CHAPTER

3

Trigonometric Equations

- Trigonometric Equations
- General Solutions of Some Standard Equations
- Problems Based on Extreme Values of Functions
- Inequalities

TRIGONOMETRIC EQUATIONS

An equation involving one or more trigonometrical ratios of unknown angle is called a trigonometric equation, e.g., $\cos^2 x - 4 \sin x = 1$. It is to be noted that a trigonometrical identity is satisfied for every value of the unknown angle, whereas a trigonometric equation is satisfied only for some values (finite or infinite in number) of unknown angle, e.g., $\sin^2 x + \cos^2 x = 1$ is a trigonometrical identity as it is satisfied for every value of $x \in R$.

Solution or Root of a Trigonometric Equation

The value of an unknown angle which satisfies the given trigonometric equation is called a solution or root of the equation. For example, $2 \sin\theta = 1$, clearly $\theta = 30°$ and $\theta = 150°$ satisfies the equation; therefore, 30° and 150° are solutions of the equation $2 \sin\theta = 1$ between 0° and 360°.

Principal Solution of a Trigonometric Equation

The solutions of a trigonometric equation lie in the interval $[0, 2\pi)$. For example, $\sin\theta = 1/2$, then the two values of θ between 0 and 2π are $\pi/6$ and $5\pi/6$. Thus, $\pi/6$ and $5\pi/6$ are the principal solutions of equation $\sin \theta = 1/2$.

General Solution of a Trigonometric Equation

It is known that trigonometric ratios are periodic functions. In fact, $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are periodic functions with a period 2π, and $\tan x$ and $\cot x$ are periodic functions with a period π. Therefore, solutions of trigonometric equations can be generalized with the help of period of trigonometric functions. The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

Clearly, general solution of a trigonometric equation will involve integral $n \in Z$. General solution of a trigonometric equation is also called a 'solution'.

Here set of all integers is denoted by Z. $n \in Z$ means $n = 0, \pm 1, \pm 2, \ldots$. For example, general solution of the equation $\cos\theta = 1$ is $\theta = 2n\pi$.

Some Important General Solutions of Equations

Equation	Solution
$\sin\theta = 0$	$\theta = n\pi, n \in Z$
$\cos\theta = 0$	$\theta = (2n+1)\dfrac{\pi}{2}, n \in Z$
$\tan\theta = 0$	$\theta = n\pi, n \in Z$
$\sin\theta = 1$	$\theta = (4n+1)\dfrac{\pi}{2}, n \in Z$
$\sin\theta = -1$	$\theta = (4n-1)\dfrac{\pi}{2}, n \in Z$

Equation	Solution
$\cos\theta = 1$	$\theta = 2n\pi, n \in Z$
$\cos\theta = -1$	$\theta = (2n+1)\pi, n \in Z$
$\cot\theta = 0$	$\theta = (2n+1)\frac{\pi}{2}, n \in Z$

Points to Remember

1. While solving a trigonometric equation, squaring the equation at any step should be avoided as far as possible. If squaring is necessary, check the solution for extraneous values. Also see Example 3.1 for explanation.
2. Never cancel terms containing unknown terms on the two sides which are in product. It may cause the loss of a genuine solution.
3. The answer should not contain such values of angles which make any of the terms undefined or infinite. Also see Example 3.2 for explanation.
4. Domain should not change while simplifying the equation. If it changes, necessary corrections must be made.
5. Check that denominator is not zero at any stage while solving the equations.

Example 3.1 **Solve the equation $\sin x + \cos x = 1$.**

Sol. If we square we have $(\sin x + \cos x)^2 = 1$

$\Rightarrow \quad 1 + \sin 2x = 1$

$\Rightarrow \quad \sin 2x = 0$

$\Rightarrow \quad 2x = n\pi, n \in Z$

$\Rightarrow \quad x = n\pi/2, n \in Z$

But for $n = 2, 6, 10, \ldots$

$\sin x + \cos + = -1$ which contradicts the given equation.

Also for $x = 3, 7, 11, \ldots$

$\sin x + \cos x = -1$

Hence, the solution is $x = 2n\pi$ or $x = (4n+1)\frac{\pi}{2}$.

Example 3.2 **Solve $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$.**

Sol. $\tan(3x - 2x) = \tan x = 1$

Therefore, $x = n\pi + (\pi/4)$ but this values does not define $\tan 2x$. Hence, there is no solution.

Example 3.3 **Find the values of θ which satisfy $r\sin\theta = 3$ and $r = 4(1 + \sin\theta)$, $0 \le \theta \le 2\pi$.**

Sol. $0 \le \theta \le 2\pi$

Eliminating r, we have $4\sin^2\theta + 4\sin\theta - 3 = 0$

$\Rightarrow \quad \sin\theta = \frac{1}{2}, -\frac{3}{2}$ (not possible) $\qquad \Rightarrow \quad \theta = \frac{\pi}{6}, \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Example 3.4 **Solve $16^{\sin^2 x} + 16^{\cos^2 x} = 10, 0 \le x < 2\pi$.**

Sol. $16^{\sin^2 x} + 16^{1-\sin^2 x} = 10$ (i)

If $16^{\sin^2 x} = t$, then $t + \frac{16}{t} = 10$

Then Eq. (i) becomes

$\Rightarrow \quad t^2 - 10t + 16 = 0$

$\Rightarrow \quad t = 2, 8$

$\Rightarrow \quad 16^{\sin^2 x} = 16^{1/4}$ or $16^{3/4}$

$\Rightarrow \quad \sin x = \pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}$

Now $\sin x = \frac{1}{2}$, then $x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\sin x = -\frac{1}{2}$, then $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

$\sin x = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad x = \frac{\pi}{3}, \frac{2\pi}{3}$

$\sin x = -\frac{\sqrt{3}}{2} \quad \Rightarrow \quad x = \frac{4\pi}{3}, \frac{5\pi}{3}$

Hence, there will be eight solutions in all.

Example 3.5 **Find general value of θ which satisfies both $\sin\theta = -1/2$ and $\tan\theta = 1/\sqrt{3}$, simultaneously.**

Sol. Here $\sin\theta < 0$ and $\tan\theta > 0$, then θ lies in the third quadrant.

Now $\sin\theta = -\frac{1}{2} \quad \Rightarrow \quad \theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

Generalizing, we have $\theta = 2n\pi + \frac{7\pi}{6}, n \in Z$.

Example 3.6 **If $\sin A = \sin B$ and $\cos A = \cos B$, then find the value of A in terms of B.**

Sol. $\sin A - \sin B = 0$ and $\cos A - \cos B = 0$

$\Rightarrow \quad 2\sin\frac{A-B}{2}\cos\frac{A+B}{2} = 0$ and $2\sin\frac{A+B}{2}\sin\frac{B-A}{2} = 0$

We observe that the common factor gives $\sin\frac{A-B}{2} = 0$

$\Rightarrow \quad \frac{A-B}{2} = n\pi, n \in Z$

$\Rightarrow \quad A - B = 2n\pi, n \in Z$

$\Rightarrow \quad A = 2n\pi + B, n \in Z$

Example 3.7 **Find the number of solutions of $\sin^2 x - \sin x - 1 = 0$ in $[-2\pi, 2\pi]$.**

Sol. $\sin^2 x - \sin x - 1 = 0$

$$\Rightarrow \sin x = \frac{1 \pm \sqrt{5}}{2}$$

$$= \frac{1-\sqrt{5}}{2} \quad \left[\sin x = \frac{1+\sqrt{5}}{2} > 1 \text{ not possible}\right]$$

$\Rightarrow$ x can attain two values in $[0, 2\pi]$ and two more values in $[-2\pi, 0)$. Thus, there are four solutions.

Example 3.8 **Find the number of solutions of the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$.**

Sol. Put $e^{\sin x} = t \Rightarrow t^2 - 4t - 1 = 0$

$\Rightarrow t = e^{\sin x} = 2 \pm \sqrt{5}$

Now $\sin x \in [-1, 1]$

$\Rightarrow e^{\sin x} \in [e^{-1}, e^{1}]$ and $2 \pm \sqrt{5} \notin [e^{-1}, e^{1}]$

Hence, there does not exist any solution.

Example 3.9 **If the equation $a \sin x + \cos 2x = 2a - 7$ possesses a solution, then find the values of a.**

Sol. The given equation can be written as $a\sin x + (1 - 2\sin^2 x) = 2a - 7$

$\Rightarrow 2\sin^2 x - a\sin x + 2a - 8 = 0$

$$\Rightarrow \sin x = \frac{a \pm \sqrt{a^2 - 8(2a-8)}}{4} = \frac{a \pm (a-8)}{4}$$

$= (a-4)/2$ (∵ $\sin x = 2$ is not possible)

Equation has solution if $-1 \le (a-4)/2 \le 1$

$\Rightarrow -2 \le (a-4) \le 2$

$\Rightarrow 2 \le a \le 6$

Concept Application Exercise 3.1

1. Solve $\sin^2\theta - \cos\theta = \frac{1}{4}, 0 \le \theta \le 2\pi$.
2. Solve $\cos^7 x + \sin^4 x = 1$ in the interval $(-\pi, \pi)$.
3. Find the general solution of $(1 - 2\cos\theta)^2 + (\tan\theta + \sqrt{3})^2 = 0$.
4. Solve $\sin 3\theta - \sin\theta = 4\cos^2\theta - 2$.

GENERAL SOLUTION OF SOME STANDARD EQUATIONS

General Solution of the Equation $\sin\theta = \sin\alpha$

Given, $\sin\theta = \sin\alpha \Rightarrow \sin\theta - \sin\alpha = 0$

$$\Rightarrow 2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$$

$$\Rightarrow \cos\frac{\theta+\alpha}{2} = 0 \text{ or } \sin\frac{\theta-\alpha}{2} = 0$$

$$\Rightarrow \frac{\theta+\alpha}{2}=(2m+1)\frac{\pi}{2}, \frac{\theta-\alpha}{2}=m\pi, m\in Z$$

$$\Rightarrow \theta=(2m+1)\pi-\alpha \text{ or } \theta=2m\pi+\alpha, m\in Z$$

$$\Rightarrow \theta=(2m+1)\pi+(-1)^{2m+1}\alpha, m\in Z \quad \text{(i)}$$

$$\text{or} \quad \theta=2m\pi+(-1)^{2m}\alpha, m\in Z \quad \text{(ii)}$$

Combining Eqs. (i) and (ii), we have

$$\theta=n\pi+(-1)^n\alpha, n\in Z$$

Note:

- *For general solution of the equation* $\sin\theta = k$, *where* $-1 \le k \le 1$. *We have* $\sin\theta = \sin(\sin^{-1} k)$ $\Rightarrow \theta = n\pi+(-1)^n(\sin^{-1}k), n\in Z$.

Example 3.10 **Solve $2\cos^2\theta+3\sin\theta=0$.**

Sol. We have $2\cos^2\theta+3\sin\theta=0$

$$\Rightarrow 2(1-\sin^2\theta)+3\sin\theta=0$$

$$\Rightarrow 2\sin^2\theta-3\sin\theta-2=0$$

$$\Rightarrow (\sin\theta-2)(2\sin\theta+1)=0$$

$$\Rightarrow 2\sin\theta+1=0 \quad [\because \sin\theta\neq 2]$$

$$\Rightarrow \sin\theta=-\frac{1}{2}=\sin\left(\frac{-\pi}{6}\right)$$

$$\Rightarrow \theta=n\pi+(-1)^n\left(\frac{-\pi}{6}\right), n\in Z$$

$$=n\pi+(-1)^{n+1}\frac{\pi}{6}, n\in Z$$

Example 3.11 **Solve $4\cos\theta-3\sec\theta=\tan\theta$.**

Sol. We have $4\cos\theta-3\sec\theta=\tan\theta$

$$\Rightarrow 4\cos\theta-\frac{3}{\cos\theta}=\frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow 4\cos^2\theta-3=\sin\theta$$

$$\Rightarrow 4(1-\sin^2\theta)-3=\sin\theta$$

$$\Rightarrow 4\sin^2\theta+\sin\theta-1=0$$

$$\Rightarrow \sin\theta=\frac{-1\pm\sqrt{1+16}}{8}$$

$$=\frac{-1\pm\sqrt{17}}{8}$$

$$=\frac{-1+\sqrt{17}}{8} \text{ or } =\frac{-1-\sqrt{17}}{8}$$

Now, $\sin\theta = \frac{-1+\sqrt{17}}{8}$

$\Rightarrow \quad \sin\theta = \sin\alpha$, where $\sin\alpha = \frac{-1+\sqrt{17}}{8}$

$\Rightarrow \quad \theta = n\pi + (-1)^n\alpha$, where $\sin\alpha = \frac{-1+\sqrt{17}}{8}$ and $n \in Z$

and $\sin\theta = \frac{-1-\sqrt{17}}{8}$

$\Rightarrow \quad \sin\theta = \sin\beta$, where $\sin\beta = \frac{-1-\sqrt{17}}{8}$

$\Rightarrow \quad \theta = n\pi + (-1)^n\beta$, where $\sin\beta = \frac{-1-\sqrt{17}}{8}$

Example 3.12 **Solve $\sin^3\theta\cos\theta - \cos^3\theta\sin\theta = 1/4$.**

Sol. $\sin^3\theta\cos\theta - \cos^3\theta\sin\theta = 1/4$

$\Rightarrow \quad 4\sin\theta\cos\theta(\sin^2\theta - \cos^2\theta) = 1$

$\Rightarrow \quad 2\sin 2\theta(-\cos 2\theta) = 1$

$\Rightarrow \quad -\sin 4\theta = 1$

$\Rightarrow \quad \sin 4\theta = -1$

$\Rightarrow \quad 4\theta = 2n\pi - \frac{\pi}{2}$

$\Rightarrow \quad \theta = (\pi/2) + (-\pi/8), n \in Z$

Concept Application Exercise 3.2

1. Solve $2\sin\theta + 1 = 0$.
2. Solve $\sin^2 n\theta - \sin^2(n-1)\theta = \sin^2\theta$.

General Solution of Equation $\cos\theta = \cos\alpha$

Given, $\cos\theta = \cos\alpha$

$\Rightarrow \quad \cos\alpha - \cos\theta = 0$

$\Rightarrow \quad 2\sin\frac{\alpha+\theta}{2}\sin\frac{\theta-\alpha}{2} = 0$

$\Rightarrow \quad \sin\frac{\alpha+\theta}{2} = 0$ or $\sin\frac{\theta-\alpha}{2} = 0$

$\Rightarrow \quad \frac{\alpha+\theta}{2} = n\pi$ or $\frac{\theta-\alpha}{2} = n\pi, n \in Z$

$\Rightarrow \quad \theta = 2n\pi - \alpha$ or $\theta = 2n\pi + \alpha, n \in Z$

$\quad = 2n\pi \pm \alpha, n \in Z$

Note:

- *For general solution of the equation* $\sin\theta = k$, *where* $-1 \le k \le 1$. *We have* $\cos\theta = \cos(\cos^{-1}k)$ $\Rightarrow \theta = 2n\pi \pm (\cos^{-1}k), n \in Z$.

Example 3.13 **Solve $\sqrt{3}\sec 2\theta = 2$.**

Sol. We have $\sqrt{3}\sec 2\theta = 2$

$$\Rightarrow \cos 2\theta = \frac{\sqrt{3}}{2}$$

$$= \cos\frac{\pi}{6}$$

$$\Rightarrow 2\theta = 2n\pi \pm \frac{\pi}{6}, n \in Z$$

$$\Rightarrow \theta = n\pi \pm \frac{\pi}{12}, n \in Z$$

Example 3.14 **Solve $\sin 2\theta + \cos\theta = 0$.**

Sol. We have $\sin 2\theta + \cos\theta = 0$

$$\Rightarrow \cos\theta = -\sin 2\theta$$

$$= \cos\left(\frac{\pi}{2} + 2\theta\right)$$

$$\Rightarrow \theta = 2n\pi \pm \left(\frac{\pi}{2} + 2\theta\right) n \in Z$$

Taking positive sign, we have

$$\theta = 2n\pi + \frac{\pi}{2} + 2\theta$$

$$= 2n\pi - \frac{\pi}{2}, n \in Z.$$

Taking negative sign, we have

$$\theta = 2n\pi - \left(\frac{\pi}{2} + 2\theta\right) \quad \Rightarrow \quad \theta = \frac{2n\pi}{3} - \frac{\pi}{6}, n \in Z$$

Example 3.15 **Solve $\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$.**

Sol. We have $\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$

$$\Rightarrow 2\cos 2\theta\cos\theta - 2\cos 2\theta = 0$$

$$\Rightarrow 2\cos 2\theta(\cos\theta - 1) = 0$$

$$\Rightarrow \cos 2\theta = 0 \text{ or, } \cos\theta - 1 = 0$$

$$\Rightarrow 2\theta = (2n+1)\frac{\pi}{2}, n \in Z \text{ or } \theta = 2m\pi, m \in Z$$

$$\Rightarrow \theta = (2n+1)\frac{\pi}{4}, n \in Z \text{ or } \theta = 2m\pi, m \in Z$$

Example 3.16 **Solve $\sec 4\theta - \sec 2\theta = 2$.**

Sol. $\dfrac{1}{\cos 4\theta} - \dfrac{1}{\cos 2\theta} = 2$

$\Rightarrow \cos 2\theta - \cos 4\theta = 2\cos 2\theta \cos 4\theta = \cos 2\theta + \cos 6\theta$

$\Rightarrow \cos 6\theta + \cos 4\theta = 0$

$\Rightarrow 2\cos 5\theta \cos \theta = 0$

$\Rightarrow \cos 5\theta = 0$ or $\cos\theta = 0$

$\Rightarrow 5\theta = (2n+1)\dfrac{\pi}{2}$ or $\theta = (2n+1)\dfrac{\pi}{2}, n \in Z$

$\Rightarrow \theta = \left(n + \dfrac{1}{2}\right)\dfrac{\pi}{5}$ or $\theta = \left(n + \dfrac{1}{2}\right)\pi, n \in Z$

Example 3.17 **Solve $5\cos 2\theta + 2\cos^2 \dfrac{\theta}{2} + 1 = 0, -\pi < \theta < \pi$.**

Sol. Changing all the values in terms of $\cos\theta$, we get

$5(2\cos^2\theta - 1) + (1 + \cos\theta) + 1 = 0 \Rightarrow 10\cos^2\theta + \cos\theta - 3 = 0$

$\Rightarrow (5\cos\theta + 3)(2\cos\theta - 1) = 0$

$\Rightarrow \theta = \dfrac{\pi}{3}, -\dfrac{\pi}{3}, \cos^{-1}\left(-\dfrac{3}{5}\right) = \pi - \cos^{-1}\dfrac{3}{5}$ and $-\pi + \cos^{-1}\dfrac{3}{5}$ $[\because -\pi < \theta < \pi]$

Example 3.18 **Solve $\cos x \cos 2x \cos 3x = 1/4$.**

Sol. $\cos x \cos 2x \cos 3x = 1/4$

$\Rightarrow 2(2\cos x \cos 3x)\cos 2x = 1$

$\Rightarrow 2(\cos 4x + \cos 2x)\cos 2x = 1$

$\Rightarrow 2(2\cos^2 2x - 1 + \cos 2x)\cos 2x = 1$

$\Rightarrow 4\cos^3 2x + 2\cos^2 2x - 2\cos 2x - 1 = 0$

$\Rightarrow (2\cos^2 2x - 1)(2\cos 2x + 1) = 0$

$\Rightarrow \cos 4x(2\cos 2x + 1) = 0$

$\Rightarrow \cos 4x = 0$ or $\cos 2x = -1/2$

$\Rightarrow 4x = (2n+1)\dfrac{\pi}{2}$ or $2x = 2m\pi \pm \dfrac{2\pi}{3}$, $m, n \in Z$

$\Rightarrow x = (2n+1)\dfrac{\pi}{8}$ or $x = m\pi \pm \dfrac{\pi}{3}$

Concept Application Exercise 3.3

1. Solve $\cos\theta = 1/3$.
2. Solve $\tan\theta \tan 4\theta = 1$ for $0 < \theta < \pi$.
3. Solve $\cot(x/2) - \operatorname{cosec}(x/2) = \cot x$.
4. Solve $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta$.
5. Solve $\sin 6\theta = \sin 4\theta - \sin 2\theta$.
6. Solve $\cos\theta + \cos 2\theta + \cos 3\theta = 0$.
7. Solve $\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0, 0 \le \theta \le \pi$.

General Solutions of the Equation $\tan\theta = \tan\alpha$

Given, $\tan\theta = \tan\alpha$

$\Rightarrow \quad \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sin\alpha}{\cos\alpha}$

$\Rightarrow \quad \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$

$\Rightarrow \quad \sin(\theta - \alpha) = 0$

$\Rightarrow \quad \theta - \alpha = n\pi$

$\Rightarrow \quad \theta = n\pi + \alpha$, where $n \in Z$

Note:

- *For general solution of the equation* $\tan\theta = k$, *where* $k \in R$. *We have* $\tan\theta = \tan(\tan^{-1}k)$

  $\Rightarrow \quad \theta = n\pi + (\tan^{-1}k),\ n \in Z$

Example 3.19 **Solve $\tan 3\theta = -1$.**

Sol. $\tan 3\theta = -1$

$= \tan\left(\dfrac{-\pi}{4}\right)$

$\Rightarrow \quad 3\theta = n\pi + \left(\dfrac{-\pi}{4}\right),\ n \in Z$

$\Rightarrow \quad \theta = \dfrac{n\pi}{3} - \dfrac{\pi}{12},\ n \in Z$

Example 3.20 **Solve $2\tan\theta - \cot\theta = -1$.**

Sol. $2\tan\theta - \cot\theta = -1$

$\Rightarrow \quad 2\tan\theta - \dfrac{1}{\tan\theta} = -1$

$\Rightarrow \quad 2\tan^2\theta + \tan\theta - 1 = 0$

$\Rightarrow \quad (\tan\theta + 1)(2\tan\theta - 1) = 0$

$\Rightarrow \quad \tan\theta = -1$ or $\tan\theta = \dfrac{1}{2}$

$\Rightarrow \quad \tan\theta = \tan\left(\dfrac{-\pi}{4}\right)$ or $\tan\theta = \tan\left(\tan^{-1}\dfrac{1}{2}\right)$

$\Rightarrow \quad \theta = n\pi + \left(\dfrac{-\pi}{4}\right)$ or $\theta = m\pi + \alpha$, where $m, n \in Z$ and $\tan\alpha = \dfrac{1}{2}$

Example 3.21 **Solve $\tan 5\theta = \cot 2\theta$.**

Sol. $\tan 5\theta = \cot 2\theta = \tan\left(\dfrac{\pi}{2} - 2\theta\right)$

$\Rightarrow \quad 5\theta = n\pi + \dfrac{\pi}{2} - 2\theta$

$\Rightarrow\ 7\theta = n\pi + \frac{\pi}{2}$

$\Rightarrow\ \theta = \frac{n\pi}{7} + \frac{\pi}{14}$, where $n \in Z$, but $n \neq 3, 10, 17, \ldots$ where $\tan 5\theta$ is not defined

Example 3.22 **Solve $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$.**

Sol. $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$

$\Rightarrow\ \tan\theta + \tan 2\theta = \sqrt{3}(1 - \tan\theta\tan 2\theta)$

$\Rightarrow\ \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = \sqrt{3}$

$\Rightarrow\ \tan 3\theta = \sqrt{3}$

$= \tan\frac{\pi}{3}$

$\Rightarrow\ 3\theta = n\pi + \frac{\pi}{3}, n \in Z$

$\Rightarrow \theta = \frac{n\pi}{3} + \frac{\pi}{9}, n \in Z$

Concept Application Exercise 3.4

1. If $\tan a\theta - \tan b\theta = 0$, then prove that the values of θ forms an A.P.
2. Solve $\tan^2\theta + 2\sqrt{3}\tan\theta = 1$.
3. Solve $\tan^2 x + (1 - \sqrt{3})\tan x - \sqrt{3} = 0$.
4. Solve $3\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta - 3\sin^2\theta = 0$.
5. Solve $\tan\theta + \tan(\theta + \pi/3) + \tan(\theta + 2\pi/3) = 3$.
6. Solve $2\sin^3 x = \cos x$.

General Solutions of the Equation $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$

Here the both given the equations are same as $\cos^2\theta = \cos^2\alpha$

$\Rightarrow\ (1 - \sin^2\theta) - (1 - \sin^2\alpha) = 0$

$\Rightarrow\ \sin^2\theta = \sin^2\alpha$

$\Rightarrow\ \sin(\theta + \alpha)\sin(\theta - \alpha) = 0$

$\Rightarrow\ \sin(\theta + \alpha) = 0$ or $\sin(\theta - \alpha) = 0$

$\Rightarrow\ \theta + \alpha = n\pi$ or $\theta - \alpha = n\pi, n \in Z$

$\Rightarrow\ \theta = n\pi \pm \alpha, n \in Z$

General Solutions of the Equation $\tan^2\theta = \tan^2\alpha$

$\tan^2\theta = \tan^2\alpha \quad \Rightarrow \quad \tan\theta = \pm\tan\alpha \quad \Rightarrow \quad \tan\theta = \tan(\pm\alpha) \quad \Rightarrow \quad \theta = n\pi \pm \alpha, n \in Z$

Example 3.23 **Solve $7\cos^2\theta + 3\sin^2\theta = 4$.**

Sol. We have $7\cos^2\theta + 3\sin^2\theta = 4$

$\Rightarrow\ 7(1 - \sin^2\theta) + 3\sin^2\theta = 4$

$\Rightarrow \quad 4\sin^2\theta = 3$

$\Rightarrow \quad \sin^2\theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$

$\Rightarrow \quad \sin^2\theta = \sin^2\frac{\pi}{3}$

$\Rightarrow \quad \theta = n\pi \pm \frac{\pi}{3}, n \in Z$

Example 3.24 **Solve $2\sin^2 x + \sin^2 2x = 2$.**

Sol. We have $2\sin^2 x + \sin^2 2x = 2$

$\Rightarrow \quad 2\sin^2 x + (2\sin x\cos x)^2 = 2$

$\Rightarrow \quad 2\sin^2 x\cos^2 x + \sin^2 x = 1$

$\Rightarrow \quad 2\sin^2 x\cos^2 x - (1 - \sin^2 x) = 0$

$\Rightarrow \quad 2\sin^2 x\cos^2 x - \cos^2 x = 0$

$\Rightarrow \quad \cos^2 x(2\sin^2 x - 1) = 0$

$\Rightarrow \quad \cos^2 x = 0 \text{ or } \sin^2 x = \frac{1}{2}$

$\Rightarrow \quad x = 2n\pi + \frac{\pi}{2} \text{ or } \sin^2 x = \sin^2\frac{\pi}{4}$

$= 2n\pi + \frac{\pi}{2} \text{ or } x = m\pi \pm \frac{\pi}{4}, m \in Z$, where $m, n \in Z$

Example 3.25 **Solve $4\cot 2\theta = \cot^2\theta - \tan^2\theta$.**

Sol. $\frac{4}{\tan 2\theta} = \frac{1}{\tan^2\theta} - \tan^2\theta$

$\Rightarrow \quad \frac{4(1-\tan^2\theta)}{2\tan\theta} = \frac{1-\tan^4\theta}{\tan^2\theta} \quad \left[\text{put } \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}\right]$

$\Rightarrow \quad (1-\tan^2\theta)[2\tan\theta - (1+\tan^2\theta)] = 0$

$\Rightarrow \quad (1-\tan^2\theta)(\tan^2\theta - 2\tan\theta + 1) = 0$

$\Rightarrow \quad (1-\tan^2\theta)(\tan\theta - 1)^2 = 0$

$\Rightarrow \quad \tan\theta = \pm 1$

$\Rightarrow \quad \theta = n\pi \pm \frac{\pi}{4}, n \in Z$

Example 3.26 **Find the most general solution of $2^{1 + |\cos x| + \cos^2 x + |\cos x|^3 + \cdots \infty} = 4$.**

Sol. We have $2^{1 + |\cos x| + \cos^2 x + |\cos x|^3 + \cdots \infty} = 4$

$\Rightarrow \quad 2^{1 + |\cos x| + |\cos x|^2 + |\cos x|^3 + \cdots \infty} = 4$

$\Rightarrow \quad 2^{\frac{1}{1-|\cos x|}} = 2^2$

$\Rightarrow \quad \dfrac{1}{1-|\cos x|}=2$

$\Rightarrow \quad |\cos x|=\dfrac{1}{2}$ or $\cos x=\pm\dfrac{1}{2}$

$\Rightarrow x=n\pi\pm\dfrac{\pi}{3}, n\in Z$

Concept Application Exercise 3.5

1. Solve $\tan^2\theta+\cot^2\theta=2$.
2. Solve $3(\sec^2\theta+\tan^2\theta)=5$.
3. Solve $4\cos^2x+6\sin^2x=5$.

Solutions of Equations of the Form $a\cos\theta + b\sin\theta = c$

To solve equation, let us convert the equation to the form $\cos\theta=\cos\alpha$ or $\sin\theta=\sin\alpha$, etc.

For this let us suppose that $\left.\begin{array}{l} a=r\cos\phi \\ b=r\sin\phi \end{array}\right\} \Rightarrow \left\{\begin{array}{l} r=\sqrt{a^2+b^2} \\ \tan\phi=\dfrac{b}{a} \end{array}\right.$ and

Substituting these values in the equation $a\cos\theta+b\sin\theta=c$, we have
$r\cos\phi\cos\theta+r\sin\phi\sin\theta=c$

$\Rightarrow \quad r\cos(\theta-\phi)=c$

$\Rightarrow \quad \cos(\theta-\phi)=\dfrac{c}{r}=\dfrac{c}{\sqrt{a^2+b^2}}=\cos\beta$ (suppose)

$\Rightarrow \quad \theta-\phi=2n\pi\pm\beta$

$\Rightarrow \quad \theta=2n\pi+\phi\pm\beta, n\in Z$

Here ϕ and β are known as a, b and c are given.
Hence, we can solve the equation of this type by putting

$a=r\cos\phi$ and $b=r\sin\phi$, provided $\left|\dfrac{c}{\sqrt{a^2+b^2}}\right|\le 1$ [$\because$ $\cos\beta$ lies between -1 and 1]

or $\dfrac{|c|}{\sqrt{a^2+b^2}}\le 1$ or $|c|\le\sqrt{a^2+b^2}$

WORKING RULES for solving such equations

1. First of all check whether $|c|\le\sqrt{a^2+b^2}$ or not.
2. If $|c|>\sqrt{a^2+b^2}$, then the given equation has no real solution.
3. If $|c|\le\sqrt{a^2+b^2}$, then divide both sides of the equation by $\sqrt{a^2+b^2}$.

4. Take $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$, then the given equation will become $\cos(\theta-\alpha) = \cos\beta$, where $\tan\alpha = \frac{b}{a}$ and $\cos\beta = \frac{c}{\sqrt{a^2+b^2}}$.

We can also take $\sin\alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\cos\alpha = \frac{b}{\sqrt{a^2+b^2}}$ and $\sin\beta = \frac{c}{\sqrt{a^2+b^2}}$, then the given equation will reduce to the form $\sin(\theta+\alpha) = \sin\beta$.

Example 3.27 **Solve $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$.**

Sol. We have $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$ (i)

This is of the form $a\cos\theta + b\sin\theta = c$, where $a = \sqrt{3}$, $b = 1$ and $c = \sqrt{2}$

Let $\sqrt{3} = r\cos\alpha$ and $1 = r\sin\alpha$

$$\Rightarrow \quad r = \sqrt{a^2+b^2} = \sqrt{(\sqrt{3})^2+1^2} = 2 \text{ and } \tan\alpha = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \alpha = \frac{\pi}{6}$$

Substituting $\sqrt{3} = r\cos\alpha$ and $1 = r\sin\alpha$ in Eq. (i), it reduces to $r\cos\alpha\cos\theta + r\sin\theta = \sqrt{2}$

$$\Rightarrow \quad r\cos(\theta-\alpha) = \sqrt{2}$$

$$\Rightarrow \quad 2\cos\left(\theta-\frac{\pi}{6}\right) = \sqrt{2}$$

$$\Rightarrow \quad \cos\left(\theta-\frac{\pi}{6}\right) = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$$

$$\Rightarrow \quad \theta-\frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}, n \in Z$$

$$\Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}, n \in Z$$

$$= 2n\pi + \frac{\pi}{4} + \frac{\pi}{6} \text{ or, } \theta = 2n\pi - \frac{\pi}{4} + \frac{\pi}{6}$$

$$= 2n\pi + \frac{5\pi}{12} \text{ or, } \theta = 2n\pi - \frac{\pi}{12}, \text{ where } n \in Z$$

Example 3.28 **Solve $\sqrt{3}\cos\theta - 3\sin\theta = 4\sin 2\theta\cos 3\theta$.**

Sol. We have $\sqrt{3}\cos\theta - 3\sin\theta = 2(\sin 5\theta - \sin\theta)$

$$\Rightarrow \quad (\sqrt{3}/2)\cos\theta - (1/2)\sin\theta = \sin 5\theta$$

$\Rightarrow \cos(\theta + \pi/6) = \sin 5\theta = \cos(\pi/2 - 5\theta)$

$\Rightarrow \theta + \pi/6 = 2n\pi \pm (\pi/2 - 5\theta)$

$\Rightarrow \theta = n\pi/3 + \pi/18$ or $\theta = -n\pi/2 + \pi/6, \forall\ n \in Z$

Example 3.29 **Find the total number of integral values of n so that $\sin x(\sin x + \cos x) = n$ has at least one solution.**

Sol. $\sin x(\sin x + \cos x) = n$

$\Rightarrow \sin^2 x + \sin x \cos x = n$

$$\Rightarrow \frac{1-\cos 2x}{2} + \frac{\sin 2x}{2} = n$$

$\Rightarrow \sin 2x - \cos 2x = 2n - 1$

$\Rightarrow -\sqrt{2} \le 2n - 1 \le \sqrt{2}$

$$\Rightarrow \frac{1-\sqrt{2}}{2} \le n \le \frac{1+\sqrt{2}}{2}$$

$\Rightarrow n = 0, 1$

Concept Application Exercise 3.6

1. Solve $\cot\theta + \operatorname{cosec}\theta = \sqrt{3}$.
2. Solve $\sin\theta + \cos\theta = \sqrt{2}\cos A$.
3. Solve $\sqrt{2}\sec\theta + \tan\theta = 1$.
4. Find the number of integral values of k for which the equation $7\cos x + 5\sin x = 2k + 1$ has at least one solution.

PROBLEMS BASED ON EXTREME VALUES OF FUNCTIONS

Example 3.30 **If $x, y \in [0, 2\pi]$, then find the total number of ordered pairs (x, y) satisfying the equation $\sin x \cos y = 1$.**

Sol. $\sin x \cos y = 1$

$\Rightarrow \sin x = 1, \cos y = 1$ or $\sin x = -1, \cos y = -1$

If $\sin x = 1, \cos y = 1 \Rightarrow x = \pi/2, y = 0, 2\pi$

If $\sin x = -1, \cos y = -1 \Rightarrow x = 3\pi/2, y = \pi$

Thus, the possible ordered pairs are $\left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 2\pi\right)$ and $\left(\frac{3\pi}{2}, \pi\right)$.

Example 3.31 **If $3\sin x + 4\cos ax = 7$ has at least one solution, then find the possible values of a.**

Sol. We have $3\sin x + 4\cos ax = 7$ which is possible only when $\sin x = 1$ and $\cos ax = 1$

$\Rightarrow x = (4n+1)\frac{\pi}{2}$ and $ax = 2m\pi$; $m, n \in Z$

$$\Rightarrow (4n+1)\frac{\pi}{2} = \frac{2m\pi}{a}$$

$$\Rightarrow a = \frac{4m}{4n+1}$$

Example 3.32 **Solve $\cos^{50} x - \sin^{50} x = 1$.**

Sol. $\cos^{50} x - \sin^{50} x = 1 \quad \Rightarrow \quad \cos^{50} x = 1 + \sin^{50} x$

L.H.S. ≤ 1 and R.H.S. ≥ 1

Hence, we must have $\cos^{50} x = 1 + \sin^{50} x = 1 \quad \Rightarrow \quad \sin x = 0 \quad \Rightarrow \quad x = n\pi$

Example 3.33 **Solve $\sin^2 x + \cos^2 y = 2\sec^2 z$ for x, y and z.**

Sol. L.H.S. $= \sin^2 x + \cos^2 y \le 2$ $\qquad [\because \sin^2 x \le$ and $\cos^2 y \le 1]$

R.H.S. $= 2\sec^2 z \ge 2$

Hence, L.H.S. = R.H.S. only when $\sin^2 x = 1$, $\cos^2 y = 1$ and $2\sec^2 z = 2$

$\Rightarrow \quad \cos^2 x = 0, \sin^2 y = 0, \cos^2 z = 1$

$\Rightarrow \quad \cos x = 0, \sin y = 0, \sin z = 0$

$x = (2m+1)\dfrac{\pi}{2}$, $y = n\pi$ and $z = t\pi$, where m, n and t are integers.

Example 3.34 **Solve $1 + \sin x \sin^2 \dfrac{x}{2} = 0$.**

Sol. $1 + \sin x \sin^2 \dfrac{x}{2} = 0$

$\Rightarrow \quad 2 + 2\sin x \sin^2 \dfrac{x}{2} = 0$

$\Rightarrow \quad 2 + \sin x (1 - \cos x) = 0$

$\Rightarrow \quad 4 + 2\sin x - \sin 2x = 0$

$\Rightarrow \quad \sin 2x = 2\sin x + 4$

Above is not possible for any value of x as L.H.S. has maximum value 1 and R.H.S. has minimum value 2.

Hence, there is no solution.

Example 3.35 **Solve $\cos 4\theta + \sin 5\theta = 2$.**

Sol. The equation $\cos 4\theta + \sin 5\theta = 2$ is valid only when $\cos 4\theta = 1$ and $\sin 5\theta = 1$.

$\Rightarrow \quad 4\theta = 2n\pi$ and $5\theta = 2m\pi + \pi/2$, $n, m \in Z$

$\Rightarrow \quad \theta = \dfrac{2n\pi}{4}$ and $\theta = \dfrac{2m\pi}{5} + \dfrac{\pi}{10}$, $n, m \in Z$

Putting $n; m = 0, \pm 1, \pm 2, \ldots$, the common value in $[0, 2\pi]$ is $\theta = \pi/2$.

Therefore, the solution is $\theta = 2k\pi + \pi/2$, $k \in Z$.

Concept Application Exercise 3.7

1. Show that $x = 0$ is the only solution satisfying the equation $1 + \sin^2 ax = \cos x$, where a is irrational.
2. Solve $\sin^4 x = 1 + \tan^8 x$.
3. Solve $\sin x\left(\cos\dfrac{x}{4} - 2\sin x\right) + \left(1 + \sin\dfrac{x}{4} - 2\cos x\right)\cos x = 0$.
4. Solve $12\sin x + 5\cos x = 2y^2 - 8y + 21$, to get the values of n and y.

INEQUALITIES

Trigonometric Inequations

To solve the trigonometric inequation of the type $f(x) \le a$, or $f(x) \ge a$, where $f(x)$ is some trigonometric ratio, the following steps should be taken:

1. Draw the graph of $f(x)$ in an interval length equal to the fundamental period of $f(x)$.
2. Draw the line $y = a$.
3. Take the portion of the graph for which the inequation is satisfied.
4. To generalize, add nT $(n \in I)$ and take union over the set of integers, where T is fundamental period of $f(x)$.

Example 3.36 **Solve $\sin x > -\frac{1}{2}$.**

Sol. As the function $\sin x$ has least positive period 2π; therefore, it is sufficient to solve the inequality of the form $\sin x > a$, $\sin x \ge a$, $\sin x < a$, and $\sin x \le a$ first on the interval of length 2π. Then get the solution set by adding numbers of the form $2\pi n$, $n \in z$, to each of the solutions obtained on that interval.

Thus, let us solve this inequality on the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

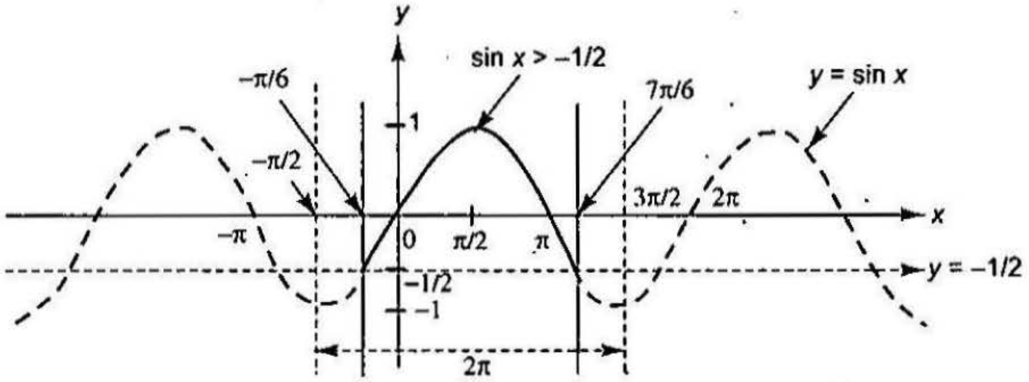


Fig. 3.1

From Fig. 3.1, $\sin x > -\frac{1}{2}$, when $-\frac{\pi}{6} < x < \frac{7\pi}{6}$

Thus, on generalizing, the above solution is $2n\pi - \frac{\pi}{6} < x < 2n\pi + \frac{7\pi}{6}$; $n \in Z$.

Example 3.37 **Solve $2\cos^2\theta + \sin\theta \le 2$, where $\pi/2 \le \theta \le 3\pi/2$.**

Sol. $2\cos^2\theta + \sin\theta \le 2$

$\Rightarrow$ $2(1 - \sin^2\theta) + \sin\theta \le 2$

$\Rightarrow$ $-2\sin^2\theta + \sin\theta \le 0$

$\Rightarrow$ $2\sin^2\theta - \sin\theta \ge 0$

$\Rightarrow$ $\sin\theta(2\sin\theta - 1) \ge 0$

$\Rightarrow$ $\sin\theta(\sin\theta - 1/2) \ge 0$,

which is possible if $\sin\theta \le 0$ or $\sin\theta \ge 1/2$.

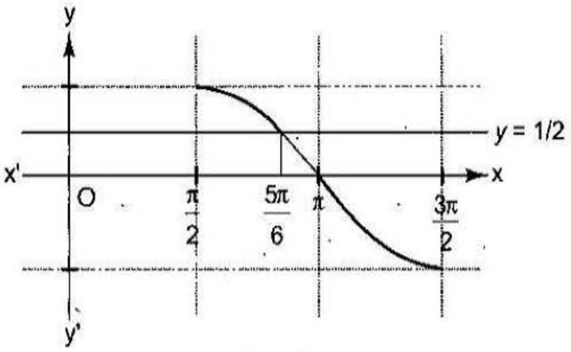


Fig. 3.2

From the graph,
Now $\sin\theta \geq 1/2 \Rightarrow \pi/2 \leq \theta \leq 5\pi/6$
and $\sin\theta \leq 0 \Rightarrow \pi \leq \theta \leq 3\pi/2$
Hence, the required values of θ are given by
$\theta \in [\pi/2, 5\pi/6] \cup [\pi, 3\pi/2]$

Example 3.38 **Solve $\sin\theta + \sqrt{3}\cos\theta \geq 1, -\pi < \theta \leq \pi$.**

Sol. The given inequation is

$\sin\theta + \sqrt{3}\cos\theta \geq 1, -\pi < \theta \leq \pi$

$\Rightarrow \frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta \geq \frac{1}{2}$

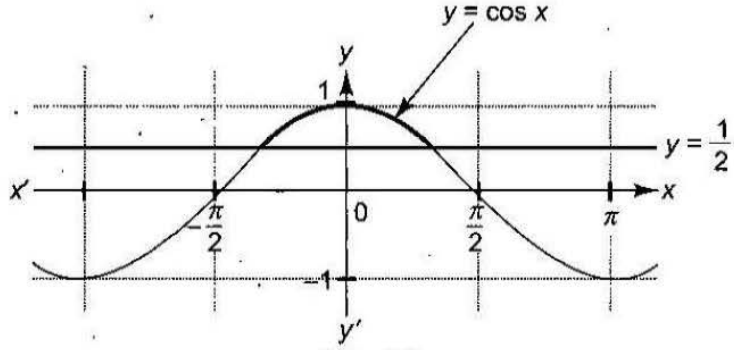


Fig. 3.3

$\Rightarrow \cos\left(\theta - \frac{\pi}{6}\right) \geq \frac{1}{2} = \cos\frac{\pi}{3} \Rightarrow -\frac{\pi}{3} \leq \theta - \frac{\pi}{6} \leq \frac{\pi}{3} \Rightarrow -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$

Example 3.39 **Solve $\cos 2x > |\sin x|, x \in \left(-\frac{\pi}{2}, \pi\right)$.**

Sol. Draw the graph of $y = \cos 2x$ and $y = |\sin x|$

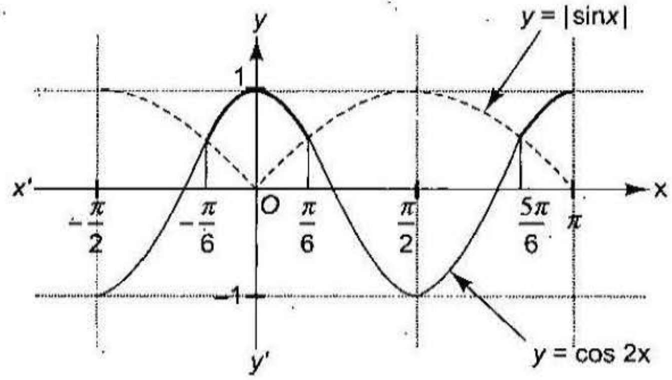


Fig. 3.4

Let $\cos 2x = \sin x$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0 \Rightarrow \sin x = -1, \frac{1}{2}$

But $\sin x \neq -1 \Rightarrow \sin x = \frac{1}{2}$

Clearly from the graph, graphs of $y = |\sin x|$ and $y = \cos 2x$ intersect at $x = \pm\frac{\pi}{6}, \frac{5\pi}{6}$.

Thus, the solution set is $x \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$.

Example 3.40 **Find the number of solutions of $\sin x = \frac{x}{10}$.**

Sol. Here, let $f(x) = \sin x$ and $g(x) = \frac{x}{10}$. Also, we know that $-1 \leq \sin x \leq 1$.

$\Rightarrow -1 \leq \frac{x}{10} \quad \Rightarrow -10 \leq x \leq 10$

Thus, sketch both curves when $x \in [-10, 10]$.

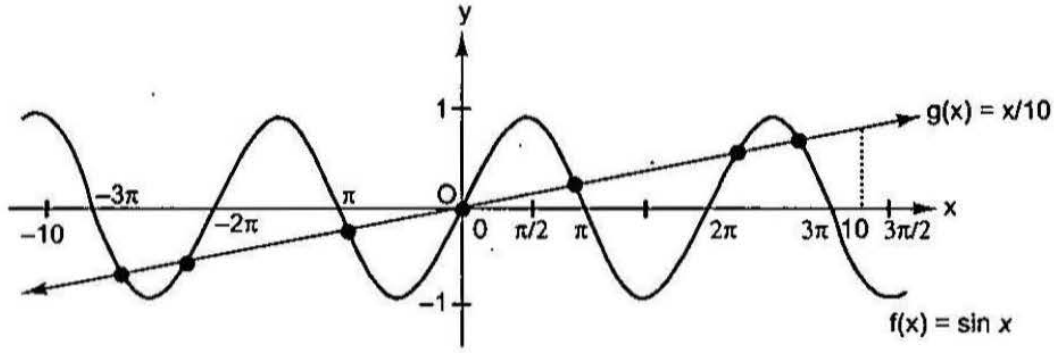


Fig. 3.5

From Fig. 3.5, $f(x) = \sin x$ and $g(x) = x/10$ intersect at seven points. So, the number of solutions is 7.

Concept Application Exercise 3.8

1. Solve $\sin^2\theta > \cos^2\theta$.
2. Find the number of solutions of the equation $\sin x = x^2 + x + 1$.
3. Solve $\tan x < 2$.
4. Prove that the least positive value of x, satisfying $\tan x = x + 1$, lies in the interval $(\pi/4, \pi/2)$.

EXERCISES

Subjective Type

Solutions on page 3.34

1. Solve $3\tan 2x - 4\tan 3x = \tan^2 3x \tan 2x$.
2. For which values of a, does the equation $4\sin(x + \pi/3)\cos(x - \pi/6) = a^2 + \sqrt{3}\sin 2x - \cos 2x$ have solutions? Find the solutions for $a = 0$, if exist.

3. Solve $\sin^4(x/3) + \cos^4(x/3) > 1/2$.
4. Solve $\sin x + \sin y = \sin(x+y)$ and $|x| + |y| = 1$.
5. Solve the equation $\tan^4 x + \tan^4 y + 2\cot^2 x \cot^2 y = 3 + \sin^2(x+y)$ for the values of x and y.
6. Find the smallest positive root of the equation $\sqrt{\sin(1-x)} = \sqrt{\cos x}$.
7. Solve the equation $2\sin x + \cos y = 2$ for the values of x and y.
8. Prove that the equation $2\sin x = |x| + a$ has no solution for $a \in \left(\frac{3\sqrt{3}-\pi}{3}, \infty\right)$.
9. Solve $\tan\left(\frac{\pi}{2}\cos\theta\right) = \cot\left(\frac{\pi}{2}\sin\theta\right)$.
10. Solve $\sin x + \sin\left(\frac{\pi}{8}\sqrt{(1-\cos 2x)^2 + \sin^2 2x}\right) = 0, x \in \left[\frac{5\pi}{2}, \frac{7\pi}{2}\right]$.
11. Solve $\sin^2 x + \frac{1}{4}\sin^2 3x = \sin x \sin^2 3x$.

Objective Type

Solutions on page 3.38

Each question has four choices a, b, c, and d, out of which *only one* answer is correct.

1. If $\sin\theta = \frac{1}{2}$ and $\cos\theta = -\frac{\sqrt{3}}{2}$, then the general value of θ is $(n \in Z)$

   a. $2n\pi + \frac{5\pi}{6}$ b. $2n\pi + \frac{\pi}{6}$ c. $2n\pi + \frac{7\pi}{6}$ d. $2n\pi + \frac{\pi}{4}$

2. The most general value for which $\tan\theta = -1$, $\cos\theta = \frac{1}{\sqrt{2}}$ is $(n \in Z)$

   a. $n\pi + \frac{7\pi}{4}$ b. $n\pi + (-1)^n \frac{7\pi}{4}$ c. $2n\pi + \frac{7\pi}{4}$ d. none of these

3. If $\cos p\theta + \cos q\theta = 0$, then the different values of θ are in A.P. where the common difference is

   a. $\frac{\pi}{p+q}$ b. $\frac{\pi}{p-q}$ c. $\frac{2\pi}{p+q}$ d. $\frac{3\pi}{p \pm q}$

4. If $\cos\theta + \cos 7\theta + \cos 3\theta + \cos 5\theta = 0$, then θ is equal to $(n \in Z)$

   a. $n\pi$ b. $n\pi/2$ c. $n\pi/4$ d. $n\pi/8$

5. If $3\tan^2\theta - 2\sin\theta = 0$, then θ is equal to $(n \in Z)$

   a. $2n\pi \pm \frac{\pi}{4}$ b. $n\pi + (-1)^n \frac{\pi}{6}$ c. $n\pi - (-1)^n \frac{\pi}{6}$ d. $n\pi + \frac{\pi}{3}$

6. If $\sin\theta$, 1, $\cos 2\theta$ are in G.P., then θ is equal to $(n \in Z)$

   a. $n\pi + (-1)^n \frac{\pi}{2}$ b. $n\pi + (-1)^{n-1} \frac{\pi}{2}$ c. $2n\pi$ d. none of these

7. The sum of all the solutions of the equation $\cos\theta \cos\left(\frac{\pi}{3}+\theta\right)\cos\left(\frac{\pi}{3}-\theta\right) = \frac{1}{4}$, $\theta \in [0, 6\pi]$

   a. 15π b. 30π c. $\frac{100\pi}{3}$ d. none of these

8. If $\sec\theta - 1 = (\sqrt{2}-1)\tan\theta$, then θ is equal to $(n \in Z)$

a. $(2n-1)\pi$ b. $2n\pi + \frac{\pi}{4}$ c. $2n\pi - \frac{\pi}{4}$ d. $2n\pi + \frac{\pi}{3}$

9. The total number of solution of $\sin^4 x + \cos^4 x = \sin x \cos x$ in $[0, 2\pi]$ is equal to

a. 2 b. 4 c. 6 d. none of these

10. Number of solutions of $\sin 5x + \sin 3x + \sin x = 0$ for $0 \le x \le \pi$ is

a. 1 b. 2 c. 3 d. none of these

11. The sum of all the solution of $\cot\theta = \sin 2\theta$, $(\theta \ne n\pi$, n integer), $0 \le \theta \le \pi$ is

a. $3\pi/2$ b. π c. $3\pi/4$ d. 2π

12. The number of solutions of $12\cos^3 x - 7\cos^2 x + 4\cos x = 9$ is

a. 0 b. 2 c. infinite d. none of these

13. Which of the following is not the general solution of $2^{\cos 2x} + 1 = 3.2^{-\sin^2 x}$?

a. $n\pi, n \in Z$ b. $\left(n + \frac{1}{2}\right)\pi, n \in Z$ c. $\left(n - \frac{1}{2}\right)\pi, n \in Z$ d. none of these

14. The general solution of $\cos x \cos 6x = -1$ is

a. $x = (2n+1)\pi, n \in Z$ b. $x = 2n\pi, n \in Z$

c. $x = n\pi, n \in Z$ d. none of these

15. The equation $\cos x + \sin x = 2$ has

a. only one solution b. two solutions

c. no solution d. infinite number of solutions

16. If $0 \le x \le 2\pi$, then the number of solutions of $3(\sin x + \cos x) - 2(\sin^3 x + \cos^3 x) = 8$ is

a. 0 b. 1 c. 2 d. 4

17. If $\frac{1}{6}\sin\theta$, $\cos\theta$, $\tan\theta$ are in G.P., then θ is equal to $(n \in Z)$

a. $2n\pi \pm \frac{\pi}{3}$ b. $2n\pi \pm \frac{\pi}{6}$ c. $n\pi + (-1)^n \frac{\pi}{3}$ d. $n\pi + \frac{\pi}{3}$

18. The number of solutions of $2\sin^2 x + \sin^2 2x = 2$, $x \in [0, 2\pi]$ is

a. 4 b. 5 c. 7 d. 6

19. General solution of $\sin^2 x - 5\sin x \cos x - 6\cos^2 x = 0$ is

a. $x = n\pi - \pi/4, n \in Z$ only b. $n\pi + \tan^{-1} 6, n \in Z$ only

c. both (a) and (b) d. none of these

20. General solution of $\tan\theta + \tan 4\theta + \tan 7\theta = \tan\theta \tan 4\theta \tan 7\theta$ is

a. $\theta = n\pi/12$, where $n \in Z$

b. $\theta = n\pi/9$, where $n \in Z$

c. $\theta = n\pi + \pi/12$, where $n \in Z$

d. none of these

21. The number of solutions of $\sec^2\theta + \text{cosec}^2\theta + 2\,\text{cosec}^2\theta = 8$, $0 \le \theta \le \pi/2$ is

a. 4 b. 3 c. 0 d. 2

22. The total number of solutions of $\tan x + \cot x = 2\,\text{cosec}\, x$ in $[-2\pi, 2\pi]$ is

a. 2 b. 4 c. 6 d. 8

23. Which of the following is true for $z = (3 + 2i\sin\theta)/(1 - 2i\sin\theta)$, where $i = \sqrt{-1}$

a. z is purely real for $\theta = n\pi \pm \pi/3, n \in Z$

b. z is purely imaginary for $\theta = n\pi \pm \pi/2, n \in Z$

c. z is purely real for $\theta = n\pi, n \in Z$

d. none of these

24. Number of roots of $\cos^2 x + \frac{\sqrt{3}+1}{2}\sin x - \frac{\sqrt{3}}{4} - 1 = 0$ which lie in the interval $[-\pi, \pi]$ is

a. 2 b. 4 c. 6 d. 8

25. The complete solution of $7\cos^2 x + \sin x\cos x - 3 = 0$ is given by

a. $n\pi + \frac{\pi}{2}$ $(n \in Z)$ b. $n\pi - \frac{\pi}{2}$ $(n \in Z)$

c. $n\pi + \tan^{-1}\left(\frac{3}{4}\right)$ $(n \in Z)$ d. $n\pi + \frac{3\pi}{4}, k\pi + \tan^{-1}\left(\frac{4}{3}\right)$ $(k, n \in Z)$

26. Let $\theta \in [0, 4\pi]$ satisfy the equation $(\sin\theta + 2)(\sin\theta + 3)(\sin\theta + 4) = 6$. If the sum of all the values of θ is of the form $k\pi$, then the value of k is

a. 6 b. 5 c. 4 d. 2

27. If the inequality $\sin^2 x + a\cos x + a^2 > 1 + \cos x$ holds for any $x \in R$, then the largest negative integral value of a is

a. -4 b. -3 c. -2 d. -1

28. The number of solution of $\sin^4 x - \cos^2 x\sin x + 2\sin^2 x + \sin x = 0$ in $0 \le x \le 3\pi$ is

a. 3 b. 4 c. 5 d. 6

29. If $x, y \in [0, 2\pi]$ and $\sin x + \sin y = 2$, then the value of $x + y$ is

a. π b. $\pi/2$ c. 3π d. none of these

30. For $n \in Z$, the general solution of $(\sqrt{3} - 1)\sin\theta + (\sqrt{3} + 1)\cos\theta = 2$ is $(n \in Z)$

a. $\theta = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$ b. $\theta = n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{12}$

c. $\theta = 2n\pi \pm \frac{\pi}{4}$ d. $\theta = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{12}$

31. If $\sin 6\theta + \sin 4\theta + \sin 2\theta = 0$, then θ is equal to $(n \in Z)$

a. $\frac{n\pi}{4}$ or $n\pi \pm \frac{\pi}{3}$ b. $\frac{n\pi}{4}$ or $n\pi \pm \frac{\pi}{6}$ c. $\frac{n\pi}{4}$ or $2n\pi \pm \frac{\pi}{6}$ d. none of these

32. The value of $\cos y\cos\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - y\right)\cos x + \sin y\cos\left(\frac{\pi}{2} - x\right) + \cos x\sin\left(\frac{\pi}{2} - y\right)$ is zero if

a. $x = 0$ b. $y = 0$ c. $x = y$ d. $n\pi + y - \frac{\pi}{4}$ $(n \in Z)$

33. The number of solution of the equation $\tan x\tan 4x = 1$ for $0 < x < \pi$ is.

a. 1 b. 2 c. 5 d. 8

34. One root of the equation $\cos x - x + \frac{1}{2} = 0$ lies in the interval

a. $\left(0, \frac{\pi}{2}\right)$ b. $\left(-\frac{\pi}{2}, 0\right)$ c. $\left(\frac{\pi}{2}, \pi\right)$ d. $\left(\pi, \frac{3\pi}{2}\right)$

35. $\tan\left(\frac{p\pi}{4}\right) = \cot\left(\frac{q\pi}{4}\right)$ if $(n \in Z)$

a. $p + q = 0$ b. $p + q = 2n + 1$ c. $p + q = 2n$ d. $p + q = 2(2n + 1)$

36. The range of y such that the equation in x, $y + \cos x = \sin x$ has a real solution is

a. $[-2, 2]$ b. $[-\sqrt{2}, \sqrt{2}]$ c. $[-1, 1]$ d. $[-1/2, 1/2]$

37. One of the general solutions of $4\sin^4 x + \cos^4 x = 1$ is

a. $n\pi \pm \alpha/2$, $\alpha = \cos^{-1}(1/5)$, $\forall\ n \in Z$ b. $n\pi \pm \alpha/2$, $\alpha = \cos^{-1}(3/5)$, $\forall\ n \in Z$

c. $2n\pi \pm \alpha/2$, $\alpha = \cos^{-1}(1/3)$, $\forall\ n \in Z$ d. none of these

38. Number of roots of $(1 - \tan\theta)(1 + \sin 2\theta) = 1 + \tan\theta$ for $\theta \in [0, 2\pi]$ is

a. 3 b. 4 c. 5 d. none of these

39. The number of solutions of $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$, $0 \le x \le 2\pi$, is

a. 7 b. 5 c. 4 d. 6

40. The number of values of θ which satisfy the equation $\sin 3\theta - \sin\theta = 4\cos^2\theta - 2$, $\forall\ \theta \in [0, 2\pi]$, is

a. 4 b. 5 c. 7 d. 0

41. One of the general solutions of $4\sin\theta \sin 2\theta \sin 4\theta = \sin 3\theta$ is

a. $(3n \pm 1)\pi/12$, $\forall\ n \in Z$ b. $(4n \pm 1)\pi/9$, $\forall\ n \in Z$

c. $(3n \pm 1)\pi/9$, $\forall\ n \in Z$ d. $(3n \pm 1)\pi/3$, $\forall\ n \in Z$

42. The general solution of $\tan\theta + \tan 2\theta + \tan 3\theta = 0$ is

a. $\theta = n\pi/6$, $n \in Z$ b. $\theta = n\pi \pm \alpha$, $n \in Z$, where $\tan\alpha = 1/\sqrt{2}$

c. Both a and b d. none of these

43. The general solution of $\sin 3\alpha = 4\sin\alpha \sin(x + \alpha)\sin(x - \alpha)$ is

a. $n\pi \pm \pi/4$, $\forall\ n \in Z$ b. $n\pi \pm \pi/3$, $\forall\ n \in Z$

c. $n\pi \pm \pi/9$, $\forall\ n \in Z$ d. $n\pi \pm \pi/12$, $\forall\ n \in Z$

44. One of the general solutions of $\sqrt{3}\cos\theta - 3\sin\theta = 4\sin 2\theta \cos 3\theta$ is

a. $m\pi + \pi/18$, $m \in Z$ b. $m\pi/2 + \pi/6$, $\forall\ m \in Z$

c. $m\pi/3 + \pi/18$, $m \in Z$ d. none of these

45. The equation $\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$ is solvable for

a. $-5/2 \le \alpha \le 1/2$ b. $-3 \le \alpha \le 1$ c. $-3/2 \le \alpha \le 1/2$ d. $-1 \le \alpha \le 1$

46. Consider the system of linear equations in x, y and z:

$$(\sin 3\theta)x - y + z = 0$$
$$(\cos 2\theta)x + 4y + 3z = 0$$
$$2x + 7y + 7z = 0$$

then which of the following can be the values of θ for which the system has a non-trivial solution

a. $n\pi + (-1)^n \pi/6$, $\forall\ n \in Z$ b. $n\pi + (-1)^n \pi/3$, $\forall\ n \in Z$

c. $n\pi + (-1)^n \pi/9$, $\forall\ n \in Z$ d. none of these

47. The smallest +ve x satisfying the equation $\log_{\cos x}\sin x + \log_{\sin x}\cos x = 2$ is

a. $\pi/2$ b. $\pi/3$ c. $\pi/4$ d. $\pi/6$

48. Number of ordered pairs which satisfy the equation $x^2 + 2x\sin(xy) + 1 = 0$ are (where $y \in [0, 2\pi]$)

a. 1 b. 2 c. 3 d. 0

49. The general solution of the equation $8\cos x \cos 2x \cos 4x = \sin 6x/\sin x$ is

a. $x = (n\pi/7) + (\pi/21)$, $\forall\ n \in Z$ b. $x = (2\pi/7) + (\pi/14)$, $\forall\ n \in Z$

c. $x = (n\pi/7) + (\pi/14)$, $\forall\ n \in Z$ d. $x = (n\pi) + (\pi/14)$, $\forall\ n \in Z$

50. If $\cos 3x + \sin\left(2x - \frac{7\pi}{6}\right) = -2$, then x is equal to ($k \in Z$)

a. $\frac{\pi}{3}(6k + 1)$ b. $\frac{\pi}{3}(6k - 1)$ c. $\frac{\pi}{3}(2k + 1)$ d. none of these

51. If $(1-\tan\theta)(1+\tan\theta)\sec^2\theta + 2^{\tan^2\theta} = 0$, then the number of values of θ in the interval $(-\pi/2, \pi/2)$ are

a. 1 b. 2 c. 3 d. 4

52. Number of solutions of $\tan\left(\frac{\pi}{2}\sin\theta\right) = \cot\left(\frac{\pi}{2}\cos\theta\right)$, $\theta \in [0, 6\pi]$, is

a. 5 b. 7 c. 4 d. 5

53. The total number of solutions of $\cos x = \sqrt{1-\sin 2x}$ in $[0, 2\pi]$ is equal to

a. 2 b. 3 c. 5 d. none of these

54. The number of solutions of $\sum_{r=1}^{5} \cos rx = 5$ in the interval $[0, 2\pi]$ is

a. 0 b. 2 c. 5 d. 10

55. The number of values of x for which $\sin 2x + \cos 4x = 2$ is

a. 0 b. 1 c. 2 d. infinite

56. Let α and β be any two positive values of x for which $2\cos x$, $|\cos x|$ and $1 - 3\cos^2 x$ are in GP. The minimum value of $|\alpha - \beta|$ is

a. $\frac{\pi}{3}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{2}$ d. none of these

57. The general solution of the equation $\sin^{100} x - \cos^{100} x = 1$ is

a. $2n\pi + \frac{\pi}{3}, n \in I$ b. $n\pi + \frac{\pi}{2}, n \in I$ c. $n\pi + \frac{\pi}{4}, n \in I$ d. $2n\pi - \frac{\pi}{3}, n \in I$

58. The total number of solutions of $|\cot x| = \cot x + \frac{1}{\sin x}$, $x \in [0, 3\pi]$ is equal to

a. 1 b. 2 c. 3 d. 0

59. If $\tan(A-B) = 1$ and $\sec(A+B) = 2/\sqrt{3}$, then the smallest positive values of A and B, respectively, are

a. $\frac{25\pi}{24}, \frac{19\pi}{24}$ b. $\frac{19\pi}{24}, \frac{25\pi}{24}$ c. $\frac{31\pi}{24}, \frac{13\pi}{24}$ d. $\frac{13\pi}{24}, \frac{31\pi}{24}$

60. If $3\tan(\theta - 15°) = \tan(\theta + 15°)$, then θ is equal to $(n \in Z)$

a. $n\pi + \frac{\pi}{4}$ b. $n\pi + \frac{\pi}{8}$ c. $n\pi + \frac{\pi}{3}$ d. none of these

61. If $\tan 3\theta + \tan\theta = 2\tan 2\theta$, then θ is equal to $(n \in Z)$

a. $n\pi$ b. $\frac{n\pi}{4}$ c. $2n\pi$ d. none of these

62. The set of all x in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ satisfying $|4\sin x - 1| < \sqrt{5}$ is given by

a. $\left(-\frac{\pi}{10}, \frac{3\pi}{10}\right)$ b. $\left(\frac{\pi}{10}, \frac{3\pi}{10}\right)$ c. $\left(\frac{\pi}{10}, \frac{3\pi}{10}\right)$ d. none of these

63. $\sin x + \cos x = y^2 - y + a$ has no value of x for any value of y if a belongs to

a. $(0, \sqrt{3})$ b. $(-\sqrt{3}, 0)$ c. $(-\infty, -\sqrt{3})$ d. $(\sqrt{3}, \infty)$

64. The solution of $4\sin^2 x + \tan^2 x + \operatorname{cosec}^2 x + \cot^2 x - 6 = 0$ is

a. $n\pi \pm \frac{\pi}{4}$ b. $2n\pi \pm \frac{\pi}{4}$ c. $n\pi + \frac{\pi}{3}$ d. $n\pi - \frac{\pi}{6}$

65. The number of solutions of $[\sin x + \cos x] = 3 + [-\sin x] + [-\cos x]$ (where [.] denotes the greatest integer function), $x \in [0, 2\pi]$, is

a. 0 b. 4 c. infinite d. 1

66. The equation $\cos^8 x + b\cos^4 x + 1 = 0$ will have a solution if b belongs to

a. $(-\infty, 2]$ b. $[2, \infty)$ c. $(-\infty, -2]$ d. none of these

67. The number of values of y in $[-2\pi, 2\pi]$ satisfying the equation $|\sin 2x| + |\cos 2x| = |\sin y|$ is

a. 3 b. 4 c. 5 d. 6

68. If both the distinct roots of the equation $|\sin x|^2 + |\sin x| + b = 0$ in $[0, \pi]$ are real, then the values of b are

a. $[-2, 0]$ b. $(-2, 0)$ c. $[-2, 0)$ d. none of these

69. If $|2\sin\theta - \operatorname{cosec}\theta| \geq 1$ and $\theta \neq \frac{n\pi}{2}$, $n \in I$, then

a. $\cos 2\theta \geq 1/2$ b. $\cos 2\theta \geq 1/4$ c. $\cos 2\theta \leq 1/2$ d. $\cos 2\theta \leq 1/4$

70. The number of solutions of the equation $\sin^3 x\cos x + \sin^2 x\cos^2 x + \sin x\cos^3 x = 1$, in the interval $[0, 2\pi]$, is

a. 4 b. 2 c. 1 d. 0

71. $e^{|\sin x|} + e^{-|\sin x|} + 4a = 0$ will have exactly four different solutions in $[0, 2\pi]$ if

a. $a \in R$ b. $a \in \left[-\frac{e}{4}, -\frac{1}{4}\right]$ c. $a \in \left[\frac{-1-e^2}{4e}, \infty\right]$ d. none of these

72. The total number of solutions of $\ln|\sin x| = -x^2 + 2x$ in $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ is equal to

a. 1 b. 2 c. 4 d. none of these

73. The total number of ordered pairs (x, y) satisfying $|x| + |y| = 4$, $\sin\left(\frac{\pi x^2}{3}\right) = 1$ is equal to

a. 2 b. 3 c. 4 d. 6

74. The total number of solutions of $\sin\{x\} = \cos\{x\}$ (where {.} denotes the fractional part) in $[0, 2\pi]$ is equal to

a. 5 b. 6 c. 8 d. none of these

75. If $a, b \in [0, 2\pi]$ and the equation $x^2 + 4 + 3\sin(ax + b) - 2x = 0$ has at least one solution, then the value of $(a + b)$ can be

a. $\frac{7\pi}{2}$ b. $\frac{5\pi}{2}$ c. $\frac{9\pi}{2}$ d. none of these

76. The equation $\tan^4 x - 2\sec^2 x + a = 0$ will have at least one solution if

a. $1 < a \leq 4$ b. $a \geq 2$ c. $a \leq 3$ d. none of these

77. Complete the set of values of x in $(0, \pi)$ satisfying the equation $1 + \log_2 \sin x + \log_2 \sin 3x \geq 0$ is

a. $\left(\frac{2\pi}{3}, \frac{3\pi}{4}\right]$ b. $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$ c. $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{2\pi}{3}, \pi\right)$ d. $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

78. The equation $\sin^2\theta - \frac{4}{\sin^3\theta - 1} = 1 - \frac{4}{\sin^3\theta - 1}$ has

a. no root b. one root c. two roots c. infinite roots

79. The sum of all roots of $\sin\left(\pi \log_3\left(\frac{1}{x}\right)\right) = 0$ in $(0, 2\pi)$ is

a. 3/2 b. 4 c. 9/2 d. 13/3

80. The number of pairs of integer (x, y) that satisfy the following two equations

$\begin{cases} \cos(xy) = x \\ \tan(xy) = y \end{cases}$ is

a. 1 b. 2 c. 4 d. 6

81. Sum of all the solutions in $[0, 4\pi]$ of the equation $\tan x + \cot x + 1 = \cos\left(x + \frac{\pi}{4}\right)$ is

a. 3π b. $\pi/2$ c. $7\pi/2$ d. 4π

82. Number of solutions the equation $\cos(\theta) \cdot \cos(\pi\theta) = 1$ has

a. 0 b. 2 c. 1 d. infinite

83. The general value of x satisfying the equation $2\cot^2 x + 2\sqrt{3}\cot x + 4 \operatorname{cosec} x + 8 = 0$ is

a. $n\pi - \frac{\pi}{6}, n \in Z$ b. $n\pi + \frac{\pi}{6}, n \in Z$ c. $2n\pi - \frac{\pi}{6}, n \in Z$ d. $2n\pi + \frac{\pi}{6}, n \in Z$

84. Assume that θ is a rational multiple of π such that $\cos\theta$ is a distinct rational. Number of values of $\cos\theta$ is

a. 3 b. 4 c. 5 d. 6

85. Number of ordered pair(s) (a, b) for each of which the equality $a(\cos x - 1) + b^2 = \cos(ax + b^2) - 1$ holds true for all $x \in R$ are

a. 1 b. 2 c. 3 d. 4

Multiple Correct Answers Type

Solutions on page 3.54

Each question has four choices a, b, c, and d, out of which *one or more* answers are correct.

1. If $4\sin^4 x + \cos^4 x = 1$, then x is equal to $(n \in Z)$

a. $n\pi$ b. $n\pi \pm \sin^{-1}\sqrt{\frac{2}{5}}$ c. $\frac{2n\pi}{3}$ d. $2n\pi \pm \frac{\pi}{4}$

2. If $\sin^3\theta + \sin\theta\cos\theta + \cos^3\theta = 1$, then θ is equal to $(n \in Z)$

a. $2n\pi$ b. $2n\pi + \frac{\pi}{2}$ c. $2n\pi - \frac{\pi}{2}$ d. $n\pi$

3. A general solution of $\tan^2\theta + \cos 2\theta = 1$ is $(n \in Z)$

a. $n\pi - \frac{\pi}{4}$ b. $2n\pi + \frac{\pi}{4}$ c. $n\pi + \frac{\pi}{4}$ d. $n\pi$

4. If $\sin x + \cos x = \sqrt{y + \frac{1}{y}}$ for $x \in [0, \pi]$, then

a. $x = \pi/4$ b. $y = 0$ c. $y = 1$ d. $x = 3\pi/4$

5. $\sin\theta + \sqrt{3}\cos\theta = 6x - x^2 - 11, 0 \le \theta \le 4\pi, x \in R$, holds for

a. no values of x and θ b. one value of x and two values of θ
c. two values of x and two values of θ d. two point of values of (x, θ)

6. If $\sin^2 x - 2\sin x - 1 = 0$ has exactly four different solutions in $x \in [0, n\pi]$, then value/values of n is/are $(n \in N)$

a. 5 b. 3 c. 4 d. 6

7. For the smallest positive values of x and y, the equation $2(\sin x + \sin y) - 2\cos(x - y) = 3$ has a solution, then which of the following is/are true?

a. $\sin\frac{x+y}{2}=1$
b. $\cos\left(\frac{x-y}{2}\right)=\frac{1}{2}$
c. number of ordered pairs (x, y) is 2
d. number of ordered pairs (x, y) is 3

8. For the equation $1-2x-x^2=\tan^2(x+y)+\cot^2(x+y)$
a. exactly one value of x exists
b. exactly two values of x exists
c. $y=-1+n\pi+\pi/4, n\in Z$
d. $y=1+n\pi+\pi/4, n\in Z$

9. If $x+y=\pi/4$ and $\tan x+\tan y=1$, then $(n\in Z)$
a. $\sin x=0$ always
b. when $x=n\pi+\pi/4$ then $y=-n\pi$
c. when $x=n\pi$ then $y=n\pi+(\pi/4)$
d. when $x=n\pi+\pi/4$ then $y=n\pi-(\pi/4)$

10. If $x+y=2\pi/3$ and $\sin x/\sin y=2$, then
a. the number of values of $x\in[0, 4\pi]$ are 4
b. number of values of $x\in[0, 4\pi]$ are 2
c. number of values of $y\in[0, 4\pi]$ are 4
d. number of values of $y\in[0, 4\pi]$ are 8

11. Let $\tan x-\tan^2 x>0$ and $|2\sin x|<1$. Then the intersection of which of the following two sets satisfies both the inequalities?
a. $x>n\pi, n\in Z$
b. $x>n\pi-\pi/6, n\in Z$
c. $x<n\pi-\pi/4, n\in Z$
d. $x<n\pi+\pi/6, n\in Z$

12. If $\cos(x+\pi/3)+\cos x=a$ has real solutions, then
a. number of integral values of a are 3
b. sum of number of integral values of a is 0
c. when $a=1$, number of solutions for $x\in[0, 2\pi]$ are 3
d. when $a=1$, number of solutions for $x\in[0, 2\pi]$ are 2

13. For $0\le x\le 2\pi$, then $2^{\operatorname{cosec}^2 x}\sqrt{\frac{1}{2}y^2-y+1}\le\sqrt{2}$
a. is satisfied by exactly one value of y
b. is satisfied by exactly two value of x
c. is satisfied by x for which $\cos x=0$
d. is satisfied by x for which $\sin x=0$

14. If $\sin^2 x-a\sin x+b=0$ has only one solution in $(0, \pi)$, then which of the following statements are correct?
a. $a\in(-\infty, 1]\cup[2, \infty)$
b. $b\in(-\infty, 0]\cup[1, \infty)$
c. $a=1+b$
d. none of these

15. If $(\operatorname{cosec}^2\theta-4)x^2+(\cot\theta+\sqrt{3})x+\cos^2\frac{3\pi}{2}=0$ holds true for all real x, then the most general values of θ can be given by $(n\in Z)$
a. $2n\pi+\frac{11\pi}{6}$
b. $2n\pi+\frac{5\pi}{6}$
c. $2n\pi\pm\frac{7\pi}{6}$
d. $n\pi\pm\frac{11\pi}{6}$

16. If $(\sin\alpha)x^2-2x+b\ge 2$ for all the real values of $x\le 1$ and $\alpha\in(0, \pi/2)\cup(\pi/2, \pi)$, then the possible real values of b is/are
a. 2
b. 3
c. 4
d. 5

17. The value of x in $(0, \pi/2)$ satisfying $\frac{\sqrt{3}-1}{\sin x}+\frac{\sqrt{3}+1}{\cos x}=4\sqrt{2}$ is
a. $\frac{\pi}{12}$
b. $\frac{5\pi}{12}$
c. $\frac{7\pi}{24}$
d. $\frac{11\pi}{36}$

18. If $\cos 3\theta=\cos 3\alpha$, then the value of $\sin\theta$ can be given by
a. $\pm\sin\alpha$
b. $\sin\left(\frac{\pi}{3}\pm\alpha\right)$
c. $\sin\left(\frac{2\pi}{3}+\alpha\right)$
d. $\sin\left(\frac{2\pi}{3}-\alpha\right)$

19. Which of the following sets can be the subset of the general solution of $1 + \cos 3x = 2 \cos 2x$ ($n \in Z$)?

a. $n\pi + \frac{\pi}{3}$ b. $n\pi + \frac{\pi}{6}$ c. $n\pi - \frac{\pi}{6}$ d. $2n\pi$

20. In a right-angled triangle, the hypotenuse is $2\sqrt{2}$ times the perpendicular drawn from the opposite vertex. Then the other acute angles of the triangle are

a. $\frac{\pi}{3}$ b. $\frac{\pi}{8}$ c. $\frac{3\pi}{8}$ d. $\frac{\pi}{6}$

Reasoning Type

Solutions on page 3.59

Each question has four choices a, b, c, and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. if both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1
b. if both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1
c. if STATEMENT 1 is TRUE and STATEMENT 2 is FALSE
d. if STATEMENT 1 is FALSE and STATEMENT 2 is TRUE

1. **Statement 1:** The value of x for which $(\sin x + \cos x)^{1 + \sin 2x} = 2$, when $0 \le x \le \pi$ is $\pi/4$ only.
Statement 2: The maximum value of $\sin x + \cos x$ occurs when $x = \pi/4$.
2. **Statement 1:** The equation $\sin^2 x + \cos^2 y = 2 \sec^2 z$ is solvable when only $\sin x = 1$; $\cos y = 1$ and $\sec z = 1$, where $x, y, z \in R$.
Statement 2: The maximum value of $\sin x$ and $\cos y$ is 1 and minimum value of $\sec z$ is 1.
3. **Statement 1:** Equation $x \sin x = 1$ has four roots for $x \in (-\pi, \pi)$.
Statement 2: The graph of $y = \sin x$ and $y = 1/x$ cuts exactly two times for $x \in (0, \pi)$.
4. **Statement 1:** $\sin x = a$, where $-1 < a < 0$, then for $x \in [0, n\pi]$ has $2(n-1)$ solutions $\forall n \in N$.
Statement 2: $\sin x$ takes value a exactly two times when we take one complete rotation covering all the quadrant starting from $x = 0$.
5. **Statement 1:** Equation $\sqrt{1 - \sin 2x} = \sin x$ has 1 solution for $x \in [0, \pi/4]$.
Statement 2: $\cos x > \sin x$ when $x \in [0, \pi/4]$.
6. **Statement 1:** The number of solution of the equation $|\sin x| = |x|$ is only one.
Statement 2: $|\sin x| \ge 0 \ \forall x \in R$.
7. **Statement 1:** General solution of $\frac{\tan 4x - \tan 2x}{1 + \tan 4x \tan 2x} = 1$ is $x = \frac{n\pi}{2} + \frac{\pi}{8}, n \in I$.
Statement 2: General solution of $\tan \alpha = 1$ is $\alpha = n\pi + \frac{\pi}{4}, n \in I$.
8. **Statement 1:** The equation $\sin(\cos x) = \cos(\sin x)$ has no real solution.
Statement 2: $\sin x \pm \cos x \in \left[-\sqrt{2}, \sqrt{2}\right]$.
9. **Statement 1:** Equation $\sin x = e^x$ has infinite solutions.
Statement 2: $y = e^x$ is an unbounded function.
10. **Statement 1:** Number of solution of $n|\sin x| = m|\cos x|$ (where $m, n \in Z$) in $[0, 2\pi]$ is independent of m and n.
Statement 2: Multiplying trigonometric functions by constant changes only range of the function but period remains same.

Linked Comprehension Type

Solutions on page 3.61

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c, and d, out of which *only one* is correct.

For Problems 1 – 3

Consider the cubic equation

$x^3 - (1 + \cos\theta + \sin\theta)x^2 + (\cos\theta\sin\theta + \cos\theta + \sin\theta)x - \sin\theta\cos\theta = 0$

whose roots are x_1, x_2 and x_3.

1. The value of $x_1^2 + x_2^2 + x_3^2$ equals
   a. 1 b. 2 c. $2\cos\theta$ d. $\sin\theta(\sin\theta + \cos\theta)$
2. Number of values of θ in $[0, 2\pi]$ for which at least two roots are equal
   a. 3 b. 4 c. 5 d. 6
3. Greatest possible difference between two of the roots if $\theta \in [0, 2\pi]$ is
   a. 2 b. 1 c. $\sqrt{2}$ d. $2\sqrt{2}$

For Problems 4 – 6

Consider the equation $\sec\theta + \operatorname{cosec}\theta = a$, $\theta \in (0, 2\pi) - \{\pi/2, \pi, 3\pi/2\}$

4. If the equation has four real roots, then
   a. $|a| \geq 2\sqrt{2}$ b. $|a| < 2\sqrt{2}$ c. $a \geq -2\sqrt{2}$ d. none of these
5. If the equation has two real roots, then
   a. $|a| \geq 2\sqrt{2}$ b. $a < 2\sqrt{2}$ c. $|a| < 2\sqrt{2}$ d. none of these
6. If the equation has no real roots, then
   a. $|a| \geq 2\sqrt{2}$ b. $a < 2\sqrt{2}$ c. $|a| < 2\sqrt{2}$ d. none of these

For Problems 7 – 9

Consider the system of equations

$\sin x \cos 2y = (a^2 - 1)^2 + 1,$

$\cos x \sin 2y = a + 1$

7. Number of values of a for which the system has a solution is
   a. 1 b. 2 c. 3 d. infinite
8. Number of values of $x \in [0, 2\pi]$, when the system has solution for permissible values of a, is/are
   a. 1 b. 2 c. 3 d. 4
9. Number of values of $y \in [0, 2\pi]$, when the system has solution for permissible values of a, are
   a. 2 b. 3 c. 4 d. 5

For Problems 10 – 12

Consider the equation $\int_0^x (t^2 - 8t + 13)\,dt = x\sin(a/x)$

10. The number of real values of x for which the equation has solution is
    a. 1 b. 2 c. 3 d. infinite
11. If x takes the values for which the equation has a solution, then the number of values of $a \in [0, 100]$ is
    a. 2 b. 1 c. 5 d. 3
12. One of the solutions of $|y - \cos a| < x$, where x and a are values that satisfy the given equation, is
    a. $y \in [-5, 7]$ b. $y \in [-7, 5]$ c. $y \in [5, 7]$ d. none of these

For Problems 13 – 15

Consider the system of equations

$x\cos^3 y + 3x\cos y\sin^2 y = 14$

$x\sin^3 y + 3x\cos^2 y\sin y = 13$

13. The value/values of x is/are

a. $\pm 5\sqrt{5}$ b. $\pm\sqrt{5}$ c. $\pm 1/\sqrt{5}$ d. none of these

14. The number of values of $y \in [0, 6\pi]$ is

a. 5 b. 3 c. 4 d. 6

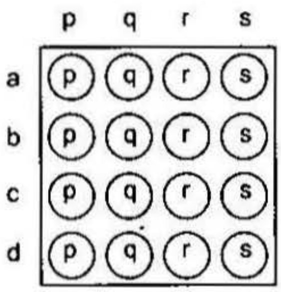
15. The value of $\sin^2 y + 2\cos^2 y$ is

a. 4/5 b. 9/5 c. 2 d. none of these

Matrix-Match Type

Solutions on page 3.64

Each question contains statements given in two columns which have to be matched. Statements (a, b, c, d) in column I have to be matched with statements (p, q, r, s) in column II. If the correct match are a → p, a → s, b → q, c → p, c → q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:



1.

Column I (Equation)	Column II (Solution)
a. $\cos^2 2x + \cos^2 x = 1$	p. $x = \left\{n\pi + \frac{\pi}{4}\right\} \cup \left\{n\pi + \frac{\pi}{6}\right\} n \in Z$
b. $\cos x + \sqrt{3}\sin x = \sqrt{3}$	q. $x = \frac{n\pi}{3}, n \in Z$
c. $1 + \sqrt{3}\tan^2 x = \left(1 + \sqrt{3}\right)\tan x$	r. $x = (2n - 1)\frac{\pi}{6}, n \in Z$
d. $\tan 3x - \tan 2x - \tan x = 0$	s. $x = \left\{2n\pi + \frac{\pi}{2}\right\} \cup \left\{2n\pi + \frac{\pi}{6}\right\}, n \in Z$

2.

Column I (Equation)	Column II (Number of solutions)
a. $x^3 + x^2 + 4x + 2\sin x = 0$ in $0 \le x \le 2\pi$	p. 4
b. $\sin e^x \cos e^x = 2^{x-2} + 2^{-x-2}$	q. 1
c. $\sin 2x + \cos 4x = 2$	r. 2
d. $30\,\lvert\sin x\rvert = x$ in $0 \le x \le 2\pi$	s. 0

3.

Column I (Equation)	Column II (Solution)
a. $\max_{\theta \in R} \{5\sin\theta + 3\sin(\theta - \alpha)\} = 7$ then the set of possible values of α is	p. $2n\pi + 3\pi/4$, $n \in Z$
b. $x \neq \frac{n\pi}{2}$ and $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1$	q. $2n\pi \pm \frac{\pi}{3}; n \in Z$
c. $\sqrt{(\sin x)} + 2^{1/4}\cos x = 0$	r. $2n\pi + \cos^{-1}(1/3)$, $n \in Z$
d. $\log_5 \tan x = (\log_5 4)(\log_4 (3\sin x)$	s. no solution

Integer Type

Solutions on page 3.66

1. Number of values of p for which equation $\sin^3 x + 1 + p^3 - 3p\sin x = 0$ $(p > 0)$ has a root is ________.
2. Number of roots of the equation $|\sin x \cos x| + \sqrt{2 + \tan^2 x + \cot^2 x} = \sqrt{3}$, $x \in [0, 4\pi]$, are ________.
3. Number of roots of the equation $(3 + \cos x)^2 = 4 - 2\sin^8 x$, $x \in [0, 5\pi]$ are ________.
4. Number of solution(s) of the equation $\frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x} = 0$ in the interval $\left(0, \frac{\pi}{4}\right)$ is ________.
5. Number of solutions of the equation $\left(\sqrt{3}+1\right)^{2x} + \left(\sqrt{3}-1\right)^{2x} = 2^{3x}$ is ________.
6. Number of integral value(s) of m for the equation $\sin x - \sqrt{3}\cos x = \frac{4m-6}{4-m}$ has solutions $x \in [0, 2\pi]$ is ________.
7. The value of a for which system of equations $\sin^2 x + \cos^2 y = \frac{3a}{2}$ and $\cos^2 x + \sin^2 y = \frac{a^2}{2}$ has a solution is ________.
8. If $\cos 4x = a_0 + a_1\cos^2 x + a_2\cos^4 x$ is true for all values of $x \in R$, then the value of $5a_0 + a_1 + a_2$ is ________.
9. Number of integral values of a for which the equation $\cos^2 x - \sin x + a = 0$ has roots when $x \in (0, \pi/2)$ is ________.
10. The maximum integral value of a for which the equation $a\sin x + \cos 2x = 2a - 7$ has a solution is ________.
11. Number of roots the equation $2^{\tan\left(x - \frac{\pi}{4}\right)} - 2(0.25)^{\frac{\sin^2\left(x - \frac{\pi}{4}\right)}{\cos 2x}} + 1 = 0$ is ________.
12. Number of solution of the equation $\sin^4 x - \cos^2 x \sin x + 2\sin^2 x + \sin x = 0$ in $0 \le x \le 3\pi$ is ________.

Archives

Solutions on page 3.69

Subjective

1. Find the coordinates of the points of intersection of the curves $y = \cos x$, $y = \sin 3x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
(IIT-JEE, 1982)
2. Find all the solution of $4\cos^2 x \sin x - 2\sin^2 x = 3\sin x$. **(IIT-JEE, 1983)**
3. Find the values of $x \in (-\pi, \pi)$ which satisfy the equation $8^{(1+|\cos x|+|\cos^2 x|+|\cos^3 x|+\ldots)} = 4^3$.
(IIT-JEE, 1984)
4. Find all values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying the equation
$(1-\tan\theta)(1+\tan\theta)\sec^2\theta + 2^{\tan^2\theta} = 0$. **(IIT-JEE, 1996)**
5. Find the number of all possible value of θ, where $0 < \theta < \pi$, for which the system of equations
$$(y+z)\cos 3\theta = (xyz)\sin 3\theta$$
$$x\sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$$
$$(xyz)\sin 3\theta = (y+2z)\cos 3\theta + y\sin 3\theta$$
have a solution (x_0, y_0, z_0) with $y_0 z_0 \ne 0$. **(IIT-JEE, 2010)**
6. Find the number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\theta \ne \frac{n\pi}{5}$ for $n = 0, \pm 1, \pm 2$ and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$. **(IIT-JEE, 2010)**

Objective

Fill in the blanks

1. The solution set of the system of equations $x + y = \frac{2\pi}{3}$, $\cos x + \cos y = \frac{3}{2}$, where x and y are real, is ______. **(IIT-JEE, 1986)**
2. The set of all x in the interval $[0, \pi]$ for which $2\sin^2 x - 3\sin x + 1 \ge 0$ is ______. **(IIT-JEE, 1987)**
3. General value of θ satisfying the equation $\tan^2\theta + \sec 2\theta = 1$ is ______. **(IIT-JEE, 1996)**
4. The real roots of the equation $\cos^7 x + \sin^4 x = 1$ in the interval $(-\pi, \pi)$ are ______, ______ and ______.
(IIT-JEE, 1997)

True or false

1. There exists a value of θ between 0 and 2π that satisfies the equation $\sin^4\theta - 2\sin^2\theta - 1 = 0$.
(IIT-JEE, 1984)

Multiple choice questions with one correct answer

1. The equation $2\cos^2\frac{x}{2}\sin^2 x = x^2 + x^{-2};\ 0 < x \le \frac{\pi}{2}$ has **(IIT-JEE, 1980)**
   a. no real solution
   b. one real solution
   c. more than one solution
   d. none of these
2. The general solution of the trigonometric equation $\sin x + \cos x = 1$ is given by **(IIT-JEE, 1981)**
   a. $x = 2n\pi;\ n = 0, \pm 1, \pm 2, \ldots$
   b. $x = 2n\pi + \pi/2;\ n = 0, \pm 1, \pm 2, \ldots$
   c. $x = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}\ n = 0, \pm 1, \pm 2, \ldots$
   d. none of these
3. The general solution of the equation $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$ is $(n \in Z)$
(IIT-JEE, 1989)

a. $n\pi+\frac{\pi}{8}$ b. $\frac{n\pi}{2}+\frac{\pi}{8}$ c. $(-1)^n\frac{n\pi}{2}+\frac{\pi}{8}$ d. $2n\pi+\cos^{-1}\frac{2}{3}$

4. The equation $(\cos p-1)x^2+(\cos p)x+\sin p=0$ in the variable x has real roots. Then p can take any value in the interval **(IIT-JEE, 1990)**

a. $(0, 2\pi)$ b. $(-\pi, 0)$ c. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ d. $(0, \pi)$

5. The number of solutions of the equation $\tan x+\sec x=2\cos x$ lying in the interval $[0, 2\pi]$ is **(IIT-JEE, 1993)**

a. 0 b. 1 c. 2 d. 3

6. The general values of θ satisfying the equation $2\sin^2\theta-3\sin\theta-2=0$ is $(n\in Z)$ **(IIT-JEE, 1995)**

a. $n\pi+(-1)^n\pi/6$ b. $n\pi+(-1)^n\pi/2$ c. $n\pi+(-1)^n 5\pi/6$ d. $n\pi+(-1)^n 7\pi/6$

7. The number of distinct real roots of $\begin{vmatrix}\sin x & \cos x & \cos x\\ \cos x & \sin x & \cos x\\ \cos x & \cos x & \sin x\end{vmatrix}=0$ in the interval $-\frac{\pi}{4}\le x\le\frac{\pi}{4}$ is **(IIT-JEE, 2001)**

a. 0 b. 2 c. 1 d. 3

8. The number of integral values of k for which the equation $7\cos x+5\sin x=2k+1$ has a solution is

a. 4 b. 8 c. 10 d. 12 **(IIT-JEE, 2002)**

9. $\cos(\alpha-\beta)=1$ and $\cos(\alpha+\beta)=1/e$ where $\alpha, \beta\in[-\pi, \pi]$. Pairs of α, β which satisfy both the equations is/are **(IIT-JEE, 2005)**

a. 0 b. 1 c. 2 d. 4

10. The value of $\theta\in(0, 2\pi)$ for which the equation is $2\sin^2\theta-5\sin\theta+2>0$ is **(IIT-JEE, 2006)**

a. $\left(0, \frac{\pi}{6}\right)\cup\left(\frac{5\pi}{6}, 2\pi\right)$ b. $\left(\frac{\pi}{8}, \frac{5\pi}{6}\right)$ c. $\left(0, \frac{\pi}{8}\right)\cup\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ d. $\left(\frac{41\pi}{48}, \pi\right)$

11. The number of solutions of the pair of equations **(IIT-JEE, 2007)**
$2\sin^2\theta-\cos 2\theta=0$
$2\cos^2\theta-3\sin\theta=0$
in the interval $[0, 2\pi]$ is

a. 0 b. 1 c. 2 d. 4

Multiple choice questions with one or more than one correct answers

1. The number of all the possible triplets (a_1, a_2, a_3) such that $a_1+a_2\cos(2x)+a_3\sin^2(x)=0$ for all x is **(IIT-JEE, 1987)**

a. 0 b. 1 c. 3 d. infinite

2. The values of θ lying between $\theta=0$ and $\theta=\theta/2$ and satisfying the equation

$$\begin{vmatrix}1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta\\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta\\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta\end{vmatrix}=0 \text{ are}$$ **(IIT-JEE, 1988)**

a. $7\pi/24$ b. $5\pi/24$ c. $11\pi/24$ d. $\pi/24$

3. The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x-7\sin x+2=0$ is

a. 0 b. 5 c. 6 d. 10 **(IIT-JEE, 1998)**

4. Let $2\sin^2 x+3\sin x-2>0$ and $x^2-x-2<0$ (x is measured in radians). Then x lies in the interval

a. $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ b. $\left(-1, \frac{5\pi}{6}\right)$ c. $(-1, 2)$ d. $\left(\frac{\pi}{6}, 2\right)$

(IIT-JEE, 1994)

5. If $\frac{\sin^4 x}{2}+\frac{\cos^4 x}{3}=\frac{1}{5}$, then

a. $\tan^2 x=\frac{2}{3}$ b. $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{1}{125}$ c. $\tan^2 x=\frac{1}{3}$ d. $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{2}{125}$

(IIT-JEE, 2009)

ANSWERS AND SOLUTIONS

Subjective Type

1. We have $3(\tan 2x-\tan 3x)=\tan 3x(1+\tan 3x\tan 2x)$

$\Rightarrow$ $3(\tan 2x-\tan 3x)/(1+\tan 3x\tan 2x)=\tan 3x$

$\Rightarrow$ $3\tan(2x-3x)=\tan 3x$

$\Rightarrow$ $3\tan x+(3\tan x-\tan^3 x)/(1-3\tan^2 x)=0$

$\Rightarrow$ $\tan x[3(1-3\tan^2 x)+3-\tan^2 x]=0$

$\Rightarrow$ $\tan x(6-10\tan^2 x)=0$

$\Rightarrow$ $\tan x=0$ or $\tan^2 x=3/5$

If $\tan x=0 \Rightarrow x=n\pi, n\in Z$

and if $\tan^2 x=3/5 \Rightarrow x=m\pi\pm\alpha=m\pi\pm\tan^{-1}\sqrt{3/5}, m\in Z$

Hence, $x=n\pi, m\pi\pm\tan^{-1}\sqrt{3/5}, \forall\, m, n\in Z$.

2. The given equation can be rewritten as $2[\sin(2x+\pi/6)+\sin\pi/2]=a^2+\sqrt{3}\sin 2x-\cos 2x$

$\Rightarrow$ $\cos 2x=(a^2-2)/2$

$\Rightarrow$ $2\cos^2 x=a^2/2$ or $\cos^2 x=(a/2)^2$ (i)

$\Rightarrow$ $a^2\le 4$ or $-2\le a\le 2$ (ii)

For $a=0$, the given equation is reduced to

$\cos x=0$, i.e., $x=n\pi+(\pi/2), n\in Z$ (iii)

3. $\sin^4(x/3)+\cos^4(x/3)>1/2\ (n\in Z)$

$\Rightarrow$ $1-2\sin^2(x/3)\cos^2(x/3)>1/2$

$\Rightarrow$ $1-\frac{1}{2}\sin^2(2x/3)>\frac{1}{2}$

$\Rightarrow$ $\sin^2(2x/3)<1$

which is always true except when $\sin^2(2x/3)=1$

This means $2x/3=n\pi\pm(\pi/2)$ or $x=(3n\pi/2)\pm(3\pi/4), n\in Z$

Hence, solution set of the inequality is $R-\{x: x=(3n\pi/2)\pm(3\pi/4), n\in Z\}$.

4. $\sin x+\sin y=\sin(x+y)$

$\Rightarrow$ $2\sin\frac{x+y}{2}\left[\cos\frac{x-y}{2}-\cos\frac{x+y}{2}\right]=0$

$\Rightarrow$ $4\sin\frac{x+y}{2}\sin\frac{x}{2}\sin\frac{y}{2}=0$

a. $\sin\frac{x+y}{2}=0 \Rightarrow x+y=2n\pi, n\in Z \Rightarrow x+y=0\ (\because |x|+|y|=1\Rightarrow -1\le x, y\le 1)$

b. $\sin\frac{x}{2}=0 \Rightarrow x=2m\pi, m\in Z \Rightarrow x=0$

c. $\sin\frac{y}{2}=0 \Rightarrow y=2p\pi, p\in Z \Rightarrow y=0$

From $|x|+|y|=1$

If $x=0$, then $|y|=1 \Rightarrow y=\pm 1$

If $y=0$, then $|x|=1 \Rightarrow x=\pm 1$

If $y=-x$, then $|x|+|-x|=2 \Rightarrow x=\pm\frac{1}{2}$ and $y=\mp\frac{1}{2}$

Hence, solutions are $(0, 1), (0, -1), (1, 0), (-1, 0), \left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

5. $\tan^4 x+\tan^4 y+2\cot^2 x\cot^2 y=3+\sin^2(x+y)$

$\Rightarrow \tan^4 x+\tan^4 y+2\cot^2 x\cot^2 y-2=1+\sin^2(x+y)$

$\Rightarrow (\tan^2 x-\tan^2 y)^2+2(\tan x\tan y-\cot x\cot y)^2=-1+\sin^2(x+y)$

Now L.H.S. ≥ 0 and R.H.S. ≤ 0

$\Rightarrow$ L.H.S. = R.H.S. = 0

$\Rightarrow \tan^2 x=\tan^2 y$ and $\tan^2 x\tan^2 y=1$ and $\sin^2(x+y)=0$

$=\tan^2 y=1$ and $x+y=n\pi, n\in Z$

$\Rightarrow x=n\pi\pm\frac{\pi}{4}, n\in Z$ and $y=n\pi\mp\frac{\pi}{4}, n\in Z$.

6. The given equation is possible if $\sin(1-x)\geq 0$ and $\cos x\geq 0$.

On squaring, we get $\sin(1-x)=\cos x$

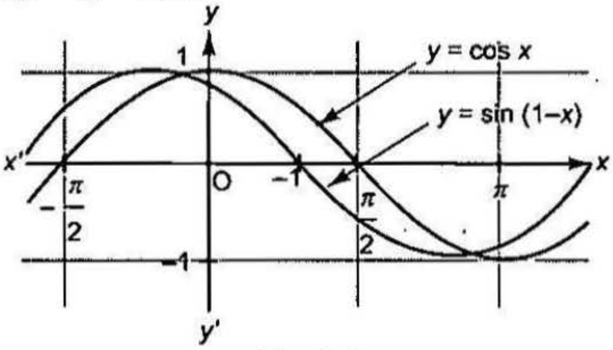


Fig. 3.6

$\Rightarrow \cos\left(\frac{\pi}{2}-(1-x)\right)=\cos x$

$\Rightarrow \frac{\pi}{2}-1+x=2n\pi\pm x, n\in Z$

$\Rightarrow x=\dfrac{2n\pi-\frac{\pi}{2}+1}{2}, n\in Z$

For $n=2$, $x=\frac{7\pi}{4}+\frac{1}{2}$ which is the smallest positive root of the given equation.

7. $2\sin x+\cos y=2$

$\Rightarrow \cos y=2(1-\sin x)$, we have $\cos y\in[-1, 1]$

$\Rightarrow -\frac{1}{2}\leq 1-\sin x\leq\frac{1}{2} \Rightarrow \frac{1}{2}\leq\sin x\leq\frac{3}{2} \Rightarrow \frac{1}{2}\leq\sin x\leq 1$

Let $t = \sin x \quad \Rightarrow \quad x = \sin^{-1}(t), t \in [1/2, 1]$

$$\Rightarrow \quad \cos y = 2(1-t) \quad \Rightarrow \quad \left.\begin{array}{l} y = 2n_1\pi \pm \cos^{-1} 2(1-t) \\ \text{and } x = n_2\pi + (-1)^{n_2} \sin^{-1}(t) \end{array}\right\} t \in [1/2, 1]$$

8. $\sin x = \frac{1}{2}|x| + \frac{a}{2}$ or $2 \sin x = |x| + a$. Consider graphs of $y = 2\sin x$ and $y = |x|$.

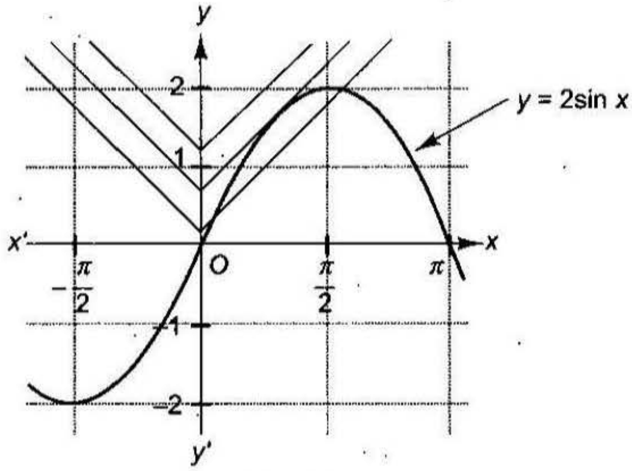


Fig. 3.7

Equation $2 \sin x = |x| + a$ will have a solution so long as the line $y = |x| + a$ intersects or at least touches the curve, $y = 2\sin x$. In this case, we must have $dy/dx = 2\cos x = 1$ = the slope of the line $\Rightarrow \quad x = \pi/3$.

Hence, the solution exists if $\frac{\pi}{3} + a > 2\sin \frac{\pi}{3} \Rightarrow a > \frac{3\sqrt{3} - \pi}{3}$

9. $\tan\left(\frac{\pi}{2} \cos \theta\right) = \cot\left(\frac{\pi}{2} \sin \theta\right) = \tan\left(\frac{\pi}{2} - \frac{\pi}{2} \sin \theta\right)$

$\Rightarrow \quad \frac{\pi}{2} \cos\theta = n\pi + \frac{\pi}{2} - \frac{\pi}{2} \sin\theta, n \in Z$

$\Rightarrow \quad \frac{\pi}{2} (\sin\theta + \cos\theta) = n\pi + \frac{\pi}{2} = \left(n + \frac{1}{2}\right) \pi$

$\Rightarrow \quad \sin\theta + \cos\theta = (2n+1)$

$\Rightarrow \quad \sqrt{2} \sin\left(\theta + \frac{\pi}{4}\right) = (2n+1)$

$\Rightarrow \quad n = 0, -1$ are the only possibilities

So, $\sin\left(\theta + \frac{\pi}{4}\right) = \pm\frac{1}{\sqrt{2}} = \sin\left(\pm\frac{\pi}{4}\right)$

$\Rightarrow \quad \theta + \frac{\pi}{4} = m\frac{\pi}{2} + \frac{\pi}{4}, m \in Z$

$\Rightarrow \quad \theta = m\frac{\pi}{2}, m \in Z$

However, for the values of $m = 2k, k \in Z$, the equation is not defined.

Hence, $\theta = (2k+1)\frac{\pi}{2}$, where $k \in Z$.

10. $\sin x+\sin\left(\frac{\pi}{8}\sqrt{(1-\cos 2x)^2+\sin^2 2x}\right)=0$

$$\sin\left(\frac{\pi}{8}\sqrt{(1-\cos 2x)^2+\sin^2 2x}\right)=\sin\frac{\pi}{8}\sqrt{2-2\cos 2x}=\sin\left(\frac{\pi}{4}|\sin x|\right)$$

Now $\sin x+\sin\left(\frac{\pi}{4}|\sin x|\right)=0$ (i)

The equation has a solution only when $\sin x\le 0$.
The graph of $f(x)=\sin x\le 0$ is shown in Fig. 3.8.

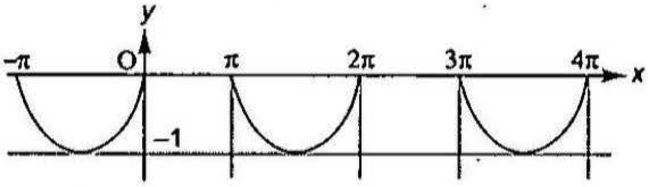


Fig. 3.8

The graph $y=\sin[\pi/4\ |\sin x|]$ is as shown in Fig. 3.9.

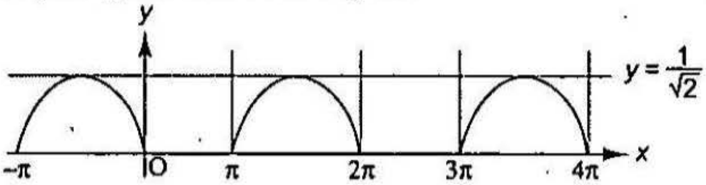


Fig. 3.9

Hence, Eq. (i) has general solution $x=n\pi,\ n\in Z$.

11. $\sin^2 x+\frac{1}{4}\sin^2 3x=\sin x\sin^2 3x$

$\Rightarrow\quad \sin^2 x-\sin x\sin^2 3x+\frac{1}{4}\sin^2 3x=0$

$\Rightarrow\quad \left(\sin x-\frac{1}{2}\sin^2 3x\right)^2+\frac{1}{4}\sin^2 3x\left(1-\sin^2 3x\right)=0$

$\Rightarrow\quad \left(\sin x-\frac{1}{2}\sin^2 3x\right)^2+\frac{1}{4}\sin^2 3x\cos^2 3x=0$

$\Rightarrow\quad \left(\sin x-\frac{1}{2}\sin^2 3x\right)^2+\frac{1}{16}\sin^2 6x=0$

$\Rightarrow\quad \sin x-\frac{1}{2}\sin^2 3x=0$ and $\sin 6x=0$

$\Rightarrow\quad 2\sin x=\sin^2 3x$ and $\sin 6x=0\quad\Rightarrow$ From $\sin 6x=0,\ x=k\pi/6,\ k\in Z$

From here, we choose those values which satisfy the equation, $2\sin x=\sin^2 3x$

Now $\sin^2 3\left(\frac{k\pi}{6}\right)=+\sin^2\frac{k\pi}{2}=\begin{Bmatrix}1, \text{ if } k \text{ is odd}\\ 0, \text{ if } k \text{ is even}\end{Bmatrix}$

$\Rightarrow\quad \sin x=0$ or $1/2$

$\Rightarrow\quad x=n\pi$ or $x=n\pi+\frac{\pi}{6}(-1)^n,\ n\in Z$

Objective Type

1. a. $\sin\theta = 1/2$ and $\cos\theta = -\sqrt{3}/2$

$\Rightarrow$ θ lies in the second quadrant.

$\Rightarrow$ $\sin\theta = \sin 5\pi/6$; $\cos\theta = \cos 5\pi/6$;

$\therefore \theta = 2n\pi + (5\pi/6)$

2. c. Since $\tan\theta < 0$ and $\cos\theta > 0$, θ lies in the fourth quadrant. Then $\theta = 7\pi/4$. Hence, the general value of θ is $2n\pi + 7\pi/4, n \in Z$.

3. c. $\cos p\theta = -\cos q\theta = \cos(\pi - q\theta)$

$\Rightarrow$ $p\theta = 2n\pi \pm (\pi - q\theta)$

$\Rightarrow$ $(p \mp q)\,\theta = (2n \pm 1)\,\pi$

$$\Rightarrow \quad \theta = \frac{(2n \pm 1)\pi}{(p \mp q)}, n \in Z$$

$$\Rightarrow \quad \theta = \frac{r\pi}{p \pm q}, \text{ where } r = -3, -1, 1, 3, \ldots$$

$$\Rightarrow \quad \theta = \ldots, \frac{-3\pi}{p \pm q}, \frac{-\pi}{p \pm q}, \frac{\pi}{p \pm q}, \frac{3\pi}{p \pm q}, \ldots$$

Shown above is an A.P. of common difference $\dfrac{2\pi}{p \pm q}$.

4. d. $(\cos\theta + \cos 7\theta) + (\cos 3\theta + \cos 5\theta) = 0$

$\Rightarrow$ $2\cos 4\theta(\cos 3\theta + \cos\theta) = 0$

$\Rightarrow$ $4\cos 4\theta \cos 2\theta \cos\theta = 0$

$$\Rightarrow \quad 4 \times \frac{1}{2^3 \sin\theta}\left(\sin 2^3\,\theta\right) = 0$$

$\Rightarrow$ $\sin 8\theta = 0$ or $\theta = n\pi/8, n \in Z$

5. b. $$3\frac{\sin^2\theta}{\cos^2\theta} - 2\sin\theta = 0, \cos\theta \neq 0$$

$\Rightarrow$ $3\sin^2\theta - 2\sin\theta(1 - \sin^2\theta) = 0$

$\Rightarrow$ $\sin\theta(2\sin^2\theta + 3\sin\theta - 2) = 0$

$\Rightarrow$ $\sin\theta(2\sin\theta - 1)(\sin\theta + 2) = 0$

$\Rightarrow$ $\sin\theta = 0, 1$

$\Rightarrow$ $\theta = n\pi, n\pi + (-1)^n(\pi/6), n \in Z$

6. b. We have $1^2 = \sin\theta\cos 2\theta$

$\Rightarrow$ $1 - \sin\theta(1 - 2\sin^2\theta) = 0$

$\Rightarrow$ $2\sin^3\theta - \sin\theta + 1 = 0$

$\Rightarrow$ $(\sin\theta + 1)(2\sin^2\theta - 2\sin\theta + 1) = 0$

$\Rightarrow$ $\sin\theta = -1$

The other factor gives imaginary roots.

$\Rightarrow \quad \theta = n\pi + (-1)^n \left(-\frac{\pi}{2}\right) = n\pi - (-1)^n \frac{\pi}{2} = n\pi + (-1)^{n-1} \frac{\pi}{2},\ n \in Z.$

7. b. $2\cos\theta[\cos 120° + \cos 2\theta] = 1$

$\Rightarrow \quad 2\cos\theta\left(-\frac{1}{2} + 2\cos^2\theta - 1\right) = 1$

$\Rightarrow \quad 4\cos^3\theta - 3\cos\theta - 1 = 0$

$\Rightarrow \quad \cos 3\theta = 1 = \cos 0$

$\Rightarrow \quad 3\theta = 2n\pi \text{ or } \theta = \frac{2n\pi}{3},\ n \in Z$

Given the values so that $2n$ does not exceed 18.

$\therefore \quad n = 0, 1, 2, 3, \ldots, 9$

Hence, the sum $= \frac{2\pi}{3}\sum_{1}^{9} n = \frac{2\pi}{3} \times \frac{9(9+1)}{2} = 30\pi.$

8. b. $\sec\theta - 1 = (\sqrt{2} - 1)\tan\theta \Rightarrow \frac{1 - \cos\theta}{\cos\theta} = \frac{(\sqrt{2} - 1)\sin\theta}{\cos\theta}$

$\Rightarrow \quad 2\sin^2(\theta/2) = (\sqrt{2} - 1)\, 2\sin(\theta/2)\cos(\theta/2)$

$\Rightarrow \quad \sin(\theta/2) = 0 \text{ or } \tan(\theta/2) = (\sqrt{2} - 1) = \tan(\pi/8)$

$\Rightarrow \quad \theta/2 = n\pi \text{ or } \theta/2 = n\pi + (\pi/8),\ n \in Z$

$\Rightarrow \quad \theta = 2n\pi \text{ or } \theta = 2n\pi + (\pi/4),\ n \in Z$

9. a. $\sin^4 x + \cos^4 x = \sin x \cos x$

$\Rightarrow \quad (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x = \sin x\cos x$

$\Rightarrow \quad 1 - \frac{\sin^2 2x}{2} = \frac{\sin 2x}{2}$

$\Rightarrow \quad \sin^2 2x + \sin 2x - 2 = 0$

$\Rightarrow \quad (\sin 2x + 2)(\sin 2x - 1) = 0$

$\Rightarrow \quad \sin 2x = 1$

$\Rightarrow \quad 2x = (4n+1)\frac{\pi}{2},\ n \in Z$

$\Rightarrow \quad x = (4n+1)\frac{\pi}{4},\ n \in Z$

$= \frac{\pi}{4}, \frac{5\pi}{4} \quad (\because x \in [0, 2\pi])$

Thus, there are two solutions.

10. c. $\sin 3x + (\sin 5x + \sin x) = 0$

$\Rightarrow \quad \sin 3x + (2\sin 3x\cos 2x) = 0$

$\Rightarrow \quad \sin 3x = 0 \text{ or } \cos 2x = -\frac{1}{2} = \cos\frac{2\pi}{3}$

$\Rightarrow \quad x = n\pi/3 \text{ or } x = n\pi \pm \pi/3,\ n \in Z$

Then $x = 0, \pi/3$, and $2\pi/3$, Hence, there are three solutions.

11. a. From the given relation

$\cos\theta = (2\sin\theta\cos\theta)\sin\theta$, $\sin\theta \neq 0$

$\Rightarrow \quad \sin\theta = \pm\dfrac{1}{\sqrt{2}}$ or $\cos\theta = 0$

$\Rightarrow \quad \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{\pi}{2}$ $(\because \theta \in [0, \pi])$

Then the sum of roots is $\dfrac{3\pi}{2}$.

12. c. The given equation is $(\cos x - 1)(12\cos^2 x + 5\cos x + 9) = 0$

$\Rightarrow \quad \cos x = 1$ only as the other factor gives imaginary roots

$= 1 \Rightarrow x = 2n\pi, n \in Z$

Hence, it has infinite solutions as $n \in Z$.

13. d. $\cos 2x = 1 - 2\sin^2 x$ and put $2^{-\sin^2 x} = t$

$\Rightarrow \quad 2^{\cos 2x} = 2^{1-2\sin^2 x} = 2\left(2^{-\sin^2 x}\right)^2 = 2t^2$

$\Rightarrow \quad 2t^2 - 3t + 1 = 0$

$\Rightarrow \quad t = 1, 1/2$

$\Rightarrow \quad 2^{-\sin^2 x} = 1 = 2^0$

$\Rightarrow \quad \sin^2 x = 0$ or $x = n\pi, n \in Z$

From $2^{-\sin^2 x} = \dfrac{1}{2} = 2^{-1}$, we get

$\sin^2 x = 1$ or $x = n\pi \pm \dfrac{\pi}{2}, n \in Z$

14. a. $\cos x \cos 6x = -1$

$\Rightarrow \quad 2\cos x\cos 6x = -2 \quad \Rightarrow \quad \cos 7x + \cos 5x = -2 \quad \Rightarrow \quad \cos 7x = -1$ and $\cos 5x = -1$

The value of x satisfying these two equations simultaneously and lying between 0 and 2π is π.

Therefore, the general solution is $x = 2n\pi + \pi, n \in Z$.

$\Rightarrow \quad x = (2n+1)\pi, n \in Z$

15. c. This is possible only when $\sin x = \cos x = 1$, which does not hold simultaneously.

Hence, there is no solution.

16. a. The given equation is $3(\sin x + \cos x) - 2(\sin x + \cos x)(1 - \sin x\cos x) = 8$

$\Rightarrow \quad (\sin x + \cos x)[3 - 2 + 2\sin x\cos x] = 8$

$\Rightarrow \quad (\sin x + \cos x)[\sin^2 x + \cos^2 x + 2\sin x\cos x] = 8$

$\Rightarrow \quad (\sin x + \cos x)^3 = 8$

$\Rightarrow \quad \sin x + \cos x = 2$

Above solution is not possible. Hence, the given equation has no solution.

17. a. $\cos^2\theta = \dfrac{1}{6}\sin\theta\tan\theta$

$\Rightarrow \quad 6\cos^3\theta = 1 - \cos^2\theta$

$\Rightarrow \quad 6\cos^3\theta + \cos^2\theta - 1 = 0$

$\Rightarrow \quad (2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}, n \in Z$

The other factor gives imaginary roots.

18. d. $(1-\cos 2x) + (1-\cos^2 2x) = 2$

$\Rightarrow \quad \cos 2x(\cos 2x + 1) = 0$

$\Rightarrow \quad \cos 2x = 0$ or $\cos 2x = -1$

$\Rightarrow \quad 2x = (2n+1)\pi/2$ or $2x = (2n \pm 1)\pi, n \in Z$

$\Rightarrow \quad x = (2n+1)\pi/4$ or $x = (2n \pm 1)\pi/2, n \in Z$

Hence, the solutions are $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4, \pi/2, 3\pi/2$.

19. c. Dividing the given equation by $\cos^2 x$, as $\cos x = 0$ does not satisfy the equation, we have

$\tan^2 x - 5\tan x - 6 = 0$

$\Rightarrow \quad (\tan x + 1)(\tan x - 6) = 0$

$\Rightarrow \quad \tan x = -1$ or $\tan x = 6$

If $\tan x = -1 = \tan(-\pi/4)$, then $x = n\pi - \pi/4, \forall n \in Z$

and, if $\tan x = 6 = \tan\alpha$ (say)

$\Rightarrow \quad \alpha = \tan^{-1} 6$, then $x = n\pi + \alpha = n\pi + \tan^{-1} 6, \forall n \in Z$

Hence, $x = n\pi - (\pi/4), n\pi + \tan^{-1} 6, n \in Z$.

20. d. From the given equation, we have $\frac{\tan\theta + \tan 4\theta}{1 - \tan\theta \tan 4\theta} = -\tan 7\theta$

$\Rightarrow \quad \tan(\theta + 4\theta) = -\tan 7\theta$

$\Rightarrow \quad \tan 5\theta = \tan(-7\theta)$

$\Rightarrow \quad 5\theta = n\pi - 7\theta$

$\Rightarrow \quad \theta = n\pi/12$, where $n \in Z$, but $n \neq 6, 18, 30, \ldots$

21. d. We have $\frac{1}{\sin^2\theta\cos^2\theta} + \frac{2}{\sin^2\theta} = 8, \sin\theta \neq 0, \cos\theta \neq 0$

$\Rightarrow \quad 1 + 2\cos^2\theta = 8\sin^2\theta\cos^2\theta = 8\cos^2\theta(1-\cos^2\theta)$

$\Rightarrow \quad 8\cos^4\theta - 6\cos^2\theta + 1 = 0$

$\Rightarrow \quad (4\cos^2\theta - 1)(2\cos^2\theta - 1) = 0$

$\Rightarrow \quad \cos^2\theta = 1/4 = \cos^2(\pi/3)$ or $\cos^2\theta = 1/2 = \cos^2(\pi/4)$

$\Rightarrow \quad \theta = n\pi \pm (\pi/3)$ or $\theta = n\pi \pm (\pi/4), n \in Z$

Hence, for $0 \leq \theta \leq \pi/2, \theta = \pi/3, \theta = \pi/4$

22. b. $\tan x + \cot x = 2\operatorname{cosec} x$

$\Rightarrow \quad \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{2}{\sin x}$

$\Rightarrow \quad \frac{1}{\sin x\cos x} = \frac{2}{\sin x}$

$\Rightarrow \quad \cos x = \frac{1}{2}$

$\Rightarrow \quad x = \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}$

Thus, there are four solutions.

23. c. Let $Z = \frac{3+2i\sin\theta}{1-2i\sin\theta} = \frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)} = \frac{(3-4\sin^2\theta)+8i\sin\theta}{1+4\sin^2\theta}$

Therefore, the real part of $Z = \frac{3-4\sin^2\theta}{1+4\sin^2\theta}$ and the imaginary part of $Z = \frac{8\sin\theta}{1+4\sin^2\theta}$

Z is real, if imaginary part $= \frac{8\sin\theta}{1+4\sin^2\theta} = 0$ or $\sin\theta = 0$ or $\theta = n\pi, \forall n \in I$

Z is purely imaginary, if real part $(3-4\sin^2\theta)/(1+4\sin^2\theta) = 0$
or $\sin^2\theta = 3/4 = \sin^2(\pi/3)$ or $\theta = n\pi \pm \pi/3, \forall n \in I$

24.b. $1 - \sin^2 x + \frac{\sqrt{3}+1}{2}\sin x - \frac{\sqrt{3}}{4} - 1 = 0$

$\Rightarrow \sin^2 x - \frac{\sqrt{3}+1}{2}\sin x + \frac{\sqrt{3}}{4} = 0$

$\Rightarrow 4\sin^2 x - 2\sqrt{3}\sin x - 2\sin x + \sqrt{3} = 0$

On solving, we get $\sin x = 1/2, \sqrt{3}/2$

$\Rightarrow x = \pi/6, 5\pi/6; \pi/3, 2\pi/3$

25. d. Since, $7\cos^2 x + \sin x\cos x - 3 = 0$,
Dividing the equation by $\cos^2 x$, we get
$7 + \tan x - 3\sec^2 x = 0$

$\Rightarrow \quad 7 + \tan x - 3(1 + \tan^2 x) = 0$

$\Rightarrow \quad 3\tan^2 x - \tan x - 4 = 0$

$\Rightarrow \quad (\tan x + 1)(3\tan x - 4) = 0$

$\Rightarrow \quad \tan x = -1$ or $\tan x = \frac{4}{3}$

$\Rightarrow \quad x = n\pi + \frac{3\pi}{4}$ or $x = k\pi + \tan^{-1}\left(\frac{4}{3}\right)$, where $(k, n \in Z)$

26. b. $(\sin\theta + 2)(\sin\theta + 3)(\sin\theta + 4) = 6$
L.H.S. > 6 and R.H.S. 6
Therefore, equality only holds if $\sin\theta = -1 \Rightarrow \theta = 3\pi/2, 7\pi/2$
Therefore, sum $= 5\pi \Rightarrow k = 5$

27. b. $\sin^2 x + a\cos x + a^2 > 1 + \cos x$
Putting $x = 0$, we get

$\Rightarrow \quad a + a^2 > 2$

$\Rightarrow \quad a^2 + a - 2 > 0$

$\Rightarrow \quad (a+2)(a-1) > 0$

$\Rightarrow \quad a < -2$ or $a > 1$

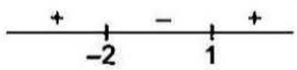


Fig. 3.10

Therefore, we have the largest negative integral value of $a = -3$.

28. b. $\sin^4 x - \cos^2 x \sin x + 2\sin^2 x + \sin x = 0$

$\Rightarrow \sin x [\sin^3 x - \cos^2 x + 2\sin x + 1] = 0$

$\Rightarrow \sin x [\sin^3 x - 1 + \sin^2 x + 2\sin x + 1] = 0$

$\Rightarrow \sin x [\sin^3 x + \sin^2 x + 2\sin x] = 0$

$\Rightarrow \sin^2 x = 0$ or $\sin^2 x + \sin x + 2 = 0$ (not possible for real x)

$\Rightarrow \sin x = 0$

Hence, the solutions are $x = 0, \pi, 2\pi, 3\pi$.

29. a. Since, $x \in [0, 2\pi]$ and $y \in [0, 2\pi]$,
and $\sin x + \sin y = 2$
This is possible only, when $\sin x = 1$ and $\sin y = 1$
$\Rightarrow x = \pi/2$ and $y = \pi/2$
Hence, $x + y = \pi$.

30. a. $(\sqrt{3} - 1)\sin\theta + (\sqrt{3} + 1)\cos\theta = 2$

$$\Rightarrow \frac{(\sqrt{3}-1)}{2\sqrt{2}}\sin\theta + \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\cos\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \sin\frac{\pi}{12}\sin\theta + \cos\frac{\pi}{12}\cos\theta = \cos\frac{\pi}{4}$$

$$\Rightarrow \cos\left(\theta - \frac{\pi}{12}\right) = \cos\frac{\pi}{4}$$

$$\Rightarrow \theta - \frac{\pi}{12} = 2n\pi \pm \frac{\pi}{4}, n \in Z$$

$$= 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$$

31. a. $\sin 6\theta + \sin 4\theta + \sin 2\theta = 0$

$\Rightarrow (\sin 6\theta + \sin 2\theta) + \sin 4\theta = 0$

$\Rightarrow 2\sin 4\theta \cos 2\theta + \sin 4\theta = 0$

$\Rightarrow \sin 4\theta\,(2\cos 2\theta + 1) = 0$

$$\Rightarrow \sin 4\theta = 0 \text{ or } \cos 2\theta = -\frac{1}{2} = \cos\frac{2\pi}{3}$$

$$\Rightarrow 4\theta = n\pi \text{ or } 2\theta = 2n\pi \pm \frac{2\pi}{3}, n \in Z$$

$$\Rightarrow \theta = \frac{n\pi}{4} \text{ or } \theta = n\pi \pm \frac{\pi}{3}$$

32. d. The given expression $\cos y \sin x - \sin y \cos x + \sin y \sin x + \cos x \cos y$ is
$\sin(x - y) + \cos(x - y)$
$\therefore \sin(x - y) + \cos(x - y) = 0$

$$\Rightarrow \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin(x - y) + \frac{1}{\sqrt{2}}\cos(x - y)\right) = 0$$

$$\Rightarrow \sin\left(x - y + \frac{\pi}{4}\right) = 0$$

$\Rightarrow \quad \frac{\pi}{4} + x - y = n\pi, n \in Z$

$\Rightarrow \quad x - y = n\pi - \frac{\pi}{4}, n \in Z$

$\Rightarrow \quad x = n\pi - \frac{\pi}{4} + y$ where $n \in Z$

33.c. $\tan x \tan 4x = 1$

$\Rightarrow \cos 4x \cos x - \sin 4x \sin x = 0$

$\Rightarrow \cos 5x = 0$

$\Rightarrow 5x = \left(n + \frac{1}{2}\right)\pi, n \in Z$

$\Rightarrow x = \frac{(2n+1)}{10}\pi, 0 < x < \pi$

$= \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$

Thus, there are only five solutions.

34. a. Let $f(x) = \cos x - x + \frac{1}{2}$

$f(0) = 1 + \frac{1}{2} > 0$

$f\left(\frac{\pi}{2}\right) = 0 - \frac{\pi}{2} + \frac{1}{2} = \frac{1-\pi}{2} < 0$

Therefore, one root lies in the interval $\left(0, \frac{\pi}{2}\right)$.

35. d. $\tan\left(\frac{p\pi}{4}\right) = \cot\left(\frac{q\pi}{4}\right) = \tan\left(\frac{\pi}{2} - \frac{q\pi}{4}\right)$

$\Rightarrow \quad \frac{p\pi}{4} = n\pi + \frac{\pi}{2} - \frac{q\pi}{4}$

$\Rightarrow \quad \frac{p}{4} = n + \frac{1}{2} - \frac{q}{4}$

$\Rightarrow \quad \frac{p+q}{4} = \frac{2n+1}{2}$

$\Rightarrow \quad p + q = 2(2n+1)$

36. b. $y = \sin x - \cos x = \sqrt{2}\left[\frac{1}{\sqrt{2}} \sin \; - \frac{1}{\sqrt{2}} \cos x\right]$

$= \sqrt{2} \sin\left(x - \frac{\pi}{4}\right) \quad \Rightarrow \quad -\sqrt{2} \le y \le \sqrt{2} \quad \Rightarrow$ Range of y is $[-\sqrt{2}, \sqrt{2}]$

37. a. $4\sin^4 x + \cos^4 x = 1$

$\Rightarrow \quad (2\sin^2 x)^2 + \frac{1}{4}(2\cos^2 x)^2 = 1$

$\Rightarrow \quad (1 - \cos 2x)^2 + \frac{1}{4}(1 + \cos 2x)^2 = 1$

$\Rightarrow \quad 5\cos^2 2x - 6\cos 2x + 1 = 0$

$\Rightarrow \quad (\cos 2x - 1)(5\cos 2x - 1) = 0$

$\Rightarrow$ $\cos 2x = 1$ or $\cos 2x = 1/5$

$\Rightarrow$ $2x = 2n\pi$ or $2x = 2n\pi \pm \alpha$, where $\alpha = \cos^{-1}(1/5)$, $\forall n \in Z$

38. c. $(1 - \tan\theta)[1 + 2\tan\theta/(1 + \tan^2\theta)] = 1 + \tan\theta$

$\Rightarrow$ $(1 - \tan\theta)(1 + \tan\theta)^2 = (1 + \tan\theta)(1 + \tan^2\theta)$

$\Rightarrow$ $(1 + \tan\theta)[(1 - \tan^2\theta) - (1 + \tan^2\theta)] = 0$

$\Rightarrow$ $-2\tan^2\theta = 0$, $(1 + \tan\theta) = 0$

$\Rightarrow$ $\tan\theta = 0$, or $\tan\theta = -1$

$\Rightarrow$ $\theta = n\pi$ or $n\pi - \pi/4$, $\forall n \in Z$, for $\theta \in [0, 2\pi]$ $\theta = 0, \pi, 2\pi, 3\pi/4, 7\pi/4$

39. d. We have $(\sin x + \sin 3x) + \sin 2x = (\cos x + \cos 3x) + \cos 2x$

$\Rightarrow$ $2\sin 2x\cos x + \sin 2x = 2\cos 2x\cos x + \cos 2x$

$\Rightarrow$ $\sin 2x(2\cos x + 1) = \cos 2x(2\cos x + 1)$

$\Rightarrow$ $(2\cos x + 1)(\sin 2x - \cos 2x) = 0$

$\Rightarrow$ $\cos x = -1/2$ or $\sin 2x - \cos 2x = 0$

$\Rightarrow$ $x = 2n\pi \pm (2\pi/3)$ or $\tan 2x = 1 = \tan(\pi/4)$

$= 2n\pi \pm (2\pi/3)$ or $x = (4n + 1)\pi/8$, $n \in Z$

But here $0 \le x \le 2\pi$

Hence, $x = \pi/8, 5\pi/8, 2\pi/3, 9\pi/8, 4\pi/3, 13\pi/8$.

40. b. $3\sin\theta - 4\sin^3\theta - \sin\theta = 2(2\cos^2\theta - 1)$

$\Rightarrow$ $2\sin\theta(1 - 2\sin^2\theta) = 2\cos 2\theta$

$\Rightarrow$ $2\cos 2\theta(\sin\theta - 1) = 0$

$\Rightarrow$ $\cos 2\theta = 0$ or $\sin\theta = 1$

$\Rightarrow$ $2\theta = (2n + 1)\pi/2$ or $\theta = 2n\pi + \pi/2$, $\forall n \in Z$

$\Rightarrow$ $\theta = (2n + 1)\pi/4$, or $\theta = (4n + 1)\pi/2$, $\forall n \in Z$

Hence, $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4, \pi/2$. $(\because \theta \in [0, 2\pi])$

41. c. We have $4\sin\theta\sin 2\theta\sin 4\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow$ $\sin\theta[4\sin 2\theta\sin 4\theta - 3 + 4\sin^2\theta] = 0$

$\Rightarrow$ $\sin\theta[2(\cos 2\theta - \cos 6\theta) - 3 + 2(1 - \cos 2\theta)] = 0$

$\Rightarrow$ $\sin\theta(-2\cos 6\theta - 1) = 0$

$\Rightarrow$ $\sin\theta = 0$ or $\cos 6\theta = -1/2$

$\Rightarrow$ $\theta = n\pi$ or $6\theta = 2n\pi \pm 2\pi/3$, $\forall n \in Z$

$= n\pi$ or $\theta = (3n \pm 1)\pi/9$, $\forall n \in Z$

42. b. From the given equation, we have $\tan\theta + \tan 2\theta + \tan(\theta + 2\theta) = 0$

$$\Rightarrow (\tan\theta + \tan 2\theta) + \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = 0$$

$$\Rightarrow (\tan\theta + \tan 2\theta)\left[1 + \frac{1}{1 - \tan\theta\tan 2\theta}\right] = 0$$

$\Rightarrow$ $(\tan\theta + \tan 2\theta)(2 - \tan\theta\tan 2\theta) = 0$

$\Rightarrow$ $\tan\theta = \tan(-2\theta)$ or $2 - \tan\theta[(2\tan\theta)/(1 - \tan^2\theta)] = 0$

$\Rightarrow$ $\theta = n\pi - 2\theta$ or $1 - 2\tan^2\theta = 0$

$= n\pi/3$ or $\tan^2\theta = 1/2 = \tan^2\alpha$ (say)

Therefore, $\theta = n\pi \pm \alpha$, where $\tan\alpha = 1/\sqrt{2}$.

43. b. We have $\sin 3\alpha = 4\sin\alpha(\sin^2 x - \sin^2\alpha)$

$\Rightarrow$ $3\sin\alpha - 4\sin^3\alpha = 4\sin\alpha\sin^2 x - 4\sin^3\alpha$

$\Rightarrow$ $3\sin\alpha = 4\sin\alpha\sin^2 x$ or $\sin\alpha = 0$

If $\sin\alpha \neq 0$, $\sin^2 x = 3/4 = (\sqrt{3}/2)^2 = \sin^2(\pi/3)$, therefore $x = n\pi \pm \pi/3$, $\forall\, n \in Z$

If $\sin\alpha = 0$, i.e., $\alpha = n\pi$, equation becomes an identity.

44. c. We have $\sqrt{3}\cos\theta - 3\sin\theta = 2(\sin 5\theta - \sin\theta)$

$\Rightarrow \left(\sqrt{3}/2\right)\cos\theta - (1/2)\sin\theta = \sin 5\theta$

$\Rightarrow \cos(\theta + \pi/6) = \sin 5\theta = \cos(\pi/2 - 5\theta)$

$\Rightarrow \theta + \pi/6 = 2n\pi \pm (\pi/2 - 5\theta)$

$\Rightarrow \theta = (n\pi/3) + (\pi/18)$ or $\theta = (-n\pi/2) + (\pi/6)$, $\forall\, n \in Z$

45. c. $\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$

$\Rightarrow (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x + \sin 2x + \alpha = 0$

$\Rightarrow \sin^2 2x - 2\sin 2x - 2 - 2\alpha = 0$

Let $\sin 2x = y$. Then the given equation becomes $y^2 - 2y - 2(1+\alpha) = 0$ where $-1 \le y \le 1$, $(\because -1 \le \sin 2x \le 1)$

For real, discriminant ≥ 0

$\Rightarrow 3 + 2\alpha \ge 0 \quad \Rightarrow \alpha \ge -\dfrac{3}{2}$

Also $-1 \le y \le 1 \quad \Rightarrow -1 \le 1 - \sqrt{3 + 2\alpha} \le 1$

$\Rightarrow 3 + 2\alpha \le 4 \quad \Rightarrow \alpha \le \dfrac{1}{2}$. Thus $-\dfrac{3}{2} \le \alpha \le \dfrac{1}{2}$

46. a. Since the system has a non-trivial solution, the determinant of coefficients = 0

$$\Rightarrow \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

$\Rightarrow \sin 3\theta(28 - 21) - \cos 2\theta(-7 - 7) + 2(-3 - 4) = 0$

$\Rightarrow \sin 3\theta + 2\cos 2\theta - 2 = 0$

$\Rightarrow (3\sin\theta - 4\sin^3\theta) + 2(1 - 2\sin^2\theta) - 2 = 0$

$\Rightarrow 4\sin^3\theta + 4\sin^2\theta - 3\sin\theta = 0$

$\Rightarrow \sin\theta(2\sin\theta - 1)(2\sin\theta + 3) = 0$

$\Rightarrow \sin\theta = 0$ or $\theta = n\pi + (-1)^n \pi/6$, $\forall\, n \in Z$

47. c. Let $\log_{\cos x}\sin x = t$, then the given equation is $t + \dfrac{1}{t} = 2$

$\Rightarrow (t-1)^2 = 0 \quad \Rightarrow t = 1 \quad \Rightarrow \log_{\cos x}\sin x = 1$ or $\sin x = \cos x \quad \Rightarrow \tan x = 1$

$\Rightarrow x = \pi/4$

48. b. Given $x^2 + 2x\sin(xy) + 1 = 0$

$\Rightarrow [x + \sin(xy)]^2 + [1 - \sin^2(xy)] = 0$

$\Rightarrow x + \sin(xy) = 0$ and $\sin^2(xy) = 1$

$\sin^2(xy) = 1$ gives $\sin(xy) = 1$ or -1

If $\sin(xy) = 1 \quad \Rightarrow x = -1 \quad \Rightarrow \sin(-y) = 1 \quad \Rightarrow \sin y = -1$, then the ordered pair is $(1, 3\pi/2)$.

If $\sin(xy) = -1 \quad \Rightarrow x = 1 \quad \Rightarrow \sin y = -1 \quad \Rightarrow (-1, 3\pi/2)$

Thus, there are two ordered pairs.

49. c. The given equation is $8\sin x\cos x\cos 2x\cos 4x = \sin 6x$ $(\sin x \neq 0)$

$\Rightarrow \sin 8x = \sin 6x \quad \Rightarrow 2\cos 7x\sin x = 0$

As $\sin x \neq 0$, $\cos 7x = 0$ or $7x = n\pi + \pi/2$, $n \in Z$

i.e., $x = n\pi/7 + \pi/14$; $n \in Z$

50. d. We have $\cos 3x + \sin\left(2x - \frac{7\pi}{6}\right) = -2$

$\Rightarrow \quad 1 + \cos 3x + 1 + \sin\left(2x - \frac{7\pi}{6}\right) = 0$

$\Rightarrow \quad (1 + \cos 3x) + 1 - \cos\left(2x - \frac{2\pi}{3}\right) = 0$

$\Rightarrow \quad 2\cos^2\frac{3x}{2} + 2\sin^2\left(x - \frac{\pi}{3}\right) = 0$

$\Rightarrow \quad \cos\frac{3x}{2} = 0$ and $\sin\left(x - \frac{\pi}{3}\right) = 0$

$\Rightarrow \quad \frac{3x}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots,$ and $x - \frac{\pi}{3} = 0, \pi, 2\pi, \ldots \quad \Rightarrow \quad x = \frac{\pi}{3}$

Therefore, the general solution of $\cos\frac{3x}{2} = 0$ and $\sin\left(x - \frac{\pi}{3}\right) = 0$ is $x = 2k\pi + \frac{\pi}{3} = \frac{\pi}{3}(6k+1)$ where $k \in Z$.

51. b. Let $\tan^2\theta = t$

$\Rightarrow \quad 1 - t^2 + 2^t = 0$

It is clearly satisfied by $t = 3$. By inspection, we get $\tan^2\theta = 3$.

Therefore, $\theta = \pm\pi/3$ in the given interval.

52. b. $\tan\left(\frac{\pi}{2}\sin\theta\right) = \cot\left(\frac{\pi}{2}\cos\theta\right)$

$\therefore \tan\left(\frac{\pi}{2}\sin\theta\right) = \tan\left(\frac{\pi}{2} - \frac{\pi}{2}\cos\theta\right)$

$\therefore \frac{\pi}{2}\sin\theta = n\pi + \frac{\pi}{2} - \frac{\pi}{2}\cos\theta$

$\sin\theta + \cos\theta = 2n + 1$

$\Rightarrow \sin\theta + \cos\theta = 1 \pm 1$

$\Rightarrow 1 + \sin 2\theta = 1$

$\Rightarrow \sin 2\theta = 0$

$\Rightarrow \theta = n\pi$

53. a. $\cos x = \sqrt{1 - \sin 2x} = |\sin x - \cos x|$

(a) $\sin x < \cos x \quad \Rightarrow \quad x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$ (i)

Then the given equation is $\cos x = \cos x - \sin x \quad \Rightarrow \quad \sin x = 0 \quad \Rightarrow \quad x = \pi, 2\pi$

$\Rightarrow \quad x = 2\pi$ [from Eq. (i)]

(b) $\sin x \geq \cos x \quad \Rightarrow \quad x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

$\Rightarrow \quad \cos x = \sin x - \cos x$

$\Rightarrow \quad \tan x = 2$

$\Rightarrow \quad x = \tan^{-1}$ (ii)

Hence, there are two solutions.

54. b. $\sum_{r=1}^{5} \cos rx = 5$

$\Rightarrow \cos x + \cos 2x + \cos 3x + \cos 4x + \cos 5x = 5$

which is possible only, when $\cos x = \cos 2x = \cos 3x = \cos 4x = \cos 5x = 1$ and is satisfied by $x = 0$ and $x = 2\pi$.

55. a. $\sin 2x + \cos 4x = 2$

It is possible only, when $\sin 2x = 1$ and $\cos 4x = 1$

$\Rightarrow \sin 2x = 1$ and $1 - 2\sin^2 2x = 1$

$\Rightarrow \sin 2x = 1$ and $\sin 2x = 0$

Hence, there is no solution.

56. d. $\cos^2 x = 2\cos x(1 - 3\cos^2 x)$

$\Rightarrow 6\cos^3 x + \cos^2 x - 2\cos x = 0$

$\Rightarrow \cos x = 0, \frac{1}{2}, -\frac{2}{3}$

$\Rightarrow x = \frac{\pi}{2}, \frac{\pi}{3}, \cos^{-1}\left(-\frac{2}{3}\right)$ ($\because \alpha, \beta$ are + ve)

If $\alpha = \frac{\pi}{2}; \beta = \frac{\pi}{3}$, then we have $|\alpha - \beta| = \frac{\pi}{6}$.

57. b. We have $\sin^{100} x - \cos^{100} x = 1$

$\Rightarrow \sin^{100} x = 1 + \cos^{100} x$

Since the L.H.S. never exceeds 1, R.H.S. exceeds 1 unless $\cos x = 0$

Then, $x = n\pi + \frac{\pi}{2}, n \in I$

58. b. $|\cot x| = \cot x + \frac{1}{\sin x}$

If $\cot x > 0 \Rightarrow \cot x = \cot x + \frac{1}{\sin x} = 0$

$\Rightarrow \frac{1}{\sin x} = 0$, which is not possible.

If $\cot x \le 0 \Rightarrow -\cot x = \cot x + \frac{1}{\sin x}$

$\Rightarrow -2\cot x = \frac{1}{\sin x}$

$\Rightarrow \cos x = -\frac{1}{2}$

$\Rightarrow x = \frac{2\pi}{3}, \frac{8\pi}{3}$

59. a. $\tan(A - B) = 1$

$\Rightarrow A - B = n_1\pi + \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, -\frac{3\pi}{4}, \ldots$

$\sec(A + B) = \frac{2}{\sqrt{3}} \Rightarrow A + B = 2n_2\pi \pm \frac{\pi}{6} = \frac{\pi}{6}, \frac{11\pi}{6}, \ldots$

For the least positive values of A and B,

$$A+B=\frac{11\pi}{6}, A-B=\frac{\pi}{4} \quad \Rightarrow \quad B=\frac{19\pi}{24}, A=\frac{25\pi}{24}$$

60. a. Let $A=\theta+15°$, $B=\theta-15°$

$\Rightarrow$ $A+B=2\theta$ and $A-B=30°$

Now $\frac{\tan A}{\tan B}=\frac{3}{1}$

$\Rightarrow$ $\frac{\tan A+\tan B}{\tan A-\tan B}=\frac{3+1}{3-1}$ (applying componendo and dividendo rule)

$\Rightarrow$ $\frac{\sin(A+B)}{\sin(A-B)}=2$

$\Rightarrow$ $\sin 2\theta=2\sin 30°=1$

$\Rightarrow$ $2\theta=2n\pi+\frac{\pi}{2}$ or $\theta=n\pi+\frac{\pi}{4}$ $n\in Z$

61. a. $\tan 3\theta+\tan\theta=2\tan 2\theta$

$\Rightarrow$ $\tan 3\theta-\tan 2\theta=\tan 2\theta-\tan\theta$

$\Rightarrow$ $\frac{\sin(3\theta-2\theta)}{\cos 3\theta\cos 2\theta}=\frac{\sin(2\theta-\theta)}{\cos 2\theta\cos\theta}$

$\Rightarrow$ $\sin\theta(2\sin\theta\sin 2\theta)=0$

$\Rightarrow$ $\sin\theta=0$ or $\sin 2\theta=0$

$\Rightarrow$ $\theta=n\pi$ or $2\theta=n\pi$, $n\in Z$

But $\theta=n\pi/2$ is rejected as when n is odd, $\tan\theta$ is not defined and when n is even, i.e., $2r$, then $\theta=r\pi$.

Then $\theta=n\pi$, $n\in I$ is the only solution.

62. a. We have $|4\sin x-1|<\sqrt{5}$

$\Rightarrow$ $-\sqrt{5}<4\sin x-1<\sqrt{5}$

$\Rightarrow$ $-\left(\frac{\sqrt{5}-1}{4}\right)<\sin x<\left(\frac{\sqrt{5}+1}{4}\right)$

$\Rightarrow$ $-\sin\frac{\pi}{10}<\sin x<\cos\frac{\pi}{5}$

$\Rightarrow$ $\sin\left(-\frac{\pi}{10}\right)<\sin x<\sin\left(\frac{\pi}{2}-\frac{\pi}{5}\right)$

$\Rightarrow$ $\sin\left(-\frac{\pi}{10}\right)<\sin x<\sin\frac{3\pi}{10}$

$\Rightarrow$ $x\in\left(-\frac{\pi}{10},\frac{3\pi}{10}\right)$

63. d. $y^2-y+a=\left(y-\frac{1}{2}\right)^2+a-\frac{1}{4}$

Since $-\sqrt{2}\le\sin x+\cos x\le\sqrt{2}$, the given equation will have no real value x for any y if $a-\frac{1}{4}>\sqrt{2}$

i.e., $a \in \left(\sqrt{2}+\frac{1}{4}, \infty\right) \Rightarrow a \in (\sqrt{3}, \infty)$ (as $\sqrt{2}+\frac{1}{4}<\sqrt{3}$)

64. a. $(2\sin x - \operatorname{cosec} x)^2 + (\tan x - \cot x)^2 = 0$

$\Rightarrow \sin^2 x = \frac{1}{2}$ and $\tan^2 x = 1$

$\Rightarrow x = n\pi \pm \frac{\pi}{4}, n \in Z$

65. a. $[\sin x + \cos x] = 3 + [-\sin x] + [-\cos x]$

Maximum value of left-hand side is 1 and minimum of right hand side is also 1

$\Rightarrow [\sin x + \cos x] = 3 + [-\sin x] + [-\cos x] = 1 \Rightarrow x \in \pi \pm \frac{\pi}{4}$

$\Rightarrow [\sin x + \cos x] = 1, [-\sin x] = -1, [-\cos x] = -1$

which is not possible.

66. c. $\cos^8 x + b\cos^4 x + 1 = 0$

$\Rightarrow b = -\left(\cos^4 x + \frac{1}{\cos^4 x}\right) \le -2 \; \forall x \in R$

$\Rightarrow b \in (-\infty, -2]$

67. b. Here $1 \le |\sin 2x| + |\cos 2x| \le \sqrt{2}$ and $|\sin y| \le 1$

so solution is possible only when $|\sin y| = 1$

$\Rightarrow \sin y = \pm 1 \quad \Rightarrow y = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

68. b. Given that $|\sin x|^2 + |\sin x| + b = 0$

$\Rightarrow |\sin x| = \frac{-1 \pm \sqrt{1-4b}}{2} \quad \Rightarrow 0 \le \frac{-1 \pm \sqrt{1-4b}}{2} < 1 \quad \Rightarrow -2 < b < 0$

69. a. $|2\sin\theta - \operatorname{cosec}\theta| \ge 1$

$\Rightarrow |2\sin^2\theta - 1| \ge |\sin\theta|$

$\Rightarrow |\cos 2\theta| \ge |\sin\theta|$

$\Rightarrow 2\cos^2 2\theta \ge 1 - \cos 2\theta$

$\Rightarrow 2\cos^2 2\theta + \cos 2\theta - 1 \ge 0$

$\Rightarrow (2\cos 2\theta - 1)(\cos 2\theta + 1) \ge 0$

$\Rightarrow \cos 2\theta \ge \frac{1}{2}$ [as $\cos\theta \ne 0$, i.e., $\cos 2\theta \ne -1$]

70. d. The given equation can be written as

$\sin x \cos x\,[\sin^2 x + \sin x\cos x + \cos^2 x] = 1$

$\Rightarrow \sin x\cos x\,[1 + \sin x\cos x] = 1$

$\Rightarrow \sin 2x\,[2 + \sin 2x] = 4$

$\Rightarrow \sin 2x = \frac{-2 \pm \sqrt{4+16}}{2} = -1 \pm \sqrt{5}$

which is not possible.

71. d. $e^{|\sin x|} + e^{-|\sin x|} + 4a = 0$, let $t = e^{|\sin x|}$

$\Rightarrow t \in [1, e]$

$\Rightarrow t + \frac{1}{t} + 4a = 0$

$\Rightarrow \quad t^2 + 4at + 1 = 0$

This quadratic expression should have two distinct roots in [1, e]

$\Rightarrow \quad 16a^2 - 4 > 0, f(1) = 1 + 4a + 1 \geq 0, f(e) = e^2 + 4ae + 1 \geq 0, 1 < -2a < e$

$\Rightarrow \quad |a| > \dfrac{1}{2}, a \geq -\dfrac{1}{2}, a \geq \dfrac{-1-e^2}{4e}, -\dfrac{e}{2} < a < -\dfrac{1}{2}$

Clearly, there is no value of a satisfying the above inequalities simultaneously.

72. b.

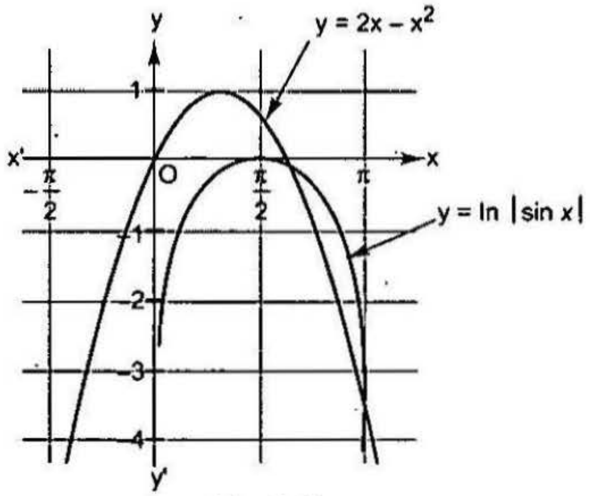


Fig. 3.11

$\ln|\sin x| = -x(x-2)$

Graphs of $y = \ln|\sin x|$ and $y = -x(x-2)$ meet exactly two times in $\left[-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$.

73. c. $|x| + |y| = 4, \sin\left(\dfrac{\pi x^2}{3}\right) = 1$

$\Rightarrow \quad |x|, |y| \in [0, 4], \dfrac{\pi x^2}{3} = (4n+1)\dfrac{\pi}{2}$

$\Rightarrow \quad x^2 = \dfrac{(4n+1)3}{2} = \dfrac{3}{2}, \text{ as } |x| \leq 4$

$\Rightarrow \quad |x| = \sqrt{\dfrac{3}{2}}, |y| = 4 - \sqrt{\dfrac{3}{2}}$

Thus, there are four ordered pairs.

74. b. $\sin\{x\} = \cos\{x\}$

$\tan\{x\} = 1$

$\tan(\pi/4) = 1 < \tan 1$

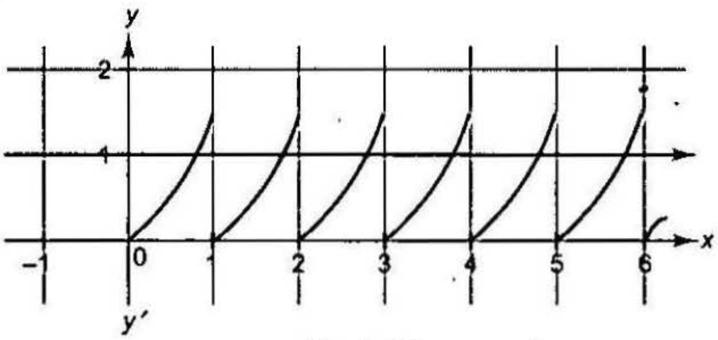


Fig. 3.12

Graphs of $y = \tan\{x\}$ and $y = 1$ meet exactly six times in $[0, 2\pi]$.

75. a. $x^2 + 4 - 2x + 3\sin(ax + b) = 0$

$(x - 1)^2 + 3 = -3\sin(ax + b)$

L.H.S. ≥ 3 and R.H.S. ≤ 3

$\Rightarrow$ L.H.S = R.H.S = 3

$(x - 1)^2 + 3 + 3\sin(ax + b) = 0$

$\Rightarrow x = 1, \sin(ax + b) = -1$

$\Rightarrow \sin(a + b) = -1$

$\Rightarrow a + b = (4n - 1)\dfrac{\pi}{2}, n \in I \Rightarrow a + b = \dfrac{7\pi}{2}$ (from the given options)

76. c. $\tan^4 x - 2\sec^2 x + a = 0$

$\Rightarrow \tan^4 x - 2(1 + \tan^2 x) + a = 0$

$\Rightarrow \tan^4 x - 2\tan^2 x + 1 = 3 - a$

$\Rightarrow (\tan^2 x - 1)^2 = 3 - a$

$\Rightarrow 3 - a \geq 0 \Rightarrow a \leq 3$

77. a. $1 + \log_2 \sin x + \log_2 \sin 3x \geq 0$

(where $\sin x, \sin 3x > 0$)

$\Rightarrow \log_2(2\sin x \sin 3x) \geq 0$

$\Rightarrow 2\sin x \sin 3x \geq 1$

For $\sin x > 0$

$\Rightarrow x \in (0, \pi)$ (i)

$\Rightarrow \sin 3x > 0$ (ii)

$\Rightarrow 3x \in (0, \pi) \cup (2\pi, 3\pi)$

$\Rightarrow x \in \left(0, \dfrac{\pi}{3}\right) \cup \left(\dfrac{2\pi}{3}, \pi\right)$

For $2\sin x \sin 3x \geq 1$

$\Rightarrow 2\sin^2 x(3 - 4\sin^2 x) \geq 1$

$\Rightarrow 8\sin^4 x - 6\sin^2 + 1 \leq 0$

$\Rightarrow (2\sin^2 x - 1)(4\sin^2 x - 1) \leq 0$

$\Rightarrow \dfrac{1}{2} \leq \sin x \leq \dfrac{1}{\sqrt{2}}$

$\Rightarrow x \in \left[\dfrac{\pi}{3}, \dfrac{\pi}{4}\right] \cup \left[\dfrac{2\pi}{3}, \dfrac{3\pi}{4}\right]$ (iii)

Thus, $x \in \left(\dfrac{2\pi}{3}, \dfrac{3\pi}{4}\right]$. [From Eqs. (i), (ii), (iii)]

78. d. $\sin^2\theta = 1$ [$\sin\theta \neq 1$]

$\Rightarrow \sin\theta = -1 \Rightarrow \theta = 2n\pi - (\pi/2) \Rightarrow$ infinite roots

79. c. $\pi \log_3\left(\dfrac{1}{x}\right) = k\pi, k \in I$

$\log_3\left(\dfrac{1}{x}\right) = k \Rightarrow x = 3^{-k}$

The possible values of k are $-1, 0, 1, 2, 3, \ldots$

$S = 3 + 1 + \dfrac{1}{3} + \dfrac{1}{3^2} + \dfrac{1}{3^3} + \cdots \infty = \dfrac{3}{1 - \dfrac{1}{3}} = \dfrac{9}{2}$

80. a. $\dfrac{\sin(xy)}{\cos(xy)} = y$

$\Rightarrow \sin(xy) = xy$

$\Rightarrow xy = 0$

$\Rightarrow x = 0$ or $y = 0$

But $x = 0$ is not possible

$\therefore y = 0$ and $x = 1$, i.e., $(1, 0)$

81. c. $\tan x + \cot x + 1 = \cos\left(x + \dfrac{\pi}{4}\right)$

$\Rightarrow \tan x + \cot x = \cos\left(x + \dfrac{\pi}{4}\right) - 1$

Now $\tan x + \cot x \le -2$ and $\cos\left(x + \dfrac{\pi}{4}\right) - 1 \ge -2$

It implies that equality holds when both are -2.

$\Rightarrow \cos\left(x + \dfrac{\pi}{4}\right) = -1$

$\Rightarrow x + \dfrac{\pi}{4} = (2m+1)\pi,\ m \in Z$

$\Rightarrow x = \dfrac{3\pi}{4}$ or $\dfrac{11\pi}{4}$

Therefore, the sum of the solutions is $\dfrac{3\pi}{4} + \dfrac{11\pi}{4} = \dfrac{7\pi}{2}$.

82. c. $\cos(\theta)\cos(\pi\theta) = 1$

$\Rightarrow \cos(\theta) = 1$ and $\cos(\pi\theta) = 1$ (i)

or $\cos(\theta) = -1$ and $\cos(\pi\theta) = -1$ (ii)

If $\cos(\theta) = 1 \Rightarrow \theta = 2m\pi$ and $\cos(\pi\theta) = 1 \Rightarrow \theta = 2\pi$ which is possible only when $\theta = 0$.

Equation (ii) is not possible for any θ as for $\cos(\theta) = -1$, θ should be odd multiple of $\pi \Rightarrow$ irrational and for $\cos(\pi\theta) = -1 \Rightarrow \theta$ should be odd integer $\Rightarrow$ rational

Both the conditions cannot be satisfied.

Therefore, $\theta = 0$ is the only solution.

83. c. $(\cot x + \sqrt{3})^2 + \cot^2 x + 4\operatorname{cosec} x + 5 = 0$

$\Rightarrow (\cot x + \sqrt{3})^2 + \operatorname{cosec}^2 x + 4\operatorname{cosec} x + 4 = 0$

$\Rightarrow (\cot x + \sqrt{3})^2 + (\operatorname{cosec} x + 2)^2 = 0$

$\Rightarrow \cot x = -\sqrt{3}$ or $\operatorname{cosec} x = -2$

$\Rightarrow x = 2n\pi - \dfrac{\pi}{6},\ n \in Z$ ($\because x \in$ 4th quadrant)

84. c. $\theta = k\pi, k = \frac{p}{q}, p, q \in I, q \neq 0$

$\cos k\pi$ is a rational

Hence, $k = 0, 1, 1/2, 1/3, 2/3$

There are five values of $\cos\theta$ for which $\cos\theta$ is rational.

85. b. Putting $x = 0 \Rightarrow b^2 = \cos b^2 - 1 \Rightarrow \cos b^2 = 1 + b^2 \Rightarrow b = 0$

For $b = 0$, we have $a(\cos x - 1) = \cos ax - 1$

$\Rightarrow 2a\sin^2\frac{x}{2} = 2\sin^2\frac{ax}{2}$

$\Rightarrow a = 0$ or $a = 1$.

Hence, ordered pairs are $(a, b) \equiv (0, 0)$ or $(1, 0)$.

Multiple Correct Answers Type

1. a, b. We have $4\sin^4 x + \cos^4 x = 1$

$\Rightarrow \quad 4\sin^4 x = 1 - \cos^4 x = (1 - \cos^2 x)(1 + \cos^2 x) = \sin^2 x(2 - \sin^2 x)$

$\Rightarrow \quad \sin^2 x[5\sin^2 x - 2] = 0$

$\Rightarrow \quad \sin x = 0$ or $\sin x = \pm\sqrt{2/5}$

$\Rightarrow \quad x = n\pi$ or $x = n\pi \pm \sin^{-1}\sqrt{\frac{2}{5}}, n \in Z$

2. a, b. $(\sin^3\theta + \cos^3\theta) - (1 - \sin\theta\cos\theta) = 0$

$\Rightarrow \quad (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta) - (1 - \sin\theta\cos\theta) = 0$

$\Rightarrow \quad (1 - \sin\theta\cos\theta)(\sin\theta + \cos\theta - 1) = 0$

If $\sin\theta\cos\theta = 1 \Rightarrow 2\sin\theta\cos\theta = 2 \Rightarrow \sin 2\theta = 2$ (not possible)

$\Rightarrow \quad \sin\theta + \cos\theta = 1$

$\Rightarrow \quad \theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}, n \in Z$

$\Rightarrow \theta = 2n\pi$ or $2n\pi + \frac{\pi}{2}$

3. a, c, d. We have $\tan^2\theta = 1 - \cos 2\theta = 2\sin^2\theta$ or $\operatorname{cosec}^2\theta\tan^2\theta = 2$
or $(1 + \cot^2\theta)\tan^2\theta = 2$ or $\tan^2\theta + 1 = 2$

$\Rightarrow \quad \tan^2\theta = 1 \quad \Rightarrow \quad \tan\theta = \pm 1 \quad \Rightarrow \quad \theta = n\pi \pm \frac{\pi}{4}, n \in Z$

Moreover, $\tan^2\theta = 2\sin^2\theta \quad \Rightarrow \quad \sin^2\theta = 0 \Rightarrow \theta = n\pi$

4. a, c. $y + \frac{1}{y} \geq 2 \Rightarrow \sqrt{y + \frac{1}{y}} \geq \sqrt{2}$

But $\sin x + \cos x \leq \sqrt{2}$

$\Rightarrow \quad y + \frac{1}{y} = 2$ and $\sin x + \cos x = \sqrt{2}$

$\Rightarrow \quad y = 1$ and $\sin\left(x + \frac{\pi}{4}\right) = 1$ or $y = 1$ and $x = \frac{\pi}{4}$

5. b, d. $\sin\theta + \sqrt{3}\cos\theta = -2-(x^2-6x+9) = -2-(x-3)^2$

$\therefore \sin\theta + \sqrt{3}\cos\theta \ge -2$ and $-2-(x-3)^2 \le -2$

As a result, we have $\sin\theta + \sqrt{3}\cos\theta = -2$ and then $x = 3$

$\therefore x = 3$ and $\cos\left(\theta - \dfrac{\pi}{6}\right) = -1$, i.e., $\theta - \dfrac{\pi}{6} = \pi, 3\pi$

6. a, c. $\sin^2 x - 2\sin x - 1 = 0$

$\Rightarrow \quad (\sin x - 1)^2 = 2 \qquad \Rightarrow \quad \sin x - 1 = \pm\sqrt{2} \qquad \Rightarrow \quad \sin x = 1-\sqrt{2}$ as $\sin x \not> 1$

There are two solutions in $[0, 2\pi]$ and two more in $[2\pi, 4\pi]$.

Thus, $n = 4, 5$.

7. a, b, c. The given equation is $2(\sin x + \sin y) - 2\cos(x-y) = 3$

$$\Rightarrow \quad 2\times 2\sin\frac{x+y}{2}\cos\frac{x-y}{2} - 2\left[2\cos^2\frac{x-y}{2} - 1\right] = 3$$

$$\Rightarrow \quad 4\cos^2\left(\frac{x-y}{2}\right) - 4\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) + 1 = 0$$

$$\Rightarrow \quad \cos\left(\frac{x-y}{2}\right) = \frac{4\sin\left(\frac{x+y}{2}\right) \pm \sqrt{16\sin^2\left(\frac{x+y}{2}\right) - 16}}{8}$$

$\therefore \sin^2\left(\dfrac{x+y}{2}\right) \ge 1 \qquad \Rightarrow \quad \sin\dfrac{x+y}{2} = \pm 1$

Since x and y are smallest and positive, we have

$\sin\dfrac{x+y}{2} = 1$ and $\dfrac{x+y}{2} = \dfrac{\pi}{2}$

i.e., $x + y = \pi$ (i)

Also, $\cos\left(\dfrac{x-y}{2}\right) = \dfrac{1}{2}$

$\Rightarrow \quad x - y = 2\pi/3$ or $-2\pi/3$ (ii)

From Eqs. (i) and (ii), we get $(x = 5\pi/6,\ y = \pi/6)$ or $(x = \pi/6,\ y = 5\pi/6)$

8. a, d. $1 - 2x - x^2 = \tan^2(x+y) + \cot^2(x+y)$

$\Rightarrow \quad -(x+1)^2 = [\tan(x+y) - \cot(x+y)]^2$

Now L.H.S. ≤ 0 and R.H.S. ≥ 0

$\Rightarrow \quad -(x+1)^2 = [\tan(x+y) - \cot(x+y)]^2 = 0$

$\Rightarrow \quad x = -1$ and $\tan(x+y) = \cot(x+y)$

$\Rightarrow \quad x = -1$ and $\tan^2(-1+y) = 1$

$\Rightarrow \quad x = -1$ and $-1 + y = n\pi \pm (\pi/4), n \in Z$

9. b, c. From $\tan x + \tan y = 1$, we have $\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y} = 1$

$\Rightarrow \sin x \cos y + \sin y \cos x = \cos x \cos y$

$\Rightarrow 2 \sin (x + y) = 2 \cos x \cos y$

$\Rightarrow 2 \sin (x + y) = \cos (x + y) + \cos (x - y)$

$\Rightarrow 2 \sin (\pi/4) = \cos (\pi/4) + \cos (x - y)$

$\Rightarrow \cos (x - y) = 1/\sqrt{2} = \cos (\pi/4)$

$\Rightarrow x - y = 2n\pi \pm (\pi/4), \forall n \in Z$ (i)

Also we have $x + y = \pi/4$ (ii)

From Eqs. (i) and (ii), we have

$x = n\pi + (\pi/4)$ and $y = -n\pi, \forall n \in Z$

or $x = n\pi$ and $y = -n\pi + \pi/4, \forall n \in Z$

10. a, c. $x + y = 2\pi/3 \Rightarrow y = (2\pi/3) - x$

$\therefore \sin x = 2 \sin (2\pi/3 - x)$

$= 2[(\sqrt{3}/2) \cos x + (1/2) \sin x] = \sqrt{3} \cos x + \sin x$

$\Rightarrow \cos x = 0 \Rightarrow x = n\pi + \pi/2, n \in Z \Rightarrow y = \frac{2\pi}{3} - n\pi - \frac{\pi}{2} = \frac{\pi}{6} - n\pi$

Hence, for $x \in [0, 4\pi]$, $x = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2$

and for $y \in [0, 4\pi]$, $y = \pi/6, 7\pi/6, 13\pi/6, 19\pi/6$

11. a, d. $\tan x - \tan^2 x > 0$

$\Rightarrow \tan x (\tan x - 1) < 0$

$\Rightarrow 0 < \tan x < 1$

$\Rightarrow 0 < x < \pi/4$

$\Rightarrow n\pi < x < n\pi + \pi/4, n \in Z$ (generalizing)

$|\sin x| < \frac{1}{2} \Rightarrow -\frac{1}{2} < \sin x < \frac{1}{2}$

$\Rightarrow -\pi/6 < x < \pi/6 \Rightarrow -\pi/6 + n\pi < x < \pi/6 + n\pi, n \in Z$ (generalizing)

Then the common values are $n\pi < x < n\pi + \pi/6$.

12. a, b, d. $\cos (x + \pi/3) + \cos x = a$

$\Rightarrow \frac{1}{2} \cos x - (\sqrt{3}/2) \sin x + \cos x = a$

$\Rightarrow (3/2) \cos x - (\sqrt{3}/2) \sin x = a$

$\Rightarrow -\sqrt{\left(\frac{9}{4} + \frac{3}{4}\right)} \le a \le \sqrt{\left(\frac{9}{4} + \frac{3}{4}\right)}$

$\Rightarrow -\sqrt{3} \le a \le \sqrt{3}$ (i)

Hence, there are three integral values of $a = -1, 0, 1$ whose sum is 0.

For $a = 1$, the given equation is $(\sqrt{3}/2) \cos x - (1/2) \sin x = 1/\sqrt{3}$

$\Rightarrow \cos (x + \pi/6) = 1/\sqrt{3}$

$\Rightarrow x + \pi/6 = 2n\pi \pm \alpha$, where $\alpha = \cos^{-1} (1/\sqrt{3})$

$\Rightarrow x = 2n\pi - \pi/6 \pm \alpha$

Hence, the solutions for $a = 1$ in $[0, 2\pi]$ are $\cos^{-1} (1/\sqrt{3}) - \pi/6$, $11/6 - \cos^{-1} (1/\sqrt{3})$.

13. a, b, c. The given inequality can be written as

$$2^{\operatorname{cosec}^2 x}\sqrt{(y-1)^2+1} \le \sqrt{2} \qquad \text{(i)}$$

Since $\operatorname{cosec}^2 x \ge 1$ for all real x, we have

$$2^{\operatorname{cosec}^2 x} \ge 2 \qquad \text{(ii)}$$

Also $(y-1)^2+1 \ge 1 \quad \Rightarrow \quad \sqrt{(y-1)^2+1} \ge 1$ (iii)

From Eqs. (i) and (ii), we get

$$2^{\operatorname{cosec}^2 x}\sqrt{(y+1)^2+1} \ge 2 \qquad \text{(iv)}$$

Therefore, from Eqs. (i) and (iv), equality holds only when $2^{\operatorname{cosec}^2 x} = 2$ and $\sqrt{(y-1)^2+1} = 1$

$\Rightarrow \quad \operatorname{cosec}^2 x = 1$ and $(y-1)^2+1=1 \quad \Rightarrow \quad \sin x = \pm 1$ and $y = 1$

$\Rightarrow \quad x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ and $y = 1$

Hence, the solution of the given inequality is $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ and $y = 1$.

14. a, b, c. $\sin^2 x - a\sin x + b = 0$ has only one solution in $(0, \pi)$.

$\Rightarrow \quad \sin x = 1$ gives one solution and $\sin x = \alpha$ gives other solution such that $\alpha > 1$ or $\alpha \le 0$

$\Rightarrow \quad (\sin x - 1)(\sin x - \alpha) = 0$ is the same equation as $\sin^2 x - a\sin x + b = 0$

$\Rightarrow \quad 1 + \alpha = a$ and $\alpha = b$

$\Rightarrow \quad 1 + b = a$ and $b > 1$ or $b \le 0$

$\Rightarrow \quad b \in (-\infty, 0] \cup [1, \infty)$ and $a \in (-\infty, 1] \cup [2, \infty)$

15. a, b. Given that the quadratic equation is an identity

$\therefore \operatorname{cosec}^2\theta = 4$ and $\cot\theta = -\sqrt{3}$

$\Rightarrow \quad \operatorname{cosec}\theta = 2$ or -2 and $\tan\theta = -\dfrac{1}{\sqrt{3}}$

$\theta = \dfrac{5\pi}{6}$ or $\dfrac{11\pi}{6}$

16. c, d. Abscissa corresponding to the vertex is given by

$x = \dfrac{1}{\sin\alpha} > 1$ is the vertex

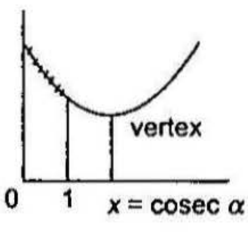


Fig. 3.13

The graph of $f(x) = (\sin\alpha)x^2 - 2x + b$ is shown as $\forall\, x \le 1$

Therefore, the minimum of $f(x) = (\sin\alpha)x^2 - 2x + b - 2$ must be greater than zero but minimum is at $x = 1$, i.e., $\sin\alpha - 2 + b - 2 \ge 0$

$\Rightarrow \quad b \ge 4 - \sin\alpha,\ \alpha \in (0, \pi) \quad \Rightarrow \quad b \ge 4$ as $\sin\alpha > 0$ in $(0, \pi)$

17.a, d. $\dfrac{\sqrt{3}-1}{2\sqrt{2}\sin x}+\dfrac{\sqrt{3}+1}{2\sqrt{2}\cos x}=2$

$$\sin\frac{\pi}{12}\cos x+\cos\frac{\pi}{12}\sin x=\sin 2x$$

$$\sin 2x=\sin\left(x+\frac{\pi}{12}\right)$$

$$\therefore 2x=x+\frac{\pi}{12} \text{ or } 2x=\pi-x-\frac{\pi}{12}$$

$$x=\frac{\pi}{12} \text{ or } 3x=\frac{11\pi}{12}$$

$$\therefore x=\frac{\pi}{12} \text{ or } \frac{11\pi}{36}$$

18. a, c, d. $\cos 3\theta=\cos 3\alpha$

Putting $n=0, 1$, we have

$3\alpha=2n\pi\pm 3\alpha$

$\therefore 3\theta=3\alpha$ or -3α or $2\pi+3\alpha$ or $2\pi-3\alpha$

$\theta=\alpha$ or $-\alpha$ or $\dfrac{2\pi}{3}+\alpha$ or $\dfrac{2\pi}{3}-\alpha \Rightarrow$ (a), (c), (d) are correct

If $n=-1$, then $3\theta=-2\pi\pm 3\alpha$

$$\Rightarrow \theta=-\frac{2\pi}{3}\pm\alpha$$

$$\Rightarrow \sin\theta=\sin\left(-\frac{2\pi}{3}\pm\alpha\right)=-\sin\left(\frac{2\pi}{3}\pm\alpha\right)=-\sin\left(\pi-\frac{\pi}{3}\pm\alpha\right)$$

$$=-\sin\left(\pi-\left(\frac{\pi}{3}\pm\alpha\right)\right)=-\sin\left(\frac{\pi}{3}\pm\alpha\right)$$

Hence, (b) is not correct.

19. b, c, d. $1+\cos 3x=2\cos 2x$

$\Rightarrow 1+4\cos^3 x-3\cos x=2(2\cos^2 x-1)$

$\Rightarrow 4\cos^3 x-4\cos^2 x-3\cos x+3=0$

Let $\cos x=y$, we have

$4y^3-4y^2-3y+3=0$

$\Rightarrow 4y^2(y-1)-3(y-1)=0$

$\Rightarrow (y-1)(4y^2-3)=0$

$\Rightarrow y=1$ or $y^2=\dfrac{3}{4}$

$\Rightarrow \cos x=1$ or $\cos^2 x=\dfrac{3}{4}$

$\Rightarrow \cos x = 1$ or $\cos^2 x = \cos^2 \pi/6$

$\Rightarrow x = 2n\pi$ or $x = n\pi \pm (\pi/6), n \in Z$

20. b, c. $p^2\sec^2\theta + p^2\text{cosec}^2\theta = (2\sqrt{2})^2 \ p^2$

$\Rightarrow \dfrac{1}{\sin^2\theta\cos^2\theta} = 8$

$\Rightarrow \sin^2 2\theta = 1/2 = \left(1/\sqrt{2}\right)^2$

$\Rightarrow 2\theta = (n\pi) + (\pi/4), n \in Z$

$\Rightarrow \theta = (n\pi/2) + (\pi/8)$

for $n = 0, \theta = \pi/8$

for $n = 1, \theta = 3\pi/8$

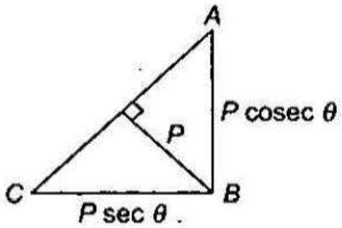


Fig. 3.14

Reasoning Type

1. a. $(\sin x + \cos x)^{1+\sin 2x} = 2 \Rightarrow \quad (\sin x + \cos x)^{(\sin x + \cos x)^2} = 2$

Now we know that the maximum value of $\sin x + \cos x$ is $\sqrt{2}$ which occurs at $x = \pi/4$, for $0 \le x \le \pi/4$.

Also, the given equation has roots only if $\sin x + \cos x = \sqrt{2}$.

Hence, there is only one solution for $0 \le x \le \pi$.

Thus, the correct answer is (*a*).

2. d. We know that $\sin^2 x \le 1$ and $\cos^2 y \le 1$, then $\sin^2 x + \cos^2 y \le 2$

Also $\sec^2 z \ge 1$, then $2\sec^2 z \ge 2$.

Hence, the given equation is solvable only if $\sin^2 x + \cos^2 y = 2$ and $2\sec^2 z = 2$, for which $\sin^2 x$, $\cos^2 y$, $\sec^2 z = 1$.

Then $\sin x, \cos y, \sec z = \pm 1$

Hence, statement 1 is false and statement 2 is true.

3. a.

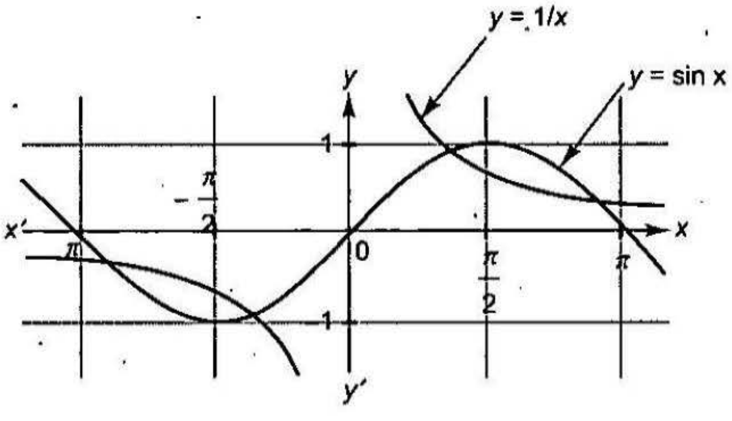


Fig. 3.15

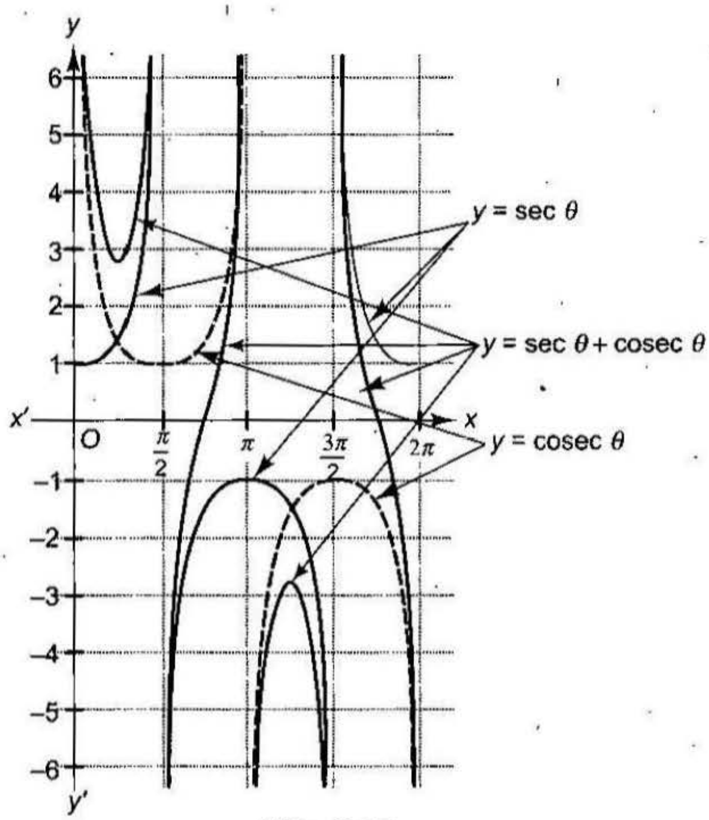


Fig. 3.17

$\sec\theta + \operatorname{cosec}\theta = a$ has solution where graphs of $y = a$ and $y = \sec\theta + \operatorname{cosec}\theta$ intersect. (i)

Graphs of $y = \sec\theta$, $y = \operatorname{cosec}\theta$ and $y = \sec\theta + \operatorname{cosec}\theta$ are as shown in Fig. 3.17.

Clearly, Eq. (i) has two solutions if $-2\sqrt{2} < y < 2\sqrt{2}$.

Equation (i) has four solutions if $y \le -2\sqrt{2}$ or $y \ge 2\sqrt{2}$.

In any case, Eq. (i) has two roots always.

For Problems 7 – 9

7. a, 8. b, 9. d.

Sol. The given system is $\sin x \cos 2y = (a^2 - 1)^2 + 1$, and $\cos x \sin 2y = a + 1$ (i)

Since the L.H.S. of the equations does not exceed 1, the given system may have solutions only for a's such that

$(a^2 - 1)^2 + 1 \le 1$ and $-1 \le a + 1 \le 1$ (ii)

$(a^2 - 1)^2 + 1 \le 1 \quad \Rightarrow \quad (a^2 - 1)^2 \le 0 \quad \Rightarrow \quad (a^2 - 1)^2 = 0 \quad \Rightarrow \quad a = 1$

But $a = 1$ does not satisfy $\cos x \sin 2y = a + 1$

Thus, $a = -1$ only and we get

$\sin x \cos 2y = 1$

$\cos x \sin 2y = 0$ (iii)

$\sin x \cos 2y = 1$

$\Rightarrow \sin x = 1, \cos 2y = 1$

or $\sin x = -1, \cos 2y = -1$

for which $\cos x \sin 2y = 0$

For Problems 10 – 12

10. a, 11. d, 12. b.

Sol. Given that $\int_0^x (t^2 - 8t + 13)\, dt = x \sin(a/x)$ (i)

R.H.S. shows that $x \neq 0$

Integrating L.H.S., we get

$$\left[\frac{t^3}{3} - 4t^2 + 13t\right]_0^x = x \sin(a/x)$$

or $(1/3)[x^3 - 12x^2 + 39x] = x \sin(a/x)$

or $\sin(a/x) = (1/3)[x^2 - 12x + 39]$ $\{\because x \neq 0\}$

$= (1/3)\{(x-6)^2 + 3\}$

$= (1/3)(x-6)^2 + 1$

But $\sin(a/x) \leq 1$, so $\sin(a/x) = 1$, which is possible only for $x = 6$

then we have $\sin(a/6) = 1$

or $a/6 = 2n\pi + \pi/2$ or $a = 12n\pi + 3\pi, n \in Z$

Hence, $x = 6$, $a = 12n\pi + 3\pi, n \in Z$.

For $a \in [0, 100]$, there are exactly three values of $a = 3\pi$, 15π and 27π, i.e.,

$|y - \cos a| < x \Rightarrow |y + 1| < 6 \Rightarrow y \in [-7, 5]$

For Problems 13 – 15

13. a, 14. d, 15. b

Sol. The given equations are

$x \cos^3 y + 3x \cos y \sin^2 y = 14$ and (i)

$x \sin^3 y + 3x \cos^2 y \sin y = 13$ (ii)

Adding Eqs. (i) and (ii), we have

$x(\cos^3 y + 3 \cos y \sin^2 y + 3 \cos^2 y \sin y \sin^3 y) = 27$

$\Rightarrow x(\cos y + \sin y)^3 = 27$

$\Rightarrow x^{1/3}(\cos y + \sin y) = 3$ (iii)

Subtracting Eq. (ii) from Eq. (i), we have

$x(\cos^3 y + 3 \cos y \sin^2 y - 3 \cos^2 y \sin y - \sin^3 y) = 1$

$\Rightarrow x(\cos y - \sin y)^3 = 1$

$\Rightarrow x^{1/3}(\cos y - \sin y) = 1$ (iv)

Dividing Eq. (iii) by (iv), we get

$\cos y + \sin y = 3\cos y - 3\sin y$

$\Rightarrow \tan y = 1/2$

Case I:

$\sin y = 1/\sqrt{5}$ and $\cos y = 2/\sqrt{5}$

$y = 2n\pi + \alpha$, where $0 < \alpha < \pi/2$ and $\sin \alpha = 1/\sqrt{5}$

i.e., y lies in the first quadrant

From Eqs. (iii), $x^{1/3}(3/\sqrt{5}) = 3$ or $x = 5/\sqrt{5}$

Case II:

$\sin y = -1/\sqrt{5}$ and $\cos y = -2/\sqrt{5}$

Draw the graphs of $y = \sin x$ and $y = 1/x$ and verify.

4. d. When $n = 1$, we have interval $[0, \pi]$, which covers only first and second quadrant, in which $\sin x = -1/2$ is not possible. Hence, the number of solutions is zero. Also from $2(n-1)$, we have zero solution when $n = 1$.
For $n = 2$, we have interval $[0, 2\pi]$ which covers all the quadrant only once. Hence, the number of solutions is two.
Also from $2(n-1)$, we have two solutions, when $n = 2$.
For $n = 3$, we have interval $[0, 3\pi]$, which covers third and fourth quadrant only once. Hence, the number of solutions is two. But from $2(n-1)$, we have four solutions which contradict.
Hence, statement 1 is false, statement 2 is true.

5. a. $\sqrt{1-\sin 2x} = \sin x$

$\Rightarrow \sqrt{(\sin x - \cos x)^2} = \sin x$
$\Rightarrow |\sin x - \cos x| = \sin x$
$\Rightarrow \cos x - \sin x = \sin x$
$\Rightarrow 2\sin x = \cos x$

$\Rightarrow \tan x = \frac{1}{2}$ which has only one solution for $x \in [0, \pi/4]$ for these values of x.

6. b. Draw the graphs of $y = |\sin x|$ and $y = |x|$ and verify that $|\sin x| = |x|$ has only one solution $x = 0$. But statement 2 is not the correct explanation of statement 1.

7. d. Given $\tan 2x = 1$

$\therefore 2x = n\pi + \frac{\pi}{4}, n \in Z$ [note that $\tan 4x$ is not defined for these values of x]

Hence, the given equation has no solution.
Therefore, statement 1 is false and statement 2 is true.

8. a. $\cos(\sin x) = \sin(\cos x)$
$\Rightarrow \cos(\sin x) = \cos[(\pi/2) - \cos x]$
$\Rightarrow \sin x = 2n\pi \pm (\pi/2 - \cos x), n \in Z$ (i)
Taking + ive sign, we get $\sin x = 2n\pi + \pi/2 - \cos x$

or $(\cos x + \sin x) = \frac{1}{2}(4n+1)\pi$

Now L.H.S. $\in [-\sqrt{2}, \sqrt{2}]$, hence $-\sqrt{2} \le \frac{1}{2}(4n+1)\pi \sqrt{2}$.

$\Rightarrow \frac{-2\sqrt{2}-\pi}{4\pi} \le n \le \frac{2\sqrt{2}-\pi}{4\pi}$, which is not satisfied by any integer n.

Similarly, taking − ve sign, we have $\sin x - \cos x = (4n-1)\pi/2$, which is also not satisfied for any integer n.
Hence, there is no solution.

9. b. Statement 1 is true.
Also statement 2 is true but does not explain statement 1.
Consider the equation $\sin x = x^3$.
Here, $y = x^3$ is an unbounded function but equation has finite number of solutions.

10. a. Let $y = n|\sin x| = m|\cos x|$

The curve $y = n|\sin x|$ and $y = m|\cos x|$ intersect at four points in $[0, 2\pi]$.

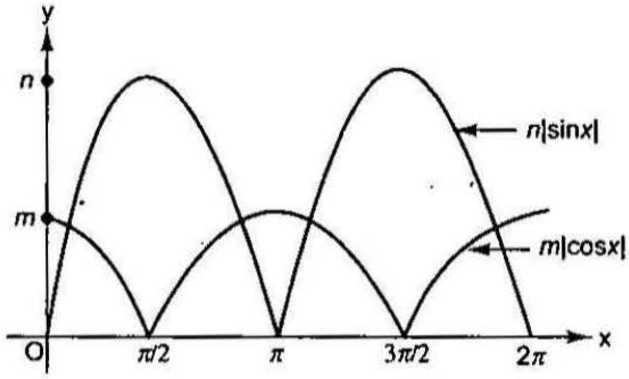


Fig. 3.16

Linked Comprehension Type

For Problems 1 – 3

1. b., 2. c., 3. a

Sol.

1. b. $x^3 - (1 + \cos\theta + \sin\theta)x^2 + (\cos\theta\sin\theta + \cos\theta + \sin\theta)x - \sin\theta\cos\theta = 0$

Given cubic function is

$f(x) = (x-1)(x-\cos\theta)(x-\sin\theta)$

Therefore, roots are 1, $\sin\theta$ and $\cos\theta$

Hence, $x_1^2 + x_2^2 + x_3^2 = 1 + \sin^2\theta + \cos^2\theta = 2$

2. c. Now if $1 = \sin\theta$, we get $\theta = \pi/2$

If $1 = \cos\theta$, then $\theta = 0, 2\pi$

and if $\sin\theta = \cos\theta$, we get $\tan\theta = 1$

$$\therefore \theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

Therefore, the number of values of θ in $[0, 2\pi]$ is 5.

3. a. Again the maximum possible difference between the two roots is 2.

$1 - \sin\theta = 2$ when $\theta = 3\pi/2$ or $1 - \cos\theta = 2$ when $\theta = \pi$

For Problems 4 – 6

4., a, 5. c, 6. d.

Sol. See Fig. 3.17 for the solution

$y = 2n\pi + (\pi + \alpha)$, where $0 < \alpha < \pi/2$ and $\sin\alpha = -1/\sqrt{5}$
i.e., y lies in the 3rd quadrant

Therefore, from Eq. (3), $x^{1/3}(-3/\sqrt{5}) = 3$ or $x = -5\sqrt{5}$.
Thus, $\sin^2 y + 2\cos^2 y = 1/5 + 8/5 = 9/5$.
Also there are exactly six values of $y \in [0, 6\pi]$, three in 1st quadrant and three in 3rd quadrant.

Matrix-Match Type

1. a → r, b → s, c → p, d → q.

a. $\cos^2 2x - \sin^2 x = 0$

$\Rightarrow \cos 3x \cos x = 0$

$\Rightarrow \cos 3x = 0$ or $\cos x = 0$

$\Rightarrow 3x = (2n-1)\dfrac{\pi}{2}$ or $x = (2n-1)\dfrac{\pi}{2}, n \in Z$

$\Rightarrow x = (2n-1)\dfrac{\pi}{6}$ or $x = (2n-1)\dfrac{\pi}{2}$

Hence, the general solution is $(2n-1)\dfrac{\pi}{6}$ as $(2n-1)\dfrac{\pi}{2}$ is contained in $(2n-1)\dfrac{\pi}{6}$.

b. $\cos x + \sqrt{3}\sin x = \sqrt{3}$

$\Rightarrow \dfrac{\cos x}{2} + \dfrac{\sqrt{3}}{2}\sin x = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \cos\left(x - \dfrac{\pi}{3}\right) = \cos\dfrac{\pi}{6}$

$\Rightarrow x - \dfrac{\pi}{3} = 2n\pi \pm \dfrac{\pi}{6}, n \in Z$

$\Rightarrow x - \dfrac{\pi}{3} = 2n\pi + \dfrac{\pi}{6}$ or $2n\pi - \dfrac{\pi}{6}$

$\Rightarrow x = 2n\pi + \dfrac{\pi}{2}$ or $x = 2n\pi + \dfrac{\pi}{6}$

c. $\sqrt{3}\tan^2 x - (\sqrt{3}+1)\tan x + 1 = 0$

$\Rightarrow \sqrt{3}\tan x(\tan x - 1) - (\tan x - 1) = 0$

$\Rightarrow (\tan x - 1)(\sqrt{3}\tan x - 1) = 0$

$\therefore \tan x = 1 \quad \Rightarrow \quad x = n\pi + \frac{\pi}{4}, n \in Z$

or $\tan x = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad x = n\pi + \frac{\pi}{6}, n \in Z$

d. $\tan 3x - \tan 2x - \tan x = 0$
or $\tan x \tan 2x \tan 3x = 0$

$x = n\pi$ or $n\pi/2$ (rejected) or $n\pi/3$

Therefore, the general solution is $n\pi/3$ as $n\pi$ is contained in $n\pi/3$.

2. **a → q ; b → s ; c → s; d → p.**

a. Here, $x^3 + (x+2)^2 + 2\sin x = 4$.
Clearly, $x = 0$ satisfies the equation
If $0 < x \le \pi, x^3 + (x+2)^2 + 2\sin x > 4$
If $\pi < x \le 2\pi, x^3 + (x+2)^2 + 2\sin x > 27 + 25 - 2$
So, $x = 0$ is the only solution.

b. Here, $\frac{1}{2}\sin(2e^x) = \frac{1}{4}\left(2^x + 2^{-x}\right) \ge \frac{1}{4}2 = \frac{1}{2}$ ($\because$ A.M. ≥ G.M.)

$\Rightarrow \quad \sin(2e^x) \ge 1 \quad \Rightarrow \quad \sin(2e^x) = 1$

But equality can hold when $2^x = 2^{-x} = 1$, i.e., $x = 0$.
Then $\sin(2.\, e^0) = 1$, which is not true.
Hence, there is no solution.

c. $\sin 2x + \cos 4x = 2$

$\Rightarrow \quad \sin 2x = 1, \cos 4x = 1$

$\therefore 1 - 2\sin^2 2x = 1$ or $1 - 2 = 1$ (absurd)

d. The given solution is $|\sin x| = x/30$.
Therefore, the solution of this equation is the point of intersections of the curves, i.e., $y = |\sin x|$ and $y = x/30$.

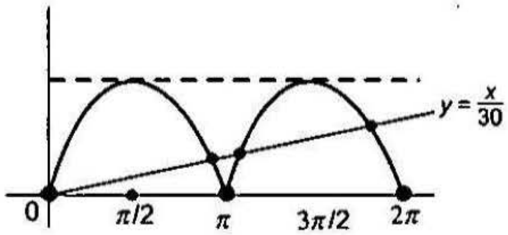


Fig. 3.18

Since there are four points of intersection in $0 \le x \le 2\pi$, the equation has four solutions.

3. **a → q; b → s; c → p; d → r.**

a. $5\sin\theta + 3(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$
$= (5 + 3\cos\alpha)\sin\theta - 3\sin\alpha\cos\theta$

$$\Rightarrow \quad \max_{\theta \in R}\{5\sin\theta + 3\sin(\theta - \alpha)\} = \sqrt{(5 + 3\cos\alpha)^2 + 9\sin^2\alpha}$$

$$= \sqrt{34 + 30\cos\alpha}$$

Therefore, the given equation is $34 + 30\cos\alpha = 49$.

$\Rightarrow \quad \cos\alpha = 1/2 \quad \Rightarrow \quad \alpha = 2n\pi \pm \pi/3, n \in Z$

b. $(\cos x)^{\sin^2 x-3\sin x+2}=1$

$\Rightarrow \quad \sin^2 x-3\sin x+2=0 \quad \Rightarrow \quad (\sin x-2)(\sin x-1)=0 \quad \Rightarrow \quad \sin x=1$

but this does not satisfy the equation because $0^{\circ}=1$ is absurd.

c. $\sqrt{(\sin x)}+2^{1/4}\cos x=0$ (i)

$\therefore \sqrt{(\sin x)}>0$ and so $\cos x<0$,

Also $\sin x>0 \Rightarrow x$ lies in 2nd quadrant

Equation (i) can be rewritten as $2^{1/4}\cos x=-\sqrt{(\sin x)}$

On squaring, we get $\sqrt{2}\cos^2 x=\sin x$

$\Rightarrow \quad \sqrt{2}\sin^2 x+\sin x-\sqrt{2}=0 \quad \Rightarrow \quad (\sqrt{2}\sin x-1)(\sin x+\sqrt{2})=0$

$\sin x\neq-\sqrt{2}$, therefore $\sin x=1/\sqrt{2}$ (ii)

$\Rightarrow x=3\pi/4$ and so the general value of x is given by $x=2n\pi+3\pi/4,\ n\in Z$

d. $\log_5\tan x=(\log_5 4)(\log_4(3\sin x))$

$\Rightarrow \quad \log_5\tan x=\log_5(3\sin x)$ (i)

Since $\log x$ is real when $x>0$, we have

$\tan x>0,\ \sin x>0$

Therefoe, x lies in the first quadrant. Now Eq. (i) gives

$\tan x=3\sin x$ or $\cos x=1/3$

$\therefore x=2n\pi+\cos^{-1}(1/3),\ n\in Z$

Integer Type

1. (1) $\sin^3 x+p^3+1=3p\sin x$

$\Rightarrow (\sin x+p+1)(\sin^2 x+1+p^2-\sin x-p-p\sin x)=0$

Therefore, either $\sin x+p+1=0 \Rightarrow p=-(1+\sin x)$, or

$\sin x=1=p$

Hence, only one value of $p(p>0)$ is possible which is given by $p=1$.

2. (0) $|\sin x\cos x|+|\tan x+\cot x|=\sqrt{3}$

$$\Rightarrow |\sin x\cos x|+\frac{1}{|\sin x\cos x|}=\sqrt{3}$$

$$|\sin x\cos x|+\frac{1}{|\sin x\cos x|}\geq 2$$

Hence, there is no solution.

3. (3) $4\leq \text{L.H.S.}\leq 16$

$2\leq \text{R.H.S.}\leq 4$

Hence, equality can occur only when both sides are 4, which is possible if $x=\pi, 3\pi, 5\pi$.

That is, there are three solutions.

4. (6) $\frac{\sin x}{\cos 3x}+\frac{\sin 3x}{\cos 9x}+\frac{\sin 9x}{\cos 27x}=0$

$\Rightarrow \frac{2\sin x\cos x}{2\cos 3x\cos x}+\frac{2\sin 3x\cos 3x}{2\cos 9x\cos 3x}+\frac{2\sin 9x\cos 9x}{2\cos 27x\cos 9x}=0$

$\Rightarrow \frac{\sin(3x-x)}{2\cos 3x\cos x}+\frac{\sin(9x-3x)}{2\cos 9x\cos 3x}+\frac{\sin(27x-9x)}{2\cos 27x\cos 9x}=0$

$\Rightarrow (\tan 3x-\tan x)+(\tan 9x-\tan 3x)+(\tan 27x-\tan 9x)=0$

$\Rightarrow \tan 27x-\tan x=0$

$\Rightarrow \tan x=\tan 27x$

$\Rightarrow 27x=n\pi+x, n\in I$

$\Rightarrow x=\frac{n\pi}{26}, n\in I.$

$\Rightarrow x=\frac{\pi}{26},\frac{2\pi}{26},\frac{3\pi}{26},\frac{4\pi}{26},\frac{5\pi}{26},\frac{6\pi}{26}$

Hence, there are six solutions.

5. (1) $\left(\sqrt{3}+1\right)^{2x}+\left(\sqrt{3}-1\right)^{2x}=2^{3x}=\left(2\sqrt{2}\right)^{2x}$

$\Rightarrow \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^{2x}+\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^{2x}=1$

$\Rightarrow (\sin 75°)^{2x}+(\cos 75°)^{2x}=1$

$\Rightarrow x=1$

6. (4) Since $-2\le \sin x-\sqrt{3}\cos x\le 2$

$\Rightarrow -2\le \frac{4m-6}{4-m}\le 2$

or $-1\le \frac{2m-3}{4-m}\le 1$

if $\frac{2m-3}{4-m}\le 1$.

$\Rightarrow \frac{(2m-3)-(4-m)}{4-m}\le 0$

$\Rightarrow \frac{3m-7}{m-4}\ge 0$ (i)

Also, $-1\le \frac{2m-3}{4-m}$

$\Rightarrow \frac{m+1}{m-4}\le 0$ (ii)

From Eq. (i) and (ii), we get $m\in\left[-1,\frac{7}{3}\right]$

Therefore, the possible integers are $-1, 0, 1, 2$.

7. (1) Adding given equations, we get

$$2 = \frac{3a}{2} + \frac{a^2}{2}$$
$$\Rightarrow a^2 + 3a - 4 = 0$$
$$\Rightarrow (a+4)(a-1) = 0$$
$$\Rightarrow a = 1 \text{ (as } a = -4 \text{ is rejected)}$$

8. (5) $\cos 4x = 2\cos^2 2x - 1$

$$= 2(2\cos^2 x - 1)^2 - 1$$
$$= 2(4\cos^4 x + 1 - 4\cos^2 x) - 1$$
$$= 8\cos^4 x - 8\cos^2 x + 1$$

$\therefore a_0 = 1;\ a_1 = -8,\ a_2 = 8$

$\therefore 5a_0 + a_1 + a_2 = 5$

9. (1) $1 - \sin^2 x - \sin x + a = 0$

$\Rightarrow \sin^2 x + \sin x - (a+1) = 0$ (i)

From Eq. (i), we get

$\sin^2 x + \sin x = (a+1)$

For $x \in (0, \pi/2)$, the range of $\sin^2 x + \sin x$ is $(0, 2)$.

$\Rightarrow 0 < (a+1) < 2 \Rightarrow a \in (-1, 1)$

10. (6) $a\sin x + 1 - 2\sin^2 x = 2a - 7$

$\Rightarrow 2\sin^2 x - a\sin x + (2a-8) = 0$

$$\Rightarrow \sin x = \frac{a \pm \sqrt{a^2 - 8(2a-8)}}{4} = \frac{a \pm (a-8)}{4} = 2 \text{ or } \frac{a-4}{2}$$

For a solution $-1 \le \frac{a-4}{2} \le 1$, we have $2 \le a \le 6$.

11. (0) $$\frac{\sin^2\left(x - \frac{\pi}{4}\right)}{\cos 2x} = \frac{\frac{1}{2}(\sin x - \cos x)^2}{\cos^2 x - \sin^2 x} = \frac{-\frac{1}{2}(\sin x - \cos x)}{\cos x + \sin x} = -\frac{1}{2}\tan\left(x - \frac{\pi}{4}\right)$$

Given equation reduces to $2^{\tan\left(x - \frac{\pi}{4}\right)} - 2(0.25)^{\frac{1}{2}\tan\left(x - \frac{\pi}{4}\right)} + 1 = 0$

$$\Rightarrow 2^{\tan\left(x - \frac{\pi}{4}\right)} = 1$$

$\Rightarrow x = \pi/4$ which is not possible as $\cos 2x = 0$ for this value of x, which is not defining the original equation.

12. (4) $\sin^4 x - \cos^2 x \sin x + 2\sin^2 x + \sin x = 0$

$\Rightarrow \sin x\,[\sin^3 x - \cos^2 x + 2\sin x + 1] = 0$

$\Rightarrow \sin x\,[\sin^3 x + \sin^2 x + 2\sin x] = 0$

$\Rightarrow \sin^2 x[\sin^2 x + \sin x + 2] = 0$

$\Rightarrow \sin x = 0$, where $x = 0, \pi, 2\pi, 3\pi$

Hence, there are four solutions.

Archives

Subjective

1. At the intersection point of $y = \cos x$ and $y = \sin 3x$, we have $\cos x = \sin 3x$

$\Rightarrow \quad \cos x = \cos\left(\frac{\pi}{2} - 3x\right) \qquad \Rightarrow \quad x = 2n\pi \pm \left(\frac{\pi}{2} - 3x\right), n \in Z$

$\Rightarrow \quad x = \frac{\pi}{4}, \frac{\pi}{8}, -\frac{3\pi}{8}$ $\qquad [\because -\pi/2 \le x \le \pi/2]$

Thus, the points are $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right), \left(\frac{\pi}{8}, \cos\frac{\pi}{8}\right)$ and $\left(-\frac{3\pi}{8}, \cos\frac{3\pi}{8}\right)$

2. The given equation is

$4\cos^2 x \sin x - 2\sin^2 x = 3\sin x$

$\Rightarrow \quad 4\cos^2 x \sin x - 2\sin^2 x - 3\sin x = 0$

$\Rightarrow \quad 4(1 - \sin^2 x)\sin x - 2\sin^2 x - 3\sin x = 0$

$\Rightarrow \quad \sin x\,[4\sin^2 x + 2\sin x - 1] = 0$

$\Rightarrow \quad$ either $\sin x = 0$ or $4\sin^2 x + 2\sin x - 1 = 0$

If $\sin x = 0 \quad \Rightarrow \quad x = n\pi, n \in Z$

If $4\sin^2 x + 2\sin x - 1 = 0$

$\Rightarrow \quad \sin x = \frac{-1 \pm \sqrt{5}}{4}$

$\therefore x = m\pi + (-1)^m \sin^{-1}\left(\frac{-1 \pm \sqrt{5}}{4}\right), m \in Z$

Thus, $x = n\pi, m\pi \pm (-1)^m \sin^{-1}\left(\frac{-1 \pm \sqrt{5}}{4}\right)$.

where m and n are integers.

3. The given equation is $8^{(1 + |\cos x| + |\cos^2 x| + \cos^3 x| + \cdots)} = 4^3$

$\Rightarrow \quad 2^{3(1 + |\cos x| + |\cos^2 x| + \cos^3 x| + \cdots\infty)} = 2^6$

$\Rightarrow \quad 3(1 + |\cos x| + |\cos^2 x| + \cos^3 x| + \cdots) = 6$

$\Rightarrow \quad 1 + |\cos x| + |\cos^2 x| + |\cos^3 x| + \cdots = 2$

$\Rightarrow \quad \frac{1}{1 - |\cos x|} = 2$

$\Rightarrow \quad |\cos x| = 1/2$

$\Rightarrow \quad |\cos x| = \frac{1}{2}$

$\Rightarrow \quad x = \pi/3, -\pi/3, 2\pi/3, -2\pi/3, \ldots$

The values of $x \in (-\pi, \pi)$ are $\pm\pi/3, \pm 2\pi/3$.

4. Given that $(1-\tan\theta)(1+\tan\theta)\sec^2\theta + 2^{\tan^2\theta} = 0$

$\Rightarrow \quad (1-\tan^2\theta)(1+\tan^2\theta) + 2^{\tan^2\theta} = 0$

Let us put $\tan^2\theta = t$

$\therefore (1-t)(1+t)+2^t=0 \quad \Rightarrow \quad 1-t^2+2^t=0$

It is clearly satisfied by $t=3$

as $-8+8=0$, we get $\tan^2\theta=3$

$\therefore \theta = \pm\pi/3$ in the given interval.

5. $(y+z)\cos 3\theta - (xyz)\sin 3\theta = 0$ (i)

$xyz\sin 3\theta = 2\cos(3\theta)z + (2\sin 3\theta)y$ (ii)

$\therefore (y+z)\cos 3\theta = (2\cos 3\theta)z + (2\sin 3\theta)y = (y+2z)\cos 3\theta + y\sin 3\theta$

$\therefore y(\cos 3\theta - 2\sin 3\theta) = z\cos 3\theta$ and $y(\sin 3\theta - \cos 3\theta) = 0$

$\Rightarrow \sin 3\theta - \cos 3\theta = 0$

$\therefore \sin 3\theta = \cos 3\theta$

$\therefore 3\theta = n\pi + (\pi/4), n \in Z \Rightarrow \theta = \dfrac{(4n+1)\pi}{3} \Rightarrow \theta = \dfrac{\pi}{3}$ only

6. $\tan\theta = \cot 5\theta$

$\Rightarrow \cos 6\theta = 0$

$\Rightarrow 4\cos^3 2\theta - 3\cos 2\theta = 0$

$\Rightarrow \cos 2\theta = 0$ or $\pm\dfrac{\sqrt{3}}{2}$ (i)

$\sin 2\theta = \cos 4\theta$

$\Rightarrow 2\sin^2 2\theta + \sin 2\theta - 1 = 0$

$\Rightarrow 2\sin^2 2\theta + 2\sin 2\theta - \sin 2\theta - 1 = \theta$

$\Rightarrow \sin 2\theta = -1$ or $\sin 2\theta = \dfrac{1}{2}$

$\Rightarrow \cos 2\theta = 0$ and $\sin 2\theta = -1$

$\Rightarrow 2\theta = -\dfrac{\pi}{2} \Rightarrow \theta = -\dfrac{\pi}{4}$

or $\cos 2\theta = \pm\dfrac{\sqrt{3}}{2}, \sin 2\theta = \dfrac{1}{2}$

$\Rightarrow 2\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6} \Rightarrow \theta = \dfrac{\pi}{12}, \dfrac{5\pi}{12}$

$\therefore \theta = -\dfrac{\pi}{4}, \dfrac{\pi}{12}, \dfrac{5\pi}{12}$

Objective

Fill in the blanks

1. We have $\cos x + \cos y = \dfrac{3}{2}$

$\Rightarrow \quad 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right) = \dfrac{3}{2}$

$\Rightarrow \quad 2\cos\frac{\pi}{3}\cos\left(\frac{x-y}{2}\right)=\frac{3}{2}$ [using : $x+y=2\pi/3$]

$\Rightarrow \quad \cos\left(\frac{x-y}{2}\right)=\frac{3}{2}$, which is not possible.

Hence, the system of equations has no solution.

2. We have $2\sin^2 x-3\sin x+1\geq 0$

$\Rightarrow \quad (2\sin x-1)(\sin x-1)\geq 0$

$\Rightarrow \quad \left(\sin x-\frac{1}{2}\right)(\sin x-1)\geq 0$

$\Rightarrow \quad \sin x\leq\frac{1}{2}$ or $\sin x\geq 1$

But we know that $\sin x\leq 1$ and $\sin x\geq 0$ for $x\in[0,\pi]$.

Therefore, either $\sin x=1$ or $0\leq\sin x\leq\frac{1}{2}$ $\Rightarrow$ either $x=\pi/2$ or $x\in[0,\pi/6]\cup[5\pi/6,\pi]$

Combining, we get $x\in\left[0,\frac{\pi}{6}\right]\cup\left\{\frac{\pi}{2}\right\}\cup\left[\frac{5\pi}{6},\pi\right]$.

3. $\tan^2\theta+\sec 2\theta=1$

$t^2+\frac{1+t^2}{1-t^2}=1$, where $t=\tan\theta$

$\Rightarrow \quad t^2(t^2-3)=0 \quad \Rightarrow \quad \tan\theta=0,\pm\sqrt{3} \quad \Rightarrow \quad \theta=n\pi$ and $\theta=n\pi\pm\pi/3,\ n\in Z$.

4. $\cos^7 x=1-\sin^4 x$

$=(1-\sin^2 x)(1+\sin^2 x)$

$=\cos^2 x(1+\sin^2 x)$

$\therefore \cos x=0$ or $x=\pi/2,-\pi/2$, or $\cos^5 x=1+\sin^2 x$

$\cos^5 x\leq 1$ but $1+\sin^2 x\geq 1$

$\Rightarrow \quad \cos^5 x=1+\sin^2 x=1$

$\Rightarrow \quad \cos x=1$ and $\sin x=0$. [both these imply $x=0$]

Hence, $x=-\frac{\pi}{2},\frac{\pi}{2}$ and 0.

True or false

1. Given that equation is $\sin^4\theta-2\sin^2\theta-1=0$

$\therefore \sin^2\theta=\frac{2\pm\sqrt{8}}{2}=1\pm\sqrt{2}$

But $\sin^2\theta$ cannot be –ve

$\therefore \sin^2\theta=\sqrt{2}+1$

But as $-1\leq\sin\theta\leq 1$, $\sin^2\theta\neq\sqrt{2}+1$

Thus there is no value of θ which satisfies the given equation.

Therefore, statement is false.

Multiple choice questions with one correct answer

1. a. The given equation is

$$2\cos^2\left(\frac{x}{2}\right)\sin^2 x=x^2+\frac{1}{x^2}$$

Now $\cos(\alpha-\beta)=1 \Rightarrow \alpha-\beta=0 \Rightarrow \alpha=\beta$

$\therefore \cos(\alpha+\beta)=1/e \Rightarrow \cos 2\alpha=1/e$

$\because 0<1/e<1$ and $2\alpha\in[-2\pi, 2\pi]$

There will be two values of 2α satisfying $\cos 2\alpha=1/e$ in $[0, 2\pi]$ and two in $[-2\pi, 0]$.

Therefore, there will be four values of a in $[-\pi, \pi]$ and correspondingly four values of β. Hence, there are four sets of (α, β).

10. a. $2\sin^2\theta-5\sin\theta+2>0$

$\Rightarrow (\sin\theta-2)(2\sin\theta-1)>0 \Rightarrow \sin\theta<1/2$ $[\because -1\le\sin\theta\le 1]$

$\Rightarrow x\in(0, \pi/6)\cup(5\pi/6, 2\pi)$

11. c. $2\sin^2\theta-\cos 2\theta=0$

$\Rightarrow 1-2\cos 2\theta=0$

$\Rightarrow \cos 2\theta=\frac{1}{2}$

$\Rightarrow 2\theta=\frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$\Rightarrow \theta=\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ (i)

where $\theta\in[0, 2\pi]$.

Also $2\cos^2\theta-3\sin\theta=0$

$\Rightarrow 2\sin^2\theta+3\sin\theta-2=0$

$\Rightarrow (2\sin\theta-1)(\sin\theta+2)=0$

$\Rightarrow \sin\theta=1/2$ $[\because \sin\theta\neq-2]$

$\Rightarrow \theta=\pi/6, 5\pi/6$ where $\theta\in[0, 2\pi]$ (ii)

Combining Eqs. (i) and (ii), we get $\theta=\frac{\pi}{6}, \frac{5\pi}{6}$

Therefore, there are two solutions.

Multiple choice questions with one or more than one correct answer

1. d. Since $a_1+a_2\cos 2x+a_3\sin^2 x=0$ for all x

Putting $x=0$ and $x=\pi/2$, we get

$a_1+a_2=0$, and $a_1-a_2+a_3=0$

$\Rightarrow a_2=-a_1$ and $a_3=-2a_1$

Therefore, the given equation becomes

$a_1-a_1\cos 2x-2a_1\sin^2 x=0, \forall x$

$\Rightarrow a_1(1-\cos 2x-2\sin^2 x)=0, \forall x \Rightarrow a_1(2\sin^2 x-2\sin^2 x)=0, \forall x$

The above is satisfied for all values of a_1.

Hence, the infinite number of triplets $(a_1, -a_1, -2a_1)$ is possible.

2. a, c. We have

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$

Operating $C_1 \to C_1 + C_2$, we get

$$\begin{vmatrix} 2 & \cos^2\theta & 4\sin 4\theta \\ 2 & 1+\cos^2\theta & 4\sin 4\theta \\ 1 & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$

Operating $R_1 \to R_1 - R_2$; $R_2 \to R_2 - R_3$, we get

$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$

Expanding along R_1, we get $1 + 4\sin 4\theta + 1 = 0$

$\Rightarrow\ 2(1 + 2\sin 4\theta) = 0 \quad \Rightarrow\ \sin 4\theta = -1/2 \quad \Rightarrow\ 4\theta = \pi + \pi/6$ or $2\pi - \pi/6$

$\Rightarrow\ 4\theta = 7\pi/6$ or $11\pi/6 \quad \Rightarrow\ \theta = 7\pi/24$ or $11\pi/24$. Hence, there are two correct options.

3. c. $3\sin^2 x - 7\sin x + 2 = 0$

$\Rightarrow\ (\sin x - 2)(3\sin x - 1) = 0 \quad \Rightarrow\ \sin x = 1/3 = \sin\alpha$, say, ($\sin x = 2$, not possible)

$x = n\pi + (-1)^n\alpha, n = 0, 1, 2, 3, 4, 5$ in $(0, 5\pi)$

4. d. $2\sin^2 x + 3\sin x - 2 > 0$

$(2\sin x - 1)(\sin x + 2) > 0$

$\Rightarrow\ 2\sin x - 1 > 0$ $\qquad [\because -1 \le \sin x \le 1]$

$\Rightarrow\ \sin x > 1/2 \quad \Rightarrow\ x \in (\pi/6, 5\pi/6)$ $\qquad$ (i)

Also $x^2 - x - 2 < 0 \quad \Rightarrow\ (x-2)(x+1) < 0 \quad \Rightarrow\ -1 < x < 2$ $\qquad$ (ii)

Combining Eqs. (i) and (ii), we get

$x \in (\pi/6, 2)$

5. a, b. $\dfrac{(\sin x)^4}{2} + \dfrac{(\cos x)^4}{3} = \dfrac{1}{5}$

$3 - 6\cos^2 x + 5(\cos x)^4 = \dfrac{6}{5}$. Let $\cos x = t$

$\Rightarrow\ 25t^4 - 30t^2 + 9 = 0$

$\Rightarrow\ t^2 = \dfrac{3}{5}$

$\Rightarrow\ \tan^2 x = \dfrac{2}{3}$

$\Rightarrow\ (\sin x)^8 = \left(\dfrac{2}{5}\right)^4 = \dfrac{16}{625}$

$\Rightarrow\ (\cos x)^8 = \left(\dfrac{\sqrt{3}}{\sqrt{5}}\right)^4 = \dfrac{81}{625}$

$\Rightarrow\ \dfrac{(\sin x)^8}{8} + \dfrac{(\cos x)^8}{27} = \dfrac{1}{125}$

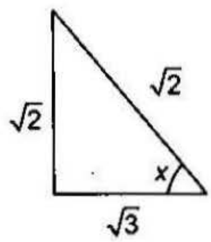


Fig. 3.19

where $0 < x \le \dfrac{\pi}{2}$

L.H.S. $= 2\cos^2\left(\dfrac{x}{2}\right)\sin^2 x = (1+\cos x)\sin^2 x$

$\because$ $1 + \cos x < 2$ and $\sin^2 x \le 1$ for $0 < x < \dfrac{\pi}{2}$

$\therefore$ $(1 + \cos x)\sin^2 x < 2$

and R.H.S. $= x^2 + \dfrac{1}{x^2} \ge 2$

$\therefore$ For $0 < x \le \dfrac{\pi}{2}$, given equation is not possible for any real value of x.

2. c. $\sin x + \cos x = 1$

$\Rightarrow \quad \dfrac{1}{\sqrt{2}}\sin x + \dfrac{1}{\sqrt{2}}\cos x = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad \sin x \cos(\pi/4) + \cos x \sin(\pi/4) = \sin \pi/4$

$\Rightarrow \quad \sin(x + \pi/4) = \sin \pi/4$

$\Rightarrow \quad x + (\pi/4) = n\pi + (-1)^n \pi/4, n \in Z$

$\Rightarrow \quad x = n\pi + [(-1)^n \pi/4] - \pi/4$

where $n = 0, \pm 1, \pm 2, \ldots$

3. b. The given equation is

$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

$\Rightarrow \quad 2\sin 2x \cos x - 3\sin 2x = 2\cos 2x \cos x - 3\cos 2x$

$\Rightarrow \quad \sin 2x(2\cos x - 3) = \cos 2x(2\cos x - 3)$

$\Rightarrow \quad \sin 2x = \cos 2x$ (as $\cos x \ne 3/2$)

$\Rightarrow \quad \tan 2x = 1 \Rightarrow 2x = n\pi + \pi/4 \Rightarrow x = \dfrac{n\pi}{2} + \dfrac{\pi}{8}, n \in Z$

4. d. The given equation is

$(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$

For this equation to have real roots $D \ge 0$

$\Rightarrow \quad \cos^2 p - 4\sin p(\cos p - 1) \ge 0$

$\Rightarrow \cos^2 p - 4\sin p \cos p + 4\sin^2 p + 4\sin p - 4\sin^2 p \ge 0$

$\Rightarrow \quad (\cos p - 2\sin p)^2 + 4\sin p(1 - \sin p) \ge 0$

For every real value of p, we have

$(\cos p - 2\sin p)^2 \ge 0$ and $\sin p(1 - \sin p) \ge 0$

$\therefore D \ge 0, \forall \sin p \in (0, \pi)$

5. c. The given equation is

$\tan x + \sec x = 2\cos x$

$\Rightarrow \quad \dfrac{\sin x}{\cos x} + \dfrac{1}{\cos x} = 2\cos x$

$\Rightarrow \quad \sin x + 1 = 2\cos^2 x = 2 - 2\sin^2 x$

$\Rightarrow \quad 2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow \quad (2\sin x - 1)(\sin x + 1) = 0$

$\Rightarrow \quad \sin x = \frac{1}{2}, -1$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2} \in [0, 2\pi]$

But for $x = 3\pi/2$, $\tan x$ and $\sec x$ are not defined.
Therefore, there are only two solutions.

6. d. The given equation is
$2\sin^2\theta - 3\sin\theta - 2 = 0$
$\Rightarrow \quad (2\sin\theta + 1)(\sin\theta - 2) = 0$

$\Rightarrow \quad \sin = -\frac{1}{2}$ [$\because \sin\theta - 2 = 0$ is not possible].

$\Rightarrow \quad \sin\theta = \sin(-\pi/6) = \sin(7\pi/6)$
$\Rightarrow \quad \theta = n\pi + (-1)^n(-\pi/6) = n\pi + [(-1)^n 7\pi/6]$
$\Rightarrow$ Thus, $\theta = n\pi + (-1)^n 7\pi/6, n \in Z$

7. c. To simplify the determinant, let $\sin x = a$; $\cos x = b$. Then the equation becomes

$\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix} = 0$. Operating $C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_2$, we get

$$\begin{vmatrix} a & b-a & 0 \\ b & a-b & b-a \\ b & 0 & a-b \end{vmatrix} = 0.$$

$\Rightarrow \quad a(a-b)^2 - (b-a)[b(a-b) - b(b-a)] = 0$
$\Rightarrow \quad a(a-b)^2 - 2b(b-a)(a-b) = 0$
$\Rightarrow \quad (a-b)^2(a-2b) = 0$
$\Rightarrow \quad a = b$ or $a = 2b$

$\Rightarrow \quad \frac{a}{b} = 1$ or $\frac{a}{b} = 2$

$\Rightarrow \quad \tan x = 1$ or $\tan x = 2$
But we have $-\pi/4 \le x \le \pi/4$
$\Rightarrow \quad \tan(\pi/4) \le \tan x \le \tan(\pi/4)$
$\Rightarrow \quad -1 \le \tan x \le 1$
$\therefore \tan x = 1 \quad \Rightarrow \quad x = \pi/4$
Therefore, there is only one real root.

8. b. We know that

$-\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2} \quad \Rightarrow \quad -\sqrt{74} \le 2k+1 \le \sqrt{74}$
$\Rightarrow \quad -8 \le 2k+1 \le 8 \quad \Rightarrow \quad -4.5 \le k \le 3.5$
(considering only integral values) $\Rightarrow$ k can take eight integral values.

9. d. Given that $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = 1/e$
where $\alpha, \beta \in [-\pi, \pi]$