\alpha, 3\beta$ be the roots of $x^2 - bx + 6 = 0$.
Then, $4\alpha\beta = a$ and $3\alpha\beta = 6$ which give $\alpha\beta = 2$ and $a = 8$.
The roots of $x^2 - 6x + 8 = 0$ are $2, 4$.
If $\alpha = 2$, then $\beta = 1$
If $\alpha = 4$, then $\beta = \frac{1}{2}$ which is not an integer.
Hence, the common root is 2.

Example 2.23 Find the values of p for which the difference between the roots of the equation $x^2 + px + 8 = 0$ is 2.

Solution:
Let α and β be the roots of the equation $x^2 + px + 8 = 0$.
Then, $\alpha + \beta = -p$, $\alpha\beta = 8$ and $|\alpha - \beta| = 2$.
Now, $(\alpha + \beta)^2 - 4\alpha\beta = (\alpha - \beta)^2$, which gives $p^2 - 32 = 4$. Thus, $p = \pm 6$.

Exercise - 2.7

1. Factorize: $x^4 + 1$. (*Hint:* Try completing the square.)
2. If $x^2 + x + 1$ is a factor of the polynomial $3x^3 + 8x^2 + 8x + a$, then find the value of a.

2.7 Rational Functions

A rational expression of x is defined as the ratio of two polynomials in x, say $P(x)$ and $Q(x)$ where $Q(x) \neq 0$. Examples of rational expressions are $\dfrac{2x+1}{x^2+x+1}$, $\dfrac{x^4+1}{x^2+1}$ and $\dfrac{x^2+x}{x^2-5x+6}$.

If the degree of the numerator $P(x)$ is equal to or larger than that of the denominator $Q(x)$, then we can write $P(x) = f(x)Q(x) + r(x)$ where $r(x) = 0$ or the degree of $r(x)$ is less than that of $Q(x)$. So $\dfrac{P(x)}{Q(x)} = f(x) + \dfrac{r(x)}{Q(x)}$.

2.7.1 Rational Inequalities

Example 2.24 Solve $\dfrac{x+1}{x+3} < 3$.

Solution:
Subtracting 3 from both sides we get $\dfrac{x+1}{x+3} - 3 < 0$.

$$\begin{aligned} \frac{x+1-3(x+3)}{x+3} &< 0 \\ \frac{-2x-8}{x+3} &< 0 \\ \frac{x+4}{x+3} &> 0 \end{aligned}$$

Thus, $x+4$ and $x+3$ are both positive or both negative.

So let us find out the signs of $x+3$ and $x+4$ as follows

x	$x+3$	$x+4$	$\frac{x+4}{x+3}$
$x<-4$	$-$	$-$	$+$
$-4<x<-3$	$-$	$+$	$-$
$x>-3$	$+$	$+$	$+$
$x=-4$	$-$	0	0

So the solution set is given by $(-\infty,-4)\cup(-3,\,\infty)$.

The above type of rational inequality problem can also be solved by plotting the signs of various factors on the intervals of the number line.

Exercise - 2.8

1. Find all values of x for which $\dfrac{x^3(x-1)}{(x-2)}>0$.
2. Find all values of x that satisfies the inequality $\dfrac{2x-3}{(x-2)(x-4)}<0$.
3. Solve $\dfrac{x^2-4}{x^2-2x-15}\leq 0$.

2.7.2 Partial Fractions

A rational expression $\dfrac{f(x)}{g(x)}$ is called a proper fraction if the degree of $f(x)$ is less than degree of $g(x)$, where $g(x)$ can be factored into linear factors and quadratic factors without real zeros. Now $\frac{f(x)}{g(x)}$ can be expressed in simpler terms, namely, as a sum of expressions of the form

(i) $\dfrac{A_1}{(x-a)}+\dfrac{A_2}{(x-a)^2}+\cdots+\dfrac{A_k}{(x-a)^k}$ if $x-a$ divides $g(x)$ and

(ii) $\dfrac{(B_1x+C_1)}{(x^2+ax+b)}+\dfrac{(B_2x+C_2)}{(x^2+ax+b)^2}+\cdots+\dfrac{(B_kx+C_k)}{(x^2+ax+b)^k}$ if x^2+ax+b has no real zeros and (x^2+ax+b) divides $g(x)$.

The resulting expression of $\dfrac{f(x)}{g(x)}$ is is called the *partial fraction decomposition*. Such a decomposition is *unique* for a given rational function.

This method is useful in doing Integral calculus. So let us discuss some examples.

Example 2.25 Resolve into partial fractions: $\dfrac{x}{(x+3)(x-4)}$.

Solution:

Let $\dfrac{x}{(x+3)(x-4)} = \dfrac{A}{x+3} + \dfrac{B}{x-4}$ where A and B are constants.

Then, $\dfrac{x}{(x+3)(x-4)} = \dfrac{A(x-4)+B(x+3)}{(x+3)(x-4)}$, which gives $x = A(x-4)+B(x+3)$.

When $x = 4$, we have $B = \dfrac{4}{7}$.

When $x = -3$, we have $A = \dfrac{3}{7}$

Hence, $\dfrac{x}{(x+3)(x-4)} = \dfrac{3}{7(x+3)} + \dfrac{4}{7(x-4)}$.

The above procedure can be carried out if the denominator has all its zeros in $\mathbb{R}$ which are all distinct.

Example 2.26 Resolve into partial fractions: $\dfrac{2x}{(x^2+1)(x-1)}$.

Solution:

In this case, note that the denominator has a factor x^2+1 which does not have real zeros.

Let $\dfrac{2x}{(x^2+1)(x-1)} = \dfrac{A}{(x-1)} + \dfrac{Bx+C}{x^2+1}$
where A, B, C are constants.

We have, $2x = A(x^2+1) + (Bx+C)(x-1)$.

When $x = 1$, we get $A = 1$.

When $x = 0$, we have $A - C = 0$ and hence $A = C = 1$.

When $x = -1$, we have $2A - 2(C-B) = -2$, which gives $B = -1$.

Thus, $\dfrac{2x}{(x^2+1)(x-1)} = \dfrac{1}{(x-1)} + \dfrac{1-x}{x^2+1}$

We now illustrate the situation when denominator has a real zeros with multiplicity more than one.

Example 2.27 Resolve into partial fractions: $\dfrac{x+1}{x^2(x-1)}$.

Solution:

Let $\dfrac{x+1}{x^2(x-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1}$.

Then, $x+1 = Ax(x-1) + B(x-1) + Cx^2$.

When $x = 0$, we have $B = -1$ and when $x = 1$, we get $C = 2$.

When $x = -1$, we have $2A - 2B + C = 0$ which gives $A = -2$.

Thus, $\dfrac{x+1}{x^2(x-1)} = \dfrac{-2}{x} - \dfrac{1}{x^2} + \dfrac{2}{x-1}$.

Exercise - 2.9

Resolve the following rational expressions into partial fractions.

1. $\dfrac{1}{x^2-a^2}$
2. $\dfrac{3x+1}{(x-2)(x+1)}$
3. $\dfrac{x}{(x^2+1)(x-1)(x+2)}$
4. $\dfrac{x}{(x-1)^3}$
5. $\dfrac{1}{x^4-1}$
6. $\dfrac{(x-1)^2}{x^3+x}$
7. $\dfrac{x^2+x+1}{x^2-5x+6}$
8. $\dfrac{x^3+2x+1}{x^2+5x+6}$
9. $\dfrac{x+12}{(x+1)^2(x-2)}$
10. $\dfrac{6x^2-x+1}{x^3+x^2+x+1}$
11. $\dfrac{2x^2+5x-11}{x^2+2x-3}$
12. $\dfrac{7+x}{(1+x)(1+x^2)}$

2.7.3 Graphical Representation of Linear Inequalities

A straight line $ax+by=c$ divides the Cartesian plane into two parts. Each part is an half plane. A vertical line $x=c$ will divide the plane in left and right half planes and a horizontal line $y=k$ will divide the plane into upper and lower half planes.

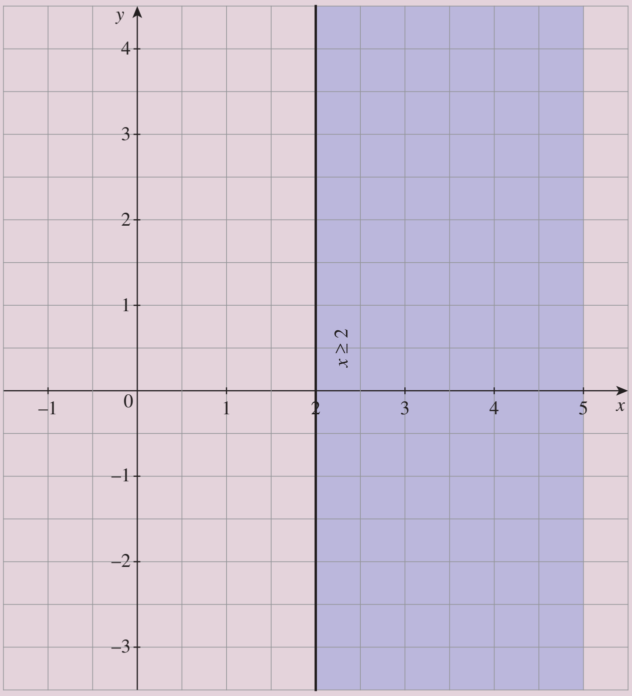
A point in the cartesian plane which is not on the line $ax+by=c$ will lie in exactly one of the two half planes determined by the line and satisfies one of the inequalities $ax+by<c$ or $ax+by>c$.

To identify the half plane represented by $ax+by<c$, choose a point P in any one of the half planes and substitute the coordinates of P in the inequality.

If the inequality is satisfied, then the required half plane is the one that contains P; otherwise the required half plane is the one that does not contain P. When $c \neq 0$, it is most convenient to take P to be the origin.

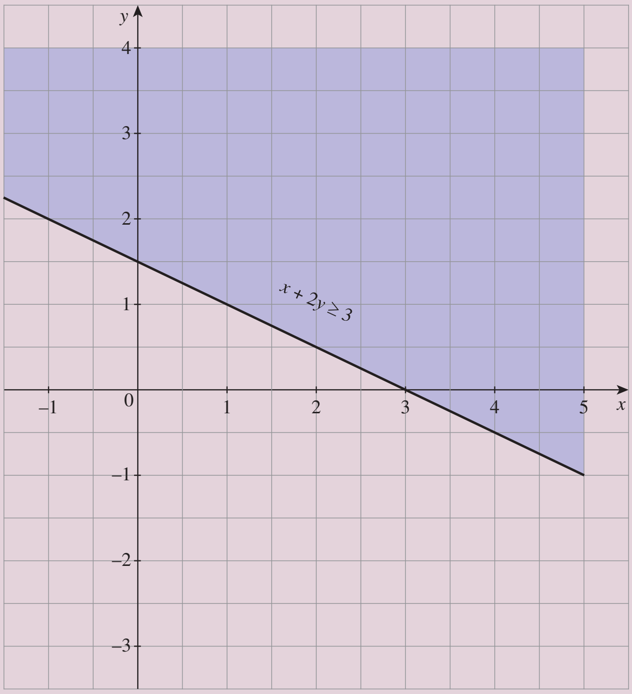
Example 2.28 Shade the region given by the inequality $x \geq 2$.

Solution:
First we consider equation $x=2$. It is a line parallel to y axis at a distance of 2 units from it. This line divides the cartesian plane into two parts. Substituting $(0,0)$ in the inequality we get $0 \geq 2$ which is false. Hence the region which does not contain the origin is represented by the inequality $x \geq 2$. The shaded region is the required solution set of the given inequality. Since $x \geq 2$, the points on the line $x=2$ are also solutions.



Example 2.29 Shade the region given by the linear inequality $x + 2y > 3$.

Proof. The line $x + 2y = 3$ divides the cartesian plane into two half planes. To find the half plane represented by $x + 2y > 3$ substitute a point in one of the half planes in the inequality and check whether it is satisfied. Let us substitute $(0, 0)$ in the inequality. We get $0 > 3$ which is false. Hence, the region given by $x + 2y > 3$ is the half plane which does not contain the origin. □



Example 2.30 Solve the linear inequalities and exhibit the solution set graphically:

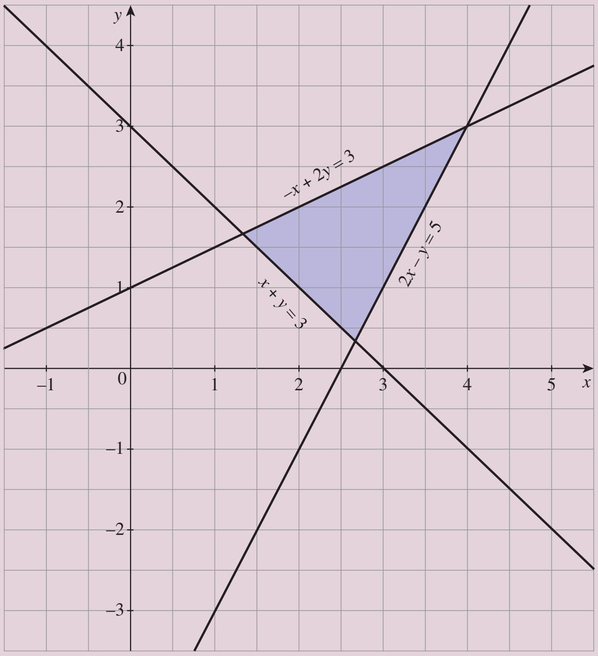
$$x + y \geq 3,\ 2x - y \leq 5,\ -x + 2y \leq 3.$$

Solution:
Observe that a straight line can be drawn if we identify any two points on it. For example, $(3, 0)$ and $(0, 3)$ can be easily identified as two points on the straight line $x + y = 3$.
Draw the three straight lines $x + y = 3, 2x - y = 5$ and $-x + 2y = 3$.
Now $(0, 0)$ does not satisfy $x + y \geq 3$. Thus, the half plane bounded by $x + y = 3$, not containing the origin, is the solution set of $x + y \geq 3$.

Similarly, the half-plane bounded by $2x - y \leq 5$ containing the origin represents the solution set of the $2x - y \leq 5$.

The region represented by $-x + 2y \leq 3$ is the half space bounded by the straight line the line $-x + 2y = 3$ that contains the origin.

The region common to the above three half planes represents the solution set of the given linear inequalities.



Exercise - 2.10

Determine the region in the plane determined by the inequalities:

(1) $x \leq 3y, \quad x \geq y.$
(2) $y \geq 2x, \quad -2x + 3y \leq 6.$
(3) $3x + 5y \geq 45, x \geq 0, y \geq 0.$
(4) $2x + 3y \leq 35, y \geq 2, x \geq 5.$
(5) $2x + 3y \leq 6, \ x + 4y \leq 4, \ x \geq 0, y \geq 0.$
(6) $x - 2y \geq 0, \ 2x - y \leq -2, \ x \geq 0, y \geq 0.$
(7) $2x + y \geq 8, x + 2y \geq 8, x + y \leq 6.$

2.8 Exponents and Radicals

First we shall consider exponents.

2.8.1 Exponents

Let $n \in \mathbb{N}$, $a \in \mathbb{R}$. Then $a^n = a \cdot a \cdots a$ (n times). If m is a negative integer and the real number $a \neq 0$, then $a^m = \dfrac{1}{a^{-m}}$.

Note that for any $a \neq 0$, we have $\dfrac{a}{a} = a^{1-1} = a^0 = 1$. It is also easy to see the following properties.

Properties of Exponents

(i) For $m, n \in \mathbb{Z}$ and $a \neq 0$, we have $a^m a^n = a^{m+n}$.
(ii) For $m, n \in \mathbb{Z}$ and $a \neq 0$, we have $\dfrac{a^m}{a^n} = a^{m-n}$.

2.8.2 Radicals

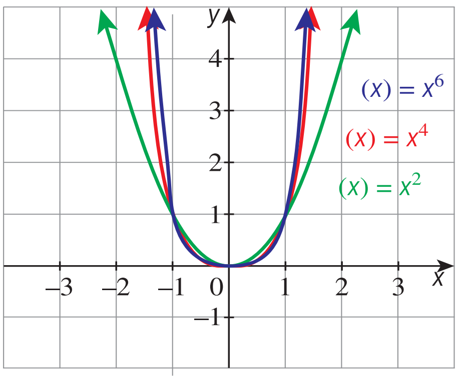
Question:

For $a \neq 0$ and $r \in \mathbb{Q}$, is it possible to define a^r?

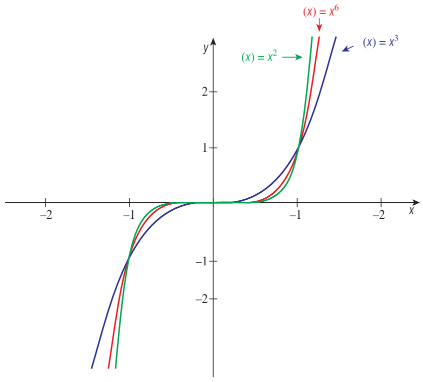
First let us consider the case when $r = \dfrac{1}{n}$, $n \in \mathbb{N}$. Suppose there is a real number $y \in \mathbb{R}$ such that $y = a^{\frac{1}{n}}$. Then we must have $y^n = a$.

This problem is basically to finding inverse function of $y = x^n$. In order to understand better let us consider the graphs of the following functions:

(i) $f(x) = x^{2n}, n \in \mathbb{N}$



(ii) $g(x) = x^{2n+1}, n \in \mathbb{N}$



From these two figures it is clear that the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^{2n+1}, n \in \mathbb{N}$ is one-to-one and onto and hence its inverse function from $\mathbb{R}$ onto $\mathbb{R}$ exists. But $f : \mathbb{R} \to [0, \infty)$ given by $f(x) = x^{2n}, n \in \mathbb{N}$ is onto but not one-to-one. However, f is one-to-one and onto if we restrict its domain to $[0, \infty)$. This is helpful in understanding nth root of a real number. So we have two cases;

Case 1 When n is even.

In this case $y^n = a$ is not meaningful when $a < 0$. So no such y exists when $a < 0$.
Assume that $a > 0$. If y is a solution to $x^n = a$, then $-y$ is also solution to $x^n = a$.

Case 2 When n is odd.

In this case no such problem arises as in Case 1. For $y \in \mathbb{R}$, there is a unique $x \in \mathbb{R}$ such that $y = x^n$.

Based on the above observation we define radicals as follow.

Definition 2.2

(i) For $n \in \mathbb{N}$, n even, and $b > 0$, there is a unique $a > 0$ such that $a^n = b$.
(ii) For $n \in \mathbb{N}$, n odd, $b \in \mathbb{R}$, there is a unique $a \in \mathbb{R}$ such that $a^n = b$. In both cases a is called the nth root of b or radical and is denoted by $b^{1/n}$ or $\sqrt[n]{b}$

(i) If $n = 2$, then nth root is called the square root; if $n = 3$, then it is called cube root.
(ii) Observe that the equation $x^2 = a^2$, has two solutions $x = a$, $x = -a$; but $\sqrt{a^2} = |a|$.
(iii) Properties of exponents given above are still valid for radicals provided each of the individual terms are defined.
(iv) Note that for $n \in \mathbb{N}$ and $a \neq 0$ we have

$$(a^n)^{1/n} = \begin{cases} |a| & \text{if } n \text{ is even }, \\ a & \text{if } n \text{ is odd }. \end{cases}$$

For example, $\sqrt[4]{(-2)^4} = 16^{1/4} = 2$, $343^{1/3} = 7$ and $(-1000)^{\frac{1}{3}} = -10$.

For any rational $r = \dfrac{m}{n}$, $m \in \mathbb{Z}, n \in \mathbb{N}$, with $\gcd(m, n) = 1$ and for $a > 0$ we define $a^r = a^{\frac{m}{n}} = (a^{1/n})^m$.

For example, $49^{3/2} = (49^{1/2})^3 = 7^3 = 343$. But $(-49)^{3/2}$ has no meaning in real number system because there is no real number x such that $x^2 = -49$.

It is clear that, for $x, y \geq 0$ we have $(x^{1/2}y^{-3})^{1/2} = x^{1/4}/y^{3/2}$.

Also, note that $\sqrt{x^2 - 10x + 25} = \sqrt{(x-5)^2} = |x - 5|$.

2.8.3 Exponential Function

Observe that for any $a > 0$ and $x \in \mathbb{R}$, a^x can be defined. If $a = 1$, we define $1^x = 1$. So we shall consider a^x, $x \in \mathbb{R}$ for $0 < a \neq 1$. Here a^x is called **exponential function with base** a. Note that a^x may not be defined if $a < 0$ and $x = \dfrac{1}{m}$ for even $m \in \mathbb{N}$. This is why we restrict to $a > 0$. Also, $a^x > 0$ for all $x \in \mathbb{R}$. It does also satisfy the following:

Properties of Exponential Function

For $a, b > 0$ and $a \neq 1 \neq b$

(i) $a^{x+y} = a^x\, a^y \quad$ for all $x,\ y \in \mathbb{R}$,
(ii) $\dfrac{a^x}{a^y} = a^{x-y} \quad$ for all $x,\ y \in \mathbb{R}$,

(iii) $(a^x)^y = a^{xy}$ for all $x,\ y \in \mathbb{R}$,
(iv) $(ab)^x = a^x b^x$ for all $x \in \mathbb{R}$,
(v) $a^x = 1$ if and only if $x = 0$.

1. Let us consider $f(x) = a^x, x \in \mathbb{R}$ where $a = 2$.
   Now $f(x) = 2^x,\ x \in \mathbb{R}$. Let us show that f is one-to-one and onto.
   Suppose $f(u) = f(v)$ for some $u, v \in \mathbb{R}$. Then, we have $2^u = 2^v$, which implies that $\dfrac{2^u}{2^v} = 1$,
   $\Rightarrow 2^{u-v} = 1$.
   So, $u - v = 0$ and hence $u = v$. Thus f is a one-to-one function.

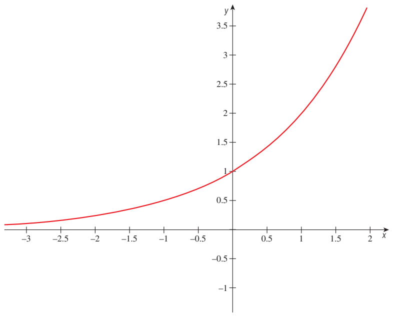


Figure 2.7: $f(x) = 2^x$

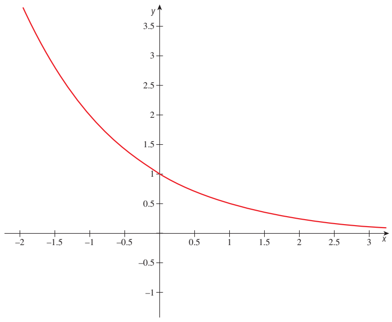


Figure 2.8: $f(x) = \dfrac{1}{2^x}$

From the graph it is clear that values of $f(x) = 2^x$ increase as x values increase and the range of f is $(0,\ \infty)$. So as $2^0 = 1$, we have $2^x > 1$ for all $x > 0$ and $2^x < 1$ for all $x < 0$. Observe that $f : \mathbb{R} \to (0, \infty)$ is onto.

2. Let us consider $a = \dfrac{1}{2}$. Let $g(x) = \left(\dfrac{1}{2}\right)^x = \dfrac{1}{2^x},\ x \in \mathbb{R}$.
   From the graph it is clear that the values of $g(x) = \left(\dfrac{1}{2}\right)^x$ decrease as x values increase and $g(\mathbb{R}) = (0,\ \infty)$. Also, $g(0) = 1$ we have $g(x) > 1$ for all $x < 0$ and $g(x) < 1$ for all $x > 0$.

Remark: Exactly same arguments as above would show that an exponential function $f(x) = a^x$, for any base $0 < a \neq 1$, is one-to-one and onto with domain $\mathbb{R}$ and codomain $(0,\ \infty)$.

A Special Exponential Function

Among all exponential functions, $f(x) = e^x,\ x \in \mathbb{R}$ is the most important one as it has applications in many areas like mathematics, science and economics. Then what is this e? The following illustration from compounding interest problem leads to the constant e.

Illustration

2.8.3.1 Compound Interest

Recall that if P is the principal, $r = \dfrac{\text{interest rate}}{100}$, n is the number of compounding periods in a year and t is the number of years, then $A = P\left(1 + \frac{r}{n}\right)^{nt}$ gives the total amount after t years. If $n = 4$, then it is compounded quarterly (the interest is added to the existing principal for three months in a year). If $n = 12$, then compounded monthly, $n = 365$ means compounded daily. We can compound every hour, every minute etc. We know that if P and r are fixed and the number of compounding periods in a year increases, then the total amount also increases. Let us consider the case with $P = 1, r = 1$ and $t = 1$. Then, we have $A_n = \left(1 + \frac{1}{n}\right)^n$. We want to understand how big it gets as n gets really large. Let us make a table with different values of $n = 10,\ 100,\ 10000,\ 100000,\ 100000000$.

n	10	100	10000	100000	100000000
A_n	2.593742460	2.704813829	2.718145927	2.718268237	2.718281815

We notice that as n gets really large, A_n values seem to be getting closer to 2.718281815..... Actually A_n values approach a real number e, an irrational number. 2.718281815 is an approximation to e. So the compound interest formula becomes $A = Pe^{rt}$, where r is the interest rate and P is the principal and t is the number of years. This is called Continuous Compounding.

Example 2.31 (i) Simplify: $(x^{1/2}y^{-3})^{1/2}$; where $x, y \geq 0$.
(ii) Simplify: $\sqrt{x^2 - 10x + 25}$.

Solution:
(i) Since $x, y \geq 0$, we have $(x^{1/2}y^{-3})^{1/2} = x^{1/4}/y^{3/2}$.
(ii) Observe that $\sqrt{x^2 - 10x + 25} = \sqrt{(x-5)^2} = |x-5|$.

(i) $(x^{1/4})^4 = x$ but $(y^4)^{1/4} = |y|$.
Observe that $x^{1/4}$ is defined only when x is positive. But y^4 is defined even when $y < 0$.
Now $(y^4)^{1/4}$ is a positive number whose fourth power equals y^4. So it has to be $|y|$.
(ii) $(x^8.y^4)^{1/4} = x^2|y|$.
(iii) Let u, v, b be rational numbers where b is positive.
Let us suppose they are not squares of rational numbers.
Then $u + v\sqrt{b}, u - v\sqrt{b}$ are called **conjugates**.
Observe that $(u + v\sqrt{b})(u - v\sqrt{b}) = u^2 - bv^2$ is now rational.
Thus, if an expression such as $u + v\sqrt{b}$ appears in the denominator we can multiply both the numerator and denominator by its conjugate, namely, $u - v\sqrt{b}$, to get a rational number in the denominator.
(iv) Using $(u\sqrt{a} - v\sqrt{b})(u\sqrt{a} + v\sqrt{b}) = u^2a - v^2b$, it is possible to simplify expressions when $u\sqrt{a} + v\sqrt{b}$ occurs in the denominator.

Example 2.32 Rationalize the denominator of $\dfrac{\sqrt{5}}{(\sqrt{6}+\sqrt{2})}$.

Solution:
Multiplying both numerator and denominator by $(\sqrt{6}-\sqrt{2})$,we get

$$\frac{\sqrt{5}}{(\sqrt{6}+\sqrt{2})} = \frac{\sqrt{5}(\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})} = \frac{(\sqrt{30}-\sqrt{10})}{4}.$$

Example 2.33 Find the square root of $7 - 4\sqrt{3}$.

Solution:

Let $\sqrt{7 - 4\sqrt{3}} = a + b\sqrt{3}$ where a, b are rationals.
Squaring on both sides, we get $7 - 4\sqrt{3} = a^2 + 3b^2 + 2ab\sqrt{3}$. So, $a^2 + 3b^2 = 7$ and $2ab = -4$.

Therefore $a = -2/b$.
From $a^2 + 3b^2 = 7$, we get $(-2/b)^2 + 3b^2 = 7$, which gives $4/b^2 + 3b^2 = 7$ or $3b^4 - 7b^2 + 4 = 0$.
Solving for b^2 we get $b^2 = \frac{(7 \pm \sqrt{49-48})}{6}$, which gives $b^2 = 1$ or $b^2 = \frac{4}{3}$.
Thus, $b = \pm 1$ or $b = \pm \frac{2}{\sqrt{3}}$.
Since b is rational, we have $b = \pm 1$ and hence the corresponding values of a are ∓ 2.
Since $\sqrt{7 - 4\sqrt{3}} > 0$, we have $\sqrt{7 - 4\sqrt{3}} = 2 - \sqrt{3}$.

It is not always possible to express square roots of $u + v\sqrt{b}$ where u, v are rationals, in the form $x + y\sqrt{b}$ with x, y rationals. For example, the square root of $1 + \sqrt{2}$ is not of the form $a + b\sqrt{2}$ with a, b rationals.

Exercise - 2.11

1. Simplify:

   (i) $(125)^{\frac{2}{3}}$, (ii) $16^{\frac{-3}{4}}$, (iii) $(-1000)^{\frac{-2}{3}}$, (iv) $(3^{-6})^{\frac{1}{3}}$, (v) $\dfrac{27^{\frac{-2}{3}}}{27^{\frac{-1}{3}}}$.
2. Evaluate $\left(\left((256)^{-1/2}\right)^{\frac{-1}{4}}\right)^3$.
3. If $(x^{1/2} + x^{-1/2})^2 = 9/2$, then find the value of $(x^{1/2} - x^{-1/2})$ for $x > 1$.
4. Simplify and hence find the value of n: $3^{2n}9^2 3^{-n}/3^{3n} = 27$.
5. Find the radius of the spherical tank whose volume is $32\pi/3$ units.
6. Simplify by rationalising the denominator. $\dfrac{7 + \sqrt{6}}{3 - \sqrt{2}}$.
7. Simplify $\dfrac{1}{3 - \sqrt{8}} - \dfrac{1}{\sqrt{8} - \sqrt{7}} + \dfrac{1}{\sqrt{7} - \sqrt{6}} - \dfrac{1}{\sqrt{6} - \sqrt{5}} + \dfrac{1}{\sqrt{5} - 2}$.
8. If $x = \sqrt{2} + \sqrt{3}$ find $\dfrac{x^2 + 1}{x^2 - 2}$.

2.9 Logarithm

We have seen that, with a base $0 < a \neq 1$, the exponential function $f(x) = a^x$ is defined on $\mathbb{R}$ having range $(0, \infty)$. We also observed that $f(x)$ is a bijection, hence it has an inverse. We call this inverse function as ***logarithmic function*** and is denoted by $\log_a(.)$. Let us discuss this function further. Note that if $f(x)$ takes x to $y = a^x$, then $\log_a(.)$ takes y to x. That is, for $0 < a \neq 1$, we have

$$y = a^x \qquad \text{is equivalent to} \qquad \log_a y = x.$$

For example, since $3^4 = 81$ we have $\log_3(81) = 4$. In other words, with fixed a, given a real number y, logarithm finds the exponent x satisfying $a^x = y$. This is useful in addressing practical problems like, “how long will it take for certain investment to reach a fixed amount?” Logarithm is also very useful in multiplying very small or big numbers.

(i) Note that exponential function a^x is defined for all $x \in \mathbb{R}$ and $a^x > 0$ and so $\log_a(\cdot)$ defined **only for positive real numbers**.
(ii) Also, $a^0 = 1$ for any base a and hence $\log_a(1) = 0$ for any base a.

2.9.1 Properties of Logarithm

(i) $a^{\log_a x} = x$ for all $x \in (0, \infty)$ and $\log_a(a^y) = y$ for all $y \in \mathbb{R}$.

(ii) For any $x, y > 0$, $\log_a(xy) = \log_a x + \log_a y$. (Product Rule)

(iii) For any $x, y > 0$, $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$. (Quotient Rule)

(iv) For any $x > 0$ and $r \in \mathbb{R}$, $\log_a x^r = r\log_a x$. (Power Rule)

(v) For any $x > 0$, with a and b as bases, $\log_b x = \dfrac{\log_a x}{\log_a b}$. (Change of base formula.)

Proof. Since exponential function with base a and logarithm function with base a are inverse of each other,

(i) follows by using the definitions.

(ii) For $x, y > 0$ let $\log_a x = u$, $\log_a y = v$, and $\log_a(xy) = w$. Rewriting these in the exponential form we obtain $a^u = x$, $a^v = y$, and, $a^w = xy$. So, $a^w = xy = a^u a^v = a^{u+v}$; thus $w = u + v$. Thus, we obtain $\log_a(xy) = \log_a x + \log_a y$.

(iii) Let $\log_a x = u$, $\log_a y = v$, and $\log_a \dfrac{x}{y} = w$. Then $a^u = x$, $a^v = y$ and $a^w = \dfrac{x}{y}$. Hence, $a^w = \dfrac{x}{y} = \dfrac{a^u}{a^v} = a^{u-v}$; which implies $w = u - v$.

Thus, we obtain $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$.

(iv) Let $\log_a x = u$. Then $a^u = x$ and therefore, $x^r = (a^u)^r = a^{ru}$. Thus, $\log_a x^r = nu = r\log_a x$.

(v) Let $\log_b x = v$. We have $b^v = x$. Taking logarithm with base a on both sides we get $\log_a b^v = \log_a x$.

On the other hand $\log_a b^v = v\log_a b$ by the Power rule. Therefore, $v\log_a b = \log_a x$.
Hence $\log_b x = \dfrac{\log_a x}{\log_a b}$, $b > 0$. This completes the proof.

□

Remark:

(i) If $a = 10$, then the corresponding logarithmic function $\log_{10} x$ is called the **common logarithm**.

(ii) If $a = e$,(an irrational number, approximately equal to 2.718), then the corresponding logarithmic function $\log_e x$ is called the **natural logarithm**. It is denoted by $\ln x$. These above particular cases of logarithmic functions are used very much in other sciences and engineering. Particularly, the natural logarithm occurs very naturally. When we write $\log x$ we mean $\log_e x$.

(iii) If $a = 2$, then the corresponding logarithmic function $\log_2 x$ called the binary logarithm, which is used in computer science.

(iv) Observe that $\log_a 35 = \log_a(7 * 5) = \log_a 7 + \log_a 5$; $\log_a \dfrac{50}{3} = \log_a 50 - \log_a 3$.

$\log_a 22^x = x\log_a 22$; $\log_5 50 = \dfrac{\log_{10} 50}{\log_{10} 5}$.

(v) Observe the graph of the logarithmic and exponential functions.

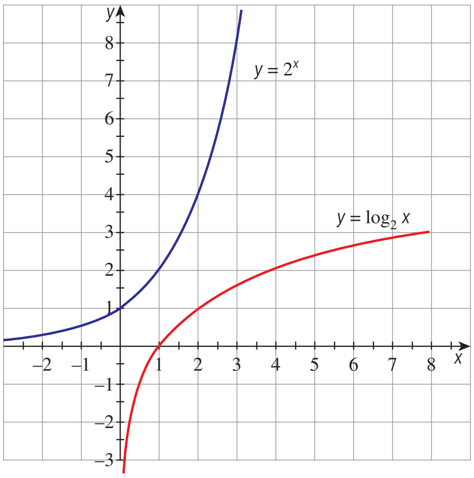


Figure 2.9

Example 2.34 Find the logarithm of 1728 to the base $2\sqrt{3}$.

Solution:
Let $\log_{2\sqrt{3}} 1728 = x$.
Then we have $(2\sqrt{3})^x = 1728 = 2^6\, 3^3 = 2^6\, (\sqrt{3})^6$.
Hence, $(2\sqrt{3})^x = (2\sqrt{3})^6$.
Therefore $x = 6$. That is, $\log_{2\sqrt{3}} 1728 = 6$.

Example 2.35 If the logarithm of 324 to base a is 4, then find a.

Solution:
We are given $\log_a 324 = 4$, which gives
$a^4 = 324 = 3^4\, (\sqrt{2})^4$. Therefore $a = 3\sqrt{2}$.

Example 2.36 Prove $\log \frac{75}{16} - 2\log \frac{5}{9} + \log \frac{32}{243} = \log 2$.

Solution:
Using the properties of logarithm, we have
$\log \frac{75}{16} - 2\log \frac{5}{9} + \log \frac{32}{243} = \log 75 - \log 16 - 2\log 5 + 2\log 9 + \log 32 - \log 243$.
(By Quotient rule.)
$= \log 3 + \log 25 - \log 16 - \log 25 + \log 81 + \log 16 + \log 2 - \log 81 - \log 3$
$= \log 2$.

Example 2.37 If $\log_2 x + \log_4 x + \log_{16} x = \frac{7}{2}$, find the value of x.

Solution:
Note that $x > 0$.
$\log_2 x + \log_4 x + \log_{16} x = \frac{7}{2}$ becomes $\frac{1}{\log_x 2} + \frac{1}{\log_x 4} + \frac{1}{\log_x 16} = \frac{7}{2}$. (change of base rule)
Thus $\frac{1}{a} + \frac{1}{2a} + \frac{1}{4a} = \frac{7}{2}$ where $a = \log_x 2$. That is $\frac{7}{4a} = \frac{7}{2}$.

Thus, $a = \frac{1}{2}$ and so, $\log_x 2 = \frac{1}{2}$ which gives $x^{\frac{1}{2}} = 2$.
Thus, $x = 2^2 = 4$.

Example 2.38 Solve $x^{\log_3 x} = 9$.

Solution:
Let $\log_3 x = y$.
Then $x = 3^y$ and so, $3^{y^2} = 9$.
Thus, $y^2 = 2$, which implies $y = \sqrt{2}, \quad -\sqrt{2}$. Hence, $x = 3^{\sqrt{2}},\ 3^{-\sqrt{2}}$.

Example 2.39 Compute $\log_3 5 \log_{25} 27$.

Solution:
$\log_3 5 \log_{25} 27 = \log_3 5 \log_{25} 3^3$.
$= \log_3 5 \times 3 \log_{25} 3$ (by exponent rule)
$= 3 \log_{25} 5 = \frac{3}{\log_5 25} = \frac{3}{2 \log_5 5} = \frac{3}{2}$.

Example 2.40 Given that $\log_{10} 2 = 0.30103$, $\log_{10} 3 = 0.47712$ (approximately), find the number of digits in $2^8.3^{12}$.

Solution:
Suppose that $N = 2^8 3^{12}$ has $n+1$ digits. Then N can be written as $10^n \times b$ where $1 \le b < 10$.
Taking logarithm to the base 10, we get
$\log N = \log(10^n\, b) = n \log 10 + \log b = n + \log b$.
On the other hand,
$\log N = \log 2^8 3^{12} = 8 \log 2 + 12 \log 3 = 8 \times 0.30103 + 12 \times 0.47712 = 8.13368$.
Thus, we get $n + \log b = 8.13368$. Since $1 \le b < 10$ the number of digits is 9.

Exercise - 2.12

1. Let $b > 0$ and $b \neq 1$. Express $y = b^x$ in logarithmic form. Also state the domain and range of the logarithmic function.
2. Compute $\log_9 27 - \log_{27} 9$.
3. Solve $\log_8 x + \log_4 x + \log_2 x = 11$.
4. Solve $\log_4 2^{8x} = 2^{\log_2 8}$.
5. If $a^2 + b^2 = 7ab$, show that $\log \frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$.
6. Prove $\log \frac{a^2}{bc} + \log \frac{b^2}{ca} + \log \frac{c^2}{ab} = 0$.
7. Prove that $\log 2 + 16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80} = 1$.
8. Prove $\log_{a^2} a\ \log_{b^2} b\ \log_{c^2} c = \frac{1}{8}$.
9. Prove $\log a + \log a^2 + \log a^3 + \cdots + \log a^n = \frac{n(n+1)}{2} \log a$.

10. If $\dfrac{\log x}{y-z} = \dfrac{\log y}{z-x} = \dfrac{\log z}{x-y}$, then prove that $xyz = 1$.
11. Solve $\log_2 x - 3\log_{\frac{1}{2}} x = 6$.
12. Solve $\log_{5-x}(x^2 - 6x + 65) = 2$.

2.10 Application of Algebra in Real Life

Algebra is used in many aspects of life. Financial planning is an area in daily life where algebra is used. Algebra concepts are used to calculate interest rates by bankers and as well as for calculating loan repayments. They are used to predict growth of money. Physical fitness is another area where calculations are made to determine the right amount of food intake for an individual taking into consideration such as the height, body mass of the person etc. Doctors use algebra in measuring drug dosage depending on age and weight of an individual. Architects depend on algebra to design buildings while civil engineers use it to design roads, bridges and tunnels. Algebra is needed to convert items to scale so that the structures designed have the correct proportions. It is used to programme computers and phones . Let us see some examples. Because of the extra-ordinary range of sensitivity of the human ear (a range of over 1000 million millions to one), it is useful to use logarithmic scale to measure sound intensity over this range. The unit of measure decibel is named after the inventor of the telephone Alexander Graham Bell.
If we know the population in the world today, the growth, which is rapid, can be measured by approximating to an exponential function. The radioactive carbon-14 is an organism which decays according to an exponential formula.

Exercise - 2.13

23CA18

Choose the correct or the most suitable answer.

1. If $|x+2| \leq 9$, then x belongs to
   (1) $(-\infty, -7)$ (2) $[-11, 7]$ (3) $(-\infty, -7) \cup [11, \infty)$ (4) $(-11, 7)$

2. Given that x, y and b are real numbers $x < y, b > 0$, then
   (1) $xb < yb$ (2) $xb > yb$ (3) $xb \leq yb$ (4) $\frac{x}{b} \geq \frac{y}{b}$

3. If $\dfrac{|x-2|}{x-2} \geq 0$, then x belongs to
   (1) $[2, \infty)$ (2) $(2, \infty)$ (3) $(-\infty, 2)$ (4) $(-2, \infty)$

4. The solution of $5x - 1 < 24$ and $5x + 1 > -24$ is
   (1) $(4, 5)$ (2) $(-5, -4)$ (3) $(-5, 5)$ (4) $(-5, 4)$

5. The solution set of the following inequality $|x-1| \geq |x-3|$ is
   (1) $[0, 2]$ (2) $[2, \infty)$ (3) $(0, 2)$ (4) $(-\infty, 2)$

6. The value of $\log_{\sqrt{2}} 512$ is
   (1) 16 (2) 18 (3) 9 (4) 12

7. The value of $\log_3 \frac{1}{81}$ is
   (1) -2 (2) -8 (3) -4 (4) -9

8. If $\log_{\sqrt{x}} 0.25 = 4$, then the value of x is
   (1) 0.5 (2) 2.5 (3) 1.5 (4) 1.25

9. The value of $\log_a b \ \log_b c \ \log_c a$ is
   (1) 2 (2) 1 (3) 3 (4) 4

10. If 3 is the logarithm of 343, then the base is
    (1) 5 (2) 7 (3) 6 (4) 9

11. Find a so that the sum and product of the roots of the equation $2x^2 + (a-3)x + 3a - 5 = 0$ are equal is
    (1) 1 (2) 2 (3) 0 (4) 4

12. If a and b are the roots of the equation $x^2 - kx + 16 = 0$ and satisfy $a^2 + b^2 = 32$, then the value of k is
    (1) 10 (2) -8 (3) $-8, 8$ (4) 6

13. The number of solutions of $x^2 + |x - 1| = 1$ is
    (1) 1 (2) 0 (3) 2 (4) 3

14. The equation whose roots are numerically equal but opposite in sign to the roots of $3x^2 - 5x - 7 = 0$ is
    (1) $3x^2 - 5x - 7 = 0$ (2) $3x^2 + 5x - 7 = 0$ (3) $3x^2 - 5x + 7 = 0$ (4) $3x^2 + x - 7$

15. If 8 and 2 are the roots of $x^2 + ax + c = 0$ and $3, 3$ are the roots of $x^2 + dx + b = 0$, then the roots of the equation $x^2 + ax + b = 0$ are
    (1) $1, 2$ (2) $-1, 1$ (3) $9, 1$ (4) $-1, 2$

16. If a and b are the real roots of the equation $x^2 - kx + c = 0$, then the distance between the points $(a, 0)$ and $(b, 0)$ is
    (1) $\sqrt{k^2 - 4c}$ (2) $\sqrt{4k^2 - c}$ (3) $\sqrt{4c - k^2}$ (4) $\sqrt{k - 8c}$

17. If $\dfrac{kx}{(x+2)(x-1)} = \dfrac{2}{x+2} + \dfrac{1}{x-1}$, then the value of k is

(1) 1 (2) 2 (3) 3 (4) 4

18. If $\frac{1-2x}{3+2x-x^2} = \frac{A}{3-x} + \frac{B}{x+1}$, then the value of $A+B$ is

(1) $\frac{-1}{2}$ (2) $\frac{-2}{3}$ (3) $\frac{1}{2}$ (4) $\frac{2}{3}$

19. The number of roots of $(x+3)^4 + (x+5)^4 = 16$ is

(1) 4 (2) 2 (3) 3 (4) 0

20. The value of $\log_3 11 \cdot \log_{11} 13 \cdot \log_{13} 15 \cdot \log_{15} 27 \cdot \log_{27} 81$ is

(1) 1 (2) 2 (3) 3 (4) 4

Summary

- π and $\sqrt{p}$, where p is a prime number, are some irrational numbers.
- $|x-a| = r$ if and only if $r \geq 0$ and $x - a = \pm r$.
- $|x-a| \leq r$ if and only if $-r \leq x - a \leq r$ or $a - r \leq x \leq a + r$.
- $|x-a| > r$ implies $x < a - r$ and $x > a + r$ (or) $x \in (-\infty, a-r) \cup (a+r, \infty)$
- inequalities, in general, have more than one solution.
- The nature of roots of $ax^2 + bx + c = 0$ is determined by the discriminant $D = b^2 - 4ac$.
- A real number a is a zero of a polynomial function $f(x)$ if and only if $(x-a)$ is a factor of $f(x)$.
- If degree of $f(x)$ is less than the degree of $g(x)$, then $\frac{f(x)}{g(x)}$ can be written as sum of its partial fractions.
- In general exponential functions and logarithmic functions are inverse functions to each other.

ICT CORNER-2(a)

Expected Outcome ⇒

Step–1

Open the Browser and type the URL Link given below (or) Scan the QR Code.

Step–2

GeoGebra Work Sheet called "Hill and Flower Puzzle" will appear. Puzzle Detail

(a)You have some flowers in your hand. If you climb up the hill the flowers will be doubled and also, when you climb down the hill it will be doubled. (b)At the top of each hill there is a idol of god where you have to put some flowers. (c)you have to climb and put flowers in all the three idols in each hill top.

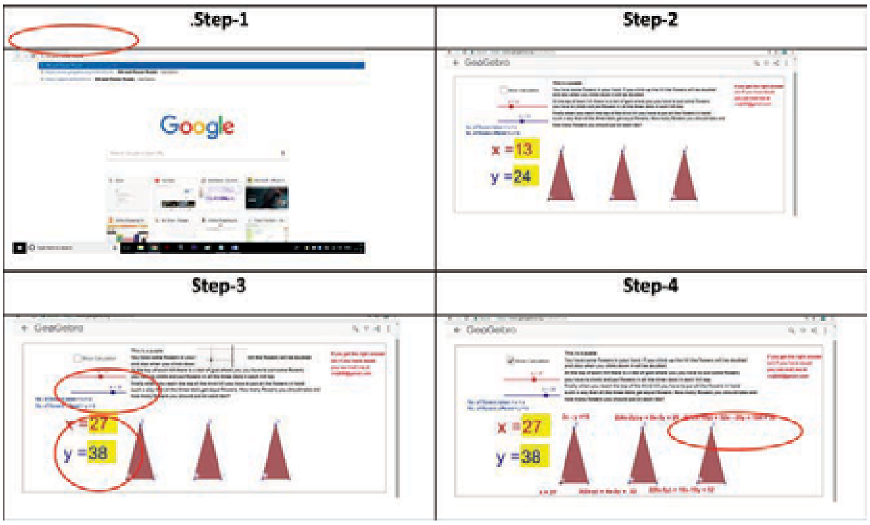
finally, when you reach the top of the third hill you have to put all the flowers in hand such a way that all the three Idols get equal flowers. How many flowers you should take and how many flowers you should put on each Idol?

Step–3

You can think of the no. of flowers taken by you as X value and no. of flowers offered to the god as Y value. And adjust the sliders in the page. Simply by thinking you cannot solve the puzzle.

Step–4

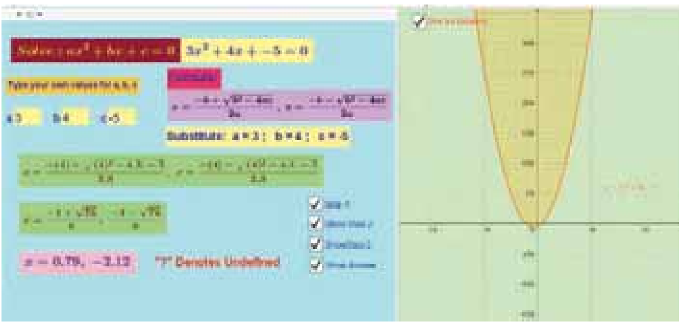
Now is the time for you to recognise the need of algebra. Think of the way to use Algebra. Otherwise Click on the box Show Calculation. Algebra calculation at each level is seen. Now you have to Identify the equation to solve the puzzle. Note: The result will be a ratio. .Step-1

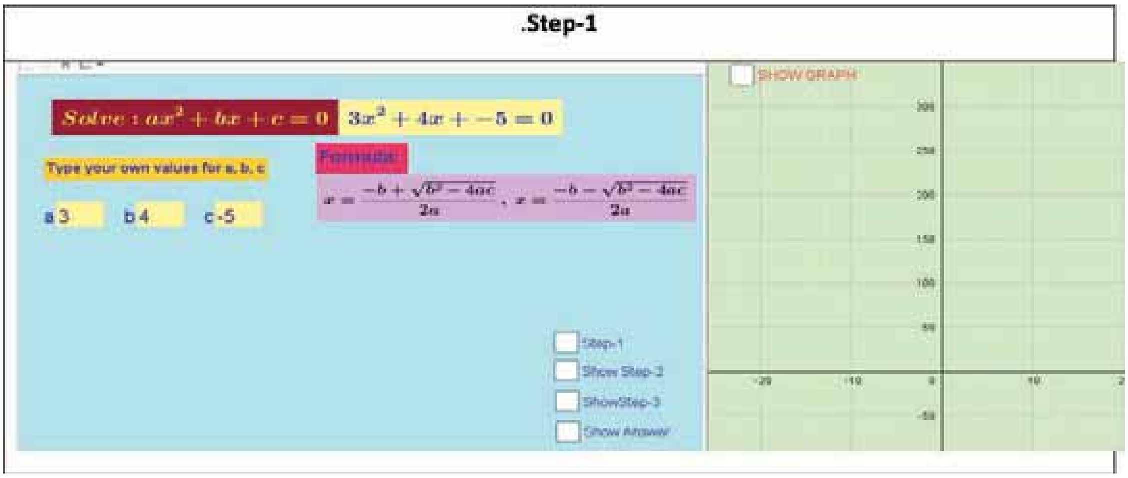


Browse in the link Hill and Flower Puzzle: https://ggbm.at/KmrE5vHsor Scan the QR Code.

ICT CORNER-2(b)

Expected Outcome ⇒



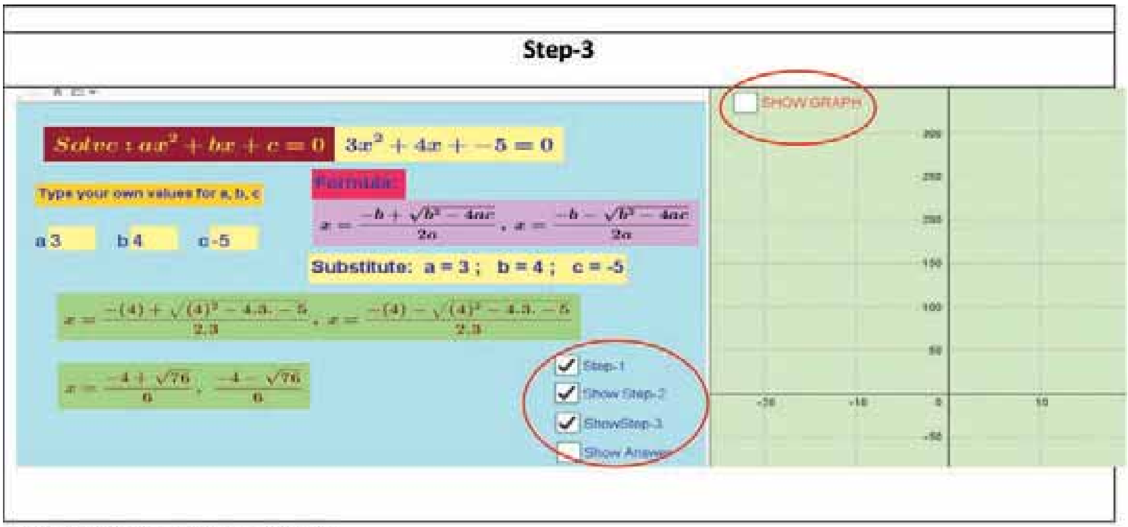


Step–1
Open the Browser and type the URL Link given below (or) Scan the QR Code. GeoGebra work book named High School Algebra will open. In that several work sheets are given, choose any worksheet you want, for example open the work sheet "Quadratic Equation" solving by formula.

In the work sheet you can enter any value between -20 and 20 for a, b and c. You yourself work out the answer using the formula given.(? mark indicates Undefined answer

Step–2
Now click on the answer to check. You can click the check box one by one to see the steps. Finally, on right hand side click Show Graph to view the graph. Compare the graph with your answer.)

(The curve where it cuts the x-axis is the answer).



*Pictures are only indicatives.

Browse in the link Hill and Flower Puzzle: https://ggbm.at/N4kX9QJqor Scan the QR Code.

B162_11_MAT_EM

Chapter 3 Trigonometry

"When I trace at my pleasure the windings to and fro of the heavenly bodies,
I no longer touch the earth with my feet"

Ptolemy

3.1 Introduction


Ptolemy of Alexandria (AD 90-168)

Trigonometry is primarily a branch of Mathematics that studies relationship involving sides and angles of triangles. The word trigonometry stems from the Greek word **trigonon** which means triangle and **metron** which means to measure. So, literally trigonometry is the study of measuring triangles. Greek mathematicians used trigonometric ratios to determine unknown distances. The Egyptians on the other hand used a primitive form of trigonometry for building Pyramids in second millinium BCE. Aristarchus (310-250 BCE) used trigonometry to determine the distances of Moon and Sun.

Eratosthenes (276-195 BCE) was the first person to calculate the earth's circumference, which he did by applying a measuring system using **stadia**, a standard unit of measurement during that period. The general principles of Trigonometry were formulated by the Greek astronomer Hipparchus (190-120 BCE) and he is credited as the founder of trigonometry. His ideas were used by Ptolemy of Alexandria (CE 100-170) leading to the development of Ptolemy theory of Astronomy. The most significant development of Trigonometry in ancient times was in India. Indian Mathematician and Astronomer Aryabhata (CE 476-550) defined sine, cosine, versine (1−cosine), inverse sine and he gave mathematical results in the form of $108\ verses$ which included a formula for the area of a triangle. Mathematicians Brahmagupta (598 CE), Bhaskara I (600 CE) and Bhaskara II (1114 CE) are other Ancient Indians who contributed significantly to develop Trigonometry. Trigonometry was developed as a separate branch of Mathematics through the works of Johann Bernoulli (1667-1748) and Leonhard Euler (1707-1783). Euler established the fundamental results connecting trigonometric functions and complex exponential. Joseph Fourier (1768-1830) made important contribution to the study of trigonometric series. His invention of Fourier series has a wide range of applications especially in vibration analysis, electrical engineering, acoustics, optics, signal processing, image processing and quantum mechanics. In modern times, trigonometric functions are developed as mathematical functions of angular magnitudes, through the medium of which many kinds of geometrical and algebraic investigations are carried out in every branch of Mathematics and applications. Our GPS system in cars and mobile phones is based on trigonometric calculations. Advanced medical scanning procedures such as CT and MRI, used in detecting tumors, involve sine and cosine functions.

Learning Objectives

At the end of this chapter, students are expected to know

- the limitations of right triangle trigonometric ratios as they involve acute angles.
- the necessity for the study of radian measure of an angle and its advantage over degree measure.
- how unit circle is used to define trigonometric functions of real numbers.
- various trigonometric identities, their relationships and applications.
- the principal solution and general solution of a trigonometric equation.
- how to solve trigonometric equations.
- law of sines, law of cosines in triangles and their applications in real life situations.
- how to solve an oblique triangle using law of sines and law of cosines.
- application of Heron's formula and how to compute area of a triangle without finding its altitude.
- the existence of inverse trigonometric functions and their domains and ranges.

Let us recapture the basics of trigonometric ratios using acute angles and their properties, which were discussed in earlier classes.

3.2 A recall of basic results

In earlier classes, we have learnt trigonometric ratios using a right triangle and proved trigonometric identities for an acute angle. One wonders, how the distance between planets, heights of Mountains, distance between far off objects like Earth and Sun, heights of tall buildings, the speed of supersonic jets are measured or calculated. Interestingly, such distances or heights are calculated applying the trigonometric ratios which were derived for acute angles. Our aim is to develop trigonometric functions defined for any real number and use them in all branches of mathematics, in particular, in calculus. First, let us recall the definition of angle and degree measure of an angle.

3.2.1 Angles

The angle AOB is a measure formed by two rays OA and OB sharing the common point O as shown in the Figure 3.1. The common point O is called the **vertex** of the angle. If we rotate the ray OA about its vertex O and takes the position OB, then OA and OB respectively are called the initial side and the terminal side of the angle produced. An anticlockwise rotation generates a positive angle (angle with positive sign), while a clockwise rotation generates a negative angle (angle with negative sign).

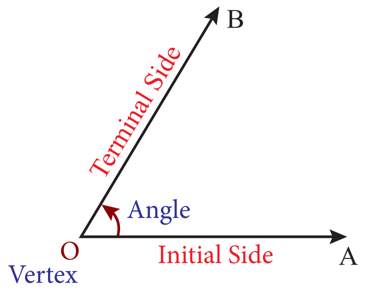


Figure 3.1

One full anticlockwise (or clockwise) rotation of OA back to itself is called one **complete rotation or revolution**.

3.2.2 Different Systems of measurement of angle

There are three different systems for measuring angles.

(i) **Sexagesimal system**

The **Sexagesimal system** is the most prevalent system of measurement where a right angle is divided into 90 equal parts called Degrees. Each degree is divided into 60 equal parts called **Minutes**, and each minute into 60 equal parts called **Seconds.**

The symbols 1°, $1'$ and $1''$ are used to denote a degree, a minute and a second respectively.

(ii) **Centesimal system**

In the **Centesimal system** , the right angle is divided into 100 equal parts, called **Grades**; each grade is subdivided into 100 Minutes, and each minute is subdivided into 100 Seconds. The symbol 1^g is used to denote a grade.

(iii) **Circular system**

In the **circular system** , the radian measure of an angle is introduced using arc lengths in a circle of radius r. Circular system is used in all branches of Mathematics and in other applications in Science. The symbol 1^c is used to denote 1 radian measure.

3.2.3 Degree Measure

The degree is a unit of measurement of angles and is represented by the symbol $^\circ$. In degrees, we split up one complete rotation into 360 equal parts and each part is one degree, denoted by 1°. Thus, 1° is 1/360 of one complete rotation. To measure a fraction of an angle and also for accuracy of measurement of angles, **minutes** and **seconds** are introduced. One minute $(1')$ corresponds to 1/60 of a degree and in turn a second $(1'')$ corresponds to 1/60 of a minute (or) 1/3600 of a degree.

We shall classify a pair of angles in the following way for better understanding and usages.

(i) Two angles that have the exact same measure are called **congruent angles**.
(ii) Two angles that have their measures adding to 90° are called **complementary angles**.
(iii) Two angles that have their measures adding to 180° are called **supplementary angles**.
(iv) Two angles between 0° and 360° are **conjugate** if their sum equals 360°.

(i) The concept of degrees, minutes and seconds, is analogous to the system of time measurement where we think of a degree representing one hour.

(ii) Observe that

$$\begin{aligned} 59.0854^\circ &= 59^\circ + 0.0854^\circ \\ 0.0854^\circ &= .0854^\circ \times \frac{60'}{1^\circ} = 5.124' \\ 5.124' &= 5' + 0.124' \\ 0.124' &= 0.124' \times \frac{60''}{1'} = 7.44'' \\ \text{Thus, } 59.0854^\circ &= 59^\circ 5' 7.44'' \end{aligned}$$

(iii) Also notice that

$$\begin{aligned} 34^\circ 51' 35'' &= 34.8597^\circ \\ \text{and } 90^\circ - 36^\circ 18' 47'' &= 53^\circ 41' 13'' \end{aligned}$$

3.2.4 Angles in Standard Position

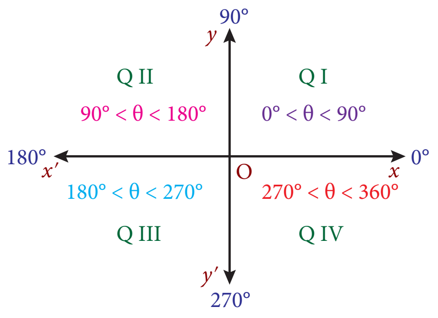


Figure 3.2

An angle is said to be in **standard position** if its vertex is at the origin and its initial side is along the positive x-axis. An angle is said to be in the first quadrant, if in the standard position, its terminal side falls in the first quadrant. Similarly, we can define for the other three quadrants. Angles in standard position having their terminal sides along the x-axis or y-axis are called **quadrantal angles**. Thus, $0^\circ, 90^\circ, 180^\circ, 270^\circ$ and 360° are quadrantal angles.

The degree measurement of a quadrantal angle is a multiple of 90°.

3.2.5 Coterminal angles

One complete rotation of a ray in the anticlockwise direction results in an angle measuring of $360°$. By continuing the anticlockwise rotation, angles larger than $360°$ can be produced. If we rotate in clockwise direction, negative angles are produced. Angles $57°$, $417°$ and $-303°$ have the same initial side and terminal side but with different amount of rotations, such angles are called **coterminal angles**. Thus, angles in standard position that have the same terminal sides are coterminal angles . Hence, if α and β are coterminal angles, then $\beta = \alpha + k(360°)$, k is an integer. The measurements of coterminal angles differ by an integral multiple of $360°$.

For example, $417°$ and $-303°$ are coterminal because $417° - (-303°) = 720° = 2\,(360°)$.

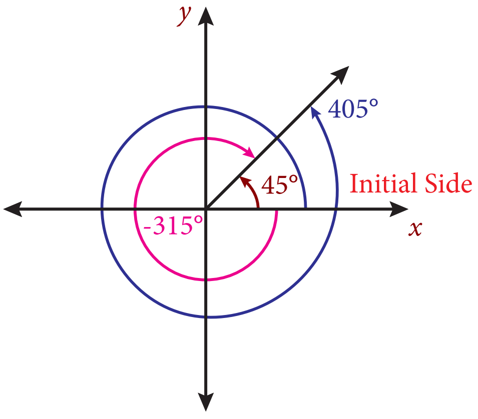


Figure 3.3

(i) Observe that $45°, -315°$ and $405°$ lie in the first quadrant.
(ii) The following pairs of angles are coterminal angles $(30°, 390°); (280°, 1000°)$ and $(-85°, 275°)$.

3.2.6 Basic Trigonometric ratios using a right triangle

We know that six ratios can be formed using the three lengths a, b, c of sides of a right triangle ABC. Interestingly, these ratios lead to the definitions of six basic trigonometric functions.

First, let us recall the trigonometric ratios which are defined with reference to a right triangle.

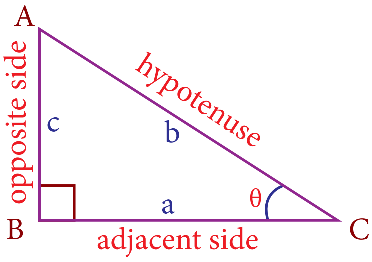


Figure 3.4

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} \quad ; \quad \cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

With the help of $\sin\theta$ and $\cos\theta$, the remaining trigonometric ratios $\tan\theta$, $\cot\theta$, cosec θ and $\sec\theta$ are determined by using the relations

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \text{cosec}\ \theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

3.2.7 Exact values of trigonometric functions of widely used angles

Let us list out the values of trigonometric functions at known angles.

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

(i) The values given above are all exact.
(ii) We observe that $\sin 30°$ and $\cos 60°$ are equal. Also $\sin 60°$ and $\cos 30°$ are equal.

(iii) The value of reciprocal ratios namely cosecant, secant and cotangent can be obtained using the above table.
(iv) The result $\cos 90° = 0$ does not allow us to define $\tan 90°$ and $\sec 90°$.
(v) Similarly $\sin 0° = 0$ does not permit us to define cosec $0°$ and $\cot 0°$.

3.2.8 Basic Trigonometric Identities

A trigonometric identity represents a relationship that is always true for all admissible values in the domain. For example $\sec\theta = \dfrac{1}{\cos\theta}$ is true for all admissible values of θ. Hence, this is an identity. However, $\sin\theta = \frac{1}{2}$ is not an identity, since the relation fails when $\theta = 60°$. Identities enable us to simplify complicated expressions. They are the basic tools of trigonometry which are being used in solving trigonometric equations. The most important part of working with identities, is to manipulate them with the help of a variety of techniques from algebra.

Let us recall the fundamental identities (Pythagorean identities) of trigonometry, namely,

$$\cos^2\theta + \sin^2\theta = 1$$
$$\sec^2\theta - \tan^2\theta = 1$$
$$\text{cosec}^2\theta - \cot^2\theta = 1$$

(i) $\sin^2\theta$ is the commonly used notation for $(\sin\theta)^2$, likewise for other trigonometric ratios.
(ii) $\sec^2\theta - \tan^2\theta = 1$ is meaningless when $\theta = 90°$. But still it is an identity and true for all values of θ for which $\sec\theta$ and $\tan\theta$ are defined. Thus, an identity is an equation that is true for all values of its domain values.
(iii) When we write $\dfrac{\sin\theta}{1+\cos\theta}$, we understand that the expression is valid for all values of θ for which $1+\cos\theta \neq 0$.

Example 3.1 Prove that $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \dfrac{1+\sin\theta}{\cos\theta}$.

Solution:

$$\begin{aligned}\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} &= \frac{\tan\theta + \sec\theta - \left(\sec^2\theta - \tan^2\theta\right)}{\tan\theta - \sec\theta + 1}\\ &= \frac{(\tan\theta + \sec\theta)\left[1 - (\sec\theta - \tan\theta)\right]}{\tan\theta - \sec\theta + 1}\\ &= \tan\theta + \sec\theta = \frac{1+\sin\theta}{\cos\theta}.\end{aligned}$$

Example 3.2 Prove that $(\sec A - \text{cosec}A)(1 + \tan A + \cot A) = \tan A \sec A - \cot A\, \text{cosec}A$.

Solution:

$$\begin{aligned}\text{L.H.S.} &= \left(\frac{1}{\cos A} - \frac{1}{\sin A}\right)\left[1 + \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right]\\ &= \frac{\sin^3 A - \cos^3 A}{\sin^2 A\cos^2 A} \qquad \text{...(i)}\\ \text{R.H.S.} &= \frac{\sin A}{\cos^2 A} - \frac{\cos A}{\sin^2 A} = \frac{\sin^3 A - \cos^3 A}{\sin^2 A\cos^2 A} \qquad \text{...(ii)}\end{aligned}$$

From (i) and (ii), we get the required result.

Example 3.3 Eliminate θ from $a\cos\theta = b$ and $c\sin\theta = d$, where a, b, c, d are constants.

Solution:

Squaring and adding $ac\cos\theta = bc$ and $ac\sin\theta = ad$, we get

$$a^2c^2 = b^2c^2 + a^2d^2.$$

Exercise - 3.1

1. Identify the quadrant in which an angle of each given measure lies

   (i) 25° (ii) 825° (iii) -55° (iv) 328° (v) -230°

2. For each given angle, find a coterminal angle with measure of θ such that $0^\circ \leq \theta < 360^\circ$

   (i) 395° (ii) 525° (iii) 1150° (iv) -270° (v) -450°

3. If $a\cos\theta - b\sin\theta = c$, show that $a\sin\theta + b\cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$.

4. If $\sin\theta + \cos\theta = m$, show that $\cos^6\theta + \sin^6\theta = \dfrac{4 - 3\left(m^2 - 1\right)^2}{4}$, where $m^2 \leq 2$.

5. If $\dfrac{\cos^4\alpha}{\cos^2\beta} + \dfrac{\sin^4\alpha}{\sin^2\beta} = 1$, prove that

   (i) $\sin^4\alpha + \sin^4\beta = 2\sin^2\alpha\sin^2\beta$ (ii) $\dfrac{\cos^4\beta}{\cos^2\alpha} + \dfrac{\sin^4\beta}{\sin^2\alpha} = 1$.

6. If $y = \dfrac{2\sin\alpha}{1 + \cos\alpha + \sin\alpha}$, then prove that $\dfrac{1 - \cos\alpha + \sin\alpha}{1 + \sin\alpha} = y$.

7. If $x = \sum_{n=0}^{\infty}\cos^{2n}\theta$, $y = \sum_{n=0}^{\infty}\sin^{2n}\theta$ and $z = \sum_{n=0}^{\infty}\cos^{2n}\theta\sin^{2n}\theta$, $0 < \theta < \dfrac{\pi}{2}$, then show that $xyz = x + y + z$.

   [Hint: Use the formula $1 + x + x^2 + x^3 + \ldots = \dfrac{1}{1 - x}$, where $|x| < 1$].

8. If $\tan^2\theta = 1 - k^2$, show that $\sec\theta + \tan^3\theta \operatorname{cosec}\theta = \left(2 - k^2\right)^{3/2}$. Also, find the values of k for which this result holds.

9. If $\sec\theta + \tan\theta = p$, obtain the values of $\sec\theta$, $\tan\theta$ and $\sin\theta$ in terms of p.

10. If $\cot\theta\,(1 + \sin\theta) = 4m$ and $\cot\theta\,(1 - \sin\theta) = 4n$, then prove that $\left(m^2 - n^2\right)^2 = mn$.

11. If $\operatorname{cosec}\theta - \sin\theta = a^3$ and $\sec\theta - \cos\theta = b^3$, then prove that $a^2b^2\left(a^2 + b^2\right) = 1$.

12. Eliminate θ from the equations $a\sec\theta - c\tan\theta = b$ and $b\sec\theta + d\tan\theta = c$.

3.3 Radian Measure

23V24I

Initially right triangles were used to define trigonometric ratios and angles were measured in degrees. But right triangles have limitations as they involve only acute angles. In degrees a full rotation corresponds to 360° where the choice of 360 dates back thousands of years to the Babylonians. They might have chosen 360 based on the number of days in a year. But it does have the nice property of breaking into smaller angles like $30^\circ, 45^\circ, 60^\circ$, 90° and 180°. In 17th century, trigonometry was extended to Physics and Chemistry where it required trigonometric functions whose domains were sets of real numbers rather than angles. This was accomplished by using correspondence between an angle and length of the arc on a unit circle. Such a measure of angle is termed as **radian measure** . For theoretical applications, the radian is the most common system of angle measurement. Radians are common unit of measurement in many technical fields, including calculus. The most important irrational number π plays a vital role in radian measures of angles. Let us introduce the radian measure of an angle.

Definition 3.1

The radian measure of an angle is the ratio of the arc length it subtends, to the radius of the circle in which it is the central angle.

Consider a circle of radius r. Let s be the arc length subtending an angle θ at the centre.

Then, $$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{s}{r} \text{ radians. Hence, } s = r\theta.$$

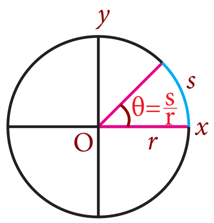


Figure 3.5

(i) All circles are similar. Thus, for a given central angle in any circle, the ratio of the intercepted arc length to the radius is always constant.

(ii) When $s = r$, we have an angle of 1 radian. Thus, one radian is the angle made at the centre of a circle by an arc with length equal to the radius of the circle.

(iii) Since the lengths s and r have same unit, θ is unitless and thus, we do not use any notation to denote radians.

(iv) $\theta = 1$ radian measure, if $s = r$
$\theta = 2$ radian measure, if $s = 2r$
Thus, in general $\theta = k$ radian measure, if $s = kr$.
Hence, radian measure of an angle tells us how many radius lengths, we need to sweep out along the circle to subtend the angle θ.

(v) Radian angle measurement can be related to the edge of the unit circle. In radian system, we measure an angle by measuring the distance travelled along the edge of the unit circle to where the terminal side of the angle intercepts the unit circle .

3.3.1 Relationship between Degree and Radian Measures

We have degree and radian units to measure angles. One measuring unit is better than another if it can be defined in a simpler and more intuitive way. For example, in measuring temperature, Celsius unit is better than Fahrenheit as Celsius was defined using 0° and 100° for freezing and boiling points of water. Radian measure is better for conversion and calculations. Radian measure is more convenient for analysis whereas degree measure of an angle is more convenient to communicate the concept between people. Greek Mathematicians observed the relation of π which arises from circumference of a circle and thus, π plays a crucial role in radian measure.

In unit circle, a full rotation corresponds to 360° whereas, a full rotation is related to 2π radians, the circumference of the unit circle. Thus, we have the following relations:

$$2\pi \text{ radians} = 360^\circ, \text{ which reduces to } \pi \text{ radians} = 180^\circ.$$
$$\text{Thus, } 1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ \quad \text{or} \quad 1^\circ = \frac{\pi}{180} \text{ radians.}$$
$$x \text{ radian} = \left(\frac{180x}{\pi}\right)^\circ \quad \text{or} \quad x^\circ = \frac{\pi x}{180} \text{ radians.}$$

Observe that the scale used in radians is much smaller than the scale in degrees. The smaller scale makes the graphs of trigonometric functions more visible and usable. The above relation gives a way to convert radians into degrees or degrees into radians.

(i) The ratio of the circumference of any circle to its diameter is always a constant. This constant is denoted by the irrational number π.

(ii) Mark a point P on a unit circle and put the unit circle on the number line so that P touches the number 0. Allow the circle to roll along the number line. The point P will touch the number 2π on the number line when the circle rolls to one complete revolution to the right.

(iii) If the unit of angle measure is not specified, then the angle is understood to be in radians.

(iv) Consider a sector of a circle with radius r. If θ is the central angle of the sector, then

$$\text{Area of the sector} = \begin{cases} \left(\dfrac{\pi r^2}{360^\circ}\right)\theta & \text{in degree measure} \\ \left(\dfrac{\pi r^2}{2\pi}\right)\theta = \dfrac{r^2\theta}{2} & \text{in radian measure} \end{cases}$$

Clearly, the calculation in radian measure is much easier to work with.

(v) The values of π and $\dfrac{22}{7}$ correct to four decimal places are 3.1416 and 3.1429 respectively. Thus, π and $\dfrac{22}{7}$ are approximately equal correct upto two decimal places. Hence, $\pi \approx \dfrac{22}{7}$.

(vi)

$$1 \text{ radian} \approx 57^\circ 17' 45'' \text{ and } 1^\circ \approx 0.017453 \text{ radian}$$

$$1' = \left(\frac{\pi}{180 \times 60}\right) \text{radian} \approx 0.000291 \text{ radian.}$$

$$1'' = \left(\frac{\pi}{180 \times 60 \times 60}\right) \text{radian} \approx 0.000005 \text{ radian.}$$

(vii) The radian measures and the corresponding degree measures for some known angles are given in the following table

Radians	0	1	0.017453	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Degrees	0°	$57^\circ 17' 45''$	1°	30°	45°	60°	90°	180°	270°	360°

(viii) $\sin 90^\circ = 1$ but $\sin 90 \neq 1$ (in radian measure).

Example 3.4 Convert (i) 18° to radians (ii)-108° to radians.

Solution:

Now, $180^\circ = \pi$ radians gives $1^\circ = \dfrac{\pi}{180}$ radians

(i) $18^\circ = \dfrac{\pi}{180} \times 18$ radians $= \dfrac{\pi}{10}$ radians

(ii) $-108^\circ = \dfrac{\pi}{180} \times (-108)$ radians $= -\dfrac{3\pi}{5}$ radians.

Example 3.5 Convert (i) $\dfrac{\pi}{5}$ radians to degrees (ii) 6 radians to degrees.

Solution:

We know that π radians $= 180^\circ$ and thus,

(i) $\dfrac{\pi}{5}$ radians $= \dfrac{180^\circ}{5} = 36^\circ$

(ii) $6 \text{ radians} = \left(\dfrac{180}{\pi} \times 6\right)^\circ \approx \left(\dfrac{7 \times 180}{22} \times 6\right)^\circ = \left(343\dfrac{7}{11}\right)^\circ.$

Example 3.6 Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring $15°$.

Solution:
Let s be the length of the arc of a circle of radius r subtending a central angle θ.
Then $s = r\theta$.

$$\text{We have, } \theta = 15° = 15 \times \frac{\pi}{180} = \frac{\pi}{12} \text{ radians}$$

$$\text{So that, } s = r\theta \text{ gives } s = 5 \times \frac{\pi}{12} = \frac{5\pi}{12}\ cm$$

In the product $r\theta$, θ must always be in radians.

Example 3.7 If the arcs of same lengths in two circles subtend central angles $30°$ and $80°$, find the ratio of their radii.

Solution:
Let r_1 and r_2 be the radii of the two given circles and l be the length of the arc.

$$\theta_1 = 30° = \frac{\pi}{6} \text{ radians}$$

$$\theta_2 = 80° = \frac{4\pi}{9} \text{ radians}$$

$$\text{Given that } l = r_1\theta = r_2\theta$$

$$\text{Thus, } \frac{\pi}{6} r_1 = \frac{4\pi}{9} r_2$$

$$\frac{r_1}{r_2} = \frac{8}{3} \text{ which implies } r_1 : r_2 = 8 : 3.$$

Exercise - 3.2

1. Express each of the following angles in radian measure:
   (i) $30°$ (ii) $135°$ (iii) $-205°$ (iv) $150°$ (v) $330°$.
2. Find the degree measure corresponding to the following radian measures
   (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{9}$ (iii) $\frac{2\pi}{5}$ (iv) $\frac{7\pi}{3}$ (v) $\frac{10\pi}{9}$.
3. What must be the radius of a circular running path, around which an athlete must run 5 times in order to describe 1 km?
4. In a circle of diameter 40 cm, a chord is of length 20 cm. Find the length of the minor arc of the chord.
5. Find the degree measure of the angle subtended at the centre of circle of radius 100 cm by an arc of length 22 cm.
6. What is the length of the arc intercepted by a central angle of measure $41°$ in a circle of radius 10 ft?
7. If in two circles, arcs of the same length subtend angles $60°$ and $75°$ at the centre, find the ratio of their radii.

8. The perimeter of a certain sector of a circle is equal to the length of the arc of a semi-circle having the same radius. Express the angle of the sector in degrees, minutes and seconds.
9. An airplane propeller rotates 1000 times per minute. Find the number of degrees that a point on the edge of the propeller will rotate in 1 second.
10. A train is moving on a circular track of $1500\ m$ radius at the rate of $66\ km/hr$. What angle will it turn in 20 seconds?
11. A circular metallic plate of radius $8\ cm$ and thickness $6\ mm$ is melted and molded into a pie (a sector of the circle with thickness) of radius $16\ cm$ and thickness $4\ mm$. Find the angle of the sector.

3.4 Trigonometric functions and their properties

3.4.1 Trigonometric Functions of any angle in terms of Cartesian coordinates

We have studied the principles of trigonometric ratios in the lower classes using acute angles . But we come across many angles which are not acute. We shall extend the acute angle idea and define trigonometric functions for any angle. The trigonometric ratios to any angle in terms of radian measure are called trigonometric functions.

Let $P(x, y)$ be a point other than the origin on the terminal side of an angle θ in standard position . Let $OP = r$. Thus, $r = \sqrt{x^2 + y^2}$

The six trigonometric functions of θ are defined as follows:

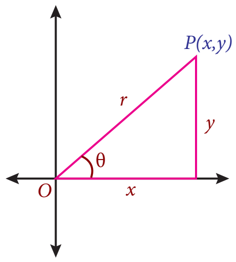


Figure 3.6

$$\sin\theta = \frac{y}{r} \text{ and } \cos\theta = \frac{x}{r}.$$

Using this, we have $\tan\theta = \dfrac{y}{x}, x \neq 0$; $\cot\theta = \dfrac{x}{y}, y \neq 0$; cosec $\theta = \dfrac{r}{y}, y \neq 0$; $\sec\theta = \dfrac{r}{x}, x \neq 0$.

(i) Since $|x| \leq r, |y| \leq r$, we have $|\sin\theta| \leq 1$ and $|\cos\theta| \leq 1$.

(ii) In the case of acute angle, the above definitions are equivalent to our earlier definitions using right triangle.

(iii) The trigonometric functions have positive or negative values depending on the quadrant in which the point $P(x, y)$ on the terminal side of θ lies.

(iv) The above definitions of trigonometric functions is independent of the points on the terminal side of the angle. (verify!)

Trigonometric ratios of Quadrantal angles

Let us recall that an angle in its standard position for which the terminal side coincides with one of the axes, is a Quadrantal angle. We shall find the trigonometric ratios for the quadrantal angles.

Consider the unit circle $x^2 + y^2 = 1$. Let $P(x, y)$ be a point on the unit circle where the terminal side of the angle θ intersects the unit circle.

Then $\cos\theta = \dfrac{x}{1} = x$ (x-coordinate of P) and

$\sin\theta = \dfrac{y}{1} = y$ (y-coordinate of P)

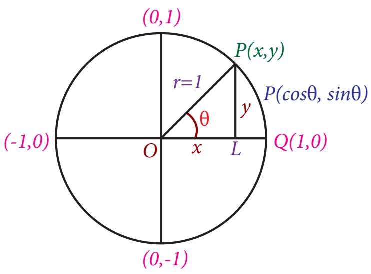


Figure 3.7

Thus, the coordinates of any point $P(x, y)$ on the unit circle is $(\cos\theta, \sin\theta)$. In this way, the angle measure θ is associated with a point on the unit circle.

The following table illustrates how trigonometric function values are determined for a Quadrantal angles using the above explanation.

Exact values of trigonometric functions of quadrantal angles.

Quadrantal angle	Corresponding point on the Unit circle $P(x, y) = P(\cos\theta, \sin\theta)$	cosine value $\cos\theta$	sine value $\sin\theta$
$\theta = 0^\circ$	$(1, 0) = (\cos 0^\circ, \sin 0^\circ)$	$\cos 0^\circ = 1$	$\sin 0^\circ = 0$
$\theta = 90^\circ$	$(0, 1) = (\cos 90^\circ, \sin 90^\circ)$	$\cos 90^\circ = 0$	$\sin 90^\circ = 1$
$\theta = 180^\circ$	$(-1, 0) = (\cos 180^\circ, \sin 180^\circ)$	$\cos 180^\circ = -1$	$\sin 180^\circ = 0$
$\theta = 270^\circ$	$(0, -1) = (\cos 270^\circ, \sin 270^\circ)$	$\cos 270^\circ = 0$	$\sin 270^\circ = -1$
$\theta = 360^\circ$	$(1, 0) = (\cos 360^\circ, \sin 360^\circ)$	$\cos 360^\circ = 1$	$\sin 360^\circ = 0$

(i) Observe that x and y coordinates of all points on the unit circle lie between -1 and 1. Hence, $-1 \leq \cos\theta \leq 1$, $-1 \leq \sin\theta \leq 1$, no matter whatever be the value of θ.

(ii) When $\theta = 360^\circ$, we have completed one full rotation. Thus, the terminal side coincides with positive x- axis. Hence, sine has equal values at 0° and at 360°. Cosine and other trigonometric functions also follow it.

(iii) If two angles differ by an integral multiple of 360° or 2π, then each trigonometric function will have equal values at both angles.

(iv) Using the values of sine and cosine at quadrantal angles, we have the following generalization geometrically.

At Quadrantal angle	Justification	Generalization
$\sin 0 = 0$ $\sin\pi = 0$	$\sin(0 + 2n\pi) = 0; n \in \mathbb{Z}$ $\sin(\pi + 2n\pi) = 0; n \in \mathbb{Z}$	$\sin\theta = 0 \Rightarrow \theta = n\pi; n \in \mathbb{Z}$
$\cos\frac{\pi}{2} = 0$ $\cos\frac{3\pi}{2} = 0$	$\cos\left(\frac{\pi}{2} + 2n\pi\right) = 0; n \in \mathbb{Z}$ $\cos\left(\frac{3\pi}{2} + 2n\pi\right) = 0; n \in \mathbb{Z}$	$\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}; n \in \mathbb{Z}$
$\tan 0 = 0$ $\tan\pi = 0$	$\tan(0 + 2n\pi) = 0; n \in \mathbb{Z}$ $\tan(\pi + 2n\pi) = 0; n \in \mathbb{Z}$	$\tan\theta = 0 \Rightarrow \theta = n\pi; n \in \mathbb{Z}$

(v) $\tan\theta$ is not defined when $\cos\theta = 0$ and so, $\tan\theta$ is not defined when $\theta = (2n+1)\dfrac{\pi}{2}$, $n \in \mathbb{Z}$.

3.4.2 Trigonometric Functions of real numbers

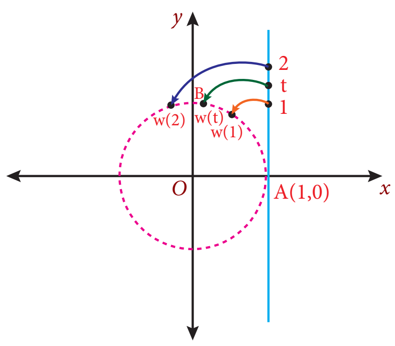


Figure 3.8

For applications of trigonometry to the problems in higher mathematics including calculus and to problems in physics and chemistry, scientists required trigonometric functions of real numbers. This was skillfully done by exhibiting a correspondence between an angle and an arc length denoting a real number on a unit circle.

Consider a unit circle with the centre at the origin. Let the angle zero (in radian measure) be associated with the point $A(1, 0)$ on the unit circle. Draw a tangent to the unit circle at the point $A(1, 0)$. Let t be a real number such that t is y- coordinate of a point on the tangent line.

For each real number t, identify a point $B(x, y)$ on the unit circle such that the arc length AB is equal to t. If t is positive, choose the point $B(x, y)$ in the anticlockwise direction, otherwise choose it in the clockwise direction. Let θ be the angle subtended by the arc AB at the centre. In this way, we have a function $\mathrm{w}(t)$ associating a real number t to a point on the unit circle. Such a function is called a **wrapping function** . Then $s = r\theta$ gives arc length $t = \theta$.

Now, define $\sin t = \sin\theta$ and $\cos t = \cos\theta$.

Clearly, $\sin t = \sin\theta = y$ and $\cos t = \cos\theta = x$.

Using $\sin t$ and $\cos t$, other trigonometric functions can be defined as functions of real numbers.

(i) $B(x, y) = B(\cos t, \sin t)$ is a point on the unit circle.
Thus, $-1 \leq \cos t \leq 1$ and $-1 \leq \sin t \leq 1$ for any real number t.

(ii) Wrapping function $\mathrm{w}(t)$ is analogous to wrapping a line around a circle.

(iii) The value of a trigonometric function of a real number t is its value at the angle t radians.

(iv) Trigonometric functions of real numbers are used to model phenomena like waves, oscillations, that occur in regular intervals.

Example 3.8 The terminal side of an angle θ in standard position passes through the point $(3, -4)$. Find the six trigonometric function values at an angle θ.

Solution:

Let $B(x, y) = B(3, -4)$, OA be the initial side and OB be the terminal side of the angle θ in the standard position.

Then $\angle AOB$ is the angle θ and θ lies in the IV quadrant. Also,

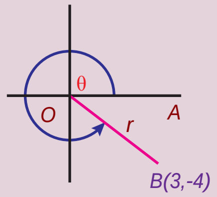


Figure 3.9

$$OB = r,\ r = \sqrt{x^2 + y^2} = \sqrt{3^2 + (-4)^2} = 5$$

$$x = 3,\ y = -4 \text{ and } r = 5, \text{ we have}$$

$$\sin\theta = \frac{y}{r} = -\frac{4}{5};\quad \cos\theta = \frac{x}{r} = \frac{3}{5};\quad \tan\theta = \frac{y}{x} = -\frac{4}{3};$$

$$\operatorname{cosec}\theta = \frac{r}{y} = -\frac{5}{4};\quad \sec\theta = \frac{r}{x} = \frac{5}{3};\quad \cot\theta = \frac{x}{y} = -\frac{3}{4}.$$

Signs of Trigonometric functions

Consider a unit circle with centre at the origin. Let θ be in standard position. Let $P(x, y)$ be the point on the unit circle corresponding to the angle θ. Then, $\cos\theta = x$, $\sin\theta = y$ and $\tan\theta = \frac{y}{x}$. The values of x and y are positive or negative depending on the quadrant in which P lies.

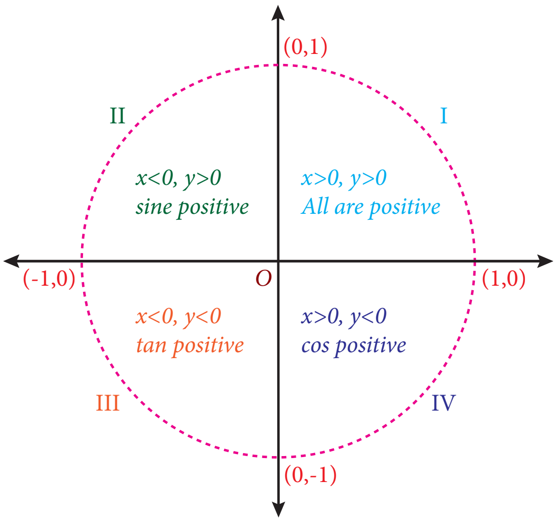


Figure 3.10

In the first quadrant:

$\cos\theta = x > 0$ (positive); $\sin\theta = y > 0$ (positive)

Thus, $\cos\theta$ and $\sin\theta$ and hence all trigonometric functions are positive in the first quadrant.

In the second quadrant:

$\cos\theta = x < 0$ (negative); $\sin\theta = y > 0$ (positive)

Thus, $\sin\theta$ and $\operatorname{cosec}\theta$ are positive and others are negative.
Similarly, we can find the sign of trigonometric functions in other two quadrants.
Let us illustrate the above discussions in Figure 3.10.

Signs of trigonometric functions in various quadrants can be remembered with the slogan

"All Students Take Chocolate". (ASTC rule)

Example 3.9 If $\sin\theta = \frac{3}{5}$ and the angle θ is in the second quadrant, then find the values of other five trigonometric functions.

Solution:

We know that $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos\theta = \pm\sqrt{1-\sin^2\theta} = \pm\sqrt{1-\frac{9}{25}} = \pm\frac{4}{5}$

Thus, $\cos\theta = -\frac{4}{5}$ as $\cos\theta$ is negative in the second quadrant.

$$\begin{aligned}
\sin\theta &= \frac{3}{5} \Rightarrow \operatorname{cosec}\theta = \frac{5}{3}; \quad \cos\theta = -\frac{4}{5} \Rightarrow \sec\theta = -\frac{5}{4} \\
\tan\theta &= \frac{\sin\theta}{\cos\theta} = -\frac{3}{4}; \quad \cot\theta = -\frac{4}{3}
\end{aligned}$$

If $\sin\theta$ and $\cos\theta$ are known, then the reciprocal identities and quotient identities can be used to find the other four trigonometric values. The Pythagorean identities can be used to find trigonometric values when one trigonometric value and the quadrant are known.

3.4.3 Allied Angles

Two angles are said to be allied if their sum or difference is a multiple of $\frac{\pi}{2}$ radians.

Thus, any two angles of θ such as, $-\theta, \frac{\pi}{2}\pm\theta, \pi\pm\theta, \frac{3\pi}{2}\pm\theta$, ..., are all allied angles .

Now we shall find the trigonometric ratios involving allied angles θ and $-\theta$.

Trigonometric ratios of $-\theta$ in terms of θ

Let $\angle AOL = \theta$ and $\angle AOM = -\theta$. Let $P(a, b)$ be a point on OL. Choose a point P' on OM such that $OP = OP'$.

Draw PN perpendicular to OA intersecting OM at P'. Since $\angle AOP = \angle AOP'$ and $\angle PON = \angle P'ON$ $\triangle PON$ and $\triangle P'ON$ are congruent.

Thus, $PN = P'N$ and hence the point P' is given by $P'(a, -b)$

Now, by the definition of trigonometric functions

$\sin\theta = \frac{b}{OP}, \ \cos\theta = \frac{a}{OP}, \ \tan\theta = \frac{b}{a}$

Thus, $\sin(-\theta) = \frac{-b}{OP'} = \frac{-b}{OP} = -\sin\theta$

and $\cos(-\theta) = \frac{a}{OP'} = \frac{a}{OP} = \cos\theta$

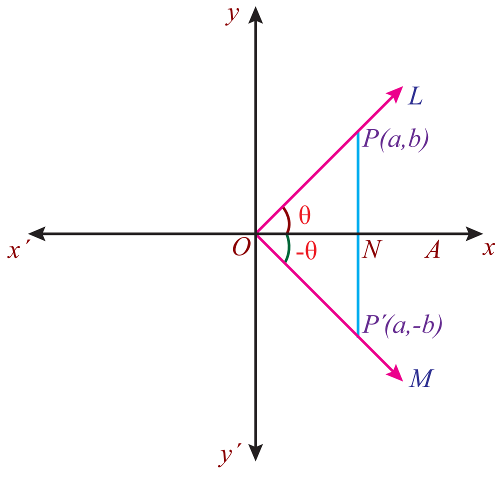


Figure 3.11

Then, it is easy to get

$\tan(-\theta) = -\tan\theta,\ \text{cosec}(-\theta) = -\text{cosec}\ \theta,\ \sec(-\theta) = \sec\theta,\ \cot(-\theta) = -\cot\theta.$

(i) $\sin(-\theta) = -\sin\theta$, and $\cos(-\theta) = \cos\theta$. These facts follow from the symmetry of the unit circle about the x-axis. The angle $-\theta$ is the same as angle θ except it is on the other side of the x- axis. Flipping a point (x, y) to the other side of the x- axis makes the point into $(x, -y)$, so the y-coordinate is negated and hence the sine is negated, but the x-coordinate remains the same and therefore the cosine is unchanged.

(ii) The negative-angle identities can be used to determine if a trigonometric function is an odd function or an even function .

Example 3.10 Find the values of (i) $\sin(-45^\circ)$ (ii) $\cos(-45^\circ)$ (iii) $\cot(-45^\circ)$

Solution:

$\sin(-45^\circ) = -\sin(45^\circ) = -\dfrac{1}{\sqrt{2}}$. We can easily see that

$\cos(-45^\circ) = \dfrac{1}{\sqrt{2}}$ and $\cot(-45^\circ) = -1$

We have already learnt the trigonometric ratios of the angle $(90^\circ - \theta)$, $\left(0 < \theta < \dfrac{\pi}{2}\right)$ in the lower class. Let us recall the trigonometric ratios of angle $(90^\circ - \theta)$;

$$\begin{aligned} \sin(90^\circ - \theta) &= \cos\theta, \quad \cos(90^\circ - \theta) = \sin\theta, \quad \tan(90^\circ - \theta) = \cot\theta \\ \text{cosec}(90^\circ - \theta) &= \sec\theta, \quad \sec(90^\circ - \theta) = \text{cosec}\ \theta, \quad \cot(90^\circ - \theta) = \tan\theta. \end{aligned}$$

Now, we will establish the corresponding trigonometric ratios for an angle of the form $(90^\circ + \theta)$.

Trigonometric ratios of an angle of the form $(90^\circ + \theta)$, $\left(0 < \theta < \dfrac{\pi}{2}\right)$ in terms of θ .

Let $\angle AOL = \theta$ and $\angle AOR = (90^\circ + \theta)$. Let $P(a, b)$ be a point on OL and choose a point P' on OR such that $OP = OP'$.

Draw perpendiculars PM and $P'N$ from P and P' on Ox and Ox' respectively.

Now, $\angle AOP' = 90^\circ + \theta$.

Clearly, $\triangle OPM$ and $\triangle P'ON$ are congruent.

$$ON = MP \text{ and } NP' = OM$$

Hence, the coordinates of P and P' are $P(a, b)$ and $P'(-b, a)$, respectively. Now

$$\sin(90^\circ + \theta) = \frac{y\text{-coordinate of } P'}{OP'} = \frac{a}{OP} = \cos\theta,$$
$$\cos(90^\circ + \theta) = \frac{x\text{-coordinate of } P'}{OP'} = \frac{-b}{OP} = -\sin\theta,$$

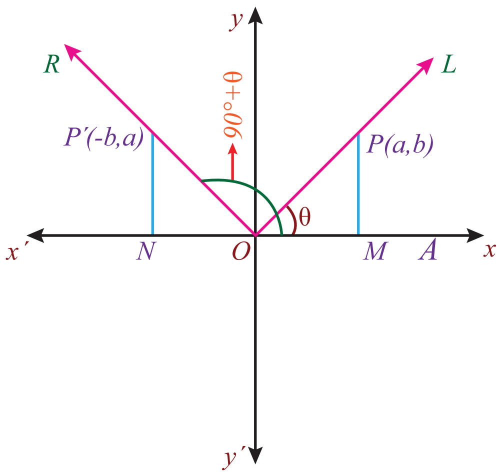


Figure 3.12

Thus, $\tan(90^\circ + \theta) = -\cot\theta$, $\text{cosec}\ (90^\circ + \theta) = \sec\theta$, $\sec(90^\circ + \theta) = -\text{cosec}\ \theta$, $\cot(90^\circ + \theta) = -\tan\theta$.

The trigonometric function of other allied angles $\pi \pm \theta$, $\dfrac{3\pi}{2} \pm \theta$, $2\pi \pm \theta$ can be obtained in a similar way.

The above results can be summarized in the following table: $\left(\text{Here } 0 < \theta < \dfrac{\pi}{2}\right)$

	$-\theta$	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\pi-\theta$	$\pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	$2\pi-\theta$	$2\pi+\theta$
sine	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
cosine	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
tangent	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$

(i) The corresponding reciprocal ratios can be written using the above table.

(ii) If the allied angles are $-\theta, \pi \pm \theta, 2\pi \pm \theta$, that is, angles of the form $2n\frac{\pi}{2} \pm \theta, n \in \mathbb{Z}$, then, the form of trigonometric ratio is unaltered (i.e., sine remains sine, cosine remains cosine etc.,)

(iii) If the allied angles are $\frac{\pi}{2} \pm \theta, \frac{3\pi}{2} \pm \theta$, that is, angles of the form $(2n+1)\frac{\pi}{2} \pm \theta, n \in \mathbb{Z}$ then, the form of trigonometric ratio is altered to its complementary ratio. $i.e.$, it is to add the prefix "co" if it is absent and remove the prefix "co" if it is already present ($i.e.$, sine becomes cosine, cosine become sine etc.,)

(iv) For determining the sign, first find out the quadrant and then attach the appropriate sign (+ or −) according to the quadrant rule "ASTC".

Example 3.11 Find the value of (i) $\sin 150°$ (ii) $\cos 135°$ (iii) $\tan 120°$.

Solution:

$$\text{(i) } \sin 150° = \sin(90° + 60°) = \cos(60°) = \frac{1}{2}$$

$$\text{(or) } \sin 150° = \sin(180° - 30°) = \sin(30°) = \frac{1}{2}$$

$$\text{(ii) } \cos 135° = \cos(90° + 45°) = -\sin(45°) = -\frac{1}{\sqrt{2}}$$

$$\text{(or) } \cos 135° = \cos(180° - 45°) = -\cos(45°) = -\frac{1}{\sqrt{2}}$$

$$\text{(iii) } \tan 120° = \tan(180° - 60°) = -\tan(60°) = -\sqrt{3}$$

(or) write $\tan 120°$ as $\tan(90° + 30°)$ and find the value.

Example 3.12 Find the value of:

(i) $\sin 765°$ (ii) cosec $(-1410°)$ (iii) $\cot\left(\frac{-15\pi}{4}\right)$.

Solution:

$$\text{(i) } \sin 765° = \sin(2 \times 360° + 45°) = \sin 45° = \frac{1}{\sqrt{2}}$$

$$\text{(ii) cosec } (-1410°) = -\text{cosec } (1410°) = -\text{cosec } (4 \times 360° - 30°) = \text{cosec } 30° = 2$$

$$\text{(iii) } \cot\left(\frac{-15\pi}{4}\right) = -\cot\left(\frac{15\pi}{4}\right) = -\cot\left(4\pi - \frac{\pi}{4}\right) = \cot\frac{\pi}{4} = 1.$$

Example 3.13 Prove that $\tan 315° \cot(-405°) + \cot 495° \tan(-585°) = 2$

Solution:

$$\begin{aligned} \text{L.H.S} &= \tan(360° - 45°)\,[-\cot(360° + 45°)] + \cot(360° + 135°)\,[-\tan(360° + 225°)] \\ &= [-\tan 45°]\,[-\cot 45°] + [-\tan 45°]\,[-\tan 45°] = (-1)(-1) + (-1)(-1) = 2. \end{aligned}$$

3.4.4 Some Characteristics of Trigonometric Functions

Trigonometric functions have some nice properties. For example,

(i) Sine and cosine functions are complementary to each other in the sense that $\sin(90^\circ - \theta) = \cos\theta$ and $\cos(90^\circ - \theta) = \sin\theta$.

(ii) As $\cos\theta$ and $\sin\theta$ are obtained as coordinates of a point on the unit circle, they satisfy the inequalities $-1 \leq \cos\theta \leq 1$ and $-1 \leq \sin\theta \leq 1$. Hence, $\cos\theta, \sin\theta \in [-1, 1]$

(iii) Trigonometric function repeats its values in regular intervals.

(iv) Sine and cosine functions have an interesting property that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

Let us discuss the last two properties.

Periodicity of Trigonometric Functions

We know that a function f is said to be a periodic function with period p, if there exists a smallest positive number p such that $f(x + p) = f(x)$ for all x in the domain.

For example, $\sin(x + 2n\pi) = \sin x, n \in \mathbb{Z}$.

$i.e., \sin(x + 2\pi) = \sin(x + 4\pi) = \sin(x + 6\pi) = \ldots = \sin x$

Thus, $\sin x$ is a periodic function with period 2π.

Similarly, $\cos x$, cosec$x$ and $\sec x$ are periodic functions with period 2π.

But $\tan x$ and $\cot x$ are periodic functions with period π.

The periodicity of $\sin x$ and $\cos x$ can be viewed best using their graphs.

(i) **The graph of the sine function**

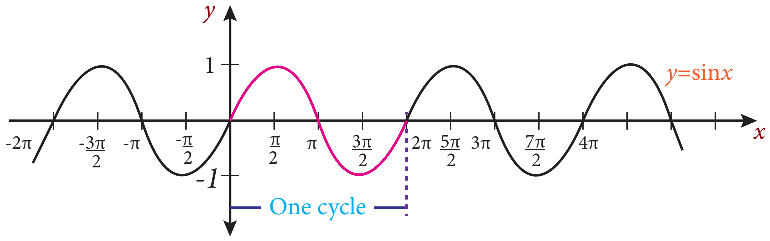


Figure 3.13: y =sinx

Here x represents a variable angle. Take the horizontal axis to be the x-axis and vertical axis to be the y-axis. Graph of the function $y = \sin x$ is shown in the Figure 3.13. First, note that it is periodic of period 2π. Geometrically it means that if you take the curve and slide it 2π either left or right, then the curve falls back on itself. Second, note that the graph is within one unit of the y-axis. The graph increases and decreases periodically. For instance, increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and decreases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$.

(ii) **The graph of the cosine function**

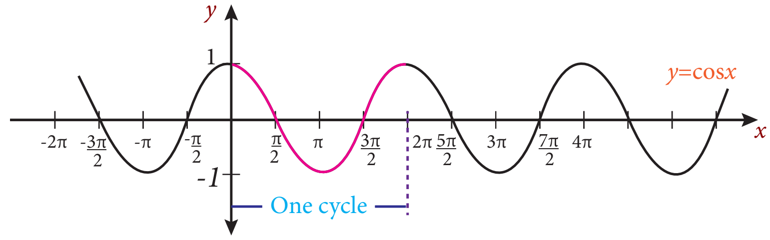


Figure 3.14: y = cosx

Observe that the graph of $y = \cos x$ looks just like the graph of $y = \sin x$ except it is being translated to the left by $\frac{\pi}{2}$. This is because of the identity $\cos x = \sin\left(\frac{\pi}{2} + x\right)$. It easily follows from the graph that $\cos x = \cos(-x) = \sin\left(\frac{\pi}{2} + x\right)$

(i) The sine and cosine functions are useful for one very important reason, since they repeat in a regular pattern (*i.e.*, they are periodic). There are a vast array of things in and around us that repeat periodically. For example, the rising and setting of the sun, the motion of a spring up and down, the tides of the ocean and so on, are repeating at regular intervals of time. All periodic behaviour can be studied through combinations of the sine and cosine functions.

(ii) Periodic functions are used throughout science to describe oscillations, waves and other phenomena that occur periodically.

Odd and Even trigonometric functions

Even and odd functions are functions satisfying certain symmetries. A real valued function $f(x)$ is an even function if it satisfies $f(-x) = f(x)$ for all real number x and an odd function if it satisfies $f(-x) = -f(x)$ for all real number x.
Basic trigonometric functions are examples of non-polynomial even and odd functions
Because $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$ for all x, it follows that $\cos x$ is an even function and $\sin x$ is an odd function.
Also note that $\sec x$ is an even function while $\tan x$, cosec x and $\cot x$ are all odd functions.
However, $f(t) = t - \cos t$ is neither even function nor odd function (why ?)

Example 3.14 Determine whether the following functions are even, odd or neither.

(i) $\sin^2 x - 2\cos^2 x - \cos x$ (ii) $\sin(\cos(x))$ (iii) $\cos(\sin(x))$ (iv) $\sin x + \cos x$

Solution:

(i) Let $f(x) = \sin^2 x - 2\cos^2 x - \cos x$

$f(-x) = f(x)$ [since $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$]

Thus, $f(x)$ is even.

(ii) Let $f(x) = \sin(\cos(x))$

$f(-x) = f(x)$, $f(x)$ is an even function.

(iii) $f(x) = \cos(\sin(x))$, $f(-x) = f(x)$, Thus, $f(x)$ is an even function.

(iv) Let $f(x) = \sin x + \cos x$

$f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$

Thus, $f(x) = \sin x + \cos x$ is neither even nor odd.

(i) In general, a function is an even function if its graph is unchanged under reflection about the y-axis. A function is odd if its graph is symmetric about the origin.

(ii) The properties of even and odd functions are useful in analyzing trigonometric functions particularly in the sum and difference formula.

(iii) The properties of even and odd functions are useful in evaluating some definite integrals, which we will see in calculus.

Exercise - 3.3

1. Find the values of (i) $\sin(480°)$ (ii) $\sin(-1110°)$ (iii) $\cos(300°)$ (iv) $\tan(1050°)$ (v) $\cot(660°)$ (vi) $\tan\left(\frac{19\pi}{3}\right)$ (vii) $\sin\left(-\frac{11\pi}{3}\right)$.
2. $\left(\frac{5}{7}, \frac{2\sqrt{6}}{7}\right)$ is a point on the terminal side of an angle θ in standard position. Determine the trigonometric function values of angle θ.
3. Find the values of other five trigonometric functions for the following:

   (i) $\cos\theta = -\frac{1}{2}$, θ lies in the III quadrant.

   (ii) $\cos\theta = \frac{2}{3}$, θ lies in the I quadrant.

   (iii) $\sin\theta = -\frac{2}{3}$, θ lies in the IV quadrant.

   (iv) $\tan\theta = -2$, θ lies in the II quadrant.

   (v) $\sec\theta = \frac{13}{5}$, θ lies in the IV quadrant.
4. Prove that $\dfrac{\cot(180° + \theta)\sin(90° - \theta)\cos(-\theta)}{\sin(270° + \theta)\tan(-\theta)\operatorname{cosec}(360° + \theta)} = \cos^2\theta\cot\theta$.
5. Find all the angles between $0°$ and $360°$ which satisfy the equation $\sin^2\theta = \frac{3}{4}$.
6. Show that $\sin^2\frac{\pi}{18} + \sin^2\frac{\pi}{9} + \sin^2\frac{7\pi}{18} + \sin^2\frac{4\pi}{9} = 2$.

3.5 Trigonometric Identities

3.5.1 Sum and difference identities or compound angles formulas

Now, compound angles are algebraic sum of two or more angles. Trigonometric functions do not satisfy the functional relations like $f(x+y) = f(x) + f(y)$ and $f(kx) = kf(x)$, k is a real number. For example, $\cos(\alpha+\beta) \neq \cos\alpha + \cos\beta$, $\sin(2\alpha) \neq 2\sin\alpha$, $\tan 3\alpha \neq 3\tan\alpha, \ldots$. Thus, we need to derive formulas for $\sin(\alpha+\beta), \cos(\alpha+\beta), \ldots$ and use them in calculations of application problems.

Music is made up of vibrations that create pressure on ear-drums. Musical tones can be modeled with sinusoidal graphs (graphs looks like that of $y = \sin x$ or $y = \cos x$). When more than one tone is played, the resulting pressure is equal to the sum of the individual pressures. In this context sum and difference trigonometric identities are used as an important application. Also, sum and difference trigonometric identities are helpful in the analysis of waves.

First we shall prove the identity for the cosine of the sum of two angles and extend it to prove all other sum or difference identities.

Identity 3.1: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Proof. Consider the unit circle with centre at O. Let $P = P(1,0)$.

Let Q, R and S be points on the unit circle such that $\angle POQ = \alpha$, $\angle POR = \alpha+\beta$ and $\angle POS = -\beta$ as shown in the Figure 3.15. Clearly, angles $\alpha, \alpha+\beta$ and $-\beta$ are in standard positions. Now, the points Q, R and S are given by $Q(\cos\alpha, \sin\alpha)$, $R(\cos(\alpha+\beta), \sin(\alpha+\beta))$ and $S(\cos(-\beta), \sin(-\beta))$.

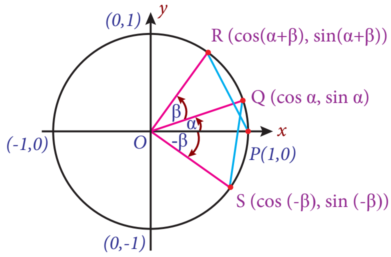


Figure 3.15

Since $\triangle POR$ and $\triangle SOQ$ are congruent. So, $PR = SQ$ which gives $PR^2 = SQ^2$

Thus, $[\cos(\alpha+\beta)-1]^2+\sin^2(\alpha+\beta) = [\cos\alpha-\cos(-\beta)]^2+[\sin\alpha-\sin(-\beta)]^2$

$$-2\cos(\alpha+\beta)+2 = 2-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta$$

Hence, $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$.

(i) In the above proof, $PR = SQ$ says that the distance between two points on a circle is determined by the radius and the central angle .

(ii) Arc lengths PR and SQ, subtends angles $\alpha+\beta$ and $\alpha+(-\beta)$ respectively at the center. Thus, PR = SQ. Thus, distance between the points $(\cos\alpha, \sin\alpha)$ and $(\cos(-\beta), \sin(-\beta))$ is same as the distance between the points $(\cos(\alpha+\beta), \sin(\alpha+\beta))$ and $(1, 0)$.

(iii) In the above derivations, $0 \le \alpha < 2\pi,\ 0 \le \beta < 2\pi$. Because of periodicity of sine and cosine, the result follows for any α and β.

Identity 3.2: $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Proof.

We know that $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$$\begin{aligned}\text{Now,}\quad \cos(\alpha-\beta) &= \cos[\alpha+(-\beta)]\\ &= \cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta)\\ \text{Hence, } \cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta.\end{aligned}$$

(i) If $\alpha=\beta$, the above identity is reduced to $\cos^2\alpha+\sin^2\alpha=1$.

(ii) If $\alpha=0$ and $\beta=x$, then $\cos(-x)=\cos x$, which shows that $\cos x$ is an even function.

Identity 3.3: $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

Proof. This formula may be proved by writing $\sin(\alpha+\beta)=\cos\left[\frac{\pi}{2}-(\alpha+\beta)\right]=\cos\left[\left(\frac{\pi}{2}-\alpha\right)-\beta\right]$ and by using Identity 3.2.

If $\alpha+\beta=\frac{\pi}{2}$, the above identity is reduced to $\cos^2\alpha+\sin^2\alpha=1$.

Identity 3.4: $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

Proof. This formula may be proved by writing $\sin(\alpha-\beta)=\sin[\alpha+(-\beta)]$ and by using Identity 3.3.

The sum and difference formulas for sine and cosine can be written in the matrix form

$$\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\cdot\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}=\begin{pmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{pmatrix}.$$

Identity 3.5: $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

Proof.

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

$$= \frac{\dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Identity 3.6: $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Proof. This result may be proved by writing $\tan(\alpha - \beta) = \tan[\alpha + (-\beta)]$ and using the Identity 3.5.

(i) Ptolemy (CE 100-170) treated the chord of an angle as his basic trigonometric function and proved the theorem: In a cyclic quadrilateral, the product of diagonals equals the sum of the products of opposite sides. That is, in a cyclic quadrilateral $ABCD$,

$$(AC)(BD) = (AB)(CD) + (AD)(BC).$$

Using this theorem, one can prove Sum and Difference identities. Hence, these identites are known as Ptolemy's sum and difference formulas.

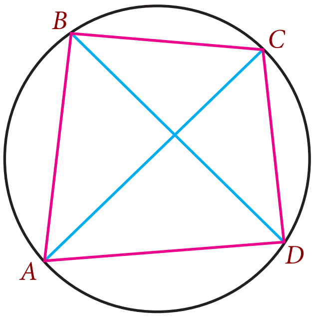


Figure 3.16

(ii) Observe that $\cos(\alpha \pm \beta) \neq \cos\alpha \pm \cos\beta$. Similarly we can observe for other trigonometric functions also.

(iii) When $\alpha = \beta$, $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ implies $\sin 0 = 0$, which we have already established.

(iv) When $\alpha = \dfrac{\pi}{2}$ and $\beta = \theta$, $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ gives $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$, which we have already proved.

(v) We can find the trigonometric function values of a given angle, if we can break it up into sum or difference of two of the special angles. For example, we can evaluate $\tan 75^\circ$ as $\tan(45^\circ + 30^\circ)$ and $\cos 135^\circ$ as $\cos(180^\circ - 45^\circ)$.

Example 3.15 Find the values of (i) $\cos 15^\circ$ and (ii) $\tan 165^\circ$.

Solution:

(i) Now, $\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Also, note that $\sin 75^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ [try yourself]

(ii) Now, $\tan 165^\circ = \tan(120^\circ + 45^\circ) = \dfrac{\tan 120^\circ + \tan 45^\circ}{1 - \tan 120^\circ \tan 45^\circ}$

But, $\tan 120^\circ = \tan(90^\circ + 30^\circ) = -\cot 30^\circ = -\sqrt{3}$ and $\tan 45^\circ = 1$

Thus, $\tan 165^\circ = \dfrac{1 - \sqrt{3}}{1 + \sqrt{3}}$.

Example 3.16 If $\sin x = \dfrac{4}{5}$ (in I quadrant) and $\cos y = \dfrac{-12}{13}$ (in II quadrant), then find (i) $\sin(x-y)$, (ii) $\cos(x-y)$.

Solution:
Given that $\sin x = \dfrac{4}{5}$.

$$\cos^2 x + \sin^2 x = 1 \text{ gives } \cos x = \pm\sqrt{1-\sin^2 x} = \pm\sqrt{1-\frac{16}{25}} = \pm\frac{3}{5}$$

In the first quadrant, $\cos x$ is always positive. Thus, $\cos x = \dfrac{3}{5}$.
Also, given that $\cos y = -\dfrac{12}{13}$ in the II quadrant. We have

$$\sin y = \pm\sqrt{1-\cos^2 y} = \pm\sqrt{1-\frac{144}{169}} = \pm\frac{5}{13}.$$

In the second quadrant, $\sin y$ is always positive. Thus, $\sin y = \dfrac{5}{13}$.

(i) $\sin(x-y) = \sin x\cos y - \cos x \sin y = \dfrac{4}{5}\left(\dfrac{-12}{13}\right) - \dfrac{3}{5}\left(\dfrac{5}{13}\right) = -\dfrac{63}{65}$.

(ii) $\cos(x-y) = \cos x \cos y + \sin x \sin y = \dfrac{3}{5}\left(\dfrac{-12}{13}\right) + \dfrac{4}{5}\left(\dfrac{5}{13}\right) = -\dfrac{16}{65}$.

Example 3.17 Prove that $\cos\left(\dfrac{3\pi}{4}+x\right) - \cos\left(\dfrac{3\pi}{4}-x\right) = -\sqrt{2}\sin x$

Solution:

$$\begin{aligned} \text{L.H.S} &= \cos\frac{3\pi}{4}\cos x - \sin\frac{3\pi}{4}\sin x - \cos\frac{3\pi}{4}\cos x - \sin\frac{3\pi}{4}\sin x \\ &= -2\sin\left(\pi - \frac{\pi}{4}\right)\sin x = -2\left(\frac{1}{\sqrt{2}}\right)\sin x = -\sqrt{2}\sin x. \end{aligned}$$

Observe that $\cos(A+x) - \cos(A-x) = -2\sin A \sin x$

Example 3.18 Point $A(9,12)$ rotates around the origin O in a plane through 60° in the anticlockwise direction to a new position B. Find the coordinates of the point B.

Solution:
Let $A(9,12) = A(r\cos\theta, r\sin\theta)$, where $r = OA$. Then $r\cos\theta = 9$ and $r\sin\theta = 12$.
Thus, $r^2 = 81 + 144 = 225 \Rightarrow r = 15$.
Hence, the point A is given by $A\,(15\cos\theta, 15\sin\theta)$.
Now, the point B is given by $B\,(15\cos(\theta+60^\circ), 15\sin(\theta+60^\circ))$.

$$\begin{aligned} 15\cos(\theta+60^\circ) &= 15\,(\cos\theta\cos 60^\circ - \sin\theta\sin 60^\circ) \\ &= (15\cos\theta)\cos 60^\circ - (15\sin\theta)\sin 60^\circ = 9\times\frac{1}{2} - 12\times\frac{\sqrt{3}}{2} = \frac{3}{2}(3-4\sqrt{3}) \end{aligned}$$

Similarly, $15\sin(\theta+60^\circ)=\frac{3}{2}(4+3\sqrt{3})$. Hence, the point B is given by

$$B\left(\frac{3}{2}(3-4\sqrt{3}),\frac{3}{2}(4+3\sqrt{3})\right).$$

Example 3.19 A ripple tank demonstrates the effect of two water waves being added together. The two waves are described by $h=8\cos t$ and $h=6\sin t$, where $t\in[0,2\pi)$ is in seconds and h is the height in millimeters above still water. Find the maximum height of the resultant wave and the value of t at which it occurs.

Solution:

Let H be the height of the resultant wave at time t. Then H is given by

$$H=8\cos t+6\sin t$$

$$\begin{aligned}
&\text{Let } 8\cos t+6\sin t=k\cos(t-\alpha)=k(\cos t\cos\alpha+\sin t\sin\alpha)\\
&\text{Hence, } k=10 \text{ and } \tan\alpha=\frac{3}{4}, \text{ so that}\\
&H=10\cos(t-\alpha)
\end{aligned}$$

Thus, the maximum of $H=10$ mm. The maximum occurs when $t=\alpha$, where $\tan\alpha=\frac{3}{4}$.

Example 3.20 Expand (i) $\sin(A+B+C)$ (ii) $\tan(A+B+C)$

Solution:

$$\begin{aligned}
\text{(i) } \sin(A+B+C) &= \sin[A+(B+C)]\\
&= \sin A\cos(B+C)+\cos A\sin(B+C)\\
&= \sin A\cos B\cos C+\cos A\sin B\cos C\\
&\quad +\cos A\cos B\sin C-\sin A\sin B\sin C\\
\text{(ii) } \tan(A+B+C) &= \tan[A+(B+C)]\\
&= \frac{\tan A+\tan(B+C)}{1-\tan A\tan(B+C)}\\
&= \frac{\tan A+\dfrac{\tan B+\tan C}{1-\tan B\tan C}}{1-\tan A\dfrac{\tan B+\tan C}{1-\tan B\tan C}}\\
&= \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}
\end{aligned}$$

(i) If A+B+C = 0 or π, we have $\tan(A+B+C)=0$ so that

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

This result is also true in the case of oblique triangles.

(ii) $\tan(x-y)+\tan(y-z)+\tan(z-x)=\tan(x-y)\tan(y-z)\tan(z-x)$

(iii) $\tan 3A - \tan 2A - \tan A = \tan 3A + \tan(-2A) + \tan(-A) = \tan 3A \tan 2A \tan A$

(iv) If $A + B + C = \dfrac{\pi}{2}$, then $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$ (How!).

Exercise - 3.4

1. If $\sin x = \dfrac{15}{17}$ and $\cos y = \dfrac{12}{13}$, $0 < x < \dfrac{\pi}{2}$, $0 < y < \dfrac{\pi}{2}$,
   find the value of (i) $\sin(x + y)$ (ii) $\cos(x - y)$ (iii) $\tan(x + y)$.
2. If $\sin A = \dfrac{3}{5}$ and $\cos B = \dfrac{9}{41}$, $0 < A < \dfrac{\pi}{2}$, $0 < B < \dfrac{\pi}{2}$,
   find the value of (i) $\sin(A + B)$ (ii) $\cos(A - B)$.
3. Find $\cos(x - y)$, given that $\cos x = -\dfrac{4}{5}$ with $\pi < x < \dfrac{3\pi}{2}$ and $\sin y = -\dfrac{24}{25}$ with $\pi < y < \dfrac{3\pi}{2}$.
4. Find $\sin(x - y)$, given that $\sin x = \dfrac{8}{17}$ with $0 < x < \dfrac{\pi}{2}$ and $\cos y = -\dfrac{24}{25}$ with $\pi < y < \dfrac{3\pi}{2}$.
5. Find the value of (i) $\cos 105^\circ$ (ii) $\sin 105^\circ$ (iii) $\tan \dfrac{7\pi}{12}$.
6. Prove that (i) $\cos(30^\circ + x) = \dfrac{\sqrt{3}\cos x - \sin x}{2}$ (ii) $\cos(\pi + \theta) = -\cos\theta$
   (iii) $\sin(\pi + \theta) = -\sin\theta$.
7. Find a quadratic equation whose roots are $\sin 15^\circ$ and $\cos 15^\circ$.
8. Expand $\cos(A + B + C)$. Hence prove that

   $\cos A \cos B \cos C = \sin A \sin B \cos C + \sin B \sin C \cos A + \sin C \sin A \cos B$,
   $if\ A + B + C = \dfrac{\pi}{2}$.
9. Prove that

   (i) $\sin(45^\circ + \theta) - \sin(45^\circ - \theta) = \sqrt{2}\sin\theta$.
   (ii) $\sin(30^\circ + \theta) + \cos(60^\circ + \theta) = \cos\theta$.
10. If $a\cos(x + y) = b\cos(x - y)$, show that $(a + b)\tan x = (a - b)\cot y$.
11. Prove that $\sin 105^\circ + \cos 105^\circ = \cos 45^\circ$.
12. Prove that $\sin 75^\circ - \sin 15^\circ = \cos 105^\circ + \cos 15^\circ$.
13. Show that $\tan 75^\circ + \cot 75^\circ = 4$.
14. Prove that $\cos(A + B)\cos C - \cos(B + C)\cos A = \sin B \sin(C - A)$.
15. Prove that $\sin(n + 1)\theta \sin(n - 1)\theta + \cos(n + 1)\theta \cos(n - 1)\theta = \cos 2\theta,\ n \in \mathbb{Z}$.
16. If $x\cos\theta = y\cos\left(\theta + \dfrac{2\pi}{3}\right) = z\cos\left(\theta + \dfrac{4\pi}{3}\right)$, find the value of $xy + yz + zx$.
17. Prove that

    (i) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$
    (ii) $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
    (iii) $\sin^2(A + B) - \sin^2(A - B) = \sin 2A \sin 2B$
    (iv) $\cos 8\theta \cos 2\theta = \cos^2 5\theta - \sin^2 3\theta$
18. Show that $\cos^2 A + \cos^2 B - 2\cos A \cos B \cos(A + B) = \sin^2(A + B)$.
19. If $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\dfrac{3}{2}$, then prove that
    $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$.

20. Show that

(i) $\tan(45^\circ + A) = \dfrac{1+\tan A}{1-\tan A}$ (ii) $\tan(45^\circ - A) = \dfrac{1-\tan A}{1+\tan A}$.

21. Prove that $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$.

22. If $\tan x = \dfrac{n}{n+1}$ and $\tan y = \dfrac{1}{2n+1}$, find $\tan(x+y)$.

23. Prove that $\tan\left(\dfrac{\pi}{4}+\theta\right)\tan\left(\dfrac{3\pi}{4}+\theta\right) = -1$.

24. Find the values of $\tan(\alpha+\beta)$, given that $\cot\alpha = \dfrac{1}{2}, \alpha \in \left(\pi, \dfrac{3\pi}{2}\right)$ and $\sec\beta = -\dfrac{5}{3}, \beta \in \left(\dfrac{\pi}{2}, \pi\right)$.

25. If $\theta + \phi = \alpha$ and $\tan\theta = k\tan\phi$, then prove that $\sin(\theta - \phi) = \dfrac{k-1}{k+1}\sin\alpha$.

3.5.2 Multiple angle identities and submultiple angle identities

In 1831, Michael Faraday discovered that when a wire is passed near a magnet, a small electric current is produced in the wire. This property is used to generate electric current for houses, institutions and business establishments throughout the world. By rotating thousands of wires near large electromagnets, massive amount of electricity can be produced.

Voltage is a quantity that can be modeled by sinusoidal graphs and functions. To model electricity and other phenomena, trigonometric functions and identities involving multiple angles or sub multiple angles are used.

If A is an angle, then $2A, 3A, \ldots$ are called **multiple angles** of A and the angle $\dfrac{A}{2}, \dfrac{A}{3}, \ldots$are called **sub-multiple angles** of A. Now we shall discuss the trigonometric ratio of multiple angles and sub-multiple angles and derive some identities.

Double Angle Identities

Let us take up the sum and difference identities and examine some of the consequences that come from them. Double angle identities are a special case of the sum identities. That is, when the two angles are equal, the sum identities are reduced to double angle identities. They are useful in solving trigonometric equations and also in the verification of trigonometric identities. Further double angle identities can be used to derive the reduction identities (power reducing identities). Also double angle identities are used to find maximum or minimum values of trigonometric expressions.

Identity 3.7: $\sin 2A = 2\sin A\cos A$

Proof.

We know that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Taking $\alpha = \beta = A$, we have $\sin(A+A) = \sin A\cos A + \sin A\cos A$

Thus, $\sin 2A = 2\sin A\cos A$.

(i) $y = \sin 2x$ and $y = 2\sin x$ are different. Draw their graphs and identify the difference.

(ii) Application of $\sin 2A = 2\sin A\cos A$: When an object is projected with speed u at an angle α to the horizontal over level ground, the horizontal distance (Range) it travels before striking the ground is given by the formula $R = \dfrac{u^2\sin 2\alpha}{g}$.

Clearly maximum of R is $\dfrac{u^2}{g}$, when $\alpha = \dfrac{\pi}{4}$.

(iii) $|\sin A\cos A| = \left|\dfrac{\sin 2A}{2}\right| \leq \dfrac{1}{2}$.

Thus, $-\dfrac{1}{2} \leq \sin A \cos A \leq \dfrac{1}{2}$.

From this, we infer that the maximum value of $\sin A \cos A$ is $\dfrac{1}{2}$ when $A = \dfrac{\pi}{4}$.

Example 3.21 A foot ball player can kick a football from ground level with an initial velocity of 80 ft/second. Find the maximum horizontal distance the football travels and at what angle? (Take $g = 32$).

Solution:

The formula for horizontal distance R is given by

$$R = \frac{u^2 \sin 2\alpha}{g} = \frac{(80 \times 80)\sin 2\alpha}{32} = 10 \times 20 \sin 2\alpha.$$

Thus, the maximum distance is 200 ft.

Hence, he has to kick the football at an angle of $\alpha = 45°$ to reach the maximum distance.

Identity 3.8: $\cos 2A = \cos^2 A - \sin^2 A$

Proof.

$$\begin{aligned} \text{We know that } \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \text{Take } \alpha = \beta = A. \text{ We have } \cos(A+A) &= \cos A \cos A - \sin A \sin A \\ \cos 2A &= \cos^2 A - \sin^2 A. \end{aligned}$$

From the identity $\cos 2A = \cos^2 A - \sin^2 A$, we also have

$$\begin{aligned} \cos 2A &= \cos^2 A - \left(1 - \cos^2 A\right) = 2\cos^2 A - 1 \text{ and} \\ \cos 2A &= (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A. \end{aligned}$$

Identity 3.9: $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

Proof.

$$\text{Now, } \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\begin{aligned} \text{Take } \alpha = \beta = A. \text{ We have, } \tan(A+A) &= \frac{\tan A + \tan A}{1 - \tan A \tan A} \\ \tan 2A &= \frac{2\tan A}{1 - \tan^2 A}. \end{aligned}$$

Identity 3.10: $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$

Proof.

$$\begin{aligned} \text{we know that } \sin 2A &= 2\sin A \cos A = \frac{2\sin A\cos A}{\sin^2 A + \cos^2 A} \\ &= \frac{\dfrac{2\sin A\cos A}{\cos^2 A}}{\dfrac{\sin^2 A + \cos^2 A}{\cos^2 A}} = \frac{2\tan A}{1 + \tan^2 A}. \end{aligned}$$

Identity 3.11: $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

Proof.

$$\begin{aligned}
\text{We know that } \cos 2A &= \cos^2 A - \sin^2 A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} \\
&= \frac{\dfrac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\dfrac{\cos^2 A + \sin^2 A}{\cos^2 A}} \\
\text{Thus, } \cos 2A &= \frac{1-\tan^2 A}{1+\tan^2 A}.
\end{aligned}$$

Power reducing identities or Reduction identities

Power reducing identities for sine, cosine and tangent can be derived using the double-angle identities. For example,

$$\cos 2A = 2\cos^2 A - 1 \Rightarrow \cos^2 A = \frac{1+\cos 2A}{2}$$

The following table is the list of power reducing identities.

Power Reducing Identities
$\sin^2 A = \dfrac{1-\cos 2A}{2}$, $\quad \cos^2 A = \dfrac{1+\cos 2A}{2}$, $\quad \tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A}$

(i) In the power reducing identities, we have reduced the square power on one side to power 1 on the other side.

(ii) Power reducing identities allow us to rewrite the even powers of sine or cosine in terms of the first power of cosine.

For example, using power reducing identities one can easily prove that

$$\cos^4 x = \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8} \text{ and } \sin^4 x = \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8} \text{ (Try it!).}$$

(iii) Power reducing formulas are important in higher level mathematics.

Triple-Angle Identities

Using double angle identities, we can derive triple angle identities.

Identity 3.12: $\sin 3A = 3\sin A - 4\sin^3 A$

Proof.

$$\begin{aligned}
\text{We have, } \sin 3A &= \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A \\
&= 2\sin A\cos^2 A + \left(1 - 2\sin^2 A\right)\sin A \\
&= 2\sin A\left(1 - \sin^2 A\right) + \left(1 - 2\sin^2 A\right)\sin A \\
&= 3\sin A - 4\sin^3 A
\end{aligned}$$

Identity 3.13: $\cos 3A = 4\cos^3 A - 3\cos A$

Proof.

$$\begin{aligned}\text{We have, } \cos 3A &= \cos(2A + A) = \cos 2A \cos A - \sin 2A \sin A \\ &= \left(2\cos^2 A - 1\right)\cos A - 2\sin A \cos A \sin A \\ &= \left(2\cos^2 A - 1\right)\cos A - 2\cos A\left(1 - \cos^2 A\right) \\ &= 4\cos^3 A - 3\cos A.\end{aligned}$$

Identity 3.14: $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Proof.

$$\begin{aligned}\text{We have, } \tan 3A &= \tan(2A + A) \\ &= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} \\ &= \frac{\dfrac{2\tan A}{1 - \tan^2 A} + \tan A}{1 - \dfrac{2\tan A}{1 - \tan^2 A}\tan A} \\ &= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.\end{aligned}$$

Double and Triple angle identities are given below:

sine	cosine	Tangent
$\sin 2A = 2\sin A \cos A$ $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$ $\sin 3A = 3\sin A - 4\sin^3 A$	$\cos 2A = \cos^2 A - \sin^2 A$ $\cos 2A = 2\cos^2 A - 1$ $\cos 2A = 1 - 2\sin^2 A$ $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$ $\cos 3A = 4\cos^3 A - 3\cos A$	$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$ $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Half-Angle Identities

Half angle identities are closely related to the double angle identities. We can use half angle identities when we have an angle that is half the size of a special angle. For example, $\sin 15^\circ$ can be computed by writing $\sin 15^\circ = \sin \dfrac{30^\circ}{2}$. Also one can find exact values for some angles using half-angle identities. If we put $2A = \theta$ or $A = \dfrac{\theta}{2}$ in the double angle identities, we get new identities in terms of angle $\dfrac{\theta}{2}$.

Let us list out the half angle identities in the following table:

Double angle identity	Half-angle identity
$\sin 2A = 2\sin A\cos A$	$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$
$\cos 2A = \cos^2 A - \sin^2 A$	$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$
$\cos 2A = 2\cos^2 A - 1$	$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$
$\cos 2A = 1 - 2\sin^2 A$	$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$
$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$	$\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$
$\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$	$\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$
$\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$	$\cos\theta = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(i) The half angle identities are often used to replace a squared trigonometric function by a non squared trigonometric function.

(ii) Half angle identities allow us to find the value of the sine and cosine of half the angle if we know the value of the cosine of the original angle.

Example 3.22 Find the value of $\sin\left(22\frac{1}{2}^\circ\right)$

Solution:

We know that $\cos\theta = 1 - 2\sin^2\frac{\theta}{2} \Rightarrow \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$. Take $\theta = 45^\circ$

we get $\sin\frac{45^\circ}{2} = \pm\sqrt{\frac{1-\cos 45^\circ}{2}}$, (taking positive sign only, since $22\frac{1}{2}^\circ$ lies in the first quadrant)

$$\text{Thus, } \sin 22\frac{1}{2}^\circ = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

Example 3.23 Find the value of $\sin 2\theta$, when $\sin\theta = \frac{12}{13}$, θ lies in the first quadrant.

Solution:

Using a right triangle, we can easily find that $\cos\theta = \frac{5}{13}$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{12}{13}\right)\left(\frac{5}{13}\right) = \frac{120}{169}.$$

Instead of constructing the triangle, we can also find the value of $\cos\theta$ using $\cos\theta = \pm\sqrt{1-\sin^2\theta}$ formula.

Example 3.24 Prove that $\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A$

Solution:

$$\begin{aligned}4\sin A\cos^3 A - 4\cos A\sin^3 A &= 4\sin A\cos A\left(\cos^2 A - \sin^2 A\right)\\ &= 4\sin A\cos A\cos 2A = 2(2\sin A\cos A)\cos 2A\\ &= 2(\sin 2A)\cos 2A = \sin 4A.\end{aligned}$$

Example 3.25 Prove that $\sin x = 2^{10}\sin\left(\frac{x}{2^{10}}\right)\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2^2}\right)\ldots\cos\left(\frac{x}{2^{10}}\right)$

Solution:

$$\begin{aligned}\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\times 2\times\sin\frac{x}{2^2}\cos\frac{x}{2^2}\cos\frac{x}{2}\\ &= 2^2\sin\frac{x}{2^2}\cos\frac{x}{2}\cos\frac{x}{2^2}\end{aligned}$$

Applying repeatedly the half angle sine formula, we get

$$\sin x = 2^{10}\sin\left(\frac{x}{2^{10}}\right)\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2^2}\right)\ldots\cos\left(\frac{x}{2^{10}}\right).$$

The above result can be extended to any finite number of times.

Example 3.26 Prove that $\dfrac{\sin\theta + \sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$

Solution:

We have $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta} = \dfrac{\sin\theta + 2\sin\theta\cos\theta}{\cos\theta + (1+\cos 2\theta)} = \dfrac{\sin\theta(1+2\cos\theta)}{\cos\theta(1+2\cos\theta)} = \tan\theta.$

Example 3.27 Prove that $1 - \frac{1}{2}\sin 2x = \dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x}$

Solution:

$$\begin{aligned}\text{We have }\frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} &= \frac{(\sin x + \cos x)\left(\sin^2 x - \sin x\cos x + \cos^2 x\right)}{\sin x + \cos x}\\ &= 1 - \sin x\cos x = 1 - \frac{1}{2}\sin 2x.\end{aligned}$$

Example 3.28 Find x such that $-\pi \leq x \leq \pi$ and $\cos 2x = \sin x$

Solution:

We have $\cos 2x = \sin x$ which gives
$2\sin^2 x + \sin x - 1 = 0$. The roots of the equation are
$$\sin x = \frac{-1 \pm 3}{4} = -1 \text{ or } \frac{1}{2}$$
Now, $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

Also $\sin x = -1 \Rightarrow x = -\frac{\pi}{2}$

Thus, $x = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$.

Example 3.29 Find the values of (i) $\sin 18^\circ$ (ii) $\cos 18^\circ$ (iii) $\sin 72^\circ$ (iv) $\cos 36^\circ$ (v) $\sin 54^\circ$

Solution:

(i) Let $\theta = 18^\circ$. Then $5\theta = 90^\circ$

$3\theta + 2\theta = 90^\circ \Rightarrow 2\theta = 90^\circ - 3\theta$

$\sin 2\theta = \sin(90^\circ - 3\theta) = \cos 3\theta$

$2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$. Since $\cos\theta = \cos 18^\circ \neq 0$, we have

$2\sin\theta = 4\cos^2\theta - 3 = 4(1-\sin^2\theta) - 3$

$4\sin^2\theta + 2\sin\theta - 1 = 0$
$$\sin\theta = \frac{-2 \pm \sqrt{4 - 4(4)(-1)}}{2(4)} = \frac{-1 \pm \sqrt{5}}{4}$$
Thus, $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$ (positive sign is taken. Why ?)

(ii) $\cos 18^\circ = \sqrt{1 - \sin^2 18^\circ}$
$$= \sqrt{1 - \left[\frac{\sqrt{5}-1}{4}\right]^2} = \frac{1}{4}\sqrt{16 - (5 + 1 - 2\sqrt{5})} = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$$
(iii) $\sin 72^\circ = \sin(90^\circ - 18^\circ) = \cos 18^\circ = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$

(iv) $\cos 36^\circ = 1 - 2\sin^2 18^\circ = 1 - 2\left[\frac{\sqrt{5}-1}{4}\right]^2 = \frac{\sqrt{5}+1}{4}$

(v) $\sin 54^\circ = \sin(90^\circ - 36^\circ) = \cos 36^\circ = \frac{\sqrt{5}+1}{4}$.

Observe that $\sin 18^\circ = \cos 72^\circ$, $\cos 18^\circ = \sin 72^\circ$ and $\cos 36^\circ = \sin 54^\circ$

Example 3.30 If $\tan\frac{\theta}{2} = \sqrt{\frac{1-a}{1+a}}\tan\frac{\phi}{2}$, then prove that $\cos\phi = \frac{\cos\theta - a}{1 - a\cos\theta}$.

Solution:
By the half-angle identity, we have

$$\cos\phi = \frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}} = \frac{1-\left(\frac{1+a}{1-a}\right)\tan^2\frac{\theta}{2}}{1+\left(\frac{1+a}{1-a}\right)\tan^2\frac{\theta}{2}}$$

$$= \frac{\left(\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)-a}{1-a\left(\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)} = \frac{\cos\theta - a}{1-a\cos\theta}.$$

Example 3.31 Find the value of $\sqrt{3}\operatorname{cosec} 20^\circ - \sec 20^\circ$

Solution:

We have $$\sqrt{3}\operatorname{cosec} 20^\circ - \sec 20^\circ = \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ} = 4\left[\frac{\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ}{2\sin 20^\circ \cos 20^\circ}\right]$$

$$= 4\left[\frac{\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ}{\sin 40^\circ}\right] = 4.$$

Example 3.32 Prove that $\cos A \cos 2A \cos 2^2A \cos 2^3A \ldots \cos 2^{n-1}A = \dfrac{\sin 2^nA}{2^n \sin A}$

Solution:

$$\begin{aligned}
\text{L.H.S.} &= \cos A \cos 2A \cos 2^2A \cos 2^3A \ldots \cos 2^{n-1}A \\
&= \frac{1}{2\sin A} 2\sin A \cos A \cos 2A \cos 2^2A \cos 2^3A \ldots \cos 2^{n-1}A \\
&= \frac{1}{2\sin A} \sin 2A \cos 2A \cos 2^2A \cos 2^3A \ldots \cos 2^{n-1}A \\
&= \frac{1}{2^2\sin A} \sin 4A \cos 2^2A \cos 2^3A \ldots \cos 2^{n-1}A
\end{aligned}$$

Continuing the process, we get

$$= \frac{\sin 2^nA}{2^n \sin A}.$$

Exercise - 3.5

1. Find the value of $\cos 2A$, A lies in the first quadrant, when

   (i) $\cos A = \frac{15}{17}$ (ii) $\sin A = \frac{4}{5}$ (iii) $\tan A = \frac{16}{63}$.

2. If θ is an acute angle, then find

(i) $\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$, when $\sin\theta=\frac{1}{25}$.

(ii) $\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, when $\sin\theta=\frac{8}{9}$.

3. If $\cos\theta=\frac{1}{2}\left(a+\frac{1}{a}\right)$, show that $\cos 3\theta=\frac{1}{2}\left(a^3+\frac{1}{a^3}\right)$.
4. Prove that $\cos 5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta$.
5. Prove that $\sin 4\alpha=4\tan\alpha\frac{1-\tan^2\alpha}{(1+\tan^2\alpha)^2}$.
6. If $A+B=45^\circ$, show that $(1+\tan A)(1+\tan B)=2$.
7. Prove that $(1+\tan 1^\circ)(1+\tan 2^\circ)(1+\tan 3^\circ)\ldots(1+\tan 44^\circ)$ is a multiple of 4.
8. Prove that $\tan\left(\frac{\pi}{4}+\theta\right)-\tan\left(\frac{\pi}{4}-\theta\right)=2\tan 2\theta$.
9. Show that $\cot\left(7\frac{1}{2}^\circ\right)=\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$.
10. Prove that $(1+\sec 2\theta)(1+\sec 4\theta)\ldots(1+\sec 2^n\theta)=\tan 2^n\theta\ \cot\theta$.
11. Prove that $32(\sqrt{3})\sin\frac{\pi}{48}\cos\frac{\pi}{48}\cos\frac{\pi}{24}\cos\frac{\pi}{12}\cos\frac{\pi}{6}=3$.

3.5.3 Product to Sum and Sum to Product Identities

Some applications of trigonometric functions demand that a product of trigonometric functions be written as sum or difference of trigonometric functions. The sum and difference identities for the cosine and sine functions look amazingly like each other except for the sign in the middle. So, we tend to combine them to get nice identities. Thus, we use them to derive several identities that make it possible to rewrite a product as a sum or a sum as a product.

We know that

$$\sin(A+B) = \sin A\cos B+\cos A\sin B \tag{3.1}$$
$$\sin(A-B) = \sin A\cos B-\cos A\sin B \tag{3.2}$$
$$\cos(A+B) = \cos A\cos B-\sin A\sin B \tag{3.3}$$
$$\cos(A-B) = \cos A\cos B+\sin A\sin B \tag{3.4}$$

From the above identities, we can easily derive the following Product to Sum identities.

$$\sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)] \tag{3.5}$$
$$\cos A\sin B = \frac{1}{2}[\sin(A+B)-\sin(A-B)] \tag{3.6}$$
$$\cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)] \tag{3.7}$$
$$\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)] \tag{3.8}$$

The above identities are very important whenever need arises to transform the product of sine and cosine into sum. This idea is very much useful in evaluation of some integrals.

To get Sum to Product identities, let us introduce the substitutions $A+B=C$ and $A-B=D$ or equivalently $A=\frac{C+D}{2}$, $B=\frac{C-D}{2}$ in the product to sum identities (3.5) to (3.8). Then, we have the following Sum to Product identities

$$\sin C+\sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2} \tag{3.9}$$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2} \tag{3.10}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2} \tag{3.11}$$

$$\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2} \tag{3.12}$$

Identity 3.15: Prove that $\sin(60° - A)\sin A\sin(60° + A) = \frac{1}{4}\sin 3A$

Proof.

$$\begin{aligned}
\sin(60° - A)\sin A\sin(60° + A) &= \sin(60° - A)\sin(60° + A)\sin A \\
&= \frac{1}{2}\left[\cos 2A - \cos 120°\right]\sin A \\
&= \frac{1}{2}\left[\cos 2A\sin A + \frac{1}{2}\sin A\right] \\
&= \frac{1}{2}\left[\frac{1}{2}\sin 3A\right] = \frac{1}{4}\sin 3A
\end{aligned}$$

Similarly we can prove the following two important identities

Identity 3.16: $\cos(60° - A)\cos A\cos(60° + A) = \frac{1}{4}\cos 3A$

Identity 3.17: $\tan(60° - A)\tan A\tan(60° + A) = \tan 3A$

Example 3.33 Express each of the following product as a sum or difference
(i) $\sin 40°\cos 30°$ (ii) $\cos 110°\sin 55°$ (iii) $\sin\frac{x}{2}\cos\frac{3x}{2}$

Solution:

(i) We know that $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

Take $A = 40°$and $B = 30°$. We get,

$2\sin 40°\cos 30° = \sin(40° + 30°) + \sin(40° - 30°) = \sin 70° + \sin 10°$.

Thus, $\sin 40°\cos 30° = \frac{1}{2}\left[\sin 70° + \sin 10°\right]$.

(ii) We know that $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

Take $A = 110°$ and $B = 55°$. We get,

$2\cos 110°\sin 55° = \sin(110° + 55°) - \sin(110° - 55°)$.

Thus, $\cos 110°\sin 55° = \frac{1}{2}\left[\sin 165° - \sin 55°\right]$

(iii) We know that $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

Take $A = \frac{x}{2}$ and $B = \frac{3x}{2}$

We get, $2\sin\frac{x}{2}\cos\frac{3x}{2} = \sin\left(\frac{x}{2} + \frac{3x}{2}\right) + \sin\left(\frac{x}{2} - \frac{3x}{2}\right)$

Thus, $\sin\frac{x}{2}\cos\frac{3x}{2} = \frac{1}{2}\left[\sin 2x - \sin x\right]$.

Example 3.34 Express each of the following sum or difference as a product

(i) $\sin 50^\circ + \sin 20^\circ$ (ii) $\cos 6\theta + \cos 2\theta$ (iii) $\cos\dfrac{3x}{2} - \cos\dfrac{9x}{2}$.

Solution:

(i) We know that $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

Take $C = 50^\circ$ and $D = 20^\circ$. We have

$$\sin 50^\circ + \sin 20^\circ = 2\sin\frac{50^\circ + 20^\circ}{2}\cos\frac{50^\circ - 20^\circ}{2} = 2\sin 35^\circ\cos 15^\circ$$

(ii) We know that $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

Take $C = 6\theta$ and $D = 2\theta$. We have

$$\cos 6\theta + \cos 2\theta = 2\cos\frac{6\theta + 2\theta}{2}\cos\frac{6\theta - 2\theta}{2} = 2\cos 4\theta\cos 2\theta$$

(iii) We know that $\cos C - \cos D = 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$

Take $C = \dfrac{3x}{2}$ and $D = \dfrac{9x}{2}$. We have

$$\cos\frac{3x}{2} - \cos\frac{9x}{2} = 2\sin\frac{\frac{3x}{2} + \frac{9x}{2}}{2}\sin\frac{\frac{9x}{2} - \frac{3x}{2}}{2} = 2\sin 3x\sin\frac{3x}{2}.$$

Example 3.35 Find the value of $\sin 34^\circ + \cos 64^\circ - \cos 4^\circ$.

Solution:

We have
$$\begin{aligned}\sin 34^\circ + \cos 64^\circ - \cos 4^\circ &= \sin 34^\circ - 2\sin\left(\frac{64^\circ + 4^\circ}{2}\right)\sin\left(\frac{64^\circ - 4^\circ}{2}\right)\\ &= \sin 34^\circ - 2\sin 34^\circ\sin 30^\circ = 0.\end{aligned}$$

Example 3.36 Show that $\cos 36^\circ\cos 72^\circ\cos 108^\circ\cos 144^\circ = \dfrac{1}{16}$.

Solution:

$$\begin{aligned}\text{L.H.S.} &= \cos 36^\circ\cos(90^\circ - 18^\circ)\cos(90^\circ + 18^\circ)\cos(180^\circ - 36^\circ)\\ &= \sin^2 18^\circ\cos^2 36^\circ\\ &= \left(\frac{\sqrt{5}-1}{4}\right)^2\left(\frac{\sqrt{5}+1}{4}\right)^2 = \frac{1}{16}.\end{aligned}$$

Example 3.37 Simplify $\dfrac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ}$

Solution:

We have
$$\frac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \frac{2\cos\left(\frac{75^\circ + 15^\circ}{2}\right)\sin\left(\frac{75^\circ - 15^\circ}{2}\right)}{2\cos\left(\frac{75^\circ + 15^\circ}{2}\right)\cos\left(\frac{75^\circ - 15^\circ}{2}\right)}$$
$$= \frac{2\cos 45^\circ \sin 30^\circ}{2\cos 45^\circ \cos 30^\circ} = \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

Try to solve using $\sin 75^\circ = \cos 15^\circ$ and $\cos 75^\circ = \sin 15^\circ$

Example 3.38 Show that $\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ = \dfrac{3}{16}$.

Solution:

We know that $\cos(60^\circ - A)\cos A\cos(60^\circ + A) = \dfrac{1}{4}\cos 3A$

$$\begin{aligned}\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ &= \cos 30^\circ\,[\cos 10^\circ \cos 50^\circ \cos 70^\circ]\\ &= \cos 30^\circ\,[\cos(60^\circ - 10^\circ)\cos 10^\circ \cos(60^\circ + 10^\circ)]\\ &= \frac{\sqrt{3}}{2}\left[\frac{1}{4}\cos 30^\circ\right] = \frac{\sqrt{3}}{2}\left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{16}\end{aligned}$$

Exercise - 3.6

1. Express each of the following as a sum or difference

   (i) $\sin 35^\circ \cos 28^\circ$ (ii) $\sin 4x \cos 2x$ (iii) $2\sin 10\theta \cos 2\theta$ (iv) $\cos 5\theta \cos 2\theta$ (v) $\sin 5\theta \sin 4\theta$.

2. Express each of the following as a product

   (i) $\sin 75^\circ - \sin 35^\circ$ (ii) $\cos 65^\circ + \cos 15^\circ$ (iii) $\sin 50^\circ + \sin 40^\circ$ (iv) $\cos 35^\circ - \cos 75^\circ$.

3. Show that $\sin 12^\circ \sin 48^\circ \sin 54^\circ = \dfrac{1}{8}$.
4. Show that $\cos\dfrac{\pi}{15}\cos\dfrac{2\pi}{15}\cos\dfrac{3\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{5\pi}{15}\cos\dfrac{6\pi}{15}\cos\dfrac{7\pi}{15} = \dfrac{1}{128}$.
5. Show that $\dfrac{\sin 8x\cos x - \sin 6x\cos 3x}{\cos 2x\cos x - \sin 3x\sin 4x} = \tan 2x$.
6. Show that $\dfrac{(\cos\theta - \cos 3\theta)(\sin 8\theta + \sin 2\theta)}{(\sin 5\theta - \sin\theta)(\cos 4\theta - \cos 6\theta)} = 1$.
7. Prove that $\sin x + \sin 2x + \sin 3x = \sin 2x\,(1 + 2\cos x)$.
8. Prove that $\dfrac{\sin 4x + \sin 2x}{\cos 4x + \cos 2x} = \tan 3x$.
9. Prove that $1 + \cos 2x + \cos 4x + \cos 6x = 4\cos x\cos 2x\cos 3x$.

10. prove that $\sin\frac{\theta}{2}\sin\frac{7\theta}{2}+\sin\frac{3\theta}{2}\sin\frac{11\theta}{2}=\sin 2\theta\sin 5\theta$.
11. Prove that $\cos(30^\circ-A)\cos(30^\circ+A)+\cos(45^\circ-A)\cos(45^\circ+A)=\cos 2A+\frac{1}{4}$.
12. Prove that $\dfrac{\sin x+\sin 3x+\sin 5x+\sin 7x}{\cos x+\cos 3x+\cos 5x+\cos 7x}=\tan 4x$.
13. Prove that $\dfrac{\sin(4A-2B)+\sin(4B-2A)}{\cos(4A-2B)+\cos(4B-2A)}=\tan(A+B)$.
14. Show that $\cot(A+15^\circ)-\tan(A-15^\circ)=\dfrac{4\cos 2A}{1+2\sin 2A}$.

3.5.4 Conditional Trigonometric Identities

We know that trigonometric identities are true for all admissible values of the angle involved. There are some trigonometric identities which satisfy the given additional conditions. Such identities are called conditional trigonometric identities.

In this section, we shall make use of the relations obtained in the earlier sections to establish some conditional identities based on some relationship.

Example 3.39 If $A+B+C=\pi$, prove the following

(i) $\cos A+\cos B+\cos C=1+4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

(ii) $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\leq\frac{1}{8}$

(iii) $1<\cos A+\cos B+\cos C\leq\frac{3}{2}$

Solution:

(i)
$$\begin{aligned}
\cos A+\cos B+\cos C &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C\\
&= 2\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\left(\frac{A}{2}-\frac{B}{2}\right)+\cos C \qquad \left(\frac{A+B+C}{2}=\frac{\pi}{2}\right)\\
&= 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{A}{2}-\frac{B}{2}\right)+1-2\sin^2\left(\frac{C}{2}\right)\\
&= 1+2\sin\left(\frac{C}{2}\right)\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\sin\left(\frac{C}{2}\right)\right]\\
&= 1+2\sin\left(\frac{C}{2}\right)\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\right]\\
&= 1+2\sin\left(\frac{C}{2}\right)\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\cos\left(\frac{A}{2}+\frac{B}{2}\right)\right]\\
&= 1+4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)
\end{aligned}$$

(ii) Let
$$\begin{aligned}
u &= \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\\
&= -\frac{1}{2}\left[\cos\left(\frac{A+B}{2}\right)-\cos\left(\frac{A-B}{2}\right)\right]\sin\frac{C}{2}\\
&= -\frac{1}{2}\left[\cos\left(\frac{A+B}{2}\right)-\cos\left(\frac{A-B}{2}\right)\right]\cos\frac{A+B}{2}
\end{aligned}$$

$\cos^2 \dfrac{A+B}{2} - \cos \dfrac{A-B}{2} \cos \dfrac{A+B}{2} + 2u = 0$, which is a quadratic in $\cos \dfrac{A+B}{2}$.

Since $\cos \dfrac{A+B}{2}$ is real number, the above equation has a solution.

Thus, the discriminant $b^2 - 4ac \geq 0$, which gives

$\cos^2 \dfrac{A-B}{2} - 8u \geq 0 \Rightarrow u \leq \dfrac{1}{8} \cos^2 \dfrac{A-B}{2} \leq \dfrac{1}{8}$

Hence, $\sin\left(\dfrac{A}{2}\right) \sin\left(\dfrac{B}{2}\right) \sin\left(\dfrac{C}{2}\right) \leq \dfrac{1}{8}$

(iii) From (i) and (ii), we have $\cos A + \cos B + \cos C > 1$ and

$\cos A + \cos B + \cos C \leq 1 + 4 \times \dfrac{1}{8}$

Thus, we get $1 < \cos A + \cos B + \cos C \leq \dfrac{3}{2}$.

$\sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2} > 0$, if $A + B + C = \pi$

Example 3.40 Prove that

$\sin \dfrac{A}{2} + \sin \dfrac{B}{2} + \sin \dfrac{C}{2} = 1 + 4 \sin\left(\dfrac{\pi - A}{4}\right) \sin\left(\dfrac{\pi - B}{4}\right) \sin\left(\dfrac{\pi - C}{4}\right)$, if $A + B + C = \pi$

Solution:

$$
\begin{aligned}
\text{L.H.S.} &= \cos\left(\frac{\pi}{2} - \frac{A}{2}\right) + \cos\left(\frac{\pi}{2} - \frac{B}{2}\right) + \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) \\
&= \left[2\cos\left(\frac{\pi}{2} - \frac{A+B}{4}\right)\cos\left(\frac{B-A}{4}\right)\right] + \left[1 - 2\sin^2\left(\frac{\pi}{4} - \frac{C}{4}\right)\right] \\
&= 1 + 2\sin\left(\frac{\pi}{4} - \frac{C}{4}\right)\cos\left(\frac{B-A}{4}\right) - 2\sin^2\left(\frac{\pi}{4} - \frac{C}{4}\right) \\
&= 1 + 2\sin\left(\frac{\pi}{4} - \frac{C}{4}\right)\left[\cos\left(\frac{B-A}{4}\right) - \sin\left(\frac{\pi}{4} - \frac{C}{4}\right)\right] \\
&= 1 + 2\sin\left(\frac{\pi}{4} - \frac{C}{4}\right)\left[\cos\left(\frac{B-A}{4}\right) - \sin\left(\frac{A+B}{4}\right)\right] \\
&= 1 + 2\sin\left(\frac{\pi}{4} - \frac{C}{4}\right)\left[\cos\left(\frac{B-A}{4}\right) - \cos\left(\frac{\pi}{2} - \frac{A+B}{4}\right)\right] \\
&= 1 + 2\sin\left(\frac{\pi}{4} - \frac{C}{4}\right)\left[2\sin\left(\frac{\frac{B-A}{4} + \frac{\pi}{2} - \frac{A+B}{4}}{2}\right)\sin\left(\frac{\frac{\pi}{2} - \frac{A+B}{4} - \frac{B-A}{4}}{2}\right)\right] \\
&= 1 + 4\sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)
\end{aligned}
$$

Example 3.41 If $A+B+C=\pi$, prove that $\cos^2 A+\cos^2 B+\cos^2 C = 1-2\cos A\cos B\cos C$.

Solution:

$$\begin{aligned}
\cos^2 A+\cos^2 B+\cos^2 C &= \frac{1}{2}\left[2\cos^2 A+2\cos^2 B+2\cos^2 C\right]\\
&= \frac{1}{2}[(1+\cos 2A)+(1+\cos 2B)+(1+\cos 2C)]\\
&= \frac{3}{2}+\frac{1}{2}[(\cos 2A+\cos 2B)+\cos 2C]\\
&= \frac{3}{2}+\frac{1}{2}\left[2\cos(A+B)\cos(A-B)+(2\cos^2 C-1)\right]\\
&= \frac{3}{2}+\frac{1}{2}\left[-2\cos C\cos(A-B)+2\cos^2 C-1\right] \quad (A+B=\pi-C)\\
&= \frac{3}{2}-\frac{1}{2}+\frac{1}{2}[-2\cos C\,(\cos(A-B)-\cos C)]\\
&= 1-\cos C\,[\cos(A-B)-\cos C]\\
&= 1-\cos C\,[\cos(A-B)+\cos(A+B)]\\
&= 1-\cos C\,[2\cos A\cos B]\\
&= 1-2\cos A\cos B\cos C
\end{aligned}$$

Exercise - 3.7

1. If $A+B+C=180^\circ$, prove that
   (i) $\sin 2A+\sin 2B+\sin 2C = 4\sin A\sin B\sin C$
   (ii) $\cos A+\cos B-\cos C = -1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
   (iii) $\sin^2 A+\sin^2 B+\sin^2 C = 2+2\cos A\cos B\cos C$
   (iv) $\sin^2 A+\sin^2 B-\sin^2 C = 2\sin A\sin B\cos C$
   (v) $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$
   (vi) $\sin A+\sin B+\sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
   (vii) $\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C) = 4\sin A\sin B\sin C$.
2. If $A+B+C=2s$, then prove that $\sin(s-A)\sin(s-B)+\sin s\sin(s-C)=\sin A\sin B$.
3. If $x+y+z=xyz$, then prove that $\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\frac{2y}{1-y^2}\frac{2z}{1-z^2}$.
4. If $A+B+C=\frac{\pi}{2}$, prove the following
   (i) $\sin 2A+\sin 2B+\sin 2C = 4\cos A\cos B\cos C$
   (ii) $\cos 2A+\cos 2B+\cos 2C = 1+4\sin A\sin B\cos C$.
5. If $\triangle ABC$ is a right triangle and if $\angle A=\frac{\pi}{2}$, then prove that
   (i) $\cos^2 B+\cos^2 C=1$
   (ii) $\sin^2 B+\sin^2 C=1$
   (iii) $\cos B-\cos C = -1+2\sqrt{2}\cos\frac{B}{2}\sin\frac{C}{2}$.

3.6 Trigonometric equations

The equations containing trigonometric functions of unknown angles are known as trigonometric equations. A solution of trigonometric equation is the value of unknown angle that satisfies the equation. Unless the domain is restricted, the trigonometric equations have infinitely many solutions, a fact due to the periodicity of trigonometric functions. Some of the equations may not have a solution at all.

For example, the equation $\sin\theta = \frac{3}{2}$, does not have solution, since $-1 \leq \sin\theta \leq 1$.

The equation $\sin\theta = 0$ has infinitely many solutions given by $\theta = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ and note that these solutions occur periodically.

General Solution

The solution of a trigonometric equation giving all the admissible values obtained with the help of periodicity of a trigonometric function is called the **general solution** of the equation.

Principal Solution

The smallest numerical value of unknown angle satisfying the equation in the interval $[0, 2\pi]$ (or) $[-\pi, \pi]$ is called a **principal solution.** We shall take the interval $[-\pi, \pi]$ for defining the principal solution. Further, in this interval we may have two solutions. Even though both are valid solutions, we take only the numerically smaller one. This helps us to define the principal domain of the trigonometric functions in order to have their inverses.

Principal value of sine function lies in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and hence lies in I quadrant or IV quadrant. Principal value of cosine function is in $[0, \pi]$ and hence in I quadrant or II quadrant. Principal value of tangent function is in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and hence in I quadrant or IV quadrant.

(i) Trigonometric equations are different from trigonometric identities, since trigonometric identities are true for all admissible values of unknown angle θ. But trigonometric equations are valid only for particular values of unknown angle.

(ii) There is no general method for solving trigonometric equations. However, one may notice that some equations may be factorisable; some equations may be expressed in terms of single function; some equations may be squared.

(iii) To find solutions to trigonometric equations, some times one may go for the technique of squaring both sides. One has to take care as it can also produce false solutions (**extraneous solutions**).

For example, to find solution for $\sin x - \cos x = 1$ in $0 \leq x < 360°$, we do squaring on both sides to get $(\sin x - 1)^2 = \cos^2 x$, which gives $2\sin x(\sin x - 1) = 0$. So, we get $x = 0, \frac{\pi}{2}, \pi$. Clearly $x = 0$ is a false solution. Thus, we have to check for correct solutions, in the squaring process.

(iv) Mostly we write the solutions of trigonometric equations in radians.

Now, we find the solutions to different forms of trigonometrical equations.

(i) **To solve an equation of the form** $\sin\theta = k \;\; (-1 \le k \le 1)$:

Let α be the numerically smallest angle such that $\sin\alpha = k$. Thus,

$$\begin{aligned} \sin\theta &= \sin\alpha \\ \sin\theta - \sin\alpha &= 0 \\ 2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} &= 0 \text{ , which gives either} \\ \cos\frac{\theta+\alpha}{2} = 0 \text{ or } \sin\frac{\theta-\alpha}{2} &= 0. \end{aligned}$$

Now, $\cos\frac{\theta+\alpha}{2} = 0$	Now, $\sin\frac{\theta-\alpha}{2} = 0,$
$\frac{\theta+\alpha}{2} = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$	$\frac{\theta-\alpha}{2} = n\pi, n \in \mathbb{Z}$
$\theta = (2n+1)\pi - \alpha, n \in \mathbb{Z}$ ...(i)	$\theta = 2n\pi + \alpha, n \in \mathbb{Z}$ ...(ii)

Combining (i) and (ii), we have $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha, n \in \mathbb{Z}.$ (3.13)

(ii) **To solve an equation of the form** $\cos\theta = k \;\; (-1 \le k \le 1)$:

Let α be the numerically smallest angle such that $\cos\alpha = k$. Thus,

$$\begin{aligned} \cos\theta &= \cos\alpha \\ \cos\theta - \cos\alpha &= 0 \\ 2\sin\frac{\theta+\alpha}{2}\sin\frac{\alpha-\theta}{2} &= 0 \\ \sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} &= 0 \text{ , which gives either} \\ \sin\frac{\theta+\alpha}{2} = 0 \text{ or } \sin\frac{\theta-\alpha}{2} &= 0 \end{aligned}$$

$\sin\frac{\theta+\alpha}{2} = 0$ gives	$\sin\frac{\theta-\alpha}{2} = 0$ gives
$\frac{\theta+\alpha}{2} = n\pi, n \in \mathbb{Z}$	$\frac{\theta-\alpha}{2} = n\pi, n \in \mathbb{Z}$
$\theta = 2n\pi - \alpha, n \in \mathbb{Z}.$ ...(i)	$\theta = 2n\pi + \alpha, n \in \mathbb{Z}$ ...(ii)

Combining (i) and (ii), we have $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha, n \in \mathbb{Z}.$ (3.14)

(iii) **To solve an equation of the form** $\tan\theta = k \;\; (-\infty < k < \infty)$:

Let α be the numerically smallest angle such that $\tan\alpha = k$. Thus,

$$\begin{aligned} \tan\theta &= \tan\alpha \\ \frac{\sin\theta}{\cos\theta} &= \frac{\sin\alpha}{\cos\alpha} \Rightarrow \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0 \\ \sin(\theta-\alpha) &= 0 \Rightarrow \theta - \alpha = n\pi \\ \theta &= n\pi + \alpha, n \in \mathbb{Z} \\ \text{Thus, } \tan\theta &= \tan\alpha \Rightarrow \theta = n\pi + \alpha, n \in \mathbb{Z} \end{aligned} \tag{3.15}$$

(iv) **To solve an equation of the form** $a\cos\theta + b\sin\theta = c$:

$$\begin{aligned}
\text{Take } a &= r\cos\alpha,\ \ b = r\sin\alpha.\ \text{Then } r = \sqrt{a^2+b^2};\ \ \tan\alpha = \frac{b}{a},\ a \neq 0\\
a\cos\theta + b\sin\theta &= c \Rightarrow r\cos\alpha\cos\theta + r\sin\alpha\sin\theta = c\\
r\cos(\theta-\alpha) &= c\\
\cos(\theta-\alpha) &= \frac{c}{r} = \frac{c}{\sqrt{a^2+b^2}} = \cos\phi\ \text{(say)}\\
\theta - \alpha &= 2n\pi \pm \phi, n \in \mathbb{Z}\\
\theta &= 2n\pi + \alpha \pm \phi, n \in \mathbb{Z}.
\end{aligned}$$

The above method can be used only when $c \leq \sqrt{a^2+b^2}$.
If $c > \sqrt{a^2+b^2}$, then the equation $a\cos\theta + b\sin\theta = c$ has no solution.

Now we summarise the general solution of trigonometric equations.

Trigonometric Equation	General solution
$\sin\theta = 0$	$\theta = n\pi; n \in \mathbb{Z}$
$\cos\theta = 0$	$\theta = (2n+1)\frac{\pi}{2}; n \in \mathbb{Z}$
$\tan\theta = 0$	$\theta = n\pi; n \in \mathbb{Z}$
$\sin\theta = \sin\alpha$, where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\theta = n\pi + (-1)^n \alpha, n \in \mathbb{Z}$
$\cos\theta = \cos\alpha$, where $\alpha \in [0, \pi]$	$\theta = 2n\pi \pm \alpha, n \in \mathbb{Z}$
$\tan\theta = \tan\alpha$, where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\theta = n\pi + \alpha, n \in \mathbb{Z}$

Example 3.42 Find the principal solution of (i) $\sin\theta = \frac{1}{2}$ (ii) $\sin\theta = -\frac{\sqrt{3}}{2}$
(iii) $\text{cosec}\theta = -2$ (iv) $\cos\theta = \frac{1}{2}$

Solution:

(i) $\sin\theta = \frac{1}{2} > 0$ so principal value lies in the I quadrant.

$\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$

Thus, $\theta = \frac{\pi}{6}$ is the principal solution.

(ii) $\sin\theta = -\frac{\sqrt{3}}{2}$

We know that principal value of $\sin\theta$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Since, $\sin\theta = -\frac{\sqrt{3}}{2} < 0$, the principal value of $\sin\theta$ lies in the IV quadrant.

$\sin\theta = -\frac{\sqrt{3}}{2} = -\sin\left(\frac{\pi}{3}\right) = \sin\left(-\frac{\pi}{3}\right)$.

Hence, $\theta = -\frac{\pi}{3}$ is the principal solution.

(iii) $\operatorname{cosec}\theta = -2$

$\operatorname{cosec}\theta = -2 \Rightarrow \frac{1}{\sin\theta} = -2 \Rightarrow \sin\theta = -\frac{1}{2}$

Since $\sin\theta = -\frac{1}{2} < 0$, the principal value of $\sin\theta$ lies in the IV quadrant.

$\sin\theta = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(-\frac{\pi}{6}\right)$

Thus, $\theta = -\frac{\pi}{6}$ is the principal solution.

(iv) $\cos\theta = \frac{1}{2}$

Principal value of $\cos\theta$ lies in the I and II quadrant.

Since $\cos\theta = \frac{1}{2} > 0$, the principal value of $\cos\theta$ lies in the interval $\left[0, \frac{\pi}{2}\right]$.

$\cos\theta = \frac{1}{2} = \cos\frac{\pi}{3}$

Thus, $\theta = \frac{\pi}{3}$ is the principal solution.

Example 3.43 Find the general solution of $\sin\theta = -\frac{\sqrt{3}}{2}$

Solution:

The general solution of $\sin\theta = \sin\alpha, \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, is $\theta = n\pi + (-1)^n\alpha, n \in \mathbb{Z}$

$$\sin\theta = -\frac{\sqrt{3}}{2} = \sin\left(-\frac{\pi}{3}\right),$$

Thus the general solution is

$$\theta = n\pi + (-1)^n\left(-\frac{\pi}{3}\right) = n\pi + (-1)^{n+1}\left(\frac{\pi}{3}\right); n \in \mathbb{Z} \quad \text{... (i)}$$

In arriving at the above general solution, we took the principal value as $-\left(\frac{\pi}{3}\right)$ with our convention that principal value is the numerically smallest one in the interval $[-\pi, \pi]$. Now through this example, we shall justify that the principal value may also be taken in $[0, 2\pi]$, as we mentioned in the definition of principle solution. If we take the principal solution in the interval $[0, 2\pi]$, then the principal solution is $\theta = \dfrac{4\pi}{3}$ and the general solution is

$$\theta = n\pi + (-1)^n \left(\frac{4\pi}{3}\right), n \in \mathbb{Z}, \frac{4\pi}{3} \in [0, 2\pi] \quad \text{... (ii)}$$

From (ii), for $n = 0, -1, 1, -2, 2, \ldots$
the corresponding solutions are $\dfrac{4\pi}{3}, \dfrac{-7\pi}{3}, \dfrac{-\pi}{3}, \dfrac{-2\pi}{3}, \dfrac{10\pi}{3}, \ldots$
From (i), for $n = 0, -1, 1, -2, 2, \ldots$
the corresponding solutions are $\dfrac{-\pi}{3}, \dfrac{-2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{-7\pi}{3}, \dfrac{5\pi}{3}, \ldots$

In both the cases, we get the same set of solutions, but in different order. Thus, we have justified that the principal solution can be taken either in $[0, 2\pi]$ or in $[-\pi, \pi]$.

Example 3.44 Find the general solution of
(i) $\sec\theta = -2$ (ii) $\tan\theta = \sqrt{3}$

Solution:

(i) $\sec\theta = -2$

$\sec\theta = -2 \Rightarrow \cos\theta = -\dfrac{1}{2}$

We know that the general solution of $\cos\theta = \cos\alpha, \alpha \in [0, \pi]$, is $\theta = 2n\pi \pm \alpha, n \in \mathbb{Z}$

Let us find $\alpha \in [0, \pi]$ such that $\cos\alpha = -\dfrac{1}{2} = \cos\left(\pi - \dfrac{\pi}{3}\right) = \cos\dfrac{2\pi}{3}$.

so, $\alpha = \dfrac{2\pi}{3}$.

Thus, the general solution is $\theta = 2n\pi \pm \dfrac{2\pi}{3}, n \in \mathbb{Z}$.

(ii) $\tan\theta = \sqrt{3}$

$\tan\theta = \sqrt{3} = \tan\dfrac{\pi}{3}$

We know that the general solution of $\tan\theta = \tan\alpha$, $\alpha \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is $\theta = n\pi + \alpha, n \in \mathbb{Z}$

Thus, $\theta = n\pi + \dfrac{\pi}{3}, n \in \mathbb{Z}$, is the general solution.

Example 3.45 Solve $3\cos^2\theta = \sin^2\theta$

Solution:

$3\cos^2\theta = 1 - \cos^2\theta \Rightarrow \cos^2\theta = \dfrac{1}{4}$

$\dfrac{\cos 2\theta + 1}{2} = \dfrac{1}{4} \Rightarrow \cos 2\theta = -\dfrac{1}{2} = \cos\left(\pi - \dfrac{\pi}{3}\right) = \cos\left(\dfrac{2\pi}{3}\right)$

$2\theta = 2n\pi \pm \dfrac{2\pi}{3}, n \in \mathbb{Z} \Rightarrow \theta = n\pi \pm \dfrac{\pi}{3}, n \in \mathbb{Z}$.

Try to solve by writing $\tan^2\theta = 3$.

Example 3.46 Solve $\sin x + \sin 5x = \sin 3x$

Solution:

$$\sin x + \sin 5x = \sin 3x \Rightarrow 2\sin 3x \cos 2x = \sin 3x$$
$$\sin 3x\,(2\cos 2x - 1) = 0$$
Thus, either $\sin 3x = 0$ (or) $\cos 2x = \dfrac{1}{2}$

If $\sin 3x = 0$, then $3x = n\pi \Rightarrow x = \dfrac{n\pi}{3}, n \in \mathbb{Z}$ ... (i)

If $\cos 2x = \dfrac{1}{2} \Rightarrow \cos 2x = \cos\dfrac{\pi}{3}$ ... (ii)

$$2x = 2n\pi \pm \frac{\pi}{3} \Rightarrow x = n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$$

From (i) and (ii), we have the general solution $x = \dfrac{n\pi}{3}$ (or) $x = n\pi \pm \dfrac{\pi}{6}, n \in \mathbb{Z}$.

Example 3.47 Solve $\cos x + \sin x = \cos 2x + \sin 2x$

Solution:

$$\cos x + \sin x = \cos 2x + \sin 2x \Rightarrow \cos x - \cos 2x = \sin 2x - \sin x$$
$$2\sin\left(\frac{x+2x}{2}\right)\sin\left(\frac{2x-x}{2}\right) = 2\cos\left(\frac{2x+x}{2}\right)\sin\left(\frac{2x-x}{2}\right)$$
$$2\sin\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right) = 2\cos\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right)$$
$$\sin\left(\frac{x}{2}\right)\left[\sin\left(\frac{3x}{2}\right) - \cos\left(\frac{3x}{2}\right)\right] = 0$$
Thus, either $\sin\left(\dfrac{x}{2}\right) = 0$ (or) $\sin\left(\dfrac{3x}{2}\right) - \cos\left(\dfrac{3x}{2}\right) = 0.$

When $\sin\left(\dfrac{x}{2}\right) = 0$	When $\sin\left(\dfrac{3x}{2}\right) - \cos\left(\dfrac{3x}{2}\right) = 0$
$\dfrac{x}{2} = n\pi \Rightarrow x = 2n\pi, n \in \mathbb{Z}.$	$\Rightarrow \tan\left(\dfrac{3x}{2}\right) = 1 = \tan\left(\dfrac{\pi}{4}\right)$
	$\dfrac{3x}{2} = n\pi + \dfrac{\pi}{4} \Rightarrow x = \dfrac{2n\pi}{3} + \dfrac{\pi}{6}.$

Thus, the general solution is $x = 2n\pi$ (or) $x = \dfrac{2n\pi}{3} + \dfrac{\pi}{6}, n \in \mathbb{Z}$.

If $\sin\theta = \cos\theta$, then $\theta \neq (2n+1)\dfrac{\pi}{2}, n \in \mathbb{Z}$. So, we have $\dfrac{\sin\theta}{\cos\theta} = 1$

Example 3.48 Solve the equation $\sin 9\theta = \sin\theta$

Solution:

$$\sin 9\theta = \sin\theta \Rightarrow \sin 9\theta - \sin\theta = 0$$
$$2\cos 5\theta \sin 4\theta = 0$$
Either $\cos 5\theta = 0$ (or) $\sin 4\theta = 0$

When $\cos 5\theta = 0 \Rightarrow 5\theta = (2n+1)\dfrac{\pi}{2}$

$\Rightarrow \theta = (2n+1)\dfrac{\pi}{10}, n \in \mathbb{Z}$

When, $\sin 4\theta = 0 \Rightarrow 4\theta = n\pi$

$\Rightarrow \theta = \dfrac{n\pi}{4}, n \in \mathbb{Z}$

Thus, the general solution of the given equation is $\theta = (2n+1)\dfrac{\pi}{10},\quad \theta = \dfrac{n\pi}{4}, n \in \mathbb{Z}$.

Example 3.49 Solve $\tan 2x = -\cot\left(x + \dfrac{\pi}{3}\right)$

Solution:

$$\begin{aligned} \tan 2x &= -\cot\left(x+\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{2}+x+\frac{\pi}{3}\right) = \tan\left(\frac{5\pi}{6}+x\right) \\ 2x &= n\pi + \frac{5\pi}{6} + x, n \in \mathbb{Z} \Rightarrow x = n\pi + \frac{5\pi}{6}, n \in \mathbb{Z}. \end{aligned}$$

Example 3.50 Solve $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

Solution:

$$\begin{aligned} &\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x \\ &\sin 3x + \sin x - 3\sin 2x = \cos 3x + \cos x - 3\cos 2x \\ &2\sin 2x\cos x - 3\sin 2x = 2\cos 2x\cos x - 3\cos 2x \\ &(\sin 2x - \cos 2x)(2\cos x - 3) = 0 \\ &\text{Then, } \sin 2x - \cos 2x = 0 \;\text{ since }\; 2\cos x - 3 \neq 0 \\ &\sin 2x = \cos 2x \Rightarrow \tan 2x = 1 \Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{8}, n \in \mathbb{Z}. \end{aligned}$$

Example 3.51 Solve $\sin x + \cos x = 1 + \sin x\cos x$

Solution:

$$\begin{aligned} &\text{Let } \sin x + \cos x = t \\ &\Rightarrow 1 + 2\sin x\cos x = t^2 \Rightarrow \sin x\cos x = \frac{t^2-1}{2} \end{aligned}$$

Thus, the given equation becomes $t^2 - 2t + 1 = 0 \Rightarrow t = 1$

$$\begin{aligned} &\text{Hence, } \sin x + \cos x = 1 \\ &\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right) = 1 \\ &\sqrt{2}\cos\left(\frac{\pi}{4} - x\right) = 1 \Rightarrow \cos\left(\frac{\pi}{4} - x\right) = \frac{1}{\sqrt{2}} \\ &\text{Thus, we have } x - \frac{\pi}{4} = \pm\frac{\pi}{4} + 2n\pi, n \in \mathbb{Z} \\ &x = \frac{\pi}{2} + 2n\pi, \;\text{ or }\; x = 2n\pi,\; n \in \mathbb{Z}. \end{aligned}$$

Example 3.52 Solve $2\sin^2 x + \sin^2 2x = 2$

Solution:

$$2\sin^2 x + \sin^2 2x = 2 \Rightarrow 2\sin^2 x + (2\sin x\cos x)^2 = 2$$
$$\cos^2 x\left(2\sin^2 x - 1\right) = 0$$
$$\text{Either}\quad \cos x = 0 \quad\text{or}\quad \sin^2 x = \frac{1}{2}$$

$\cos x = 0$	$\sin^2 x = \frac{1}{2} = \sin^2\frac{\pi}{4}$
$\Rightarrow x = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$	$x = n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z}$ How

Thus the solution is $x = (2n+1)\frac{\pi}{2},\quad x = n\pi \pm \frac{\pi}{4}, n \in \mathbb{Z}$.

Example 3.53 Prove that for any a and b, $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$

Solution:

Now, $a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta\right]$

$= \sqrt{a^2+b^2}\,[\cos\alpha\sin\theta + \sin\alpha\cos\theta]$

where $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}},\quad \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$

$= \sqrt{a^2+b^2}\quad \sin(\alpha+\theta)$

Thus, $|a\sin\theta + b\cos\theta| \le \sqrt{a^2+b^2}$

Hence, $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$.

Example 3.54 Solve $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$

Solution:

$\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$

Here $a = -1$; $b = \sqrt{3}$; $c = \sqrt{2}$; $r = \sqrt{a^2+b^2} = 2$.

Thus, the given equation can be rewritten as

$$\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta = \frac{1}{\sqrt{2}}$$
$$\sin\theta\cos\frac{\pi}{6} - \cos\theta\sin\frac{\pi}{6} = \sin\frac{\pi}{4}$$
$$\sin\left(\theta - \frac{\pi}{6}\right) = \sin\frac{\pi}{4}$$
$$\theta - \frac{\pi}{6} = n\pi \pm (-1)^n\frac{\pi}{4}, n \in \mathbb{Z}$$

Thus, $\theta = n\pi + \frac{\pi}{6} \pm (-1)^n\frac{\pi}{4}, n \in \mathbb{Z}$.

Example 3.55 Solve $\sqrt{3}\tan^2\theta + \left(\sqrt{3}-1\right)\tan\theta - 1 = 0$

Solution:

$$\sqrt{3}\tan^2\theta + \left(\sqrt{3}-1\right)\tan\theta - 1 = 0$$
$$\sqrt{3}\tan^2\theta + \sqrt{3}\tan\theta - \tan\theta - 1 = 0$$
$$\left(\sqrt{3}\tan\theta - 1\right)(\tan\theta + 1) = 0$$

Thus, either $\sqrt{3}\tan\theta - 1 = 0$ (or) $\tan\theta + 1 = 0$

If $\sqrt{3}\tan\theta - 1 = 0$, then

$$\tan\theta = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$$
$$\Rightarrow \theta = n\pi + \frac{\pi}{6}, n \in \mathbb{Z} ...\text{(i)}$$

If $\tan\theta + 1 = 0$ then

$$\tan\theta = -1 = \tan\left(\frac{-\pi}{4}\right)$$
$$\Rightarrow \theta = n\pi - \frac{\pi}{4}, n \in \mathbb{Z} ...\text{(ii)}$$

From (i) and (ii) we have the general solution.

Exercise - 3.8

1. Find the principal solution and general solutions of the following:

   (i) $\sin\theta = -\frac{1}{\sqrt{2}}$ (ii) $\cot\theta = \sqrt{3}$ (iii) $\tan\theta = -\frac{1}{\sqrt{3}}$.

2. Solve the following equations for which solutions lies in the interval $0^\circ \le \theta < 360^\circ$

   (i) $\sin^4 x = \sin^2 x$

   (ii) $2\cos^2 x + 1 = -3\cos x$

   (iii) $2\sin^2 x + 1 = 3\sin x$

   (iv) $\cos 2x = 1 - 3\sin x$.

3. Solve the following equations:

   (i) $\sin 5x - \sin x = \cos 3x$

   (ii) $2\cos^2\theta + 3\sin\theta - 3 = 0$

   (iii) $\cos\theta + \cos 3\theta = 2\cos 2\theta$

   (iv) $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

   (v) $\sin 2\theta - \cos 2\theta - \sin\theta + \cos\theta = 0$

   (vi) $\sin\theta + \cos\theta = \sqrt{2}$

   (vii) $\sin\theta + \sqrt{3}\cos\theta = 1$

   (viii) $\cot\theta + \text{cosec}\theta = \sqrt{3}$

   (ix) $\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = \sqrt{3}$

   (x) $\cos 2\theta = \frac{\sqrt{5}+1}{4}$

   (xi) $2\cos^2 x - 7\cos x + 3 = 0$

3.7 Properties of Triangle

One important use of trigonometry is to solve practical problems that can be modeled by a triangle. Determination of all the sides and angles of a triangle is referred as solving the triangle. In any triangle, the three sides and three angles are called basic elements of a triangle. Pythagorean theorem plays a vital role in finding solution of the right triangle. The law of sines and the law of cosines are important tools that can be used effectively in solving an oblique triangle (a triangle with no right angle). In this section, we shall discuss the relationship between the sides and angles of a triangle and derive the law of sines and the law of cosines.

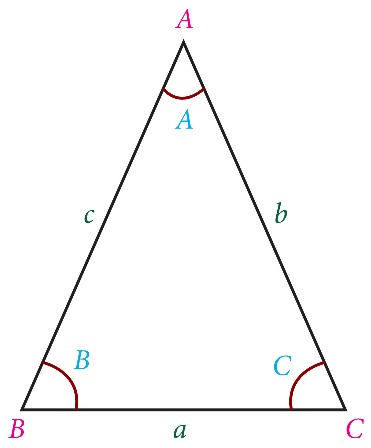


Figure 3.17

Notation: Let ABC be a triangle. The angles of $\triangle ABC$ corresponding to the vertices A, B, C are denoted by A, B, C themselves. The sides opposite to the angles A, B, C are denoted by a, b, c respectively. Also we use the symbol $\triangle$ to denote the area of a triangle.

The circle which passes through the three vertices of a triangle is called circumcircle of the triangle. The centre and the radius R of the circumcircle are known as circumcentre and circumradius respectively.

In $\triangle ABC$, we have $A+B+C=\pi$ and $a+b>c, \quad b+c>a, \quad a+c>b$.

The Law of Sines or Sine Formula

3.7.1 Law of Sines

The Law of Sines is a relationship between the angles and the sides of a triangle. While solving a triangle, the law of sines can be effectively used in the following situations:

(i) To find an angle if two sides and one angle which is not included, by them are given.
(ii) To find a side, if two angles and one side which is opposite to one of given angles, are given.

Theorem 3.1 (Law of Sines)**:** In any triangle, the lengths of the sides are proportional to the sines of the opposite angles. That is, in $\triangle ABC$, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$ where R is the circumradius of the triangle.

Proof. The angle A of the $\triangle ABC$ is either acute or right or obtuse. Let O be the centre of the circumcircle of $\triangle ABC$ and R, its radius.

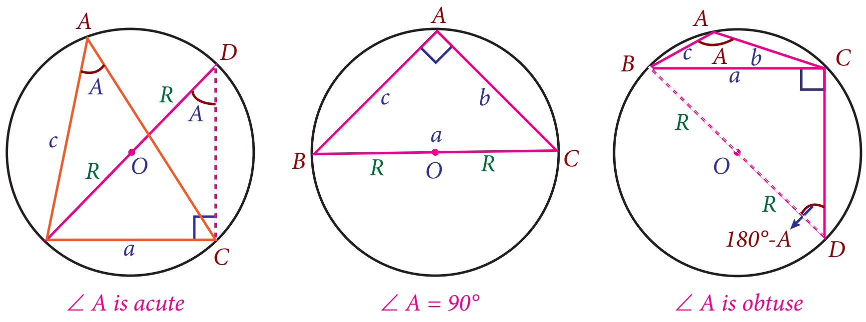


Figure 3.18

Case I: $\angle A$ is acute.

Produce BO to meet the circle at D.

$\angle BDC = \angle BAC = A$

$\angle BCD = 90°$

$\sin \angle BDC = \dfrac{BC}{BD}$ or $\sin A = \dfrac{a}{2R} \Rightarrow \dfrac{a}{\sin A} = 2R$

Case II: $\angle A$ is right angle.

In this case O must be on the side BC of the $\triangle ABC$.

Now, $\dfrac{a}{\sin A} = \dfrac{BC}{\sin 90°} = \dfrac{2R}{1} = 2R \Rightarrow \dfrac{a}{\sin A} = 2R$

Case III: $\angle A$ is obtuse .

Produce BO to meet the circle at D.

$\angle BDC + \angle BAC = 180°$

$\angle BDC = 180° - \angle BAC = 180° - A$

$\angle BCD = 90°$

$\sin \angle BDC = \dfrac{BC}{BD}$ or $\sin(180° - A) = \sin A = \dfrac{a}{2R} \Rightarrow \dfrac{a}{\sin A} = 2R$

In each case, we have $\dfrac{a}{\sin A} = 2R$

Similarly, by considering angles B and C, we can prove that $\dfrac{b}{\sin B} = 2R$ and $\dfrac{c}{\sin C} = 2R$ respectively.

Thus, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R.$

(i) The Law of Sines can be written as a collection of three equations

$$\frac{a}{b} = \frac{\sin A}{\sin B}; \qquad \frac{a}{c} = \frac{\sin A}{\sin C}; \qquad \frac{b}{c} = \frac{\sin B}{\sin C};$$

(ii) The Law of Sines says that the sides of a triangle are proportional to the sines of their opposite angles.

(iii) Using the Law of Sines, it is impossible to find the solution to a triangle given two sides and the included angle.

(iv) An interesting geometric consequence of the Law of Sines is that the largest side of any triangle is opposite to the largest angle. (Prove)

Napier's Formula

Theorem 3.2: In $\triangle ABC$, we have

(i) $\tan \dfrac{A-B}{2} = \dfrac{a-b}{a+b} \cot \dfrac{C}{2}$

(ii) $\tan \dfrac{B-C}{2} = \dfrac{b-c}{b+c} \cot \dfrac{A}{2}$

(iii) $\tan \dfrac{C-A}{2} = \dfrac{c-a}{c+a} \cot \dfrac{B}{2}$

Proof.

We know the sine formula: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$$\begin{aligned}
\text{Now, } \frac{a-b}{a+b}\cot\frac{C}{2} &= \frac{2R\sin A - 2R\sin B}{2R\sin A + 2R\sin B}\cot\frac{C}{2}\\
&= \frac{\sin A - \sin B}{\sin A + \sin B}\cot\frac{C}{2}\\
&= \frac{2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}}{2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}}\cot\frac{C}{2}\\
&= \cot\frac{A+B}{2}\tan\frac{A-B}{2}\cot\frac{C}{2}\\
&= \cot\left(90^\circ - \frac{C}{2}\right)\tan\frac{A-B}{2}\cot\frac{C}{2}\\
&= \tan\frac{C}{2}\tan\frac{A-B}{2}\cot\frac{C}{2} = \tan\frac{A-B}{2}
\end{aligned}$$

Similarly we can prove the other two results.

3.7.2 Law of Cosines

When two sides and included angle or the three sides of a triangle are given, the triangle cannot be solved using the sine formula. In such a situation, the law of cosines can be used to solve the triangle. Also, the Law of Cosines is used to derive a formula for finding the area of a triangle given two sides and the included angle.

Theorem 3.3 (The Law of Cosines)**:** In $\triangle ABC$, we have

$$\cos A = \frac{b^2+c^2-a^2}{2bc};\ \cos B = \frac{c^2+a^2-b^2}{2ca};\ \cos C = \frac{a^2+b^2-c^2}{2ab}.$$

Proof. In $\triangle ABC$, draw $AD \perp BC$.
In $\triangle ABD$, we have $AB^2 = AD^2 + BD^2 \Rightarrow c^2 = AD^2 + BD^2$.
Now, we find the values of AD and BD in terms of the elements of $\triangle ABC$.

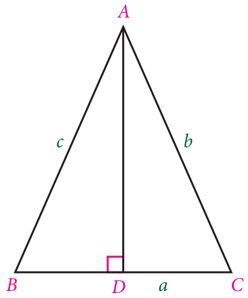


Figure 3.19

$$\begin{aligned}
&\frac{AD}{AC} = \sin C \Rightarrow AD = b\sin C\\
&BD = BC - DC = a - b\cos C\\
&c^2 = (b\sin C)^2 + (a - b\cos C)^2\\
&\quad = b^2\sin^2 C + a^2 + b^2\cos^2 C - 2ab\cos C\\
&\quad = a^2 + b^2\left(\sin^2 C + \cos^2 C\right) - 2ab\cos C\\
&\quad = a^2 + b^2 - 2ab\cos C
\end{aligned}$$

Thus, $c^2 = a^2 + b^2 - 2ab\cos C$ or $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$.

Similarly, we can prove the other two results, namely

$$a^2 = b^2 + c^2 - 2bc\cos A \text{ or } \cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$b^2 = c^2 + a^2 - 2ca\cos B \text{ or } \cos B = \frac{c^2+a^2-b^2}{2ca}$$

(i) $a^2 = b^2 + c^2 - 2bc\cos A$ says that the square of a side is the sum of squares of other two sides diminished by twice the product of those two sides and the cosine of the included angle. Also one formula will give the other formula by cycling through the letters a, b, c.

(ii) The Laws of Cosine for right triangles reduce to Pythagorean theorem. Thus, the Law of cosines can be viewed as a generalisation of Pythagorean theorem.

(iii) The advantage of using law of cosines over law of sines is that unlike the sine function, the cosine function distinguishes between acute and obtuse angles. If cosine of an angle is positive, then it is acute. Otherwise, it is obtuse.

(iv) **The Law of Cosines says** : The direct route is the shortest. Let us explain this. In a $\triangle ABC$, $c^2 = a^2 + b^2 - 2ab\cos C$. Since $-\cos C < 1$ we have $c^2 < a^2 + b^2 + 2ab$. Thus, we have $c < a + b$. Hence, In $\triangle ABC$, we have $a < b + c,\ \ b < c + a,\ \ c < a + b$

(v) When using the law of cosines, it is always best to find the measure of the largest unknown angle first, since this will give us the obtuse angle of the triangle if there is one such angle.

3.7.3 Projection Formula

Theorem 3.4: In a $\triangle ABC$, we have

(i) $a = b\cos C + c\cos B$, (ii) $b = c\cos A + a\cos C$, (iii) $c = a\cos B + b\cos A$

Proof. In $\triangle ABC$, we have $a = BC$. Draw $AD \perp BC$.

$$\begin{aligned} a &= BC = BD + DC \\ &= \frac{BD}{AB}AB + \frac{DC}{AC}AC \\ &= (\cos B)c + (\cos C)b \\ a &= b\cos C + c\cos B \end{aligned}$$

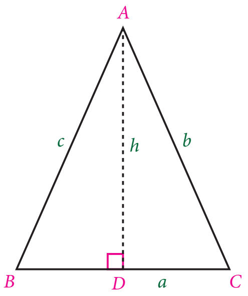


Figure 3.20

Similarly, one can prove the other two projection formulas.

$a = b\cos C + c\cos B$ says that $a =$ projection of b on $a +$ projection of c on a
Thus, a side of triangle is equal to sum of the projections of other two sides on it.

3.7.4 Area of the Triangle

We shall use some elements of an oblique triangle and the sine function to find the area of the triangle. Recall that area formula for $\triangle ABC$ is $\frac{1}{2}bh$ where b is the base and h is the height. For oblique triangle, we must find h before using the area formula.

Theorem 3.5: In $\triangle ABC$, area of the triangle is

$$\triangle = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$$

Proof. In $\triangle ABC$, draw $AD \perp BC$

In $\triangle ADC$, $\dfrac{AD}{AC} = \sin C \Rightarrow AD = b\sin C$

Thus, $\triangle = \dfrac{1}{2} \times$ base $\times$ height $= \dfrac{1}{2}ab\sin C$.

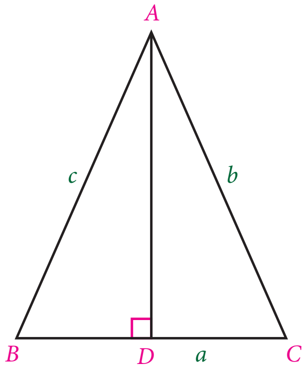


Figure 3.21

Similarly we can derive the other two results.

(i) The formula for the area of an oblique triangle says that the area is equal to one half of the product of two sides and the sine of their included angle.

(ii) The area formula is used to compute the area of the segment of a circle. Segment of a circle is the region between a chord and the arc it cuts off.

Let r be the radius of a circle and θ be the angle subtended by the chord AB at the centre.

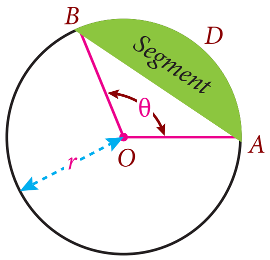


Figure 3.22

$$\begin{aligned}\text{Area of the segment } ABD &= \text{Area of the sector} - \text{Area of the } \triangle OAB \\ &= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta \\ &= \frac{1}{2}r^2(\theta - \sin\theta)\end{aligned}$$

(iii) The area formula of a triangle is viewed as generalisation of area formula of a right triangle.

(iv) In the above formula, it is clear that the measure of third side is not required in finding the area of the triangles. Also there is no need of finding the altitude of the triangle in order to find its area.

Example 3.56

The Government plans to have a circular zoological park of diameter $8\ km$. A separate area in the form of a segment formed by a chord of length $4\ km$ is to be allotted exclusively for a veterinary hospital in the park. Find the area of the segment to be allotted for the veterinary hospital.

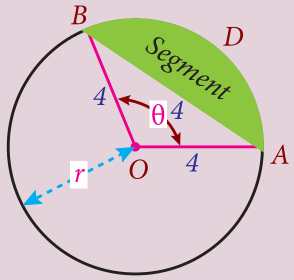


Figure 3.23

Solution:

Let AB be the chord and O be the centre of the circular park.

Let $\angle AOB = \theta$.

$$\begin{aligned}\text{Area of the segment} &= \text{Area of the sector} - \text{Area of} \triangle OAB. \\ &= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta \\ &= \left(\frac{1}{2}\times 4^2\right)[\theta - \sin\theta] = 8\,[\theta-\sin\theta] \qquad \text{...(i)}\end{aligned}$$

But $\cos\theta = \dfrac{4^2+4^2-4^2}{2(4)(4)} = \dfrac{1}{2}$

Thus, $\theta = \dfrac{\pi}{3}$

From (i), area of the the segment to be allotted for the veterinary hospital

$$= 8\left[\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right] = \frac{4}{3}\left[2\pi - 3\sqrt{3}\right]\ m^2$$

3.7.5 Half-Angle formula

Theorem 3.6: In $\triangle ABC$

(i) $\sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$, (ii) $\cos\dfrac{A}{2}=\sqrt{\dfrac{s(s-a)}{bc}}$,

(iii) $\tan\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$, where s is the semi-perimeter of $\triangle ABC$ given by $s=\dfrac{a+b+c}{2}$

Proof. (i)

$$\begin{aligned}
\sin\frac{A}{2} &= +\sqrt{\sin^2\frac{A}{2}}=\sqrt{\frac{1-\cos A}{2}}=\sqrt{\frac{1}{2}\left(1-\frac{b^2+c^2-a^2}{2bc}\right)}\\
&=\sqrt{\frac{2bc-b^2-c^2+a^2}{4bc}}=\sqrt{\frac{a^2-(b-c)^2}{4bc}}\\
&=\sqrt{\frac{(a+b-c)(a-b+c)}{4bc}}=\sqrt{\frac{(a+b+c-2c)(a+b+c-2b)}{4bc}}\\
&=\sqrt{\frac{(2s-2b)(2s-2c)}{4bc}}=\sqrt{\frac{(s-b)(s-c)}{bc}}
\end{aligned}$$

Thus, $$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Similarly, we can prove the other two results.

The other half-angle formulas are

$$\begin{aligned}
\sin\frac{B}{2} &= \sqrt{\frac{(s-c)(s-a)}{ac}}, & \sin\frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}}\\
\cos\frac{B}{2} &= \sqrt{\frac{s(s-b)}{ac}}, & \cos\frac{C}{2} &= \sqrt{\frac{s(s-c)}{ab}}\\
\tan\frac{B}{2} &= \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, & \tan\frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}},
\end{aligned}$$

Corollary: $\sin A=2\sin\dfrac{A}{2}\cos\dfrac{A}{2}=2\sqrt{\dfrac{(s-b)(s-c)}{bc}}\sqrt{\dfrac{s(s-a)}{bc}}$

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

Area of a triangle (Heron's Formula)

Heron's formula is named after Hero of Alexendria, a Greek Engineer and Mathematician in (CE 10 - 70). This formula is used only when all the three sides are known.

Theorem 3.7: In $\triangle ABC$, $\triangle=\sqrt{s(s-a)(s-b)(s-c)}$ where s is the semi-perimeter of $\triangle ABC$.

Proof.

$$\begin{aligned}
\triangle &= \frac{1}{2}ab\sin C=\frac{1}{2}ab\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)\\
&= ab\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{s(s-c)}{ab}}=\sqrt{s(s-a)(s-b)(s-c)}
\end{aligned}$$

(i) Using Heron's formula , Pythagorean theorem can be proved for right triangle and conversely, using Pythagorean theorem for right triangle one can establish Heron's area formula.

(ii) If area of a triangle is given as an integer, then Heron's formula is useful in finding triangles with integer sides.

(iii) If the perimeter of a triangle is fixed, then Heron's formula is useful for finding triangles having integer area and integer sides.

For example, if the perimeter of a triangle is $100m$, then there is a triangle with sides $32\ m, 34\ m, 34\ m$ and area 480 sq.meter.

Example 3.57 In a $\triangle ABC$, prove that $b^2\sin 2C + c^2\sin 2B = 2bc\sin A$.

Solution:

The Sine formula is, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

Thus, $a = 2R\sin A;\quad b = 2R\sin B;\quad c = 2R\sin C$

$$\begin{aligned}
b^2\sin 2C + c^2\sin 2B &= 4R^2\sin^2 B\sin 2C + 4R^2\sin^2 C\sin 2B \\
&= 4R^2\left(2\sin^2 B\sin C\cos C + 2\sin^2 C\sin B\cos B\right) \\
&= 8R^2\sin B\sin C\left(\sin B\cos C + \sin C\cos B\right) \\
&= 8R^2\sin B\sin C\sin(B+C) \\
&= 8R^2\sin B\sin C\sin\left(\pi - A\right) = 8R^2\sin B\sin C\sin A \\
&= 8R^2\left(\frac{b}{2R}\right)\left(\frac{c}{2R}\right)\sin A = 2bc\sin A.
\end{aligned}$$

Example 3.58 In a $\triangle ABC$, prove that $\sin\left(\dfrac{B-C}{2}\right) = \dfrac{b-c}{a}\cos\dfrac{A}{2}$.

Solution:

The Sine formula is, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$$\begin{aligned}
\text{Now,}\quad \frac{b-c}{a}\cos\frac{A}{2} &= \frac{2R\sin B - 2R\sin C}{2R\sin A}\cos\frac{A}{2} \\
&= \frac{2\sin\left(\dfrac{B-C}{2}\right)\cos\left(\dfrac{B+C}{2}\right)}{2\sin\dfrac{A}{2}\cos\dfrac{A}{2}}\cos\frac{A}{2} \\
&= \frac{\sin\left(\dfrac{B-C}{2}\right)\cos\left(90^\circ - \dfrac{A}{2}\right)}{\sin\dfrac{A}{2}} \\
&= \frac{\sin\left(\dfrac{B-C}{2}\right)\sin\left(\dfrac{A}{2}\right)}{\sin\dfrac{A}{2}} \\
&= \sin\left(\frac{B-C}{2}\right).
\end{aligned}$$

Example 3.59 If the three angles in a triangle are in the ratio $1 : 2 : 3$, then prove that the corresponding sides are in the ratio $1 : \sqrt{3} : 2$.

Solution:
Let the angles be $\theta, 2\theta, 3\theta$.

Then $\theta + 2\theta + 3\theta = 180°$

Thus, $\theta = 30°$

Using the sine formula, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, we have,

$$\frac{a}{\sin 30°} = \frac{b}{\sin 60°} = \frac{c}{\sin 90°}$$

$$a : b : c = \sin 30° : \sin 60° : \sin 90°$$

$$= \frac{1}{2} : \frac{\sqrt{3}}{2} : 1 = 1 : \sqrt{3} : 2$$

Example 3.60 In a $\triangle ABC$, prove that

$$(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c$$

Solution:

$$\begin{aligned} \text{L.H.S} &= b\cos A + c\cos A + c\cos B + a\cos B + a\cos C + b\cos C \\ &= b\cos C + c\cos B + c\cos A + a\cos C + b\cos A + a\cos B \\ &= a+b+c \quad \text{[by projection formula]} \end{aligned}$$

Example 3.61 In a triangle ABC, prove that $\dfrac{a^2+b^2}{a^2+c^2} = \dfrac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}$

Solution:
The sine formula is, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$$\begin{aligned} \text{L.H.S} &= \frac{a^2+b^2}{a^2+c^2} = \frac{(2R\sin A)^2 + (2R\sin B)^2}{(2R\sin A)^2 + (2R\sin C)^2} \\ &= \frac{\sin^2 A + \sin^2 B}{\sin^2 A + \sin^2 C} = \frac{1-\cos^2 A + \sin^2 B}{1-\cos^2 A + \sin^2 C} \\ &= \frac{1-\left(\cos^2 A - \sin^2 B\right)}{1-\left(\cos^2 A - \sin^2 C\right)} = \frac{1-\cos(A+B)\cos(A-B)}{1-\cos(A+C)\cos(A-C)} \\ &= \frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}. \end{aligned}$$

Example 3.62 Derive cosine formula using the law of sines in a $\triangle ABC$.

Solution:

The law of Sines: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$$\begin{aligned}
\frac{b^2+c^2-a^2}{2bc} &= \frac{(2R\sin B)^2 + (2R\sin C)^2 - (2R\sin A)^2}{2\,(2R\sin B)\,(2R\sin C)} \\
&= \frac{\sin^2 B + \sin(C+A)\sin(C-A)}{2\sin B\sin C} \\
&= \frac{\sin B\,[\sin B + \sin(C-A)]}{2\sin B\sin C} \\
&= \frac{\sin(C+A) + \sin(C-A)}{2\sin C} = \cos A
\end{aligned}$$

Thus, $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$. Similarly we can derive other two cosine formulas.

Example 3.63 Using Heron's formula, show that the equilateral triangle has the maximum area for any fixed perimeter. [Hint: In $xyz \le k$, maximum occurs when $x = y = z$].

Solution:

Let ABC be a triangle with constant perimeter $2s$. Thus s is constant.
We know that $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$
Observe that $\triangle$ is maximum, when $(s-a)(s-b))(s-c)$ is maximum.

$$\text{Now, } (s-a)(s-b)(s-c) \le \left(\frac{(s-a)+(s-b)+(s-c)}{3}\right)^3 = \frac{s^3}{27} \qquad [G.M \le A.M.]$$

$$\text{Thus, we get } (s-a)(s-b)(s-c) \le \frac{s^3}{27}$$

Equality occurs when $s-a = s-b = s-c$. That is, when $a = b = c$, maximum of $(s-a)(s-b))(s-c)$ is $\dfrac{s^3}{27}$

Thus, for a fixed perimeter $2s$, the area of a triangle is maximum when $a = b = c$.

Hence, for a fixed perimeter, the equilateral triangle has the maximum area and the maximum area is given by $\triangle = \sqrt{\dfrac{s\,(s^3)}{27}} = \dfrac{s^2}{3\sqrt{3}}$ sq.units.

Exercise - 3.9

1. In a $\triangle ABC$, if $\dfrac{\sin A}{\sin C} = \dfrac{\sin(A-B)}{\sin(B-C)}$, prove that a^2, b^2, c^2 are in Arithmetic Progression.
2. The angles of a triangle ABC, are in Arithmetic Progression and if $b : c = \sqrt{3} : \sqrt{2}$, find $\angle A$.

3. In a $\triangle ABC$, if $\cos C = \dfrac{\sin A}{2\sin B}$, show that the triangle is isosceles.
4. In a $\triangle ABC$, prove that $\dfrac{\sin B}{\sin C} = \dfrac{c - a\cos B}{b - a\cos C}$.
5. In a $\triangle ABC$, prove that $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$.
6. In a $\triangle ABC$, $\angle A = 60°$. Prove that $b + c = 2a\cos\left(\dfrac{B-C}{2}\right)$.
7. In a $\triangle ABC$, prove the following
   (i) $a\sin\left(\dfrac{A}{2} + B\right) = (b+c)\sin\dfrac{A}{2}$
   (ii) $a(\cos B + \cos C) = 2(b+c)\sin^2\dfrac{A}{2}$
   (iii) $\dfrac{a^2 - c^2}{b^2} = \dfrac{\sin(A-C)}{\sin(A+C)}$
   (iv) $\dfrac{a\sin(B-C)}{b^2 - c^2} = \dfrac{b\sin(C-A)}{c^2 - a^2} = \dfrac{c\sin(A-B)}{a^2 - b^2}$
   (v) $\dfrac{a+b}{a-b} = \tan\left(\dfrac{A+B}{2}\right)\cot\left(\dfrac{A-B}{2}\right)$.
8. In a $\triangle ABC$, prove that $(a^2 - b^2 + c^2)\tan B = (a^2 + b^2 - c^2)\tan C$.
9. An Engineer has to develop a triangular shaped park with a perimeter $120\ m$ in a village. The park to be developed must be of maximum area. Find out the dimensions of the park.
10. A rope of length $12\ m$ is given. Find the largest area of the triangle formed by this rope and find the dimensions of the triangle so formed.
11. Derive Projection formula from (i) Law of sines, (ii) Law of cosines.

3.8 Application to Triangle

Much of architecture and engineering relies on triangular support on any structure where stability is desired. Trigonometry helps to calculate the correct angle for the triangular support. Also trigonometry envisages the builders to correctly layout a curved structure. For a right triangle, any two information with atleast one side say SS, SA are sufficient to find the remaining elements of the triangle. But, to find the solution of an oblique triangle we need three elements with atleast one side. If any three elements with atleast one side of a triangle are given, then the Law of Sines, the Law of Cosines, the Projection formula can be used to find the other three elements.

Working Rule:

- In a right triangle, two sides determine the third side via the Pythagorean theorem and one acute angle determine the other by using the fact that acute angles in a right triangle are complementary.
- If all the sides of a triangle are given, then we can use either cosine formula or half-angle formula to calculate all the angles of the triangle.
- If any two angles and any one of the sides opposite to given angles are given, then we can use sine formula to calculate the other sides.
- If any two sides of a triangle and the included angle are given, we cannot use the Law of sines; but then we can use the law of cosines to calculate other side and other angles of the triangle. In this case we have a unique triangle.
- All methods of solving an oblique triangle require that the length of atleast one side must be provided.

Let us summarise the working rule to solve an oblique triangle in the following table

Oblique triangle (all angles are acute or or one angle is obtuse	Given Information				
	SAA	SSA*	SAS	SSS	AAA
Details and Application for solutions	Law of sines	#(ambiguity arises)	The given angle must be included angle; Either law of Cosines or tangents	Law of Cosines; First find the largest angle	Infinite number of triangles

\# Angle is not included; We may have more than one triangle;
Application of law of sines yields three cases: No solution or one triangle or two triangles.
Suppose a, b, A are known. Let $h = b \sin A$
If $a < h$, there is no triangle, If $a = h$, then it is right triangle.
If $a > h$ and $a < b$, we have two triangles.
If $a \geq b$, we have only one triangle.

* SSA means Side, Side and Angle

Example 3.64 In a $\triangle ABC$, $a = 3, b = 5$ and $c = 7$.
Find the values of $\cos A, \quad \cos B$ and $\cos C$.

Solution:
By Cosine formula,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{5^2 + 7^2 - 3^2}{2(5)(7)} = \frac{13}{14}$$

Similarly, $\cos B = \dfrac{11}{14}, \quad \cos C = -\dfrac{1}{2}$.

Example 3.65 In $\triangle ABC$, $A = 30°, B = 60°$ and $c = 10$, Find a and b.

Solution:
Given that $A = 30°, B = 60°, C = 180° - (A + B) = 90°$
Using sine formula,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 30°} = \frac{b}{\sin 60°} = \frac{10}{\sin 90°}$$

$$a = \frac{10 \sin 30°}{\sin 90°} = \frac{10\left(\frac{1}{2}\right)}{1} = 5$$

$$b = \frac{10 \sin 60°}{\sin 90°} = \frac{10\left(\frac{\sqrt{3}}{2}\right)}{1} = 5\sqrt{3}.$$

Example 3.66 In a $\triangle ABC$, if $a = 2\sqrt{2}, b = 2\sqrt{3}$ and $C = 75°$, find the other side and the angles.

Solution:
Given that $a = 2\sqrt{2}, b = 2\sqrt{3}$ and $C = 75°$.

Using cosine formula, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

$$\cos 75^\circ = \frac{8+12-c^2}{8\sqrt{6}} \Rightarrow \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{8+12-c^2}{8\sqrt{6}} \Rightarrow c = \sqrt{2}\left(\sqrt{3}+1\right)$$

Also, $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{1}{\sqrt{2}}$, Thus $A = 60^\circ$, $B = 180^\circ - (A+C) = 45^\circ$.

Example 3.67 Find the area of the triangle whose sides are 13 cm, 14 cm and 15 cm.

Solution:
We know that $s = \dfrac{a+b+c}{2} \Rightarrow s = \dfrac{13+14+15}{2} = 21\ cm$.
Area of a triangle is $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$
$= \sqrt{21(21-13)(21-14)(21-15)} = 84$ sq.cm.

Example 3.68 In any $\triangle ABC$, prove that $a\cos A + b\cos B + c\cos C = \dfrac{8\triangle^2}{abc}$.

Solution:
We know that $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$

Thus, $a\cos A + b\cos B + c\cos C = 2a\left(\dfrac{2\triangle}{ac}\right)\left(\dfrac{2\triangle}{ab}\right) = \dfrac{8\triangle^2}{abc}$

Example 3.69
Suppose that there are two cell phone towers within range of a cell phone. The two towers are located at 6 km apart along a straight highway, running east to west and the cell phone is north of the highway. The signal is 5 km from the first tower and $\sqrt{31}$ km from the second tower. Determine the position of the cell phone north and east of the first tower and how far it is from the highway.

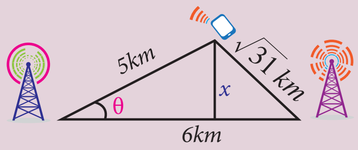


Figure 3.24

Solution:
Let θ be the position of the cell phone from north to east of the first tower.
Then, using the cosine formula, we have,

$$\begin{aligned}\left(\sqrt{31}\right)^2 &= 5^2+6^2-2\times5\times6\cos\theta\\ 31 &= 25+36-60\cos\theta\\ \cos\theta &= \frac{1}{2} \Rightarrow \theta = 60^\circ\end{aligned}$$

Let x be the distance of the cell phone's position from the highway.

Then, $\sin\theta = \dfrac{x}{5} \Rightarrow x = 5\sin\theta = 5\sin 60^\circ = \dfrac{5\times\sqrt{3}}{2}\ km$.

Example 3.70
Suppose that a boat travels 10 km from the port towards east and then turns 60° to its left. If the boat travels further 8 km, how far from the port is the boat?

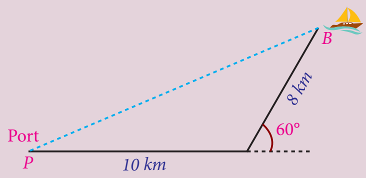


Figure 3.25

Solution:
Let BP be the required distance.
By using the cosine formula, we have,

$$BP^2 = 10^2 + 8^2 - 2 \times 10 \times 8 \times \cos 120^\circ = 244\,km \Rightarrow BP = 2\sqrt{61}\,km$$

Example 3.71
Suppose two radar stations located $100km$ apart, each detect a fighter aircraft between them. The angle of elevation measured by the first station is 30°, whereas the angle of elevation measured by the second station is 45°. Find the altitude of the aircraft at that instant.

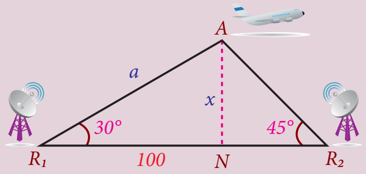


Figure 3.26

Solution:
Let R_1 and R_2 be two radar stations and A be the position of fighter aircraft at the time of detection.
Let x be the required altitude of the aircraft.
Draw $\perp AN$ from A to R_1R_2 meeting at N.

$$\angle A = 180^\circ - (30^\circ + 45^\circ) = 105^\circ$$

Thus, $$\frac{a}{\sin 45^\circ} = \frac{100}{\sin 105^\circ} \Rightarrow a = \frac{100}{\frac{\sqrt{3}+1}{2\sqrt{2}}} \times \frac{1}{\sqrt{2}} = \frac{200\left(\sqrt{3}-1\right)}{2} = 100 \times \left(\sqrt{3}-1\right) km$$

Now, $$\sin 30^\circ = \frac{x}{a} \Rightarrow x = 50 \times \left(\sqrt{3}-1\right) km.$$

Exercise - 3.10

1. Determine whether the following measurements produce one triangle, two triangles or no triangle: $\angle B = 88^\circ, a = 23, b = 2$. Solve if solution exists.
2. If the sides of a $\triangle ABC$ are $a = 4, b = 6$ and $c = 8$, then show that $4\cos B + 3\cos C = 2$.
3. In a $\triangle ABC$, if $a = \sqrt{3} - 1, b = \sqrt{3} + 1$ and $C = 60^\circ$, find the other side and other two angles.
4. In any $\triangle ABC$, prove that the area $\triangle = \dfrac{b^2 + c^2 - a^2}{4\cot A}$.
5. In a $\triangle ABC$, if $a = 12\ cm$, $b = 8\ cm$ and $C = 30^\circ$, then show that its area is 24 $sq.cm$.
6. In a $\triangle ABC$, if $a = 18\ cm$, $b = 24\ cm$ and $c = 30\ cm$, then show that its area is 216 $sq.cm$.
7. Two soldiers A and B in two different underground bunkers on a straight road, spot an intruder at the top of a hill. The angle of elevation of the intruder from A and B to the ground level in the eastern direction are 30° and 45° respectively. If A and B stand 5 km apart, find the distance of the intruder from B.
8. A researcher wants to determine the width of a pond from east to west, which cannot be done by actual measurement. From a point P, he finds the distance to the eastern-most point of the pond to

be $8\ km$, while the distance to the western most point from P to be $6\ km$. If the angle between the two lines of sight is $60°$, find the width of the pond.

9. Two Navy helicopters A and B are flying over the Bay of Bengal at same altitude from the sea level to search a missing boat. Pilots of both the helicopters sight the boat at the same time while they are apart $10\ km$ from each other. If the distance of the boat from A is $6\ km$ and if the line segment AB subtends $60°$ at the boat, find the distance of the boat from B.
10. A straight tunnel is to be made through a mountain. A surveyor observes the two extremities A and B of the tunnel to be built from a point P in front of the mountain. If $AP = 3km$, $BP = 5\ km$ and $\angle APB = 120°$, then find the length of the tunnel to be built.
11. A farmer wants to purchase a triangular shaped land with sides $120 feet$ and $60 feet$ and the angle included between these two sides is $60°$. If the land costs ₹500 per sq.ft, find the amount he needed to purchase the land. Also find the perimeter of the land.
12. A fighter jet has to hit a small target by flying a horizontal distance. When the target is sighted, the pilot measures the angle of depression to be $30°$. If after $100\ km$, the target has an angle of depression of $45°$, how far is the target from the fighter jet at that instant?
13. A plane is $1\ km$ from one landmark and $2\ km$ from another. From the planes point of view the land between them subtends an angle of $45°$. How far apart are the landmarks?
14. A man starts his morning walk at a point A reaches two points B and C and finally back to A such that $\angle A = 60°$ and $\angle B = 45°$, $AC = 4km$ in the $\triangle ABC$. Find the total distance he covered during his morning walk.
15. Two vehicles leave the same place P at the same time moving along two different roads. One vehicle moves at an average speed of $60km/hr$ and the other vehicle moves at an average speed of $80\ km/hr$. After half an hour the vehicle reach the destinations A and B. If AB subtends $60°$ at the initial point P, then find AB.
16. Suppose that a satellite in space, an earth station and the centre of earth all lie in the same plane. Let r be the radius of earth and R be the distance from the centre of earth to the satellite. Let d be the distance from the earth station to the satellite. Let $30°$ be the angle of elevation from the earth station to the satellite. If the line segment connecting earth station and satellite subtends angle α at the centre of earth, then prove that $d = R\sqrt{1 + \left(\frac{r}{R}\right)^2 - 2\frac{r}{R}\cos\alpha}$.

3.9 Inverse Trigonometric Functions

A function $f(x)$ has inverse if and only if it is one-to-one and onto. Thus, inverse of a function cannot be defined if it fails to be one-to-one. However, if we restrict the domain suitably, we can make the function to be one-to-one in the restricted domain. For example, $y = x^2$ is not one-to-one for all real numbers. But $y = x^2$ is one-to-one and onto either for $x \geq 0$ or $x \leq 0$. Hence $y = x^2, x \geq 0$ has the inverse $f^{-1}(x) = \sqrt{x}, x \geq 0$. Now, owing to their periodicity, none of six trigonometric functions is one-to-one over their natural domains. We shall restrict their domains so that trigonometric functions are one-to-one enabling the existence of their inverse functions. This restriction can be done in many ways once again due to their periodicity. The conventional choices for the restricted domains are arbitrary but they have some important characteristics. Each restricted domain includes the number 0 and some positive angles and the image of restricted domain contains the entire range.

Let us define the inverse of sine function. Consider $f(x) = \sin x, x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then, $\sin x$ is one-to-one and onto in this restricted domain. Hence, the inverse of sine function exists. Note that $f^{-1}(y) = x$ if and only if $f(x) = y$. We write $f^{-1}(x) = \sin^{-1}(x)$. Thus, inverse of sine is defined as $\sin^{-1}(y) = x$ if and only if $\sin x = y$.

Clearly, $\sin x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ and $\sin^{-1} x : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus, $\sin^{-1} t$ is an angle whose sine is equal to t and which is located in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Similarly we can define the other inverse trigonometric functions.

The inverse functions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\text{cosec}^{-1}(x)$, $\sec^{-1}(x)$, $\cot^{-1}(x)$ are called inverse circular functions. For the function $y=\sin x$, there are infinitely many angles x which satisfy $\sin x = t$, $-1 \le t \le 1$. Of these infinite set of values, there is one which lies in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. This angle is called the **principal angle** and denoted by $\sin^{-1}t$. The principal value of an inverse function is that value of the general value which is numerically least. It may be positive or negative. When there are two values, one is positive and the other is negative such that they are numerically equal, then the principal value is the positive one.

We shall illustrate below the restricted domains, ranges of trigonometric functions and the domains, ranges of the corresponding inverse functions.

(i) $\sin x : \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \to [-1,1]$; $\sin^{-1}x : [-1,1] \to \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

(ii) $\cos x : [0,\pi] \to [-1,1]$; $\cos^{-1}x : [-1,1] \to [0,\pi]$

(iii) $\tan x : \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \to (-\infty,\infty)$; $\tan^{-1}x : (-\infty,\infty) \to \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(iv) $\cot x : (0,\pi) \to (-\infty,\infty)$; $\cot^{-1}x : (-\infty,\infty) \to (0,\pi)$

(v) $\text{cosec}x : \left[-\frac{\pi}{2},\frac{\pi}{2}\right] - \{0\} \to \mathbb{R} - (-1,1)$; $\text{cosec}^{-1}x : \mathbb{R} - (-1,1) \to \left[-\frac{\pi}{2},\frac{\pi}{2}\right] - \{0\}$

(vi) $\sec x : [0,\pi] - \left\{\frac{\pi}{2}\right\} \to \mathbb{R} - (-1,1)$; $\sec^{-1}x : \mathbb{R} - (-1,1) \to [0,\pi] - \left\{\frac{\pi}{2}\right\}$

(i) $\sin^{-1}x$ does not mean $\dfrac{1}{\sin x}$.
(ii) Another notation for $\sin^{-1}x$ is $\arcsin x$ due to **Sir. John FW Herschel (1813)**.
(iii) While discussing the inverse of sine function, we confined to $y=\sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and
$x = \sin^{-1}y, -1 \le y \le 1$
(iv) The graph of inverse function f^{-1} is the reflexion of the graph of f about the line $y=x$. Thus, if $(a,b) \in f$ then $(b,a) \in f^{-1}$.

Principal values of inverse trigonometric functions are listed below:

Principal value for $x \ge 0$	Principal value for $x < 0$
$0 \le \sin^{-1}(x) \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \sin^{-1}(x) < 0$
$0 \le \cos^{-1}(x) \le \frac{\pi}{2}$	$\frac{\pi}{2} < \cos^{-1}(x) \le \pi$
$0 \le \tan^{-1}(x) < \frac{\pi}{2}$	$-\frac{\pi}{2} < \tan^{-1}(x) < 0$
$0 < \cot^{-1}(x) \le \frac{\pi}{2}$	$-\frac{\pi}{2} < \cot^{-1}(x) < 0$
$0 \le \sec^{-1}(x) < \frac{\pi}{2}$	$\frac{\pi}{2} < \sec^{-1}(x) \le \pi$
$0 < \text{cosec}^{-1}(x) \le \frac{\pi}{2}$	$-\frac{\pi}{2} < \text{cosec}^{-1}(x) < 0$

(i) Properties, graphs, theorems on inverse trigonometric functions will be studied in higher secondary second year.

(ii) Inverse trigonometric functions are much useful in the evaluation of some integrals which will be studied later.

Example 3.72 Find the principal value of (i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$, (ii) $\operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right)$, (iii) $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$.

Solution:

(i) Let $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$, where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\Rightarrow \sin y = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3} \Rightarrow y = \frac{\pi}{3}$

Thus, the principal value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

(ii) Let $\operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right) = y$, where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\Rightarrow \operatorname{cosec} y = \frac{2}{\sqrt{3}} = \sin y = \frac{\sqrt{3}}{2}$

Thus, the principal value of $\operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{3}$

(iii) Let $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = y$, where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\tan y = -\frac{1}{\sqrt{3}} \Rightarrow \tan y = \tan\left(-\frac{\pi}{6}\right) \Rightarrow y = -\frac{\pi}{6}$

Thus the principal value of $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$.

Exercise - 3.11

1. Find the principal value of (i) $\sin^{-1}\frac{1}{\sqrt{2}}$ (ii) $\cos^{-1}\frac{\sqrt{3}}{2}$ (iii) $\operatorname{cosec}^{-1}(-1)$ (iv) $\sec^{-1}\left(-\sqrt{2}\right)$ (v) $\tan^{-1}\left(\sqrt{3}\right)$.
2. A man standing directly opposite to one side of a road of width x meter views a circular shaped traffic green signal of diameter a meter on the other side of the road. The bottom of the green signal is b meter height from the horizontal level of viewer's eye. If α denotes the angle subtended by the diameter of the green signal at the viewer's eye, then prove that

$$\alpha = \tan^{-1}\left(\frac{a+b}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right).$$

Exercise - 3.12

Choose the correct answer or the most suitable answer:

1. $\dfrac{1}{\cos 80^\circ} - \dfrac{\sqrt{3}}{\sin 80^\circ} =$

   (1) $\sqrt{2}$ (2) $\sqrt{3}$ (3) 2 (4) 4

2. If $\cos 28^\circ + \sin 28^\circ = k^3$, then $\cos 17^\circ$ is equal to

   (1) $\dfrac{k^3}{\sqrt{2}}$ (2) $-\dfrac{k^3}{\sqrt{2}}$ (3) $\pm\dfrac{k^3}{\sqrt{2}}$ (4) $-\dfrac{k^3}{\sqrt{3}}$

3. The maximum value of $4\sin^2 x + 3\cos^2 x + \sin\dfrac{x}{2} + \cos\dfrac{x}{2}$ is

   (1) $4+\sqrt{2}$ (2) $3+\sqrt{2}$ (3) 9 (4) 4

4. $\left(1+\cos\dfrac{\pi}{8}\right)\left(1+\cos\dfrac{3\pi}{8}\right)\left(1+\cos\dfrac{5\pi}{8}\right)\left(1+\cos\dfrac{7\pi}{8}\right) =$

   (1) $\dfrac{1}{8}$ (2) $\dfrac{1}{2}$ (3) $\dfrac{1}{\sqrt{3}}$ (4) $\dfrac{1}{\sqrt{2}}$

5. If $\pi < 2\theta < \dfrac{3\pi}{2}$, then $\sqrt{2+\sqrt{2+2\cos 4\theta}}$ equals to

   (1) $-2\cos\theta$ (2) $-2\sin\theta$ (3) $2\cos\theta$ (4) $2\sin\theta$

6. If $\tan 40^\circ = \lambda$, then $\dfrac{\tan 140^\circ - \tan 130^\circ}{1+\tan 140^\circ \tan 130^\circ} =$

   (1) $\dfrac{1-\lambda^2}{\lambda}$ (2) $\dfrac{1+\lambda^2}{\lambda}$ (3) $\dfrac{1+\lambda^2}{2\lambda}$ (4) $\dfrac{1-\lambda^2}{2\lambda}$

7. $\cos 1^\circ + \cos 2^\circ + \cos 3^\circ + \ldots + \cos 179^\circ =$

   (1) 0 (2) 1 (3) -1 (4) 89

8. Let $f_k(x) = \dfrac{1}{k}\left[\sin^k x + \cos^k x\right]$ where $x \in R$ and $k \geq 1$. Then $f_4(x) - f_6(x) =$

   (1) $\dfrac{1}{4}$ (2) $\dfrac{1}{12}$ (3) $\dfrac{1}{6}$ (4) $\dfrac{1}{3}$

9. Which of the following is not true?

   (1) $\sin\theta = -\dfrac{3}{4}$ (2) $\cos\theta = -1$ (3) $\tan\theta = 25$ (4) $\sec\theta = \dfrac{1}{4}$

10. $\cos 2\theta \cos 2\phi + \sin^2(\theta-\phi) - \sin^2(\theta+\phi)$ is equal to

    (1) $\sin 2(\theta+\phi)$ (2) $\cos 2(\theta+\phi)$ (3) $\sin 2(\theta-\phi)$ (4) $\cos 2(\theta-\phi)$

11. $\dfrac{\sin(A-B)}{\cos A\cos B} + \dfrac{\sin(B-C)}{\cos B\cos C} + \dfrac{\sin(C-A)}{\cos C\cos A}$ is

    (1) $\sin A + \sin B + \sin C$ (2) 1 (3) 0 (4) $\cos A + \cos B + \cos C$

12. If $\cos p\theta + \cos q\theta = 0$ and if $p \neq q$, then θ is equal to (n is any integer)

    (1) $\dfrac{\pi(3n+1)}{p-q}$ (2) $\dfrac{\pi(2n+1)}{p\pm q}$ (3) $\dfrac{\pi(n\pm 1)}{p\pm q}$ (4) $\dfrac{\pi(n+2)}{p+q}$

13. If $\tan\alpha$ and $\tan\beta$ are the roots of $x^2+ax+b=0$, then $\dfrac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$ is equal to

(1) $\dfrac{b}{a}$ (2) $\dfrac{a}{b}$ (3) $-\dfrac{a}{b}$ (4) $-\dfrac{b}{a}$

14. In a triangle ABC, $\sin^2 A+\sin^2 B+\sin^2 C=2$, then the triangle is

(1) equilateral triangle (2) isosceles triangle (3) right triangle (4) scalene triangle.

15. If $f(\theta)=|\sin\theta|+|\cos\theta|, \theta\in R$, then $f(\theta)$ is in the interval

(1) $[0,2]$ (2) $\left[1,\sqrt{2}\right]$ (3) $[1,2]$ (4) $[0,1]$

16. $\dfrac{\cos 6x+6\cos 4x+15\cos 2x+10}{\cos 5x+5\cos 3x+10\cos x}$ is equal to

(1) $cos2x$ (2) $\cos x$ (3) $\cos 3x$ (4) $2\cos x$

17. The triangle of maximum area with constant perimeter $12m$

(1) is an equilateral triangle with side $4m$ (2) is an isosceles triangle with sides $2m, 5m, 5m$
(3) is a triangle with sides $3m, 4m, 5m$ (4) Does not exist.

18. A wheel is spinning at 2 radians/second. How many seconds will it take to make 10 complete rotations?

(1) 10π seconds (2) 20π seconds (3) 5π seconds (4) 15π seconds

19. If $\sin\alpha+\cos\alpha=b$, then $\sin 2\alpha$ is equal to

(1) $b^2-1,$ if $b\le\sqrt{2}$ (2) $b^2-1,$ if $b>\sqrt{2}$ (3) $b^2-1,$ if $b\ge 1$ (4) $b^2-1,$ if $b\ge\sqrt{2}$

20. In a $\triangle ABC$, if

(i) $\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}>0$

(ii) $\sin A\sin B\sin C>0$ then

(1) Both (i) and (ii) are true (2) Only (i) is true
(3) Only (ii) is true (4) Neither (i) nor (ii) is true.

Summary

Sum and Difference Identities(Ptolemy Identities):

$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$; $\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta};\quad \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Double, Triple and Half angle Identities

sine	cosine	Tangent
$\sin 2A = 2\sin A\cos A$ $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$ $\sin 3A = 3\sin A - 4\sin^3 A$ $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$ $\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\dfrac{\theta}{2}}$	$\cos 2A = \cos^2 A - \sin^2 A$ $\cos 2A = 2\cos^2 A - 1$ $\cos 2A = 1 - 2\sin^2 A$ $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$ $\cos 3A = 4\cos^3 A - 3\cos A$ $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$ $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$ $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$	$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$ $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\dfrac{\theta}{2}}$

Trigonometric Equation	General solution
$\sin\theta = \sin\alpha$, where $\alpha \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$	$\theta = n\pi + (-1)^n\alpha, n \in \mathbb{Z}$
$\cos\theta = \cos\alpha$, where $\alpha \in [0, \pi]$	$\theta = 2n\pi \pm \alpha, n \in \mathbb{Z}$
$\tan\theta = \tan\alpha$, where $\alpha \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$	$\theta = n\pi + \alpha, n \in \mathbb{Z}$

Law of sine	Law of cosine	Law of Tangent
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R.$	$\cos A = \dfrac{b^2+c^2-a^2}{2bc}$; $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$; $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$	$\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$ $\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$ $\tan\dfrac{C-A}{2} = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

In a $\triangle ABC$, we have

(i) $a = b\cos C + c\cos B$, $b = c\cos A + a\cos C$ and $c = a\cos B + b\cos A$ [**Projection formula**].

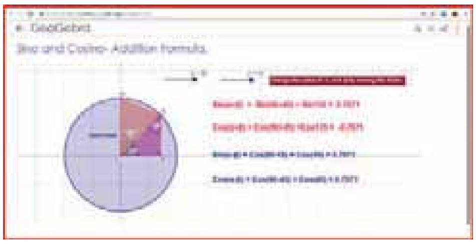
(ii) $\triangle = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$ [**Area of the triangle**].

(iii) $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$

(iv) $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$, $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$, $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$, where s is the semi-perimeter of $\triangle ABC$ given by $s = \dfrac{a+b+c}{2}$. [**Half Angle formula**].

ICT CORNER-3(a)

Expected Outcome⇒



Step–1

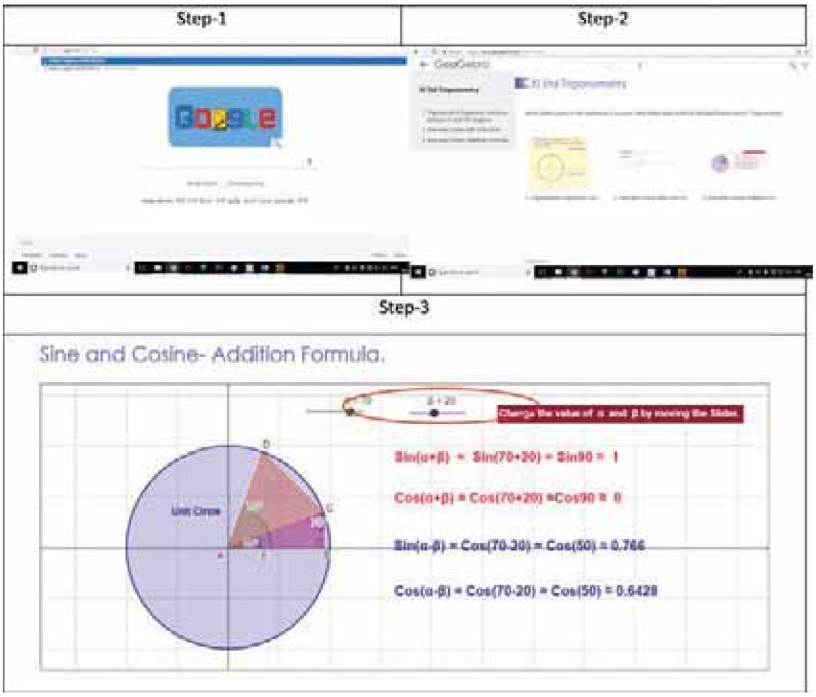
Open the Browser and type the URL Link given below (or) Scan the QR Code..

Step–2

GeoGebra Work book called "XI Std Trigonometry" will appear. In this several work sheets of different Trigonometry concepts are seen. Select one you want. For example, open "Sine and Cosine - Addition Formula"

Step–3

Move the Sliders to change α and β values. Angles in the Diagram changes and the Addition formula values change correspondingly.



Observe for each value and calculate by formulae and check the results. Similarly Open and check other work sheets for Ratios and Unit circle etc..

*Pictures are only indicatives.

Browse in the link XI Std Trigonometry GeoGebra Work book: https://ggbm.at/k5vP7pv2

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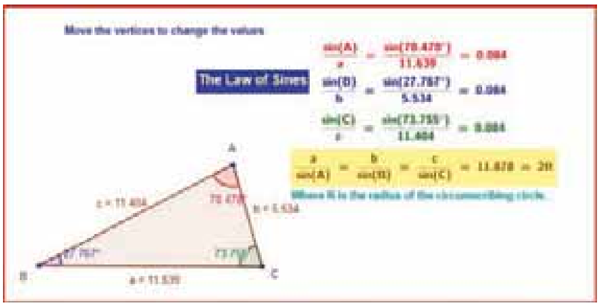
ICT CORNER 3(b)

Expected Outcome⇒



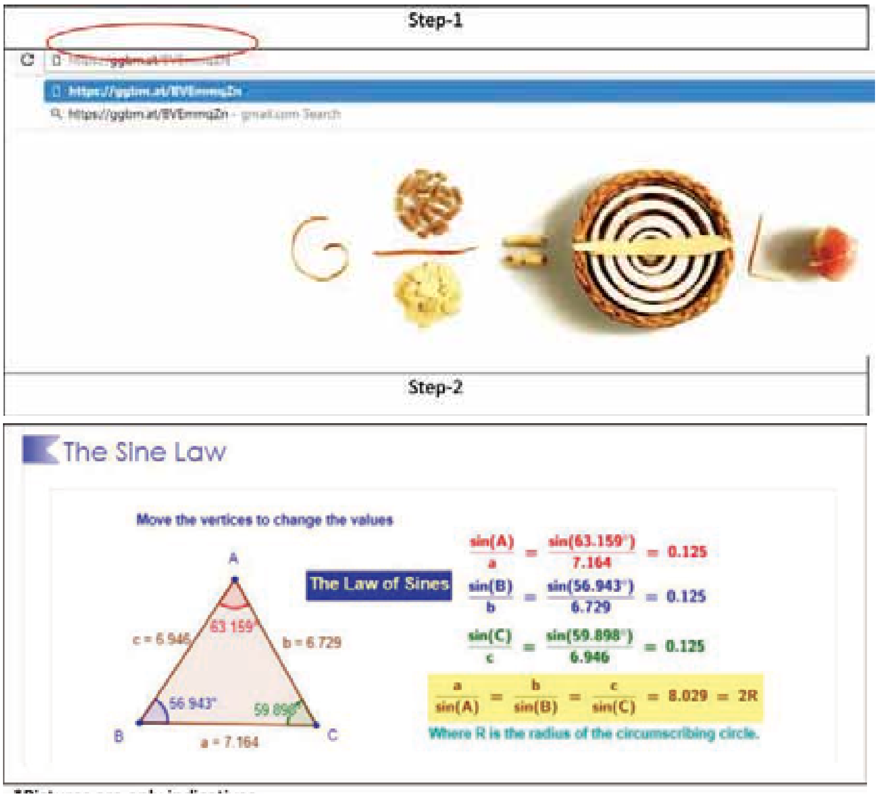
Step–1

Open the Browser andtype the URL Link given below (or) Scan the QR Code.

Step–2

GeoGebra Work sheet called "The Sine Law" will appear. In the worksheet a Triangle is seen with vertices A,B and C with sides a, b and c. Move the vertices to change the Triangle size.

For any triangle Sine Law is maintained and the working is shown on right-hand side-Observe.



*Pictures are only indicatives.

Browse in the link The Sine Law- GeoGebra Work sheet:
https://ggbm.at/BVEmmqZn

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Chapter 4 Combinatorics and Mathematical Induction

"No great discovery was ever made without a bold guess"

- Newton

4.1 Introduction

Combinatorics is the branch of mathematics which is related to counting. It deals with arrangements of objects as well as enumeration, that is, counting of objects with specific properties. The roots of the subject can be traced as far back as 2800 BCE when it was used to study magic squares and patterns within them.

English physicist and mathematician Sir Isaac Newton, most famous for his law of gravitation, was instrumental in the scientific revolution of the 17th century. Newton's belief in the "Persistance of patterns" led to his first significant mathematical discovery, the generalization of the expansion of binomial expressions.

Newton discovered Binomial Theorem which he claimed the easiest way to solve the quadratures of curves. This discovery is essential in understanding probability. The generalized version of the Binomial Theorem, the Multinomial Theorem, applies to multiple variables. It is widely used in Combinatorics and Statistics.


Newton (1643 – 1727)

He was the first to use fractional indices and to employ coordinate geometry to derive solutions to Diophantine equations. He approximated partial sums of the harmonic series by logarithms (a precursor to Euler's summation formula) and was the first to use power series with confidence and to revert power series. Newton's work on infinite series was inspired by Simon Stevin's decimals.

In 1705, he was knighted by Queen Anne of England, making him Sir Isaac Newton. Newton made discoveries in optics and theory of motions. Along with mathematician Leibnitz, Newton is credited for developing essential theories of calculus.

Combinatorics has many real life applications where counting of objects are involved. For example, we may be interested to know if there are enough mobile numbers to meet the demand or the number of allowable passwords in a computer system. It also deals with counting techniques and with optimisation methods, that is, methods related to finding the best possible solution among several possibilities in a real problem. In this chapter we shall study counting problems in terms of ordered or unordered arrangements of objects. These arrangements are referred to as permutations and combinations. Combinatorics are largely used in the counting problems of Network communications, Cryptography, Network Security and Probability theory. We shall explore their properties and apply them to counting problems.

Consider another situation: We all know that our electricity consumer card number is of the form A: B : C, where A denotes the electrical substation /larger capacity transformer number, B denotes the smaller capacity electricity transformer number and C denotes the consumer number. There may be conditions that to each substation certain maximal number of transformer can only be linked and with a particular transformer certain maximal number of consumer connection can only be linked. Now the question of deciding, whether a new Transformer/Substation needs to be erected, can be made by the count of the number of consumer connections linked with a substation transformer. How to get that count? This count can be easily arrived by the use of counting principles.

In this Chapter, the art of counting is discussed starting with the Fundamental principles of counting, travelling through Permutation and Combinations.

Learning Objectives

On completion of this chapter, the students are expected to know

- the principles of counting and applying it to various situations.
- how to compute the number of ways in arranging a set of distinct objects.
- how to compute the number of arrangements from a set containing identical objects.
- how to compute and applying the strategies to find the number of combinations of a set of different objects.
- the applications of the principle of mathematical induction.

We shall start the chapter with the section on

4.2 Fundamental principles of counting

1. **The Sum Rule** Let us consider two tasks which need to be completed. If the first task can be completed in M different ways and the second in N different ways, and if these cannot be performed simultaneously, then there are $M + N$ ways of doing either task. This is the sum rule of counting.

Example 4.1 Suppose one girl or one boy has to be selected for a competition from a class comprising 17 boys and 29 girls. In how many different ways can this selection be made?

Solution:
The first task of selecting a girl can be done in 29 ways. The second task of selecting a boy can be done in 17 ways. It follows from the sum rule, that there are 17+29 = 46 ways of making this selection.

The sum rule may be extended to more than two tasks. Thus if there are n non-simultaneous tasks $T_1, T_2, T_3, \cdots, T_n$ which can be performed in $m_1, m_2, \cdots, m_n$ ways respectively, then the number of ways of doing one of these tasks is $m_1 + m_2 + \cdots + m_n$.

2. **The Product Rule** Let us suppose that a task comprises of two procedures. If the first procedure can be completed in M different ways and the second procedure can be done in N different ways after the first procedure is done, then the total number of ways of completing the task is $M \times N$

Example 4.2 Consider the 3 cities Chennai, Trichy and Tirunelveli. In order to reach Tirunelveli from Chennai, one has to pass through Trichy. There are 2 roads connecting Chennai with Trichy and there are 3 roads connecting Trichy with Tirunelveli. What are the total number of ways of travelling from Chennai to Tirunelveli?

Solution:

There are 2 roads connecting Chennai to Trichy. Suppose these are R_1 and R_2. Further there are 3 roads connecting Trichy to Tirunelveli . Let us name them as S_1, S_2 and S_3. Suppose a person chooses R_1 to travel from Chennai to Trichy and may further choose any of the 3 roads S_1, S_2 or S_3 to travel from Trichy to Tirunelveli. Thus the possible road choices are $(R_1, S_1), (R_1, S_2), (R_1, S_3)$. Similarly, if the person chooses R_2 to travel from Chennai to Trichy, the choices would be $(R_2, S_1), (R_2, S_2), (R_2, S_3)$.

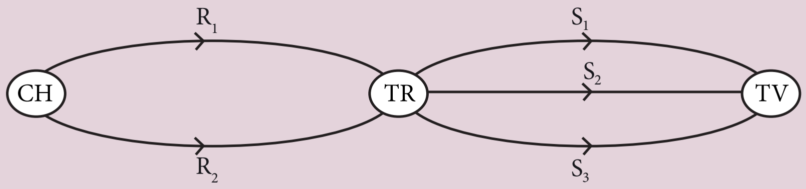


Figure 4.1

Thus there are $2 \times 3 = 6$ ways of travelling from Chennai to Tirunelveli.

An extension of the product rule may be stated as follows:
If a task comprises of n procedures $P_1, P_2, P_3, \cdots, P_n$ which can be performed in $m_1, m_2, \cdots, m_n$ ways respectively, and procedure P_i can be done after procedures $P_1, P_2, P_3, \cdots, P_{i-1}$ are done, then the number of ways of completing the task is $m_1 \times m_2 \times \cdots \times m_n$.

3. **The Inclusion-Exclusion Principle** Suppose two tasks A and B can be performed simultaneously. Let $n(A)$ and $n(B)$ represent the number of ways of performing the tasks A and B independent of each other. Also let $n(A \cap B)$ be the number of ways of performing the two tasks simultaneously. We cannot use the sum rule to count the number of ways of performing one of the tasks as that would lead to over counting. To obtain the correct number of ways we add the number of ways of performing each of the two tasks and then subtract the number of ways of doing both tasks simultaneously. This method is referred to as the principle of inclusion - exclusion. Using the notation of set theory we write it as

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Suppose we have to find the number of positive integers divisible by 2 or 7 (but not both), upto 1000. Let $n(A)$ denote the number of integers divisible by 2, $n(B)$ denote the number of integers divisible by 7 and $n(A \cap B)$ the number of integers divisible by both 2 and 7. Then the number of positive integers divisible by 2 or 7 is given by

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 500 + 142 - 71 = 571.$$

(Note that $n(A)$ will include all multiples of 2 upto 1000, $n(B)$ will include all multiples of 7 upto 1000 and so on.)

Tree Diagrams: Tree diagrams are often helpful in representing the possibilities in a counting problem. Typically in a tree the branches represent the various possibilities. For example, suppose a person wants to buy a Car for the family. There are two different branded cars and five colours are available for each brand. Each colour will have three different variant on it namely GL,SS,SL. Then the various choices for choosing a car can be represented through a tree diagram as follows:

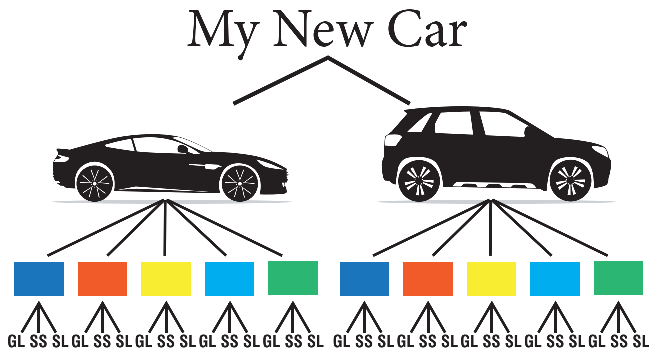


Figure 4.2

We shall now illustrate the different rules described above through examples

Example 4.3 A School library has 75 books on Mathematics, 35 books on Physics. A student can choose only one book. In how many ways a student can choose a book on Mathematics or Physics?

Solution:

(i) A student can choose a Mathematics book in "75" different ways.
(ii) A student can choose a Physics book in "35" different ways.

Hence applying the Rule of Sum, the number of ways a student can choose a book is $75 + 35 = 110$.

Now we shall discuss the problem stated in our introduction.

Example 4.4 If an electricity consumer has the consumer number say 238:110: 29, then describe the linking and count the number of house connections upto the 29th consumer connection linked to the larger capacity transformer number 238 subject to the condition that each smaller capacity transformer can have a maximal consumer link of say 100.

Solution:
The following figure illustrates the electricity distribution network.

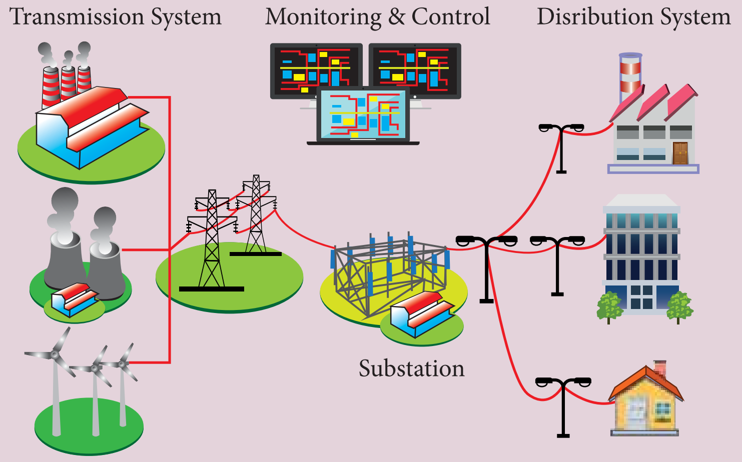


Figure 4.3

There are 110 smaller capacity transformers attached to a larger capacity transformer. As each smaller capacity transformer can be linked with only 100 consumers, we have for the 109 transformers, there will be $109 \times 100 = 10900$ links. For the 110^{th} transformer there are only 29 consumers linked. Hence, the total number of consumers linked to the 238^{th} larger capacity transformer is $10900 + 29 = 10929$.

Example 4.5 A person wants to buy a car. There are two brands of car available in the market and each brand has 3 variant models and each model comes in five different colours as in Figure 4.2 In how many ways she can choose a car to buy?

Solution:
A car can be bought by choosing a brand, then a variant model, and then a colour. A brand can be chosen in 2 ways; a model can be chosen in 3 ways and a colour can be chosen in 5 ways. By the rule of product the person can buy a car in $2 \times 3 \times 5 = 30$ different ways.

Example 4.6 A Woman wants to select one silk saree and one sungudi saree from a textile shop located at Kancheepuram. In that shop, there are 20 different varieties of silk sarees and 8 different varieties of sungudi sarees. In how many ways she can select her sarees?

Solution:
The work is done when she selects one silk saree and one sungudi saree. The Woman can select a silk saree in 20 ways and sungudi saree in 8 ways. By the rule of product, the total number of ways of selecting these 2 sarees is $20 \times 8 = 160$.

Example 4.7 In a village, out of the total number of people, 80 percentage of the people own Coconut groves and 65 percent of the people own Paddy fields. What is the minimum percentage of people own both?

Solution:
Let $n(C)$ denote the percentage of people who own the Coconut groves and $n(P)$ denote the percentage of people who own Paddy fields. We are given $n(C) = 80$ and $n(P) = 65$. By the rule of inclusion - exclusion $n(C \cap P) = n(C) + n(P) - n(C \cup P)$. The maximum value of $n(C \cup P)$ is 100. Therefore, the minimum value of $n(C \cap P)$ is $80 + 65 - 100 = 45$. That is, the minimum percentage of the people who own both is 45.

In the next problem, we use the notion of a 'string'. A string is formed by writing given letters one by one in a sequence. For instance, strings of length three formed out of the letters a,b,c & d are aaa, abb, bda, dca, cdd $\cdots$.

Example 4.8

(i) Find the number of strings of length 4, which can be formed using the letters of the word BIRD, without repetition of the letters.

(ii) How many strings of length 5 can be formed out of the letters of the word PRIME taking all the letters at a time without repetition.

Solution:

(i) There are as many strings as filling the 4 vacant places by the 4 letters, keeping in mind that repetition is not allowed. The first place can be filled in 4 different ways by any one of the letters B,I,R,D. Following which, the second place can be filled in by any one of the remaining 3 letters in 3 different ways, following which the third place can be filled in 2 different ways, following which fourth place can be filled in 1 way.

Thus the number of ways in which the 4 places can be filled, by the rule of product is $4 \times 3 \times 2 \times 1 = 24$. Hence, the required number of strings is 24.

(ii) There are 5 different letters with which 5 places are to be filled. The first place can be filled in 5 ways as any one of the five letters P,R,I,M,E can be placed there. Having filled the first place with any of the 5 letters, 4 letters are left to be placed in the second place, three letters are left for the third place and 2 letters are left to be put in the fourth place. The remaining 1 letter has to be placed in the fifth place.

Hence, the total number of ways filling up five places is $5 \times 4 \times 3 \times 2 \times 1 = 120$.

Observe the similarity between the above two cases.

Example 4.9 How many strings of length 6 can be formed using letters of the word FLOWER if

(i) either starts with F or ends with R?

(ii) neither starts with F nor ends with R?

Solution:

In any such string, each of the letters F,L,O,W,E,R is used exactly once.

(i) If such a string starts with F, then the other five positions are to be filled with the letters L, O, W, E, R.

As there cannot be any repetition of letters in the formation of the strings we can fill up the 2nd, 3rd, 4th, 5th and 6th places in 5, 4, 3, 2 and 1 ways.

F					
1 way	**5 way**	**4 way**	**3 way**	**2 way**	**1 way**

Figure 4.4

Hence, by the rule of product, the number of strings of length 6 starting with F is equal to $5 \times 4 \times 3 \times 2 \times 1 = 120$.

If such a string ends with R, then the other five positions are to be filled with the letters F,L,O,W,E.

					R
5 way	**4 way**	**3 way**	**2 way**	**1 way**	**1 way**

Figure 4.5

As in the previous case, we conclude that the number of strings of length ending with R is 120.

If a string starts with F and also ends with R, then the other 4 positions are to be filled with letters L, O, W, E.

F					**R**
1 way	**4 way**	**3 way**	**2 way**	**1 way**	**1 way**

Figure 4.6

As in the previous cases, the number of strings of length of 6 starting with F and ending with R is $4 \times 3 \times 2 \times 1 = 24$.

By the principle of inclusion - exclusion, the number of strings of length 6, either starting with F or ending with R is $120 + 120 - 24 = 216$.

(ii) A string that neither starts with F nor ends with R is one which has not been counted in (i). Together, they account for all possible strings of length 6 formed out of the letters, F,L,O,W,E,R, where no letter is repeated.

Now, the number of all such strings is formed by filling the first position by any of the 6 letters, the second by any of the remaining 5 letters and so on. That is, there are in total $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ such strings. The number of words neither starting with F nor ending with R is the same as the difference between total number of letter strings and the number of strings either starting with F or end with R which is $720 - 216 = 504$.

Example 4.10 How many licence plates may be made using either two distinct letters followed by four digits or two digits followed by 4 distinct letters where all digits and letters are distinct?

Solution:
Here we have two cases:

Case 1: The number of licence plates having two letters followed by four digits is
$26 \times 25 \times 10 \times 9 \times 8 \times 7 = 32, 76, 000.$

Case 2: The number of licence plates having two digits followed by four letters is
$10 \times 9 \times 26 \times 25 \times 24 \times 23 = 3, 22, 92, 000.$
Since either case 1 or case 2 is possible, the total number of licence plates is
$(26 \times 25 \times 10 \times 9 \times 8 \times 7) + (10 \times 9 \times 26 \times 25 \times 24 \times 23) = 3, 55, 68, 000.$

Example 4.11 Count the number of positive integers greater than 7000 and less than 8000 which are divisible by 5, provided that no digits are repeated.

Solution:
It should be a 4-digit number greater than 7000 and less than 8000. Then the 1000^{th} place will be the digit 7. Further, as the number must be divisible by 5 the unit place should be either 0 or 5.

7			**0 or 5**
1 way	**8 way**	**7 way**	**2 way**

Figure 4.7

As repetition is not permitted, the 100^{th} place can be filled in 8 ways using remaining numbers and 10^{th} place can be filled in 7 ways.

Hence, the required number of numbers is $1 \times 8 \times 7 \times 2 = 112$.

Example 4.12 How many 4 - digit even numbers can be formed using the digits $0, 1, 2, 3$ and 4, if repetition of digits are not permitted?

Solution:
There are two conditions as follows:

1. It is 4-digit number and hence its 1000^{th} place cannot be 0.
2. It is an even number and hence its unit place can be either 0, 2 or 4.

Two cases arise in this situation. Either 0 in the unit place or not.

Case 1. When the unit place is filled by 0, then the 1000^{th} place can be filled in 4 ways, 100^{th} place can be filled in 3 ways and 10^{th} place in 2 ways. Therefore, number of 4-digit numbers having 0 at unit place is
$4 \times 3 \times 2 \times 1 = 24$.

			0
4 way	3 way	2 way	1 way

Figure 4.8

Case 2. When the unit place is filled with non-zero numbers, that is 2 or 4, the number of ways is 2, the number of ways of filling the 1000^{th} place is in 3 ways (excluding '0'), 100^{th} place in 3 ways and 10^{th} place in 2 ways. Therefore, number of 4-digit numbers without 0 at unit place is $3 \times 3 \times 2 \times 2 = 36$.

$\bar{0}$			2/4
3 way	3 way	2 way	2 way

Figure 4.9

Hence, by the rule of sum, the required number of 4 digit even numbers is 24+36 = 60.

Example 4.13 Find the total number of outcomes when 5 coins are tossed once.

Solution:
When a coin is tossed, the outcomes are in two ways which are $\{Head, Tail\}$.
By the rule of product rule, the number of outcomes when 5 coins are tossed is
$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$.

More generally, if n coins are tossed then the number of outcomes is 2^n.

Example 4.14 In how many ways (i) 5 different balls be distributed among 3 boxes? (ii) 3 different balls be distributed among 5 boxes?

Solution:

(i) Each ball can be placed into any one of the three boxes in 3 different ways. Therefore, by rule of product, the number of ways of distributing 5 different balls among three boxes is $3 \times 3 \times 3 \times 3 \times 3 = 3^5 = 243$.
(ii) Each ball can be placed into any one of the five boxes in 5 different ways. Therefore, by rule of product, the number of ways of distributing 3 different balls among five boxes is $5 \times 5 \times 5 = 5^3 = 125$.

In order to avoid confusions, take the objects(balls) and distribute them in places(boxes). More generally, if n different objects are to be placed in m places, then the number of ways of placing is m^n.

Example 4.15 There are 10 bulbs in a room. Each one of them can be operated independently. Find the number of ways in which the room can be illuminated.

Solution:
Each of the 10 bulbs are operated independently means that each bulb can be operated in two ways. That is in off mode or on mode. The total number of doing this are 2^{10} which includes the case in which 10 bulbs are off. Keeping all 10 bulbs in "off" mode, the room cannot be illuminated. Hence, the total number of ways are $2^{10} - 1 = 1024 - 1 = 1023$.

Another concept which is an essential tool in a counting process which is stated as follows:

The Pigeonhole Principle:

Suppose a flock of pigeons fly into a set of pigeonholes. If there are more pigeons than pigeonholes then there must be at least one pigeonhole with at least two pigeons in it. A generalised form of this may be applied to other objects and situations as well.
If $k + 1$ or more objects are placed in k boxes, then there is at least one box containing two or more of the objects.
Here are some examples.

1. In any group of 27 English words, there must be at least two words which begin with the same letter (since there are only 26 letters in the English alphabet).
2. If six meetings are held on weekdays only, then there must be at least two meetings held on the same day.

In order to understand the Permutation and Combinations we need a concept called "Factorials" which will be discussed in the next section.

4.3 Factorials

Factorial of a natural number n is the product of the first n natural numbers. It is denoted by $n!$. That is,

$$n! = 1 \times 2 \times 3 \times \cdots \times n.$$

We read this symbol as "n factorial" or "factorial of n". The notation $n!$ was introduced by the French mathematician Christian Kramp in the year 1808. Note that for a positive integer n

$$\begin{aligned} n! &= n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1 \\ &= n(n-1)! \quad \text{for} \quad n > 1 \\ &= n(n-1)(n-2)! \quad \text{for} \quad n > 2 \\ &= n(n-1)(n-2)(n-3)! \quad \text{for} \quad n > 3 \quad \text{and so on.} \end{aligned}$$

Observe that,

$$\begin{aligned} 1! &= 1 \\ 2! &= 2 \times 1 = 2 \\ 3! &= 3 \times 2 \times 1 = 6 \\ 4! &= 4 \times 3 \times 2 \times 1 = 24 \\ 5! &= 5 \times 4 \times 3 \times 2 \times 1 = 120 \\ \ldots &= \ldots \\ 22! &= 22 \times 21 \times 20 \times \cdots \times 3 \times 2 \times 1 \quad = 1124000727777607680000 \end{aligned}$$

The number 22 (the Birth date of Ramanujan) has a special place with respect to factorial that, it is the least integer N greater than 1 whose factorial has exactly N digits.

It will be a good exercise for both students and teachers to find the next number N such that $N!$ has exactly N digits.

Note that $0! = 1$ is evident by substituting $n = 0$ in the equation $(n+1)! = (n+1) \times n!$ as $1! = (0+1) \times 0! \implies 0! = \frac{1!}{1} = 1$. This way, we talk of factorial for non-negative integers. Note that factorials can be extended to certain negative numbers and also to complex numbers, which are beyond the scope of this book.

We shall now discuss certain examples in order to familiarise the computation of factorials.

Example 4.16 Find the value of

$$\text{(i)}\ 5! \quad \text{(ii)}\ 6! - 5! \quad \text{(iii)}\ \frac{8!}{5! \times 2!}.$$

Solution:

(i) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$

(ii) $6! - 5! = 6 \times 5! - 5! = (6-1) \times 5! = 5 \times 120 = 600.$

(iii) $\dfrac{8!}{5! \times 2!} = \dfrac{8 \times 7 \times 6 \times 5!}{5! \times 2!} = \dfrac{8 \times 7 \times 6}{2} = 168.$

Example 4.17 Simplify $\frac{7!}{2!}$

Solution:

$$\frac{7!}{2!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2!}{2!} = 7 \times 6 \times 5 \times 4 \times 3 = 2520.$$

Example 4.18 Evaluate $\frac{n!}{r!(n-r)!}$ when (i) $n = 7, r = 5$ (ii) $n = 50, r = 47$ (iii) For any n with $r = 3$.

Solution:

(i) When $n = 7, r = 5$

$$\frac{n!}{r!(n-r)!} = \frac{7!}{5!(7-5)!} = \frac{7 \times 6 \times 5!}{5! \times 2!} = \frac{7 \times 6}{1 \times 2} = 21.$$

(ii) When $n = 50, r = 47$

$$\frac{n!}{r!(n-r)!} = \frac{50!}{47!(50-47)!} = \frac{50 \times 49 \times 48 \times 47!}{47! \times 3!} = \frac{50 \times 49 \times 48}{1 \times 2 \times 3} = 19600.$$

(iii) For any n and $r = 3$

$$\frac{n!}{r!(n-r)!} = \frac{n!}{3!(n-3)!} = \frac{n \times (n-1) \times (n-2)(n-3)!}{1 \times 2 \times 3 \times (n-3)!} = \frac{n(n-1)(n-2)}{6}.$$

Example 4.19 Let N denote the number of days. If the value of $N!$ is equal to the total number of hours in N days then find the value of N?

Solution:
We need to solve the equation $N! = 24 \times N$.
For $N = 1, 2, 3, 4,\quad N! < 24 \times N$.
For $N = 5$, we have $N! = 5! = 4! \times 5 = 24N$.
For $N > 5$, we have $N! \geq 5!N > 24 \times N$. Hence $N = 5$.

Example 4.20 If $\frac{6!}{n!} = 6$, then find the value of n.

Solution:

$\frac{6!}{n!} = \frac{1.2.3.4.5.6.}{1.2.3...n} = 6$. As $n < 6$ we get, $n = 5$.

Example 4.21 If $n! + (n-1)! = 30$, then find the value of n.

Solution:
Now, $30 = 6 \times 5$. As $n! + (n-1)! = (n+1)(n-1)!$, equating $(n-1)! = 6 = 3!$, we get $n = 4$.

Example 4.22 What is the unit digit of the sum $2! + 3! + 4! + ... + 22!$?

Solution:
From $5!$ onwards for all $n!$ the unit digit is zero and hence the contribution to the unit digit is through $2! + 3! + 4!$ only. which is $2 + 6 + 24 = 32$. Therefore the required unit digit is 2.

Example 4.23 If $\frac{1}{7!} + \frac{1}{8!} = \frac{A}{9!}$ then find the value of A.

Solution:
We have, $\frac{A}{9 \times 8 \times 7!} = \frac{1}{7!} + \frac{1}{8 \times 7!}$

Therefore, $\frac{1}{7!} \times \frac{A}{9 \times 8} = \frac{1}{7!} \times \left[1 + \frac{1}{8}\right]$ equivalently, $\frac{A}{72} = \frac{9}{8}$, which imply $A = 81$.

Example 4.24 Prove that $\frac{(2n)!}{n!} = 2^n(1.3.5 \cdots (2n-1))$.

Solution:

$$\begin{aligned}
\frac{(2n)!}{n!} &= \frac{1.2.3.4 \cdots (2n-2).(2n-1)2n}{n!} \\
&= \frac{(1.3.5. \cdots (2n-1))\,(2.4.6. \cdots (2n-2).2n)}{n!} \\
&\quad \text{(Grouping the odd and even numbers separately)} \\
&= \frac{(1.3.5. \cdots (2n-1)) \times 2^n \times (1.2.3. \cdots (n-1).n)}{n!} \text{(taking out the 2' s)} \\
&= \frac{(1.3.5. \cdots (2n-1)) \times 2^n \times n!}{n!} \\
&= 2^n\,(1.3.5. \cdots (2n-1))\,.
\end{aligned}$$

Exercise - 4.1

1. (i) A person went to a restaurant for dinner. In the menu card, the person saw 10 Indian and 7 Chinese food items. In how many ways the person can select either an Indian or a Chinese food?
   (ii) There are 3 types of toy car and 2 types of toy train available in a shop. Find the number of ways a baby can buy a toy car and a toy train?
   (iii) How many two-digit numbers can be formed using 1,2,3,4,5 without repetition of digits?
   (iv) Three persons enter in to a conference hall in which there are 10 seats. In how many ways they can take their seats?
   (v) In how many ways 5 persons can be seated in a row?
2. (i) A mobile phone has a passcode of 6 distinct digits. What is the maximum number of attempts one makes to retrieve the passcode?

(ii) Given four flags of different colours, how many different signals can be generated if each signal requires the use of three flags, one below the other?

3. Four children are running a race.
   (i) In how many ways can the first two places be filled?
   (ii) In how many different ways could they finish the race?
4. Count the number of three-digit numbers which can be formed from the digits 2,4,6,8 if
   (i) repetitions of digits is allowed. (ii) repetitions of digits is not allowed
5. How many three-digit numbers are there with 3 in the unit place? (i) with repetition (ii) without repetition.
6. How many numbers are there between 100 and 500 with the digits 0, 1, 2, 3, 4, 5 ? if (i) repetition of digits allowed (ii) the repetition of digits is not allowed.
7. How many three-digit odd numbers can be formed by using the digits 0, 1, 2, 3, 4, 5 ? if (i) the repetition of digits is not allowed (ii) the repetition of digits is allowed.
8. Count the numbers between 999 and 10000 subject to the condition that there are (i) no restriction. (ii) no digit is repeated. (iii) at least one of the digits is repeated.
9. How many three-digit numbers, which are divisible by 5, can be formed using the digits 0, 1, 2, 3, 4, 5 if (i) repetition of digits are not allowed? (ii) repetition of digits are allowed?
10. To travel from a place A to place B, there are two different bus routes B_1, B_2, two different train routes T_1, T_2 and one air route A_1. From place B to place C there is one bus route say $B_1^{'}$, two different train routes say $T_1^{'}, T_2^{'}$ and one air route $A_1^{'}$. Find the number of routes of commuting from place A to place C via place B without using similar mode of transportation.
11. How many numbers are there between 1 and 1000 (both inclusive) which are divisible neither by 2 nor by 5?
12. How many strings can be formed using the letters of the word LOTUS if the word
    (i) either starts with L or ends with S? (ii) neither starts with L nor ends with S?
13. (i) Count the total number of ways of answering 6 objective type questions, each question having 4 choices.
    (ii) In how many ways 10 pigeons can be placed in 3 different pigeon holes ?
    (iii) Find the number of ways of distributing 12 distinct prizes to 10 students?
14. Find the value of

    (i) $6!$ (ii) $4!+5!$ (iii) $3!-2!$ (iv) $3!\times 4!$ (v) $\dfrac{12!}{9!\times 3!}$ (vi) $\dfrac{(n+3)!}{(n+1)!}$

15. Evaluate $\dfrac{n!}{r!(n-r)!}$ when

    (i) $n=6, r=2$ (ii) $n=10, r=3$ (iii) For any n with $r=2$.

16. Find the value of n if

    (i) $(n+1)! = 20(n-1)!$ (ii) $\dfrac{1}{8!}+\dfrac{1}{9!}=\dfrac{n}{10!}$.

Factorials can be generalised as double factorial as follows:

Double Factorial of n:

Factorial of an integer n, denoted by $n!$ can be viewed as a function $f : \mathbb{N} \cup \{0\} \to \mathbb{N}$, where $\mathbb{N}$ is the set of all Natural integers, defined as

$$f(n) = \begin{cases} 1 & \text{for } n = 0, \\ n \times (n-1) \times \ (n-2) \times .... \times 3 \times 2 \times 1 & \text{for } n \neq 0. \end{cases}$$

One can define $n!!$ (double factorial of n) as

$$g(n) = \begin{cases} 1 & \text{for } n = 0, \\ n \times (n-2) \times \ (n-4) \times .... \times 4 \times 2 & \text{for n is even} \\ n \times (n-2) \times \ (n-4) \times .... \times 3 \times 1 & \text{for n is odd} \end{cases}$$

Accordingly, $5!! = 5 \times 3 \times 1 = 15$ and $8!! = 8 \times 6 \times 4 \times 2 = 376$.
Note that $n!! \neq (n!)!$ as $4!! = 8$ where as $(4!)! = (24)!$

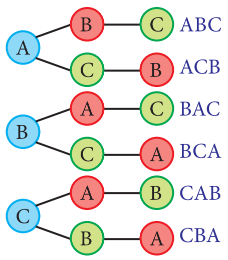
2C4NZB

4.4 Permutations

What is a permutation ?

Permuations come in various disguises.

Suppose three friends A, B and C have to stand in line for a photograph. In how many order can they stand? Some of the possible arrangements (from left to right) are



A, B, C: A, C, B: B, A, C
B, C, A: C, B, A: C, A, B.

Thus there are six possible ways in which they can arrange themselves for the photograph.

Thus if 3 objects have to be arranged in a row there are $3 \times 2 \times 1 = 3!$ possible permutations. The number of permutations of 4 objects taken all at a time is $4 \times 3 \times 2 \times 1 = 4!$ Thus if n objects have to be arranged in a line there are $n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1 = n!$ possible arrangements or permutations.

Suppose you have 7 letters A,B,C,D,E,F and G. We want to make a 4 letter string. We have 7 choices for the 1st letter. Having chosen the first letter, we have 6 choices for the second letter. Proceeding this way, we have 4 choices for the 4th letter.
Hence, the number of permutations of 4 letters chosen from 7 letters is

$$7 \times 6 \times 5 \times 4 = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{7!}{3!} = \frac{7!}{(7-4)!}.$$

More generally, the number of distinct permutations of r objects which can be made from n distinct objects is $\dfrac{n!}{(n-r)!}$. It is denoted by ${}^{n}P_{r}$. The formal proof of this result will be proved in this section.

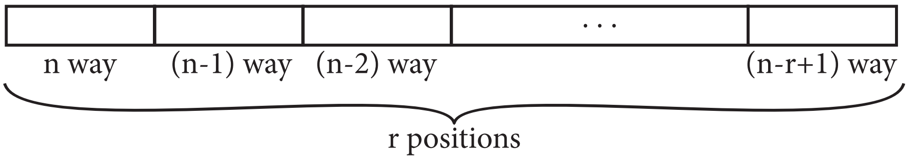
4.4.1 Permutations of distinct objects

In terms of function on any finite set say $S = \{x_1, x_2, ...x_n\}$, a permutation can be defined as a bijective mapping on the set S onto itself. The number of permutation on the set S is the same as the total number of bijective mappings on the set S.

We denote the number of permutations by ${}^nP_r$.

Theorem 4.1: If n, r are positive integers and $r \leq n$, then the number of permutations of n distinct objects taken r at a time is $n\,(n-1)\,(n-2)\cdots(n-r+1)$.

Proof. A permutation is an ordering. A permutation of n distinct objects taken r at a time is formed by filling of r positions, in a row with objects chosen from the given n distinct objects.



There are n objects that can be filled in the first position. For the second position there are remaining $n-1$ objects. There are $n-2$ objects for the third position. Continuing like this until finally we place one of the $(n-(r-1))$ possible objects in the r^{th} position. By the rule of product we conclude ${}^nP_r = n\,(n-1)\,(n-2)\cdots(n-r+1)$.

□

Theorem 4.2: If $n \geq 1$, and $0 \leq r \leq n$, then ${}^nP_r = \dfrac{n!}{(n-r)!}$.

Proof. By Theorem 4.1, we have,

$$\begin{aligned} {}^nP_r &= n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) \\ &= \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) \times (n-r) \times (n-r-1)\cdots 2 \times 1}{(n-r) \times (n-r-1) \times \cdots \times 2 \times 1} \\ &= \frac{n!}{(n-r)!}. \end{aligned}$$

□

To be specific, if n is a positive integer, and r is any non-negative integer, we can represent,

$${}^nP_r = \begin{cases} \dfrac{n!}{(n-r)!} & \text{for } r \leq n, \\ 0 & \text{for } r > n. \end{cases}$$

$${}^nP_r = \begin{cases} {}^nP_n = n! & \text{for } r = n, \\ {}^nP_0 = 1 & \text{for } r = 0. \end{cases}$$

The n different objects arranged in a row is ${}^nP_n = n!$ ways.

Theorem 4.3: The number of permutations of n different objects taken r at a time where repetition is allowed, is n^r.

Proof. As in Theorem 4.1.

$$\underbrace{\underset{\text{n way}}{\boxed{}}\ \underset{\text{n way}}{\boxed{}}\ \underset{\text{n way}}{\boxed{}}\ \boxed{\quad\cdots\quad}\ \underset{\text{n way}}{\boxed{}}}_{\text{r positions}}$$

We can fill the first position with n objects. For the second position (still we can use the object used in first position), there are n objects, and so on the r^{th} position can be filled with n objects.By the rule of product, The number of permutations of n different objects taken r at a time when repetition allowed is $n \times n \times n \times \cdots n$ (r times) $= n^r$. □

4.4.2 Properties of Permutations

1.
$$^nP_n = ^nP_{n-1}.$$

Proof. We have,

$$^nP_{n-1} = \frac{n!}{(n-(n-1))!} = \frac{n!}{1!} = n! = \frac{n!}{(n-n)!} = {^nP_n}$$ □

2.
$$^nP_r = n \times ^{n-1}P_{r-1}$$

Proof. We have,

$$n \times {^{n-1}P_{r-1}} = n \times \frac{(n-1)!}{((n-1)-(r-1))!} = \frac{n!}{(n-r)!} = {^nP_r}.$$

Continuing this process, we arrive at

$$\begin{aligned} ^nP_r &= n \times {^{n-1}P_{r-1}} = n \times (n-1) \times ^{n-2} P_{r-2} \\ &= n \times (n-1) \times (n-2) \times ^{n-3} P_{r-3} \times \cdots \times (n-(r-1))^{n-r}P_0 \\ &= n \times (n-1) \times (n-2) \times \cdots \times (n-r+1). \end{aligned}$$

$$\boxed{^nP_r = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1).}$$

□

3. $^nP_r = ^{n-1}P_r + r \times {^{n-1}P_{r-1}}$

Proof. We have,

$$\begin{aligned} ^{n-1}P_r + r \times {^{n-1}P_{r-1}} &= \frac{(n-1)!}{((n-1)-r)!} + r\frac{(n-1)!}{(n-r)!} \\ &= \frac{(n-1)!}{(n-1-r)!} + r\frac{(n-1)!}{(n-r)!} \\ &= \frac{(n-1)! \times (n-r)}{(n-1-r)! \times (n-r)} + r\frac{(n-1)!}{(n-r)!} \\ &= \frac{(n-1)! \times (n-r)}{(n-r)!} + r\frac{(n-1)!}{(n-r)!} \\ &= \frac{(n-1)!\,((n-r)+r)}{(n-r)!} = \frac{(n-1)!n}{(n-r)!} \\ &= \frac{n!}{(n-r)!} = {^nP_r} \end{aligned}$$

□

Example 4.25 Evaluate:

$$\text{(i)}\ \ {}^4P_4 \quad \text{(ii)}\ \ {}^5P_3 \quad \text{(iii)}{}^8P_4\ \ \text{(iv)}\ \ {}^6P_5.$$

Solution:

(i) ${}^4P_4 = 4 \times 3 \times 2 \times 1 = 4! = 24.$

(ii) ${}^5P_3 = 5 \times 4 \times 3 = 60.$

(iii) ${}^8P_4 = 8 \times 7 \times 6 \times 5 = 1680.$

(iv) ${}^6P_5 = 6 \times 5 \times 4 \times 3 \times 2 = 6! = 720.$

Example 4.26 If ${}^{(n+2)}P_4 = 42\ \times\ {}^nP_2$, find n.

Solution:

$$\begin{aligned} {}^{(n+2)}P_4 &= 42\ \times\ {}^nP_2 \\ \Rightarrow \frac{{}^{(n+2)}P_4}{{}^nP_2} &= 42 \\ \Rightarrow \frac{(n+2)(n+1)(n)(n-1)}{n(n-1)} &= 42 \\ \Rightarrow (n+2)(n+1) &= 42 = 7 \times 6 \\ \Rightarrow n+2 &= 7 \Rightarrow n = 5. \end{aligned}$$

Example 4.27 If ${}^{10}P_r ={}^7 P_{r+2}$ find r.

Solution:

$$\begin{aligned} {}^{10}P_r &= {}^7P_{r+2} \\ \frac{10!}{(10-r)!} &= \frac{7!}{(5-r)!} \end{aligned}$$

i.e.,

$$\begin{aligned} \frac{10 \times 9 \times 8 \times 7!}{(10-r) \times (9-r) \times (8-r) \times (7-r) \times (6-r) \times (5-r)!} &= \frac{7!}{(5-r)!} \\ (10-r) \times (9-r) \times (8-r) \times (7-r) \times (6-r) &= 10 \times 9 \times 8 = 6 \times 5 \times 4 \times 3 \times 2. \end{aligned}$$

Therefore, $10 - r = 6 \Rightarrow r = 4.$

Example 4.28 How many 'letter strings' together can be formed with the letters of the word "VOWELS" so that

(i) the strings begin with E

(ii) the strings begin with E and end with W.

Solution:
The given strings contains 6 letters (V,O,W,E,L,S).

(i) Since all strings must begin with E, we have the remaining 5 letters which can be arranged in ${}^5P_5 = 5!$ ways
Therefore the total number of strings with E as the starting letter is $5! = 120$.

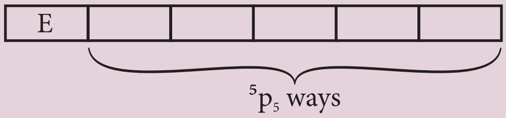


Figure 4.10

(ii) Since all strings must begin with E, and end with W, we need to fix E and W. The remaining 4 letters can be arranged in ${}^4P_4 = 4!$ ways.

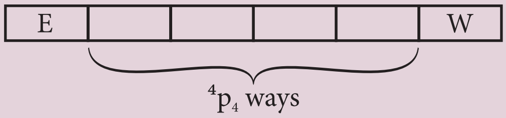


Figure 4.11

Therefore the total number of strings with E as the starting letter and W as the final letter is $4! = 24$.

Example 4.29 A number of four different digits is formed with the use of the digits 1,2,3,4 and 5 in all possible ways. Find the following

(i) How many such numbers can be formed?
(ii) How many of these are even?
(iii) How many of these are exactly divisible by 4?

Solution:

(i) The solution for this is the same as the number of permutations taking four digits out of 5 digits is ${}^5P_4 = 5\times4\times3\times2 = 120$.

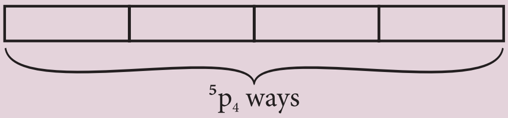


Figure 4.12

(ii) For even number last digits must be 2 or 4 which is filled in ${}^2P_1$ ways and remaining 3 places filled from remaining 4 digits in ${}^4P_3$ ways. Therefore the required number of ways is ${}^2P_1 \times {}^4P_3 = 2 \times 24 = 48$.

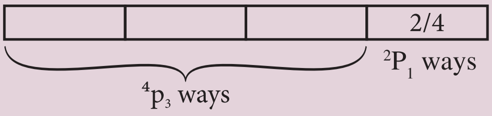


Figure 4.13

(iii) Since the number divisible by 4, then last two digit must be divisible by 4. The Last two digits become 12,24,32,52 (4 ways). The remaining first two places filled from remaining 3 digits in ${}^3P_2$ ways. The required number of numbers which are divisible by 4 is ${}^4P_1 \times {}^3P_2 = 4 \times 6 = 24$.

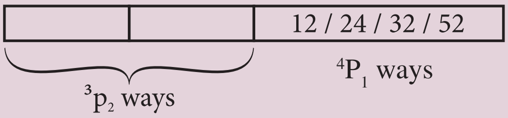


Figure 4.14

4.4.3 Objects always together (String method)

The number of permutations of n different objects, taken all at a time, when m specified objects are always together,

- Consider a string of m specified objects as a single unit
- Then we have $(n-m+1)$ objects. Permute this $(n-m+1)$ objects in $(n-m+1)!$ ways.
- Then permute the m specified objects between themselves in $m!$ ways.
- Finally, the answer is $m! \times (n-m+1)!$.

4.4.4 No two things are together (Gap method)

To obtain the number of permutations of n different objects when no two of k given objects occur together and there are no restrictions on the remaining $m = n-k$ objects, we follow the procedure as follows:

- First of all, arrange the m objects on which there is no restriction in a row. These m objects can be permuted in ${}^{m}P_{m} = m!$ ways.
- Then count the number of gaps between every two of m objects on which there is no restriction including the end positions. Number of such gaps will be one more than m that is $(m+1)$. In this $m+1$ gaps, we can permute the k objects in ${}^{m+1}P_k$ ways.
- Then the required number of ways are $m! \times {}^{(m+1)}P_k$.

Example 4.30 How many different strings can be formed together using the letters of the word "EQUATION" so that

(i) the vowels always come together?
(ii) the vowels never come together?

Solution:

(i) There are 8 letters in the word "EQUATION" which includes 5 vowels (E,U,A,I,O) and 3 consonants (Q,T,N). Considering 5 vowels as one letter, we have 4 letters which can be arranged in ${}^{4}P_{4} = 4!$ ways. But corresponding each of these arrangements, the vowels E,U,A,I,O can be put in ${}^{5}P_{5} = 5!$ ways.

Hence, by the rule of product required number of words is $4! \times 5! = 24 \times 120 = 2880$.

(ii) The total number of strings formed by using all the eight letters of the word "EQUATION" is

$$ {}^{8}P_{8} = 8! = 40320. $$

So, the total number of strings in which vowels are never together is the same as the difference between the total number of strings and the number of strings in which vowels are together is $40320 - 2880 = 37440$.

Example 4.31 There are 15 candidates for an examination. 7 candidates are appearing for mathematics examination while the remaining 8 are appearing for different subjects. In how many ways can they be seated in a row so that no two mathematics candidates are together?

Solution:

Let us arrange the 8-non-mathematics candidates in ${}^{8}P_{8} = 8!$ ways. Each of these arrangements create 9 gaps. Therefore, the 7 mathematics candidates can be placed in these 9 gaps in ${}^{9}P_{7}$ ways.

$$ \_O_1\_O_2\_O_3\_O_4\_O_5\_O_6\_O_7\_O_8\_ $$

By the rule of product, the required number of arrangements is

$$ 8! \times {}^{9}P_{7} = 8! \times \frac{9!}{2!} = \frac{8! \times 9!}{2!}. $$

Example 4.32 In how many ways 5 boys and 4 girls can be seated in a row so that no two girls are together.

Solution:

$$\_B_1\_B_2\_B_3\_B_4\_B_5\_$$

The 5 boys can be seated in the row in ${}^5P_5 = 5!$ ways. In each of these arrangements 6 gaps are created. Since no two girls are to sit together, we may arrange 4 girls in this 6 gaps. This can be done in ${}^6P_4$ ways. Hence, the total number of seating arrangements is

$$5! \times {}^6P_4 = 120 \times 360 = 43200.$$

Example 4.33 4 boys and 4 girls form a line with the boys and girls alternating. Find the number of ways of making this line.

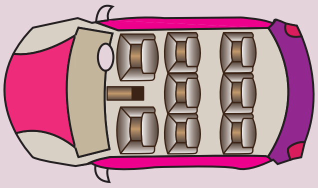
Solution:
4 boys can be arranged in a line in ${}^4P_4 = 4!$ ways. By keeping boys as first in each of these arrangements, 4 gaps are created. In these 4 gaps, 4 girls can be arranged in ${}^4P_4 = 4!$ ways.

$$B_1\_B_2\_B_3\_B_4\_ \text{ or } G_1\_G_2\_G_3\_G_4\_$$

Therefore, keeping boys as first, the total number of arrangements are $4! \times 4!$. Similarly, keeping girls as first, by a similar argument, the total number of arrangements are $4! \times 4!$. Hence, by the rule of sum, keeping either a boy or a girl first, the total number of arrangements are

$$(4! \times 4!) + (4! \times 4!) = 2(4!)^2 = 1152.$$

Example 4.34 A van has 8 seats. It has two seats in the front with two rows of three seats behind. The van belongs to a family, consisting of seven members, $F, M, S_1, S_2, S_3, D_1, D_2$. How many ways can the family sit in the van if



(i) There are no restriction?
(ii) Either F or M drives the van?
(iii) D_1, D_2 sits next to a window and F is driving?

Solution:

(i) As there 8 seats to be occupied out of which one seat is for the one who drives. Since there are no restrictions any one can drive the van. Hence the number of ways of occupying the driver seat is ${}^7P_1 = 7$ ways . The number of ways of occupying the remaining 7 seats by the remaining 6 people is ${}^7P_6 = 5040$. Hence the total number of ways the family can be seated in the car is $7 \times 5040 = 35280$.
(ii) As the driver seat can be occupied by only F or M, there are only two ways it can be occupied. Hence the total number of ways the family can be seated in the car is $2 \times 5040 = 10080$.
(iii) As there are only 5 window seats available for $D_1 \& D_2$ to occupy the number of ways of seated near the windows by the two family members is ${}^5P_2 = 20$. As the driver seat is occupied by F, the remaining 4 people can be seated in the available 5 seats in ${}^5P_4 = 120$. Hence the total number of ways the family can be seated in the car is
$20 \times 1 \times 120 = 2400$.

To understand the next problem we now define, **The Rank of a word in the dictionary.**
It is the place at which the given word comes when writing all the strings formed by the letters of the given word in the dictionary order or lexicographic order.

Example 4.35 If the letters of the word TABLE are permuted in all possible ways and the words thus formed are arranged in the dictionary order (alphabetical order), find the ranks of the words (i) TABLE, (ii) BLEAT

Solution:
The dictionary order of the letters of given word is A, B, E, L, T. In the dictionary order of the words which begin with A come first. If we fill the first place with A, remaining 4 letters (B, E, L, T) can be arranged in $4!$ ways. On proceeding like this we get

(i) The rank of the word TABLE

$A - - - - = 4! = 24$ ways
$B - - - - = 4! = 24$ ways
$E - - - - = 4! = 24$ ways
$L - - - - = 4! = 24$ ways
$TABEL = 1$ way
$TABLE = 1$ way

The rank of the word TABLE is $4 \times 4! + 1 + 1 = 98$.

(ii) The rank of the word BLEAT

$A - - - - = 4! = 24$ ways
$BA - -- = 3! = 6$ ways
$BE - -- = 3! = 6$ ways
$BLA - - = 2! = 2$ ways
$BLEAT = 1$ way

The rank of the word BLEAT is $24 + 6 + 6 + 2 + 1 = 39$.

4.4.5 Permutations of not all distinct objects

Consider permuting the letters of the word JEE. In this case the letters of the word are not different. There are 2 E's, which are of same kind. Let us treat, temporarily, the 2 E's as different, say E_1 and E_2. The number of permutations of 3 different letters taken all at a time is 3!.

Permutations when E_1, E_2 are different	Permutations when E_1, E_2 are the same.
JE_1E_2, JE_2E_1	JEE
E_1JE_2, E_2JE_1	EJE
E_1E_2J, E_2E_1J	EEJ

It is because of the two E_1, E_2 permuted internally will give rise to the same permutations. Since they are same, the required number of permutations is $\dfrac{3!}{2!} = 3$.

Theorem 4.4: The number of permutations of n objects, where p objects are of the same kind and rest are all different is $\dfrac{n!}{p!}$.

Generally, the number of permutations of n objects, where p_1 objects are one kind, p_2 objects are of second kind, $\cdots$ p_k, are of k^{th} kind and the rest of it are of different kind is $\dfrac{n!}{p_1! \times p_2! \times \cdots \times p_k!}$.

Example 4.36 Find the number of ways of arranging the letters of the word BANANA.

Solution:
This word has 6 letters in which there are 3 A'S, 2 N's and one B. The number of ways of arrangements is $\dfrac{6!}{3! \times 2!} = 60$.

Example 4.37 Find the number of ways of arranging the letters of the word RAMANUJAN so that the relative positions of vowels and consonants are not changed.

Solution:
In the word RAMANUJAN there are 4 vowels (A,A,U,A)in that 3 A's, 1 U and 5 consonants (R,M,N,J,N) in that two N's and rest are distinct. The 4 vowels (A,A,A,U) can be arranged themselves in $\frac{4!}{3!} = 4$ ways. The 5 consonants (R,M,N,J,N) can be arranged themselves in $\frac{5!}{2!} = 60$ ways. Therefore the number of required arrangements are $\frac{4!}{3!} \times \frac{5!}{2!} = 4 \times 60 = 240$.

Example 4.38 Three twins pose for a photograph standing in a line. How many arrangements are there (i). when there are no restrictions. (ii). when each person is standing next to his or her twin?

Solution:

(i) The six persons without any restriction may be arranged in ${}^6P_6 = 6! = 720$ ways.
(ii) Let us consider three twins as T_1, T_2, T_3. Each twin is considered as a single unit and these three can be permuted in 3! ways. Again each twin can be permuted between themselves in 2! ways. Hence, the total number of arrangements is $3! \times 2! \times 2! \times 2! = 48$ ways.

Example 4.39 How many numbers can be formed using the digits 1,2,3,4,2,1 such that, even digits occupies even place?

Solution:
There are 6 places in that there are 3 even places we have 2,4,2 as even numbers. The number of ways of permuting 2,4,2 in the 3 even places in $\frac{3!}{2!} = 3$ ways. The remaining numbers 1,3,1 can be permuted in the remaining 3 places in $\frac{3!}{2!} = 3$ ways. Hence, the required number of numbers is $3 \times 3 = 9$.

Example 4.40 How many paths are there from start to end on a 6×4 grid as shown in the picture?

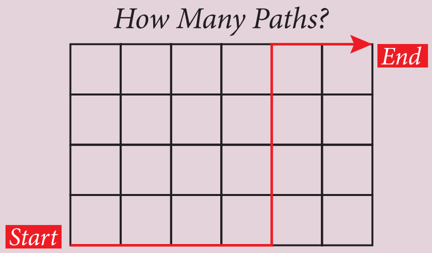


Figure 4.15

Solution:
Note that any such path comprises 6 horizontal unit lengths and 4 vertical unit lengths. This each path consists of 10 unit lengths where 6 are of one kind (horizontal) and 4 are of another kind (vertical).

Thus the total number of paths is $\dfrac{10!}{4! \times 6!} = 210.$

Example 4.41 If the different permutations of all letters of the word BHASKARA are listed as in a dictionary, how many strings are there in this list before the first word starting with B?

Solution:
The required number of strings is the total number of strings starting with A and using the letters A,A,B,H,K,R,S is $\dfrac{7!}{2!} = 2520.$

Example 4.42 If the letters of the word IITJEE are permuted in all possible ways and the strings thus formed are arranged in the lexicographic order, find the rank of the word IITJEE

Solution:
The lexicographic order of the letters of given word is E, E, I, I, J, T. In the lexicographic order, the strings which begin with E come first. If we fill the first place with E, remaining 5 letters (E, I, I, J, T) can be arranged in $\dfrac{5!}{2!}$ ways. On proceeding like this we get,

$E - - - -- = \dfrac{5!}{2!} = 60$ ways

$IIE - -- = 3! = 6$ ways

$IIJ - -- = \dfrac{3!}{2!} = 3$ ways

$IITE - - = 2! = 2$ ways

$IITJEE = 1$ way

The rank of the word IITJEE is 60 + 6 + 3 + 2 + 1 = 72.

Example 4.43 Find the sum of all 4-digit numbers that can be formed using the digits 1, 2, 4, 6, 8.

Solution:
The number of 4-digit numbers that can be formed using the given 5 digits is ${}^5P_4 = 120$. We first find the sum of the digits in the unit place of all these 120 numbers. By filling the 1 in unit place,

the remaining three places can be filled with remaining 4 digits in ${}^4P_3 = 24$ ways. This means, the number of 4-digit numbers having 1 in units place is ${}^4P_3 = 24$. Similarly, each of the digits 2, 4, 6, 8 appear 24 times in units place. An addition of all these digits gives the sum of all the unit digits of all 120 numbers. Therefore,

$$({}^4P_3 \times 1) + ({}^4P_3 \times 2) + ({}^4P_3 \times 4) + ({}^4P_3 \times 6) + ({}^4P_3 \times 8)$$

$$\begin{aligned} &= {}^4P_3 \times (1+2+4+6+8) \\ &= {}^4P_3 \times (\text{sum of the digits}) \\ &= {}^4P_3 \times 21. \end{aligned}$$

Similarly, we get the sum of the digits in 10^{th} place as ${}^4P_3 \times 21$. Since it is in 10^{th} place, its value is ${}^4P_3 \times 21 \times 10$. Similarly, the values of the sum of the digits in 100^{th} place and 1000^{th}place are ${}^4P_3 \times 21 \times 100$ and ${}^4P_3 \times 21 \times 1000$ respectively. Hence the sum of all the 4 digit numbers formed by using the digits 1, 2, 4, 6, 8 is

$$({}^4P_3 \times 21) + ({}^4P_3 \times 21 \times 10) + ({}^4P_3 \times 21 \times 100) + ({}^4P_3 \times 21 \times 1000)$$

$$\begin{aligned} &= {}^4P_3\,(21 \times 1111) \\ &= 24 \times 21 \times 1111 \\ &= 559944. \end{aligned}$$

Deduction 4.1: The sum of all r-digit numbers that can be formed using the given n non zero digits is ${}^{(n-1)}P_{(r-1)} \times$ (sum of the digits) $\times\, 111\cdots1$(r times)

Deduction 4.2: If 0 is one digit among the given n digits, then we get that the sum of the r-digits numbers that can be formed using the given n digits (including 0) is

$\{{}^{(n-1)}P_{(r-1)} \times$ (sum of the digits) $\times\, 111\cdots1$(r times)$\}\, -$
$\{{}^{(n-2)}P_{(r-2)} \times$ (sum of the digits) $\times\, 111\cdots1$((r-1) times)$\}$.

Permutation as Function

Permutation on any finite set $S_n = \{x_1, x_2, x_3, \cdots, x_n\}$ is a bijective function from $S_n \to S_n$. Therefore the set of all permutations on a finite set with n elements is the same as the total number of bijective functions on the set. This is precisely $n!$. Hence the study of permutation is the same as the study of the bijective mappings on the set. Few representations for a permutation on S_3 are given by

$$\begin{bmatrix} A & B & C \\ B & C & A \end{bmatrix}, \begin{bmatrix} A & B & C \\ C & A & B \end{bmatrix}, \begin{bmatrix} A & B & C \\ C & B & A \end{bmatrix}, \cdots$$

Exercise - 4.2

1. If ${}^{(n-1)}P_3 :{}^{n} P_4 = 1 : 10$, find n.
2. If ${}^{10}P_{r-1} = 2 \times {}^6 P_r$, find r.
3. (i) Suppose 8 people enter an event in a swimming meet. In how many ways could the gold, silver and bronze prizes be awarded?
   (ii) Three men have 4 coats, 5 waist coats and 6 caps. In how many ways can they wear them?

4. Determine the number of permutations of the letters of the word SIMPLE if all are taken at a time?
5. A test consists of 10 multiple choice questions. In how many ways can the test be answered if
   (i) Each question has four choices?
   (ii) The first four questions have three choices and the remaining have five choices?
   (iii) Question number n has $n+1$ choices?
6. A student appears in an objective test which contain 5 multiple choice questions. Each question has four choices out of which one correct answer.
   (i) What is the maximum number of different answers can the students give?
   (ii) How will the answer change if each question may have more than one correct answers?
7. How many strings can be formed from the letters of the word ARTICLE, so that vowels occupy the even places?
8. 8 women and 6 men are standing in a line.
   (i) How many arrangements are possible if any individual can stand in any position?
   (ii) In how many arrangements will all 6 men be standing next to one another?
   (iii) In how many arrangements will no two men be standing next to one another?
9. Find the distinct permutations of the letters of the word MISSISSIPPI?
10. How many ways can the product $a^2b^3c^4$ be expressed without exponents?
11. In how many ways 4 mathematics books, 3 physics books, 2 chemistry books and 1 biology book can be arranged on a shelf so that all books of the same subjects are together.
12. In how many ways can the letters of the word SUCCESS be arranged so that all Ss are together?
13. A coin is tossed 8 times,
   (i) How many different sequences of heads and tails are possible?
   (ii) How many different sequences containing six heads and two tails are possible?
14. How many strings are there using the letters of the word INTERMEDIATE, if
   (i) The vowels and consonants are alternative (ii) All the vowels are together
   (iii) Vowels are never together (iv) No two vowels are together.
15. Each of the digits 1, 1, 2, 3, 3 and 4 is written on a separate card. The six cards are then laid out in a row to form a 6-digit number.
   (i) How many distinct 6-digit numbers are there?
   (ii) How many of these 6-digit numbers are even?
   (iii) How many of these 6-digit numbers are divisible by 4?
16. If the letters of the word GARDEN are permuted in all possible ways and the strings thus formed are arranged in the dictionary order, then find the ranks of the words (i) GARDEN (ii) DANGER.
17. Find the number of strings that can be made using all letters of the word THING. If these words are written as in a dictionary, what will be the 85^{th} string?
18. If the letters of the word FUNNY are permuted in all possible ways and the strings thus formed are arranged in the dictionary order, find the rank of the word FUNNY.
19. Find the sum of all 4-digit numbers that can be formed using digits 1, 2, 3, 4, and 5 repetitions not allowed?
20. Find the sum of all 4-digit numbers that can be formed using digits 0, 2, 5, 7, 8 without repetition?

4.5 Combinations

Let us suppose there are four persons A, B, C and D (actual names may be used here) and we have to select three of them to be a part of a committee. In how many ways can we make this selection? For example, A, B, C is one possible choice. Here the order of selection is immaterial. Thus A, B, C is the same as B, A, C or C, A, B as long as the same three persons are selected. Thus the possible distinct choices or selections are A, B, C; A, B, D; A, C, D and B, C, D. We may thus conclude that there are 4 ways of selecting 3 people out of 4. Each choice or selection is referred to as a combination of 4 different objects taken 3 at a time.

Suppose two persons are to be selected from four persons. The possible choices are: A, B : A, C: A, D : B, C : B, D : C, D. Thus the number of combinations of 4 different objects taken 2 at a time is 6. The number of combinations of n different objects taken r at a time is represented by ${}^nC_r$. From the above we may conclude that ${}^4C_3 = 4$ and ${}^4C_2 = 6$. Now, ${}^4C_3$ is the number of combinations of 4 objects taken 3 at a time. Note that in each combination, the three objects may be arranged in 3! ways. Thus the total number of permutations of 4 objects taken 3 at a time is ${}^4C_3 \times 3!$. This is also equal to ${}^4P_3$. Hence ${}^4P_3 = {}^4C_3 \times 3!$.

In general, this leads to an important relationship between permutations and combinations as,

$$ {}^nP_r = {}^nC_r \times r!. $$

Normally for any reader there may be a confusion between permutation and combination. The following table with an example may be helpful in clearing the confusion.

S.No	Description	Permutation	Combination
1	What is a	Number of Arrangement or Listing of objects	Number of Selections or Grouping of objects
2	Where to use	If the ordering of objects matters	If the ordering of objects does not matter
3	Representation	${}^nP_r$	${}^nC_r$
		Examples	
4	In a game of cricket	The number of batting line up of 11 players out of the 15 players	The number of teams consisting of 11 players out of 15 players
5	In a process of prize distribution	The number of ways of distributing 3 distinct prizes	The number of ways of distributing 3 identical prizes
6	In a committee formation	The number of ways of choosing a President and a Vice-President for a committee of 13 members	The number of ways of forming a committee of 2 persons from 13 members
7	In a process of choosing objects	The number of ways of choosing 3 out of 15 distinct objects one after another	The number of ways of choosing 3 out of 15 distinct objects simutaneously

Theorem 4.5: The number of combinations of n distinct objects taken r at a time is given by

$$ {}^nC_r = \frac{n!}{r!(n-r)!}, 0 \leq r \leq n. $$

Proof. By the relation between permutations and combinations, we have

$$^nC_r = \frac{^nP_r}{r!} = \frac{n!}{r!(n-r)!} \text{ (by a result from permutation).}$$

□

4.5.1 Properties of Combinations

Property 1: (i) $^nC_0 = 1$, (ii) $^nC_n = 1$, (iii) $^nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$.

Proof.

$$\begin{aligned}
(i)^nC_0 &= \frac{n!}{0!(n-0)!} = 1. \\
(ii)^nC_n &= \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = 1. \\
(iii)^nC_r &= \frac{n(n-1)(n-2)\cdots(n-(r-1))}{r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.
\end{aligned}$$

□

In view of,

$$^nC_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r} =^n C_r,$$

we have,

Property 2:

$$^nC_r =^n C_{n-r}.$$

Property 3: If $^nC_x =^n C_y$ then either $x = y$ or $x + y = n$.

Proof. By the property 2 we have, $^nC_y =^n C_{n-y}$

Therefore, $^nC_x =^n C_y =^n C_{n-y}$ gives us $x = y$ or $x = n - y$ equivalently, $x = y$ or $x + y = n$. □

Property 4: $^nC_r +^n C_{r-1} =^{n+1} C_r$.

Proof. Using the expressions for the "combination" we have,

$$\begin{aligned}
^nC_r +^n C_{r-1} &= \frac{n!}{r!.(n-r)!} + \frac{n!}{(r-1)!.(n-(r-1))!} \\
&= \frac{n!}{r! \times (n-r)!} + \frac{n!}{(r-1)! \times (n-r+1)} \\
&= \frac{n!}{r.(r-1)! \times (n-r)!} + \frac{n!}{(r-1)!.(n-r)!(n-r+1)} \\
&= \frac{n!}{(r-1)! \times (n-r)!} \times \left(\frac{1}{r} + \frac{1}{(n-r+1)}\right) \\
&= \frac{n!}{(r-1)! \times (n-r)!} \times \frac{(n-r+1+r)}{r(n-r+1)} \\
&= \frac{n!}{(r-1)! \times (n-r)!} \times \frac{(n+1)}{r(n-r+1)} \\
&= \frac{(n+1)!}{r! \times (n+1-r)!} =^{n+1} C_r.
\end{aligned}$$

□

Property 5:

$$^{n}C_{r} = \frac{n}{r} \times^{(n-1)} C_{(r-1)}.$$

Proof.

$$\begin{aligned} \frac{n}{r} \times^{(n-1)} C_{(r-1)} &= \frac{n}{r}\frac{(n-1)!}{(r-1)! \times ((n-1)-(r-1))!} \\ &= \frac{n(n-1)!}{r(r-1)!(n-r)!} =^{n} C_{r} \end{aligned}$$

□

Example 4.44 Evaluate the following: (i) $^{10}C_3$ (ii) $^{15}C_{13}$ (iii) $^{100}C_{99}$ (iv) $^{50}C_{50}$.

Solution:

(i)

$$\begin{aligned} ^{10}C_3 &= \frac{10!}{7! \times 3!} = \frac{10 \times 9 \times 8 \times 7!}{7! \times 3 \times 2 \times 1} \\ &= \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120 \end{aligned}$$

(ii)

$$\begin{aligned} ^{15}C_{13} &= \frac{15!}{2! \times 13!} = \frac{15 \times 14 \times 13!}{2 \times 1 \times 13!} \\ &= \frac{15 \times 14}{2 \times 1} = 105 \end{aligned}$$

(iii)

$$^{100}C_{99} = \frac{100 \times 99!}{99!} = 100$$

(iv)

$$^{50}C_{50} = \frac{50!}{50! \times 0!} = 1.$$

Example 4.45 Find the value of $^5C_2$ and $^7C_3$ using the property 5

Solution:

$$^{n}C_{r} = \frac{n}{r} \times {}^{n-1}C_{r-1}.$$

Substituting $n = 5$ and $r = 2$, we get

$$\begin{aligned} ^5C_2 &= \frac{5}{2} \times {}^4C_1 = \frac{5}{2} \times \frac{4}{1} \\ &= \frac{5 \times 4}{2 \times 1} = 10. \end{aligned}$$

Substituting $n = 7$ and $r = 3$, we get

$$\begin{aligned} ^7C_3 &= \frac{7}{3} \times {}^6C_2 = \frac{7}{3} \times \frac{6}{2} \times {}^5C_{2-1} \\ &= \frac{7}{3} \times \frac{6}{2} \times {}^5C_1 = \frac{7}{3} \times \frac{6}{2} \times \frac{5}{1} \\ &= \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35 \end{aligned}$$

Example 4.46 If ${}^nC_4 = 495$, What is n?.

Solution:
We know that,

$$ {}^nC_4 = 495. $$

Therefore,

$$ \frac{n \times (n-1) \times (n-2) \times (n-3)}{4 \times 3 \times 2 \times 1} = 495 $$

$$ \implies n \times (n-1) \times (n-2) \times (n-3) = 495 \times 4 \times 3 \times 2 \times 1 $$

Factoring $495 = 3 \times 3 \times 5 \times 11$, and writing this product as a product of 4 consecutive numbers in the descending order we get, $n \times (n-1) \times (n-2) \times (n-3) = 12 \times 11 \times 10 \times 9$. Equating n with the maximum number, we obtain $n = 12$.

Example 4.47 If ${}^nP_r = 11880$ and ${}^nC_r = 495$, Find n and r.

Solution:
We know that,

$$ \frac{{}^nP_r}{{}^nC_r} = r!. $$

Therefore,

$$ r! = \frac{11880}{495} = 24 = 4!, $$

gives $r = 4$. Using this $r = 4$, in ${}^nc_4 = 495$, and applying the result of the Example (4.46) we get, $n = 12$.

Example 4.48 Prove that ${}^{24}C_4 + \sum_{r=0}^{4} {}^{(28-r)}C_3 = {}^{29}C_4$

Solution:

$$ \begin{aligned} {}^{24}C_4 + \sum_{r=0}^{4} {}^{(28-r)}C_3 &= {}^{24}C_4 + {}^{28}C_3 + {}^{27}C_3 + {}^{26}C_3 + {}^{25}C_3 + {}^{24}C_3 \\ &= {}^{24}C_4 + {}^{24}C_3 + {}^{25}C_3 + {}^{26}C_3 + {}^{27}C_3 + {}^{28}C_3 \\ &= {}^{25}C_4 + {}^{25}C_3 + {}^{26}C_3 + {}^{27}C_3 + {}^{28}C_3 \\ &= {}^{26}C_4 + {}^{26}C_3 + {}^{27}C_3 + {}^{28}C_3 \\ &= {}^{27}C_4 + {}^{27}C_3 + {}^{28}C_3 \\ &= {}^{28}C_4 + {}^{28}C_3 \\ &= {}^{29}C_4 \end{aligned} $$

Example 4.49 Prove that ${}^{10}C_2 + 2 \times {}^{10}C_3 + {}^{10}C_4 = {}^{12}C_4$

Solution:

$$\begin{aligned}
{}^{10}C_2 + 2 \times {}^{10}C_3 + {}^{10}C_4 &= {}^{10}C_2 + ({}^{10}C_3 + {}^{10}C_3) + {}^{10}C_4 \\
&= ({}^{10}C_2 + {}^{10}C_3) + ({}^{10}C_3 + {}^{10}C_4) \\
&= {}^{11}C_3 + {}^{11}C_4 \\
&= {}^{12}C_4
\end{aligned}$$

Example 4.50 If ${}^{(n+2)}C_7 : {}^{(n-1)}P_4 = 13 : 24$ find n.

Solution:

$$\begin{aligned}
{}^{(n+2)}C_7 : {}^{(n-1)}P_4 &= 13 : 24. \\
\frac{{}^{(n+2)}C_7}{{}^{(n-1)}P_4} &= \frac{13}{24}. \\
\frac{(n+2)!}{(n-5)! \times 7!} \times \frac{(n-5)!}{(n-1)!} &= \frac{13}{24}. \\
\frac{(n+2)(n+1)n(n-1)!}{(n-1)! \times 7!} &= \frac{13}{24}. \\
(n+2)(n+1)(n) &= \frac{13}{24} \times 7! = \frac{13}{24} \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\
(n+2)(n+1)(n) &= 13 \times 14 \times 15 \\
n+2 &= 15 \Longrightarrow n = 13.
\end{aligned}$$

Example 4.51 A salad at a certain restaurant consists of 4 of the following fruits: apple, banana, guava, pomegranate, grapes, papaya and pineapple. Find the total possible number of fruit salads.

Solution:
There are seven fruits and we have to select four fruits for the fruit salad. Hence, the total number of possible ways of making a fruit salad is ${}^7C_4 = {}^7C_3 = 35$.

Example 4.52 A Mathematics club has 15 members. In that 8 are girls. 6 of the members are to be selected for a competition and half of them should be girls. How many ways of these selections are possible?

Solution:
There are 8 girls and 7 boys in the mathematics club. The number of ways of selecting 6 members in that half of them girls (3 girls and 3 others) is ${}^8C_3 \times {}^7C_3 = 56 \times 35 = 1960$.

Example 4.53 In rating 20 brands of cars, a car magazine picks a first, second, third, fourth and fifth best brand and then 7 more as acceptable. In how many ways can it be done?

Solution:
The picking of 5 brands for a first, second, third, fourth and fifth best brand from 20 brands in ${}^{20}P_5$ ways. From the remaining 15 we need to select 7 acceptable in ${}^{15}C_7$ ways. By the rule of product it can be done in ${}^{20}P_5 \times {}^{15}C_7$ ways.

Example 4.54 From a class of 25 students, 10 students are to be chosen for an excursion party. There are 4 students who decide that either all of them will join or none of them will join. In how many ways can the excursion party be chosen?

Solution:
There are two possibilities (i) All the 4 students will got to the excursion party then, we need to select 6 students out of 21 students. It can be done in ${}^{21}C_6 = \dfrac{21!}{6! \times 15!}$ ways.

(ii) All the 4 students will not go to the excursion party then, we need to select 10 students out of 21 students. It can be done in ${}^{21}C_{10} = \dfrac{21!}{10! \times 11!}$. Hence, the total number of ways is ${}^{21}C_6 + {}^{21}C_{10} = \dfrac{21!}{6! \times 15!} + \dfrac{21!}{10! \times 11!}$.

Example 4.55 A box of one dozen apple contains a rotten apple. If we are choosing 3 apples simultaneously, in how many ways, one can get only good apples.

Solution:
The total number of ways of selecting 3 apples from 12 apples is ${}^{12}C_3 = 220$.

The total number of ways of getting a rotten apple when selecting 3 apples from 12 apples is equal to selecting 1 rotten apple and remaining 2 apples can be selected from 11 apples is ${}^{11}C_2 = 55$.

Therefore, the total number of ways of getting only good apples is

$$ {}^{12}C_3 - {}^{11}C_2 = 220 - 55 = 165 $$

.

Example 4.56 An exam paper contains 8 questions, 4 in Part A and 4 in Part B. Examiners are required to answer 5 questions. In how many ways can this be done if
(i) There are no restrictions of choosing a number of questions in either parts.
(ii) At least two questions from Part A must be answered.

Solution:

(i) **There are no restrictions.** Totally there are 8 questions in both Part A and Part B. The total number of ways of attempting 5 questions from 8 questions is ${}^{8}C_5 = {}^{8}C_3 = 56$.
(ii) **At least two questions from Part A needs to be answered.**Accordingly, various choices are tabulated as follows.

Part A	Part B	Number of selections
2	3	${}^4C_2 \times {}^4C_3$
3	2	${}^4C_3 \times {}^4C_2$
4	1	${}^4C_4 \times {}^4C_1$

Therefore, the required number of ways of answering is

$$ {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_2 + {}^4C_4 \times {}^4C_1 = 24 + 24 + 4 = 52. $$

Example 4.57 Out of 7 consonants and 4 vowels, how many strings of 3 consonants and 2 vowels can be formed?

Solution:
Number of ways of selecting (3 consonants out of 7) and (2 vowels out of 4) is

$$^7C_3 \times ^4 C_2$$

Each string contains 5 letters. Number of ways of arranging 5 letters among themselves is $5! = 120$
Hence required number of ways is,

$$^7C_3 \times ^4 C_2 \times 5! = 35 \times 6 \times 120 = 25200$$

.

Example 4.58 Find the number of strings of 5 letters that can be formed with the letters of the word PROPOSITION.

Solution:
There are 11 letters in the word, with respect to number of repetitions of letters there are 4 distinct letters (R, S, T, N), 2 sets of two alike letters (PP,II), 1 set of three alike letters (OOO). The following table will illustrate the combination of these sets and the number of words

S.No	Letter Options	Selections	Arrangements
1	5 distinct (R, S, T, N, P, I, O)	$^7C_5$	$^7C_5 \times 5! = 2520$
2	1 set of 3 alike (OOO), 1 set of 2 alike (PP, II)	$^1C_1 \times ^2 C_1$	$^1C_1 \times ^2 C_1 \times \frac{5!}{3! \times 2!} = 20$
3	1 set of 3 alike (OOO), 2 distinct (R, S, T, N, P, I)	$^1C_1 \times ^6 C_2$	$^1C_1 \times ^6 C_2 \times \frac{5!}{3!} = 300$
4	2 sets of 2 alike (PP, II, OO), 1 distinct $(R, S, T, N$ and remaining one in 2 alike)	$^3C_2 \times ^5 C_1$	$^3C_2 \times ^5 C_1 \times \frac{5!}{2! \times 2!} = 450$
5	1 set of 2 alike (PP, II, OO), 3 distinct $(R, S, T, N$ and remaining two in 2 alike)	$^3C_1 \times ^6 C_3$	$^3C_1 \times ^6 C_3 \times \frac{5!}{2!} = 3600$

Hence, the total number of strings are $2520 + 20 + 300 + 450 + 3600 = 6890$.

Example 4.59 If a set of m parallel lines intersect another set of n parallel lines (not parallel to the lines in the first set), then find the number of parallelograms formed in this lattice structure.

Solution:
Whenever we select 2 lines from the first set of m lines and 2 lines from the second set of n lines, one parallelogram is formed. Thus the number of parallelograms formed is $^mC_2 \times ^n C_2$.

Example 4.60 How many diagonals are there in a polygon with n sides?

Solution:
A polygon of n sides has n vertices. By joining any two vertices of a polygon, we obtain either a side or a diagonal of the polygon. Number of line segments obtained by joining the vertices of a n sided polygon taken two at a time is ${}^{n}C_2 = \dfrac{n(n-1)}{2}$. Out of these lines, there are n sides of polygon. Therefore, number of diagonals of the polygon is

$$\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}.$$

In particular for a pentagon and heptagon (Septagon), number of diagonals respectively are $\frac{5(5-3)}{2} = 5$ and $\frac{7(7-3)}{2} = 14$.

Exercise - 4.3

1. If ${}^{n}C_{12} = {}^{n}C_9$ find ${}^{21}C_n$.
2. If ${}^{15}C_{2r-1} = {}^{15}C_{2r+4}$, find r.
3. If ${}^{n}P_r = 720$, and ${}^{n}C_r = 120$, find n, r.
4. Prove that ${}^{15}C_3 + 2 \times {}^{15}C_4 + {}^{15}C_5 = {}^{17}C_5$.
5. Prove that ${}^{35}C_5 + \sum_{r=0}^{4} {}^{(39-r)}C_4 = {}^{40}C_5$.
6. If ${}^{(n+1)}C_8 : {}^{(n-3)}P_4 = 57 : 16$, find the value of n.
7. Prove that ${}^{2n}C_n = \dfrac{2^n \times 1 \times 3 \times \cdots (2n-1)}{n!}$.
8. Prove that if $1 \leq r \leq n$ then $n \times {}^{(n-1)}C_{r-1} = (n-r+1)\,{}^{n}C_{r-1}$.
9. (i) A Kabaddi coach has 14 players ready to play. How many different teams of 7 players could the coach put on the court?
   (ii) There are 15 persons in a party and if each 2 of them shakes hands with each other, how many handshakes happen in the party?
   (iii) How many chords can be drawn through 20 points on a circle?
   (iv) In a parking lot one hundred , one year old cars, are parked. Out of them five are to be chosen at random for to check its pollution devices. How many different set of five cars can be chosen?
   (v) How many ways can a team of 3 boys,2 girls and 1 transgender be selected from 5 boys, 4 girls and 2 transgenders?
10. Find the total number of subsets of a set with

    [Hint: ${}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \cdots + {}^{n}C_n = 2^n$]
    (i) 4 elements (ii) 5 elements (iii) n elements.
11. A trust has 25 members.

    (i) How many ways 3 officers can be selected?
    (ii) In how many ways can a President, Vice President and a Secretary be selected?

12. How many ways a committee of six persons from 10 persons can be chosen along with a chair person and a secretary?
13. How many different selections of 5 books can be made from 12 different books if,
    (i) Two particular books are always selected?
    (ii) Two particular books are never selected?
14. There are 5 teachers and 20 students. Out of them a committee of 2 teachers and 3 students is to be formed. Find the number of ways in which this can be done. Further find in how many of these committees
    (i) a particular teacher is included?
    (ii) a particular student is excluded?
15. In an examination a student has to answer 5 questions, out of 9 questions in which 2 are compulsory. In how many ways a student can answer the questions?
16. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly three aces in each combination.
17. Find the number of ways of forming a committee of 5 members out of 7 Indians and 5 Americans, so that always Indians will be the majority in the committee.
18. A committee of 7 peoples has to be formed from 8 men and 4 women. In how many ways can this be done when the committee consists of
    (i) exactly 3 women?
    (ii) at least 3 women?
    (iii) at most 3 women?
19. 7 relatives of a man comprises 4 ladies and 3 gentlemen, his wife also has 7 relatives; 3 of them are ladies and 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relative and 3 of the wife' s relatives?
20. A box contains two white balls, three black balls and four red balls. In how many ways can three balls be drawn from the box, if at least one black ball is to be included in the draw?
21. Find the number of strings of 4 letters that can be formed with the letters of the word EXAMINATION?.
22. How many triangles can be formed by joining 15 points on the plane, in which no line joining any three points?
23. How many triangles can be formed by 15 points, in which 7 of them lie on one line and the remaining 8 on another parallel line?
24. There are 11 points in a plane. No three of these lies in the same straight line except 4 points, which are collinear. Find,
    (i) the number of straight lines that can be obtained from the pairs of these points?
    (ii) the number of triangles that can be formed for which the points are their vertices?
25. A polygon has 90 diagonals. Find the number of its sides?

4.6 Mathematical induction

Let us consider the sum of the first n positive odd numbers. These are $1, 3, 5, 7, \cdots, 2n-1$. The first odd number 1 which is equal to 1. The first two odd numbers are 1 and 3 and their sum is 4. Writing these as follows helps us to see a pattern.

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \\ 1+3+5+7+9 &= 25 \end{aligned}$$

and so on. We note that the right hand side of the expressions are the perfect squares $1, 4, 9, 16, 25 \cdots$. This pattern compels us to make the conjecture that the sum of the first n odd numbers is equal to n^2. Symbolically, we express this as,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

However we have only made a conjecture. In order to prove the conjecture we shall use the Principle of Mathematical Induction. Mathematical Induction is a method or technique of proving mathematical results or theorems of the above kind. This technique relies upon making conjectures by observing all possible cases of a specific result. It is well suited for proving results in algebra or in other disciplines of mathematics where results or theorems are stated in terms of n, n being a positive integer. The process of Mathematical Induction may be compared to that of climbing an infinite staircase.

In order to ensure that we complete the climb, it is sufficient to ensure the following.

(a) We can climb the first step.
(b) Once we have reached a particular step of the staircase, we can climb to the next step.

Being sure of (a) and (b) will enable us to climb all the steps in the staircase. Similarly, when we apply this method to prove a mathematical statement $P(n)$, the process of induction involves the following steps.

Figure 4.16

Step 1: Verify that the statement is true for n = 1, that is, verify that P(1) is true. This is akin to climbing the first step of the staircase and is referred to as the **initial step**.

Step 2: Verify that the statement is true for $n = k + 1$ whenever it is true for $n = k$, where k is a positive integer. This means that we need to prove that $P(k + 1)$ is true whenever $P(k)$ is true. This is referred to as the **inductive step**.

Step 3: If steps 1 and 2 have been established then the statement $P(n)$ is true for all positive integers n.

One of the interesting method of proof in Mathematics is by the Mathematical induction. We shall illustrate the method through problems. As an illustration of the process let us revisit a well known result through an example below:

Example 4.61 By the principle of mathematical induction, prove that, for all integers $n \geq 1$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Solution:
Let,

$$P(n) := 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Substituting the value of $n = 1$, in the statement we get, $P(1) = \dfrac{1(1+1)}{2} = 1.$ Hence, $P(1)$ is true.

Let us assume that the statement is true for $n = k$. Then

$$P(k) = 1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}.$$

We need to show that $P(k+1)$ is true. Consider,

$$P(k+1) = \underbrace{1 + 2 + 3 + \cdots + k} + (k+1) = \frac{k(k+1)}{2} + (k+1).$$

That is,

$$P(k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

This implies, $P(k+1)$ is true. The validity of $P(k+1)$ follows from that of $P(k)$. Therefore by the principle of mathematical induction, for all integers $n \geq 1$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Example 4.62 Prove that the sum of first n positive odd numbers is n^2.

Solution:
Let $P(n) = 1 + 3 + 5 + \cdots + (2n-1)$. Therefore $P(1) = 1 = 1^2$ is true.

We assume that $P(k) = 1 + 3 + 5 + ...(2k-1)$ is true for $n = k$. That is $P(k) = k^2$

We need to prove $P(k+1) = (k+1)^2$.

$$\begin{aligned} P(k+1) &= 1 + 3 + 5 + ...(2(k+1) - 1) \\ &= \underbrace{1 + 3 + 5 + 7 + ... + (2k-1)} + 2k + 1 \\ &= P(k) + 2k + 1 \\ &= k^2 + 2k + 1 = (k+1)^2 \end{aligned}$$

This implies, $P(k+1)$ is true. Hence, by the principle of mathematical induction, $P(n)$ is true for all natural numbers.

Example 4.63 By the principle of mathematical induction, prove that, for all integers $n \geq 1$,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution:
Let,

$$P(n) := 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Substituting $n = 1$ in the statement we get, $P(1) = \frac{1(1+1)(2(1)+1)}{6} = 1$. Hence, $P(1)$ is true. Let us assume that the statement is true for $n = k$. Then

$$P(k) = 1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

We need to show that $P(k+1)$ is true. Consider

$$\begin{aligned}
P(k+1) &= \underbrace{1^2 + 2^2 + 3^2 + \cdots + k^2} + (k+1)^2 \\
&= P(k) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)\left(k(2k+1) + 6(k+1)\right)}{6} \\
&= \frac{(k+1)\left(2k^2 + 7k + 6\right)}{6} \\
&= \frac{(k+1)\left[(k+2)(2k+3)\right]}{6} \\
&= \frac{(k+1)\left[((k+1)+1)(2(k+1)+1)\right]}{6}.
\end{aligned}$$

That is,

$$P(k+1) = \frac{(k+1)((k+1)+1)\left(2(k+1)+1\right)}{6}.$$

This implies, $P(k+1)$ is true. The validity of $P(k+1)$ follows from that of $P(k)$. Therefore by the principle of mathematical induction,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}, \text{ for all } n \geq 1.$$

Example 4.64 Using the Mathematical induction, show that for any natural number n,

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Solution:
Let $P(n) := \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$. Substituting the value of $n = 1$, in the statement we get, $P(1) = \frac{1}{1.2} = \frac{1}{2}$. Hence, $P(1)$ is true. Let us assume that the statement is true for $n = k$. Then

$$P(k) = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}.$$

We need to show that $P(k+1)$ is true. Consider,

$$\begin{aligned} P(k+1) &= \underbrace{\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{k(k+1)}} + \frac{1}{(k+1)(k+2)} \\ &= P(k) + \frac{1}{(k+1)(k+2)} \\ &= \frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)} \\ &= \frac{1}{k+1}\left(\frac{k}{1} + \frac{1}{k+2}\right) \\ &= \frac{1}{k+1}\left(\frac{k^2+2k+1}{k+2}\right) \\ &= \frac{1}{k+1}\left(\frac{(k+1)^2}{k+2}\right) = \frac{(k+1)}{(k+2)}. \end{aligned}$$

This implies, $P(k+1)$ is true. The validity of $P(k+1)$ follows from that of $P(k)$. Therefore, by the principle of mathematical induction, for any natural number n,

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Example 4.65 Prove that for any natural number n, $a^n - b^n$ is divisible by $a - b$, where $a > b$

Solution:
Let

$$P(n) := a^n - b^n, \text{ is divisible by } a - b.$$

Substituting the value of $n = 1$, in the statement we get,

$$P(1) = a - b,$$

which is divisible by $a - b$. Hence, $P(1)$ is true. Let us assume that the statement is true for $n = k$. Then $P(k) = a^k - b^k$, is divisible by $a - b$. We can write

$$P(k) = a^k - b^k = \lambda(a-b),\ \lambda \in \mathbb{N}.$$

We need to show that $P(k+1) = a^{k+1} - b^{k+1}$, is divisible by $a - b$.

$$\begin{aligned} P(k+1) &= a^{k+1} - b^{k+1} \\ &= a^{k+1} - ab^k + ab^k - b^{k+1} \\ &= a(a^k - b^k) + b^k(a-b) \\ &= a(\lambda(a-b)) + b^k(a-b) \\ &= (a-b)\left(a\lambda + b^k\right) \\ &= (a-b)\lambda_1,\ \ \lambda_1 = a\lambda + b^k,\ \ \lambda_1 \in \mathbb{N}. \end{aligned}$$

which is divisible by $a-b$. This implies that $P(k+1)$ is true. The validity of $P(k+1)$ follows from that of $P(k)$. Therefore by the principle of mathematical induction, $a^n - b^n$ is divisible by $a-b$, where $a > b$, for all natural numbers n.

Example 4.66 Prove that $3^{2n+2} - 8n - 9$ is divisible by 8 for all $n \geq 1$.

Solution:
Let

$$P(n) := 3^{2n+2} - 8n - 9, \text{ is divisible by } 8.$$

Substituting the value of $n = 1$, in the statement we get,

$$P(1) = 3^{2+2} - 8(1) - 9 = 64,$$

which is divisible by 8. Hence, $P(1)$ is true. Let us assume that the statement is true for $n = k$. Then $P(k) = 3^{2k+2} - 8k - 9$, is divisible by 8. We can write

$$\begin{aligned} P(k) &= 3^{2k+2} - 8k - 9 = 8k_1, \quad k_1 \in \mathbb{N} \text{ and therefore,} \\ 3^{2k+2} &= 8k_1 + 8k + 9. \end{aligned}$$

We need to show that $P(k+1) = 3^{2(k+1)+2} - 8(k+1) - 9$, is divisible by 8.
Consider,

$$\begin{aligned} P(k+1) &= 3^{2(k+1)+2} - 8(k+1) - 9 \\ &= 3^2 3^{2k+2} - 8k - 8 - 9 \\ &= 3^2\left(8k_1 + 8k + 9\right) - 8k - 17 \; (by (4.1)) \\ &= 72k_1 + 64k + 64 \\ &= 8\left(9k_1 + 8k + 1\right) \\ &= 8k_2, \; k_2 = 9k_1 + 8k + 1 \in \mathbb{N} \end{aligned}$$

which is divisible by 8. This implies that $P(k+1)$ is true. This means that the validity of $P(k+1)$ follows from that of $P(k)$. Therefore by the principle of mathematical induction, $3^{2n+2} - 8n - 9$ is divisible by 8 for all $n \geq 1$.

Example 4.67 Using the Mathematical induction, show that for any integer

$$n \geq 2, \, 3n^2 > (n+1)^2$$

Solution:
Let $P(n)$ be the statement that $3n^2 > (n+1)^2$ with $n \geq 2$. Therefore the first stage is $n = 2$.
Now, $P(2) = 3 \times 2^2 = 12$ and $3^2 = 9$. As $12 > 9$ we get $P(2)$ is true.
We assume that $P(n)$ is true for $n = k$.

Now,

$$\begin{aligned} P(k+1) &= 3(k+1)^2 = 3k^2 + 6k + 3 \\ &= P(k) + 6k + 3 \\ &> (k+1)^2 + 6k + 3 \\ &= k^2 + 8k + 4 \\ &= k^2 + 4k + 4 + 4k \\ &= (k+2)^2 + 4k \\ &> (k+2)^2 \;\text{ since } k > 0. \end{aligned}$$

This is the statement $P(k+1)$. The validity of $P(k+1)$ follows from that of $P(k)$. Therefore by the principle of mathematical induction, for all $n \geq 2$, $3n^2 > (n+1)^2$.

Example 4.68 Using the Mathematical induction, show that for any integer

$$n \geq 2, \qquad 3^n > n^2$$

Solution:
Let $P(n)$ be the statement that $3^n > n^2$ with $n \geq 2$. Therefore the first stage is $n = 2$. Now, $P(2) = 3^2 = 9$ and $2^2 = 4$. As $9 > 4$, we get $P(2)$ is true
We assume that $P(n)$ is true for $n = k$. That is $P(k) > k^2$. Now,

$$\begin{aligned} P(k+1) &= 3^{k+1} = 3 \times 3^k = 3 \times P(k) \\ &> 3k^2 \\ &> (k+1)^2, \text{ by Example 4.67.} \end{aligned}$$

Hence, for any integer $n \geq 2$, $3^n > n^2$.

Example 4.69 By the principle of mathematical induction, prove that, for $n \in \mathbb{N}$,

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(n-1)\beta) = \cos\left(\alpha + \frac{(n-1)\beta}{2}\right) \times \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Solution:
Let $P(n) := \cos(\alpha) + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + ... + \cos(\alpha+(n-1)\beta)$. Then,

$$P(1) = \cos(\alpha) = \frac{\cos(\alpha).\sin(\frac{\beta}{2})}{\sin(\frac{\beta}{2})},$$

which shows $P(1)$ is true. We now assume that $P(n)$ is true for $n = k$. That is,

$$\cos(\alpha) + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(k-1)\beta) = \cos\left(\alpha + \frac{(k-1)\beta}{2}\right) \times \frac{\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}.$$

We need to prove $P(k+1)$ is true. Now,

$$P(k+1) = \underbrace{\cos(\alpha) + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(k-1)\beta)} + \cos(\alpha+k\beta)$$

Then,

$P(k+1) = P(k) + \cos(\alpha + k\beta)$

$$= \frac{\cos\left(\alpha + \frac{(k-1)\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)}{\sin(\frac{\beta}{2})} + \cos(\alpha + k\beta)$$

$$= \frac{1}{\sin(\frac{\beta}{2})}\left[\cos\left(\alpha + \frac{(k-1)\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right) + \cos(\alpha + k\beta)\sin\left(\frac{\beta}{2}\right)\right]$$

$$= \frac{1}{\sin(\frac{\beta}{2})}\left[\cos\left(\left(\alpha + \frac{k\beta}{2}\right) - \frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right) + cos(\alpha + k\beta)\sin\left(\frac{\beta}{2}\right)\right]$$

$$= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\left(\cos\left(\alpha + \frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\right.\right.$$
$$\left.\left. + \sin\left(\alpha + \frac{k\beta}{2}\right)\sin\left(\frac{\beta}{2}\right)\right)\sin\left(\frac{k\beta}{2}\right) + \cos(\alpha + k\beta)\sin\left(\frac{\beta}{2}\right)\right]$$

$$= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha + \frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)\right.$$
$$\left. + \sin\frac{\beta}{2}\left(\sin\left(\alpha + \frac{k\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right) + \cos(\alpha + k\beta)\right)\right]$$

$$= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha + \frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)\right.$$
$$\left. + \frac{\sin\frac{\beta}{2}}{2}\left(2\sin\left(\alpha + \frac{k\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right) + 2\cos(\alpha + k\beta)\right)\right]$$

$$= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha + \frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right)\right.$$
$$\left. + \frac{\sin\frac{\beta}{2}}{2}\left(\cos\alpha - \cos\left(\alpha + k\beta\right) + 2\cos(\alpha + k\beta)\right)\right]$$

$$= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha + \frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right) + \frac{\sin\frac{\beta}{2}}{2}\left(\cos\alpha + \cos(\alpha + k\beta)\right)\right]$$

$$= \frac{1}{\sin\left(\frac{\beta}{2}\right)}\left[\cos\left(\alpha + \frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{k\beta}{2}\right) + \frac{\sin\frac{\beta}{2}}{2}\left(2\cos\left(\alpha + \frac{k\beta}{2}\right)\cos\left(\frac{-k\beta}{2}\right)\right)\right]$$

$$= \frac{\cos\left(\alpha + \frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\left[\sin\left(\frac{k\beta}{2}\right)\cos\left(\frac{\beta}{2}\right) + \sin\frac{\beta}{2}\cos\left(\frac{k\beta}{2}\right)\right]$$

$$= \frac{\cos\left(\alpha + \frac{k\beta}{2}\right)\sin\left(\frac{(k+1)\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

That is,

$$\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \cdots + \cos(\alpha + (k-1)\beta) + \cos(\alpha + k\beta)$$

$$= \cos\left(\alpha + \frac{k\beta}{2}\right) \times \frac{\sin\left(\frac{(k+1)\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

This implies that $P(k+1)$ is true. The validity of $P(k+1)$ follows from that of $P(k)$. Therefore by the principle of mathematical induction,

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(n-1)\beta)$$

$$= \cos\left(\alpha + \frac{(n-1)\beta}{2}\right) \times \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Example 4.70 Using the Mathematical induction, show that for any natural number n, with the assumption $i^2 = -1$,

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta),$$

Solution:
Let,

$$P(n) = (r(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta).$$

Substituting the value of $n = 1$, in the statement we get,

$$P(1) = (r(\cos\theta + i\sin\theta))^1 = r(\cos\theta + i\sin\theta).$$

Hence, $P(1)$ is true. Let us assume that the statement is true for $n = k$. Then

$$(r(\cos\theta + i\sin\theta))^k = r^k(\cos k\theta + i\sin k\theta),$$

We need to show that $P(k+1)$ is true. Consider,

$$\begin{aligned}
P(k+1) &= (r(\cos\theta + i\sin\theta))^{k+1} \\
&= (r(\cos\theta + i\sin\theta))^k \times r(\cos\theta + i\sin\theta) \\
&= r^k(\cos k\theta + i\sin k\theta) \times r(\cos\theta + i\sin\theta) \\
&= r^{k+1} \times \big((\cos k\theta\cos\theta + i^2\sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)\big) \\
&= r^{k+1} \times ((\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)) \\
&= r^{k+1} \times (\cos(k+1)\theta + i\sin(k+1)\theta).
\end{aligned}$$

This implies that $P(k+1)$ is true. The validity of $P(k+1)$ follows from that of $P(k)$. Therefore by the principle of mathematical induction, for any natural number n,

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

.

What we have proved in Example 4.70 is called the **Demoivre's theorem** for natural numbers, which will be studied in detail in the second year of Higher Secondary course.

Exercise - 4.4

1. By the principle of mathematical induction, prove that, for $n \geq 1$
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$
2. By the principle of mathematical induction, prove that, for $n \geq 1$
$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}.$$
3. Prove that the sum of the first n non-zero even numbers is $n^2 + n$.
4. By the principle of Mathematical induction, prove that, for $n \geq 1$
$$1.2 + 2.3 + 3.4 + \cdots + n.(n+1) = \frac{n(n+1)(n+2)}{3}.$$
5. Using the Mathematical induction, show that for any natural number $n \geq 2$,
$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$
6. Using the Mathematical induction, show that for any natural number $n \geq 2$,
$$\frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \cdots + \frac{1}{1+2+3+\cdots+n} = \frac{n-1}{n+1}.$$
7. Using the Mathematical induction, show that for any natural number n,
$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \cdots + \frac{1}{n.(n+1).(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}.$$
8. Using the Mathematical induction, show that for any natural number n,
$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}.$$
9. Prove by Mathematical Induction that
$$1! + (2 \times 2!) + (3 \times 3!) + ... + (n \times n!) = (n+1)! - 1.$$
10. Using the Mathematical induction, show that for any natural number n, $x^{2n} - y^{2n}$ is divisible by $x + y$.
11. By the principle of Mathematical induction, prove that, for $n \geq 1$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 > \frac{n^3}{3}.$$
12. Use induction to prove that $n^3 - 7n + 3$, is divisible by 3, for all natural numbers n.
13. Use induction to prove that $5^{n+1} + 4 \times 6^n$ when divided by 20 leaves a remainder 9, for all natural numbers n.

14. Use induction to prove that $10^n + 3 \times 4^{n+2} + 5$, is divisible by 9, for all natural numbers n.
15. Prove that using the Mathematical induction

$$\sin(\alpha) + \sin\left(\alpha + \frac{\pi}{6}\right) + \sin\left(\alpha + \frac{2\pi}{6}\right) + \cdots$$
$$+ \sin\left(\alpha + \frac{(n-1)\pi}{6}\right) = \frac{\sin\left(\alpha + \frac{(n-1)\pi}{12}\right) \times \sin\left(\frac{n\pi}{12}\right)}{\sin\left(\frac{\pi}{12}\right)}.$$

2CDK1Y

Exercise - 4.5

Choose the correct or the most suitable answer

1. The sum of the digits at the 10^{th} place of all numbers formed with the help of 2, 4, 5, 7 taken all at a time is

 (1) 432 (2) 108 (3) 36 (4) 18

2. In an examination there are three multiple choice questions and each question has 5 choices . Number of ways in which a student can fail to get all answer correct is

 (1) 125 (2) 124 (3) 64 (4) 63

3. The number of ways in which the following prize be given to a class of 30 boys first and second in mathematics, first and second in physics, first in chemistry and first in English is

 (1) $30^4 \times 29^2$ (2) $30^3 \times 29^3$ (3) $30^2 \times 29^4$ (4) 30×29^5.

4. The number of 5 digit numbers all digits of which are odd is

 (1) 25 (2) 5^5 (3) 5^6 (4) 625.

5. In 3 fingers, the number of ways four rings can be worn is $\cdots\cdots\cdots$ ways.

 (1) $4^3 - 1$ (2) 3^4 (3) 68 (4) 64

6. If ${}^{(n+5)}P_{(n+1)} = (\frac{11(n-1)}{2})^{(n+3)}P_n$, then the value of n are

 (1) 7 and 11 (2) 6 and 7 (3) 2 and 11 (4) 2 and 6.

7. The product of r consecutive positive integers is divisible by

 (1) $r!$ (2) $(r-1)!$ (3) $(r+1)!$ (4) r^r.

8. The number of five digit telephone numbers having at least one of their digits repeated is

 (1) 90000 (2) 10000 (3) 30240 (4) 69760.

9. If ${}^{a^2-a}C_2 = {}^{a^2-a}C_4$ then the value of 'a' is

 (1) 2 (2) 3 (3) 4 (4) 5

10. There are 10 points in a plane and 4 of them are collinear. The number of straight lines joining any two points is

 (1) 45 (2) 40 (3) 39 (4) 38.

11. The number of ways in which a host lady invite 8 people for a party of 8 out of 12 people of whom two do not want to attend the party together is

 (1) $2 \times {}^{11}C_7 + {}^{10}C_8$ (2) ${}^{11}C_7 + {}^{10}C_8$ (3) ${}^{12}C_8 - {}^{10}C_6$ (4) ${}^{10}C_6 + 2!$.

12. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines.

(1) 6 (2) 9 (3) 12 (4) 18

13. Everybody in a room shakes hands with everybody else. The total number of shake hands is 66. The number of persons in the room is $\cdots\cdots\cdots\cdots$

(1) 11 (2) 12 (3) 10 (4) 6

14. Number of sides of a polygon having 44 diagonals is $\cdots\cdots$

(1) 4 (2) 4! (3) 11 (4) 22

15. If 10 lines are drawn in a plane such that no two of them are parallel and no three are concurrent, then the total number of points of intersection are

(1) 45 (2) 40 (3)10! (4) 2^{10}

16. In a plane there are 10 points are there out of which 4 points are collinear, then the number of triangles formed is

(1) 110 (2) $^{10}C_3$ (3) 120 (4) 116

17. In $^{2n}C_3 :^n C_3 = 11 : 1$ then n is

(1) 5 (2) 6 (3)11 (4)7

18. $^{(n-1)}C_r +^{(n-1)} C_{(r-1)}$ is

(1) $^{(n+1)}C_r$ (2) $^{(n-1)}C_r$ (3) ^nC_r (4)$^nC_{r-1}$.

19. The number of ways of choosing 5 cards out of a deck of 52 cards which include at least one king is

(1) $^{52}C_5$ (2) $^{48}C_5$ (3)$^{52}C_5 +^{48} C_5$ (4)$^{52}C_5 -^{48} C_5$.

20. The number of rectangles that a chessboard has $\cdots$

(1) 81 (2) 9^9 (3)1296 (4) 6561

21. The number of 10 digit number that can be written by using the digits 2 and 3 is

(1) $^{10}C_2 +^9 C_2$ (2) 2^{10} (3)$2^{10} - 2$ (4) 10!

22. If P_r stands for ^rP_r then the sum of the series $1 + P_1 + 2P_2 + 3P_3 + \cdots + nP_n$ is

(1) P_{n+1} (2) $P_{n+1} - 1$ (3) $P_{n-1} + 1$ (4)$^{(n+1)}P_{(n-1)}$

23. The product of first n odd natural numbers equals

(1) $^{2n}C_n \times^n P_n$ (2) $(\frac{1}{2})^n \times {}^{2n}C_n \times^n P_n$ (3)$(\frac{1}{4})^n \times {}^{2n}C_n \times^{2n} P_n$ (4)$^nC_n \times^n P_n$

24. If $^nC_4$, $^nC_5$, $^nC_6$ are in AP the value of n can be

(1) 14 (2) 11 (3)9 (4)5

25. $1 + 3 + 5 + 7 + \cdots + 17$ is equal to

(1) 101 (2) 81 (3) 71 (4) 61

Summary

In this chapter, we acquired the knowledge of

- Factorial of a natural number n is the product of the first n natural numbers.
- $n! = n(n-1)!$, for any integer $n \geq 1$.
- The number of ways of arranging n unlike objects is $n!$.
- The number of distinct permutations of r objects which can be made from n distinct objects is
$$^nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1).$$
- The number of permutations of n objects taken all at a time where P_1 objects one of first kind, P_2 objects one of second kind, $\cdots$ P_k objects one of the k^{th} kind and the rest, if any are all different and is given by
$$\frac{n!}{P_1!P_2!\cdots P_k!}.$$
- Order matters for a permutation where as order does not matter for a combination.
- The number of combinations of n different objects taken r at a time denoted by ^nC_r is given by
$$^nC_r = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

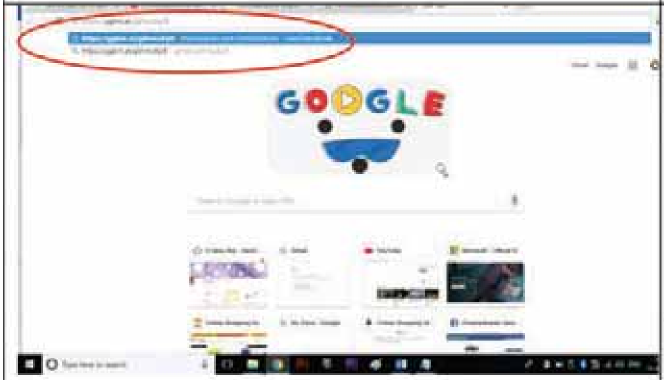
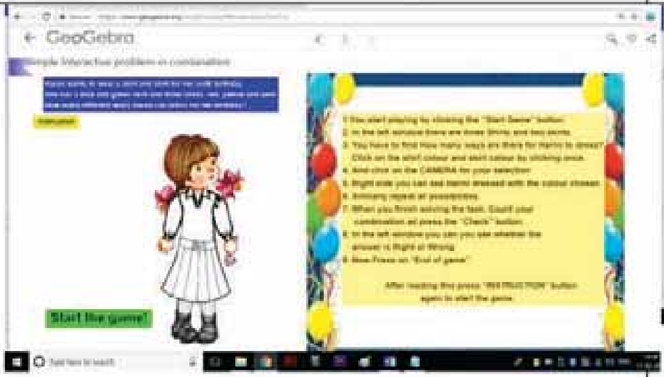
ICT CORNER-4(a)

Expected Outcome $\Rightarrow$

Step–1: Open the Browser and type the URL Link given below (or) Scan the QR Code. GeoGebra worksheet Permutations and Combination will appear. You can select the worksheet you want to open, for example open Combination Game

Step–2: Press Instruction button and read how to play. After reading once again press Instruction button and press Start the game.

Step–3: Find the number of combination to dress up the girl with 3 Shirts and 2 Skirts. After selecting each pair, click on Camera picture, so that the selected will appear in right space. After completing the selection press check button to see the result.

.Step-1	Step-2
	
Step-3	**Step-4**
	
	
Go through Remaining work sheets to gain clear idea of Permutations and Combinations	

*Pictures are only indicatives.

Browse in the link

Permutations and Combinations: https://ggbm.at/gVnmcKp9

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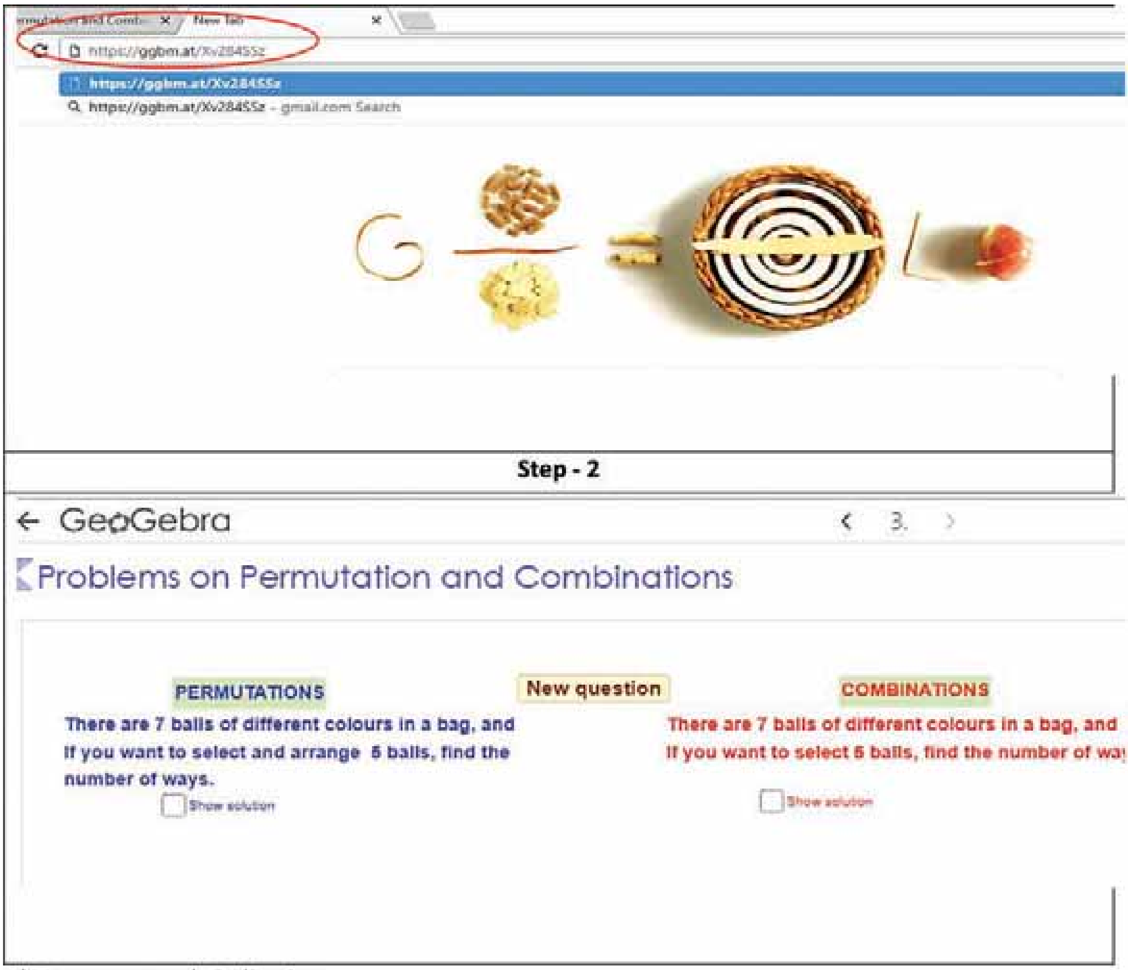
ICT CORNER-4(b)

Expected Outcome ⇒

Step–1: Open the Browser and type the URL Link given below (or) Scan the QR Code.

Step–2: GeoGebra worksheet "Problems on Permutations and Combination" will appear. By clicking on "New Problem" new question will be generated as many times you want to do. You have to work out the problem yourself and find the answer. Now click on "Show Solution" on each Permutation and Combination to get the respective solution and check with your answer.



*Pictures are only indicatives.

Browse in the link

Permutations and Combinations: https://ggbm.at/Xv284SSz

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Chapter 5

Binomial Theorem, Sequences and Series

"Life stands before me like an eternal spring with new and brilliant clothes"

Johann Carl Friedrich Gauss

5.1 Introduction

Binomial theorem facilitates the algebraic expansion of the binomial $(a + b)$ for a positive integral exponent n. Binomial theorem is used in all branches of Mathematics and also in other Sciences. Using the theorem, for example one can easily find the coefficient of x^{20} in the expansion of $(2x-7)^{23}$. If one wants to know the maturity amount after 10 years on a sum of money deposited in a nationalised bank at the rate of 8% compound interest per year or to know the size of population of our country after 15 years if the annual growth rate and present population size are known, Binomial theorem helps us in finding the above quantities. The coefficients that appear in the binomial expansion of $(a + b)^n$, $n \in \mathbb{N}$, are called binomial coefficients. Binomial theorem plays a vital role in determining the probabilities of events when the random experiment involves finite sample space and each outcome is either success or failure. In this chapter we learn binomial theorem and some of its applications.

Greek Mathematician Euclid mentioned the special case of binomial theorem for exponent 2. Binomial theorem for exponent 3 was known by 6^{th} century in India. In 1544, Michael Stifel (German Mathematician) introduced the term **binomial coefficient** and expressed $(1 + x)^n$ in terms of $(1 + x)^{n-1}$.

The German Mathematician Johann Carl Friedrich Gauss is one of the most renowned Mathematicians in history. Many have referred to him as the "Prince of Mathematics ". He has contributed in the areas of Number theory, Physics, Astronomy etc., Number Theory was Gauss's favourite field and he referred to Number theory as the "Queen of Mathematics". Anecdote involves his school teachers to wanted take rest and asked the students to sum the integers from 1 to 100. Within a few seconds Gauss shown the answer has 5050. Nobody is sure which method of summing an arithmetic sequence Gauss used as a child.

Johann Carl Friedrich Gauss (1777–1855)

Over the period of thousand years, legends have developed mathematical problems involving sequences and series. One of the famous legends about series concerns the invention of chess, where the cells of chess board were related to $1, 2, 4, 8, \ldots$ (imagine the number related to 64^{th} cell). There are many applications of arithmetic and geometric progressions to real life situations.

In the earlier classes we have learnt about sequences, series. Roughly speaking a sequence is an arrangement of objects in some order and a series is the sum of the terms of a sequence of numbers. The concept of infinite series helps us to compute many values, like $\sin \frac{9}{44}\pi$, $\log 43$ and e^{20} to a desired level of approximation. Sequences are important in differential equations and analysis. We learn more about sequences and series.

Learning Objectives

On completion of this chapter, the students are expected to know

- the concept of Binomial Theorem, to compute binomial coefficients and their applications
- the concepts of sequences and series
- how to compute arithmetic, geometric and harmonic means
- how to find the sum of finite and infinite series of real numbers
- how to add series using telescopic summation
- how to apply binomial, exponential and logarithmic series

5.2 Binomial Theorem

The prefix **bi** in the words bicycle, binocular, binary and in many more words means **two**. The word ***binomial*** stands for expressions having two terms. For examples $(1+x)$, $(x+y)$, (x^2+xy) and $(2a+3b)$ are some binomial expressions.

5.2.1 Binomial Coefficients

In Chapter 4 we have learnt and used the symbol ${}^n\mathrm{C}_r$ which is defined as

$$ {}^n\mathrm{C}_r = \frac{n(n-1)(n-2)\dots(n-(r-1))}{r(r-1)(r-2)\dots 1} = \frac{n!}{(n-r)!\,r!}. $$

Since ${}^n\mathrm{C}_r$ occurs as the coefficients of x^r in $(1+x)^n$ $n \in \mathbb{N}$ and as the coefficients of $a^r b^{n-r}$ in $(a+b)^n$, they are called binomial coefficients. Though the values of ${}^n\mathrm{C}_r$ can be computed by formula, there is an interesting simple way to find ${}^n\mathrm{C}_r$ without doing cumbersome multiplications.

Pascal Triangle

The Pascal triangle is an arrangement of the numbers ${}^n\mathrm{C}_r$ in a triangular form. The $(k+1)^{st}$ row consists of the numbers

$$ {}^k\mathrm{C}_0\,,\ {}^k\mathrm{C}_1\,,\ {}^k\mathrm{C}_2\,,\ {}^k\mathrm{C}_3\,,\dots,\ {}^k\mathrm{C}_k\,. $$

In fact, the Pascal triangle is

$$
\begin{array}{ccccccccccc}
 & & & & {}^0\mathrm{C}_0 & & & & & & \\
 & & & {}^1\mathrm{C}_0 & & {}^1\mathrm{C}_1 & & & & & \\
 & & {}^2\mathrm{C}_0 & & {}^2\mathrm{C}_1 & & {}^2\mathrm{C}_2 & & & & \\
 & {}^3\mathrm{C}_0 & & {}^3\mathrm{C}_1 & & {}^3\mathrm{C}_2 & & {}^3\mathrm{C}_3 & & & \\
{}^4\mathrm{C}_0 & & {}^4\mathrm{C}_1 & & {}^4\mathrm{C}_2 & & {}^4\mathrm{C}_3 & & {}^4\mathrm{C}_4 & & \\
\dots & & \dots & & \dots & & \dots & & & & \\
\dots & & \dots & & \dots & & \dots & & & &
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
\dots & & \dots & & \dots & & \dots & & \\
\dots & & \dots & & \dots & & \dots & &
\end{array}
$$

Recall the expansion and observe the coefficients of each term of the identities $(a+b)^0$, $(a+b)^1$, $(a+b)^2$, $(a+b)^3$. There is a pattern in the arrangements of coefficents

$$
\begin{array}{rcl}
(a+b)^0 = 1 & \qquad & 1 \\
(a+b)^1 = a+b & & 1 \quad 1 \\
(a+b)^2 = a^2+2ab+b^2 & & 1 \quad 2 \quad 1 \\
(a+b)^3 = a^3+3a^2b+3ab^2+b^3 & & 1 \quad 3 \quad 3 \quad 1
\end{array}
$$

If we observe carefully the Pascal triangle, we may notice that each row starts and ends with 1 and other entries are the sum of the two numbers just above it. For example '3' is the sum of 1 and 2 above it; '10' is the sum of 4 and 6 above it. We will prove in a short while that

$$(a+b)^n = {}^n\mathrm{C}_0\, a^n b^0 + {}^n\mathrm{C}_1\, a^{n-1} b^1 + \cdots + {}^n\mathrm{C}_r\, a^{n-r} b^r + \cdots + {}^n\mathrm{C}_n\, a^0 b^n.$$

which is the ***binomial expansion*** of $(a+b)^n$. The binomial expansion of $(a+b)^n$ for any $n \in \mathbb{N}$ can be written using Pascal triangle. For example, from the fifth row we can write down the expansion of $(a+b)^4$ and from the sixth row we can write down the expansion of $(a+b)^5$ and so on. We know the terms (without coefficients) of $(a+b)^5$ are

$$a^5,\ a^4b,\ a^3b^2,\ a^2b^3,\ ab^4,\ b^5$$

and the sixth row of the Pascal triangle is

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1.$$

Using these two we can write

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

The Pascal triangle can be constructed using addition alone, without using any multiplication or division. So without multiplication we can write down the binomial expansion for $(a+b)^n$ for any $n \in \mathbb{N}$.

The above pattern resembling a triangle, is credited in the name of the seventeenth century French Mathematician Blaise Pascal, who studied mathematical properties of this structure and used this concept effectively in Probability Theory.

5.2.2 Binomial theorem for positive integral index

Now we prove the most celebrated theorem called Binomial Theorem.

Theorem 5.1 (Binomial theorem for positive integral index): If n is any positive integer, then

$$(a+b)^n = {}^n\mathrm{C}_0\, a^n b^0 + {}^n\mathrm{C}_1\, a^{n-1} b^1 + \cdots + {}^n\mathrm{C}_r\, a^{n-r} b^r + \cdots + {}^n\mathrm{C}_n\, a^0 b^n.$$

Proof. We prove the theorem by using mathematical induction. For any positive integer n, let $P(n)$ be the statement

$$(a+b)^n = {}^n\mathrm{C}_0\, a^n b^0 + {}^n\mathrm{C}_1\, a^{n-1} b^1 + \cdots + {}^n\mathrm{C}_r\, a^{n-r} b^r + \cdots + {}^n\mathrm{C}_n\, a^0 b^n.$$

Since

$${}^1\mathrm{C}_0 = 1 \ \text{ and } \ {}^1\mathrm{C}_1 = 1,$$

the expression in the right hand side of $P(1)$ is $a^1b^0 + a^0b^1$ which is same as $a+b$; the left hand side is $(a+b)^1$. Hence $P(1)$ is true.

We assume that for a positive integer k, $P(k)$ is true. That is,

$$(a+b)^k = {}^k\mathrm{C}_0\, a^k b^0 + {}^k\mathrm{C}_1\, a^{k-1}b^1 + \cdots + {}^k\mathrm{C}_r\, a^{k-r}b^r + \cdots + {}^k\mathrm{C}_k\, a^0 b^k.$$

Let us use the identity

$${}^n\mathrm{C}_r + {}^n\mathrm{C}_{r-1} = {}^{n+1}\mathrm{C}_r$$

in the proof. Now

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)(a+b)^k\\
&= (a+b)\left[{}^k\mathrm{C}_0\, a^k b^0 + {}^k\mathrm{C}_1\, a^{k-1}b^1 + \cdots + {}^k\mathrm{C}_r\, a^{k-r}b^r + \cdots + {}^k\mathrm{C}_k\, a^0 b^k\right]\\
&= \left[{}^k\mathrm{C}_0\, a^{k+1} b^0 + {}^k\mathrm{C}_1\, a^{k}b^1 + \cdots + {}^k\mathrm{C}_r\, a^{k-r+1}b^r + \cdots + {}^k\mathrm{C}_k\, a^1 b^k\right]\\
&\quad + \left[{}^k\mathrm{C}_0\, a^{k} b^1 + {}^k\mathrm{C}_1\, a^{k-1}b^2 + \cdots + {}^k\mathrm{C}_r\, a^{k-r}b^{r+1} + \cdots + {}^k\mathrm{C}_k\, a^0 b^{k+1}\right]\\
&= {}^k\mathrm{C}_0\, a^{k+1}b^0 + \left[{}^k\mathrm{C}_1 + {}^k\mathrm{C}_0\right] a^k b^1 + \cdots + \left[{}^k\mathrm{C}_r + {}^k\mathrm{C}_{r-1}\right] a^{k-r+1}b^r\\
&\quad + \cdots + \left[{}^k\mathrm{C}_k + {}^k\mathrm{C}_{k-1}\right] a^1 b^k + {}^k\mathrm{C}_k\, a^0 b^{k+1}\\
&= {}^{k+1}\mathrm{C}_0\, a^{k+1}b^0 + {}^{k+1}\mathrm{C}_1\, a^k b^1 + {}^{k+1}\mathrm{C}_2\, a^{k-1}b^2 + \cdots + {}^{k+1}\mathrm{C}_r\, a^{k-r+1}b^r\\
&\quad + \cdots + {}^{k+1}\mathrm{C}_k\, a^1 b^k + {}^{k+1}\mathrm{C}_{k+1}\, a^0 b^{k+1}\\
(a+b)^{k+1} &= {}^{k+1}\mathrm{C}_0\, a^{(k+1)}b^0 + {}^{k+1}\mathrm{C}_1\, a^{(k+1)-1}b^1 + {}^{k+1}\mathrm{C}_2\, a^{(k+1)-2}b^2 + \cdots\\
&\quad + {}^{k+1}\mathrm{C}_r\, a^{(k+1)-r}b^r + \cdots + {}^{k+1}\mathrm{C}_k\, a^1 b^{(k+1)-1} + {}^{k+1}\mathrm{C}_{k+1}\, a^0 b^{k+1}.
\end{aligned}$$

This shows that $P(k+1)$ is true whenever $P(k)$ is true. Thus, by the principle of mathematical induction, $P(n)$ is true for all natural numbers n. Hence,

$$(a+b)^n = {}^n\mathrm{C}_0\, a^n b^0 + {}^n\mathrm{C}_1\, a^{n-1}b^1 + \cdots + {}^n\mathrm{C}_r\, a^{n-r}b^r + \cdots + {}^n\mathrm{C}_n\, a^0 b^n, n \in \mathbb{N}.$$

(i) The expansion of $(a+b)^n, n \in \mathbb{N}$ can also be written as

$$(a+b)^n = \Sigma_{k=0}^{n}\, {}^n\mathrm{C}_k\, a^{n-k}b^k \text{ or } \Sigma_{k=0}^{n}\, {}^n\mathrm{C}_k\, a^k b^{n-k}.$$

(ii) The expansion of $(a+b)^n, n \in \mathbb{N}$, contains exactly $(n+1)$ terms.

(iii) In $(a+b)^n = \Sigma_{k=0}^{n}\, {}^n\mathrm{C}_k\, a^{n-k}b^k$, the powers of a decreases by 1 in each term, whereas the powers of b increases by 1 in each term. However, the sum of powers of a and b in each term is always n.

(iv) The $(r+1)^{th}$ term in the expansion of $(a+b)^n, n \in \mathbb{N}$, is

$$T_{r+1} = {}^n\mathrm{C}_r\, a^{n-r}b^r, \;\; r = 0, 1, 2, \cdots, n.$$

(v) In the product $(a+b)(a+b)\cdots(a+b), n$ times, to get b^r, we need any r factors out of these n factors. This can be done in ${}^n\mathrm{C}_r$ ways. That is why, we have ${}^n\mathrm{C}_r$ as the coefficient of $a^{n-r}b^r$.

(vi) In the expansion of $(a+b)^n, n \in \mathbb{N}$, the coefficients at equidistant from the beginning and from the end are equal due to the fact that ${}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}$.

(vii) In the expansion of $(a+b)^n, n \in \mathbb{N}$, the greatest coefficient is ${}^n\mathrm{C}_{\frac{n}{2}}$ if n is even and the greatest coefficients are ${}^n\mathrm{C}_{\frac{n-1}{2}}$ or ${}^n\mathrm{C}_{\frac{n+1}{2}}$, if n is odd.

(viii) In the expansion of $(a+b)^n, n \in \mathbb{N}$, if n is even, the middle term is $T_{\frac{n}{2}+1} = {}^n\mathrm{C}_{\frac{n}{2}}\, a^{n-\frac{n}{2}} b^{\frac{n}{2}}$. If n is odd, then the two middle terms are $T_{\frac{n-1}{2}+1}$ and $T_{\frac{n+1}{2}+1}$.

5.3 Particular cases of Binomial Theorem

(i) Replacing b by $(-b)$, in the binomial expansion of $(a+b)^n, n \in \mathbb{N}$, we get

$$\begin{aligned}(a-b)^n &= {}^n\mathrm{C}_0\, a^n b^0 - {}^n\mathrm{C}_1\, a^{n-1} b^1 + {}^n\mathrm{C}_2\, a^{n-2} b^2 - \cdots \\ &\qquad + (-1)^r\, {}^n\mathrm{C}_r\, a^{n-r} b^r + \cdots + (-1)^n\, {}^n\mathrm{C}_n\, a^0 b^n.\end{aligned}$$

Observe that the sign '$+$' and '$-$' appear alternately in the binomial expansion of $(a-b)^n$.

(ii) Replacing a by 1 and b by x, in the binomial expansion of $(a+b)^n$, we get

$$(1+x)^n = {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1\, x + {}^n\mathrm{C}_2\, x^2 + \cdots + {}^n\mathrm{C}_r\, x^r + \cdots + {}^n\mathrm{C}_n\, x^n.$$

In particular, when $x = 1$, ${}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 + \cdots + {}^n\mathrm{C}_n = 2^n$.

If X is a set containing n elements, then we know that ${}^n\mathrm{C}_r$ is the number of subsets of X having exactly r elements. So by adding ${}^n\mathrm{C}_r$ for $r = 0, 1, 2, \ldots, n$ we get the number of subsets of X. So by using the above identity we see that **a set of n elements has 2^n subsets.**

(iii) $(1-x)^n = {}^n\mathrm{C}_0 - {}^n\mathrm{C}_1\, x + {}^n\mathrm{C}_2\, x^2 - \cdots + (-1)^r\, {}^n\mathrm{C}_r\, x^r + \cdots + (-1)^n x^n$. In particular, when $x = 1$, ${}^n\mathrm{C}_0 + {}^n\mathrm{C}_2 + {}^n\mathrm{C}_4 + \cdots = {}^n\mathrm{C}_1 + {}^n\mathrm{C}_3 + {}^n\mathrm{C}_5 + \cdots = 2^{n-1}$

Example 5.1 Find the expansion of $(2x+3)^5$.

Solution:
By taking $a = 2x$, $b = 3$ and $n = 5$ in the binomial expansion of $(a+b)^n$ we get

$$\begin{aligned}(2x+3)^5 &= (2x)^5 + 5(2x)^4 3 + 10(2x)^3 3^2 + 10(2x)^2 3^3 + 5(2x)3^4 + 3^5 \\ &= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243.\end{aligned}$$

Example 5.2 Evaluate 98^4.

Solution:
By taking $a = 100$, $b = 2$ and $n = 4$ in the binomial expansion of $(a-b)^n$ we get

$$\begin{aligned}98^4 &= (100-2)^4 \\ &= {}^4\mathrm{C}_0\, 100^4 - {}^4\mathrm{C}_1\, 100^3 2 + {}^4\mathrm{C}_2\, 100^2 2^2 - {}^4\mathrm{C}_3\, 100^1 2^3 + {}^4\mathrm{C}_4\, 100^0 2^4 \\ &= 100000000 - 8000000 + 240000 - 3200 + 16 \\ &= 92236816.\end{aligned}$$

Example 5.3 Find the middle term in the expansion of $(x+y)^6$.

Solution:
Here $n = 6$; which is even. Thus the middle term in the expansion of $(x+y)^6$ is the term containing $x^{\frac{6}{2}}y^{\frac{6}{2}}$, that is the term ${}^6\mathrm{C}_3\, x^3y^3$ which is equal to $20x^3y^3$.

Example 5.4 Find the middle terms in the expansion of $(x+y)^7$.

Solution:
As $n = 7$ which is odd, the terms containing x^4y^3 and x^3y^4 are the two middle terms. They are ${}^7\mathrm{C}_3\, x^4y^3$ and ${}^7\mathrm{C}_4\, x^3y^4$ which are equal to $35x^4y^3$ and $35x^3y^4$.

Example 5.5 Find the coefficient of x^6 in the expansion of $(3+2x)^{10}$.

Solution:
Let us take $a = 3$ and $b = 2x$ in the binomial expansion of $(a+b)^{10}$. Then, x^6 will appear in the term containing $(2x)^6$ and nowhere else. So the term containing x^6 is

$${}^{10}\mathrm{C}_6\, a^4b^6 = {}^{10}\mathrm{C}_4\, a^4b^6 = \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1}\, 3^4(2x)^6 = 210\times 3^4\times 2^6x^6. \quad [\because\ {}^{10}\mathrm{C}_6 = {}^{10}\mathrm{C}_4\,]$$

So coefficient of x^6 in the expansion of $(3+2x)^{10}$ is $210\times 3^4 2^6$.

Example 5.6 Find the coefficient of x^3 in the expansion of $(2-3x)^7$.

Solution:
Let us take $a = 2$ and $b = -3x$ in the binomial expansion of $(a+b)^7$. Then, x^3 will appear in the term containing $(-3x)^3$ and nowhere else. So the term containing x^3 is

$${}^7\mathrm{C}_3\, a^4b^3 = {}^7\mathrm{C}_4\, a^4b^3 = \frac{7\times 6\times 5}{3\times 2\times 1}\, 2^4(-3x)^3 = 35\times 2^4\times(-3)^3x^3. \quad [\because\ {}^7\mathrm{C}_4 = {}^7\mathrm{C}_3\,]$$

So coefficient of x^3 in the expansion of $(2-3x)^7$ is $35\times 16\times(-27) = -15120$.

Example 5.7 The 2^{nd}, 3^{rd} and 4^{th} terms in the binomial expansion of $(x+a)^n$ are 240, 720 and 1080 for a suitable value of x. Find x, a and n.

Solution:
It is given that $T_2 = 240$, $T_3 = 720$ and $T_4 = 1080$.

$$\begin{aligned} T_2 &= {}^n\mathrm{C}_1\, x^{n-1}a &&= 240 \qquad (1)\\ T_3 &= {}^n\mathrm{C}_2\, x^{n-2}a^2 &&= 720 \qquad (2)\\ T_4 &= {}^n\mathrm{C}_3\, x^{n-3}a^3 &&= 1080 \qquad (3) \end{aligned}$$

Dividing (2) by (1) and (3) by (2) we get

$$\frac{a}{x} = \frac{6}{n-1} \qquad (4)$$
$$\frac{a}{x} = \frac{9}{2(n-2)} \qquad (5)$$

From (4) and (5)

$$\frac{6}{n-1} = \frac{9}{2(n-2)}$$

Thus $n = 5$. Substituting $n = 5$ in (1), (4) and dividing (1) by (4)

$$\frac{5x^4a}{\frac{a}{x}} = \frac{240}{\frac{6}{4}}$$

Thus $5x^5 = 160$ and hence $x = 2$. Substituting in (4) we get $a = 3$.

Example 5.8 Expand $\left(2x - \frac{1}{2x}\right)^4$.

Solution:
We have

$$\begin{aligned}
\left(2x - \frac{1}{2x}\right)^4 &= {}^4C_0\,(2x)^4\left(-\frac{1}{2x}\right)^0 + {}^4C_1\,(2x)^3\left(-\frac{1}{2x}\right)^1 + {}^4C_2\,(2x)^2\left(-\frac{1}{2x}\right)^2 \\
&\quad + {}^4C_3\,(2x)^1\left(-\frac{1}{2x}\right)^3 + {}^4C_4\,(2x)^0\left(-\frac{1}{2x}\right)^4 \\
&= (2x)^4 - 4(2x)^3\left(\frac{1}{2x}\right) + 6(2x)^2\left(\frac{1}{2x}\right)^2 - 4(2x)\left(\frac{1}{2x}\right)^3 + \left(\frac{1}{2x}\right)^4 \\
&= 16x^4 - 16x^2 + 6 - \frac{1}{x^2} + \frac{1}{16x^4}
\end{aligned}$$

Example 5.9 Expand $\left(x^2 + \sqrt{1-x^2}\right)^5 + \left(x^2 - \sqrt{1-x^2}\right)^5$.

Solution:
We have

$$\begin{aligned}
\left(x^2 + \sqrt{1-x^2}\right)^5 &= {}^5C_0\,(x^2)^5\left(\sqrt{1-x^2}\right)^0 + {}^5C_1\,(x^2)^4\left(\sqrt{1-x^2}\right)^1 \\
&\quad + {}^5C_2\,(x^2)^3\left(\sqrt{1-x^2}\right)^2 + {}^5C_3\,(x^2)^2\left(\sqrt{1-x^2}\right)^3 \\
&\quad + {}^5C_4\,(x^2)^1\left(\sqrt{1-x^2}\right)^4 + {}^5C_5\,(x^2)^0\left(\sqrt{1-x^2}\right)^5 \\
&= x^{10} + 5x^8\sqrt{1-x^2} + 10x^6(1-x^2) + 10x^4(1-x^2)\sqrt{1-x^2} \\
&\quad +5x^2(1-x^2)^2 + (1-x^2)^2(\sqrt{1-x^2})
\end{aligned}$$

$$\begin{aligned}
\left(x^2-\sqrt{1-x^2}\right)^5 &= {}^5\mathrm{C}_0\,(x^2)^5\left(\sqrt{1-x^2}\right)^0 - {}^5\mathrm{C}_1\,(x^2)^4\left(\sqrt{1-x^2}\right)^1 \\
&\quad + {}^5\mathrm{C}_2\,(x^2)^3\left(\sqrt{1-x^2}\right)^2 - {}^5\mathrm{C}_3\,(x^2)^2\left(\sqrt{1-x^2}\right)^3 \\
&\quad + {}^5\mathrm{C}_4\,(x^2)^1\left(\sqrt{1-x^2}\right)^4 - {}^5\mathrm{C}_5\,(x^2)^0\left(\sqrt{1-x^2}\right)^5 \\
&= x^{10} - 5x^8\sqrt{1-x^2} + 10x^6(1-x^2) - 10x^4(1-x^2)\sqrt{1-x^2} \\
&\qquad +5x^2(1-x^2)^2 - (1-x^2)^2(\sqrt{1-x^2})
\end{aligned}$$

Thus

$$\begin{aligned}
\left(x^2+\sqrt{1-x^2}\right)^5 + \left(x^2-\sqrt{1-x^2}\right)^5 &= 2[x^{10} + 10x^6(1-x^2) + 5x^2(1-x^2)^2] \\
&= 2[x^{10} + 10x^6 - 10x^8 + 5x^2(1-2x^2+x^4)] \\
&= 2[x^{10} - 10x^8 + 15x^6 - 10x^4 + 5x^2]
\end{aligned}$$

Example 5.10 Using Binomial theorem, prove that $6^n - 5n$ always leaves remainder 1 when divided by 25 for all positive integer n.

Solution:
To prove this it is enough to prove, $6^n - 5n = 25k + 1$ for some integer k. We first consider the expansion

$$(1+x)^n = {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1\,x + {}^n\mathrm{C}_2\,x^2 + \cdots + {}^n\mathrm{C}_{n-1}\,x^{n-1} + {}^n\mathrm{C}_n\,x^n, n \in \mathbb{N}$$

Taking $x = 5$ we get $(1+5)^n = {}^n\mathrm{C}_0 + {}^n\mathrm{C}_1\,5 + {}^n\mathrm{C}_2\,5^2 + \cdots + {}^n\mathrm{C}_{n-1}\,5^{n-1} + {}^n\mathrm{C}_n\,5^n$. The above equality reduces to $6^n = 1 + 5n + 25({}^n\mathrm{C}_2 + 5\,{}^n\mathrm{C}_3 + \cdots + {}^n\mathrm{C}_n\,5^{n-2})$.
That is,

$$6^n - 5n = 1 + 25({}^n\mathrm{C}_2 + 5\,{}^n\mathrm{C}_3 + \cdots + {}^n\mathrm{C}_n\,5^{n-2}) = 1 + 25k, k \in \mathbb{N}.$$

Thus $6^n - 5n$ always leaves remainder 1 when divided by 25 for all positive integer n.

Example 5.11 Find the last two digits of the number 7^{400}.

Solution:
We have

$$\begin{aligned}
7^{400} &= (7^2)^{200} = (50-1)^{200} \\
&= {}^{200}\mathrm{C}_0\,50^{200} - {}^{200}\mathrm{C}_1\,50^{199} + \cdots \\
&\qquad + {}^{200}\mathrm{C}_{198}\,50^2(-1)^{198} + {}^{200}\mathrm{C}_{199}\,50(-1)^{199} + {}^{200}\mathrm{C}_{200}\,(-1)^{200} \\
&= 50^2\left({}^{200}\mathrm{C}_0\,50^{198} - {}^{200}\mathrm{C}_1\,50^{197} + \cdots + {}^{200}\mathrm{C}_{198}\,(-1)^{198}\right) - 200\times 50 + 1.
\end{aligned}$$

As 50^2 and 200 are divisible by 100, the last two digits: 0 1.

Exercise - 5.1

1. Expand (i) $\left(2x^2 - \frac{3}{x}\right)^3$ (ii) $\left(2x^2 - 3\sqrt{1-x^2}\right)^4 + \left(2x^2 + 3\sqrt{1-x^2}\right)^4$.
2. Compute (i) 102^4 (ii) 99^4 (iii) 9^7.
3. Using binomial theorem, indicate which of the following two number is larger: $(1.01)^{1000000}$, 10000.
4. Find the coefficient of x^{15} in $\left(x^2 + \dfrac{1}{x^3}\right)^{10}$.
5. Find the coefficient of x^6 and the coefficient of x^2 in $\left(x^2 - \dfrac{1}{x^3}\right)^6$.
6. Find the coefficient of x^4 in the expansion of $(1+x^3)^{50}(x^2 + \dfrac{1}{x})^5$.
7. Find the constant term of $\left(2x^3 - \dfrac{1}{3x^2}\right)^5$.
8. Find the last two digits of the number 3^{600}.
9. If n is a positive integer, show that, $9^{n+1} - 8n - 9$ is always divisible by 64.
10. If n is an odd positive integer, prove that the coefficients of the middle terms in the expansion of $(x+y)^n$ are equal.
11. If n is a positive integer and r is a nonnegative integer, prove that the coefficients of x^r and x^{n-r} in the expansion of $(1+x)^n$ are equal.
12. If a and b are distinct integers, prove that $a - b$ is a factor of $a^n - b^n$, whenever n is a positive integer. [Hint: write $a^n = (a - b + b)^n$ and expand]
13. In the binomial expansion of $(a+b)^n$, the coefficients of the 4^{th} and 13^{th} terms are equal to each other, find n.
14. If the binomial coefficients of three consecutive terms in the expansion of $(a+x)^n$ are in the ratio $1 : 7 : 42$, then find n.
15. In the binomial coefficients of $(1+x)^n$, the coefficients of the 5^{th}, 6^{th} and 7^{th} terms are in AP. Find all values of n.
16. Prove that $C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = \dfrac{2n!}{(n!)^2}$.

5.4 Finite Sequences

A sequence is a list of elements with a particular order. While the idea of a sequence of numbers, $a_1, a_2, \cdots$, is straight forward, it is useful to think of a sequence as a function whose domain is either the set of first n natural numbers or $\mathbb{N}$. Throughout this chapter, we consider only sequences of real numbers and we will refer to them as sequences. The arithmetic sequences and geometric sequences are also known as arithmetic progressions(AP) and geometric progressions (GP). Let us recall, the basic definitions of sequences and series.

- If X is any set and $n \in \mathbb{N}$, then any function $f : \{1, 2, 3, \ldots, n\} \to X$ is called a ***finite sequence*** on X and any function $g : \mathbb{N} \to X$ is called an ***infinite sequence*** on X. The value $f(n)$ of the function f at n is denoted by a_n and the sequence itself is denoted by (a_n).
- If the set X happens to be a set of real numbers, the sequence is called a numerical sequence or a sequence of real numbers.
- Though every sequence is a function, a function is not necessarily a sequence.
- Unlike sets, where elements are not repeated, the terms in a sequence may be repeated. In particular, a sequence in which all terms are same is called a constant sequence.
- A useful way to visualise a sequence (a_n) is to plot the graph of $\{(n, a_n) : n \in \mathbb{N}\}$ which gives some details about the sequence.

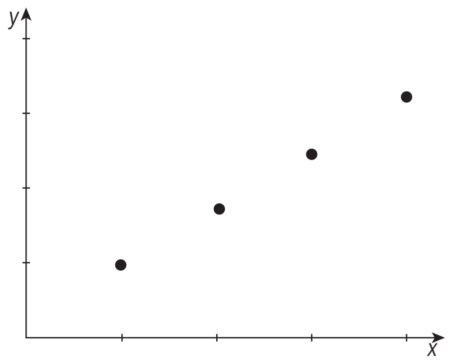

Figure 5.1

5.4.1 Arithmetic and Geometric Progressions

Progressions are some special cases of sequences where the terms of the sequences are either in increasing form or decreasing form.

We recall some definitions and results we discussed in earlier classes on arithmetic and geometric progressions.

Arithmetic Progression (AP)

- A sequence of the form

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots,\ a+(n-1)d,\ a+nd,\ \ldots$$

  is called an ***arithmetic progression*** or an ***arithmetic sequence***. In other words, each term (other than the first term) of the sequence is obtained by adding a constant to its previous term; the constant d is called ***common difference*** and the term a is called the ***initial term*** or ***first term***.
- The n^{th} term of an arithmetic progression is given by $T_n = a + (n-1)d$.
- The sequences $\sqrt{2},\ \sqrt{2}+\sqrt{3},\ \sqrt{2}+2\sqrt{3},\ \sqrt{2}+3\sqrt{3},\ \ldots$ and $12,\ 9,\ 6,\ 3,\ \ldots$ are arithmetic sequences with common differences $\sqrt{3}$ and -3 respectively.
- It is interesting to observe that $3, 7, 11$ are three prime numbers which form an AP.
- For $n \in \mathbb{N}, T_n = an + b$ where a and b are relatively prime, form an AP which contains infinitely many prime numbers along with infinitely many composite numbers.

Geometric Progression (GP)

- A sequence of the form

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots,\ ar^{n-1},\ ar^n,\ \ldots$$

  with $a \neq 0$, and $r \neq 0$ is called a ***geometric progression*** or a ***geometric sequence***. In other words, each term (other than the first term) of the sequence is obtained by multiplying its previous term by a constant; the constant r is called ***common ratio*** and the term a is called the ***initial term*** or ***first term***.
- The n^{th} term of a geometric progression is given by $T_n = ar^{n-1}$.
- The sequences $1,\ 2,\ 4,\ 8,\ 16,\ \ldots$ and $\sqrt{2},\ 2,\ 2\sqrt{2},\ 4,\ 4\sqrt{2},\ 16,\ \ldots$ are geometric sequences with common ratios 2 and $\sqrt{2}$ respectively.
- Taking logarithm of each term in a geometric progression with positive common ratio yields an arithmetic progression. *i.e.*, If $a, ar, ar^2, \ldots$ is a GP with $r > 0$, then $\log a, \log(ar), \log(ar^2), \ldots$ is an AP with common difference $\log r$.

It is interesting to note that the constant sequence $c,\ c,\ c, \ldots$ is an arithmetic sequence and is also a geometric sequence if $c \neq 0$.

Let us consider the special constant sequence $0, 0, 0, 0, \ldots$. We have no problem in seeing this as an arithmetic sequence. But when we try to see this as a geometric sequence clearly the initial term a must be 0. What can we say about the common ratio r? If we take r as 1, 2 or any other number we get the same sequence $0, 0, 0, 0, \ldots$. We are left with the situation where a geometric sequence has infinitely many common ratios. To overcome these confusions some mathematicians exclude this sequence from the class of geometric sequences by assuming $a \neq 0$ in the definition. (We made this assumption)

5.4.2 Arithmetico-Geometric Progression (AGP)

Combining arithmetic and geometric progressions, a new progression called arithmetico geometric progression is formed. As we use the abbreviations AP and GP for arithmetic progressions and geometric progressions, we use the abbreviation AGP for arithmetico geometric progression. AGP's arise in various applications, such as the computation of expected value in probability theory.

Definition 5.1

A sequence of the form

$$a,\ (a+d)r,\ (a+2d)r^2,\ (a+3d)r^3,\ \ldots,\ (a+(n-1)d)r^{n-1},\ (a+nd)r^n,\ \ldots$$

is called an ***arithmetico-geometric progression*** or an ***arithmetico-geometric sequence***.

Consider an AP: $a, a+d, a+2d, \ldots$
GP: $1, r, r^2, \ldots$

Then the AGP is $a, (a+d)r, (a+2d)r^2, \ldots$

Here, a is the ***initial term***, d is the ***common difference*** and r is the ***common ratio*** of the AGP.

If we take $r = 1$, then the AGP will become an AP and if we take $d = 0$, then it will become a GP. So the arithmetic and Geometric progressions become particular cases of AGP. This is a nice situation to know the concept of generalization in mathematics.

We note that the n^{th} term of an AGP is given by $T_n = (a+(n-1)d)r^{n-1}$. All APs and all GPs are AGPs.

For example, the AP $0, 1, 2, 3, 4, \ldots$ and the GP $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ give the AGP $\frac{0}{1}, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \ldots$

The sequence $4, 14, 40, 104, 256, 608, \ldots$ is also an example of an arithmetico-geometric progression. For this sequence $a = 4$, $d = 3$ and $r = 2$.

5.4.3 Harmonic Progression (HP)

Harmonic progression is one of many important sequences and is closely related to the arithmetic progression. Harmonic progression is widely used.

Definition 5.2

A sequence $h_1, h_2, h_3, \ldots$ is said to a ***harmonic sequence*** or a ***harmonic progression*** if $\frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \frac{1}{h_4}, \ldots$ is an arithmetic sequence.

Note that a sequence is in harmonic progression if its reciprocals are in arithmetic progression. But we should not say that harmonic progressions are reciprocals of arithmetic progressions; in fact, if an arithmetic sequence contains a zero term, then its reciprocal is not meaningful. Of course, if an arithmetic progression contains no zero term, then its reciprocal is a harmonic progression. So a general harmonic progression will be of the form

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \ldots$$

As the denominator of a fraction cannot be 0, $a+kd \neq 0$ for any integer $k \geq 0$. So the condition $\frac{-a}{d}$ is not a whole number is essential. Problems in harmonic progression can be converted into problems in arithmetic progression and be solved using arithmetic progression techniques and formulae.

(i) The sequence $\left(\frac{1}{n}\right) = 1, \frac{1}{2}, \frac{1}{3}, \cdots$ is a harmonic sequence. One can draw the graph of
$\{(n, \frac{1}{n}) : n \in \mathbb{N}\}$ and visualize the harmonic sequence $\left(\frac{1}{n}\right)$.

(ii) If a, b, c are in HP, then $b = \dfrac{2ac}{a+c}$.

(iii) In a triangle, if the altitudes are in AP, then the sides are in HP

Example 5.12 Prove that if a, b, c are in HP, if and only if $\frac{a}{c} = \frac{a-b}{b-c}$.

Solution:
If a, b, c are in HP, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP. Thus, we have $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$, which gives $ab - ac = ac - bc$. So, $a(b-c) = c(a-b)$, which gives $\frac{a}{c} = \frac{a-b}{b-c}$. On the other hand, if $\frac{a}{c} = \frac{a-b}{b-c}$, then $a(b-c) = c(a-b)$. Dividing each terms of both sides by abc, we get $\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}$. Thus, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP and hence a, b, c are in HP.

Example 5.13 If the 5^{th} and 9^{th} terms of a harmonic progression are $\frac{1}{19}$ and $\frac{1}{35}$, find the 12^{th} term of the sequence.

Solution:
Let h_n be the harmonic progression and let $a_n = \dfrac{1}{h_n}$. Then $a_5 = 19$ and $a_9 = 35$. As a_n's from an arithmetic progression, we have $a + 4d = 19$ and $a + 8d = 35$. Solving these two equations, we get $a = 3$ and $d = 4$. Thus $a_{12} = a + 11d = 47$. Thus the 12^{th} term of the harmonic progression is $\dfrac{1}{47}$.

What can we say about constant sequences? All constant sequences other than the zero sequence are harmonic progressions also.

5.4.4 Arithmetic, Geometric and Harmonic Mean

We know the concept of "*average*". There are many "*averages*". Arithmetic mean (AM), Geometric mean(GM) and Harmonic mean(HM) are also some averages. Let us now recall the definitions of arithmetic mean and geometric mean, where the terms need not in AP or GP.

Arithmetic Mean and Geometric Mean

Definition 5.3

Let n be any positive integer. Let $a_1, a_2, a_3, \ldots, a_n$ be n numbers . Then the number

$$\frac{a_1 + a_2 + a_3 + \cdots + a_n}{n}$$

is called the ***arithmetic mean*** of the numbers $a_1, a_2, a_3, \ldots, a_n$.

The numbers $a_1, a_2, a_3, \ldots, a_n$ need not be distinct and it is not necessary that the numbers are positive. It easily follows from the definition that 16 is the arithmetic mean of the numbers $14, 14, 17, 20, 15$.

Taking the multiplication in place of addition and n^{th} root in place of division by n in the definition of arithmetic mean we get the definition of geometric mean.

Definition 5.4

Let n be any positive integer. Let $a_1, a_2, a_3, \ldots, a_n$ be n non-negative numbers. Then the number

$$\sqrt[n]{a_1 a_2 a_3 \ldots a_n}$$

is called the ***geometric mean*** of the numbers $a_1, a_2, a_3, \ldots, a_n$.

Here also the numbers $a_1, a_2, a_3, \ldots, a_n$ need not be distinct but it is necessary that the numbers are non-negative. The geometric mean of the numbers $4, 6, 9$ is $\sqrt[3]{216} = 6$. The arithmetic mean of these three numbers is $\frac{19}{3} = 6\frac{1}{3}$. Observe that the arithmetic mean is greater than the geometric mean in this case. Is this true always?

It can be proved that "*For any set of n non-negative numbers, the arithmetic mean is greater than or equal to the geometric mean*". That is, if AM denotes the arithmetic mean and GM denotes the geometric mean, then $AM \geq GM$.

Let us prove this inequality $AM \geq GM$ for two non-negative numbers.

Theorem 5.2: If AM and GM denote the arithmetic mean and the geometric mean of two nonnegative numbers, then $AM \geq GM$. The equality holds if and only if the two numbers are equal.

Proof. Let a and b be any two nonnegative numbers. Then

$$AM = \frac{a+b}{2} \quad \text{and} \quad GM = \sqrt{ab}.$$

We have, $(a+b)^2 - 4ab = (a-b)^2 \geq 0$ Thus, $(a+b)^2 - 4ab \geq 0$ which gives $(a+b) \geq 2\sqrt{ab}$. Hence $\frac{a+b}{2} \geq \sqrt{ab}$.
In other words, $AM \geq GM$.

Moreover, the equality holds if and only if $(a+b)^2 - 4ab = 0$. This holds if and only if $(a-b)^2 = 0$ which holds if and only if $a = b$. Thus $AM = GM$ if and only if $a = b$. □

Geometrical Proof for $AM \geq GM$

Let a and b be any two nonnegative real numbers. If at least one of them is zero, then GM is 0 and hence we have nothing to prove. So let us assume that $a > 0$ and $b > 0$. We draw a straight line segment AB of length $a+b$ and a semi-circle having AB as diameter. Let M be the midpoint of AB. Then M is the center of the semi-circle drawn. Since M is the midpoint of AB, we have $AM = MB = \frac{a+b}{2}$. So the radius of the circle is $\frac{a+b}{2}$. Let D be the point on AB so that $AD = a$; then $DB = b$.

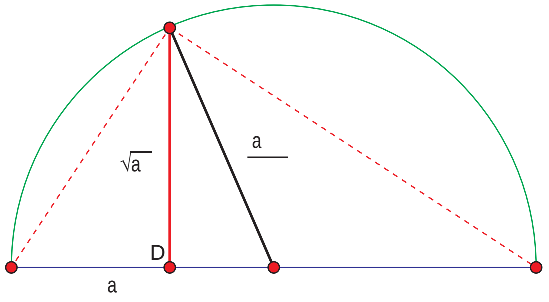

Figure 5.2

Through D we draw the perpendicular to AB and let it to meet the semi-circle at C. We draw straight lines CA, CB and CM. Since M is the center $CM =$ radius $= \frac{a+b}{2}$. Clearly $MD = \frac{a+b}{2} - a$. Using the similar triangles ΔACD and ΔCBD we have $\frac{CD}{AD} = \frac{BD}{CD}$ and hence $CD^2 = AD \times BD = ab$. So $CD = \sqrt{ab}$. (Using Pythagoras theorem also we can prove that $CD = \sqrt{ab}$.) Since the length of any half chord is less than or equal to the radius, we have $CD \leq CM$. In other words, $\sqrt{ab} \leq \frac{a+b}{2}$. This means $GM \leq AM$.

The length of the half chord DC is equal to the radius if and only if $D = M$. Thus equality $AM = GM$ holds if and only if $a = b$.

Result 5.1: If $a_1, a_2, a_3, \ldots, a_n$ is an arithmetic progression, every term $a_k \;\; (k > 1)$ is the arithmetic mean of its immediate predecessor a_{k-1} and immediate successor a_{k+1}.

Proof. Let $a_1, a_2, a_3, \ldots, a_n$ be an arithmetic progression with initial term a and common difference d. Then

$$a_k = a + (k-1)d, \quad a_{k-1} = a + (k-2)d \;\text{ and }\; a_{k+1} = a + kd.$$

Thus

$$\frac{a_{k-1} + a_{k+1}}{2} = \frac{a + (k-2)d + a + kd}{2} = \frac{2a + (2k-2)d}{2} = a + (k-1)d = a_k.$$

Therefore, a_k is the arithmetic mean of a_{k-1} and a_{k+1}. □

Result 5.2: If $a_1, a_2, a_3, \ldots, a_n$ is a geometric progression, every term $a_k \;\; (k > 1)$ is the geometric mean of its immediate predecessor a_{k-1} and immediate successor a_{k+1}.

Proof. Let $a_1, a_2, a_3, \ldots, a_n$ be a geometric progression with initial term a and common ratio r. Then

$$a_k = ar^{k-1}, \quad a_{k-1} = ar^{k-2} \;\text{ and }\; a_{k+1} = ar^k.$$

Thus

$$\sqrt{a_{k-1}\, a_{k+1}} = \sqrt{ar^{k-2}ar^k} = \sqrt{a^2r^{2k-2}} = ar^{k-1} = a_k.$$

Therefore, a_k is the geometric mean of a_{k-1} and a_{k+1}. □

Example 5.14 Find seven numbers $A_1, A_2, \ldots, A_7$ so that the sequence $4, A_1, A_2, \ldots, A_7, 7$ is in arithmetic progression and also 4 numbers G_1, G_2, G_3, G_4 so that the sequence $12, G_1, G_2, G_3, G_4, \frac{3}{8}$ is in geometric progression.

Solution:
Since $a = 4$ and $4 + 8d = 7$ we get $d = \frac{3}{8}$. So the required 7 numbers are $4\frac{3}{8},\ 4\frac{6}{8},\ 5\frac{1}{8},\ 5\frac{4}{8},\ 5\frac{7}{8},\ 6\frac{2}{8},\ 6\frac{5}{8}$.

Since $a = 12$ and $ar^5 = \frac{3}{8}$ we get $r^5 = \frac{1}{32}$ and hence $r = \frac{1}{2}$. Thus the required 4 numbers are $6, 3, 1\frac{1}{2}, \frac{3}{4}$.

Example 5.15 If the product of the 4^{th}, 5^{th} and 6^{th} terms of a geometric progression is 4096 and if the product of the 5^{th}, 6^{th} and 7^{th}-terms of it is 32768, find the sum of first 8 terms of the geometric progression.

Solution:
Let $a, ar, ar^2, \ldots$ be the geometric series having the given properties. Since the 4^{th}, 5^{th} and 6^{th} terms are ar^3, ar^4 and ar^5, their product is a^3r^{12}. Thus $a^3r^{12} = 4096$. Similarly $a^3r^{15} = 32768$.

Therefore $\dfrac{a^3r^{15}}{a^3r^{12}} = \dfrac{32768}{4096}$. Hence $r^3 = 8$. This implies that $r = 2$.
$a^3r^{12} = 4096$ we have $a^3 = 1$. Therefore $a = 1$.

The sum of the first 8 terms is $\dfrac{a(1-r^8)}{1-r} = \dfrac{1-2^8}{1-2} = 255$.

Harmonic Mean

The harmonic mean of a set of positive numbers is the reciprocal of the arithmetic mean of the reciprocals of the set of numbers. That is, if $h_1, h_2, \ldots, h_n$ are positive numbers, then their reciprocals are $\frac{1}{h_1}, \frac{1}{h_2}, \ldots, \frac{1}{h_n}$; the arithmetic mean of the reciprocals is

$$\frac{\frac{1}{h_1} + \frac{1}{h_2} + \cdots + \frac{1}{h_n}}{n}$$

and the reciprocal of this arithmetic mean, that is the harmonic mean of the numbers $h_1, h_2, \ldots, h_n$ is

$$\frac{n}{\frac{1}{h_1} + \frac{1}{h_2} + \cdots + \frac{1}{h_n}}.$$

Definition 5.5

The harmonic mean of a set $\{h_1, h_2, \ldots, h_n\}$ of positive numbers is defined as

$$\frac{n}{\frac{1}{h_1} + \frac{1}{h_2} + \cdots + \frac{1}{h_n}}.$$

In particular, the harmonic mean of two positive numbers a and b is $\dfrac{2}{\frac{1}{a}+\frac{1}{b}}$ which is equal to $\frac{2ab}{a+b}$.

It can be proved that "*For any set of n positive numbers, the geometric mean is greater than or equal to the harmonic mean*". That is, $GM \geq HM$.

Let us prove this inequality $GM \geq HM$ for two non-negative numbers.

Theorem 5.3: If GM and HM denote the geometric mean and the harmonic mean of two non-negative numbers, then $GM \geq HM$. The equality holds if and only if the two numbers are equal.

Proof. Let a and b be any two positive numbers. Then

$$\begin{aligned}
GM &= \sqrt{ab} \quad \text{and} \quad HM = \frac{2ab}{a+b}. \\
GM - HM &= \sqrt{ab} - \frac{2ab}{a+b} \\
&= \frac{\sqrt{ab}(a+b) - 2ab}{a+b} \\
&= \frac{\sqrt{ab}((a+b) - 2\sqrt{ab})}{a+b} \\
&= \frac{\sqrt{ab}(\sqrt{a}-\sqrt{b})^2}{a+b} \\
&\geq 0
\end{aligned}$$

Thus $GM - HM \geq 0$ and hence $GM \geq HM$. □

We have already proved that $AM \geq GM$ and now we have $GM \geq HM$. Combining these two, we have an important inequality $AM \geq GM \geq HM$.

Result 5.3: For any two positive numbers, the three means AM, GM and HM are in geometric progression.

Proof. Let a and b be any two positive real numbers. Then

$$AM = \frac{a+b}{2}, \quad GM = \sqrt{ab}, \text{ and } HM = \frac{2ab}{a+b}.$$

Now

$$AM \times HM = \left(\frac{a+b}{2}\right)\left(\frac{2ab}{a+b}\right) = ab = (\sqrt{ab})^2 = GM^2.$$

That is, $AM \times HM = GM^2$ and hence AM, GM and HM are in geometric progression. □

We note the following interesting results.

- If b is the arithmetic mean of a and c, then a, b, c is an arithmetic progression.
- If b is the geometric mean of a and c, then a, b, c is a geometric progression.
- If b is the harmonic mean of a and c, then a, b, c is a harmonic progression.

If a vehicle travels at a speed of x kmph. covering certain distance and it returns the same distance with a speed of y kmph., then the average speed of the vehicle in the whole travel is the harmonic mean of the upward and downward speeds. Indeed, if d is the distance, then time taken for upward journey is $\frac{d}{x}$ and the time taken for downward journey is $\frac{d}{y}$.

Thus average speed is $\dfrac{2d}{\dfrac{d}{x}+\dfrac{d}{y}} = \dfrac{2xy}{x+y}$.

For example if a vehicle travels at a speed of 60 kmph. covering certain distance and it returns the same distance with a speed of 40 kmph., then the average speed of the vehicle in the whole travel is the harmonic mean of 60 and 40. That is $\frac{2\times 60\times 40}{60+40} = 48$ kmph speed.

Exercise - 5.2

1. Write the first 6 terms of the sequences whose n^{th} terms are given below and classify them as arithmetic progression, geometric progression, arithmetico-geometric progression, harmonic progression and none of them.

   (i) $\frac{1}{2^{n+1}}$ (ii) $\dfrac{(n+1)(n+2)}{n+3(n+4)}$ (iii) $4\left(\frac{1}{2}\right)^n$ (iv) $\frac{(-1)^n}{n}$ (v) $\frac{2n+3}{3n+4}$ (vi) 2018 (vii) $\frac{3n-2}{3^{n-1}}$

2. Write the first 6 terms of the sequences whose n^{th} term a_n is given below.

   (i) $a_n = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$ (ii) $a_n = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ a_{n-1} + a_{n-2} & \text{if } n > 2 \end{cases}$

   (iii) $a_n = \begin{cases} n & \text{if } n \text{ is } 1, 2 \text{ or } 3 \\ a_{n-1} + a_{n-2} + a_{n-3} & \text{if } n > 3 \end{cases}$

3. Write the n^{th} term of the following sequences.

$$\begin{array}{llll} \text{(i)} & 2,\ 2,\ 4,\ 4,\ 6,\ 6,\ \ldots & \text{(ii)} & \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \frac{5}{6},\ \ldots \\ \text{(iii)} & \frac{1}{2},\ \frac{3}{4},\ \frac{5}{6},\ \frac{7}{8},\ \frac{9}{10},\ \ldots & \text{(iv)} & 6,\ 10,\ 4,\ 12,\ 2,\ 14,\ 0, 16,\ -2, \ldots \end{array}$$

4. The product of three increasing numbers in GP is 5832. If we add 6 to the second number and 9 to the third number, then resulting numbers form an AP. Find the numbers in GP.
5. Write the n^{th} term of the sequence $\frac{3}{1^2 2^2}, \frac{5}{2^2 3^2}, \frac{7}{3^2 4^2}, \ldots$ as a difference of two terms.
6. If t_k is the k^{th} term of a GP, then show that t_{n-k}, t_n, t_{n+k} also form a GP for any positive integer k.
7. If a, b, c are in geometric progression, and if $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$, then prove that x, y, z are in arithmetic progression.
8. The AM of two numbers exceeds their GM by 10 and HM by 16. Find the numbers.
9. If the roots of the equation $(q-r)x^2 + (r-p)x + p - q = 0$ are equal, then show that p, q and r are in AP.
10. If a, b, c are respectively the p^{th}, q^{th} and r^{th} terms of a GP, show that $(q-r)\log a + (r-p)\log b + (p-q)\log c = 0$.

5.5 Finite Series

Roughly speaking a series is the sum of terms of a sequence of numbers; a finite series is the sum of terms of a finite sequence of numbers. If (a_n) is a sequence of numbers, then the expression $a_1 + a_2 + \cdots + a_n$ is called a ***finite series***. The expression $a_1 + a_2 + \cdots + a_n$ is denoted as $\sum\limits_{k=1}^{n} a_k$. Sometimes, depending upon the problem under consideration and for simplicity a series may be given as $a_0 + a_1 + a_2 + \cdots$ with first term as a_0.

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5.5.1 Sum of Arithmetic, Geometric and Arithmetico-Geometric Progressions

In the earlier classes we studied about the sum of a few terms, like sum of first n terms, of arithmetic and geometric progressions. We now recall them.

Sum of Arithmetic and Geometric Progressions

- A series is said to be an ***arithmetic series*** if the terms of the series form an arithmetic sequence. A series is said to be a ***geometric series*** if the terms of the series form a geometric sequence.
- The sum S_n of the first n terms of the arithmetic sequence $(a + (n-1)d)$ is given by $S_n = na + \frac{(n-1)n}{2}d = \frac{n}{2}[2a + (n-1)d]$.
- The sum S_n of the first n terms of the geometric sequence (ar^{n-1}) is given by $S_n = \frac{a(1-r^n)}{1-r}$ provided $r \neq 1$. If $r = 1$, then the sequence is nothing but the constant sequence $a, a, a, \ldots$ and the sum of the first n terms is clearly na. Thus, if $r \neq 1$, then $1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$.

Sum of Arithmetico-Geometric Progressions

- A series is said to be an ***arithmetico-Geometric series*** if the terms of the series form an arithmetico-Geometric sequence.
- The sum S_n of the first n terms of the arithmetico-Geometric sequence $((a + (n-1)d)r^{n-1})$ is given by

$$S_n = \frac{a - (a + (n-1)d)r^n}{1-r} + dr\left(\frac{1 - r^{n-1}}{(1-r)^2}\right)$$

for $r \neq 1$.

Example 5.16 Find the sum up to n terms of the series: $1 + \frac{6}{7} + \frac{11}{49} + \frac{16}{343} + \cdots$.

Solution:
Here $a = 1$, $d = 5$ and $r = \frac{1}{7}$.

$$\begin{aligned} S_n &= \frac{a - (a + (n-1)d)r^n}{1 - r} + dr\left(\frac{1 - r^{n-1}}{(1-r)^2}\right) \\ &= \frac{1 - (1 + 5(n-1))(\frac{1}{7})^n}{1 - \frac{1}{7}} + 5 \times \frac{1}{7}\left(\frac{(1 - \frac{1}{7})^{n-1}}{(1 - \frac{1}{7})^2}\right) \\ &= \frac{1 - \frac{5n-4}{7^n}}{\frac{6}{7}} + \frac{\frac{5}{7}(7^{n-1} - 1)}{7^{n-1}(\frac{6}{7})^2} \\ &= \frac{7^n - 5n + 4}{7^{n-1}6} + \frac{5(7^{n-1} - 1)}{7^{n-2}36} \end{aligned}$$

5.5.2 Telescopic Summation for Finite Series

Telescopic summation is a more general method used for summing a series either for finite or infinite terms. This technique expresses sum of n terms of a given series just in two terms, usually first and last term, by making the intermediate terms cancel each other. After canceling intermediate terms, we bring the last term which is far away from the first term very close to the first term. So this process is called “Telescopic Summation”.

Example 5.17 Find the sum of the first n terms of the series $\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \ldots$

Solution:
Let t_k denote the k^{th} term of the given series. Then $t_k = \frac{1}{\sqrt{k}+\sqrt{k+1}}$. If we are successful in writing the k^{th} term as a difference of two expressions, then we can solve using this technique. We have

$$t_k = \frac{1}{\sqrt{k} + \sqrt{k+1}} = \frac{\sqrt{k} - \sqrt{k+1}}{(\sqrt{k} + \sqrt{k+1})(\sqrt{k} - \sqrt{k+1})} = \frac{\sqrt{k} - \sqrt{k+1}}{k - (k+1)} = \sqrt{k+1} - \sqrt{k}$$

Thus

$$t_1 + t_2 + \cdots + t_n = (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{n+1} - \sqrt{n}) = \sqrt{n+1} - 1$$

Example 5.18 Find $\sum\limits_{k=1}^{n} \frac{1}{k(k+1)}$.

Solution:
Let t_k denote the k^{th} term of the given series. Then $t_k = \frac{1}{k(k+1)}$. By using partial fraction we get

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Thus

$$t_1 + t_2 + \cdots + t_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

Exercise - 5.3

1. Find the sum of the first 20-terms of the arithmetic progression having the sum of first 10 terms as 52 and the sum of the first 15 terms as 77.
2. Find the sum up to the 17^{th} term of the series $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \cdots$
3. Compute the sum of first n terms of the following series:

   i. $8 + 88 + 888 + 8888 + \cdots$ ii. $6 + 66 + 666 + 6666 + \cdots$

4. Compute the sum of first n terms of $1 + (1+4) + (1+4+4^2) + (1+4+4^2+4^3) + \cdots$
5. Find the general term and sum to n terms of the sequence $1, \frac{4}{3}, \frac{7}{9}, \frac{10}{27}, \ldots$.
6. Find the value of n, if the sum to n terms of the series $\sqrt{3} + \sqrt{75} + \sqrt{243} + \cdots$ is $435\sqrt{3}$.
7. Show that the sum of $(m+n)^{th}$ and $(m-n)^{th}$ term of an AP. is equal to twice the m^{th} term.
8. A man repays an amount of Rs.3250 by paying Rs.20 in the first month and then increases the payment by Rs.15 per month. How long will it take him to clear the amount?
9. In a race, 20 balls are placed in a line at intervals of 4 meters, with the first ball 24 meters away from the starting point. A contestant is required to bring the balls back to the starting place one at a time. How far would the contestant run to bring back all balls?
10. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2^{nd} hour, 4^{th} hour and n^{th} hour?
11. What will Rs.500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?
12. In a certain town, a viral disease caused severe health hazards upon its people disturbing their normal life. It was found that on each day, the virus which caused the disease spread in Geometric Progression. The amount of infectious virus particle gets doubled each day, being 5 particles on the first day. Find the day when the infectious virus particles just grow over 1,50,000 units?

5.5.3 Some Special Finite Series

In this section we give some of the important formulas of summing up finitely many terms which follows either an AP, GP, or any specific series.

1. Summation of first n natural numbers:

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

   [Treating this as an AP, one can find the sum.]

2. Summation of the squares of first n natural numbers:

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

   [Use the identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ and try to prove this result.]

3. Summation of the cubes of first n natural numbers:

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

   [Use the identity $k^4 - (k-1)^4 = (4k^3 - 6k^2 + 4k - 1)$ and try to prove this result.]

Note that the above three results were proved in the earlier classes.

5.6 Infinite Sequences and Series

A finite sum of real numbers is well defined by the properties of real numbers, but in order to make sense of an infinite series, we need to consider the concept of convergence. Consider the infinite sum: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ with each term positive. Can we assign a numerical value to the sum? While at first it may seem difficult or impossible, we can certainly do something similar where we have one quantity getting closer and closer to a fixed quantity.

Let us discuss an interesting problem. Let there be two plates A and B. Let a full cake be placed on the plate A and let B as empty. Let us cut the cake in A into exactly two equal parts and place one part on B leaving the other part in A. Let us cut the remaining part of the cake in A into exactly two equal parts and place one part on B leaving the other part in A. Let us again cut the remaining part of the cake in A into exactly two equal parts and place one part on B leaving the other part in A. If we go on doing this what will happen? What will be the amount of cake "*finally*" in A and in B? Let us list the stage by stage status:

Stage	Plate A	Plate B
0	1	0
1	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{4}$	$\frac{1}{2}+\frac{1}{4}$
3	$\frac{1}{8}$	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$
4	$\frac{1}{16}$	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$
5	$\frac{1}{32}$	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$
...	...	...
n	$\frac{1}{2^n}$	$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots+\frac{1}{2^n}$
...	...	...

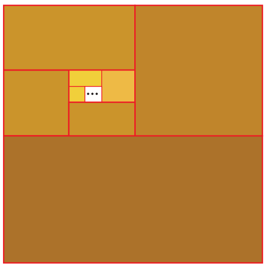

Figure 5.3

Intuitively we feel that "*finally*" nothing will remain in plate A and the full cake will be in plate B. In other words, the cake available in A is 0 and the cake available in B is 1. That is, intuitively we feel that

$$1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32},\ \ldots$$

"*goes*" to 0 and

$$\frac{1}{2},\ \frac{1}{2}+\frac{1}{4},\ \frac{1}{2}+\frac{1}{4}+\frac{1}{8},\ \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16},\ \ldots$$

"*goes*" to 1 or equivalently

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots \qquad \text{is } 1.$$

In this section let us learn the sense in which the words "*finally*" and "*goes*" are used and also let us learn the addition of infinitely many numbers.

We intuitively feel that $1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32},\ \ldots$ "*goes*" to 0. Similarly we feel that the sequence 1, $\frac{1}{10},\ \frac{1}{100},\ \frac{1}{1000},\ \frac{1}{10000},\ \frac{1}{100000},\ \ldots$ also "*goes*" to 0.

If (a_n) is a sequence and a is a number so that for any given small positive number, there is a stage after which the distance between a_n and a is smaller than that positive number, then we may say that a_n goes to a as n goes to infinity. In technical terms we may say that a_n *tends to a as n tends to infinity.* In other words, in the limiting case a_n becomes a or the limit of a_n is a as n tends to ∞. We also say that the sequence (a_n) converges to a. If (a_n) converges to a, then we write $\lim\limits_{n\to\infty} a_n = a$.

At the same time we cannot say that the sequence

$$1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ \ldots$$

goes to some number. In other words, this sequence do not converge to any limit. So a sequence may not have a limit. But we can prove that a sequence cannot converge to more than one limit; that is, **if a sequence converges to a limit, then it is unique**.

5.6.1 Fibonacci Sequence

The Fibonacci sequence is a sequence of numbers where a number other than first two terms, is found by adding up the two numbers before it. Starting with 1, the sequence goes 1, 1, 2, 3, 5, 8, 13, 21, 34, and so forth. Written as a rule, the expression is $x_n = x_{n-1} + x_{n-2}, n \geq 3$ with $x_0 = 1, x_1 = 1$

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Named after **Fibonacci**, also known as Leonardo of Pisa or Leonardo Pisano, Fibonacci numbers were first introduced in his Liber abaci in 1202. The son of a Pisan merchant, Fibonacci traveled widely and traded extensively. Mathematics was incredibly important to those in the trading industry, and his passion for numbers was cultivated in his youth.

Knowledge of numbers is said to have first originated in the Hindu-Arabic arithmetic system, which Fibonacci studied while growing up in North Africa. Prior to the publication of Liber abaci, the Latin-speaking world had yet to be introduced to the decimal number system. He wrote many books about geometry, commercial arithmetic and irrational numbers. He also helped in the development of the concept of zero.

$$\begin{array}{ccccccccccccccccc} n & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \cdots \\ x_n & = & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 89 & 144 & 233 & 377 & \cdots \end{array}$$

For example, The 8^{th} term is sum of 6^{th} term and 7^{th} term. Thus, $x_8 = 8 + 13 = 21$.

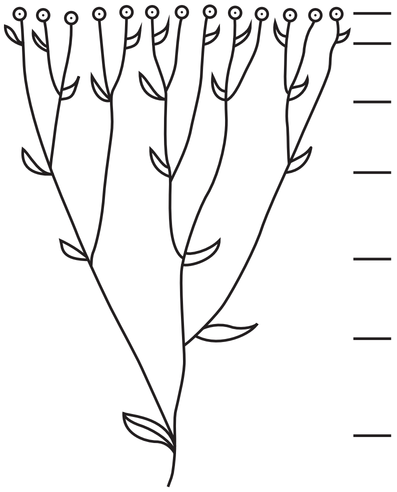


Figure 5.4

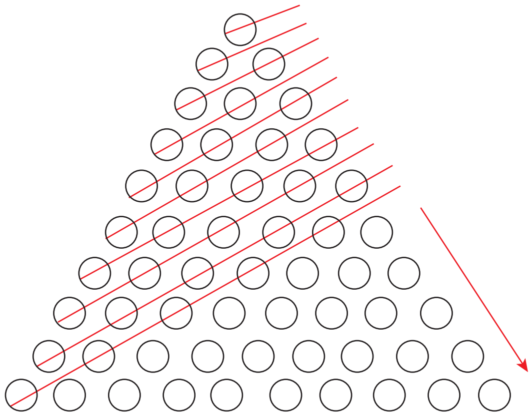


Figure 5.5

There is an interesting pattern in the Fibonacci sequence.

Observe that

(i) every third number is a multiple of 3^{rd} term ($t_3 = 2$).
(ii) every fourth number is a multiple of 4^{th} term ($t_4 = 3$).
(iii) every fifth number is a multiple of 5^{th} term ($t_5 = 5$).
(iv) So, every n^{th} number is a multiple of n^{th} term.

Infinite Series

If (a_n) is an infinite sequence of numbers, then the formal expression $a_1+a_2+\cdots$ is called an ***infinite series*** and is denoted as $\sum\limits_{k=1}^{\infty} a_k$.

In the beginning of the section we have seen the infinite series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

and intuitively felt that "*its sum*" is 1 (in the cake problem). Actually in the cake problem, the stage by stage availability of the cake in the plate B is given in the following sequence.

$$\frac{1}{2}, \frac{1}{2}+\frac{1}{4}, \frac{1}{2}+\frac{1}{4}+\frac{1}{8}, \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} \cdots$$

The n^{th} term of this sequence is

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}$$

which is equal to $\frac{2^n-1}{2^n}$. If s_n denote this sum, then $\lim\limits_{n\to\infty} s_n = 1$. This is one of the reasons for us to feel that the sum

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots \qquad \text{is } 1.$$

Motivated by this we may define the sum of infinitely many numbers. Let (a_n) be a sequence of real numbers and let $s_n = a_1+a_2+a_3+\cdots+a_n$. If the sequence (s_n) converges to a limit s, then it is meaningful to say that the sequence (a_n) is "*summable*" and the sum is s. In this case, we write

$$a_1+a_2+a_3+\cdots = s$$

It is customary to say that the series converges to s.

Definition 5.6

Let $\sum\limits_{n=1}^{\infty} a_n$ be a series of real numbers and let

$$s_n = a_1+a_2+a_3+\cdots+a_n, n \in \mathbb{N}$$

The sequence (s_n) is called the ***partial sum sequence*** of $\sum\limits_{n=1}^{\infty} a_n$. If (s_n) converges and if $\lim\limits_{n\to\infty} s_n = s$, then the series is said to be a ***convergent series*** and s is called the ***sum*** of the series.

We write $\sum\limits_{n=1}^{\infty} a_n = s$. Let us see some examples. The series $\sum\limits_{n=1}^{\infty} (-1)^{n+1}$ does not converge because the partial sum sequence $1, 0, 1, 0, 1, 0, \ldots$ does not converge.

We cannot apply algebraic rules meant for finite series to an infinite series blindly. Consider $\sum\limits_{n=1}^{\infty} (-1)^{n+1} = 1-1+1-1+\cdots$ If $S = 1-1+1-1+\cdots$ then one may argue that $S = 0$ or 1 or $\frac{1}{2}$ according to $S = (1-1)+(1-1)+\cdots$, $S = 1+(-1+1)+(-1+1)+\cdots$ or $1-S = 1-(+1-1+1-1+\cdots = S)$ respectively.

The series $\sum\limits_{n=0}^{\infty} x^n$ converges if $x = \frac{1}{2}$ whereas it does not converge if $x = 2$. This shows that series like $\sum\limits_{n=0}^{\infty} a_n x^n$ converges for some values of x and does not converge for some values of x. The problem of finding the values of x for which sequences of this form converges is beyond the scope of this book. However in the rest of the chapter we list some series with the appropriate values of x for which the series converges and the sum of the series whenever it converges.

5.6.2 Infinite Geometric Series

The series Σx^n is called a geometric series or geometric progression. Let us start with the series: $\sum\limits_{n=0}^{\infty} x^n, x \neq 1$. If $s_n = x_0 + x_1 + x_2 + \cdots + x_n$, then $s_n = \frac{1-x^n}{1-x}$. As x^n tends to 0 if $|x| < 1$, we say that s_n tends to $\frac{1}{1-x}$ if $|x| < 1$.

- $\sum\limits_{n=0}^{\infty} x^n$ converges for all real number x with $|x| < 1$ and the sum is $\frac{1}{1-x}$. That is, for all real numbers x satisfying $|x| < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

- $\sum\limits_{n=0}^{\infty} (-1)^n x^n$ converges for all real number x with $|x| < 1$ and the sum is $\frac{1}{1+x}$. That is, for all real numbers x satisfying $|x| < 1$,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

- $\sum\limits_{n=0}^{\infty} (2x)^n$ converges for all real number x with $|x| < \frac{1}{2}$ and the sum is $\frac{1}{1-2x}$. That is, for real numbers x satisfying $|x| < \frac{1}{2}$,

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

- $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all real number x and the sum is e^x. That is,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

  for all real numbers x.
- $\sum\limits_{n=0}^{\infty} (-1)^n x$ converges only for $x = 0$.

Let us discuss some special series. By assuming the convergence of those series let us solve some problems.

5.6.3 Infinite Arithmetico-Geometric Series

- The sum of the arithmetico-Geometric series $\Sigma((a + (n-1)d))r^{n-1}$ is given by

$$S = \lim_{n\to\infty} S_n = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

  for $-1 < r < 1$.

Example 5.19 Find the sum: $1+\frac{4}{5}+\frac{7}{25}+\frac{10}{125}+\cdots$.

Solution:
Here $a=1$, $d=3$ and $r=\frac{1}{5}$.

$$\begin{aligned} S_\infty &= \frac{a}{1-r}+\frac{dr}{(1-r)^2} \\ &= \frac{1}{1-\frac{1}{5}}+\frac{3\times\frac{1}{5}}{(1-\frac{1}{5})^2} \\ &= \frac{5}{4}+\left(\frac{3}{5}\right)\left(\frac{25}{16}\right) \\ &= \frac{35}{16} \end{aligned}$$

5.6.4 Telescopic Summation for Infinite Series

We discussed about summing terms of a finite sequence using telescopic summation technique in Section 5.5.2. The same applies for infinite series also.

Example 5.20 Find $\sum\limits_{n=1}^{\infty}\frac{1}{n^2+5n+6}$.

Solution:
Let a_n denote the n^{th} term of the given series. Then $a_n=\frac{1}{n^2+5n+6}$. By using partial fraction, we get

$$a_n=\frac{1}{n+2}-\frac{1}{n+3}.$$

Let s_n denote the sum of first n terms of the given series. Then

$$s_n=a_1+a_2+\cdots+a_n=\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots+\left(\frac{1}{n+2}-\frac{1}{n+3}\right)=\frac{1}{3}-\frac{1}{n+3}.$$

But as n tends to infinity, $\frac{1}{n+3}$ tends to zero and hence $\frac{1}{3}-\frac{1}{n+3}$ tends to $\frac{1}{3}$. In other words s_n tends to $\frac{1}{3}$. Thus $\sum\limits_{n=1}^{\infty}\frac{1}{n^2+5n+6}=\frac{1}{3}$.

5.6.5 Binomial Series

In the discussion on geometric series we have seen that

$$\frac{1}{1-x}=1+x+x^2+\cdots,\ \frac{1}{1+x}=1-x+x^2-\cdots \text{ and } \frac{1}{1-2x}=1+2x+4x^2+\cdots$$

for some suitable values of x. But the expressions $\frac{1}{1-x}$, $\frac{1}{1+x}$ and $\frac{1}{1-2x}$ can be written as $(1-x)^{-1}$, $(1+x)^{-1}$ and $(1-2x)^{-1}$. This suggests us a possibility of having negative exponents, that is negative powers, for $(1+x)$, $(1-x)$ and so on. Yes. This is possible. We can have any power, positive or negative, integer or rational. We can even have irrational exponent for $(1+x)$. The binomial theorem

(Theorem 5.1) we proved already is for positive integral exponent (integral exponent means integer power). Now let us state the binomial theorem for rational power.

Binomial Theorem for Rational Exponent

Theorem 5.4 (Binomial Theorem for Rational Exponent): For any rational number n,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

for all real numbers x satisfying $|x| < 1$.

As the proof involves higher mathematical concepts, let us assume the theorem without proof and see some particular cases and solve some problems. In the theorem

1. By taking $-x$ in the place of x, we get

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \qquad (|x| < 1)$$

2. By taking $-n$ in the place of n, we get

$$(1+x)^{-n} = 1 + (-n)x + \frac{(-n)(-n-1)}{2!}x^2 + \frac{(-n)(-n-1)(-n-2)}{3!}x^3 + \cdots$$

Hence

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots \qquad (|x| < 1)$$

3. by taking $-x$ and $-n$ in the places of x and n, we get

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots \qquad (|x| < 1)$$

Even though we have explicitly mentioned that n is a rational number in the theorem, some of us may hesitate to use n in place of a general rational number. So we give the theorem using the representation $\frac{p}{q}$ for a rational number.

$$\begin{aligned}
(1+x)^{\frac{p}{q}} &= 1 + \frac{p}{q}x + \frac{\frac{p}{q}(\frac{p}{q}-1)}{2!}x^2 + \frac{\frac{p}{q}(\frac{p}{q}-1)(\frac{p}{q}-2)}{3!}x^3 + \cdots \\
&= 1 + \frac{p}{q}x + \frac{p(p-q)}{q^2 2!}x^2 + \frac{p(p-q)(p-2q)}{q^3 3!}x^3 + \cdots \qquad (|x| < 1) \\
(1-x)^{\frac{p}{q}} &= 1 - \frac{p}{q}x + \frac{p(p-q)}{q^2 2!}x^2 - \frac{p(p-q)(p-2q)}{q^3 3!}x^3 + \cdots \qquad (|x| < 1)
\end{aligned}$$

Though the theorem gives a formula to compute $(1+x)^n$, to solve numerical problems quickly we must remember and able to write certain expansions directly. Observation of the coefficient in each of such expansions will be very helpful in solving problems. Let us list some of them: (Try yourself!).

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
2. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
3. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \cdots$
4. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \cdots$

All the above expansions are valid only when $|x| < 1$.

Example 5.21 Expand $(1+x)^{\frac{2}{3}}$ up to four terms for $|x| < 1$.

Solution:
Here $n = \frac{2}{3}$.

$$\begin{aligned}
\frac{n(n-1)}{2!} &= \frac{\frac{2}{3}(\frac{2}{3}-1)}{2!} \\
&= \frac{\frac{2}{3}(\frac{-1}{3})}{2} \\
&= \frac{-1}{9} \\
\frac{n(n-1)(n-2)}{2!} &= \frac{\frac{2}{3}(\frac{2}{3}-1)(\frac{2}{3}-2)}{3!} \\
&= \frac{\frac{2}{3}(\frac{-1}{3})(\frac{-4}{3})}{6} \\
&= \frac{4}{81}
\end{aligned}$$

Thus

$$(1+x)^{\frac{2}{3}} = 1 + \frac{2}{3}x - \frac{1}{9}x^2 + \frac{4}{81}x^3 + \cdots$$

Example 5.22 Expand $\dfrac{1}{(1+3x)^2}$ in powers of x. Find a condition on x for which the expansion is valid.

Solution:
If we take $3x = y$, then

$$\frac{1}{(1+3x)^2} = \frac{1}{(1+y)^2}.$$

Now $\dfrac{1}{(1+y)^2}$ can be expanded using binomial theorem in powers of y. The expansion is valid only for values of y satisfying $|y| < 1$. Replacing y by $3x$ we can get an expansion of $\dfrac{1}{(1+3x)^2}$. The expansion is valid only for values of x satisfying $|3x| < 1$; that is the expansion is valid only for values of x satisfying $|x| < \frac{1}{3}$.

$$\begin{aligned}
\frac{1}{(1+3x)^2} &= (1+3x)^{-2} \\
&= 1 - 2(3x) + \frac{2(2+1)}{2!}(3x)^2 - \frac{2(2+1)(2+2)}{3!}(3x)^3 \\
&\quad + \frac{2(2+1)(2+2)(2+3)}{4!}(3x)^4 - \cdots
\end{aligned}$$

Hence, $\frac{1}{(1+3x)^2} = 1 - 6x + 27x^2 - 108x^3 + 405x^4 - \cdots, |x| < \frac{1}{3}$.

Example 5.23 Expand $\dfrac{1}{(3+2x)^2}$ in powers of x. Find a condition on x for which the expansion is valid.

Solution:
(Clearly we have to use the expansion of $(1+x)^{-2}$. So, we have to write $(3+2x)$ as $3\left(1+\dfrac{2x}{3}\right)$ and proceed.)

$$\begin{aligned}
\frac{1}{(3+2x)^2} &= \frac{1}{3^2\left(1+\frac{2x}{3}\right)^2}\\
&= \frac{1}{9}\left(1+\frac{2x}{3}\right)^{-2}\\
&= \frac{1}{9}(1+y)^{-2} \qquad\qquad \left(\text{where } \frac{2x}{3}=y\right)\\
&= \frac{1}{9}(1-2y+3y^2-4y^3+5y^4-\cdots), if|y|<1\\
&= \frac{1}{9}\left(1-2\left(\frac{2x}{3}\right)+3\left(\frac{2x}{3}\right)^2-4\left(\frac{2x}{3}\right)^3+5\left(\frac{2x}{3}\right)^4-\cdots\right), |\frac{2x}{3}|<1\\
&= \frac{1}{9}\left(1-\frac{4}{3}x+\frac{4}{3}x^2-\frac{32}{27}x^3+\frac{80}{81}x^4-\cdots\right)\\
\text{Thus, } \frac{1}{(3+2x)^2} &= \frac{1}{9}-\frac{4}{27}x+\frac{4}{27}x^2-\frac{32}{243}x^3+\frac{80}{729}x^4-\cdots, |x|<\frac{3}{2}
\end{aligned}$$

The expansion is valid if $|y|<1$. So, the expansion is valid if $|x|<\frac{3}{2}$.

We can find square root, cube root and other roots of any positive number by using binomial theorem. Let us see one such problem.

Example 5.24 Find $\sqrt[3]{65}$.

Solution:
We know that for $|x|<1$,

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots$$

$$\begin{aligned}
\sqrt[3]{65} &= 65^{\frac{1}{3}}\\
&= (64+1)^{\frac{1}{3}}\\
&= 64^{\frac{1}{3}}\left(1+\frac{1}{64}\right)^{\frac{1}{3}}\\
&= 4\left(1+\frac{1}{64}\right)^{\frac{1}{3}}
\end{aligned}$$

$$\begin{aligned}
&= 4\left(1+\frac{1}{3}\times\frac{1}{64}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\times\left(\frac{1}{64}\right)^2+\cdots\right)\\
&= 4+\frac{1}{48}-4\times\frac{1}{9}\times\frac{1}{64}\times\frac{1}{64}+\cdots\\
&= 4+\frac{1}{48}-\frac{1}{36864}+\cdots\\
&\approx 4+0.02 \qquad \left(\text{since } \frac{1}{36864}+\cdots \text{ is very small}\right)\\
\sqrt[3]{65} &= 4.02 \ \text{(approximately).}
\end{aligned}$$

Example 5.25 Prove that $\sqrt[3]{x^3+7}-\sqrt[3]{x^3+4}$ is approximately equal to $\frac{1}{x^2}$ when x is large.

Solution:

$$\begin{aligned}
\sqrt[3]{x^3+7} &= (x^3+7)^{\frac{1}{3}}\\
&= \left[x^3\left(1+\frac{7}{x^3}\right)\right]^{\frac{1}{3}}, \qquad (|\frac{7}{x^3}|<1 \text{ as } x \text{ is large})\\
&= x\left(1+\frac{7}{x^3}\right)^{\frac{1}{3}}\\
&= x\left(1+\frac{1}{3}\times\frac{7}{x^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{7}{x^3}\right)^2+\cdots\right)\\
&= x\left(1+\frac{7}{3}\times\frac{1}{x^3}-\frac{49}{9}\times\frac{1}{x^6}+\cdots\right)\\
&= x+\frac{7}{3}\times\frac{1}{x^2}-\frac{49}{9}\times\frac{1}{x^6}+\cdots\\
\sqrt[3]{x^3+4} &= (x^3+4)^{\frac{1}{3}}\\
&= \left[x^3\left(1+\frac{4}{x^3}\right)\right]^{\frac{1}{3}}\\
&= x\left(1+\frac{4}{x^3}\right)^{\frac{1}{3}} \qquad (|\frac{4}{x^3}|<1)\\
&= x\left(1+\frac{1}{3}\times\frac{4}{x^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{4}{x^3}\right)^2+\cdots\right)\\
&= x+\frac{4}{3}\times\frac{1}{x^2}-\frac{16}{9}\times\frac{1}{x^6}+\cdots
\end{aligned}$$

Since x is large, $\frac{1}{x}$ is very small and hence higher powers of $\frac{1}{x}$ are negligible. Thus $\sqrt[3]{x^3+7}=x+\frac{7}{3}\times\frac{1}{x^2}$ and $\sqrt[3]{x^3+4}=x+\frac{4}{3}\times\frac{1}{x^2}$. Therefore

$$\sqrt[3]{x^3+7}-\sqrt[3]{x^3+4} = \left(x+\frac{7}{3}\times\frac{1}{x^2}\right)-\left(x+\frac{4}{3}\times\frac{1}{x^2}\right)=\frac{1}{x^2}$$

The binomial theorem is true for all real numbers n. For example, when $n = \sqrt{2}$, we have

$$(1+x)^{\sqrt{2}} = 1 + \sqrt{2}x + \frac{\sqrt{2}(\sqrt{2}-1)}{2!}x^2 + \frac{\sqrt{2}(\sqrt{2}-1)(\sqrt{2}-2)}{3!}x^3 + \cdots, \quad |x| < 1.$$

5.6.6 Exponential Series

The series $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ is called an ***exponential series***. It can be proved that this series converges for all values of x.

For any real number x, $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ where

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots.$$

We have

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{5.1}$$

for all values of x. By taking $-x$ in place of x in (5.1) we get

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots. \tag{5.2}$$

In particular,

$$\frac{1}{e} = e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots.$$

From (5.1) and (5.2) we get

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \text{ and } \frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

In particular we have

$$\frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots \text{ and } \frac{e - e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots$$

By taking $2x$ in place of x in (5.1) we get

$$e^{2x} = 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots.$$

On simplification we see that

$$e^{2x} = 1 + \frac{2x}{1!} + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots.$$

5.6.7 Logarithmic Series

The series $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ is called a ***logarithmic series***. This series converges for all values of x satisfying $|x| < 1$. This series converges when $x = 1$ also.

For all values of x satisfying $|x| < 1$, the sum of the series is $\log(1 + x)$. Thus

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

for all values of x satisfying $|x| < 1$. By taking $-x$ in place of x we get

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

for all values of x satisfying $|x| < 1$.

Now $\log\left(\frac{1+x}{1-x}\right) = \log(1 + x) - \log(1 - x)$. Using this we get

$$\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right].$$

Suppose we want to write $\log(1+2x)$ in a series, then we can replace $2x$ by y and use the expansion

$$\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots$$

for all values of y satisfying $|y| < 1$. But if $|y| < 1$, then $|2x| < 1$ and hence $|x| < \frac{1}{2}$. So if $|x| < \frac{1}{2}$, then

$$\log(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \cdots.$$

Thus $\log(1 + 2x) = 2x - \frac{4x^2}{2} + \frac{8x^3}{3} - \frac{16x^4}{4} + \cdots$ for all values of x satisfying $|x| < \frac{1}{2}$.

Exercise - 5.4

1. Expand the following in ascending powers of x and find the condition on x for which the binomial expansion is valid.
   (i) $\dfrac{1}{5+x}$ (ii) $\dfrac{2}{(3+4x)^2}$ (iii) $(5+x^2)^{\frac{2}{3}}$ (iv) $(x+2)^{-\frac{2}{3}}$
2. Find $\sqrt[3]{1001}$ approximately (two decimal places).
3. Prove that $\sqrt[3]{x^3+6} - \sqrt[3]{x^3+3}$ is approximately equal to $\frac{1}{x^2}$ when x is sufficiently large.
4. Prove that $\sqrt{\frac{1-x}{1+x}}$ is approximately equal to $1 - x + \frac{x^2}{2}$ when x is very small.
5. Write the first 6 terms of the exponential series (i) e^{5x} (ii) e^{-2x} (iii) $e^{\frac{1}{2}x}$.
6. Write the first 4 terms of the logarithmic series (i) $\log(1 + 4x)$ (ii) $\log(1 - 2x)$ (iii) $\log\left(\frac{1+3x}{1-3x}\right)$ (iv) $\log\left(\frac{1-2x}{1+2x}\right)$. Find the intervals on which the expansions are valid.
7. If $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$, then show that $x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} + \cdots$.
8. If $p - q$ is small compared to either p or q, then show that $\sqrt[n]{\frac{p}{q}} \simeq \frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}$. Hence find $\sqrt[8]{\frac{15}{16}}$.
9. Find the coefficient of x^4 in the expansion of $\frac{3-4x+x^2}{e^{2x}}$.
10. Find the value of $\sum\limits_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{1}{9^{n-1}} + \frac{1}{9^{2n-1}}\right)$.

Exercise - 5.5

Choose the correct or the most suitable answer.

1. The value of $2+4+6+\cdots+2n$ is

   (1) $\frac{n(n-1)}{2}$ (2) $\frac{n(n+1)}{2}$ (3) $\frac{2n(2n+1)}{2}$ (4) $n(n+1)$

2. The coefficient of x^6 in $(2+2x)^{10}$ is

   (1) ${}^{10}C_6$ (2) 2^6 (3) ${}^{10}C_6\, 2^6$ (4) ${}^{10}C_6\, 2^{10}$.

3. The coefficient of x^8y^{12} in the expansion of $(2x+3y)^{20}$ is

   (1) 0 (2) $2^8 3^{12}$ (3) $2^8 3^{12} + 2^{12} 3^8$ (4) ${}^{20}C_8\, 2^8 3^{12}$.

4. If ${}^nC_{10} > {}^nC_r$ for all possible r, then a value of n is

   (1) 10 (2) 21 (3) 19 (4) 20.

5. If a is the arithmetic mean and g is the geometric mean of two numbers, then

   (1) $a \leq g$ (2) $a \geq g$ (3) $a = g$ (4) $a > g$.

6. If $(1+x^2)^2(1+x)^n = a_0 + a_1x + a_2x^2 + \cdots + x^{n+4}$ and if a_0, a_1, a_2 are in AP, then n is

   (1) 1 (2) 2 (3) 3 (4) 4.

7. If $a, 8, b$ are in AP, $a, 4, b$ are in GP, and if a, x, b are in HP then x is

   (1) 2 (2) 1 (3) 4 (4) 16.

8. The sequence $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}+\sqrt{2}}, \frac{1}{\sqrt{3}+2\sqrt{2}}, \cdots$ form an

   (1) AP (2) GP (3) HP (4) AGP.

9. The HM of two positive numbers whose AM and GM are 16, 8 respectively is

   (1) 10 (2) 6 (3) 5 (4) 4.

10. If S_n denotes the sum of n terms of an AP whose common difference is d, the value of $S_n - 2S_{n-1} + S_{n-2}$ is

    (1) 0 (2) 2d (3) 4d (4) d^2.

11. The remainder when 38^{15} is divided by 13 is

    (1) 12 (2) 1 (3) 11 (4) 5.

12. The n^{th} term of the sequence $1, 2, 4, 7, 11, \cdots$ is

    (1) n^3+3n^2+2n (2) n^3-3n^2+3n (3) $\frac{n(n+1)(n+2)}{3}$ (4) $\frac{n^2-n+2}{2}$.

13. The sum up to n terms of the series $\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \cdots$ is

    (1) $\sqrt{2n+1}$ (2) $\frac{\sqrt{2n+1}}{2}$ (3) $\sqrt{2n+1}-1$ (4) $\frac{\sqrt{2n+1}-1}{2}$.

14. The n^{th} term of the sequence $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \cdots$ is

    (1) 2^n-n-1 (2) $1-2^{-n}$ (3) $2^{-n}+n-1$ (4) 2^{n-1}.

15. The sum up to n terms of the series $\sqrt{2}+\sqrt{8}+\sqrt{18}+\sqrt{32}+\cdots$ is

    (1) $\frac{n(n+1)}{2}$ (2) $2n(n+1)$ (3) $\frac{n(n+1)}{2}$ (4) 1.

16. The value of the series $\frac{1}{2} + \frac{7}{4} + \frac{13}{8} + \frac{19}{16} + \cdots$ is

(1) 14 (2) 7 (3) 4 (4) 6.

17. The sum of an infinite GP is 18. If the first term is 6, the common ratio is

(1) $\frac{1}{3}$ (2) $\frac{2}{3}$ (3) $\frac{1}{6}$ (4) $\frac{3}{4}$.

18. The coefficient of x^5 in the series e^{-2x} is

(1) $\frac{2}{3}$ (2) $\frac{3}{2}$ (3) $\frac{-4}{15}$ (4) $\frac{4}{15}$.

19. The value of $\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots$ is

(1) $\frac{e^2+1}{2e}$ (2) $\frac{(e+1)^2}{2e}$ (3) $\frac{(e-1)^2}{2e}$ (4) $\frac{e^2+1}{2e}$.

20. The value of $1 - \frac{1}{2}\left(\frac{2}{3}\right) + \frac{1}{3}\left(\frac{2}{3}\right)^2 - \frac{1}{4}\left(\frac{2}{3}\right)^3 + \cdots$ is

(1) $\log\left(\frac{5}{3}\right)$ (2) $\frac{3}{2}\log\left(\frac{5}{3}\right)$ (3) $\frac{5}{3}\log\left(\frac{5}{3}\right)$ (4) $\frac{2}{3}\log\left(\frac{2}{3}\right)$.

Summary

In this chapter we have acquired the knowledge of

- Binomial theorem for any $n \in \mathbb{N}$,

$$(a+b)^n = {}^n\mathrm{C}_0\, a^n b^0 + {}^n\mathrm{C}_1\, a^{n-1} b^1 + \cdots + {}^n\mathrm{C}_n\, a^0 b^n.$$

- ${}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + \cdots + {}^n\mathrm{C}_n = 2^n.$
- ${}^n\mathrm{C}_1 + {}^n\mathrm{C}_3 + + {}^n\mathrm{C}_5 \cdots + = {}^n\mathrm{C}_0 + {}^n\mathrm{C}_2 + + {}^n\mathrm{C}_4 \cdots + = 2^{n-1}$
- $AM \geq GM \geq HM$
- The n^{th} term of an AP is given by $T_n = a + (n-1)d$.
- The n^{th} term of an GP is given by $T_n = ar^{n-1}$.
- The n^{th} term of an AGP is given by $T_n = (a + (n-1)d)r^{n-1}$.
- For any positive numbers a and b, we have

$$\mathrm{AM} = \frac{a+b}{2}, \quad \mathrm{GM} = \sqrt{ab}, \quad \mathrm{HM} = \frac{2ab}{a+b}.$$

- The sum of first n terms of an AP is given by $S_n = \dfrac{n}{2}(2a + (n-1)d)$.
- The sum of first n terms of an GP is given by $S_n = \dfrac{a(1-r^n)}{1-r}$ for $r \neq 1$.
- The sum of first n terms of an AGP is given by $S_n = \dfrac{a - (a + (n-1)d)r^n}{1-r} + dr\left(\frac{1-r^{n-1}}{(1-r)^2}\right)$ for $r \neq 1$.
- $\sum\limits_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.
- $\sum\limits_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

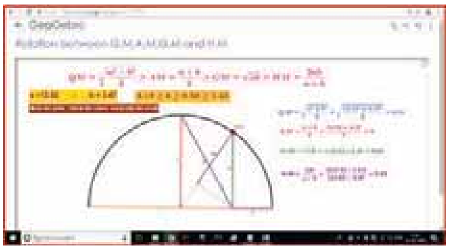
- $\sum\limits_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.
- Fibonacci sequence $1, 1, 2, 3, 5, \cdots$
- Binomial theorem for rational exponent

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

  for all real numbers x satisfying $|x| < 1$.
- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
- $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \cdots$
- $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \cdots$
- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$.
- $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$.
- $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$ and $\frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$.
- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ for all values of x satisfying $|x| < 1$.
- $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$ for all values of x satisfying $|x| < 1$.
- $\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right]$.

ICT CORNER-5(a)

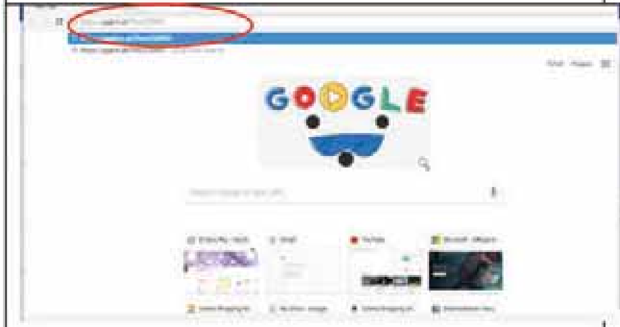
Expected Outcome $\Rightarrow$



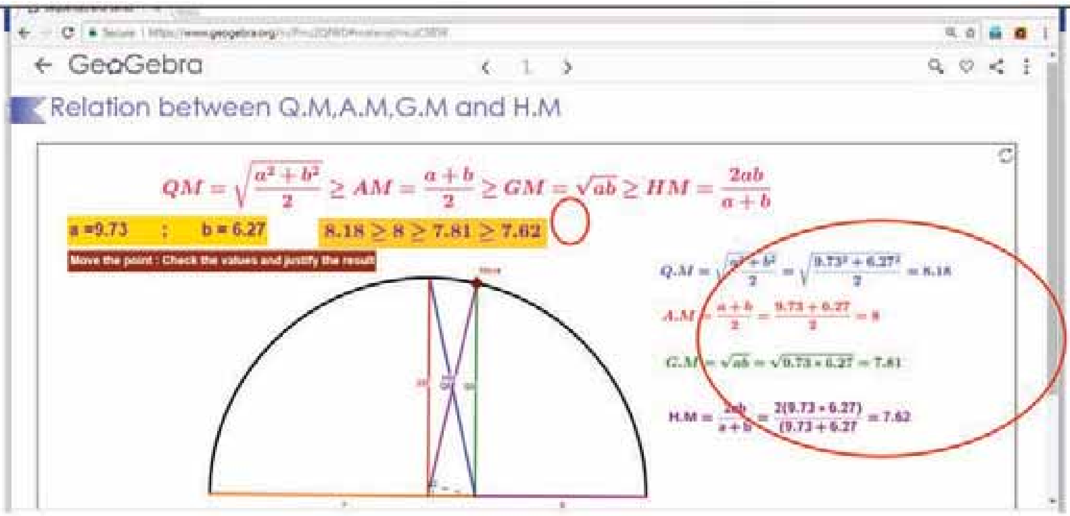
Step–1
Open the Browser and type the URL Link given below (or) Scan the QR Code..

Step–2
Sequences and Series workbook will open. Several worksheets are given in this workbook. Select a worksheet "Relation between Q.M,A.M,G.M and H.M"

Step–3
You have All the Mean formula working with varying a and b. Move the point named "Move" and see that Mean values changes corresponding to a and b.

.Step-1	Step-2
	

Step-3



Now is the time for you to find out, Why the lines are named as AM, QM, GM and HM.
Also try other worksheets to make it clear in your lesson concepts.

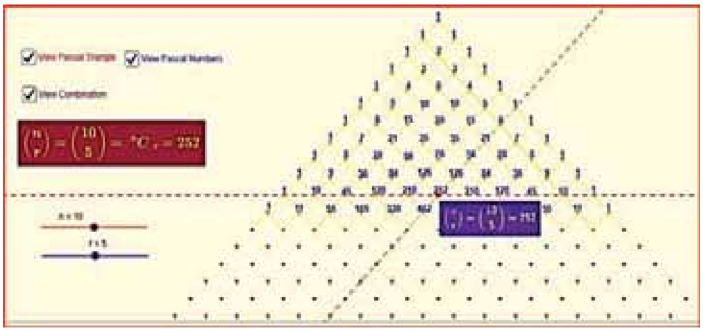
*Pictures are only indicatives.

Browse in the link Sequence and Series:
https://ggbm.at/Pmz2QfWDor Scan the QR Code.

B162_11_MAT_EM

ICT CORNER-5(b)

Expected Outcome ⇒



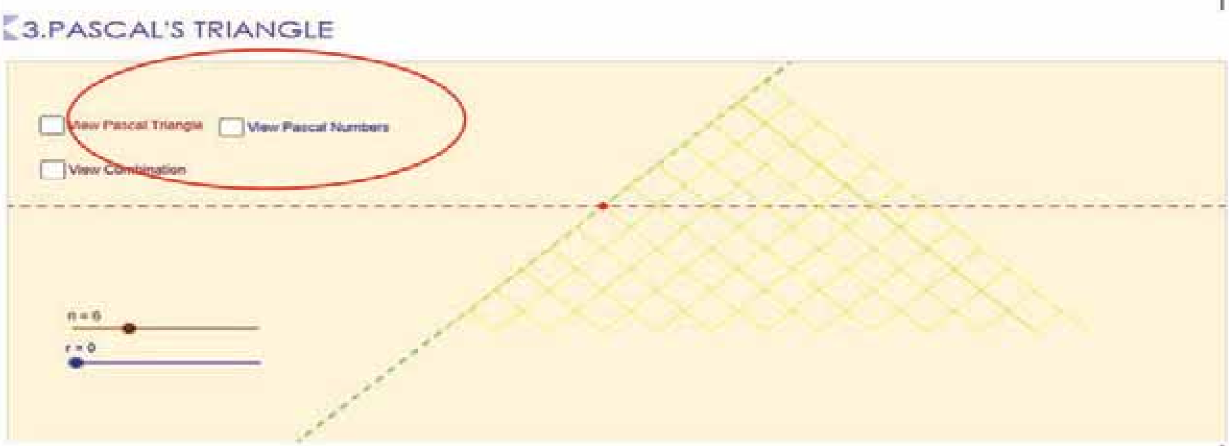
Step–1

Open the Browser and type the URL Link given below (or) Scan the QR Code. GeoGebra worksheet “PASCALS TRIANGLE” will appear. You can see pascal triangle arrangements. There are check boxes “View Pascal Triangle”, “View Pascal’s Numbers” and “View Combination”

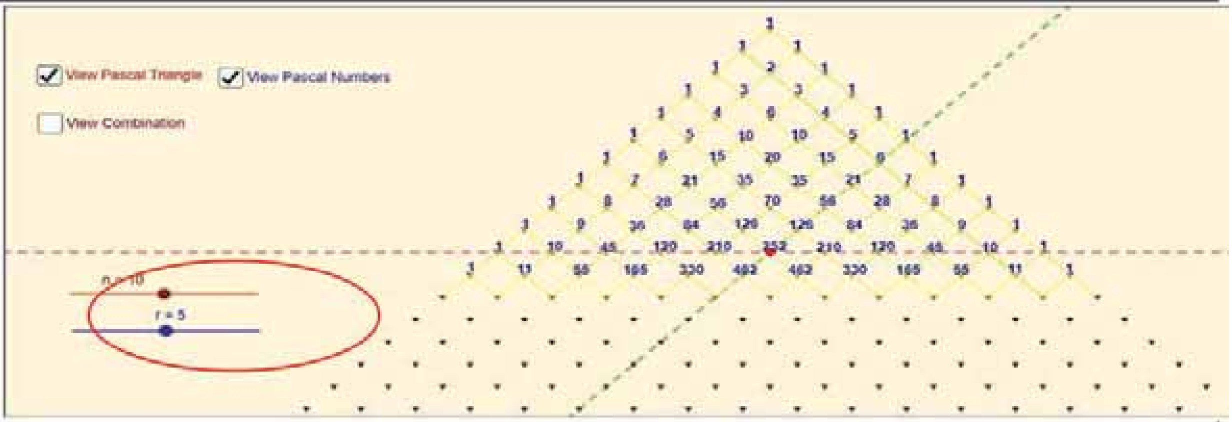
Step–2

Click on “View Pascal’s Numbers” and move the sliders “n” and “r”. The red colour point moves over the numbers. Leave on any number and click “View Calculation” to see combinatorial working. Now compare the pascal numbers with the calculation.

.Step-1



Step-2



*Pictures are only indicatives.

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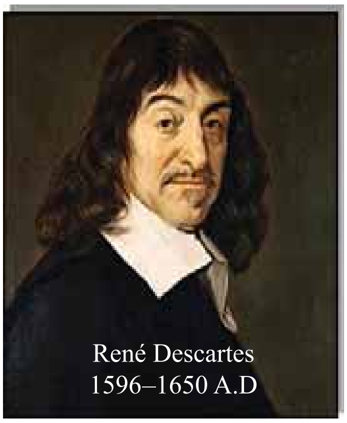
B162_11_MAT_EM

Chapter 6 Two Dimensional Analytical Geometry

"My powers are ordinary. Only my application brings me success ".

Sir Isaac Newton

6.1 Introduction


René Descartes
1596–1650 A.D

Francois viete(1540-1603) introduced the first systematic *algebraic* notation and contributed to the theory of equations.Two French mathematicians-philosophers **René Descartes** and **Pierre de Fermat** independently founded analytical geometry in the-1630s by adapting Francois viete's algebra to the study of geometric loci. Descartes established analytical geometry as"a way of visualizing algebraic formulas" and developed the coordinate system as "a device to locate points on a plane". His main achievement was to bridge the gap between algebra and geometry. With regard to algebra, he explained in detail that how algebraic equations can be expressed and explained through the use of geometrical shapes. Analytical geometry is a great invention of Descartes and Fermat. Cartesian geometry, the alternative term used for analytical geometry is named after him.

From the17th century onwards, mathematics is being developed in two directions: pure and applied mathematics. One of the first areas of applied mathematic studied in the 17th century was the motion of an object in a straight line. The straight line graphs can be used in the fields of study as diverse as business, economics, social sciences, physics, and medicine. The problem of the shortest line plays a chief and historically important role in the foundations of geometry.

Given a real-world problem, our first task is to formulate the problem using the language of mathematics. Many techniques are used in the construction of mathematical models. Let us see how linear equations (models) can be constructed from a given set of information and solved using appropriate mathematical techniques. Consider some of the real-world, simple problems as illustrated below:

Real life situation 6.1: When a student walks from his house, at an average speed of 6 kmph, reaches his school by ten minutes before school starts. When his average speed is 4 kmph, he reaches his school five minutes late. If he starts to walk to school every day at 8.00 A.M, then how to find (i) the distance between house and the school (ii) the minimum average speed to reach the school on time and time taken to reach the school (iii) the time at which the school starts (iv) the pair of straight lines of his path of walk (Combined equation of two straight lines).

Real life situation 6.2: Suppose the Government has decided to erect a new Electrical Power Transmission Substation to provide better power supply to two villages namely A and B. The substation has to be on the line l. The distances of villages A and B from the foot of the perpendiculars P and Q on the line l are 3 km and 5 km respectively and the distance between P and Q is 6 km. How to calculate **the smallest length** of cable required to connect the two villages (or the roads that connect the villages as well as the power station) from the power station and to find the equations of the cable lines (or roads) that connect the power station to two villages.

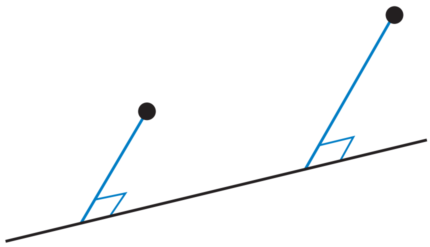


Figure 6.1

Real life situation 6.3:

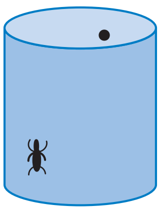


Figure 6.2

Consider a hollow cylindrical vessel, with circumference 24 cm and height 10 cm. An ant is located on the outside of vessel 4 cm from the bottom. There is a drop of honey at the diagrammatically opposite inside of the vessel, 3 cm from the top. What is the shortest distance the ant would need to crawl to get the honey? What is the equation of the path traced out by the ant. Here is a picture that illustrates the position of the ant and the honey.

Real life situation 6.4: The quantity demanded of a certain type of Compact Disk is 22,000 units when a unit price is ₹ 8. The customer will not buy the disk, at a unit price of ₹ 30 or higher. On the other side the manufacturer will not market any disk if the price is ₹ 6 or lower. However, if the price is ₹ 14 the manufacturer can supply 24,000 units. Assume that the quantity demanded and quantity supplied are linearly proportional to the price. How to find (i) the demand equation (ii) supply equation (iii) the market equilibrium quantity and price. (iv) The quantity of demand and supply when the price is ₹ 10.

The equation of the straight line for each of the problems stated above, not only solves the specific case of solutions but also helps us get many information through it. Later, in this chapter, let us try to solve these types of problems by using the concepts of straight lines. In order to understand the straight line, we need to get acquainted with some of its basic concepts. Let us discuss those in detail

Learning Objective

On completion of this chapter, the students expected to know

- the equation of a line in different forms
- whether two given lines are parallel or perpendicular;
- the distance of a given point from a given line and between two parallel lines,
- the family of straight lines for a given condition
- the equation of pair of straight lines, angle between them and angle bisectors

6.2 Locus of a point

Definition 6.1

A **point** is an exact position or location on a plane surface.

It is important to understand that a point is not a thing, but a place. We indicate the position of a point by placing a dot. In plane analytical geometry, points are defined as ordered pairs of real numbers, say, (x, y) with reference to the coordinate system.

Generally, a horizontal line is called the x-axis; and the line vertical to the x-axis is called the y-axis. Intersection of these two axes is called the origin. Any point P in the plane can be located by a unique ordered pair of numbers(x, y) where x gives the distance between the point P and the y-axis and y denote the distance between the point P and the x-axis. Note that if x is negative it lies left of $y - axis$, similarly if y is negative it lies below the x-axis. In applications, often letters other than x and y are used, and different scales are chosen in the horizontal and vertical directions.

Definition 6.2

The path traced out by a moving point under certain conditions is called the **locus** of that point. Alternatively, when a point moves in accordance with a geometrical law, its path is called locus. The plural of locus is loci.

The following illustrations shows some cases of loci and its different uses.

Illustration 6.1: In cricket, when a ball is bowled by a bowler, the path traced out by the ball is the locus of the ball. Whenever there is dispute between batsmen and the fielders for leg before wicket (LBW) decisions, the locus of the ball solves the crises, raised by the players for review, through the third umpire. The likely path of the ball can be projected forward, through the batsman's legs, to see whether it would have hit the stumps or not. Consultation of the third umpire, for conventional slow motion or Hawk-Eye, the probable decision will be taken. This method is currently sanctioned in international cricket.
https://www.hawkeyeinnovations.com/sports

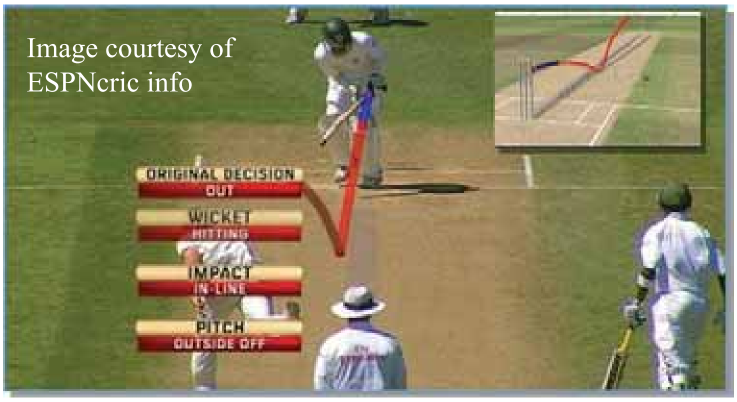


Figure 6.3

Illustration 6.2: Suppose P be a point on the rim (circumference) of a circular wheel. When the circle is rolling without slipping along a straight line, the locus of the point P on the rim is shown in figure. The path traced out by the point P is known as cycloid. (Try yourself by taking a point inside the circle. Find the names of the curve from the web site: www.mathworld.wolfram.com

https://www.geogebra.org/b/bd2ADu2I

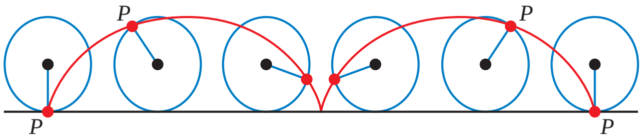


Figure 6.4

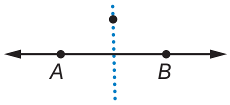
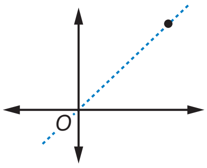
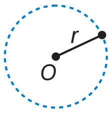
Illustration 6.3: A missile is launched from the army ship to attack and another from the land to intercept it. The loci of the missiles are shown in figure.

Locus of missiles play a vital role in many wars. During the Gulf War (2 Aug1990-28 Feb 1991), Iraq attacked Israeli cities with Scud missiles. To defend from Scud attack, Israel used Patriot missiles to shoot down enemy missiles. To launch a satellite or space shuttle successfully, the determination of path plays an crucial role in space research.

Figure 6.5

An equation in the two variables x and y will ordinarily be satisfied by infinitely many pair of real value of x and y. Every such pair is called a real solution of the equation. Each real solution of the equation will have its graph. The collection of all these graphs is called the locus of the given equation.

The following table shows some important loci in mathematics

A moving point ***P*** under the given condition	Graph	Name of the path
A point P moves such that it is equidistant from two fixed points A and B	 **Figure 6.6**	Perpendicular bisector of the line segment ***AB***
A point P moves such that it is equidistant from two fixed lines ox and oy	 **Figure 6.7**	Angle bisector of the angle$\angle xoy$
A point P moves equidistant from a fixed point O	 **Figure 6.8**	Circle

Now let us discuss the ways of finding the locus of the points. The equation of the locus is the relation that exists between the coordinates of all the points strictly lying on the path.

Procedure for finding the equation of the locus of a point

(i) If we are finding the equation of the locus of a point P, assign coordinates, say (h, k) to P

(ii) Express the given conditions as equations in terms of the known quantities and unknown parameters.

(iii) Eliminate the parameters, so that the resulting equation contains only h, k and known quantities.

(iv) Replace h by x, and k by y, in the resulting equation. The resulting equation is the equation of the locus of point P.

Example 6.1 Find the locus of a point which moves such that its distance from the x-axis is equal to the distance from the y-axis.

Solution:

Let $P\,(h,k)$ be a point on the locus.
Let A and B be the foot of the perpendiculars drawn from the point P on the x-axis and the y-axis respectively.
Therefore P is $(OA, OB) = (BP, AP) = (h,k)$
Given that $AP = BP$

$$\Rightarrow k = h$$

replacing h and k by substituting $h = x$ and $k = y$
The locus of P is, $y = x$, is a line passing through the origin

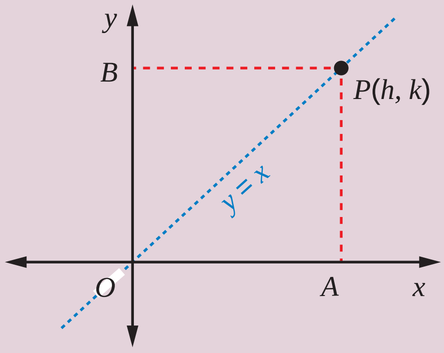


Figure 6.9

Example 6.2 Find the path traced out by the point $\left(ct, \frac{c}{t}\right)$, here $t \neq 0$ is the parameter and c is a constant

Solution:

Let $P\,(h,k)$be a point on the locus. From the given information, we have $h = ct$ and $k = \frac{c}{t}$.
To eliminate t, taking product of these two equations

$$(h)\,(k) = (ct)\left(\frac{c}{t}\right) \quad \Rightarrow \quad hk = c^2$$

Therefore, the required locus is $xy = c^2$

Example 6.3 Find the locus of a point P moves such that its distances from two fixed points $A\,(1,0)$ and $B\,(5,0)$, are always equal.

Solution:
Given that $A\,(1,0)$ and $B\,(5,0)$
Let $P\,(h,k)$ be any point on the required path.
From the information we have $AP = BP$
That is

$$\sqrt{(h-1)^2 + (k-0)^2} = \sqrt{(h-5)^2 + (k-0)^2} \Rightarrow h = 3$$

Therefore the locus of P is $x = 3$, which is a straight line parallel to the y-axis.

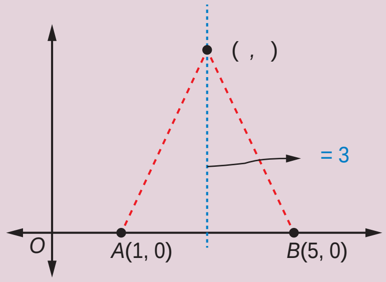


Figure 6.10

Example 6.4 If θ is a parameter, find the equation of the locus of a moving point, whose coordinates are $(a\sec\theta,\ b\tan\theta)$.

Solution:

Let $P\ (h,\ k)$ be any point on the required path. From the given information we have

$$h = a\sec\theta \text{ and } k = b\tan\theta$$
$$\Rightarrow \frac{h}{a} = \sec\theta \text{ and } \frac{k}{b} = \tan\theta$$

To eliminate the parameter θ, squaring and subtracting, we get

$$\left(\frac{h}{a}\right)^2 - \left(\frac{k}{b}\right)^2 = \sec^2\theta - \tan^2\theta$$
$$\left(\frac{h}{a}\right)^2 - \left(\frac{k}{b}\right)^2 = 1$$

Therefore the locus of the given point is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

https://www.geogebra.org/geometry

Whenever the parameters are in trigonometric form, try to use trigonometric identities to eliminate θ

$$\boxed{\sin^2\theta + \cos^2\theta = 1,\ \sec^2\theta - \tan^2\theta = 1,\ \operatorname{cosec}^2\theta - \cot^2\theta = 1.}$$

Example 6.5 A straight rod of the length 6 units, slides with its ends A and B always on the x and y axes respectively. If O is the origin, then find the locus of the centroid of ΔOAB .

Solution:

Let the coordinates of the points O, A and B are$(0,0)$, $(a,0)$ and $(0,b)$ respectively.

Observed that the points A and B are moving points.

Let(h,k) be a centroid of ΔOAB.
Centroid of ΔOAB is

$$\left(\frac{0+a+0}{3}, \frac{0+0+b}{3}\right) = (h,k).$$
$$\frac{a}{3} = h \ \Rightarrow\ a = 3h, \qquad \frac{b}{3} = k \ \Rightarrow\ b = 3k$$

From right ΔOAB, $\quad OA^2 + OB^2 = AB^2$

$$(3h)^2 + (3k)^2 = (6)^2 \quad\Rightarrow\quad h^2 + k^2 = 4$$

Locus of (h,k) is a circle, $x^2 + y^2 = 4$.

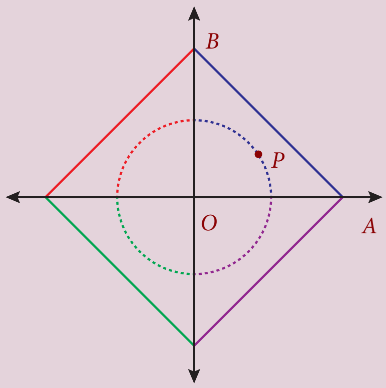


Figure 6.11

Example 6.6 If θ is a parameter, find the equation of the locus of a moving point, whose coordinates are $(a(\theta - \sin\theta)\,,\ a(1 - \cos\theta))$.

Solution:

Let $P\ (h,\ k)$ be any point on the required path. From the given information we have

$$h = a(\theta - \sin\theta) \qquad (6.1)$$
$$k = a(1 - \cos\theta) \qquad (6.2)$$

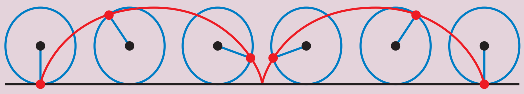


Figure 6.12

Let us find the value of θ and $\sin\theta$ from equation (6.2)

$$k = a(1 - \cos\theta)$$

$$\cos\theta = \frac{a-k}{a} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{a-k}{a}\right) \text{ and } \sin\theta = \frac{\sqrt{2ak - k^2}}{a}$$

Substituting above values in (6.1) we get $h = a\cos^{-1}\left(\frac{a-k}{a}\right) - \sqrt{2ak - k^2}$

The locus of $(h,\ k)$ is

$$x = a\cos^{-1}\left(\frac{a-y}{a}\right) - \sqrt{2ay - y^2} \qquad (6.3)$$

https://www.geogebra.org/b/bd2ADu2I#material/zCKMj8kE

Though, the parametric form given above is converted to Cartesian form, in some cases the parametric form may be more useful to work with than the cartesian form.

Exercise - 6.1

1. Find the locus of P, if for all values of α, the co-ordinates of a moving point P is

   (i) $(9\cos\alpha\,,\ 9\sin\alpha)$ (ii) $(9\cos\alpha\,,\ 6\sin\alpha)$.
2. Find the locus of a point P that moves at a constant distant of (i) two units from the x-axis (ii) three units from the y-axis.
3. If θ is a parameter, find the equation of the locus of a moving point, whose coordinates are $x = a\cos^3\theta$, $y = a\sin^3\theta$.
4. Find the value of k and b, if the points $P(-3, 1)$ and $Q(2,b)$ lie on the locus of $x^2 - 5x + ky = 0$.
5. A straight rod of length 8 units slides with its ends A and B always on the x and y axes respectively. Find the locus of the mid point of the line segmentAB
6. Find the equation of the locus of a point such that the sum of the squares of the distance from the points $(3, 5)$, $(1, -1)$ is equal to 20
7. Find the equation of the locus of the point P such that the line segment AB, joining the points $A(1, -6)$ and $B(4, -2)$, subtends a right angle at P.
8. If O is origin and R is a variable point on $y^2 = 4x$, then find the equation of the locus of the mid-point of the line segment OR.
9. The coordinates of a moving point P are $\left(\frac{a}{2}(\text{cosec}\theta + \sin\theta)\,,\ \frac{b}{2}(\text{cosec}\theta - \sin\theta)\right)$, where θ is a variable parameter. Show that the equation of the locus P is $b^2x^2 - a^2y^2 = a^2b^2$.
10. If $P(2, -7)$ is a given point and Q is a point on $2x^2 + 9y^2 = 18$, then find the equations of the locus of the mid-point of PQ.

11. If R is any point on the x-axis and Q is any point on the y-axis and P is a variable point on RQ with $RP = b$, P$Q = a$. then find the equation of locus of P.
12. If the points $P(6, 2)$ and $Q(-2, 1)$ and R are the vertices of a ΔPQR and R is the point on the locus $y = x^2 - 3x + 4$, then find the equation of the locus of centroid of ΔPQR
13. If Q is a point on the locus of $x^2 + y^2 + 4x - 3y + 7 = 0$, then find the equation of locus of P which divides segment OQ externally in the ratio 3:4,where O is origin.
14. Find the points on the locus of points that are 3 units from x-axis and 5 units from the point $(5, 1)$.
15. The sum of the distance of a moving point from the points $(4, 0)$ and $(-4, 0)$ is always 10 units. Find the equation of the locus of the moving point.

6.3 Straight Lines

Linear equations can be rewritten using the laws of elementary algebra into several different forms. These equations are often referred to as the "equations of the straight line."

In the general form the linear equation is written as:

$$ax + by + c = 0 \tag{6.4}$$

where a and b are not both equal to zero. The name "linear" comes from the fact that the set of solutions of such an equation forms a straight line in the plane. In this chapter "line", we mean a straight line unless otherwise stated.

There are many ways to write the equation of a line which can all be converted from one to another by algebraic manipulation. These forms are generally named by the type of information (data) about the line that is needed to write down the form. Some of the important data are **points, slope**, and **intercepts**

6.3.1 The relationship between the angle of inclination and slope

Definition 6.3

The angle of inclination of a straight line is the angle, say θ, made by the line with the x-axis measured in the counter clockwise (positive) direction.

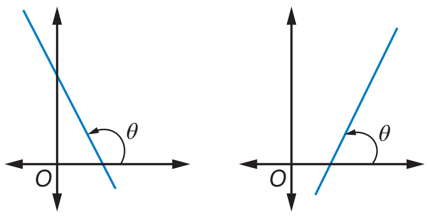


Figure 6.13

Definition 6.4

The slope or gradient of a straight line is a number that measures its "direction and steepness".

The slope of a line in the plane containing the x and y axes, is generally represented by the letter m. It can be measured in many ways as given below:

(i) When θ is the angle of inclination of the line with the x-axis measured in the counter clockwise direction then the slope

$m = \tan\theta.$
When θ is $\frac{\pi}{2}$, $\Rightarrow m = \tan\frac{\pi}{2}$ is undefined.

(ii) When (x_1, y_1) and (x_2, y_2) are any two points on the line with $x_2 \neq x_1$, then the slope is the change in the y coordinate divided by the corresponding change in the x coordinate.
This is described by the following equation.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{vertical\ change}{horizontal\ change}$$

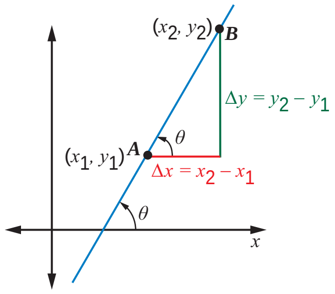


Figure 6.14

(iii) When the general form of the linear equation $ax + by + c = 0$ is given, then the slope of the line is

$$m = -\frac{a}{b}, \quad b \neq 0.$$

m is undefined when $b = 0$

The slope of a line can be a positive or negative or zero or undefined as shown below:

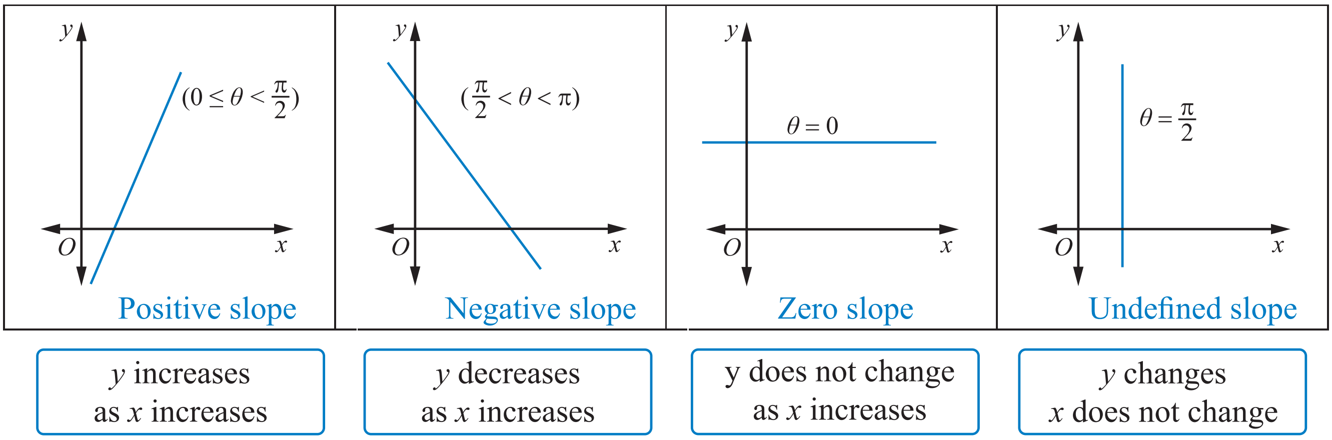


Figure 6.15

Definition 6.5
In a plane three or more points are said to be collinear if they lie on a same straight line.

Let A, $B$and C be any three points on a plane. If the slope of AB is equal to the slope of BC (or AC), then they are collinear.

6.3.2 Intercepts of a Line

Definition 6.6

The **intercept** of a line is the point at which the line crosses either the x-axis or the y-axis.

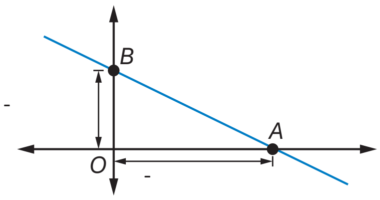


Figure 6.16

The x-intercept is a point where the y value is zero, and the y-intercept is a point where the x-value is zero.

Therefore the intercepts of a line are the points where the line intersects, or crosses, the horizontal and vertical axes.

Therefore it is clear that

(i) the equation of the y-axis is $x = 0$.
(ii) the equation of the x-axis is $y = 0$.

In the figure *OA* is the x-intercept and *OB* is the y-intercept.

Different types of *x* and *y* Intercepts:

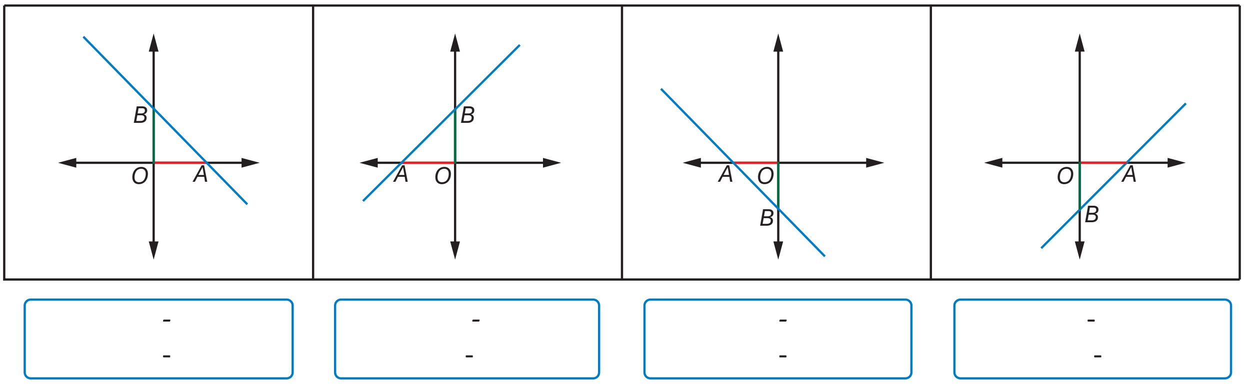


Figure 6.17

We have learnt the definition and detailed information about the points, slope and intercepts. Using these information, let us recall the different forms of an equation of a straight line.

6.3.3 Different Forms of an equation of a straight line

Two conditions are sufficient to determine uniquely the equation of a straight line. Using the combination of any two information from slope, intercepts and points, we can now form different types of straight lines such as

(i) Slope and intercept form
(ii) Point and Slope form
(iii) The two Point form
(iv) Intercepts form and two more special types are
(v) Normal form
(vi) Parametric form

Now let us look at an important way of describing the relationship between two quantities using the notion of a function.

(i) Slope and Intercept form

Proportional linear functions can be written in the form $y = mx$, where m is the slope of the line. Non proportional linear functions can be written in the form

$$y = mx + b, \quad b \neq 0 \tag{6.5}$$

This is called the slope-intercept form of a straight line because m is the slope and b is the y-intercept.

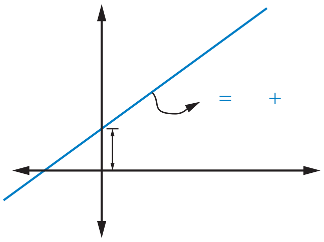


Figure 6.18

(1) when $b = 0$ and $m \neq 0$, the line passes through the origin and its equation is $y = mx$.
(2) when $b = 0$ and $m = 0$, the line coincides with the x-axis and its equation is $y = 0$.
(3) when $b \neq 0$ and $m = 0$, the line is parallel to the x-axis and its equation is $y = b$.

(ii) Point - Slope form:

Let m be the slope of the line and $A(x_1, y_1)$ be the given point on the line. Let $P(x, y)$ be any point other than A on the given line. Slope of the line joining $A(x_1, y_1)$ and $P(x, y)$ is given by $m = \dfrac{y - y_1}{x - x_1}$

$$\Rightarrow y - y_1 = m(x - x_1), \tag{6.6}$$

which is known as point-slope form.

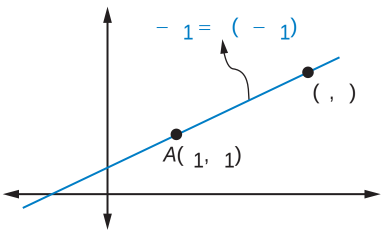


Figure 6.19

Since, the slope m is undefined for lines parallel to the y-axis, the point-slope form of the equation will not give the equation of a line through $A(x_1, y_1)$ parallel to the y-axis. However, this presents no difficulty, since for any such line the abscissa of any point on the line is x_1. Therefore, the equation of such a line is $x = x_1$.

(iii) Two Points form

If (x_1, y_1) and (x_2, y_2) are any two points on the line with $x_2 \neq x_1$ and $y_1 \neq y_2$, then the slope is $m = \dfrac{(y_2 - y_1)}{(x_2 - x_1)}$.

The equation using point- slope form, we get $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

Rewriting the above equation, we get

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \tag{6.7}$$

This equation is called two points form.

Two points form can also be represented in terms of the determinant as

$$\begin{vmatrix} x - x_1 & y - y_1 \\ x_2 - x_1 & y_2 - y_1 \end{vmatrix} = 0.$$

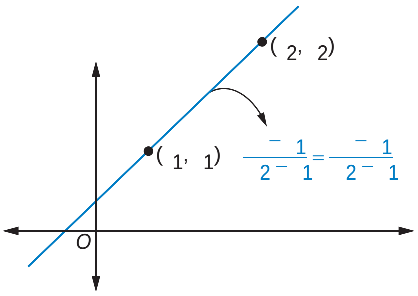


Figure 6.20

(iv) Intercepts form:

If the intercepts of a line on the x-axis and the y-axis are known then the equation of the line can also be found using intercepts. Suppose x-intercept $OA = a$ and y-intercept $OB = b$, where a and b are non-zero, then the line passes through two points $A\,(a, 0)$ and $B\,(0, b)$ is

$$\frac{y-0}{b-0} = \frac{x-a}{0-a}$$

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad (6.8)$$

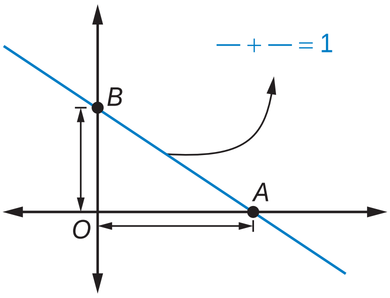


Figure 6.21

The above equation is called an intercept form.

Lines that pass through the origin or which are horizontal or vertical or violate the nonzero condition on a or b cannot be represented in this form.

In most of the cases this form is used to draw the graph of the line in easy way.

(v) Normal form:

Let A and B be the intercepts made by the line.

Let p be the length of the normal OP drawn from the origin to a line AB, which makes an angle α with the x-axis.

$$\text{In right } \Delta OPA,\ \frac{OP}{OA} = \cos\alpha \text{ and}$$

$$\text{in right } \Delta OPB,\ \frac{OP}{OB} = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha$$

$$\Rightarrow \frac{1}{OA} = \frac{\cos\alpha}{p}$$

$$\text{and } \frac{1}{OB} = \frac{\sin\alpha}{p}$$

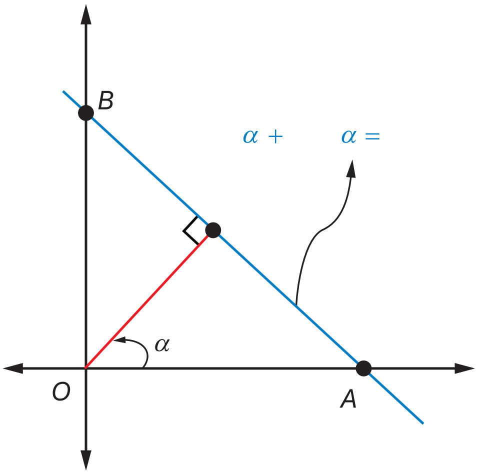


Figure 6.22

Using the above data in intercepts form

$$\frac{x}{OA} + \frac{y}{OB} = 1$$

$$\text{We get, } \frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1$$

$$\Rightarrow x\cos\alpha + y\sin\alpha = p \qquad (6.9)$$

is called the normal form of equation.

If p is positive in all positions of the line and if α is always measured from x-*axis* in the positive direction, this equation holds in every case as shown in the figure

(vi) Parametric form:

Parametric equations of a straight line is of the form

$$x = ar + x_1 \text{ and } y = br + y_1$$

where a and b are constants and r is the parameter.

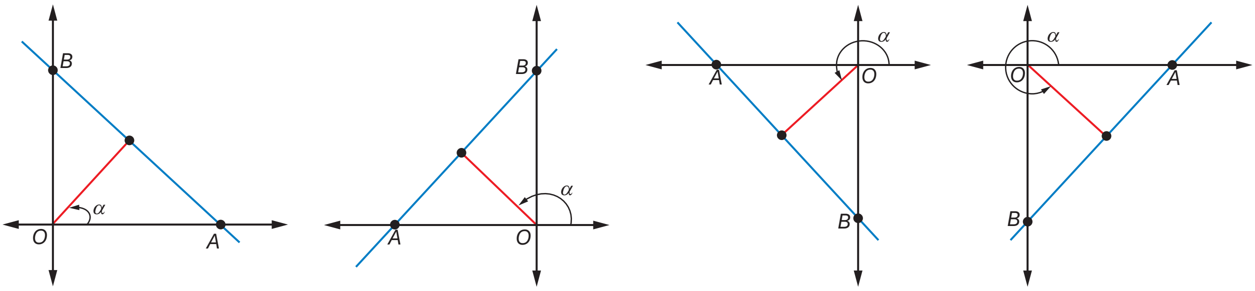


Figure 6.23

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = r, (a \neq 0, \quad b \neq 0) .$$

Suppose we have the equation of the line passing through the point $Q\,(x_1, y_1)$ and making an angle θ with x-axis. Let $P\,(x, y)$ be a point on the line at a distance r from Q. Drop perpendiculars *QN* and *PM* respectively from Q and P to the x-axis and perpendicular *QR* to *PM*.

From the right ΔQRP

$$x - x_1 = QR = PQ \, \cos\theta \; = \; r\cos\theta$$

$$\text{Therefore } \frac{x - x_1}{\cos\theta} = r \qquad (6.10)$$

Similarly, $y - y_1 = RP = QP \, \sin\theta = r\sin\theta$

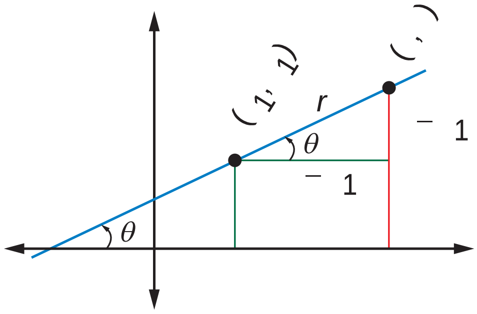


Figure 6.24

$$\Rightarrow \frac{y - y_1}{\sin\theta} = r \qquad (6.11)$$

From (6.10) and (6.11) we get

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r \qquad (6.12)$$

where the parameter r is the distance between (x_1, y_1) and any point (x, y) on the line. This is called the symmetric form or parametric form of the line.

The co-ordinates of any point on this line can be written as $(x_1 + r\cos\theta\,,\ y_1 + r\sin\theta)$. Clearly coordinates of the point depend on the value of r. This variable r is called parameter. Since r is a parameter the equations, $x = x_1 + r\cos\theta,\ y = y_1 + r\sin\theta$, is called the parametric equations of the line. The value of $'r'$ is positive for all points lying on the line one side of the given point and negative for all points lying on the line other side of the given point.

(vii) The general form of the equation of the straight line is

$$ax + by + c = 0, \quad \text{where } a,\ b \text{ and } c \text{ are all not zeros}$$

The below table summarizes the types of straight lines related to the given information.

S.No	Information given	Equation of the straight lines
1	Slope(m) and y-intercept (b)	$y = mx + b$
2	Slope (m) and point (x_1, y_1)	$y - y_1 = m(x - x_1)$
3	Two points(x_1, y_1) and (x_2, y_2)	$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$
4	x-intercept (a) and y-intercept(b)	$\frac{x}{a} + \frac{y}{b} = 1$
5	Normal length (p) , angle (α)	$x\cos\alpha + y\sin\alpha = p$
6	Parametric form: parameter-r	$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$
7	The general equation	$ax + by + c = 0$

If we have two variable quantities, then each can be represented by a variable. If the rate of change of one variable with respect to the other variable is constant, then the relationship between them is linear.

In linear equation, the choice of one as independent and other as a dependent may represent the physical reality or may be convenient fiction. The independent variable is normally plotted on the horizontal axis (x-axis), the dependent variable on the vertical axis (y-axis). That is the values of x are always independent and y is dependent on those values of x.

The number scales on the two axes need not be the same. Indeed, in many applications different quantities are represented by x and y. For example, x may represent the number of mobile phones sold and y the total revenue resulting from the sales. In such cases it is often desirable to choose different number scales to represent the different quantities. However, that the zero of both number scales coincide at the origin of the two-dimensional coordinate system.

From the given information, to solve the problem using the concepts of straight lines, we have to select suitably one of the six equations given above.

Example 6.7 Find the slope of the straight line passing through the points (5,7) and (7,5). Also find the angle of inclination of the line with the x-axis.

Solution:

Let (x_1, y_1) and (x_2, y_2) be (5,7) and (7,5) respectively. Let θ be the angle of inclination of the line with the x-axis

Slope of the line $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-7}{7-5} = -1$

We know that $m = \tan\theta$

That is, $\tan\theta = -1 \Rightarrow \theta = \frac{3\pi}{4}$ *or* $135°$

Slope and angle of inclination of the line with the x-axis are respectively $m = -1$ and $\theta = \frac{3\pi}{4}$

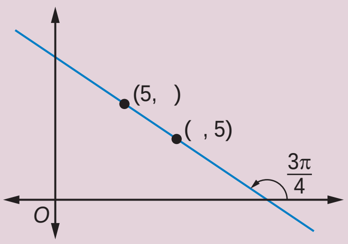


Figure 6.25

Example 6.8 Find the equation of a straight line cutting an intercept of 5 from the negative direction of the y-axis and is inclined at an angle $150°$ to the x-axis.

Solution:

Given that the negative y intercept is 5 i.e., $b = -5$ and $\theta = 150°$

slope $m = \tan 150° = \tan(180° - 30°) = -\tan 30° = -\dfrac{1}{\sqrt{3}}$

Slope and intercept form of the equation is $y = mx + b$.

That is $y = -\dfrac{1}{\sqrt{3}}x - 5$.

$$x + \sqrt{3}y + 5\sqrt{3} = 0$$

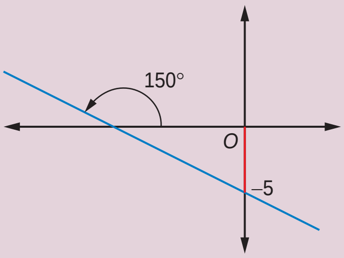


Figure 6.26

Example 6.9 Show the points $\left(0, -\frac{3}{2}\right)$, $(1, -1)$ and $\left(2, -\frac{1}{2}\right)$ are collinear.

Solution:

Let A, B and C be $\left(0, -\frac{3}{2}\right)$, $(1, -1)$ and $\left(2, -\frac{1}{2}\right)$ respectively.

The slope of *AB* is $\dfrac{-1 + \frac{3}{2}}{1 - 0} = \dfrac{1}{2}$

The slope of *BC* is $\dfrac{-\frac{1}{2} + 1}{2 - 1} = \dfrac{1}{2}$

Thus, the slope of *AB* is equal to slope of *BC*.
Hence, A, B and C are lying on the same line.

If the rate of change of one variable with respect to the other variable is constant, then this constant rate of change can be taken as slope.(such as speed, constant increase or constant decrease…). Also equations of straight lines depend on the coordinate axes how we define it. Thus in real world problems the equations are not necessarily identical but the path and distance will always be the same.

Example 6.10 The Pamban Sea Bridge is a railway bridge of length about 2065 m constructed on the PalkStrait, which connects the Island town of Rameswaram to Mandapam, the main land of India. The Bridge is restricted to a uniform speed of only 12.5 m/s. If a train of length 560 m starts at the entry point of the bridge from Mandapam, then

(i) find an equation of the motion of the train.
(ii) when does the engine touch island
(iii) when does the last coach cross the entry point of the bridge
(iv) what is the time taken by a train to cross the bridge.

Solution:

Let the x-axis be the time in seconds the y-axis be the distance in metres. Let the engine be at the origin O. Therefore the length of the train 560 m is the negative y-intercept.

The uniform speed 12.5 m/s is the slope of the motion of the train. $\left(\text{speed} = \dfrac{\text{distance}}{\text{time}}\right)$

Since we are given slope and y-intercept, the equation of the line is

$$y = mx + b \tag{6.13}$$

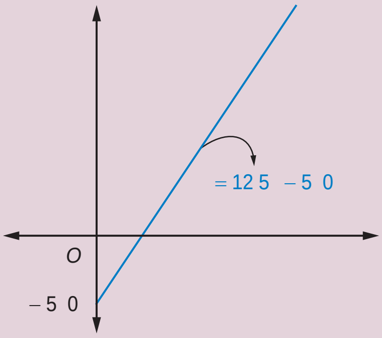


Figure 6.27

(i) The equation of the motion of the train,

$$\begin{aligned} \text{when } m &= 12.5 \text{ and } b = 560, \\ \text{is } y &= 12.5x - 560 \end{aligned}$$

(ii) When the engine touches the other side of the bridge (island)

$$\begin{aligned} y &= 2065 \text{ and } b = 0 \\ 2065 &= 12.5x \\ x &= 165.2 \text{ seconds.} \end{aligned}$$

(iii) When $y = 0$, the last coach cross the entry point of the bridge,

$$\begin{aligned} 0 &= 12.5x - 560 \\ x &= 44.8 \text{ seconds.} \end{aligned}$$

(iv) When $y = 2065$, the time taken for the train to cross the other end of the bridge is given by

$$\begin{aligned} 2065 &= 12.5x - 560 \\ x &= 210 \text{ seconds.} \end{aligned}$$

(One may take the tail of the train as the origin and can find the equation of the straight line. It need not be identical with the above equation, but the path traced out by the train, distance, time, etc.,will be the same. Try it.)

Example 6.11 Find the equations of the straight lines, making the y- intercept of 7 and angle between the line and the y-axis is 30°

Solution:

There are two straight lines making 30° with the y-axis.

From the figure, it is clear that the two lines make the angles 60° and 120° with the x-axis

$$\begin{aligned} \text{Let } m_1 \text{ be } \tan 60^\circ &= \sqrt{3} \text{ and} \\ m_2 \text{ be } \tan 120^\circ &= \tan(180^\circ - 60^\circ) = -\tan 60^\circ = -\sqrt{3} \\ m_1 = \sqrt{3}, m_2 &= -\sqrt{3} \text{ and } b = 7 \\ \text{Equations of lines are } y &= m_1 x + b \text{ and } y = m_2 x + b \\ y &= \sqrt{3}x + 7 \text{ and } y = -\sqrt{3}x + 7 \end{aligned}$$

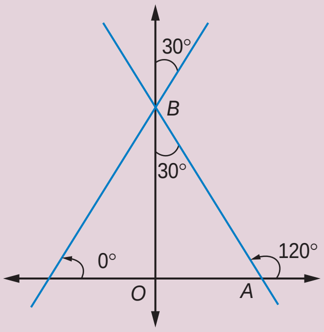


Figure 6.28

Whenever two points are given, one can apply two points form or point and slope form. Also when two intercepts are given, one can apply intercepts form or two points form The following example, is solved in chapter V, using the concepts of sequence and series. Let us solve this problem here, using the concepts of straight lines.

Example 6.12 The seventh term of an arithmetic progression is 30 and tenth term is 21.

(i) Find the first three terms of an A.P.
(ii) Which term of the A.P. is zero (if exists)
(iii) Find the relation ship between Slope of the straight line and common difference of A.P.

Solution:

Since there is a constant increase or decrease in arithmetic progression, it is a linear function.
Let the x-axis be the number of the term and the y-axis be the value of the term.
Let (x_1, y_1) and (x_2, y_2) be (7,30) and (10,21) respectively, using the equation

$$\begin{aligned} y - y_1 &= m(x - x_1) \text{ we get} \\ y - 30 &= \frac{21 - 30}{10 - 7}(x - 7) \\ y &= -3x + 51 \end{aligned} \tag{6.14}$$

(i) Substituting $x = 1, 2$ and 3 in the equation (6.14) we get the first three terms of AP as 48, 45, and 42.
(ii) Substituting $y = 0$ in equation (6.14) we get

$$0 = -3x + 51 \quad \Rightarrow x = 17.$$

That is seventeenth term of A.P. is zero.
(iii) clearly the slope of the line -3 is equal to the common difference A.P.

From this example, slope of the line is equal to common difference of A.P. (Try to prove it)

Example 6.13 The quantity demanded of a certain type of Compact Disk is 22,000 units when a unit price is ₹ 8. The customer will not buy the disk, at a unit price of ₹ 30 or higher. On the other side the manufacturer will not market any disk if the price is ₹ 6 or lower. However, if the price ₹ 14 the manufacturer can supply 24,000 units. Assume that the quantity demanded and quantity supplied are linearly proportional to the price. Find (i) the demand equation (ii) supply equation (iii) the market equilibrium quantity and price. (iv) The quantity of demand and supply when the price is ₹ 10.

Solution:

Let the x-axis represent the number of units in thousands and the y-axis represent the price in rupees per unit.

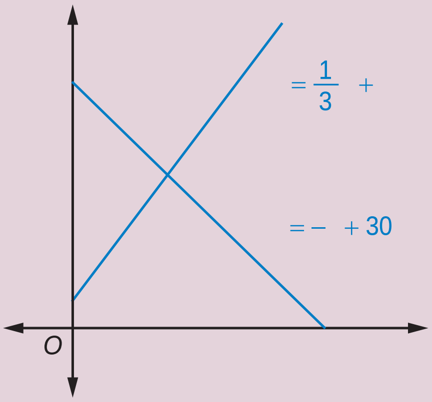


Figure 6.29

(i) For demand function, let (x_1, y_1) and (x_2, y_2) be (22,8) and (0,30) respectively.
Using two point form, we get the demand function as

$$\frac{y-8}{30-8} = \frac{x-22}{0-22}$$
$$\Rightarrow y_D = -x+30 \text{ (demand function).}$$

(ii) For supply function, let (x_1, y_1) and (x_2, y_2) be (0,6) and (24,14) respectively.
Using two point form, we get the supply function as

$$\frac{y-6}{14-6} = \frac{x-0}{24-0}$$
$$y_S = \frac{1}{3}x+6 \text{(supply function).}$$

(iii) At the market equilibrium the demand equals to supply,
That is, $y_D = y_S \quad \Rightarrow -x+30 = \frac{1}{3}x+6$

$$x = 18 \text{ and } y = 12.$$

Market equilibrium price is Rs12 and number of quantity is 18,000 units.

(iv) when the price $y = 10$, from the demand function $y_D = -x+30$, we get $x = 20$.
That is, the demand is 20,000 units.
Similarly from the supply function $y_S = \frac{1}{3}x+6$, we get $x = 12$.
Hence, the supply is 12,000 units.

Example 6.14 Find the equation of the straight line passing through (– 1, 1) and cutting off equal intercepts, but opposite in signs with the two coordinate axes.

Solution:

Let the intercepts cut off from the axes be of lengths a and $-a$.
$\therefore$ Equation of the line is of the form

$$\frac{x}{a} - \frac{y}{a} = 1 \Rightarrow \ x - y = a.$$

Since it passes through $(-1, 1)$
$\Rightarrow (-1) - (1) = a \Rightarrow a = -2.$
Equation of the line is $x - y + 2 = 0$.

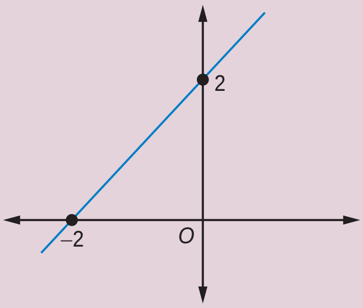


Figure 6.30

Example 6.15 A straight line L with negative slope passes through the point $(9,\ 4)$ cuts the positive coordinate axes at the points P and Q. As L varies, find the minimum value of $|OP| + |OQ|$, where O is the origin.

Solution:

Let m be the slope of the line L. Since it passes through the point $(9,\ 4)$ the equation of the line L is $y - 4 = m(x - 9)$.

The points P and Q are respectively $(9 - \frac{4}{m},\ 0)$ and $(0,\ 4 - 9m)$.

$$\begin{aligned}
(m < 0)\ |OP| + |OQ| &= \left|9 - \frac{4}{m}\right| + |4 - 9m| \\
&= \left|9 + \frac{4}{k}\right| + |4 + 9k| \quad (m < 0,\ \text{take } m = -k,\ k > 0) \\
&= (9 + \frac{4}{k}) + (4 + 9k) \quad (\text{all terms are positive}) \\
&= (4 + 9) + (\frac{4}{K} + 9K) \\
&\geq 13 + 2\sqrt{\frac{4}{K} \times 9K} \quad (\text{Arithmetic mean} \geq \text{Geometric mean}) \\
|OP| + |OQ| &\geq 25
\end{aligned}$$

Therefore, the minimum absolute value of $|OP| + |OQ|$ is 25.

Example 6.16 The length of the perpendicular drawn from the origin to a line is 12 and makes an angle $150°$ with positive direction of the x-axis. Find the equation of the line.

Solution:

Here, $p = 12$ and $\alpha = 150°$, So the equation of the required line is of the form

$$\begin{aligned}
x\cos\alpha + y\sin\alpha &= p \\
\text{That is, } x\cos 150° + y\sin 150° &= 12 \\
\Rightarrow \sqrt{3}x - y + 24 &= 0
\end{aligned}$$

Example 6.17 Area of the triangle formed by a line with the coordinate axes, is 36 square units. Find the equation of the line if the perpendicular drawn from the origin to the line makes an angle of 45° with positive the x-axis.

Solution:

Let p be the length of the perpendicular drawn from the origin to the required line.
The perpendicular makes 45° with the x-axis.
The equation of the required line is of the form,

$$\begin{aligned} x\cos\alpha + y\sin\alpha &= p \\ \Rightarrow x\cos 45^\circ + y\sin 45^\circ &= p \\ \Rightarrow x + y &= \sqrt{2}p \end{aligned}$$

This equation cuts the coordinate axes at $A(\sqrt{2}p, 0)$ and $B(0, \sqrt{2}p)$.
Area of the ΔOAB is

$$\frac{1}{2} \times \sqrt{2}p \times \sqrt{2}p = 36 \quad \Rightarrow p = 6 \quad (\because\ p \text{ is positive})$$

Therefore the equation of the required line is

$$x + y = 6\sqrt{2}$$

Example 6.18 Find the equation of the lines make an angle 60° with positive x-axis and at a distance $5\sqrt{2}$ units measured from the point $(4, 7)$, along the line $x - y + 3 = 0$.

Solution:

The angle of inclination of the line $x - y + 3 = 0$ is 45°, and a point on the line is $(4, 7)$.
Using parametric form

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r,$$

the above equation can be written as

$$\frac{x-4}{\frac{1}{\sqrt{2}}} = \frac{y-7}{\frac{1}{\sqrt{2}}} = \pm 5\sqrt{2} \text{ (for either side of (4,7) at a distance } r = \pm 5\sqrt{2})$$

That is $x - 4 = y - 7 = \pm 5$.
The points on the lines at a distance $5\sqrt{2}$ units either side of $(4, 7)$ are
$(4 + 5, 7 + 5)$ and $(4 - 5, 7 - 5)$.
The points on the lines are $(9, 12)$ and $(-1, 2)$ and the given slope is $m = \tan 60^\circ = \sqrt{3}$.
Therefore the required equations, using slope and a point form, we get

$$\begin{aligned} \sqrt{3}x - y + (12 - 9\sqrt{3}) &= 0 \text{ and} \\ \sqrt{3}x - y + (2 + \sqrt{3}) &= 0 \end{aligned}$$

6.3.4 General form to other forms

$Ax + By + C = 0$, where A, B and C being real numbers and A and B cannot be simultaneously equal to zero, can be expressed in terms of A, B and C of the general form into other forms.

(i) Slope and intercept form: $(B \neq 0)$: The given equation can be written as

$$y = -\frac{A}{B}x - \frac{C}{B} \quad \Rightarrow \text{ slope} = -\frac{A}{B} \text{ and } y\text{-intercept} = -\frac{C}{B}$$

(ii) Intercepts form: The given equation can be written as

$$\frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1 \ (A,\ B \text{ and } C \text{ are all non-zero})$$

Comparing with intercept form, we get

$$x\text{ -intercept } (a) = \frac{-C}{A} \text{ and } y\text{ -intercept } (b) = \frac{-C}{B}$$

(iii) Normal form: Here A and B are not equal to zero,
Comparing $Ax + By + C = 0$, with $x\cos\theta + y\sin\theta = p$
We get $\dfrac{-A}{\sqrt{A^2+B^2}}x + \dfrac{-B}{\sqrt{A^2+B^2}}\,y = \dfrac{|C|}{\sqrt{A^2+B^2}}$
Here $\cos\alpha = \dfrac{-A}{\sqrt{A^2+B^2}}$, $\sin\alpha = \dfrac{-B}{\sqrt{A^2+B^2}}$ and $p = \dfrac{|C|}{\sqrt{A^2+B^2}}$

Example 6.19 Express the equation $\sqrt{3}x - y + 4 = 0$ in the following equivalent form:
(i) Slope and Intercept form
(ii) Intercept form
(iii) Normal form

Solution:

(i) Slope and intercept form
It is given that

$$\begin{aligned} \sqrt{3}x - y + 4 &= 0 \\ \Rightarrow y &= \sqrt{3}x + 4 \end{aligned} \qquad (6.15)$$

Comparing the above equation with the equation $y = mx + b$, we have

$$\text{Slope} = \sqrt{3} \text{ and } y\text{-intercept} = 4$$

(ii) Intercept form

$$\begin{aligned} \sqrt{3}x - y + 4 &= 0 \Rightarrow \sqrt{3}x - y = -4 \\ \frac{-\sqrt{3}}{4}x + \frac{y}{4} &= 1 \\ \text{That is } \frac{x}{\left(-\frac{4}{\sqrt{3}}\right)} + \frac{y}{4} &= 1 \end{aligned} \qquad (6.16)$$

Comparing the above equation with the equation $\dfrac{x}{a} + \dfrac{y}{b} = 1$

$$\text{We get, } x\text{-intercept} = -\frac{4}{\sqrt{3}} \text{ and } y\text{-intercept} = 4$$

(iii) Normal form:

$$\begin{aligned} \sqrt{3}x - y + 4 &= 0 \\ (-\sqrt{3})x + y &= 4, \end{aligned} \tag{6.17}$$

Comparing the above equation with the equation $Ax + By + C = 0$.

Here $A = -\sqrt{3}$, and $B = 1$, $\sqrt{A^2 + B^2} = 2$

Therefore, dividing the above equation by 2, we get

$$\frac{-\sqrt{3}x}{2} + \frac{y}{2} = 2 \tag{6.18}$$

Comparing the above equation with the equation $x\cos\alpha + y\sin\alpha = p$

If we take

$$\begin{aligned} \cos\alpha &= \frac{-\sqrt{3}}{2} \text{ and } \sin\alpha = \frac{1}{2} \text{ and } p = 2 \\ \Rightarrow \alpha &= 150^\circ = \frac{5\pi}{6} \text{ and length of the normal } (p) = 2 \end{aligned}$$

The normal form is $x\ \cos\dfrac{5\pi}{6} + y\ \sin\dfrac{5\pi}{6} = 2$

To express the given equation to the required form, some times, it is more convenient to use property the proportionality of the coefficients of like terms.

Example 6.20 Rewrite $\sqrt{3}x + y + \ 4 = 0$ in to normal form.

Solution:

The required form $x\cos\alpha + y\sin\alpha = p$
Given form $-\sqrt{3}x - y = 4$ ($\because p$ is always positive)
Since both represent the same equation, the coefficients are proportional. We get,

$$\frac{\cos\alpha}{-\sqrt{3}} = \frac{\sin\alpha}{-1} = \frac{p}{4}$$

$$\frac{\cos\alpha}{-\sqrt{3}} = \frac{\sin\alpha}{-1} = \frac{p}{4} = \frac{\sqrt{\cos^2\alpha + \sin^2\alpha}}{\sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2}} = \frac{1}{2} \quad \left(\begin{array}{l}\text{In componendo and dividendo,}\\ \text{whenever a term is square off,}\\ \text{then it should be square root off it.}\end{array}\right)$$

$$\begin{aligned} \cos\alpha &= \frac{-\sqrt{3}}{2},\ \ \sin\alpha = \frac{-1}{2} \text{ and } p = \frac{4}{2} \\ \alpha &= 210^\circ = \frac{7\pi}{6} \text{ and } p = 2 \end{aligned}$$

Normal form of the equation is

$$x\cos\frac{7\pi}{6} + y\sin\frac{7\pi}{6} = 2$$

Finding the shortest path between two points on a curved surface can often be difficult. However, the length of a path on the surface of a cylinder is not changed if the curved surface is flattened. For the following problem, by unrolling the hollow cylinder and flattening it into a rectangle, a single reflection allows us to determine the ant's path.

Example 6.21

Consider a hollow cylindrical vessel, with circumference 24 cm and height 10 cm. An ant is located on the outside of vessel 4 cm from the bottom. There is a drop of honey at the diagrammatically opposite inside of the vessel, 3 cm from the top. (i) What is the shortest distance the ant would need to crawl to get the honey drop?. (ii) Equation of the path traced out by the ant. (iii) Where the ant enter in to the cylinder?. Here is a picture that illustrates the position of the ant and the honey.

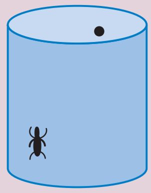


Figure 6.31

Solution:

By unrolling the hollow cylinder and flattening it into a rectangle, and with a single reflection allows us to determine the ant's path, as shown the figure. Let the base line x-axis in cm. and the vertical line through A (initial position of the ant) be the y-axis. Let H be the position of honey drop and E be the entry point of ant inside the vessel. From the given information we have

Let $A(x_1, y_1)$ and $H(x_2, y_2)$ be (0,4) and (12,13) respectively

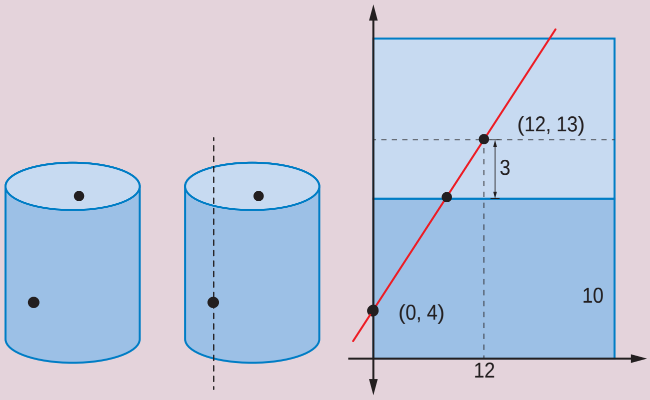


Figure 6.32

(i) The shortest distance between A and H is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{12^2 + 9^2} = 15$$

the ant would need to crawl 15 cm to get the honey.

(ii) The equation of the path AH is $\dfrac{y-4}{13-4} = \dfrac{x-0}{12-0}$

$$y = 0.75x + 4 \tag{6.19}$$

(iii) At the entry point E, $y = 10 \ \Rightarrow x = 8$

$$E = (8,\ 10)$$

Taking the orign at different location, different form of equation can be obtained, but the path and distance are the same as above.

Exercise - 6.2

1. Find the equation of the lines passing through the point (1,1)
   (i) with y-intercept (-4)
   (ii) with slope 3
   (iii) and (-2, 3)
   (iv) and the perpendicular from the origin makes an angle 60° with x- axis.
2. If $P(r, c)$ is mid point of a line segment between the axes, then show that $\frac{x}{r}+\frac{y}{c}=2$.
3. Find the equation of the line passing through the point $(1, 5)$ and also divides the co-ordinate axes in the ratio 3:10.
4. If p is length of perpendicular from origin to the line whose intercepts on the axes are a and b, then show that $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$.
5. The normal boiling point of water is 100°C or 212°F and the freezing point of water is $0°C$ or 32°F. (i) Find the linear relationship between C and F
   Find (ii) the value of C for 98.6°F and (iii) the value of F for 38°C
6. An object was launched from a place P in constant speed to hit a target. At the 15th second it was 1400 m away from the target and at the 18th second 800m away. Find (i) the distance between the place and the target (ii) the distance covered by it in 15 seconds.(iii) time taken to hit the target.
7. Population of a city in the years 2005 and 2010 are 1,35,000 and 1,45,000 respectively.
   Find the approximate population in the year 2015. (assuming that the growth of population is constant)
8. Find the equation of the line, if the perpendicular drawn from the origin makes an angle 30° with x-axis and its length is 12.
9. Find the equation of the straight lines passing through (8, 3) and having intercepts whose sum is 1
10. Show that the points (1, 3), (2, 1) and $\left(\frac{1}{2}, 4\right)$ are collinear, by using (i) concept of slope (ii) using a straight line and (iii) any other method
11. A straight line is passing through the point $A(1, 2)$ with slope $\frac{5}{12}$. Find points on the line which are 13 units away from A.
12. A 150 m long train is moving with constant velocity of 12.5 m/s. Find (i) the equation of the motion of the train, (ii) time taken to cross a pole. (iii) The time taken to cross the bridge of length 850 m is?
13. A spring was hung from a hook in the ceiling. A number of different weights were attached to the spring to make it stretch, and the total length of the spring was measured each time shown in the following table.

Weight, (kg)	2	4	5	8
Length, (cm)	3	4	4.5	6

   (i) Draw a graph showing the results.
   (ii) Find the equation relating the length of the spring to the weight on it.
   (iii) What is the actual length of the spring.
   (iv) If the spring has to stretch to 9 cm long, how much weight should be added?
   (v) How long will the spring be when 6 kilograms of weight on it?

14. A family is using Liquefied petroleum gas (LPG) of weight 14.2 kg for consumption. (Full weight 29.5kg includes the empty cylinders tare weight of 15.3kg.). If it is use with constant rate then it lasts for 24 days. Then the new cylinder is replaced (i) Find the equation relating the quantity of gas in the cylinder to the days. (ii) Draw the graph for first 96days.

15. In a shopping mall there is a hall of cuboid shape with dimension $800 \times 800 \times 720$ units, which needs to be added the facility of an escalator in the path as shown by the dotted line in the figure. Find (i)the minimum total length of the escalator. (ii) the heights at which the escalator changes its direction. (iii) the slopes of the escalator at the turning points.

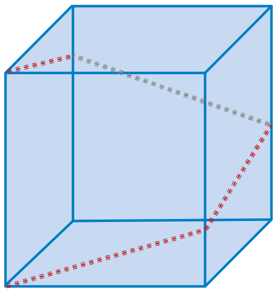


Figure 6.33

6.4 Angle between two straight lines

Two straight lines in a plane would either be parallel or coincide or intersect. Normally when two straight lines intersect, they form two angles at the point of intersection. One is an acute angle and another is an obtuse angle or equal. Both these angles would be supplements(Sum equals 180°) of each other. By definition, when we say 'angle between two straight lines' we mean the acute angle between the two lines.

$$\begin{aligned} \text{Let } y &= m_1x + c_1 \text{ and} \\ y &= m_2x + c_2 \end{aligned}$$

be the equations of two straight lines and let these two lines make angles $\theta_1$and θ_2 with x- axis.

$$\begin{aligned} \text{Then } m_1 &= \tan\theta_1 \text{ and} \\ m_2 &= \tan\theta_2 \end{aligned}$$

If ϕ (phi) is the angle between these two straight lines, then

$$\begin{aligned} \phi &= \theta_2 - \theta_1 \Rightarrow \tan\phi = \tan(\theta_2 - \theta_1) \\ \Rightarrow \tan\phi &= \frac{m_2 - m_1}{1 + m_2m_1} \\ \Rightarrow \phi &= \tan^{-1}\frac{m_2 - m_1}{1 + m_2m_1} \end{aligned}$$

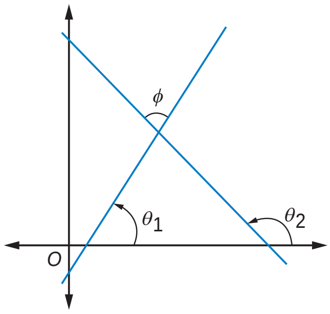


Figure 6.34

If $\dfrac{m_2 - m_1}{1 + m_2m_1}$ is positive then ϕ is the acute angle and if it is negative ϕ is the obtuse angle

between the two lines

6.4.1 Condition for Parallel Lines

Lines in the same plane that do not intersect are called parallel lines.

Let $y = m_1x + c_1$ and $y = m_2x + c_2$ be the equations of two straight lines

If these two lines are parallel, then the angle between lines is zero or π (pi)

If ϕ (phi) is the angle between the lines then, $\phi = 0$

$$\Rightarrow \tan\phi = 0$$

$$\Rightarrow \frac{m_2 - m_1}{1 + m_2m_1} = 0 \Rightarrow m_2 - m_1 = 0 \Rightarrow m_2 = m_1$$

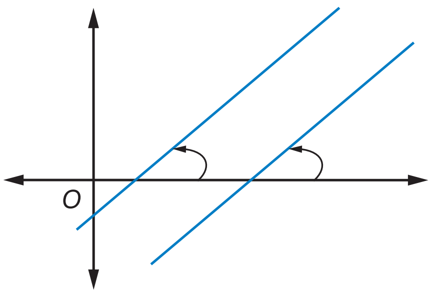


Figure 6.35

That is parallel lines have the same slope. If two non-vertical lines have the same slope, then they are parallel. All vertical lines are parallel.

If the equation of the two lines are in general form as $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then the condition for lines to be parallel is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \quad \text{or} \quad a_1b_2 = a_2b_1.$$

(i) The lines parallel to $ax + by + c = 0$ are of the form $ax + by = k$.
(ii) The line parallel to $ax + by + c = 0$ and passing through a point (x_1, y_1), then its equation is $ax + by = ax_1 + by_1$.

6.4.2 Condition for perpendicular Lines

If the lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are perpendicular, then the angle between lines is $\frac{\pi}{2}$

If ϕ is the angle between the lines then

$$\tan\phi = \frac{m_2 - m_1}{1 + m_2m_1} \Rightarrow \cot\phi = \frac{1 + m_2m_1}{m_2 - m_1}$$

$$\cot\frac{\pi}{2} = \frac{1 + m_2m_1}{m_2 - m_1} \Rightarrow 1 + m_1m_2 = 0$$

$$\Rightarrow m_1m_2 = -1$$

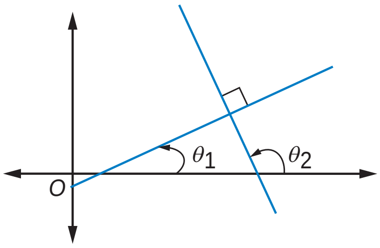


Figure 6.36

If the equation of the two lines are in general form as $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then the condition for lines to be perpendicular is $a_1a_2 + b_1b_2 = 0$

(i) The perpendicular line to $ax + by + c = 0$ are of the form $bx - ay = k$.
(ii) The perpendicular line to $ax + by + c = 0$ and passes through the point (x_1, y_1), then the required equation is $bx - ay = bx_1 - ay_1$.

Form of lines	Condition for parallel	Condition for perpendicular
$y = m_1x + c_1$ and $y = m_2x + c_2$	$m_2 = m_1$	$m_1m_2 = -1$
$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$	$a_1b_2 = a_2b_1$	$a_1a_2 + b_1b_2 = 0$

Important Note: If $m_1m_2 = -1$, then the two lines are perpendicular. But the converse is not true, because the lines are parallel to the axes, the result cannot be applied, even when the lines are perpendicular.

6.4.3 Position of a point with respect to a straight line

Any line $ax + by + c = 0$ $(c \neq 0)$,divides the whole plane in to two parts:

(i) one containing the origin called origin side of the line and

(ii) the other not containing the origin called non-origin side of the line.
A point $P\,(x_1, y_1)$ is on the origin side or non-origin side of the line $ax + by + c = 0$ $(c \neq 0)$, according as $ax_1 + by_1 + c$ and c are of the same sign or opposite sign.
If $c > 0$, then $P\,(x_1, y_1)$ is on the origin side or non orign side of the line $ax + by + c = 0$, according as $ax_1 + by_1 + c$ is positive or negative.

After rewriting the equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, such that both $c_1 > 0$ and $c_2 > 0$, and if

(i) $a_1a_2 + b_1b_2 < 0$, then the angle between them is acute

(ii) $a_1a_2 + b_1b_2 > 0$, then the angle between them is obtuse.

6.4.4 Distance Formulas

Let us develop formulas to find the distance between

(i) two points

(ii) a point to a line

(iii) two parallel lines

(i) The distance between two points (x_1, y_1) and (x_2, y_2) is given by the formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The above formula was already proved in lower class.

(ii) The distance from a point $P\,(x_1, y_1)$ to a line $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

Proof. Let AB be the given line,

$$ax + by + c = 0 \tag{6.20}$$

$P\,(x_1, y_1)$ be the given point.

Draw a line CD parallel to AB through P and drop a perpendicular from $P\,(x_1, y_1)$ to AB to meet at M. Also drop a perpendicular from the origin to the line AB to meet at R, and meeting CD at Q.

Let $\angle AOR = \alpha$

Therefore the normal form of the the line AB is

$$x \cos \alpha + y \sin \alpha = p \tag{6.21}$$

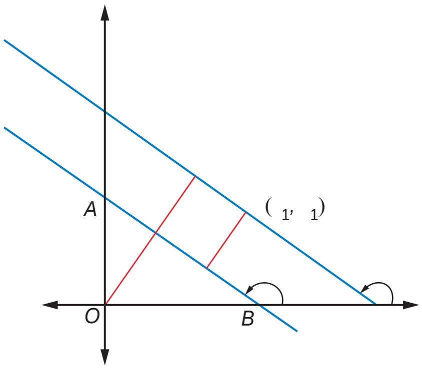


Figure 6.37

Since the equations (6.20) and (6.21) represent the same equations the coefficients of like terms are proportional.

$$\frac{\cos\alpha}{a}=\frac{\sin\alpha}{b}=\frac{p}{-c}\Rightarrow\frac{\cos\alpha}{a}=\frac{\sin\alpha}{b}=\frac{p}{-c}=\frac{\sqrt{\cos^2\alpha+\sin^2\alpha}}{\pm\sqrt{a^2+b^2}}$$

$$\cos\alpha=\frac{\pm a}{\sqrt{a^2+b^2}},\quad \sin\alpha=\frac{\pm b}{\sqrt{a^2+b^2}}\ \text{ and } p=\frac{\mp c}{\sqrt{a^2+b^2}}$$

Normal equation of CD is $\quad x\cos\alpha+y\sin\alpha=p'$ (6.22)

Since it passes through $P(x_1,y_1)$, (6.22) $\Rightarrow\ p'=x_1\cos\alpha+y_1\sin\alpha$

$$\begin{aligned}\text{Required distance} &= PM=QR=OR-OQ\ =p-p'\\ &= p-p'=p-(x_1\cos\alpha+y_1\sin\alpha)\\ &= \mp\frac{c}{\sqrt{a^2+b^2}}\mp\frac{ax_1}{\sqrt{a^2+b^2}}\mp\frac{by_1}{\sqrt{a^2+b^2}}=\pm\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\end{aligned}$$

$$\text{Required distance}=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$

(iii) The distance between two parallel lines $a_1x+b_1y+c_1=0$ and $a_1x+b_1y+c_2=0$ is

$D=\dfrac{|c_2-c_1|}{\sqrt{a^2+b^2}}$ (It can be proved using above result by taking point (x_1,y_1) as the origin)

The coordinates of the nearest point (foot of the perpendicular) on the line $ax+by+c=0$ from the point (x_1,y_1) can be found from

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=-\frac{(ax_1+by_1+c)}{a^2+b^2}\quad\text{(using parametric form)}\qquad(6.23)$$

Therefore, the coordinates of the image of the point (x_1,y_1) with respect to the line $ax+by+c=0$ are given by

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=-\frac{2(ax_1+by_1+c)}{a^2+b^2}\qquad(6.24)$$

Example 6.22 Find the equations of a parallel line and a perpendicular line passing through the point (1, 2) to the line $3x+4y=7$.

Solution:

Parallel line to $3x+4y=7$ is of the form $3x+4y=3x_1+4y_1$ Let (x_1,y_1) be (1, 2)

$$\begin{aligned}\Rightarrow 3x+4y &= 3(1)+4(2)\\ 3x+4y &= 11\end{aligned}$$

Perpendicular line to $3x+4y=7$ is of the form

$$\begin{aligned}4x-3y &= 4x_1-3y_1\\ \text{Here } (x_1,y_1) &= (1,2)\\ \Rightarrow 4x-3y &= 4(1)-3(2)\\ 4x-3y &= -2\end{aligned}$$

$\therefore$ The parallel and perpendicular lines are respectively

$$\begin{aligned} 3x + 4y &= 1 \\ 4x - 3y &= -2 \end{aligned}$$

Example 6.23 Find the distance

(i) between two points (5, 4) and (2, 0)
(ii) from a point (1, 2) to the line $5x + 12y - 3 = 0$
(iii) between two parallel lines $3x + 4y = 12$ and $6x + 8y + 1 = 0$.

Solution:

(i) Distance between two points $(x_1 y_1) = (5, 4)$ and $(x_2, y_2) = (2, 0)$ is

$$\begin{aligned} D &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{3^2 + 4^2} \\ D &= 5 \end{aligned}$$

(ii) Distance between the point (x_1, y_1) and the line $ax + by + c = 0$ is

$$D = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

$\therefore$ The distance between the point (1, 2) and the line $5x + 12y - 3 = 0$ is

$$\begin{aligned} D &= \left| \frac{5(1) + 12(2) - 3}{\sqrt{5^2 + 12^2}} \right| \\ D &= 2 \end{aligned}$$

(iii) Distance between two parallel lines $a_1 x + b_1 y + c_1 = 0$ and $a_1 x + b_1 y + c_2 = 0$ is

$$D = \frac{|c_1 - c_2|}{\sqrt{a_1^2 + b_1^2}}$$

Given lines can be written as $3x + 4y - 12 = 0$ and $3x + 4y + \frac{1}{2} = 0$
Here $a_1 = 3, b_1 = 4, c_1 = -12, c_2 = \frac{1}{2}$

$$D = \frac{|c_1 - c_2|}{\sqrt{a_2^1 + b_2^1}} = \left| \frac{-12 - \frac{1}{2}}{\sqrt{3^2 + 4^2}} \right| = \frac{25}{2 \times 5} = 2\,.5 \text{ units.}$$

Example 6.24 Find the nearest point on the line $2x + y = 5$ from the origin.

Solution:

The required point is the foot of the perpendicular from the origin on the line $2x + y = 5$.
The line perpendicular to the given line, through the origin is $x - 2y = 0$.
Solving the equations $2x + y = 5$ and $x - 2y = 0$, we get $x = 2, y = 1$
Hence the nearest point on the line from the origin is (2, 1).

Alternate method:

Using the formula (6.23)

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = -\frac{(ax_1+by_1+c)}{a^2+b^2}$$
$$\frac{x-0}{2} = \frac{y-0}{1} = -\frac{(2(0)+1(0)-5)}{2^2+1^2}$$
$$\Rightarrow \frac{x}{2} = \frac{y}{1} = 1 \Rightarrow (2,\ 1)$$

Example 6.25 Find the equation of the bisector of the acute angle between the lines $3x+4y+2=0$ and $5x+12y-5=0$.

Solution:

First, let us make the constant term positives in both the equations.
The angle bisectors of the given equations are

$$\frac{3x+4y+2}{\sqrt{3^2+4^2}} = \pm\frac{-5x-12y+5}{\sqrt{5^2+12^2}} \quad \text{(moving point is equidistance from the lines)}$$

Since $a_1a_2+b_1b_2 = -15-48 < 0$, the equation of bisector of the acute angle between the lines is
$\dfrac{3x+4y+2}{5} = +\dfrac{-5x-12y+5}{13} \Rightarrow 64x+112y+1=0$

Example 6.26 Find the points on the line $x+y=5$, that lie at a distance 2 units from the line $4x+3y-12=0$

Solution:

Any point on the line $x+y=5$ is $x=t, y=5-t$
The distance from $(t, 5-t)$ to the line $4x+3y-12=0$ is given by 2 units.

$$\therefore \frac{4(t)+3(5-t)-12}{\sqrt{4^2+3^2}} = 2$$
$$\Rightarrow \frac{|t+3|}{5} = 2$$
$$\Rightarrow t+3 = \pm 10 \Rightarrow t=-13,\ t=7$$

$\therefore$ The points (−13, 18) and (7, −2).

Example 6.27 A straight line passes through a fixed point $(6,\ 8)$. Find the locus of the foot of the perpendicular drawn to it from the origin O.

Solution:

Let the point (x_1, y_1) be (6, 8). and $P(h,k)$ be a point on the required locus.
Family of equations of the straight lines passing through the fixed point (x_1, y_1) is

$$y-y_1 = m(x-x_1) \ \Rightarrow\ y-8 = m(x-6) \tag{6.25}$$

Since OP is perpendicular to the line (6.25)

$$m \times \left(\frac{k-0}{h-0}\right) = -1 \Rightarrow m = -\frac{h}{k}$$

Also $P(h,k)$ lies on (6.25) $\Rightarrow k-8 = -\frac{h}{k}(h-6)$

$$\Rightarrow k\,(k-8) = -h(h-6) \Rightarrow h^2+k^2-6h-8k=0$$

Locus of $P(h,k)$ is $x^2+y^2-6x-8y=0$

This result shows from the fact that the angle in a semi circle is a right angle.

6.4.5 Family of lines

All lines follow a specific condition are called a family of lines. The following example shows some families of straight lines.(where $m.h,$ and k are arbitrary constants).

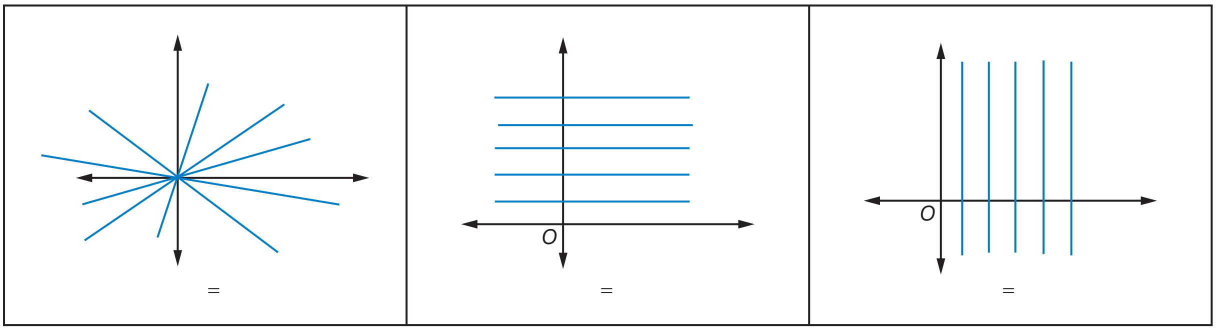


Figure 6.38

It may seem that the equation of a straight line $ax+by+c=0$ contains three arbitrary constants. In fact, it is not so. On dividing it by b (or a, whichever is non-zero), we get

$$\begin{aligned} \frac{a}{b}x+y+\frac{c}{b} &= 0, \quad \text{which can be written as} \\ Ax+y+C &= 0 \quad \text{where} \quad A=\frac{a}{b} \text{ and } C=\frac{c}{b}. \end{aligned}$$

The above equation can be written as slope and intercept form.

It follows that the equation of a straight line contains two arbitrary constants, and the number of these arbitrary constants cannot be decreased further. Thus, the equation of every straight line contains two arbitrary constants; consequently, two conditions are needed to determine the equation of a straight line uniquely.

One condition yields a linear relation among two arbitrary constants and hence each arbitrary constant determines the other. Therefore, the lines which satisfy one condition contain a **single arbitrary constant.** Such a system of lines is called **one parameter family of lines** and the unknown arbitrary constant is called, the **parameter.**

Let us now discuss the three types of families of straight lines, using $y=mx+b$. First two types are one parameter families and third one is two parameters families

(i) when m is arbitrary and b is a fixed constant.
(ii) when b is arbitrary and m is a fixed constant.
(iii) when both m and b are arbitrary

6.4.6 One parameter families

(i) when m is arbitrary and b is a fixed constant

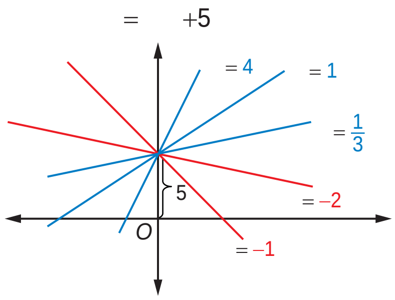


Figure 6.39

Let us find the family of equations of straight lines for the line $y = mx + b$ by considering m is arbitrary constant and b is a fixed constant say $b = 5$. Therefore the equation, for different real values of m, represents a family of lines with y-intercept 5 units. A few members of this family are shown in figure. For example, in this diagram the slope m takes $-1,\ -2,\ \frac{1}{3},\ 1$ and 4

Example 6.28 Find the equations of the straight lines in the family of the lines $y = mx + 2$, for which m and the x-coordinate of the point of intersection of the lines with $2x+3y = 10$ are integers.

Solution:

To find the equations of straight lines for the family of line $y = mx + 2$, we have to determine the values of the parameter m.

The point of intersection of the lines $y = mx + 2$ and $2x + 3y = 10$ is

$$\left(\frac{4}{3m+2}, \frac{10m+4}{3m+2}\right)$$

It is given that the slope m and the x-coordinate are integers.

$\therefore\ \dfrac{4}{3m+2}$ is an integer $\Rightarrow 3m + 2$ is a divisor of 4 $(\pm 1,\ \pm 2$ and $\pm\, 4)$

$\therefore\ 3m + 2 = \pm\, 1,\quad 3m + 2 = \pm\, 2,\quad 3m + 2 = \pm\, 4$, where m is an integer

Solving we get, $m = \{-2, -1, 0\}$

The equations are, $y = -2x + 2$, $y = -x + 2$ and $y = 2$

(ii) when b is arbitrary and m is a fixed constant.

As discussed above, suppose b is arbitrary constant and m is a fixed constant say $m = -2$,the equation $y = mx + b$ becomes $y = -2x + b$. For different real values of b, a family of lines can be obtained with slope -2. A few members of this family are shown in the figure. For example, in this diagram b can take values $-3,\ -1,\ 0,\ 1,\ 2,\ 3$ and 4.

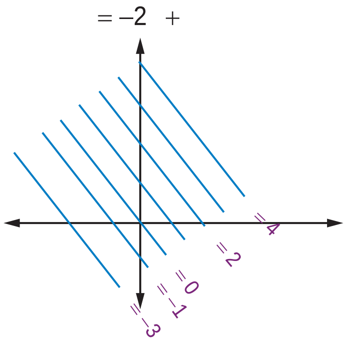


Figure 6.40

Two special cases family of parallel lines and family of perpendicular lines are given below

Family of parallel lines: Family of parallel lines to $ax+by+c=0$ is of the form $ax+by+\lambda=0$. For different values of λ (call it *lambda*), we get different lines parallel to $ax+by+c=0$.

Family of perpendicular lines: Family of perpendicular lines to $ax+by+c=0$ is of the form $bx-ay+\lambda=0$. For different values of λ, we get different lines perpendicular to $ax+by+c=0$.

6.4.7 Two parameters families

(iii) when both m and b are arbitrary

Suppose both m and b are arbitrary constants in $y=mx+b$, we cannot visualize the family easily in graph. But some cases like $y-y_1=m(x-x_1)$ for different real values of m, a family of lines can be visualized. Suppose the slope m takes -2, -4, $\frac{1}{3}$, 1 and 3, lines which pass through the fixed point (x_1, y_1) except the vertical line $x=x_1$ as shown in the diagram.

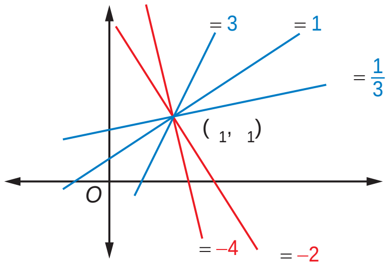


Figure 6.41

The family of equation of straight lines through the point of intersection of the two given lines

Let $L_1 \equiv a_1x+b_1y+c_1=0$ and $L_2 \equiv a_2x+b_2y+c_2=0$,

be the equation of two given lines. The family of equations of straight lines through the point of intersection of the above lines is $L_1+\lambda L_2=0$ where λ is a parameter. That is, for different real values of λ we get different equations.

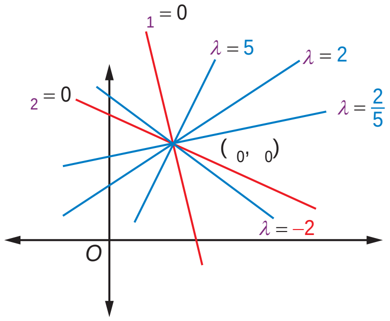


Figure 6.42

Example 6.29 Find the equation of the line through the intersection of the lines $3x+2y+5=0$ and $3x-4y+6=0$ and the point (1,1).

Solution:

The family of equations of straight lines through the point of intersection of the lines is of the form $(a_1x+b_1y+c_1)+\lambda(a_2x+b_2y+c_2)=0$

That is, $(3x+2y+5)+\lambda(3x-4y+6)=0$

Since the required equation passes through the point (1,1), the point satisfies the above equation

Therefore $\{3+2(1)+5\}+\lambda\{3(1)-4(1)+6\}=0 \Rightarrow \lambda=-2$

Substituting $\lambda=-2$ in the above equation we get the required equation as $3x-10y+7=0$

(verify the above problem by using two points form)

Example 6.30 Suppose the Government has decided to erect a new Electrical Power Transmission Substation to provide better power supply to two villages namely A and B. The substation has to to be on the line l. The distances of villages A and B from the foot of the perpendiculars P and Q on the line l are 3 km and 5 km respectively and the distance between P and Q is 6 km. (i) What is the smallest length of cable required to connect the two villages. (ii) Find the equations of the cable lines that connect the power station to two villages. (Using the knowledge in conjunction with the principle of reflection allows for approach to solve this problem.)

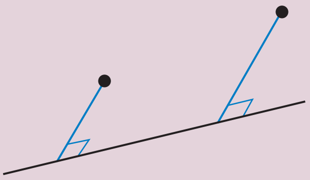


Figure 6.43

Solution:

Take conveniently PQ as x-axis, PA as y-axis and P is the origin (instead of conventional origin O). Therefore, the coordinates are $P(0,0)$, $A(0,3)$ and $B(6,5)$

If the image of A about the x-axis is $\overline{\text{A}}$, then $\overline{\text{A}}$ is $(0,-3)$.

The required R is the point of intersection of the line $\overline{\text{A}}\text{B}$ and x-axis.

AR and BR are the path of the cable (road)

The shortest length of the cable is $AR + BR = BR + R\overline{\text{A}} = B\overline{\text{A}} = \sqrt{(6-0)^2 + (5+3)^2} = 10\,\text{km}$

Equation of the line $\overline{A}B$ is $y - (-3) = \dfrac{5-(-3)}{6-0}(x-0)$

$$4x - 3y = 9$$

When $y \;=\; 0, R$ is $\left(\dfrac{9}{4}, 0\right)$

That is the substation should be located at a distance of 2.25 km from P.

The equation of AR is $4x + 3y = 9$

The equations of the cable lines (roads) of RA and RB are $4x - 3y = 9$ and $4x + 3y = 9$.

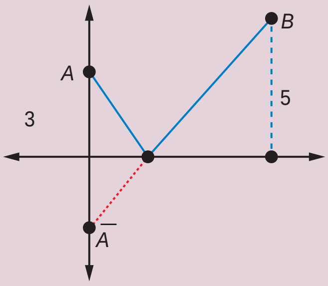


Figure 6.44

Example 6.31 A car rental firm has charges ₹ 25 with 1.8 free kilometers, and ₹ 12 for every additional kilometer. Find the equation relating the cost y to the number of kilometers x. Also find the cost to travel 15 kilometers

Solution:

Given that up to 1.8 kilometers the fixed rent is ₹ 25.

The corresponding equation is

$$y = 25, \quad 0 \le x \le 1.8 \qquad (6.26)$$

Also ₹ 12 for every additional kilometer after 1.8 kilometers

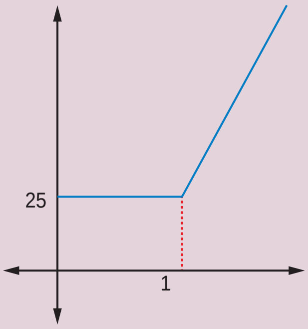


Figure 6.45

The corresponding equation is

$$y \;=\; 25 + 12\,(x - 1.8)\,, \quad x > 1.8 \qquad (6.27)$$

The combined equation of (6.26) and (6.27), we get

$$y \;=\; \begin{cases} 25, & 0 \le x \le 1.8 \\ 25 + 12(x-1.8), & x > 1.8 \end{cases} \qquad (6.28)$$

When $x = 15$, from (6.27), we get cost to travel 15 kilometers is ₹ 183.40.

If you connect from your mobile phone to book a call taxi it automatically connects you to the first available cab in your area wherever you are, as long as there's taxi service. What type of coordinate system used to locate and navigate? www.mapbox.com

Example 6.32 If a line joining two points (3, 0) and (5, 2) is rotated about the point (3, 0) in counter clockwise direction through an angle 15°, then find the equation of the line in the new position

Solution:

Let P (3, 0) and Q (5, 2) be the given points.

Slope of $PQ = \dfrac{y_2 - y_1}{x_2 - x_1} = 1 \Rightarrow$ the angle of inclination of the line $PQ = \tan^{-1}(1) = \dfrac{\pi}{4} = 45°$

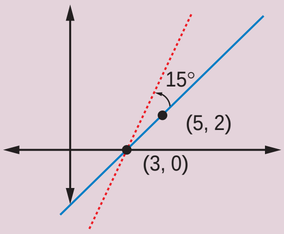


Figure 6.46

$\therefore$ The slope of the line in new position is

$$m = \tan(45° + 15°) \Rightarrow \text{Slope} = \tan(60°) = (\sqrt{3})$$

$\therefore$ Equation of the straight line passing through (3, 0) and with the slope $\sqrt{3}$ is

$$\begin{aligned} y - 0 &= \sqrt{3}(x - 3) \\ \sqrt{3}x - y - 3\sqrt{3} &= 0 \end{aligned}$$

Exercise - 6.3

1. Show that the lines are $3x + 2y + 9 = 0$ and $12x + 8y - 15 = 0$ are parallel lines.
2. Find the equation of the straight line parallel to $5x - 4y + 3 = 0$ and having x-intercept 3.
3. Find the distance between the line $4x + 3y + 4 = 0$, and a point (i) $(-2, 4)$ (ii) $(7, -3)$
4. Write the equation of the lines through the point $(1, -1)$
   (i) parallel to $x + 3y - 4 = 0$
   (ii) perpendicular to $3x + 4y = 6$
5. If $(-4, 7)$ is one vertex of a rhombus and if the equation of one diagonal is $5x - y + 7 = 0$, then find the equation of another diagonal.
6. Find the equation of the lines passing through the point of intersection lines $4x - y + 3 = 0$ and $5x + 2y + 7 = 0$, and (i) through the point $(-1, 2)$ (ii) Parallel to $x - y + 5 = 0$ (iii) Perpendicular to $x - 2y + 1 = 0$
7. Find the equations of two straight lines which are parallel to the line $12x + 5y + 2 = 0$ and at a unit distance from the point (1, − 1).
8. Find the equations of straight lines which are perpendicular to the line $3x + 4y - 6 = 0$ and are at a distance of 4 units from (2, 1).
9. Find the equation of a straight line parallel to $2x + 3y = 10$ and which is such that the sum of its intercepts on the axes is 15.

10. Find the length of the perpendicular and the co-ordinates of the foot of the perpendicular from $(-10,-2)$ to the line $x+y-2=0$
11. If p_1 and p_2 are the lengths of the perpendiculars from the origin to the straight lines $x \sec\theta + y \operatorname{cosec}\theta = 2a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$, then prove that $p_1^2+p_2^2=a^2$.
12. Find the distance between the parallel lines
    (i) $12x+5y=7$ and $12x+5y+7=0$
    (ii) $3x-4y+5=0$ and $6x-8y-15=0$.
13. Find the family of straight lines (i) Perpendicular (ii) Parallel to $3x+4y-12=0$.
14. If the line joining two points A(2,0) and B(3,1) is rotated about A in anticlockwise direction through an angle of 15°, then find the equation of the line in new position.
15. A ray of light coming from the point (1,2) is reflected at a point A on the x-axis and it passes through the point (5,3). Find the co-ordinates of the point A.
16. A line is drawn perpendicular to $5x=y+7$. Find the equation of the line if the area of the triangle formed by this line with co-ordinate axes is 10 sq. units.
17. Find the image of the point $(-2,3)$ about the line $x+2y-9=0$.
18. A photocopy store charges ₹ 1.50 per copy for the first 10 copies and ₹ 1.00 per copy after the 10th copy. Let x be the number of copies, and let y be the total cost of photocopying. (i) Draw graph of the cost as x goes from 0 to 50 copies. (ii) Find the cost of making 40 copies
19. Find atleast two equations of the straight lines in the family of the lines $y=5x+b$, for which b and the x-coordinate of the point of intersection of the lines with $3x-4y=6$ are integers.
20. Find all the equations of the straight lines in the family of the lines $y=mx-3$, for which m and the x-coordinate of the point of intersection of the lines with $x-y=6$ are integers.

6.5 Pair of Straight Lines

The equations of two or more lines can be expressed together by an equation of degree higher than one. As we see that a linear equation in x and y represents a straight line, the product of two linear equations represent two straight lines, that is a pair of straight lines. Hence we study pair of straight lines as a quadratic equations in x and y.

Let $L_1 \equiv a_1x+b_1y+c_1=0$ and $L_2 \equiv a_2x+b_2y+c_2=0$, be separate equations of two straight lines. If $P(x_1,y_1)$ is a point on L_1, then it satisfies the equaiton $L_1=0$. Similarly, if $P(x_1,y_1)$ is on L_2 then $L_2=0$. If $P(x_1,y_1)$ lies either on $L_1=0$ or $L_2=0$, then $P(x_1,y_1)$ satisfies the equation $(L_1)(L_2)=0$, and no other point satisfies $L_1\cdot L_2=0$. Therefore the equation $L_1\cdot L_2=0$ represents the pair of straight lines $L_1=0$ and $L_2=0$.
For example, consider the two equations

$$y+\sqrt{3}x=0 \text{ and } y-\sqrt{3}x=0.$$

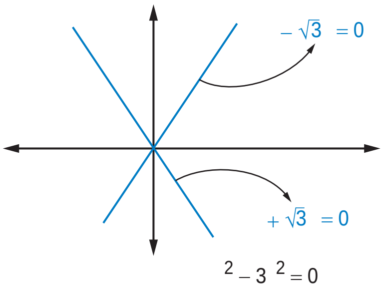


Figure 6.47

The above two equations represent the equation of two straight lines passing through the origin with slopes $-\sqrt{3}$ and $\sqrt{3}$ respectively.
Combining the above equation, we get

$$\left(y+\sqrt{3}x\right)\left(y-\sqrt{3}x\right)=0$$

$\Rightarrow y^2-3x^2=0$, represents the pair of straight lines

6.5.1 Pair of Lines Passing through the Origin

We first consider a simple case. Both the lines in this pair pass through the origin. Thus, their equations can be written as

$$y - m_1 x = 0$$
$$y - m_2 x = 0$$

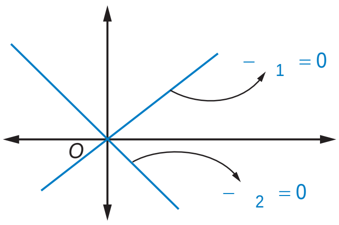


Figure 6.48

Combined equation of these two lines is

$$\begin{aligned} (y - m_1 x)(y - m_2 x) &= 0 \qquad (6.29) \\ y^2 - (m_1 + m_2)xy + m_1 m_2 x^2 &= 0 \end{aligned}$$

The above equation suggests that the general equation of a pair of straight lines passing through the origin with slopes m_1 and m_2, $ax^2 + 2hxy + by^2 = 0$ is **a homogenous equation of degree two**, implying that the degree of each term is 2.

Nature of the homogenous equations tells us whether the lines pass through the origin.

Example 6.33 Separate the equations $5x^2 + 6xy + y^2 = 0$

Solution:

We factorize this equation straight away as

$$\begin{aligned} 5x^2 + 6xy + y^2 &= 0 \\ 5x^2 + 5xy + xy + y^2 &= 0 \\ 5x(x + y) + y(x + y) &= 0 \\ (5x + y)(x + y) &= 0 \end{aligned}$$

So that the lines are $5x + y = 0,$ and $x + y = 0$

Alternate method: since the given equation is a homogeneous equation, divide the given equation

$$\begin{aligned} 5x^2 + 6xy + y^2 &= 0 \text{ by } x^2 \\ \text{We get } 5 + 6\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 &= 0 \\ \text{Substitute } \frac{y}{x} &= m \text{ (slope of the lines for homogenous equation)} \end{aligned}$$

The above equation becomes $m^2 + 6m + 5 = 0$

$$\begin{aligned} \text{Factorizing, we get} \qquad (m + 1)(m + 5) &= 0 \\ \Rightarrow m = -1, \quad m &= -5 \\ \Rightarrow \frac{y}{x} = -1, \quad \frac{y}{x} &= -5 \end{aligned}$$

That is, the lines are $x + y = 0, \qquad 5x + y = 0$

Example 6.34 If exists, find the straight lines by separating the equations $2x^2 + 2xy + y^2 = 0$.

Solution:

Since the given equation is a homogeneous equation, divide the given equation $2x^2 + 2xy + y^2 = 0$ by x^2 and substituting $\frac{y}{x} = m$

We get $m^2 + 2m + 2 = 0$

The values of m (slopes) are not real (complex number), therefore no line will exist with the joint equation $2x^2 + 2xy + y^2 = 0$

We sometimes say that this equation represents imaginary lines.

Note that in the entire plane, only (0, 0) satisfies this equation

6.5.2 Angle between Pair of Straight Lines

Consider the equation of a pair of straight lines passing through the origin as:

$$ax^2 + 2hxy + by^2 = 0 \tag{6.30}$$

Let m_1 and m_2 be the slopes of these two lines.

By dividing (6.30) by x^2 and substituting $\frac{y}{x} = m$

$$\text{we get,} \quad bm^2 + 2hm + a = 0$$

This quadratic in m will have its roots as m_1 and m_2.

Thus, $m_1+m_2 = \frac{-2h}{b}$ and $m_1 m_2 = \frac{a}{b}$

If the angle between the two lines is θ Then

$$\begin{aligned}
\tan\theta &= \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right| \\
&= \left|\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1 + m_2 m_1}\right| \\
&= \left|\frac{\sqrt{\left(\frac{-2h}{b}\right)^2 - 4\frac{a}{b}}}{1 + \frac{a}{b}}\right| \\
\tan\theta &= \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|
\end{aligned}$$

As a consequence of this formula, we can conclude that

1. The lines are real and distinct, if m_1 and m_2 are real and distinct, that is if $h^2 - ab > 0$
2. The lines are real and coincident, if m_1 and m_2 are real and equal, that is if $h^2 - ab = 0$
3. The lines are not real (imaginary), if m_1 and m_2 are not real, that is if $h^2 - ab < 0$

Also, we see that the lines represented by (6.30), are parallel (since both pass through the origin, the lines are coincident lines) if $\tan\theta = 0$, that is $h^2 - ab = 0$, and perpendicular if $\cot\theta = 0$ that is $a + b = 0$

Pair of straight lines	Condition for parallel	Condition for perpendicular
$ax^2 + 2hxy + by^2 = 0$	$h^2 - ab = 0,$	$a + b = 0$

Example 6.35 Find the equation of the pair of lines through the origin and perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$

Solution:

Let m_1 and m_2 be the slopes of these two lines.

$$y - m_1 x = 0 \text{ and } y - m_2 x = 0 \tag{6.31}$$

Combined equation of these two lines is

$$\begin{aligned} (y - m_1 x)(y - m_2 x) &= 0 \\ y^2 - (m_1 + m_2) xy + m_1 m_2 x^2 &= 0 \end{aligned} \tag{6.32}$$

Given that

$$ax^2 + 2hxy + by^2 = 0 \tag{6.33}$$

$$\text{Thus, } m_1 + m_2 = \frac{-2h}{b} \quad \text{and} \quad m_1 m_2 = \frac{a}{b} \tag{6.34}$$

The lines perpendicular to (6.31) are

$$y + \frac{1}{m_1} x = 0 \quad \text{and} \quad y + \frac{1}{m_2} x = 0$$

The combined equation is

$$\begin{aligned} (m_1 y + x)(m_2 y + x) &= 0 \\ m_1 m_2 y^2 + (m_1 + m_2) xy + x^2 &= 0 \end{aligned}$$

By using (6.34) $\quad \frac{a}{b} y^2 - \frac{2h}{b} xy + x^2 = 0$

$$\text{The required equation is } ay^2 - 2hxy + bx^2 = 0 \tag{6.35}$$

6.5.3 Equation of the bisectors of the angle between the lines $ax^2 + 2hxy + by^2 = 0$

Let the equations of the two straight lines be $y - m_1 x = 0$ and $y - m_2 x = 0$

$$\therefore\ m_1 + m_2 = -\frac{2h}{b} \quad \text{and} \quad m_1 m_2 = \frac{a}{b}$$

We know that the equation of bisectors is the locus of points from which the perpendicular drawn to the two straight lines are equal.

Let $P(p, q)$ be any point on the locus of bisectors.

The perpendiculars from $P(p, q)$ to the line $y - m_1 x = 0$ is equal to the perpendicular from $P(p, q)$ to $y - m_2 x = 0$

$$\pm \frac{q - m_1 p}{\sqrt{1 + m_1^2}} = \pm \frac{q - m_2 p}{\sqrt{1 + m_2^2}}$$

$$\text{That is,} \qquad (q - m_1 p)^2 \left(1 + m_2^2\right) = (q - m_2 p)^2 \left(1 + m_1^2\right)$$

$$\text{Simplifying we get} \qquad p^2 - q^2 = 2pq\left(\frac{1 - m_1 m_2}{m_1 + m_2}\right)$$

$$\Rightarrow p^2 - q^2 = 2pq\left(\frac{1 - \frac{a}{b}}{-\frac{2h}{b}}\right)$$

$$\text{That is} \qquad \frac{p^2 - q^2}{a - b} = \frac{pq}{h}$$

$$\therefore \text{ The locus of } P(p, q) \text{ is} \qquad \frac{x^2 - y^2}{a - b} = \frac{xy}{h} \tag{6.36}$$

Example 6.36 Show that the straight lines $x^2 - 4xy + y^2 = 0$ and $x + y = 3$ form an equilateral triangle.

Solution:

Let the line $x + y = 3$ intersects the pair of line x^2 -4$xy + y^2 = 0$ at A and B.
The angle between the lines $x^2 - 4xy + y^2 = 0$ is

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = \frac{2\sqrt{4 - 1}}{2} = \sqrt{3}$$

$$\Rightarrow \theta = \tan^{-1}\sqrt{3} = 60^\circ$$

The angle bisectors of the angle AOB are given by

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

$$\Rightarrow x^2 - y^2 = 0$$

$$\Rightarrow x + y = 0 \quad \text{and} \quad x - y = 0$$

The angle bisectors $x - y = 0$ is perpendicular to the given line through AB,

$$\text{that is } x + y = 3 \Rightarrow \Delta OAB \text{ is isosceles.}$$

$$\Rightarrow \angle ABO = \angle BAO = 60^\circ$$

therefore the given lines form an equilateral triangle.

Example 6.37 If the pair of lines represented by $x^2 - 2cxy - y^2 = 0$ and $x^2 - 2dxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that $cd = -1$.

Solution:

Given that the pair of straight lines,

$$x^2 - 2cxy - y^2 = 0 \tag{6.37}$$

$$x^2 - 2dxy - y^2 = 0 \tag{6.38}$$

The equation of the angle bisectors of (6.37) is

$$\begin{aligned} \frac{x^2-y^2}{2} &= \frac{xy}{-c} \\ \Rightarrow \quad cx^2+2xy-cy^2 &= 0 \end{aligned} \tag{6.39}$$

Given that the angle bisector of (6.37) is (6.38)
Therefore equations

$$\begin{aligned} x^2-2dxy-y^2 &= 0 \\ cx^2+2xy-cy^2 &= 0 \end{aligned}$$

represent the same equation as the angle bisector of (6.37)

Comparing the like terms of the above two equations, we get

$$\begin{aligned} \frac{1}{c} = \frac{-2d}{2} &= \frac{-1}{-c} \\ \Rightarrow \quad cd &= -1 \end{aligned}$$

6.5.4 General form of Pair of Straight Lines

Consider the equations of two arbitrary lines $l_1x+m_1y+n_1=0$ and $l_2x+m_2y+n_2=0$

The combined equation of the two lines is

$$(l_1x+m_1y+n_1)(l_2x+m_2y+n_2)=0$$

If we multiply the above two factors together, we get a more general equation to a pair of straight lines has the form

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \tag{6.40}$$

The above equation is **a non homogenous equation of degree two.**

in which
$$\begin{aligned} a &= l_1l_2 & 2g &= l_1n_2+l_2n_1 \\ b &= m_1m_2 & 2f &= m_1n_2+m_2n_1 \\ c &= n_1n_2 & 2h &= l_1m_2+l_2m_1 \end{aligned}$$

An equation of the form (6.40) will always represent a pair of straight lines, provided it must able to be factorized into two linear factors of the form $l_1x+m_1y+n_1=0$ and $l_2x+m_2y+n_2=0$.

Condition that the general second degree equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ should represent a pair of straight lines

Let us rearrange the equation of the pair of straight lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ as a quadratic in x, we have

$$ax^2+2(hy+g)x+\left(by^2+2fy+c\right)=0$$

$$x=\frac{-(hy+g)\pm\sqrt{(hy+g)^2-a\left(by^2+2fy+c\right)}}{a}$$

$$ax+hy+g=\pm\sqrt{(hy+g)^2-a\left(by^2+2fy+c\right)}$$

That is, $$ax+hy+g=\pm\sqrt{\left(h^2-ab\right)y^2+2(gh-af)y+g^2-ac}$$

Since each of the above equations represents a straight line, they must be of the first degree in x and y. Therefore the expression under the radical sign should be a perfect square and the condition for this is

$$4\left(gh - af\right)^2 - 4\left(h^2 - ab\right)\left(g^2 - ac\right) = 0$$

Simplifying and dividing by a,

We get, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ or

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$ (Expansion of Determinant will be studied in the next chapter)

Results without Proof:

(i) Two straight lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are

parallel if $\dfrac{a}{h} = \dfrac{h}{b} = \dfrac{g}{f}$ or $bg^2 = af^2$

(ii) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel straight lines, then

the distance between them is $2\sqrt{\dfrac{(g^2 - ac)}{a(a+b)}}$ or $2\sqrt{\dfrac{(f^2 - bc)}{b(a+b)}}$

The relationship between the equations of pair of straight lines

$$ax^2 + 2hxy + by^2 = 0 \qquad (6.41)$$

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0) \qquad (6.42)$$

The slope of the above pair of straight lines (6.41) and (6.42) are depending only on the coefficients of

x^2, xy and y^2.

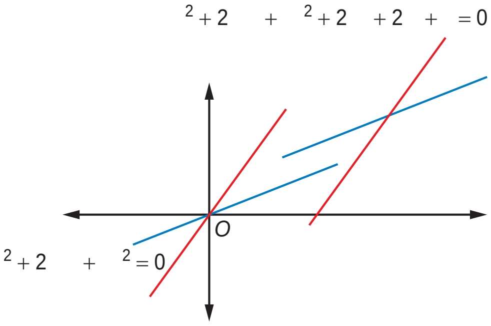


Figure 6.49

(i) If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, then $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines through the origin parallel to the first pair. The point of intersection (6.41) is (0, 0) and the point of intersection (6.42) is

$$P\left(\frac{hf - bg}{ab - h^2}, \quad \frac{gh - af}{ab - h^2}\right)$$

(ii) If θ be the angle between the straight lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then it will have the same value as the angle between the two lines represented by $ax^2 + 2hxy + by^2 = 0$

$$\text{Thus } \theta = \tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

(iii) If the two straight lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are at right angles, then the two lines represented by $ax^2 + 2hxy + by^2 = 0$ are also at right angles and the condition is $a + b = 0$

Example 6.38 If the equation $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines, find (i) the value of λ and the separate equations of the lines (ii) point of intersection of the lines (iii) angle between the lines

Solution:

(i) general equation is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

given equation is $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$

Comparing the given equation with the general equation of the second degree we have

$$a = \lambda, b = 12, c = -3, h = -5, g = \frac{5}{2}, f = -8$$

Now applying the condition for pair of straight lines

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$
$$\text{i.e. } \lambda(12)(-3) + 2(-8)\left(\frac{5}{2}\right)(-5) - \lambda(-8)^2 - 12\left(\frac{5}{2}\right)^2 - (-3)(-5)^2 = 0$$
$$\Rightarrow -36\lambda + 200 - 64\lambda - 75 + 75 = 0, \quad \Rightarrow \lambda = 2$$

$\therefore$ The equation is $2x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$.
Let us first factorize the terms of second degree terms from the above equation,

$$\text{we get} \qquad 2x^2 - 10xy + 12y^2 \equiv (x - 2y)(2x - 6y)$$
$$\therefore\ 2x^2 - 10xy + 12y^2 + 5x - 16y - 3 \equiv (x - 2y + c_1)(2x - 6y + c_2)$$

Equating like terms, we get

$$2c_1 + c_2 = 5, \quad 3c_1 + c_2 = 8, \quad c_1c_2 = -3$$

Solving first two equation, we get $c_1 = 3, c_2 = -1$.

$\therefore$ The separate equations of the lines are

$$x - 2y + 3 = 0 \quad \text{and} \quad 2x - 6y - 1 = 0$$

(ii) Point of intersection of the lines is given by solving the two equation of the lines, we get

$$(x, y) = \left(-10, -\frac{7}{2}\right)\left(\text{or use the formula}\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right)\right)$$

(iii) Angle between the lines is given by

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$
$$= \frac{2\sqrt{25 - 24}}{2 + 12} = \frac{1}{7}$$
$$\therefore\ \theta = \tan^{-1}\left(\frac{1}{7}\right)$$

Example 6.39 A student when walks from his house, at an average speed of 6 kmph, reaches his school ten minutes before the school starts. When his average speed is 4 kmph, he reaches his school five minutes late. If he starts to school every day at 8.00 A.M, then find (i) the distance between his house and the school (ii) the minimum average speed to reach the school on time and time taken to reach the school (iii) the time the school gate closes (iv) the pair of straight lines of his path of walk.

Solution:

Let x-axis be the time in hours and y-axis be the distance in kilometer.
From the given information, we have

$$y = 6(x - \frac{10}{60}) \Rightarrow y = 6x - 1 \tag{6.43}$$

$$y = 4(x + \frac{5}{60}) \Rightarrow y = 4x + \frac{1}{3} \tag{6.44}$$

Solving the above two equations, we get $(x, y) = \left(\frac{2}{3}, 3\right)$

$$x = \frac{2}{3} \text{ hour} = 40 \text{ minutes}, \ y = 3 \text{ km}$$

(i) the distance between house and the school is 3km

(ii) the minimum average speed to reach the school on time is

$$\frac{60}{40} \times 3 = 4.5 \, \text{kmph}$$

and time taken is hours $\frac{2}{3}$ or 40 minutes

(iii) the school gate closes at 8.40 AM

(iv) the pair of straight lines of his path of walk to school is

$$(6x - y - 1)\left(4x - y + \frac{1}{3}\right) = 0$$

$$72x^2 - 30xy + 3y^2 - 6x + 2y - 1 = 0 \tag{6.45}$$

Draw the graph by using the site: https://www.geogebra.org/graphing
https://www.geogebra.org/m/fhS6HUtP

Example 6.40 If one of the straight lines of $ax^2 + 2hxy + by^2 = 0$ is perpendicular to $px + qy = 0$, then show that $ap^2 + 2hpq + bq^2 = 0$.

Solution:

Let m_1 and m_2 be the slopes of the pair of lines $ax^2 + 2hxy + by^2 = 0$ and m be the slope of $px + qy = 0$

Therefore, $m_1 + m_2 = -\frac{2h}{b}$, $m_1 m_2 = \frac{a}{b}$ and $m = -\frac{p}{q}$

since one of the straight lines of $ax^2 + 2hxy + by^2 = 0$ is perpendicular to $px + qy = 0$,

$$
\begin{aligned}
m\,m_1 = -1 \quad \text{or} \quad m\,m_2 &= -1\\
\Rightarrow \quad (m\,m_1+1)=0 \ \text{ or } \ (m\,m_2+1) &= 0\\
\Rightarrow \quad (m\,m_1+1)(m\,m_2+1) &= 0\\
\Rightarrow \quad (m_1m_2)m^2 + \ (m_1+m_2)m \ + \ 1 &= 0\\
\Rightarrow \quad \left(\frac{a}{b}\right)\left(-\frac{p}{q}\right)^2 + \left(-\frac{2h}{b}\right)\left(-\frac{p}{q}\right) \ + \ 1 &= 0\\
\Rightarrow \quad ap^2+2hpq+bq^2 &= 0
\end{aligned}
$$

The pair of straight lines through the origin is a homogeneous equation of degree two

Example 6.41 Show that the straight lines joining the origin to the points of intersection of $3x-2y+2=0$ and $3x^2+5xy-2y^2+4x+5y=0$ are at right angles

Solution:

The straight lines joining the origin and the points of intersection of given equations is a second degree homogeneous equation.

Following steps show, the way of homogenizing the $3x^2+5xy-2y^2+4x+5y=0$ with $3x-2y+2=0$

$$
\begin{aligned}
3x^2+5xy-2y^2+(4x+5y)(1) &= 0 \ \text{ and } \ \frac{(3x-2y)}{-2}=1\\
3x^2+5xy-2y^2+(4x+5y)\left(\frac{3x-2y}{-2}\right) &= 0\\
(-2)(3x^2+5xy-2y^2)+(4x+5y)(3x-2y) &= 0
\end{aligned}
$$

On simplification,
We get, $2x^2+xy-2y^2=0. \quad \Rightarrow a=2,\ b=-2 \ \Rightarrow \ a+b=0$
Since sum of the co-efficient of x^2 and y^2 is equal to zero, the lines are at right angles

Exercise - 6.4

1. Find the combined equation of the straight lines whose separate equations are $x-2y-3=0$ and $x+y+5=0$.
2. Show that $4x^2+4xy+y^2-6x-3y-4=0$ represents a pair of parallel lines.
3. Show that $2x^2+3xy-2y^2+3x+y+1=0$ represents a pair of perpendicular lines.
4. Show that the equation $2x^2-xy-3y^2-6x+19y-20=0$ represents a pair of intersecting lines. Show further that the angle between them is $\tan^{-1}(5)$.
5. Prove that the equation to the straight lines through the origin, each of which makes an angle α with the straight line $y=x$ is $x^2-2xy\ \sec 2\alpha+y^2=0$
6. Find the equation of the pair of straight lines passing through the point $(1,3)$ and perpendicular to the lines $2x-3y+1=0$ and $5x+y-3=0$

7. Find the separate equation of the following pair of straight lines
   (i) $3x^2 + 2xy - y^2 = 0$
   (ii) $6(x-1)^2 + 5(x-1)(y-2) - 4(y-2)^2 = 0$
   (iii) $2x^2 - xy - 3y^2 - 6x + 19y - 20 = 0$
8. The slope of one of the straight lines $ax^2 + 2hxy + by^2 = 0$ is twice that of the other, show that $8h^2 = 9ab$.
9. The slope of one of the straight lines $ax^2 + 2hxy + by^2 = 0$ is three times the other, show that $3h^2 = 4ab$.
10. A ΔOPQ is formed by the pair of straight lines $x^2 - 4xy + y^2 = 0$ and the line PQ. The equation of PQ is $x + y - 2 = 0$. Find the equation of the median of the triangle ΔOPQ drawn from the origin O.
11. Find p and q, if the following equation represents a pair of perpendicular lines

$$6x^2 + 5xy - py^2 + 7x + qy - 5 = 0$$

12. Find the value of k, if the following equation represents a pair of straight lines. Further, find whether these lines are parallel or intersecting, $12x^2 + 7xy - 12y^2 - x + 7y + k = 0$
13. For what value of k does the equation $12x^2 + 2kxy + 2y^2 + 11x - 5y + 2 = 0$ represent two straight lines.
14. Show that the equation $9x^2 - 24xy + 16y^2 - 12x + 16y - 12 = 0$ represents a pair of parallel lines. Find the distance between them.
15. Show that the equation $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$ represents a pair of parallel lines. Find the distance between them.
16. Prove that one of the straight lines given by $ax^2 + 2hxy + by^2 = 0$ will bisect the angle between the co-ordinate axes if $(a+b)^2 = 4h^2$
17. If the pair of straight lines $x^2 - 2kxy - y^2 = 0$ bisect the angle between the pair of straight lines $x^2 - 2lxy - y^2 = 0$, Show that the later pair also bisects the angle between the former.
18. Prove that the straight lines joining the origin to the points of intersection of $3x^2 + 5xy - 3y^2 + 2x + 3y = 0$ and $3x - 2y - 1 = 0$ are at right angles.

2E7RDH

Exercise - 6.5

Choose the correct or more suitable answer

1. The equation of the locus of the point whose distance from y-axis is half the distance from origin is
   (1) $x^2 + 3y^2 = 0$ (2) $x^2 - 3y^2 = 0$ (3) $3x^2 + y^2 = 0$ (4) $3x^2 - y^2 = 0$
2. Which of the following equation is the locus of $(at^2, 2at)$
   (1) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (3) $x^2 + y^2 = a^2$ (4) $y^2 = 4ax$
3. Which of the following point lie on the locus of $3x^2 + 3y^2 - 8x - 12y + 17 = 0$
   (1) $(0,0)$ (2) $(-2,3)$ (3) $(1,2)$ (4) $(0,-1)$
4. If the point (8,−5) lies on the locus $\frac{x^2}{16} - \frac{y^2}{25} = k$, then the value of k is
   (1) 0 (2) 1 (3) 2 (4) 3

5. Straight line joining the points $(2,3)$ and $(-1,4)$ passes through the point (α,β) if

   (1) $\alpha+2\beta=7$ (2) $3\alpha+\beta=9$ (3) $\alpha+3\beta=11$ (4) $3\alpha+\beta=11$

6. The slope of the line which makes an angle 45° with the line $3x-y=-5$ are

   (1) 1, -1 (2) $\frac{1}{2}$, -2 (3) 1, $\frac{1}{2}$ (4) 2, $-\frac{1}{2}$

7. Equation of the straight line that forms an isosceles triangle with coordinate axes in the I-quadrant with perimeter $4+2\sqrt{2}$ is

   (1) $x+y+2=0$ (2) $x+y-2=0$ (3) $x+y-\sqrt{2}=0$ (4) $x+y+\sqrt{2}=0$

8. The coordinates of the four vertices of a quadrilateral are (−2,4), (−1,2), (1,2) and (2,4) taken in order.The equation of the line passing through the vertex (−1,2) and dividing the quadrilateral in the equal areas is

   (1) $x+1=0$ (2) $x+y=1$ (3) $x+y+3=0$ (4) $x-y+3=0$

9. The intercepts of the perpendicular bisector of the line segment joining (1, 2) and (3,4) with coordinate axes are

   (1) 5, −5 (2) 5, 5 (3) 5, 3 (4) 5, −4

10. The equation of the line with slope 2 and the length of the perpendicular from the origin equal to $\sqrt{5}$ is

    (1) $x+2y=\sqrt{5}$ (2) $2x+y=\sqrt{5}$ (3) $2x+y=5$ (4) $x+2y-5=0$

11. A line perpendicular to the line $5x-y=0$ forms a triangle with the coordinate axes. If the area of the triangle is 5 sq. units, then its equation is

    (1) $x+5y\pm5\sqrt{2}=0$ (2) $x-5y\pm5\sqrt{2}=0$ (3) $5x+y\pm5\sqrt{2}=0$ (4) $5x-y\pm5\sqrt{2}=0$

12. Equation of the straight line perpendicular to the line$x-y+5=0$, through the point of intersection the y-axis and the given line

    (1) $x-y-5=0$ (2) $x+y-5=0$ (3) $x+y+5=0$ (4) $x+y+10=0$

13. If the equation of the base opposite to the vertex (2, 3) of an equilateral triangle is $x+y=2$, then the length of a side is

    (1) $\sqrt{\frac{3}{2}}$ (2) 6 (3) $\sqrt{6}$ (4) $3\sqrt{2}$

14. The line $(p+2q)x+(p-3q)y=p-q$ for different values of p and q passes through the point

    (1) $\left(\frac{3}{2},\frac{5}{2}\right)$ (2) $\left(\frac{2}{5},\frac{2}{5}\right)$ (3) $\left(\frac{3}{5},\frac{3}{5}\right)$ (4) $\left(\frac{2}{5},\frac{3}{5}\right)$

15. The point on the line $2x-3y=5$ is equidistance from (1,2) and $(3,4)$ is

    (1) $(7,3)$ (2) $(4,1)$ (3) $(1,-1)$ (4) $(-2,3)$

16. The image of the point (2, 3) in the line $y=-x$ is

    (1) (−3, −2) (2) (−3, 2) (3) (−2, −3) (4) (3, 2)

17. The length of $\perp$ from the origin to the line $\frac{x}{3}-\frac{y}{4}=1$, is

    (1) $\frac{11}{5}$ (2) $\frac{5}{12}$ (3) $\frac{12}{5}$ (4) $-\frac{5}{12}$

18. The y-intercept of the straight line passing through (1,3) and perpendicular to $2x-3y+1=0$ is

    (1) $\frac{3}{2}$ (2) $\frac{9}{2}$ (3) $\frac{2}{3}$ (4) $\frac{2}{9}$

19. If the two straight lines $x + (2k - 7)y + 3 = 0$ and $3kx + 9y - 5 = 0$ are perpendicular then the value of k is

(1) $k = 3$ (2) $k = \frac{1}{3}$ (3) $k = \frac{2}{3}$ (4) $k = \frac{3}{2}$

20. If a vertex of a square is at the origin and its one side lies along the line $4x + 3y - 20 = 0$, then the area of the square is

(1) 20 sq. units (2) 16 sq. units (3) 25 sq. units (4) 4 sq.units

21. If the lines represented by the equation $6x^2 + 41xy - 7y^2 = 0$ make angles α and β with x- axis, then $\tan\alpha\tan\beta =$

(1) $-\frac{6}{7}$ (2) $\frac{6}{7}$ (3) $-\frac{7}{6}$ (4) $\frac{7}{6}$

22. The area of the triangle formed by the lines $x^2 - 4y^2 = 0$ and $x = a$ is

(1) $2a^2$ (2) $\frac{\sqrt{3}}{2}a^2$ (3) $\frac{1}{2}a^2$ (4) $\frac{2}{\sqrt{3}}a^2$

23. If one of the lines given by $6x^2 - xy + 4cy^2 = 0$ is $3x + 4y = 0$,, then c equals to

(1) -3 (2) -1 (3) 3 (4) 1

24. θ is acute angle between the lines $x^2 - xy - 6y^2 = 0$, then $\frac{2\cos\theta + 3\sin\theta}{4\sin\theta + 5\cos\theta}$ is

(1) 1 (2) $-\frac{1}{9}$ (3) $\frac{5}{9}$ (4) $\frac{1}{9}$

25. The equation of one the line represented by the equation $x^2 + 2xy\cot\theta - y^2 = 0$ is

(1) $x - y\cot\theta = 0$ (2) $x + y\tan\theta = 0$ (3) $x\cos\theta\ + y(\sin\theta + 1) = 0$

(4) $x\sin\theta\ + y(\cos\theta + 1) = 0$

Summary

The types of straight lines related to the information.

S.No	Information given	Equation of the lines
1	Slope(m) and y-intercept (b)	$y = mx + b$
2	Slope (m) and point (x_1, y_1)	$y - y_1 = m(x - x_1)$
3	Two points(x_1, y_1) and (x_2, y_2)	$\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$
4	x-intercept (a) and y-intercept (b)	$\dfrac{x}{a} + \dfrac{y}{b} = 1$
5	Normal length (p) , angle (α)	$x\cos\alpha + y\sin\alpha = p$
6	Parametric form: parameter-r	$\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$
7	The general equation	$ax + by + c = 0$

Form of lines	Condition for parallel	Condition for perpendicular
$y = m_1x + c_1$ and $y = m_2x + c_2$	$m_2 = m_1$	$m_1m_2 = -1$
$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$	$a_1b_2 = a_2b_1$	$a_1a_2 + b_1b_2 = 0$

A point $P\,(x_1, y_1)$is on the origin side or non origin side of the line $ax + by + c = 0$ $(c \neq 0)$, according as $ax_1 + by_1 + c$ and c are of the same sign or opposite sign.

The distance between two points (x_1, y_1) and (x_2, y_2) is given by the formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance from a point $P\,(x_1, y_1)$ to a line $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

The distance between two parallel lines $a_1x + b_1y + c_1 = 0$ and $a_1x + b_1y + c_2 = 0$ is $\dfrac{|c_2 - c_1|}{\sqrt{a^2 + b^2}}$

The line parallel to $ax + by + c = 0$ through a point $(x_1,\ y_1)$, is $ax + by = ax_1 + by_1$ and the perpendicular line is $bx - ay = bx_1 - ay_1$

The coordinates of the image of the point (x_1, y_1) with respect to the line $ax + by + c = 0$ can be obtained by the line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2}$$

Pair of straight lines	Condition for parallel	Condition for perpendicular
$ax^2 + 2hxy + by^2 = 0$	$h^2 - ab = 0,$	$a + b = 0$

The equation of the bisectors of the angle between the lines $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

.

The condition that the general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ should represent a pair of straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

(i) The angle between them is $\theta = \tan^{-1} \left| \dfrac{2\sqrt{h^2 - ab}}{a + b} \right|$

If $a + b = 0$, then the lines are perpendicular.

(ii) The point of intersection

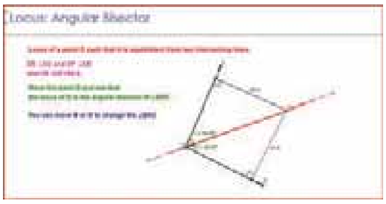
$P\left(\dfrac{hf - bg}{ab - h^2}, \quad \dfrac{gh - af}{ab - h^2}\right)$

(iii) Two straight lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are parallel if $\dfrac{a}{h} = \dfrac{h}{b} = \dfrac{g}{f}$ or $bg^2 = af^2$

(iv) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel straight lines, then the distance between them is $2\sqrt{\dfrac{(g^2 - ac)}{a(a + b)}}$ or $2\sqrt{\dfrac{(f^2 - bc)}{b(a + b)}}$

ICT CORNER-6(a)

Expected Outcome ⇒

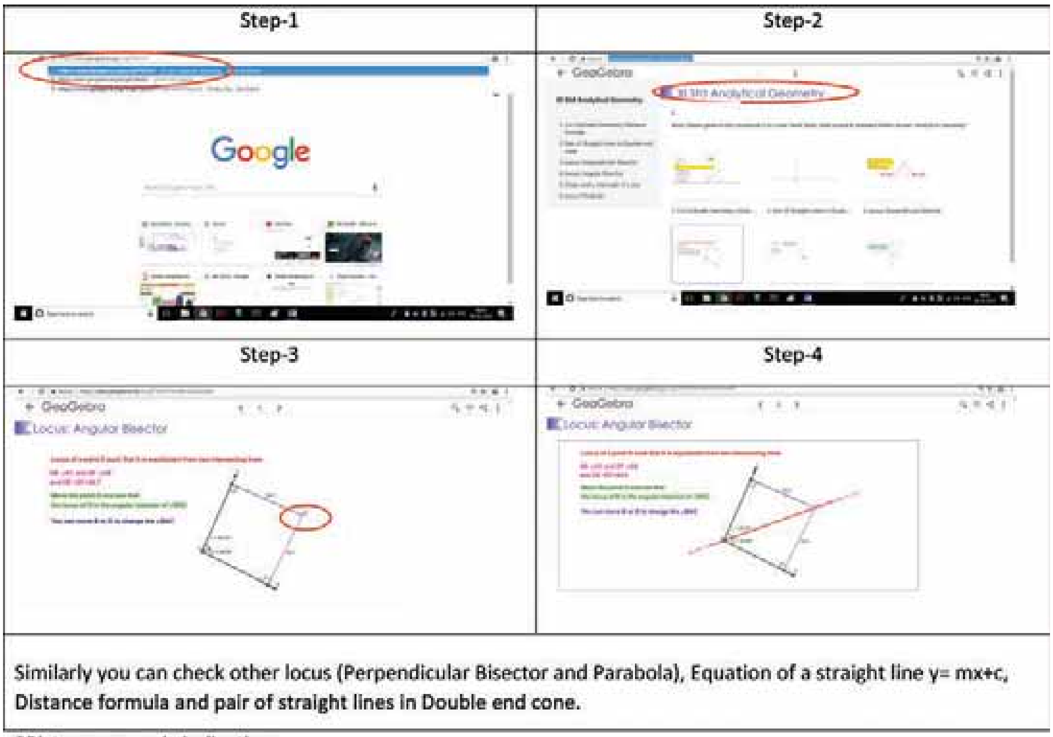


Step – 1
Open the Browser and type the URL Link given below (or) Scan the QR Code.

Step – 2
GeoGebra work book called "XI STD Analytical Geometry" will open. There are more than 5 worksheets given. Select the one you want. For example, select "Locus: **Angular Bisector**"

Step – 3
Locus: Angular Bisector work sheet will open. In the page move the point D to see the locus of the point D

Step – 4
The point D moves such that its perpendicular distance from two fixed lines are equal. Justify the locus.



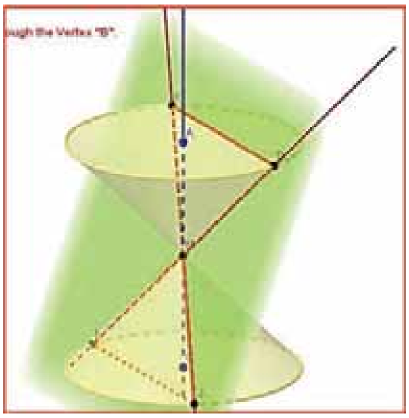
*Pictures are only indicatives.

Browse in the link
Analytical Geometry GeoGebra Workbook: `https://ggbm.at/QNaNghJZ`

B162_11_MAT_EM

ICT CORNER-6(b)

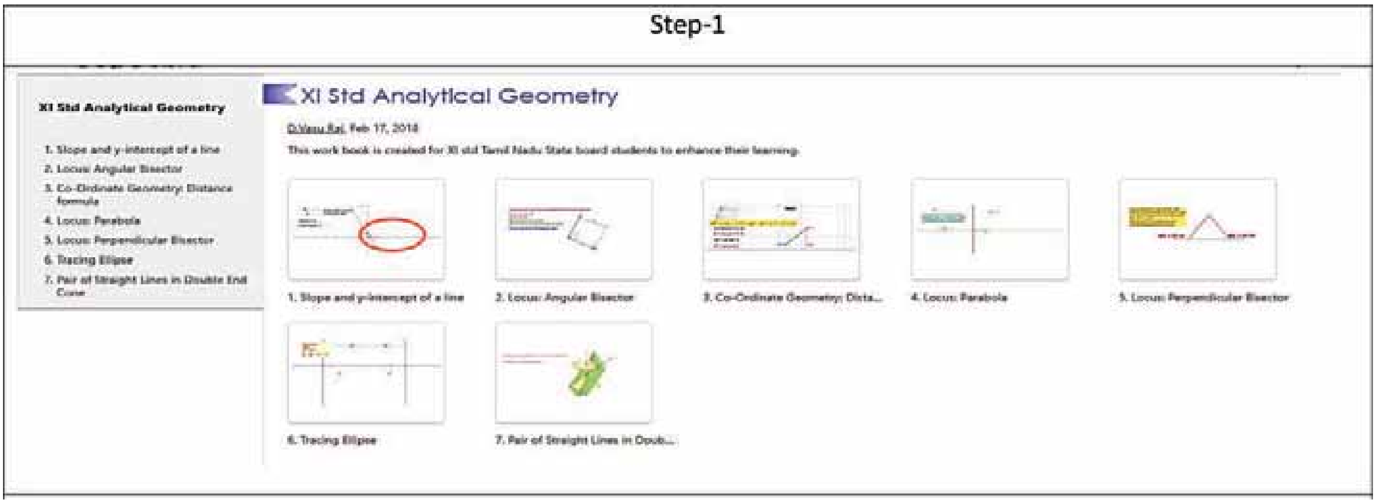
Expected Outcome ⇒



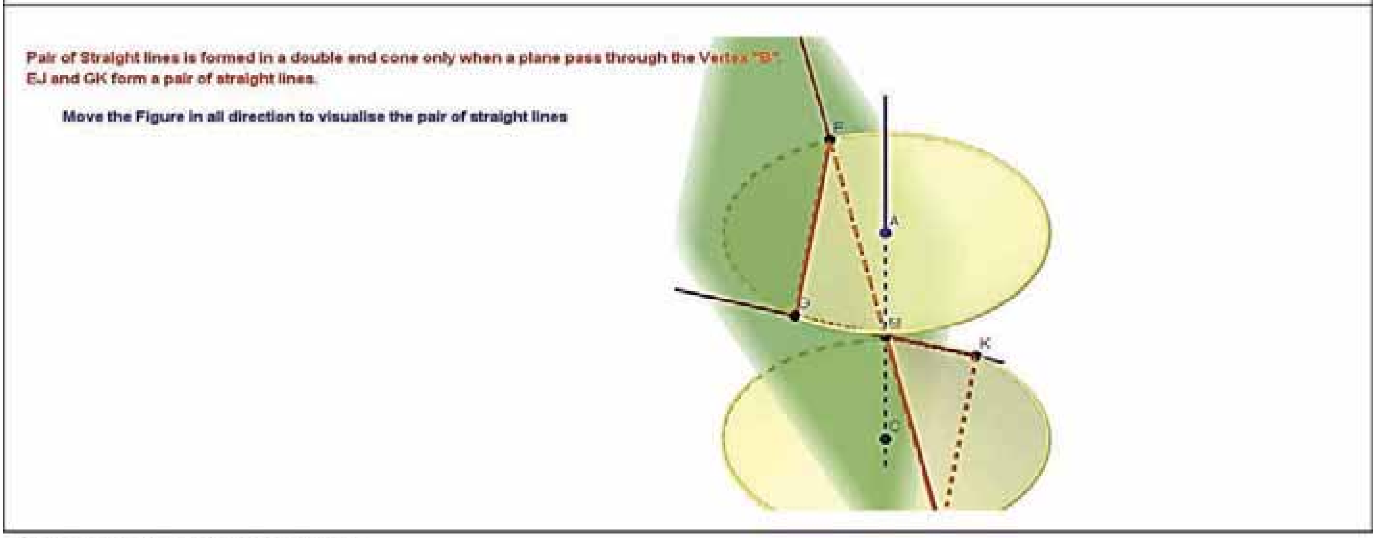
Step–1
Open the Browser and type the URL Link given below (or) Scan the QR Code.

Step–2
GeoGebra work book called "XI STD Analytical Geometry" will open. There are 7 worksheets given. Select the one you want. For example, select "Pair of Straight Line in Double End Cone"

Step–3
A plane cuts the Double end Cone at 4 points at its edges and also it passes through the Vertex. Now If you move by holding the points E or D the plane moves. You can observe at any position you get a pair of straight line. You can rotate the picture in any direction. Explore.



Step-2



*Pictures are only indicatives.

Browse in the link
Analytical Geometry GeoGebra Workbook: `https://ggbm.at/QNaNghJZ`

B162_11_MAT_EM

Answers

Exercise 1.1

(1) (i) $\{2,3,5,7\}$ (ii) $\{1\}$ (iii) $\{1,2,3,4,5,6,7,8,9,10\}$ (iv) $\{-5\}$

(2) $\{x \in \mathbb{R} \colon x^2 = 1\}$

(3) (i) finite (ii) infinite (iii) infinite (iv) infinite (v) infinite

(5) not true (6) 0 (7) 128 (8) $\{0,1,2,3\}$

(9) $A = \{x,y,z\}$ and $B = \{1,2\}$ (10) $\{(-1,0),(-1,1),(0,2),(1,2)\}$

Exercise 1.2

(1) (i) reflexive; not symmetric; transitive (ii) not reflexive; symmetric; not transitive

(iii) reflexive; not symmetric ;not transitive (iv) reflexive; symmetric; transitive

(v) R is an empty set; not reflexive; symmetric; transitive

(2) (i) (c,c) and (d,d) (ii) (c,a)

(iii) nothing to include (iv) $(c,c),(d,d),(c,a)$ to be included

(3) (i) (c,c) (ii) (c,a) (iii) nothing (iv) (c,c) and (c,a)

(5) $\{(3,8),(6,6),(9,4),(12,2)\}$; not reflexive; not symmetric; transitive; not an equivalence relation

(7) $R = \{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(4,1),$
$(4,2),(5,1)\}$; not reflexive; symmetric; not transitive; not equivalence

(8) smallest set is $\{(a,a),(b,b),(c,c)\}$; largest set is $A \times A$

Exercise 1.3

(1) Yes; inverse is not a function

(2) $f(-4) = 8,\ f(1) = 0,\ f(-2) = 6,\ f(7) = 0,\ f(0) = 0$

(3) $f(-3) = 1,\ f(5) = 38,\ f(2) = 1,\ f(-1) = -5,\ f(0) = -3$

(4) (i) a function; not one–to–one and not onto (ii) not a function

(5) (i) $\{(1,a),(2,a),(3,a),(4,a)\}$ (ii) not possible (iii) not possible

(iv) $\{(1,a),(2,b),(3,c),(4,d)\}$

(6) $\mathbb{R} - \{n\pi + (-1)^n\frac{\pi}{6}\}, n \in \mathbb{Z}$ (7) $\emptyset$

(8) $(-\infty, -\frac{1}{3}] \cup [1,\infty)$ (9) $\mathbb{R} - \{0\}, \mathbb{R} - \{0\}$

(10) for all x, $(f \circ g) = \mathbf{O}, (g \circ f) = \mathbf{O}$ (12) $f^{-1}(\mathrm{x}) = \dfrac{x+5}{3}$ (13) $x > 0$

(15) total cost $= 0.43m + 50$, airfare = ₹738

(16) $(A+S)(\mathrm{x}) = 55,000 + 0.09x$, total family income = ₹$14,05,000$

(17) $(g \circ f)(\mathrm{x}) = 62.115x$

(18) day revenue = ₹ $(200 - x)x$; total cost = ₹ $100(200 - x)$;

total profit = ₹ $(200 - x)x - 100(200 - x)$

(19) $f^{-1}(\mathrm{x}) = \frac{9x}{5} + 32$ (20) $f^{-1}(\mathrm{x}) = \frac{x+4}{3}$

Exercise 1.5

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
3	2	4	1	1	4	2	2	3	2	2	3	3	2	4

(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)
3	4	3	2	4	4	1	4	2	3

Exercise 2.1

(1) $\sqrt{7} \in \mathbb{R} - \mathbb{Q}$, $-\frac{1}{4} \in \mathbb{Q}$, $0 \in \mathbb{Z}, \mathbb{Q}$,

$3.14 \in \mathbb{Q}$, $4 \in \mathbb{N}, \mathbb{Z}, \mathbb{Q}$, $\frac{22}{7} \in \mathbb{Q}$

(3) yes, $4 + \sqrt{3}$, $2 + \sqrt{3}$ (4) yes, $2 + \sqrt{3}, 2 - \sqrt{3}$ (5) $\frac{1}{2^{1001}}$

Exercise 2.2

(1)(i) $-4 < x < 10$ (ii) $\mathbb{R}$ (iii) $\frac{11}{3} \leq x \leq \frac{13}{3}$ (iv) $-7 < x < 7$

(2) $\left(-\infty, \frac{5}{12}\right) \cup \left(\frac{7}{12}, \infty\right)$ (3) $\left(-\infty, \frac{-7}{3}\right] \cup \left[\frac{7}{3}, \infty\right)$ (4) $-\frac{15}{2} \leq x \leq \frac{11}{2}$

(5) $-\frac{3}{10} < x < \frac{7}{10}$ (6) no solution

Exercise 2.3

(1)(i) $[-1, 4)$ (ii) $[-3, 5]$ (iii) $(-\infty, 3)$ (iv) $(-\infty, 5)$

(2)(i) $1, 2, 3, 4$ (ii) $\ldots, -3, -2, -1, 0, 1, 2, 3, 4$

(3)(i) $\left(-\infty, -\frac{9}{2}\right]$ (ii) $\ldots, -7, -6, -5$ (iii) no solution (4)(i) $(-\infty, 2]$ (ii) $\left(\frac{34}{5}, \infty\right)$

(5) 93 (6) between 120ℓ and 300ℓ

(7) $(11, 13), (13, 15), (15, 17), (17, 19)$ (8) $t = 9$seconds, 11seconds

(9) less than 10 hours (10) less than ₹$21,000$ or greater than ₹33000

Exercise 2.4

(1) $x^2 - 4x - 21 = 0$ (2) $-\frac{2}{5}(x^2 - 2x - 4)$ (3) $x^2 + \frac{\sqrt{2}}{3}x + \frac{1}{3} = 0$

(6) (i) $b = 0$ (ii) $3b^2 = 16ac$ (iii) $c = a$

(8) (i) real and distinct (ii) real and distinct (iii) real and distinct

(9) (i) not intersect (ii) intersect at two points (iii) touch at only one point

(10) $\left(x+\frac{5}{2}\right)^2-\left(\frac{3}{2}\right)^2$

Exercise 2.5

(1) $\left[-3, \frac{5}{2}\right]$ (2) $[1,2]$ (4) (i) $(-\infty, 2]$ (ii) $\left(\frac{34}{5}, \infty\right)$

Exercise 2.6

(1) $-\frac{5}{2}$ and $\frac{5}{2}$ (2) $\frac{3+\sqrt{53}}{2}$ and $\frac{3-\sqrt{53}}{2}$ (3) $x=\pm 2$ (4) $x=-\frac{3}{5}$ or -1

Exercise 2.7

(1)$(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$ (2) $a=5$

Exercise 2.8

(1) $(0,1)\cup(2,\infty)$ (2) $\left(-\infty, \frac{3}{2}\right)\cup(2,4)$ (3) $(-3,-2]\cup[2,5)$

Exercise 2.9

(1) $\frac{1}{2a(x-a)}-\frac{1}{2a(x+a)}$

(2) $\frac{7}{3(x-2)}+\frac{2}{3(x+1)}$

(3) $\frac{1}{6(x-1)}+\frac{2}{15(x+2)}+\frac{-3x+1}{10(x^2+1)}$

(4) $\frac{1}{(x-1)^2}+\frac{1}{(x-1)^3}$

(5) $\frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{1}{2(x^2+1)}$

(6) $\frac{1}{x}-\frac{2}{x^2+1}$

(7) $1+\frac{13}{x-3}-\frac{7}{x-2}$

(8) $(x-5)+\frac{32}{x+3}-\frac{11}{x+2}$

(9) $-\frac{14}{9(x+1)}-\frac{11}{3(x+1)^2}+\frac{14}{9(x-2)}$

(10) $\frac{4}{x+1}+\frac{2x-3}{x^2+1}$

(11) $2+\left(\frac{2}{x+3}-\frac{1}{x-1}\right)$

(12) $\frac{3}{x+1}+\frac{4-3x}{x^2+1}$

Exercise 2.11

(1)(i) 25 (ii) $\frac{1}{8}$ (iii) $\frac{1}{100}$ (iv) $\frac{1}{9}$ (v) $\frac{1}{3}$

(2) 8 (3) $\frac{1}{\sqrt{2}}$ (4) $\frac{1}{2}$ (5) 2

(6) $\dfrac{21+7\sqrt{2}+3\sqrt{6}+2\sqrt{3}}{7}$ (7) 5 (8) $\dfrac{2+2\sqrt{6}}{5}$

Exercise 2.12

(1) $\log_b y = x,\quad (0,\infty),\quad (-\infty,\infty)$

(2) $\dfrac{5}{6}$ (3) 64 (4) 2 (11) $2\sqrt{2}$ (12) -10

Exercise 2.13

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
2	1	1	3	2	2	3	1	2	2	2	3	3	2	3

(16)	(17)	(18)	(19)	(20)
1	3	1	1	4

Exercise 3.1

(1)(i) first quadrant (ii) second quadrant (iii) fourth quadrant (iv) fourth quadrant
(v) second quadrant

(2)(i) 35° (ii) 165° (iii) 70° (iv) 90° (v) 270° (8) $k \in [-1,1]$

(9) $\sec\theta = \dfrac{p^2+1}{2p};\ \tan\theta = \dfrac{p^2-1}{2p};\ \sin\theta = \dfrac{p^2-1}{p^2+1}$ (12) $\left(c^2+bd\right)^2 = (ad+cb)^2 + \left(ac-b^2\right)^2$

Exercise 3.2

(1)(i) $\dfrac{\pi}{6}$ radians (ii) $\dfrac{3\pi}{4}$ radians (iii) $-\dfrac{41\pi}{36}$ radians (iv) $\dfrac{5\pi}{6}$ radians (v) $\dfrac{11\pi}{6}$ radians

(2)(i) 60° (ii) 20° (iii) 72° (iv) 420° (v) 200°

(3) $r \approx 31.82$ meters (4) $s = \dfrac{20\pi}{3} = 20.95\,\text{cm}$ (5) $\theta = 12^\circ 36'$ (6) $s = 7.16$feet

(7) $r_1 : r_2 = 5:4$

(8) Angle of sector $\approx 65^\circ 27' 16''$ (9) 6000° (10) 14° (11) $\dfrac{3\pi}{4}$

Exercise 3.3

(1)(i) $\dfrac{\sqrt{3}}{2}$ (ii) $-\dfrac{1}{2}$ (iii) $\dfrac{1}{2}$ (iv) $-\dfrac{1}{\sqrt{3}}$ (v) $-\dfrac{1}{\sqrt{3}}$ (vi) $\sqrt{3}$ (vii) $\dfrac{\sqrt{3}}{2}$

(2) $\sin\theta = \dfrac{2\sqrt{6}}{7};\quad \cos\theta = \dfrac{5}{7};\quad \tan\theta = \dfrac{2\sqrt{6}}{5};$

$\operatorname{cosec}\theta = \dfrac{7}{2\sqrt{6}};\quad \sec\theta = \dfrac{7}{5};\quad \cot\theta = \dfrac{5}{2\sqrt{6}}$

$$(3)(i)\ \sin\theta = -\frac{\sqrt{3}}{2};\quad \operatorname{cosec}\theta = -\frac{2}{\sqrt{3}};\quad \sec\theta = -2;\quad \tan\theta = \sqrt{3};\quad \cot\theta = \frac{1}{\sqrt{3}}$$

$$(ii)\ \sin\theta = \frac{\sqrt{5}}{3};\quad \operatorname{cosec}\theta = \frac{3}{\sqrt{5}};\quad \sec\theta = \frac{3}{2};\quad \tan\theta = \frac{\sqrt{5}}{2};\quad \cot\theta = \frac{2}{\sqrt{5}}$$

$$(iii)\ \cos\theta = \frac{\sqrt{5}}{3};\quad \operatorname{cosec}\theta = -\frac{3}{2};\quad \sec\theta = \frac{3}{\sqrt{5}};\quad \tan\theta = -\frac{2}{\sqrt{5}};\quad \cot\theta = -\frac{\sqrt{5}}{2}$$

$$(iv)\ \sec\theta = -\sqrt{5};\quad \cos\theta = -\frac{1}{\sqrt{5}};\quad \cot\theta = -\frac{1}{2};\quad \sin\theta = \frac{2}{\sqrt{5}};\quad \operatorname{cosec}\theta = \frac{\sqrt{5}}{2};$$

$$(v)\ \cos\theta = \frac{5}{13};\quad \sin\theta = -\frac{12}{13};\quad \operatorname{cosec}\theta = -\frac{13}{12};\quad \tan\theta = -\frac{12}{5};\quad \cot\theta = -\frac{5}{12};$$

(5) $\theta = 60^\circ, 120^\circ$

Exercise 3.4

(1)(i) $\sin(x+y) = \frac{220}{221}$ (ii) $\cos(x-y) = \frac{171}{221}$ (iii) $\tan(x+y) = \frac{220}{21}$

(2)(i) $\sin(A+B) = \frac{187}{205}$ (ii) $\cos(A-B) = \frac{156}{205}$ (3) $\cos(x-y) = \frac{4}{5}$

(4) $\sin(x-y) = -\frac{87}{425}$ (5) $\cos 105^\circ = \frac{1-\sqrt{3}}{2\sqrt{2}}$; $\sin 105^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$;

$\tan\left(\frac{7\pi}{12}\right) = -(2+\sqrt{3})$

(7) $4x^2 - 2\sqrt{6}x + 1 = 0$ (16) 0 (22) 1 (24) $\frac{2}{11}$

Exercise 3.5

(1)(i) $\frac{161}{289}$ (ii) $-\frac{7}{25}$ (iii) $\frac{3713}{4225}$ (2)(i) $\frac{2\sqrt{3}}{5}$, $\frac{1}{3\sqrt{2}}$

Exercise 3.6

(1)(i) $\frac{1}{2}[\sin 63^\circ + \sin 7^\circ]$ (ii) $\frac{1}{2}[\sin 6x + \sin 2x]$ (iii) $\frac{1}{2}[\sin 12\theta + \sin 8\theta]$

(iv) $\frac{1}{2}[\cos 7\theta + \cos 3\theta]$ (v) $\frac{1}{2}[\cos\theta - \cos 9\theta]$ (2)(i) $2\cos 55^\circ \sin 20^\circ$

(ii) $2\cos 40^\circ \cos 25^\circ$ (iii) $2\sin 45^\circ \cos 5^\circ$ (iv) $2\sin 55^\circ \sin 20^\circ$

Exercise 3.8

(1)(i) $\theta = -\frac{\pi}{4}$; $\theta = n\pi + (-1)^n\left(-\frac{\pi}{4}\right)$. $n \in \mathbb{Z}$

(ii) $\theta = \frac{\pi}{6}$; $\theta = n\pi + \frac{\pi}{6}$, $n \in \mathbb{Z}$

(iii) $\theta = -\frac{\pi}{6}$; $\theta = n\pi + \left(\frac{-\pi}{6}\right)$, $n \in \mathbb{Z}$

(2)(i) $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ (ii) $x = \frac{2\pi}{3}, \frac{4\pi}{3}, \pi$

(iii) $x = \frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ (iv) $x = 0, \pi$

(3)(i) $x = (2n+1)\frac{\pi}{6}$ or $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}$, $n \in \mathbb{Z}$

(ii) $\theta = n\pi + (-1)^n \frac{\pi}{6}$ or $\theta = n\pi + (-1)^n \frac{\pi}{2}$ $n \in \mathbb{Z}$

(iii) $\theta = (2n+1)\frac{\pi}{4}$ or $\theta = 2n\pi,$ $n \in \mathbb{Z}$

(iv) $\theta = \frac{n\pi}{3}$ or $\theta = n\pi \pm \frac{\pi}{3},$ $n \in \mathbb{Z}$

(v) $\theta = 2n\pi$ or $\theta = \frac{2n\pi}{3} + \left(\frac{-\pi}{6}\right),$ $n \in \mathbb{Z}$

(vi) $\theta = (8n+1)\frac{\pi}{4},$ $n \in \mathbb{Z}$

(vii) $\theta = 2n\pi + \frac{\pi}{6} \pm \frac{\pi}{3},$ $n \in \mathbb{Z}$

(viii) $\theta = 2n\pi - \frac{\pi}{3} \pm \frac{2\pi}{3},$ $n \in \mathbb{Z}$

(ix) $\theta = \frac{n\pi}{3} + \frac{\pi}{18},$ $n \in \mathbb{Z}$

(x) $\theta = n\pi \pm \frac{\pi}{10},$ $n \in \mathbb{Z}$

(xi) $x = 2n\pi \pm \frac{\pi}{3}$ $n \in \mathbb{Z}$

Exercise 3.9

(2) $\angle A = 75^\circ$ (9) $40\,m, 40\,m, 4\,m$ (10) $4\,m,\ 4\,m,\ 4\,m$ and $4\sqrt{3}\ sq.meter.$

Exercise 3.10

(1) no such triangle exist (3) $\angle A = 15^\circ, \angle B = 105^\circ$ (7) $\frac{5\sqrt{2}}{\sqrt{3}-1}$ km (8) $2\sqrt{13}$ km

(9) $3+\sqrt{73}$ km (10) 7 km

(11) Total Cost: 155800 and Perimeter $180 + 20\sqrt{27}$feet

(12) $x = 100$ km (13) $\sqrt{5-2\sqrt{2}}$ km

(14) $2\sqrt{6} + 4(\sqrt{3}+2)$ km (15) $AB = 10\sqrt{52} = 20\sqrt{13}$ km

Exercise 3.11

(1)(i) $\theta = \frac{\pi}{6}$ (ii) $\theta = \frac{\pi}{6}$ (iii) $\theta = -\frac{\pi}{2}$ (iv) $\theta = \frac{3\pi}{4}$ (v) $\theta = \frac{\pi}{3}$

Exercise 3.12

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
4	1	1	1	4	4	1	2	4	2	3	2	3	3	2
(16)	(17)	(18)	(19)	(20)										
4	1	1	1	1										

Exercise 4.1

(1)(i)	17	(ii)	6	(iii)	20	(iv)	720	(v)	120		
(2)(i)	151200	(ii)	24	(3)(i)	12	(ii)	24	(4)(i)	64	(ii)	24
(5)(i)	90	(ii)	64	(6)(i)	144	(ii)	80	(7)(i)	48	(ii)	90
(8)(i)	9000	(ii)	4536	(iii)	4464	(9)(i)	36	(ii)	60	(10)	13
(11)	400	(12)(i)	42	(ii)	78	(13)(i)	4^6	(ii)	3^{10}	(iii)	10^{12}
(14)(i)	720	(ii)	144	(iii)	4	(iv)	144	(v)	220	(vi)	$(n+3)(n+2)$
(15)(i)	15	(ii)	120	(iii)	$\frac{n(n-1)}{2}$	16.(i)	4	(ii)	100		

Exercise 4.2

(1) 10 (2) 4 (3)(i) 336 (ii) 172800 (4) 720

(5)(i) 4^{10} (ii) $3^4 \times 5^6$ (iii) $11!$ (6)(i) 4^5 (ii) 15^5

(7) 144 (8)(i) $14!$ (ii) $9! \times 6!$ (iii) $8! \times {}^9P_6$ (9) 34650

(10) 1260 (11) 6912 (12) 60 (13)(i) 2^8 (ii) 28

(14)(i) 43200 (ii) 151200 (iii) 19807200 (iv) 151200 (15)(i) 180
(ii) 60 (iii) 30 (16)(i) 379 (ii) 135 (17) 120, NIGHT

(18) 7 (19) 399960 (20) 571956

Exercise 4.3

(1) 1 (2) 3 (3) 3 (6) 20

(9) (i) ${}^{14}C_7 = 3432$ (ii) ${}^{15}C_2 = 105$ (iii) ${}^{20}C_2 = 190$ (iv) ${}^{100}C_5$ (v) ${}^5C_3 \times {}^4C_2 \times {}^2C_1 = 120$

(10) (i) $2^4 = 16$ (ii) $2^5 = 32$ (iii) 2^n

(11) (i) ${}^{25}C_3$ (ii) ${}^{25}P_3$

(12) ${}^{10}P_2 \times {}^8C_4 = 3150$ (13)(i) ${}^{10}C_3 = 120$ (ii) ${}^{10}C_5 = 252$

(14) ${}^5C_2 \times {}^{20}C_3 = 11400$ (i) ${}^4C_1 \times {}^{20}C_3 = 4560$ (ii) ${}^5C_2 \times {}^{19}C_3 = 9690$

(15) ${}^7C_3 = 35$ (16) 4512 (17) 546 (18)(i) 280 (ii) 336 (iii) 736

(19) 485 (20) 64 (21) 2454

(22) ${}^{15}C_3 = 455$ (23) 364 (24)(i) 50 (ii) 161 (25) 15

Exercise 4.5

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
2	2	1	2	4	2	1	4	2	2	3	4	2	3	1
(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)					
4	2	3	4	3	2	2	2	1	2					

Exercise 5.1

(1)(i) $8x^6 - 36x^3 + 54 - \frac{27}{x^3}$ (ii) $2[16x^8 + 216x^4(1-x^2) + 81(1-x^2)^2]$

(2)(i) 108243216 (ii) 96059601 (iii) 4782969

(3) $(1.01)^{10^6} > 10000$ (4) 10 (5) 15, x^6 term is not possible.

(6) 26235 (7) $-\frac{40}{27}$ (8) '01'

(13) $n = 15$ (14) $n = 55$ (15) $n = 7$ or 14

Exercise 5.2

(1)(i) H.P (ii) None of them. (iii) G.P

(iv) None of them (v) None of them (vi) None of them (vii) A.G.P

(2)(i) $2, 2, 4, 4, 6, 6, \ldots$ (ii) $1, 2, 3, 5, 8, 13, \ldots$ (iii) $1, 2, 3, 6, 11, 20, \ldots$

(3)(i) $a_n = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$ (ii) $a_n = \frac{n}{n+1}, \forall n \in \mathbb{N}$ (iii) $a_n = \frac{2n-1}{2n}, \forall n \in \mathbb{N}$

(iv) $a_n = \begin{cases} 7-n & \text{if } n \text{ is odd} \\ 8+n & \text{if } n \text{ is even} \end{cases}$

(4) $12, 18, 27$ (5) $t_n = \frac{1}{n^2} - \frac{1}{(n+1)^2}$ (8) $5, 45$

Exercise 5.3

(1) $a = \frac{133}{25}, d = -\frac{2}{75}, S_{20} = \frac{304}{3}$ (2) $S_{17} = 527$

(3)(i) $S_n = \frac{8}{81}[10(10^n - 1) - 9n]$ (ii) $S_n = \frac{6}{81}[10(10^n - 1) - 9n]$

(4) $S_n = \frac{4}{9}(4^n - 1) - \frac{n}{3}$ (5) $\frac{3n-2}{3^{n-1}}, \frac{3^n - (3n-2)}{2 \cdot 3^{n-1}} + \frac{3^{n-1} - 1}{8 \cdot 3^{n-3}}$

(6) $n = 15$ (8) 20 months (9) 2480 metres (10) $120, 480, 30(2)^n$

(11) $500\left(\frac{11}{10}\right)^n$, 1296.87 (12) 15^{th} day

Exercise 5.4

(1)(i) $\frac{1}{5}\left[1-\frac{x}{5}+\frac{x^2}{25}-\frac{x^3}{125}+\ldots\right]$ $\quad |x|<5$

(ii) $\frac{2}{9}\left[1-2\left(\frac{4x}{3}\right)+3\left(\frac{4x}{3}\right)^2-4\left(\frac{4x}{3}\right)^3+\ldots\right]$ $\quad |x|<\frac{3}{4}$

(iii) $5^{\left(\frac{2}{3}\right)}\left[1+\frac{2}{15}(\mathrm{x})^2-\frac{1}{225}x^4+\frac{4}{81\times 125}x^6+\ldots\right]$ $\quad x^2<5$

(iv) $2^{\left(-\frac{2}{3}\right)}\left[1-\frac{x}{3}+\frac{5}{36}x^2-\frac{5}{81}x^3+\ldots\right]$ $\quad |x|<2$

(2) $(1001)^{\left(\frac{1}{3}\right)} \approx 10.00333$

(5)(i) $1+5x+\frac{25x^2}{2}+\frac{125x^3}{6}+\frac{625x^4}{24}+\frac{625x^5}{24}+\ldots$

(ii) $1-2x+2x^2-\frac{4x^3}{3}+\frac{2x^4}{3}-\frac{4x^5}{15}+\ldots$

(iii) $1+\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{48}x^3+\frac{1}{384}x^4+\ldots$

(6)(i) $4x-8x^2+\frac{64x^3}{3}-\frac{64x^4}{3}+\ldots$ $\quad$ for $|x|<\frac{1}{4}$

(ii) $-2x-\frac{4x^2}{2}-\frac{8x^3}{3}-\frac{16x^4}{4}-\ldots$ $\quad$ for $|x|<\frac{1}{2}$

(iii) $2[3x+\frac{27x^3}{3}+\frac{243x^5}{5}+\frac{2187x^7}{7}+\ldots]$ $\quad$ for $|x|<\frac{1}{3}$

(iv) $-2[2x+\frac{8x^3}{3}+\frac{32x^5}{5}+\frac{128x^7}{7}+\ldots]$ $\quad$ for $|x|<\frac{1}{2}$

(8) $\left(\frac{15}{16}\right)^{\frac{1}{8}} \simeq 0.99196$ $\quad$ (9) $\frac{28}{3}$ $\quad$ (10) $\frac{1}{2}\log_e^{10}$

Exercise 5.5

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
4	4	4	4	2	3	1	3	4	1	1	4	4	2	3

(16)	(17)	(18)	(19)	(20)
2	2	3	3	2

Exercise 6.1

(1)(i) $x^2+y^2=81$ $\quad$ (ii) $\frac{x^2}{81}+\frac{y^2}{36}=1$ $\quad$ (2)(i) $y=2$ $\quad$ (ii) $x=3$

(3) $x^{2/3}+y^{2/3}=a^{2/3}$ $\quad$ (4) $k=-24,$ $\quad b=-\frac{1}{4}$ $\quad$ (5) $x^2+y^2=16$

(6) $x^2+y^2-4x-4y+8=0$ (7) $x^2+y^2-5x+8y+16=0$

(8) $y^2=2x$ (10) $8x^2+36y^2-16x+252y+431=0$

(11) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (12) $9x^2-33x-3y+35=0$

(13) $x^2+y^2-12x+9y+63=0$ (14) $(5-\sqrt{21},3),(5+\sqrt{21},3),(8,-3),(2,-3)$

(15) $\frac{x^2}{25}+\frac{y^2}{9}=1$

Exercise 6.2

(1)(i) $y=5x-4$ (ii) $3x-y=2$ (iii) $2x+3y=5$ (iv) $x+\sqrt{3}y=(1+\sqrt{3})$

(3) $10x+3y=25$

(5)(i) $C=\frac{5}{9}(F-32)$ (or) $F=\frac{9}{5}C+32$ (ii) $C=37°$ (iii) $F=100.4°$

(6)(i) 4400metre (ii) $D=3000$metre (iii) $T=22seconds$ (7) $P=1,55,000$

(8) $\sqrt(3)x+y=24$ (9) $3x-4y=12, x-2y=2$ (11) $(13,7),(-11,-3)$

(12)(i) $y=12.5x-150$ (ii) 12seconds (iii) 80seconds

(13)(ii) $x-2y+4=0$ (iii) $2cm$ (iv) $14kg$ (v) $5cm$

(14)(i) $y=-\frac{71}{120}x+14.2, \quad 0\le x\le 24$

(ii) $y=f(\text{x})$ is a periodic function with period 24, $f(\text{x})=f(x+24)$

(15)(i) The minimum length $=3280$units

(ii) $180, 360$ and 540units

(iii) The slope at each turning point is $\frac{9}{40}$

Exercise 6.3

(2) $5x-4y-15=0$ (3)(i) $\frac{8}{5}$ (ii) $\frac{23}{5}$

(4)(i) $x+3y+2=0$ (ii) $4x-3y-7=0$ (5) $x+5y-31=0$

(6)(i) $x+1=0$ (ii) $x-y=0$ (iii) $2x+y+3=0$

(7) $12x+5y+6=0$, and $12x+5y-20=0$ (8) $4x-3y+15=0$, and $4x-3y-25=0$ (9) $2x+3y-18=0$ (10) $7\sqrt{2}$, and $(-3,5)$

(12) (i)$\frac{14}{13}$, (ii)$\frac{5}{2}$ (13)(i) $4x-3y+k=0, k\in\mathbb{R}$ (ii) $3x+4y+k_1=0, k_1\in\mathbb{R}$

(14) $\sqrt{3}x-y-2\sqrt{3}=0$ (15) $A\left(\frac{13}{5},0\right)$ (16) $x+5y=\pm10$

(17) $(0,7)$ (18)(i) $y=\begin{cases}1.50x, & 0\le x\le 10\\ x+5, & x>10\end{cases}$ (ii) ₹45

(19) $y = 5x + 7, \quad y = 5x - 10$

(20) $y + 3 = 0,$ $\quad 2x + y + 3 = 0,$ and $\quad 2x - y - 3 = 0$

Exercise 6.4

(1) $x^2 - xy - 2y^2 + 5x - 3y - 15 = 0$

(6) $3x^2 - 13xy - 10y^2 + 33x + 73y - 126 = 0$ (7)(i) $x + y = 0, 3x - y = 0$

(ii) $3x + 4y - 11 = 0, 2x - y = 0$ (iii) $x + y - 5 = 0, 2x - 3y + 4 = 0$

(10) $y = x$

(11) $p = 6, q = 17 \text{(or)} -\dfrac{67}{6}$ (12) $k = -1$

(13) $k = -5 \text{(or)} -\dfrac{35}{4}$ (14) $\dfrac{8}{5}$ (15) $\sqrt{5}$

Exercise 6.5

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
4	4	3	4	3	2	2	4	2	3	1	2	3	4	2
(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)					
1	3	2	1	2	1	3	1	3	4					

Glossary

Chapter 1 Sets, Relations and Functions

absolute value function or modulus function	மட்டு சார்பு அல்லது எண்ணளவுச் சார்பு
basic relations	அடிப்படைத் தொடர்புகள்
bijective function	இருபுறச் சார்பு
cardinality	செவ்வெண்மை
closed interval	மூடிய இடைவெளி
co-domain	துணைச் சார்பகம்
composition of functions	சார்புகளின் சேர்ப்பு
constant	மாறிலி
constant function	மாறிலிச் சார்பு
dependent variable	சார்ந்த மாறி
dilation	விரிதல்
disjoint set	வெட்டாக் கணம்
domain	சார்பகம்
empty relation	வெற்றுத் தொடர்பு
empty set or void set	வெற்றுகணம்
equivalence relation	சமானத் தொடர்பு
even function	இரட்டைப் படைச் சார்பு
exponential function	அடுக்குக் குறிச் சார்பு
extreme relations	உச்சத் தொடர்புகள்
finite set	முடிவுறு கணம்
greatest integer function	மீப்பெரு முழுஎண் சார்பு
horizontal line test	கிடை மட்டக் கோட்டுச் சோதனை
identity function	சமனிச் சார்பு
image	பிம்பம்
improper subset	தகா உட்கணம்
independent variable	சாரா மாறி
infinite set	முடிவுறா கணம்
inverse	நேர்மாறு
invertible	நேர்மாற்றுத்தன்மை

linear function	நேரியச் சார்பு
logarithmic function	மடக்கைச் சார்பு
odd function	ஒற்றைப் படைச் சார்பு
one-to-one function	ஒன்றுக்கொன்றான சார்பு
onto function	மேற்கோர்த்தல் சார்பு
open interval	திறந்த இடைவெளி
power set	அடுக்கு கணம்
pre-image	முன் பிம்பம்
proper subset	தகு உட்கணம்
range	வீச்சகம்
rational function	விகிதமுறுச் சார்பு
real line	மெய்யெண் கோடு
real-valued function	மெய் மதிப்புச் சார்பு
reciprocal function	தலைகீழ் சார்பு
reflection	பிரதிபலிப்பு
reflexive relation	தற்சுட்டுத் தொடர்பு
set difference	கண வேறுபாடு
signum function	குறியீட்டுச் சார்பு
singleton set	ஓருறுப்பு கணம்
smallest integer function	மீச்சிறு முழு எண் சார்பு
step functions	படி நிலைச் சார்புகள்
subset	உட்கணம்
symmetric difference	சமச்சீர் வேறுபாடு
transitive relation	கடப்புத் தொடர்பு
translation	இடப்பெயர்ச்சி
trivial subset	வெள்ளிடை உட்கணம்
union of sets	சேர்ப்புக் கணம்
universal relation	அனைத்துத் தொடர்பு
variable	மாறி
vertical line test	நிலைக் குத்துக் கோட்டுச் சோதனை

zero function	பூஜ்ஜியச் சார்பு
set	கணம்
super set	மேற்கணம் அல்லது மிகைக் கணம்

Chapter 2 Basic Algebra

conjugate	இணை
discriminant	தன்மைக் காட்டி
division algorithm	வகுத்தல் கோட்பாடு
exponents	அடுக்குக் குறி
inequality	அசமன்பாடு
partial fraction	பகுதிப் பின்னம்
polynomial expression	பல்லுறுப்புக் கோவை
quartic	நான்காம் படி
quintic	ஐந்தாம் படி
radical	படிமூலம்
rational inequality	விகிதமுறு அசமன்பாடு
remainder theorem	மீதித் தேற்றம்

Chapter 3 Trigonometry

allied angles	தொடர்புடைக் கோணங்கள்
angles in standard position	திட்டநிலையில்கோணங்கள்
attitude of a triangle	முக்கோணத்தின் உயரம்
centesimal system	நூற்றின் கூறுமுறை
chord	நாண்
circular system	வட்ட முறைஅமைப்பு
circum centre	சுற்றுவட்ட மையம்
circum circle	சுற்றுவட்டம்
circum radius	சுற்றுவட்டத்தின் ஆரம்
complementary angles	நிரப்புக் கோணங்கள்
compound angles	கூட்டுக் கோணங்கள்
conjugate angles	இணையியக் கோணங்கள்

general solution	பொதுத் தீர்வு
interval	இடைவெளி
inverse trigonometric functions	நேர்மாறு முக்கோணவியல் சார்புகள்
periodicity	காலமுறைப் பண்பு
power reducing identities	அடுக்குக் குறைப்பு முற்றொருமைகள்
principal solution	முதன்மைத் தீர்வு
projection	வீழல்
quadrant	காற்பகுதி
radian	ரேடியன்
sector	வட்டக்கோணப்பகுதி
sub-multiple angles	உட்மடங்குக் கோணங்கள்
supplementary angles	மிகை நிரப்புக்கோணங்கள்
trigonometric identities	முக்கோணவியல் முற்றொருமைகள்
trigonometric ratios	முக்கோணவியல் விகிதங்கள்

Chapter 4 Combinatorics

cryptography	குறியாக்கவியல்
factorial	காரணியப் பெருக்கம்
heptagon	எழுகோணம்
inclusion-exclusion	சேர்த்தல்–நீக்கல்
inductive step	தொகுக்கும் நிலை
mathematical induction	கணிதத் தொகுத்தறிதல் முறை
pentagon	ஐங்கோணம்
permutation	வரிசைமாற்றம்
polygon	பலகோணம்
product rule	பெருக்கல்விதி
string method	கட்டுதல் முறை
sum rule	கூட்டல் விதி

Chapter 5 Sequences and Series

arithmetic mean	கூட்டுச் சராசரி
arithmetic progression	கூட்டுத் தொடர்முறை
arithmetic series	கூட்டுத் தொடர்
arithmetico–geometric progression	கூட்டுப் பெருக்குத் தொடர்முறை
arithmetico–geometric series	கூட்டுப் பெருக்குத்தொடர்
binomial coefficients	ஈருறுப்புக் கெழுக்கள்
binomial expansion	ஈருறுப்பு விரிவு
binomial series	ஈருறுப்புத் தொடர்
binomial theorem	ஈருறுப்புத் தேற்றம்
common difference	பொது வித்தியாசம்
common ratio	பொது விகிதம்
convergent series	ஒருங்குத் தொடர்
exponential series	அடுக்குக்குறித் தொடர்
finite sequence	முடிவுறுத் தொடர்முறை
finite series	முடிவுறு தொடர்
geometric mean	பெருக்குச் சராசரி
geometric progression	பெருக்குத் தொடர்முறை
geometric series	பெருக்குத் தொடர்
harmonic mean	இசைச் சராசரி
harmonic progression	இசைத் தொடர்முறை
infinite sequence	முடிவுறாத் தொடர்முறை
infinite series	முடிவுறாத் தொடர்
initial term	முதல் உறுப்பு
logarithmic series	மடக்கைத்தொடர்
partial sum	பகுதிக் கூடுதல்
rational exponent	விகிதமுறு அடுக்கு
telescopic summation	தொலைநோக்கிக் கூடுதல்

Chapter 6 Two Dimensional Analytical Geometry

angle of inclination	சாய்வுக் கோணம்
arbitrary constant	மாறத்தக்க மாறிலி
bisector	இருசமவெட்டி
equilibrium	சமநிலை
fixed constant	நிலையான மாறிலி
fixed point	நிலைப்புள்ளி
general form	பொது வடிவம்
homogenous equation	சமப்படித்தான சமன்பாடு
intercept	வெட்டுத்துண்டு
locus	நியமப் பாதை (அ) இயங்குவரை
negative intercept	குறை வெட்டுத்துண்டு
negative slope	குறை சாய்வு
non–homogenous equation	அசமப்படித்தான சமன்பாடு
normal form	செங்குத்து வடிவம்
parameter	துணையலகு
parametric form	துணையலகு வடிவம்
positive intercept	மிகை வெட்டுத்துண்டு
positive slope	மிகை சாய்வு
symmetry	சமச்சீர்
two point form	இருபுள்ளிகள் வடிவம்

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Mathematics – Class XI

List of Authors and Reviewers

Domain Experts

Dr. K. Srinivasan
Professor & Head (Rtd.),
Department of Mathematics,
Presidency College (Autonomous), Chennai.

Dr. R. Murthy
Principal (Rtd.),
Govt. Arts & Science College, Uthiramerur.

Dr. E. Chandrasekaran
Professor in Mathematics,
Vel Tech Rangarajan Dr. Sagunthala
R&D Institute of Science and Technology
(Deemed to be University), Avadi, Chennai.

Dr. G.P. Youvaraj
Director,
Ramanujam Institute for Advanced Study in
Mathematics, University of Madras, Chennai.

Dr. T.N. Shanmugam
Professor of Mathematics (Rtd.),
Anna University, Chennai.

Dr. R. Sivaraman
Associate Professor,
Department of Mathematics, D.G. Vaishnav College,
Arumbakkam, Chennai.

Content Experts

Dr. R. Vembu
Associate Professor,
SBK College, Aruppukkottai.

Dr. G. Palani
Assistant Professor,
Dr. Ambedkar Govt. Arts College, Vyasarpadi,
Chennai.

Dr. S. Sivasubramanian
Associate Professor of Mathematics,
University College of Engineering, Tindivanam
Anna University.

Advisors

Dr. P. Veeramani
Professor, Department of Mathematics,
Indian Institute of Technology – Madras,
Chennai – 600 036.

Dr. Arindama Singh
Professor, Department of Mathematics,
Indian Institute of Technology–Madras,
Chennai – 600 036.

Dr. Jonaki B. Ghosh
Professor, Department of Education,
Lady Shri Ram College, NewDelhi.

Dr. Shailesh Shirali
Director, Sahayadri School–Higher Education,
KFI (Krishnamurthi Foundation India),

Dr. Shashidhar Jagadeeshan, (NGO)
Faculty, Centre for Learning, Bengaluru.

Academic Coordinator

B. Tamilselvi
Deputy Director, SCERT, Chennai.

Authors

N. Kalaiselvam
PGT, Chennai Girls HSS,
Nungambakkam, Chennai.

A. Balamurugan
PGT, Adhiyaman GBHSS, Dharmapuri.

R. Villavankothai
PGT, GGHSS, Keelmanambedu, Tiruvallur District.

C.S. Veeraragavan
PGT, Sri Krishnaa Matric. Hr. Sec. School,
TVS Nagar, Coimbatore.

Dr. K. Venkateswaran
Academic Director,
AVB Matric HSS, Coimbatore.

P. Hemalatha
Head of the Department (Mathematics),
PGT, Velammal Matric. Hr. Sec. School,
T.S. Krishna Nagar, Mogappair East, Chennai.

Dr. P. Revathy
Vice Principal, P.S. Senior Secondary School, Mylapore,
Chennai.

R. Maheswari
PGT, Sitadevi HSS, East Tambaram, Chennai.

S. Sandhiya
PGT, Sir M.Venkatasubba Rao MHSS, T.Nagar, Chennai.

Coordinator

V.Ramaprabha
Senior Lecturer, DIET, Tirur,Tiruvallur Dt.

K.P. Sujatha
BT Asst., SCERT, Chennai.

Content Readers

Dr. M.P. Jeyaraman
Assistant Professor in Maths
L.N. Govt. Arts College, Ponneri.

Dr. N. Geetha
Assistant Professor in Maths
L.N. Govt. Arts College, Ponneri.

P. Divya Prabha
PGT Maths, Govt. HSS,
Arumbakkam, Chennai.

S.Vijayalakshmi
BT Asst,G.H.S.School,Kuvathur,Kanchipuram Dt.

Art and Design Team

Chief Co-ordinator and Creative Head

SrinivasanNatarajan

Illustrator

G. Achuthan
Art Teacher, JG Hindu Vidyalaya MHSS
West Mambalam, Chennai.

Wrapper Design

Kathir Arumugam

Layout

WinMac Solutions

In-House

QC
Manohar Radhakrishnan
Gopu Rasuvel, Tamil Kumaran,
Jerald Wilson

Coordination

Ramesh Munisamy

Typist

P. Jeyanthi
Typist, DIET, Mannargudi.

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