1. **જો $\sin(\theta + \alpha) = a$ અને $\sin(\theta + \beta) = b$ હોય, તો સાબિત કરો કે $\cos(\alpha + \beta) - 4ab\cos(\alpha - \beta) = 1 - 2a^2 - 2b^2$.**

➥ અહીં $\sin(\theta + \alpha) = a$ ....(i)

અને $\sin(\theta + \beta) = b$ ....(ii) આપેલ છે.

$\therefore \cos(\theta + \alpha) = \sqrt{1 - a^2}$ અને $\cos(\theta + \beta) = \sqrt{1 - b^2}$

$\therefore \cos(\alpha - \beta) = \cos\{\theta + \alpha - (\theta + \beta)\}$

$= \cos(\theta + \beta)\cos(\theta + \alpha) + \sin(\theta + \alpha)\sin(\theta + \beta)$

$= \sqrt{1 - a^2}\sqrt{1 - b^2} + a \cdot b = ab + \sqrt{(1 - a^2)(1 - b^2)}$

$= ab + \sqrt{1 - a^2 - b^2 + a^2b^2}$

અને $\cos(\alpha - \beta) = ab + \sqrt{1 - a^2 - b^2 + a^2b^2}$

$= \cos2(\alpha - \beta) - 4ab\cos(\alpha - \beta)$

$= 2\cos^2(\alpha - \beta) - 1 - 4ab\cos(\alpha - \beta)$

$= 2\cos(\alpha - \beta)[\cos(\alpha - \beta) - 2ab] - 1$

$= 2\left(ab + \sqrt{1 - a^2 - b^2 + a^2b^2}\right)\left(ab + \sqrt{1 - a^2 - b^2 + a^2b^2} - 2ab\right) - 1$

$= 2\left[\left(\sqrt{1 - a^2 - b^2 + a^2b^2} + 2ab\right)\left(\sqrt{1 - a^2 - b^2 + a^2b^2} - ab\right)\right] - 1$

$= 2[1 - a^2 - b^2 + a^2b^2 - a^2b^2] - 1$

$= 2 - 2a^2 - 2b^2 - 1$

$= 1 - 2a^2 - 2b^2$

2. **જો $\cos(\theta + \phi) = m\cos(\theta - \phi)$ હોય, તો સાબિત કરો કે, $\tan\theta = \frac{1 - m}{1 + m}\cot\phi$.**

➥ $\cos(\theta + \phi) = m\cos(\theta - \phi)$ આપેલ છે.

$\therefore \frac{\cos(\theta + \phi)}{\cos(\theta - \phi)} = \frac{m}{1}$

હવે યોગ વિયોગ પ્રમાણ લેતાં,

$\frac{\cos(\theta - \phi) - \cos(\theta + \phi)}{\cos(\theta - \phi) + \cos(\theta + \phi)} = \frac{1 - m}{1 + m}$

$\therefore \frac{-2\sin\left(\frac{\theta - \phi + \theta + \phi}{2}\right) \cdot \sin\left(\frac{\theta - \phi - \theta - \phi}{2}\right)}{2\cos\left(\frac{\theta - \phi + \theta + \phi}{2}\right) \cdot \cos\left(\frac{\theta - \phi - \theta - \phi}{2}\right)} = \frac{1 - m}{1 + m}$

$\therefore \frac{\sin\theta \cdot \sin\phi}{\cos\theta \cdot \cos\phi} = \frac{1 - m}{1 + m}$ $\left[\begin{array}{l} \because \sin(-\theta) = -\sin\theta \\ \text{અને } \cos(-\theta) = \cos\theta \end{array}\right]$

$\therefore \tan\theta \cdot \tan\phi = \frac{1 - m}{1 + m}$

$\therefore \tan\theta = \left(\frac{1 - m}{1 + m}\right)\cot\phi$

જે માંગેલ પરિણામ છે.

3. **નીચેના વિસ્તરણનું મૂલ્ય મેળવો.**

$$3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right].$$

➡ આપેલ સમીકરણ,

$$3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right].$$

$= 3[\cos^4\alpha + \sin^4(\pi + \alpha)] - 2[\cos^6\alpha + \sin^6(\pi - \alpha)]$

$= 3[\cos^4\alpha + \sin^4\alpha] - 2[\cos^6\alpha + \sin^6\alpha] = 3 - 1 = 1$

4. **જો $a\cos2\theta + b\sin2\theta = c$ સમીકરણના બીજ α અને β હોય તો $\tan\alpha + \tan\beta = \frac{2b}{a+c}$.**

➡ $a\cos2\theta + b\sin2\theta = c$ આપેલ સમીકરણ છે.

$\therefore a\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) + b\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = c \quad \left[\because \sin2\theta = \frac{2\tan\theta}{1+\tan^2\theta} \text{ અને } \cos2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$

$\therefore a(1 - \tan^2\theta) + 2b\tan\theta = c(1 + \tan^2\theta)$

$\therefore a - a - \tan^2\theta + 2b\tan\theta = c + c\tan^2\theta$

$\therefore (a + c)\tan^2\theta - 2b\tan\theta + c - a = 0$

ઉપરના સમીકરણના બીજ $\tan\alpha$ અને $\tan\beta$ છે.

$\tan\alpha + \tan\beta = $ બીજનો સરવાળો $= \frac{-(-2b)}{a+c} = \frac{2b}{a+c}$

5. **સમીકરણનો સામાન્ય ઉકેલ મેળવો : $5\cos^2\theta + 7\sin^2\theta - 6 = 0$.**

➡ $5\cos^2\theta + 7\sin^2\theta - 6 = 0$

$\therefore 5\cos^2 + 7(1 - \cos^2\theta) - 6 = 0$

$\therefore 5\cos^2\theta + 7 - 7\cos^2\theta - 6 = 0$

$\therefore 5\cos^2\theta + 7 - 7\cos^2\theta - 6 = 0 \Rightarrow -2\cos^2\theta + 1 = 0$

$\therefore 2\cos^2\theta - 1 = 0 \quad \begin{bmatrix}\because \cos^2\theta = \cos^2\alpha \\ \because \theta = n\pi \pm \alpha\end{bmatrix}$

$\therefore \cos^2\theta = \frac{1}{2}$

$\therefore \cos^2\theta = \cos^2\frac{\pi}{4}$

$\therefore \theta = n\pi \pm \frac{\pi}{4}, \quad n \in Z$

6. **જો $x = \sec\phi - \tan\phi$ અને $y = \mathrm{cosec}\,\phi + \cot\phi$ હોય, તો બતાવો કે, $xy + x - y + 1 = 0$.**

➡ અહીં, $x = \sec\phi - \tan\phi$ ....(i)

અને $y = \mathrm{cosec}\,\phi + \cot\phi$ આપેલ છે. ....(ii)

હવે $1 \cdot xy = (\sec\phi - \tan\phi)(\mathrm{cosec}\,\phi + \cot\phi)$

$\therefore xy = \sec\phi \cdot \mathrm{cosec}\,\phi - \mathrm{cosec}\,\phi \cdot \tan\phi + \sec\phi \cdot \cot\phi - \tan\phi \cdot \cot\phi$

$\therefore xy = \sec\phi \cdot \mathrm{cosec}\,\phi - \frac{1}{\cos\phi} + \frac{1}{\sin\phi} - 1$

$\therefore 1 + xy = \sec\phi\,\mathrm{cosec}\,\phi - \sec\phi + \mathrm{cosec}\,\phi$ .....(iii)

પરિણામ (i) અને (ii) પરથી,

$x - y = \sec\phi - \tan\phi - \mathrm{cosec}\,\phi - \cot\phi$

$\therefore x - y = \sec\phi - \mathrm{cosec}\,\phi - \frac{\sin\phi}{\cos\phi} - \frac{\cos\phi}{\sin\phi}$

$\therefore\ x - y = \sec\phi - \operatorname{cosec}\phi - \left(\dfrac{\sin^2\phi + \cos^2\phi}{\sin\phi \cdot \cos\phi}\right)$

$\therefore\ x - y = \sec\phi - \operatorname{cosec}\phi - \dfrac{1}{\sin\phi \cdot \cos\phi}$

$\therefore\ x - y = \sec\phi - \operatorname{cosec}\phi - \operatorname{cosec}\phi \cdot \sec\phi$

$\therefore\ x - y = -(\sec\phi \cdot \operatorname{cosec}\phi - \sec\phi + \operatorname{cosec}\phi)$

$\therefore\ x - y = -(xy + 1)$ પરિણામ (iii) પરથી

$\therefore\ xy + x - y + 1 = 0$ જે માંગેલ પરિણામ છે.

7. જો θ પ્રથમ ચરણમાં હોય અને $\cos\theta = \dfrac{8}{17}$ હોય, તો $\cos(30° + \theta) + \cos(45° - \theta) + \cos(120° - \theta)$ નું મૂલ્ય શોધો.

➥ અહીં, $\cos3\theta = \dfrac{8}{17}$ છે. $\Rightarrow \sin\theta = \sqrt{1 - \dfrac{64}{289}}$

$\therefore\ \sin\theta = \sqrt{\dfrac{289 - 64}{289}} \Rightarrow \sin\theta = \pm\dfrac{15}{17}$

$\therefore\ \sin\theta = \dfrac{15}{17}$ (∵ θ પ્રથમ ચરણમાં છે.)

હવે, $\cos(30° + \theta) + \cos(45° - \theta) + \cos(120° - \theta)$

$= \cos(30° + \theta) + \cos(45° - \theta) + \cos[90° + (30° - \theta)]$

$= \cos(30° + \theta) + \cos(45° - \theta) - \sin(30° - \theta)$

$= \cos30°\cos\theta - \sin30°\sin\theta + \cos45°\cos\theta + \sin45°\sin\theta - \sin30°\cos\theta + \cos30°\sin\theta$

$= \dfrac{\sqrt{3}}{2}\cos\theta - \dfrac{1}{2}\sin\theta + \dfrac{1}{\sqrt{2}}\cos\theta + \dfrac{1}{\sqrt{2}}\sin\theta - \dfrac{1}{2}\cos\theta\,\dfrac{\sqrt{3}}{2}\sin\theta$

$= \left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{2}} - \dfrac{1}{2}\right)\cos\theta + \left(\dfrac{1}{\sqrt{2}} - \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\right)\sin\theta$

$= \left(\dfrac{\sqrt{6} + 2 - \sqrt{2}}{2\sqrt{2}}\right)\cos\theta + \left(\dfrac{2 - \sqrt{2} + \sqrt{6}}{2\sqrt{2}}\right)\sin\theta$

$= \left(\dfrac{\sqrt{6} + 2 - \sqrt{2}}{2\sqrt{2}}\right)\dfrac{8}{17} + \left(\dfrac{2 - \sqrt{2} + \sqrt{6}}{2\sqrt{2}}\right)\dfrac{15}{17}$

$= \dfrac{1}{17(2\sqrt{2})}\left(8\sqrt{6} + 16 - 8\sqrt{2} + 30 - 15\sqrt{2} + 15\sqrt{6}\right)$

$= \dfrac{1}{17(2\sqrt{2})}\left(23\sqrt{6} - 23\sqrt{2} + 46\right)$

$= \dfrac{23\sqrt{6}}{17(2\sqrt{2})} - \dfrac{23\sqrt{2}}{17(2\sqrt{2})} + \dfrac{46}{17(2\sqrt{2})}$

$= \dfrac{23\sqrt{3}}{17(2)} - \dfrac{23}{17(2)} + \dfrac{23}{17\sqrt{2}}$

$= \dfrac{23}{17}\left(\dfrac{\sqrt{3} - 1}{2} + \dfrac{1}{\sqrt{2}}\right)$

8. કિંમત મેળવો : $\cos^4\left(\dfrac{\pi}{8}\right) + \cos^4\left(\dfrac{3\pi}{8}\right) + \cos^4\left(\dfrac{5\pi}{8}\right) + \cos^4\left(\dfrac{7\pi}{8}\right)$

➥ $\cos^4\left(\dfrac{\pi}{8}\right) + \cos^4\left(\dfrac{3\pi}{8}\right) + \cos^4\left(\dfrac{5\pi}{8}\right) + \cos^4\left(\dfrac{7\pi}{8}\right)$

$= \cos^4\left(\dfrac{\pi}{8}\right) + \cos^4\left(\dfrac{3\pi}{8}\right) + \cos^4\left(\pi - \dfrac{3\pi}{8}\right) + \cos^4\left(\pi - \dfrac{\pi}{8}\right)$

$= \cos^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{3\pi}{8}\right) + \cos^4\left(\frac{3\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right)$

$= 2\left[\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8}\right] = 2\left[\cos^4\frac{\pi}{8} + \cos^4\left(\frac{\pi}{2} - \frac{\pi}{8}\right)\right]$

$= 2\left[\cos^4\frac{\pi}{8} + \cos^4\frac{\pi}{8}\right]$

$= 2\left[\left(\cos^2\frac{\pi}{8} + \sin^2\frac{\pi}{8}\right)^2 - 2\cos^2\frac{\pi}{8} \cdot \sin^2\frac{\pi}{8}\right]$

($\because a^2 + b^2 = (a + b)^2 - 2ab$ થાય.)

$= 2\left[1 - 2\cos^2\frac{\pi}{8} \cdot \sin^2\frac{\pi}{8}\right] = 2 - \left(2\sin\frac{\pi}{8} \cdot \cos\frac{\pi}{8}\right)^2$

$= 2 - \left(\sin\frac{2\pi}{8}\right)^2 = 2 - \left(\frac{1}{\sqrt{2}}\right)^2$

$= 2 - \frac{1}{2} = \frac{3}{2}$

9. સમીકરણનો સામાન્ય ઉકેલ મેળવો : $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

➡ $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

$\therefore \sin 3x + \sin x - 3\sin 2x = \cos 3x + \cos x - 3\cos 2x$

$\therefore 2\sin 2x \cos x - 3\sin 2x = 2\cos 2x \cdot \cos x - 3\cos 2x$

$\therefore \sin 2x\ (2\cos x - 3) = \cos 2x\ (2\cos x - 3)$

$\therefore 2\cos x - 3 = 0$ અથવા $\sin 2x = \cos 2x$

$\therefore \cos x = \frac{3}{2}$ જે શક્ય નથી. $\therefore \cos 2x = \cos\left(\frac{\pi}{2} - 2x\right)$

$\therefore 2x = 2k\pi \pm \left(\frac{\pi}{2} - 2x\right), \quad k \in Z$

$\therefore 2x = 2k\pi + \frac{\pi}{2} - 2x, 2x = 2k\pi - \frac{\pi}{2} + 2x$

$\therefore 4x = 2k\pi + \frac{\pi}{2}$ આ શક્ય નથી.

$\therefore x = \frac{k\pi}{2} + \frac{\pi}{8}, k \in Z$

$\therefore$ ઉકેલગણ $= \left\{\frac{k\pi}{2} + \frac{\pi}{8} \middle| k \in Z\right\}$

10. નીચેના સમીકરણનો સામાન્ય ઉકેલ મેળવો : $(\sqrt{3} - 1)\cos\theta + (\sqrt{3} + 1)\sin\theta = 2$

➡ $(\sqrt{3} - 1)\cos\theta + (\sqrt{3} + 1)\sin\theta = 2$ ....(1)

હવે $\sqrt{3} - 1 = r\sin\alpha$ અને $\sqrt{3} + 1 = r\cos\alpha$ લેતાં,

$\therefore r^2 = (\sqrt{3} - 1)^2 + (\sqrt{3} + 1)^2$

$\therefore r^2 = 3 + 1 - 2\sqrt{3} + 3 + 1 + 2\sqrt{3}$

$\therefore r^2 = 8$

$\therefore r = 2\sqrt{2}$

અહીં $\tan\alpha = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3} \cdot \frac{\pi}{4}}$

$\therefore \tan\alpha = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$\therefore \tan\alpha = \tan\frac{\pi}{12}$

$\therefore \alpha = \frac{\pi}{12}$

∴ સમીકરણ (i) નું સામાન્ય સ્વરૂપ,

$r \sin\alpha \cos\theta + r \cos\alpha \sin\theta = 2$ થાય.

$r [\sin(\theta + \alpha)] = 2$

$\therefore \sin(\theta + \alpha) = \frac{2}{2\sqrt{2}}$

$\therefore \sin(\theta + \alpha) = \frac{1}{\sqrt{2}}$

$\therefore \sin(\theta + \alpha) = \sin\frac{\pi}{4} \theta + \alpha = n\pi + (-1)^n \frac{\pi}{4},\ n \in Z$

$\therefore \theta = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{12},\ n \in Z$

અન્ય રીત :

➥ $(\sqrt{3} - 1)\cos\theta + (\sqrt{3} + 1)\sin\theta = 2$ ....(1)

હવે $\sqrt{3} - 1 = r\cos\alpha$ અને $\sqrt{3} + 1 = r\sin\alpha$ મૂકતાં,

$\therefore r = 2\sqrt{2}$

અહીં $\tan\alpha = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$

➥ $(\sqrt{3} - 1)\cos\theta + (\sqrt{3} + 1)\sin\theta = 2$ ....(1)

હવે $\sqrt{3} - 1 = r\sin\alpha$ અને $\sqrt{3} + 1 = r\cos\alpha$ લેતાં,

$\therefore r^2 = (\sqrt{3} - 1)^2 + (\sqrt{3} + 1)^2$

$\therefore r^2 = 3 + 1 - 2\sqrt{3} + 3 + 1 + 2\sqrt{3}$

$\therefore r^2 = 8$

$\therefore r = 2\sqrt{2}$

અહીં $\tan\alpha = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3} \cdot \frac{\pi}{4}}$

$\therefore \tan\alpha = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$\therefore \tan\alpha = \tan\frac{\pi}{12}$

$\therefore \alpha = \frac{\pi}{12}$

∴ સમીકરણ (i) નું સામાન્ય સ્વરૂપ,

$r \sin\alpha \cos\theta + r \cos\alpha \sin\theta = 2$ થાય.

$r [\sin(\theta + \alpha)] = 2$

$\therefore \sin(\theta + \alpha) = \frac{2}{2\sqrt{2}}$

$\therefore \sin(\theta + \alpha) = \frac{1}{\sqrt{2}}$

$\therefore \sin(\theta + \alpha) = \sin\frac{\pi}{4} \theta + \alpha = n\pi + (-1)^n \frac{\pi}{4},\ n \in Z$

$\therefore \theta = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{12},\ n \in Z$

અન્ય રીત :

➥ $(\sqrt{3} - 1)\cos\theta + (\sqrt{3} + 1)\sin\theta = 2$ ....(1)

હવે $\sqrt{3} - 1 = r\cos\alpha$ અને $\sqrt{3} + 1 = r\sin\alpha$ મૂકતાં,

$\therefore\ r = 2\sqrt{2}$

અહીં $\tan\alpha = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = \dfrac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}$