Exercise 7.8

Answer 1E.

(a) Since $y = \dfrac{x}{x-1}$ has an infinite discontinuous at $x = 1, \int_1^2 \dfrac{x}{x-1}$ is an improper integral of type II

(b) Since $\int_1^\infty \dfrac{1}{1+x^3}dx$ has an infinite interval of integration, it is an improper integral of Type I

(c) Since $\int_{-\infty}^{\infty} x^2 e^{-x^2} dx$ has an infinite interval of integration, it is an improper integral of Type I.

(d) $\int_0^{\pi/4} \cot x dx$ is an improper integral of Type II, because $\cot x$ is infinite discontinuity at $x = 0$.

Answer 2E.

(a) Since $y = \tan x$ is not infinite discontinuity at any point of $\left[0, \dfrac{\pi}{4}\right]$, so it is not improper integral.

Since $\int_0^{\pi/4} \tan x dx$ has not an infinite interval of integration, so it is not improper integral.

Therefore $\int_0^{\pi/4} \tan x dx$ is not improper integral and

$$\int_0^{\pi/4} \tan x dx = \left[\ln|\sec x|\right]_0^{\pi/4}$$

$$= \left[\ln\left|\sec\frac{\pi}{4}\right| - \ln|\sec 0|\right]$$

$$= \ln\left|\sec\frac{\pi}{4}\right|$$

$$\boxed{\text{Therefore } \int_0^{\pi/4} \tan x dx = \ln\left|\sec\frac{\pi}{4}\right|}$$

(b) Since $y=\tan x$ is infinite discontinuity at $x=\frac{\pi}{2}$, so $\int_0^{\pi}\tan x\,dx$ is an improper integral of type II.

(c)
$$\int_{-1}^{1}\frac{dx}{x^2-x-2}=\int_{-1}^{1}\frac{dx}{x^2-2x+x-2}$$
$$=\int_{-1}^{1}\frac{dx}{x(x-2)+1(x-2)}$$
$$=\int_{-1}^{1}\frac{dx}{(x+1)(x-2)}$$

Since $y=\frac{1}{(x+1)(x-2)}$ is infinite discontinuity at $x=-1$,

So $\int_{-1}^{1}\frac{1}{(x+1)(x-2)}dx$ is an improper integral of type II.

(d) $\int_0^{\infty}e^{-x^3}dx$ has an infinite interval of integration, it is an improper integral of type I.

Answer 3E.

The area under the curve $y=\frac{1}{x^3}$ from x = 1 to x = t is given as

$$A(s)=\int_1^t y\,dx$$
$$=\int_1^t\frac{1}{x^3}dx$$
$$=\int_1^t x^{-3}dx$$
$$=\left[\frac{x^{-3+1}}{-3+1}\right]_1^t$$
$$=\left[-\frac{1}{2x^2}\right]_1^t$$
$$=\left[-\frac{1}{2t^2}+\frac{1}{2}\right]$$
$$=\frac{1}{2}-\frac{1}{2t^2}$$

The area under the curve when $t = 10$ is

$$A(10) = \frac{1}{2} - \frac{1}{2\times(10)^2}$$
$$= 0.5 - \frac{1}{200}$$
$$= 0.5 - 0.005$$
$$= 0.495$$

The area under the curve when $t = 100$ is

$$A(100) = \frac{1}{2} - \frac{1}{2\times(100)^2}$$
$$= 0.5 - \frac{1}{20000}$$
$$= 0.5 - 0.00005$$
$$= 0.49995$$

The area under the curve when $t = 1000$ is

$$A(1000) = \frac{1}{2} - \frac{1}{2\times(1000)^2}$$
$$= \frac{1}{2} - \frac{1}{2000000}$$
$$= 0.5 - 0.0000005$$
$$= 0.4999995$$

The total area under the curve for $x \geq 1$ is

$$A(x \geq 1) = \lim_{t\to\infty} \int_1^t \frac{1}{x^3} dx$$
$$= \lim_{t\to\infty} \left(\frac{1}{2} - \frac{1}{2t^2}\right)$$
$$= \frac{1}{2} - \lim_{t\to\infty} \frac{1}{2t^2}$$
$$= \frac{1}{2} - 0$$
$$= \frac{1}{2}$$

Answer 4E.

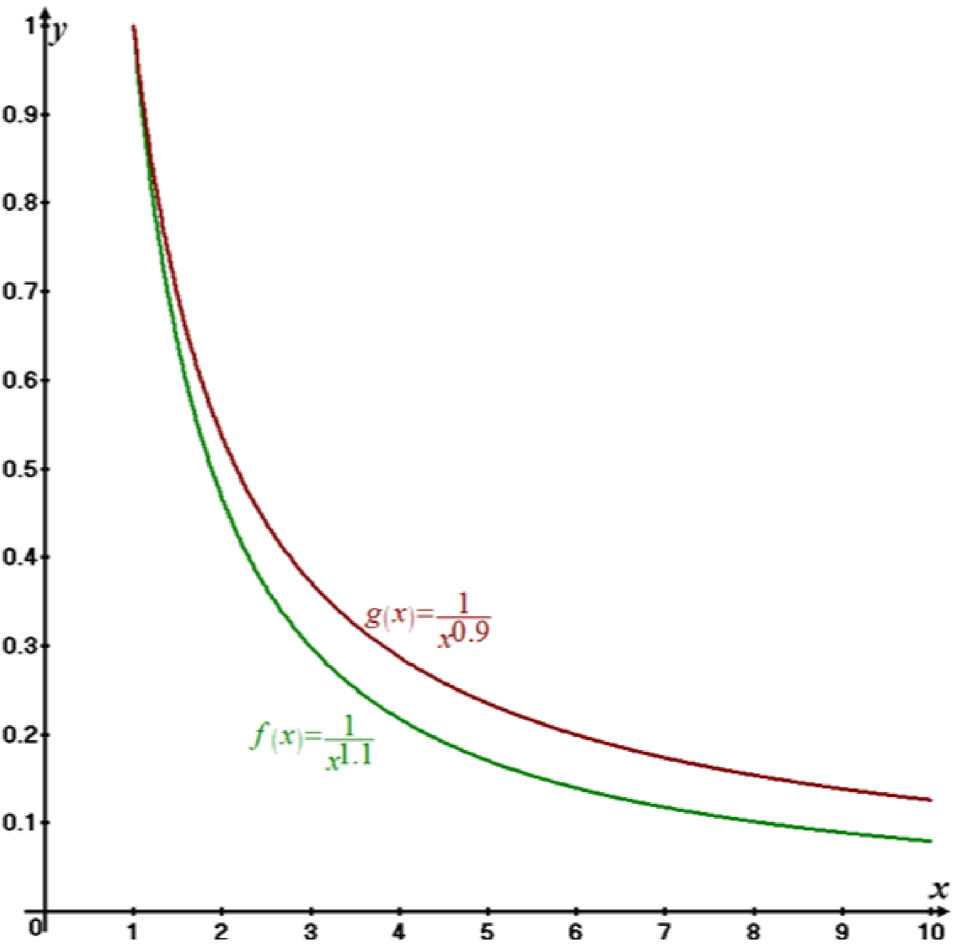
Consider the functions

$$f(x) = \frac{1}{x^{1.1}}$$

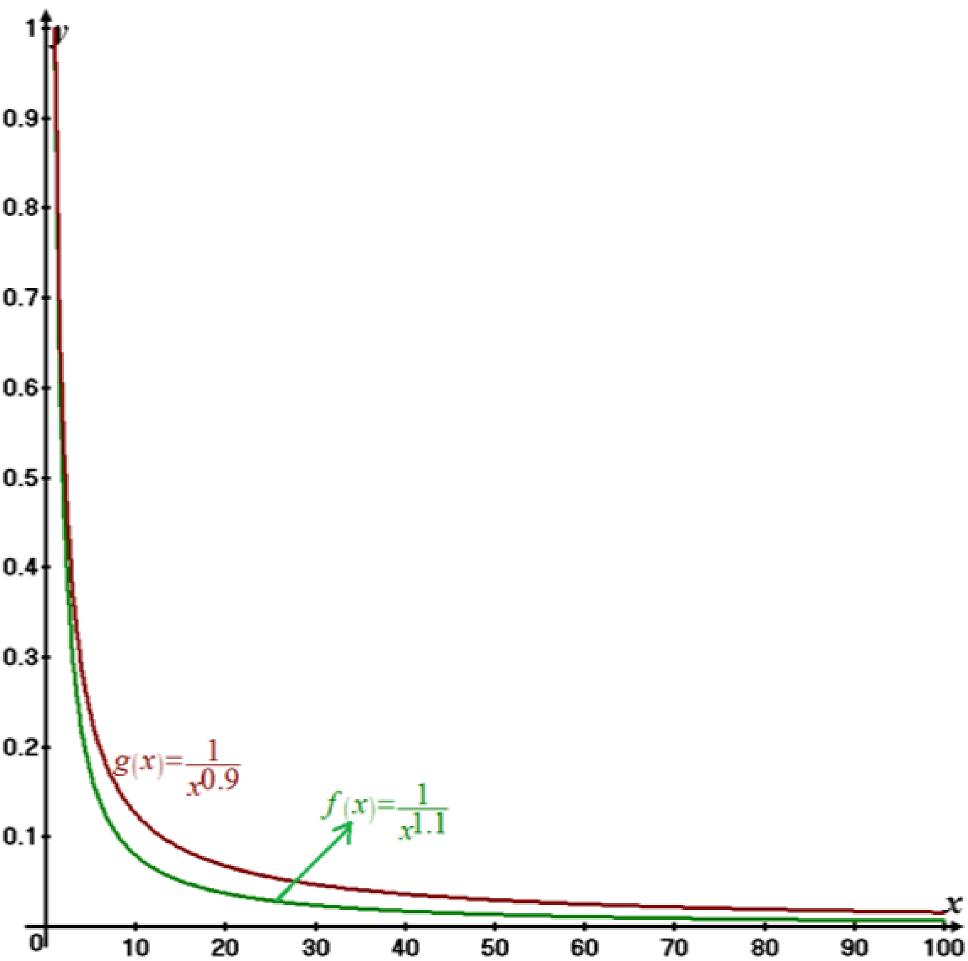
$$g(x) = \frac{1}{x^{0.9}}$$

a) Need to sketch the graphs in the viewing windows $[0,10]$ by $[0,1]$ and $[0,100]$ by $[0,1]$.

The graph of the functions in the viewing window $[0,10]$ by $[0,1]$ is as shown below:



The graph of the functions in the viewing window $[0,100]$ by $[0,1]$ is as shown below:



b) Need to find the area under the graphs of *f* and *g* from $x=1$ and $x=t$.

Area under the graphs of *f* and *g* is

$$A=\int_0^t (g(t)-f(t))\,dt$$
$$=\int_0^t \left(\frac{1}{x^{0.9}}-\frac{1}{x^{1.1}}\right)dt$$
$$=\int_0^t \left(x^{-0.9}-x^{-1.1}\right)dt$$
$$=\left[\frac{x^{-0.9+1}}{-0.9+1}-\frac{x^{-1.1+1}}{-1.1+1}\right]_0^t$$
$$=\left[\frac{x^{0.1}}{0.1}-\frac{x^{-0.1}}{-0.1}\right]_0^t$$
$$=\left[\frac{x^{0.1}}{0.1}+\frac{x^{-0.1}}{0.1}\right]_0^t$$
$$=\left[\frac{x^{0.1}}{\frac{1}{10}}+\frac{x^{-0.1}}{\frac{1}{10}}\right]_0^t$$
$$=\left[10x^{0.1}+10x^{-0.1}\right]_0^t$$

Continuation to the above steps

$$=\left[10t^{0.1}+10t^{-0.1}-10(0)^{0.1}-10(0)^{-0.1}\right]$$
$$=\left[10t^{0.1}+10t^{-0.1}-0\right]$$
$$=10\left[t^{0.1}+t^{-0.1}\right]$$

Thus, the area under the graphs is $A=10\left[t^{0.1}+t^{-0.1}\right]$.

Need to evaluate the area for $t=10,100,10^4,10^6,10^{10},10^{20}$

For $t=10,$ the area becomes

$$A=10\left[10^{0.1}+10^{-0.1}\right]$$
$$\approx 20.53253647$$

For $t=100,$ the area becomes

$$A=10\left[100^{0.1}+100^{-0.1}\right]$$
$$\approx 22.15850536$$

For $t=10^4,$ the area becomes

$$A=10\left[\left(10^4\right)^{0.1}+\left(10^4\right)^{-0.1}\right]$$
$$\approx 29.09993602$$

For $t=10^6,$ the area becomes

$$A=10\left[\left(10^6\right)^{0.1}+\left(10^6\right)^{-0.1}\right]$$
$$\approx 42.32260349$$

For $t=10^{10},$ the area becomes

$$A=10\left[\left(10^{10}\right)^{0.1}+\left(10^{10}\right)^{-0.1}\right]$$
$$\approx 101.0000000$$

For $t=10^{20},$ the area becomes

$$A=10\left[\left(10^{20}\right)^{0.1}+\left(10^{20}\right)^{-0.1}\right]$$
$$\approx 1000.100000$$

c) Need to find the total area under each curve for $x \geq 1$.

For $x \geq 1$, the curves never intersect at any place.

As *x* increases, the area under the curves tends to infinity.

Thus, the total area under each curve does not exist.

Answer 5E.

Consider the integral

$$I = \int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx$$

$$= \underset{t\to\infty}{Lt} \int_3^t \frac{1}{(x-2)^{3/2}}\,dx$$

Substitute

$$p = x-2$$
$$dp = dx$$

Substitute p value into integral we get

$$\int_3^t \frac{1}{(x-2)^{3/2}}\,dx = \int_3^t \frac{1}{p^{3/2}}\,dp$$

$$= \left[\frac{-2}{\sqrt{p}}\right]_3^t$$

$$= \left[\frac{-2}{\sqrt{x-2}}\right]_3^t$$

$$I = \underset{t\to\infty}{Lt} \left[\frac{-2}{\sqrt{x-2}}\right]_3^t$$

$$= \underset{t\to\infty}{Lt} \left[\frac{-2}{\sqrt{t-2}} - \left(\frac{-2}{\sqrt{1}}\right)\right]$$

$$= \underset{t\to\infty}{Lt} \left[\frac{-2}{\sqrt{t-2}} + 2\right]$$

$$= 2$$

The limit value is finite so the given improper integral is convergent to 2

Answer 6E.

Let $I = \int_0^\infty \frac{1}{\sqrt[4]{1+x}} dx$

$= \underset{t\to\infty}{Lt} \int_0^t \frac{1}{\sqrt[4]{1+x}} dx$ ......(1)

$\int \frac{1}{\sqrt[4]{1+x}} dx = \int \frac{1}{\sqrt[4]{p}} dp$ $\begin{bmatrix} \text{because } 1+x=p \\ \Rightarrow \quad dx = dp \end{bmatrix}$

$= \frac{p^{-\frac{1}{4}+1}}{-\frac{1}{4}+1} + c$

$= \frac{4}{3} p^{3/4} + c$

$= \frac{4}{3}(1+x)^{3/4} + c$

Therefore $I = \underset{t\to\infty}{Lt} \left[\frac{4}{3} p^{3/4}\right]_0^t$

$= \underset{t\to\infty}{Lt} \left[\frac{4}{3}(1+x)^{3/4}\right]_0^t$

$= \underset{t\to\infty}{Lt} \left[\frac{4}{3}(1+t)^{3/4} - \frac{4}{3}\right]$

$= \infty$, divergent.

Answer 7E.

Let $I = \int_{-\infty}^0 \frac{1}{3-4x} dx$

$= \underset{t\to-\infty}{Lt} \int_t^0 \frac{1}{3-4x} dx$

$= \underset{t\to-\infty}{Lt} \left[\frac{-1}{4} \ln|3-4x|\right]_t^0$

$= \underset{t\to-\infty}{Lt} \left[\frac{-1}{4} \ln 3 + \frac{1}{4} \ln|3-4t|\right]$

$= \infty$, divergent

Answer 8E.

Let $I = \int_1^\infty \frac{1}{(2x+1)^3} dx$

$= \underset{p\to\infty}{Lt} \int_1^p \frac{1}{(2x+1)^3} dx$

$\int \frac{1}{(2x+1)^3} dx = \int \frac{1}{t^3} \frac{dt}{2}$ $\begin{bmatrix} \text{because } 2x+1=t \\ \Rightarrow \quad 2dx = dt \\ \Rightarrow \quad dx = \frac{dt}{2} \end{bmatrix}$

$= \frac{1}{2} \int t^{-3} dt$

$= \frac{1}{2}\left[\frac{t^{-3+1}}{-3+1}\right] + c$

$= \frac{-1}{4} \frac{1}{t^2} + c$

$= \frac{-1}{4} \frac{1}{(2x+1)^2}$

Therefore $I = \underset{p\to\infty}{Lt}\left[\frac{-1}{4}\frac{1}{(2x+1)^2}\right]_1^p$

$= \underset{p\to\infty}{Lt}\left[\frac{-1}{4}\frac{1}{(2p+1)^2}+\frac{1}{4}\frac{1}{9}\right]$

$= \frac{1}{36}$, convergent

Answer 9E.

Let $I = \int_2^\infty e^{-5p}dp$

$= \underset{t\to\infty}{Lt}\int_2^t e^{-5p}dp$

$= \underset{t\to\infty}{Lt}\left[\frac{e^{-5p}}{-5}\right]_2^t$

$= \underset{t\to\infty}{Lt}\left[\frac{e^{-5t}}{-5}-\frac{e^{-10}}{-5}\right]$

$= \frac{e^{-10}}{5}$

$$\boxed{\text{Therefore } I = \int_2^\infty e^{-5p}dp \text{ convergent}}$$

Answer 10E.

Consider the improper integral

$I = \int_{-\infty}^{0} 2^r\, dr$

We know that

$\int_{-\infty}^{a} f(x)dx = \lim_{t\to-\infty}\int_t^a f(x)dx$

$\int_{-\infty}^{0} 2^r\, dr = \underset{t\to-\infty}{Lt}\int_t^0 2^r\, dr$

we know that $\int 2^x dx = \frac{2^x}{\ln 2}$

$= \underset{t\to-\infty}{Lt}\left[\frac{2^r}{\ln 2}\right]_t^0$

$= \underset{t\to-\infty}{Lt}\left[\frac{2^0}{\ln 2}-\frac{2^t}{\ln 2}\right]$

$= \underset{t\to-\infty}{Lt}\frac{1}{\ln 2}-\underset{t\to-\infty}{Lt}\frac{2^t}{\ln 2}$

$2^t \to 2^{-\infty} \to \frac{1}{2^\infty} \to \frac{1}{\infty} \to 0 \quad \text{as } t\to-\infty$

$= \frac{1}{\ln 2}-\frac{0}{\ln 2}$

$= \boxed{\frac{1}{\ln 2}}$

Answer 11E.

If $\int_a^t f(x)\,dx$ exists for every number $t \ge a$, then $\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$

If $\int_t^b f(x)\,dx$ exists for every number $t \le b$, then $\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$

Provided this limit exists (as a finite number).

The improper integrals $\int_a^{\infty} f(x)dx$ and $\int_{-\infty}^{b} f(x)dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

Rewrite the above integral

$$\int_0^{\infty}\frac{x^2}{\sqrt{1+x^3}}dx = \int_1^{\infty}\frac{x^2}{\sqrt{1+x^3}}dx$$
$$= \int_1^{\infty}\frac{1}{\sqrt{u}}\left(\frac{1}{3}\right)du$$
$$= \lim_{t\to\infty}\int_1^{t}\frac{1}{\sqrt{u}}\left(\frac{1}{3}\right)du$$
$$= \lim_{t\to\infty}\left[\frac{2\sqrt{u}}{3}\right]_1^t$$
$$= \lim_{t\to\infty}\left[\frac{2\sqrt{t}}{3}-\frac{2\sqrt{1}}{3}\right]$$
$$= \infty$$

Hence, the limit does not exists as a finite number and so by above definition the given improper integral is $\boxed{\text{divergent}}$.

Answer 12E.

Let $$I = \int_{-\infty}^{\infty}\left(y^3-3y^2\right)dy$$
$$= \int_{-\infty}^{0}\left(y^3-3y^2\right)dy + \int_{0}^{\infty}\left(y^3-3y^2\right)dy$$

Let $$I_1 = \int_{-\infty}^{0}\left(y^3-3y^2\right)dy$$
$$= \left[\frac{y^4}{4}-3\frac{y^3}{3}\right]_{-\infty}^{0}$$
$$= -\infty, \text{divergent}$$

Since I_1 is divergent and there is no need to evaluate $I_2 = \int_0^{\infty}\left(y^3-3y^2\right)dy$

$\boxed{\text{Therefore } I \text{ is divergent}}$

Answer 13E.

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx \quad \text{--- (1)}$$

Now in $\int xe^{-x^2}dx$ Substitute $x^2 = u \Rightarrow 2x\,dx = du$

$$\int xe^{-x^2}dx = \int e^{-u}\frac{du}{2}$$
$$= \frac{1}{2}\int e^{-u}du$$
$$= \frac{1}{2}\cdot\frac{e^{-u}}{-1} = -\frac{1}{2}e^{-x^2} \quad \ldots\ldots \quad (2)$$

Thus equation (1) becomes

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = \lim_{t_1\to-\infty}\int_{t_1}^{0} xe^{-x^2}dx + \lim_{t_2\to\infty}\int_{0}^{t_2} xe^{-x^2}dx$$

$$= \lim_{t_1\to-\infty}\left[\left(-\frac{1}{2}e^{-x^2}\right)\right]_{t_1}^{0} + \lim_{t_2\to\infty}\left[\left(-\frac{1}{2}e^{-x^2}\right)\right]_{0}^{t_2} \qquad \text{[By using result (2)]}$$

$$= -\frac{1}{2}\lim_{t_1\to-\infty}\left(e^{0}-e^{-t_1^2}\right) - \frac{1}{2}\lim_{t_2\to\infty}\left(e^{-t_2^2}-e^{0}\right)$$

$$= -\frac{1}{2}\left[\left(1-e^{-\infty}\right)\right] - \frac{1}{2}\left[\left(e^{-\infty}-1\right)\right]$$

$$= -\frac{1}{2}+\frac{1}{2} = 0 \qquad \left[e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0\right]$$

Thus $\int_{-\infty}^{\infty} xe^{-x^2}dx = 0$ and this is convergent

Answer 14E.

Consider the following improper integral:

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$$

The improper integral can be written as follows:

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \lim_{t\to\infty}\int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx \quad \cdots\cdots(1)$$

Put $\sqrt{x} = u$, then $\frac{1}{2\sqrt{x}}dx = du$. So, $\frac{1}{\sqrt{x}}dx = 2du$.

If $x=1$, then $u=1$ and if $x=t$, then $u=\sqrt{t}$.

Substitute these values in equation (1).

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \lim_{t\to\infty}\int_1^{\sqrt{t}} e^{-u}\cdot 2\,du$$

$$= \lim_{t\to\infty} 2\int_1^{\sqrt{t}} e^{-u}\,du$$

$$= \lim_{t\to\infty} 2\left(\frac{e^{-u}}{-1}\right)_1^{\sqrt{t}}$$

$$= \lim_{t\to\infty} 2\left(-e^{-\sqrt{t}}+e^{-1}\right)$$

$$= 2\left(-\lim_{t\to\infty} e^{-\sqrt{t}} + \lim_{t\to\infty} e^{-1}\right)$$

$$= 2\left(-0+e^{-1}\right)$$

$$= \frac{2}{e}$$

Therefore, the improper integral exists.

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called **convergent** if the corresponding limit exists, and **divergent** if the limit does not exist.

Thus, the improper integral $\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$ is convergent and the value of the integral is,

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \boxed{\frac{2}{e}}.$$

Answer 15E.

Consider the integral,

$$\int_0^\infty \sin^2 \alpha \, d\alpha$$

Need to determine whether the integral is convergent or divergent.

Use the definition $\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$ with $a=0$.

Note that the improper integral $\int_a^\infty f(x)dx$ is **convergent** if the limit $\lim_{t\to\infty}\int_a^t f(x)dx$ **exists**, **divergent** if the limit **does not exist**.

First use the definition

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$

$$\int_0^\infty \sin^2\alpha\,d\alpha = \lim_{t\to\infty}\int_0^t \sin^2\alpha\,d\alpha$$

$$= \lim_{t\to\infty}\int_0^t \frac{1-\cos 2\alpha}{2}d\alpha \quad \text{Use } \sin^2 x dx = \frac{1-\cos 2x}{2}$$

$$= \lim_{t\to\infty}\frac{1}{2}\int_0^t (1-\cos 2\alpha)\,d\alpha$$

$$= \lim_{t\to\infty}\frac{1}{2}\int_0^t (d\alpha - \cos 2\alpha d\alpha)$$

$$= \lim_{t\to\infty}\frac{1}{2}\left[\alpha - \frac{\sin 2\alpha}{2}\right]_0^t$$

$$= \lim_{t\to\infty}\frac{1}{2}\left[t - \frac{\sin 2t}{2} - 0 + \frac{\sin 2(0)}{2}\right] \quad \text{Apply limits}$$

$$= \lim_{t\to\infty}\frac{1}{2}\left(t - \frac{\sin 2t}{2}\right)$$

$$= \frac{1}{2}\left(\infty - \frac{\sin 2\infty}{2}\right)$$

$$= \infty$$

Since the limit does not exist, the improper integral $\int_0^\infty \sin^2\alpha\,d\alpha$ is $\boxed{\text{divergent}}$.

Answer 16E.

Consider the provided statement to determine whether integral is convergent or divergent.

Provided integral is,

$$\int_{-\infty}^{\infty}\cos(\pi t)\cdot dt \quad \text{......} \ (1)$$

An integral is said to be convergent if its limit is approaching to an argument of function as close as possible. An integral which is not convergent is called divergent.

Now consider,

$$\int \cos(\pi t)\cdot dt$$

It is assumed that $\pi t = u$ then $\pi \cdot dt = du$. Therefore,

$$\begin{aligned}\int \cos(\pi t)\cdot dt &= \frac{1}{\pi}\int \cos(u)\cdot du \\ &= \frac{1}{\pi}\sin(u) + c \\ &= \frac{1}{\pi}\sin(\pi t) + c\end{aligned}$$

Now compute the boundaries of the equation (1),

$$\lim_{t\to-\infty}\left(\frac{\sin(\pi t)}{\pi}\right) = \text{diverges}$$

$$\lim_{t\to\infty}\left(\frac{\sin(\pi t)}{\pi}\right) = \text{diverges}$$

Hence, overall this integral is **divergent**.

Answer 17E.

Consider the improper integral

$$I = \int_1^\infty \frac{1}{x^2+x}dx$$

We know that

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$

$$\int_1^\infty \frac{1}{x^2+x}dx = \underset{t\to\infty}{Lt}\int_1^t \frac{1}{x(1+x)}dx$$

$$\frac{1}{x(1+x)} = \frac{(1+x)-x}{x(1+x)}$$

$$= \frac{1+x}{x(1+x)} - \frac{x}{x(1+x)}$$

$$= \frac{1}{x} - \frac{1}{1+x}$$

$$\underset{t\to\infty}{Lt}\int_1^t \frac{1}{x(1+x)}dx = \underset{t\to\infty}{Lt}\int_1^t\left(\frac{1}{x}-\frac{1}{x+1}\right)dx$$

We know that $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|$

$$\underset{t\to\infty}{Lt}\int_1^t\left(\frac{1}{x}-\frac{1}{x+1}\right)dx = \underset{t\to\infty}{Lt}\int_1^t\left(\frac{\frac{d}{dx}(x)}{x}-\frac{\frac{d}{dx}(x+1)}{x+1}\right)dx$$

$$= \underset{t\to\infty}{Lt}\left[\ln|x| - \ln|x+1|\right]_1^t$$

Now substitute the limits

$$\underset{t\to\infty}{Lt}\left[\ln|x|-\ln|x+1|\right]_1^t = \underset{t\to\infty}{Lt}\left[\ln|t|-\ln|t+1|\right]-\left[\ln|1|-\ln|1+1|\right]$$

$$= \underset{t\to\infty}{Lt}\left[\ln\left|\frac{t}{t+1}\right|\right]-\left[0-\ln|2|\right]$$

$$= \underset{t\to\infty}{Lt}\left[\ln\left|\frac{t}{t+1}\right|+\log 2\right]$$

$$\ln\left|\frac{t}{t+1}\right| \to \ln\left|\frac{t}{t\left(1+\frac{1}{t}\right)}\right| \to \ln\left|\frac{1}{1+\frac{1}{t}}\right| \to \ln\left|\frac{1}{1+0}\right| \to \ln|1| = 0$$

$$\underset{t\to\infty}{Lt}\left[\ln\left|\frac{t}{t+1}\right|+\log 2\right] = \underset{t\to\infty}{Lt}\left[0+\ln 2\right]$$

$$= \ln 2$$

This is finite number so the given improper integral is convergent

$$\boxed{I = \int_1^\infty \frac{1}{x^2+x}dx = \ln 2}$$

Answer 18E.

So the integral can be written as,

$$\int_2^\infty \frac{dv}{v^2+2v-3} = \int_2^\infty \frac{dv}{(v+3)(v-1)}$$

$$= \int_2^\infty \left(\frac{A}{(v+3)} + \frac{B}{(v-1)} \right) dv$$

$$= \int_2^\infty \left(\frac{-1}{4(v+3)} + \frac{1}{4(v-1)} \right) dv$$

$$= \frac{1}{4}\left[\int_2^\infty \frac{-1}{(v+3)} dv + \int_2^\infty \frac{1}{(v-1)} dv \right]$$

$$= \frac{-1}{4} x \left[\lim_{t\to\infty} \int_2^t \frac{1}{(v+3)} dv - \lim_{t\to\infty} \int_2^t \frac{1}{(v-1)} dv \right]$$

$$= \frac{1}{4}\left[\lim_{t\to\infty} (\ln(v+3))_2^t - \lim_{t\to\infty} (\ln(v-1))_2^t \right]$$

$$= \frac{-1}{4}\left[\lim_{t\to\infty} (\ln(t+3) - \ln(5)) - \lim_{t\to\infty} (\ln(t-1)) \right]$$

$$= \frac{-1}{4}\left[\lim_{t\to\infty} \ln\left(\frac{t+3}{5}\right) - \lim_{t\to\infty} \ln(t-1) \right]$$

$$= \frac{-1}{4} \lim_{t\to\infty} \ln\left(\frac{t+3}{5(t-1)}\right)$$

$$= \frac{-1}{4} \lim_{t\to\infty} \ln\left(\frac{1+\frac{3}{t}}{5\left(1-\frac{1}{t}\right)}\right)$$

$$= \frac{-1}{4} \ln\left(\frac{1+\frac{3}{\lim_{t\to\infty} t}}{5\left(1-\frac{1}{\lim_{t\to\infty} t}\right)}\right)$$

$$= \frac{-1}{4} \ln\left(\frac{1}{5}\right)$$

$$= \frac{-1}{4} \ln(5)^{-1}$$

$$= \frac{1}{4} \ln(5)$$

This is a finite value.

So the integral converges.

Therefore, $\int_2^\infty \frac{dv}{v^2+2v-3} = \boxed{\frac{1}{4}\ln(5)}$.

Answer 19E.

To determine that whether the following integral is convergent or divergent:

$$I = \int_{-\infty}^{0} z e^{2z} dz$$

Integration by parts is,

$$\int u dv = uv - \int v du$$

And, a known integral formula is,

$$\int e^{az} dz = \frac{1}{a} e^{az}$$

Therefore,

$$I = \int_{-\infty}^{0} z e^{2z} dz$$

$$= \underset{t \to -\infty}{Lt} \int_{t}^{0} z e^{2z} dz$$

$$= \underset{t \to -\infty}{Lt} \int_{t}^{0} z\, d\left(\frac{1}{2} e^{2z}\right)$$

$$= \underset{t \to -\infty}{Lt} \left[z\left(\frac{e^{2z}}{2}\right) - \int \frac{1}{2} e^{2z}\, dz \right]_{t}^{0}$$

Then,

$$I = \underset{t \to -\infty}{Lt} \left[z\left(\frac{e^{2z}}{2}\right) - \int \frac{1}{2} e^{2z}\, dz \right]_{t}^{0}$$

$$= \underset{t \to -\infty}{Lt} \left[z\left(\frac{e^{2z}}{2}\right) - \frac{1}{4} e^{2z} \right]_{t}^{0}$$

$$= \underset{t \to -\infty}{Lt} \left[\left(0 - \frac{1}{4} e^{0}\right) - \left(t\left(\frac{e^{2t}}{2}\right) - \frac{1}{4} e^{2t} \right) \right]$$

$$= \underset{t \to -\infty}{Lt} \left[-\frac{1}{4} - \frac{te^{2t}}{2} + \frac{1}{4} e^{2t} \right]$$

$$= \underset{t \to -\infty}{Lt} \left[-\frac{1}{4} - \frac{t}{2e^{-2t}} + \frac{1}{4e^{-2t}} \right]$$

$$= -\frac{1}{4} - 0 + 0$$

$$= -\frac{1}{4}$$

This is the finite value, so the given integral is convergent and converges to $\boxed{-\frac{1}{4}}$

Answer 20E.

Let $I = \int_{2}^{\infty} y e^{-3y} dy$

$$= \underset{t \to \infty}{Lt} \int_{2}^{t} y e^{-3y} dy$$

$$= \underset{t \to \infty}{Lt} \left[y \frac{e^{-3y}}{-3} - (1) \frac{e^{-3y}}{9} \right]_{2}^{t} \qquad [\text{because by integration by parts}]$$

$$= \underset{t \to \infty}{Lt} \left[\frac{t\, e^{-3t}}{-3} - \frac{e^{-3t}}{9} + \frac{2}{3} e^{-6} + \frac{e^{-6}}{9} \right]$$

$$= \frac{2}{3} e^{-6} + \frac{e^{-6}}{9}$$

$$= \frac{7}{9} e^{-6}, \text{convergent}$$

Answer 21E.

$$\int_{1}^{\infty} \frac{\ln x}{x} dx = \lim_{t \to \infty} \int_{1}^{t} \ln x . \frac{1}{x} dx \qquad \text{--- (1)}$$

Substitute $\ln x = u \quad \Rightarrow \frac{1}{x} dx = du$

Then $\int \ln x . \frac{1}{x} dx = \int u . du$

$$= \frac{u^2}{2} = \frac{1}{2} (\ln x)^2$$

Therefore equation (1) becomes

$$\int_1^\infty \frac{\ln x}{x}dx = \lim_{t\to\infty}\int_1^t \ln x \cdot \frac{1}{x}dx$$
$$= \lim_{t\to\infty}\left[\frac{1}{2}(\ln x)^2\right]_1^t$$
$$= \lim_{t\to\infty}\frac{1}{2}\left[(\ln t)^2 - (\ln 1)^2\right]$$
$$= \frac{1}{2}\lim_{t\to\infty}(\ln t)^2 = \infty$$

Thus $\int_1^\infty \frac{\ln x}{x}dx$ is divergent

Answer 22E.

Consider the integral,

$$\int_{-\infty}^{\infty} x^3 e^{-x^4} dx$$

The objective is to determine whether the given integral is convergent or divergent.

Let

$$u = -x^4$$
$$du = -4x^3 dx$$

Rewrite the integral as,

$$\int_{-\infty}^{\infty} x^3 e^{-x^4} dx = \lim_{h\to+\infty}\int_{-h}^{h} x^3 e^{-x^4} dx$$
$$= \lim_{h\to+\infty}\int_{-h^4}^{-h^4}\left(-\frac{1}{4}\right)e^u du$$
$$= \lim_{h\to+\infty}\left[\left(-\frac{1}{4}\right)e^{-h^4} - \left(-\frac{1}{4}\right)e^{-h^4}\right]$$
$$= 0$$

Therefore, the integral $\int_{-\infty}^{\infty} x^3 e^{-x^4} dx$ is convergent and $\int_{-\infty}^{\infty} x^3 e^{-x^4} dx = \boxed{0}$.

Answer 23E.

Consider the integral,

$$\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}dx \quad \ldots\ldots (1)$$

Recall the definition improper integral of type 1,

If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

Here any real number a can be used.

For convenience, use $a = 0$, $f(x) = \frac{x^2}{9+x^6}$, in the above formula.

$$\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}dx = \int_{-\infty}^{0} \frac{x^2}{9+x^6}dx + \int_{0}^{\infty} \frac{x^2}{9+x^6}dx \quad \ldots\ldots (2)$$

$$= \lim_{t_1\to-\infty}\int_{t_1}^{0} \frac{x^2}{9+x^6}dx + \lim_{t_2\to\infty}\int_{0}^{t_2} \frac{x^2}{9+x^6}dx \quad \text{Use } \begin{cases}\int_{-\infty}^a f(x)dx = \lim_{t\to-\infty}\int_t^a f(x)dx \\ \int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx\end{cases}$$

Use substitution to evaluate the integrals $\int_{t_1}^{0} \frac{x^2}{9+x^6} dx$ and $\int_{0}^{t_2} \frac{x^2}{9+x^6} dx$

Substitute $x^3 = u$ in the above integrals.

Differentiate $x^3 = u$ on each side.

$$3x^2 dx = du$$

Change the limits of integration.

When $x = t_1$, $u = t_1^3$

When $x = t_2$, $u = t_2^3$

When $x = 0$, $u = 0$

Then,

$$\int_{t_1}^{0} \frac{x^2}{9+x^6} dx = \int_{t_1^3}^{0} \frac{\frac{du}{3}}{9+u^2}$$

$$= \frac{1}{3}\int_{t_1^3}^{0} \frac{du}{9+u^2}$$

$$= \left[\frac{1}{3}\cdot\frac{1}{3}\cdot\tan^{-1}\frac{u}{3}\right]_{t_1^3}^{0}$$ Use $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$= \left[\frac{1}{9}\tan^{-1}\frac{u}{3}\right]_{t_1^3}^{0}$$

$$= \frac{1}{9}\left(0 - \tan^{-1}\frac{t_1^3}{3}\right)$$

$$= -\frac{1}{9}\tan^{-1}\frac{t_1^3}{3}$$

Find the limit as $t_1 \to -\infty$

$$\lim_{t_1\to-\infty}\int_{t_1}^{0} \frac{x^2}{9+x^6} dx = \lim_{t_1\to-\infty}\left(-\frac{1}{9}\tan^{-1}\frac{t_1^3}{3}\right)$$

$$= -\frac{1}{9}\left(-\frac{\pi}{2}\right)$$

$$= \frac{1}{9}\left(\frac{\pi}{2}\right)$$

The limit exists. So it is convergent. And therefore,

$$\int_{-\infty}^{0} \frac{x^2}{9+x^6} dx = \frac{1}{9}\cdot\frac{\pi}{2} \quad \text{...... (3)}$$

And also evaluate the integral $\int_0^{t_2^3} \frac{x^2}{9+x^6} dx$

$$\int_0^{t_2^3} \frac{x^2}{9+x^6} dx = \int_0^{t_2^3} \frac{\frac{du}{3}}{9+u^2}$$

$$= \frac{1}{3}\int_0^{t_2^3} \frac{du}{9+u^2}$$

$$= \left[\frac{1}{3}\cdot\frac{1}{3}\cdot\tan^{-1}\frac{u}{3}\right]_0^{t_2^3}$$ Use $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$= \left[\frac{1}{9}\tan^{-1}\frac{u}{3}\right]_0^{t_2^3}$$

$$= \frac{1}{9}\left(\tan^{-1}\frac{t_2^3}{3} - \tan^{-1}0\right)$$ Apply the limits

$$= \frac{1}{9}\left(\tan^{-1}\frac{t_2^3}{3} - 0\right)$$

$$= \frac{1}{9}\tan^{-1}\frac{t_2^3}{3}$$

Find the limit as $t_2 \to \infty$

$$\lim_{t_2\to\infty} \int_0^{t_2^3} \frac{x^2}{9+x^6} dx = \lim_{t_2\to\infty}\left(\frac{1}{9}\tan^{-1}\frac{t_2^3}{3}\right)$$

$$= \frac{1}{9}\left(\frac{\pi}{2}\right)$$

The limit exists. So it is convergent. And therefore,

$$\int_0^{\infty} \frac{x^2}{9+x^6} dx = \frac{1}{9}\cdot\frac{\pi}{2} \quad \text{......} \; (4)$$

Substitute the evaluated integral values, (3) and (4) in (2)

$$\int_{-\infty}^{\infty} \frac{x^2}{9+x^6} dx = \frac{1}{9}\left(\frac{\pi}{2}\right) + \frac{1}{9}\left(\frac{\pi}{2}\right)$$

$$= \frac{1}{9}\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= \frac{\pi}{9}$$

Thus, the value of the improper integral is $\boxed{\frac{\pi}{9}}$.

Answer 24E.

Consider the integral,

$$\int_0^\infty \frac{e^x}{e^{2x}+3}dx$$

Determine whether the given integral $\int_0^\infty \frac{e^x}{e^{2x}+3}dx$ is convergent or divergent. If it is convergent, evaluate it.

Let $u = e^x$. Then $du = e^x dx$ and the limits of integration will change to $[1,\infty)$.

Rewrite the integral.

$$\begin{aligned}\int_0^\infty \frac{e^x}{e^{2x}+3}dx &= \int_0^\infty \frac{e^x}{e^x \cdot e^x + 3}dx \\ &= \int_1^\infty \frac{du}{(u)^2+3} \\ &= \int_1^\infty \frac{du}{u^2+\left(\sqrt{3}\right)^2}\end{aligned}$$

Recall that

$$\int \frac{dx}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

Thus,

$$\begin{aligned}\int_0^\infty \frac{e^x}{e^{2x}+3}dx &= \int_1^\infty \frac{du}{u^2+\left(\sqrt{3}\right)^2}du \\ &= \left[\frac{\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)}{\sqrt{3}}\right]_1^\infty \\ &= \frac{1}{\sqrt{3}}\left[\lim_{u\to\infty}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]\end{aligned}$$

As $u \to \infty$, $\frac{u}{\sqrt{3}} \to \infty$, and so $\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) = \frac{\pi}{2}$. Recall that $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$.

So, the given integral becomes

$$\begin{aligned}\int_0^\infty \frac{e^x}{e^{2x}+3}dx &= \frac{1}{\sqrt{3}}\left(\lim_{u\to\infty}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2}\right) - \frac{\pi}{6\sqrt{3}} \\ &= \frac{3\pi - \pi}{6\sqrt{3}} \\ &= \boxed{\frac{\pi}{3\sqrt{3}}}\end{aligned}$$

Therefore, the given integral $\int_0^\infty \frac{e^x}{e^{2x}+3}dx$ is convergent and its value is $\boxed{\frac{\pi}{3\sqrt{3}}}$.

Answer 25E.

Consider the following integral:

$$\int_e^{\infty} \frac{1}{x(\ln x)^3} dx$$

The objective is to determine whether the improper integral is convergent or divergent.

Definition of improper integral:

If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then,

$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

Now the integral $\int_e^{\infty} \frac{1}{x(\ln x)^3} dx$ can be written as,

$$\int_e^{\infty} \frac{1}{x(\ln x)^3} dx = \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^3} dx$$

Use substitution to solve the integral.

Let $u = \ln x$ then $du = \frac{1}{x} dx$

$$\lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^3} dx = \lim_{t\to\infty} \int_e^t \frac{1}{u^3} \cdot du$$

$$= \lim_{t\to\infty} \left[\frac{-1}{2u^2}\right]_e^t$$

$$= \lim_{t\to\infty} \left[\frac{-1}{2(\ln x)^2}\right]_e^t$$

$$= \lim_{t\to\infty} \left[\frac{-1}{2(\ln t)^2} + \frac{1}{2(\ln e)^2}\right]$$

$$= \lim_{t\to\infty} \left[\frac{-1}{2(\ln t)^2} + \frac{1}{2(1)^2}\right]$$

$$= \lim_{t\to\infty} \left[\frac{-1}{2(\ln t)^2} + \frac{1}{2}\right]$$

$$= 0 + \frac{1}{2}$$

$$= \frac{1}{2}$$

Hence, there exists a limit and it is convergent.

Therefore, the given integral is, convergent

Answer 26E.

$$\int_0^{\infty} \frac{x \arctan x}{(1+x^2)^2} dx = \lim_{t\to\infty} \int_0^t \frac{x \tan^{-1} x}{(1+x^2)^2} dx$$

Substitute $x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$

$$\int \frac{x \tan^{-1} x}{(1+x^2)^2} dx = \int \frac{(\tan\theta).\theta}{(1+\tan^2\theta)^2}.\sec^2\theta \, d\theta$$

$$= \int \frac{\theta \tan\theta \sec^2\theta}{\sec^4\theta} d\theta$$

$$= \int \theta \frac{\tan\theta}{\sec^2\theta} d\theta$$

$$= \int \theta \frac{\sin\theta}{\cos\theta}.\cos^2\theta \, d\theta$$

$$= \int \theta \sin\theta \cos\theta \, d\theta$$

$$= \frac{1}{2} \int \theta. \sin 2\theta \, d\theta$$

Integrate by parts

$$\frac{1}{2}\int \theta.\sin 2\theta d\theta = \frac{1}{2}\left[\theta\frac{(-\cos 2\theta)}{2} - \int 1.\frac{(-\cos 2\theta)}{2}d\theta\right]$$

$$= -\frac{1}{4}\theta\ \cos 2\theta + \frac{1}{4}.\frac{\sin 2\theta}{2}$$

$$= -\frac{1}{4}\theta.\frac{1-\tan^2\theta}{1+\tan^2\theta} + \frac{1}{8}.\frac{2\tan\theta}{1+\tan^2\theta}$$

$$= -\frac{1}{4}(\tan^{-1}x).\frac{1-x^2}{(1+x^2)} + \frac{1}{4}\frac{x}{(1+x^2)}$$

Thus $\int_0^\infty \frac{x\arctan x}{(1+x^2)^2}dx = \lim_{t\to\infty}\left(-\frac{1}{4}\right)\left[\tan^{-1}x\frac{1-x^2}{1+x^2} - \frac{x}{1+x^2}\right]_0^t$

$$= -\frac{1}{4}\lim_{t\to\infty}\left[\left(\tan^{-1}t.\frac{1-t^2}{1+t^2} - \frac{t}{1+t^2}\right) - (\tan^{-1}0 - 0)\right]$$

$$= -\frac{1}{4}\left[\left(\lim_{t\to\infty}\tan^{-1}t\right).\lim_{t\to\infty}\left(\frac{1-t^2}{1+t^2}\right) - \lim_{t\to\infty}\frac{t}{1+t^2}\right]$$

$$= -\frac{1}{4}\left[\frac{\pi}{2}.\lim_{t\to\infty}\left(-\frac{2t}{2t}\right) - \lim_{t\to\infty}\frac{1}{2t}\right] \qquad \text{[By L-hospital rule]}$$

$$= -\frac{1}{4}.\frac{\pi}{2}.(-1) - 0 = \frac{\pi}{8}$$

Therefore the integral $\boxed{\int_0^\infty \frac{x\arctan x}{(1+x^2)^2}dx = \pi/8}$ is convergent

Answer 27E.

Given the integral as $\int_0^1 \frac{3}{x^5}dx$, which is discontinuous at $x = 0$.

So, $\int_0^1 \frac{3}{x^5}dx = \lim_{t\to 0^+}\int_t^1 \frac{3}{x^5}dx$

$$= \lim_{t\to 0^+}\left[\frac{-3}{4x^4}\right]_t^1$$

$$= \lim_{t\to 0^+}\left[\frac{-3}{4} - \frac{-3}{4t^4}\right]$$

As $t\to 0^+$, $\frac{-3}{4t^4}\to -\infty$

So the limit does not exist.
As the integrand is negative, the integral diverges.

Answer 28E.

Consider the integral,

$$\int_2^3 \frac{1}{\sqrt{3-x}}dx$$

Determine whether the given integral $\int_2^3 \frac{1}{\sqrt{3-x}}dx$ is convergent or divergent. If it is convergent, evaluate it.

This definite integral is improper since $\frac{1}{\sqrt{3-x}}$ has an infinite discontinuity at $x = 3$.
However, as you will see in the next step, this improper integral still can be evaluated normally without using limits.

Let $u=3-x$. Then $du=-dx$ and the new limits of integration in terms of u are $[3-2,3-3]=[1,0]$.

This gives

$$\begin{aligned}\int_2^3 \frac{1}{\sqrt{3-x}}dx &= \int_1^0 \frac{-1}{\sqrt{u}}du \\ &= -\int_1^0 u^{\frac{-1}{2}}du \\ &= -\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^0 \quad \text{Since, } \int u^n du = \frac{u^{n+1}}{n+1} \\ &= -\left[2u^{\frac{1}{2}}\right]_1^0 \\ &= -2\left(\sqrt{0}-\sqrt{1}\right) \\ &= -2(-1) \\ &= 2\end{aligned}$$

Therefore, the given integral $\int_2^3 \frac{1}{\sqrt{3-x}}dx$ is convergent and its value is $\boxed{2}$.

Q29E.

Clearly, the function is not defined at $x=-2$ since the denominator of the function is zero at $x=-2$ so the function is continuous on the interval $(-2,14]$ and discontinuous at $x=-2.$

This result is confirmed by using above graph, that is, at $x=-2$ the function has a vertical asymptote.

So the given integral is an improper integral.

If f is continuous on $(a,b]$ and is discontinuous at a, then $\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$.

Rewrite equation (1) as:

$$\int_{-2}^{14} \frac{1}{\sqrt[4]{x+2}}dx = \lim_{t\to -2^+}\int_t^{14} \frac{1}{\sqrt[4]{x+2}}dx$$

Solve the integral $\lim_{t\to -2^+}\int_t^{14}\frac{1}{\sqrt[4]{x+2}}dx$.

$$\begin{aligned}\int_{-2}^{14}\frac{1}{\sqrt[4]{x+2}}dx &= \lim_{t\to -2^+}\int_t^{14}\frac{1}{\sqrt[4]{x+2}}dx \\ &= \lim_{t\to -2^+}\int_t^{14}(x+2)^{-\frac{1}{4}}dx \\ &= \lim_{t\to -2^+}\left[\frac{(x+2)^{-\frac{1}{4}+1}}{-\frac{1}{4}+1}\right]_t^{14}\end{aligned}$$

$$\begin{aligned}&= \lim_{t\to -2^+}\left[\frac{4}{3}(x+2)^{\frac{3}{4}}\right]_t^{14} \\ &= \frac{4}{3}\lim_{t\to -2^+}\left[(14+2)^{\frac{3}{4}}-(t+2)^{\frac{3}{4}}\right] \\ &= \frac{4}{3}\lim_{t\to -2^+}\left[\left(2^4\right)^{\frac{3}{4}}-(t+2)^{\frac{3}{4}}\right] \\ &= \frac{4}{3}\lim_{t\to -2^+}(8)-\frac{4}{3}\lim_{t\to -2^+}\left((t+2)^{\frac{3}{4}}\right) \\ &= \frac{4}{3}(8)-\frac{4}{3}\left((-2+2)^{\frac{3}{4}}\right) \\ &= \frac{4}{3}(8)-0 \\ &= \frac{32}{3}\end{aligned}$$

Therefore, the limit $\lim_{t\to -2^+}\int_t^{14}\frac{1}{\sqrt[4]{x+2}}dx$ exists and is equal to $\frac{32}{3}$, that implies that the improper integral $\int_{-2}^{14}\frac{1}{\sqrt[4]{x+2}}dx$ is convergent.

Therefore, the improper integral $\int_{-2}^{14}\frac{1}{\sqrt[4]{x+2}}dx$ is $\boxed{\text{convergent}}$.

Answer 30E.

Consider the following definite integral:

$$\int_6^8\frac{4}{(x-6)^3}dx$$

The objective is to find whether the integral is convergent or divergent.

Rewrite the integral as,

$$\int_6^8\frac{4}{(x-6)^3}dx = \lim_{t\to 6}\int_t^8\frac{4}{(x-6)^3}dx$$

Solve the integral as follows:

$$\begin{aligned}\int_6^8\frac{4}{(x-6)^3}dx &= \lim_{t\to 6}\int_t^8\frac{4}{(x-6)^3}dx \\ &= \left[\lim_{t\to 6}\left(-\frac{2}{(x-6)^2}\right)\right]_t^8 \\ &= \lim_{t\to 6}\left[\left(-\frac{2}{(8-6)^2}\right)+\left(\frac{2}{(t-6)^2}\right)\right] \\ &= -\frac{1}{2}+\lim_{t\to 6}\frac{2}{(t-6)^2}\end{aligned}$$

$$=\infty$$

Therefore, the given integral is $\boxed{\text{divergent}}$.

Answer 31E.

We have to evaluate $\int_{-2}^{3}\frac{1}{x^4}dx$

To evaluate the given integral we use the following definitions.

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 2:

(a) If f is continuous on $[a,b)$ and is discontinuous at b,

then $\int_a^t f(x)dx$ exists for every real number $t \geq a$,

then $\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$

if this limit exists (as a finite number)

(b) If f is continuous on $(a,b]$ and is discontinuous at a,

then $\int_a^t f(x)dx$ exists for every real number $t \geq a$,

then $\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$

if this limit exists (as a finite number)

(c) If f has a discontinuity at c, where $a < c < b$, and both

$\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

The given function has an asymptote at x = 0

Since it occurs in the middle of the interval [-2, 3], we use the above definition of an improper integral of type2 with c = 0.

$$\text{So } \int_{-2}^{3}\frac{1}{x^4}dx = \int_{-2}^{0}\frac{1}{x^4}dx + \int_{0}^{3}\frac{1}{x^4}dx$$

$$= \lim_{t\to 0^-}\int_{-2}^{t}\frac{1}{x^4}dx + \lim_{s\to 0^+}\int_{s}^{3}\frac{1}{x^4}dx \qquad \text{(using (a) and (b))}$$

$$\text{Now } \lim_{t\to 0^-}\int_{-2}^{t}\frac{1}{x^4}dx = \lim_{t\to 0^-}\int_{-2}^{t}x^{-4}dx$$

$$= \lim_{t\to 0^-}\left[\frac{x^{-3}}{-3}\right]_{-2}^{t}$$

$$= -\frac{1}{3}\lim_{t\to 0^-}\left(t^{-3} - (-2)^{-3}\right]$$

$$= -\frac{1}{3}\lim_{t\to 0^-}\left(\frac{1}{t^3} + \frac{1}{8}\right)$$

$$= \infty$$

Thus $\int_{-2}^{0}\frac{1}{x^4}dx = \infty$ is divergent, [we do not need to evaluate $\int_{0}^{3}\frac{1}{x^4}dx$]

Therefore $\int_{-2}^{3}\frac{1}{x^4}dx$ is divergent

Answer 32E.

We have to evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

The given function has asymptote at x = 1

Then $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\int_0^t \frac{dx}{\sqrt{1-x^2}}$

$$= \lim_{t\to 1^-}\left[\sin^{-1}x\right]_0^t$$
$$= \lim_{t\to 1^-}\left[\sin^{-1}t - \sin^{-1}0\right]$$
$$= \lim_{t\to 1^-}\sin^{-1}t - \lim_{t\to 1^-}0$$
$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Thus $\boxed{\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}}$ is convergent

Answer 33E.

Consider an integral $\int_0^9 \frac{1}{\sqrt[3]{x-1}}dx$

To determine whether each integral is convergent or divergent

Definition:

If f has a discontinuity at c, where $a<c<b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Let

$$I = \int_0^9 \frac{1}{\sqrt[3]{x-1}}dx$$

Observe that the above integral is improper because $f(x) = \frac{1}{\sqrt[3]{x-1}}$ vertical asymptote $x=1$.

Since it occurs in the middle of the interval $[0,9]$ of above definition with $c=1$:

Therefore,

$$I = \int_0^1 \frac{dx}{\sqrt[3]{x-1}} + \int_1^9 \frac{dx}{\sqrt[3]{x-1}}$$

$$= \left[\frac{(x-1)^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}\right]_0^1 + \left[\frac{(x-1)^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}\right]_1^9$$

$$= \left[\frac{3}{2}(x-1)^{\frac{2}{3}}\right]_0^1 + \left[\frac{3}{2}(x-1)^{\frac{2}{3}}\right]_1^9$$

$$= \frac{3}{2}(0) - \frac{3}{2}(-1)^{\frac{2}{3}} + \frac{3}{2}\left(8^{\frac{2}{3}}\right) - \frac{3}{2}(0)$$

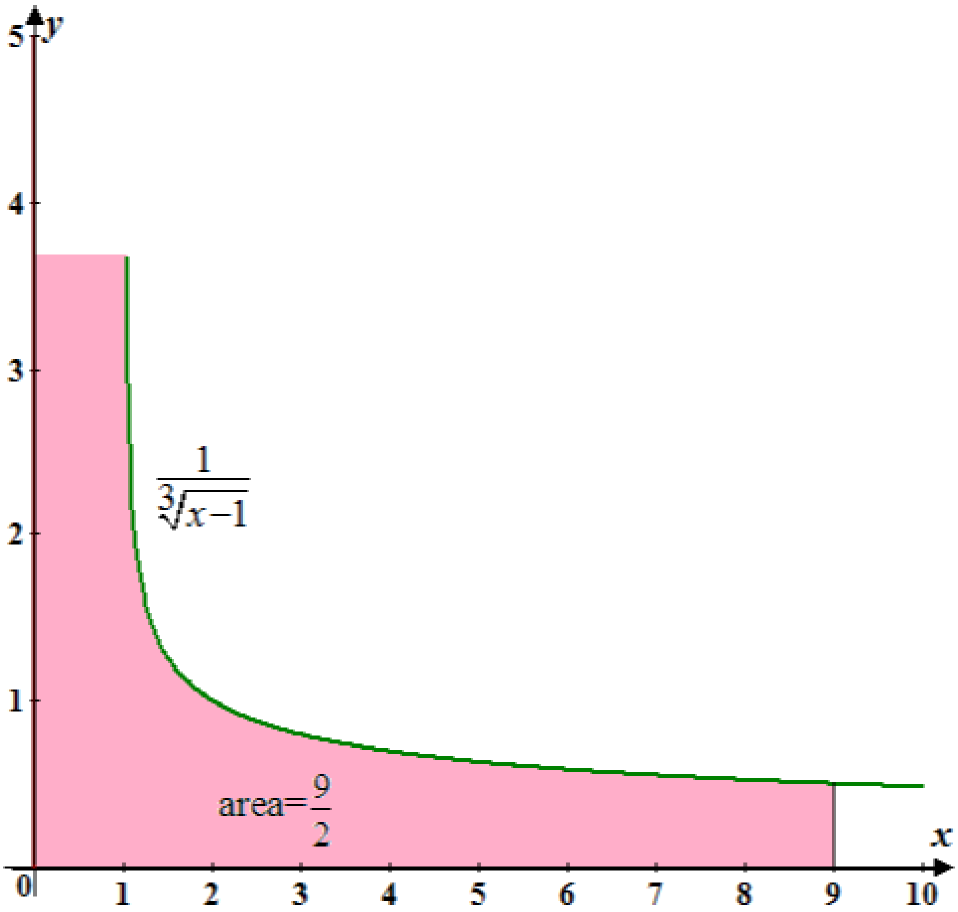
$$= -\frac{3}{2} + \frac{3}{2}(4)$$

$$= -\frac{3}{2} + 6$$

$$= -\frac{3+12}{2}$$

$$= \frac{9}{2}$$

Thus the given improper integral is convergent and since the integrand is positive, we can interpret the value of the integral as the area of the shaded region in below figure



Answer 34E.

Consider an integral $\int_0^5 \frac{w}{w-2}dw$

To determine whether each integral is convergent or divergent

Definition:

If f has a discontinuity at c, where $a<c<b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$$

Let $I=\int_0^5 \frac{w}{w-2}dw$

Observe that the above integral is improper because $f(x)=\frac{w}{w-2}$ vertical asymptote $w=2$.

Since it occurs in the middle of the interval $[0,5]$ of above definition with $c=2$:

So $I=\int_0^2 \frac{w}{w-2}dw+\int_2^5 \frac{w}{w-2}dw$

Where

$$\int_0^2 \frac{w}{w-2}dw=\lim_{t\to 2^-}\int_0^t \frac{w-2+2}{w-2}dw$$

$$=\lim_{t\to 2^-}\int_0^t\left(1+\frac{2}{w-2}\right)dw$$

$$=\lim_{t\to 2^-}\left[w+2\ln|w-2|\right]_0^t$$

$$=\lim_{t\to 2^-}\left[(t-0)+(2\ln|t-2|-2\ln|0-2|)\right]$$

$$=\lim_{t\to 2^-}\left[t+(2\ln|t-2|-2\ln|-2|)\right]$$

$$=\lim_{t\to 2^-}t+2\lim_{t\to 2^-}\ln(2-t)-2\ln 2$$

$$=2-\infty-2\ln 2$$

$$=-\infty \quad (\text{undefined})$$

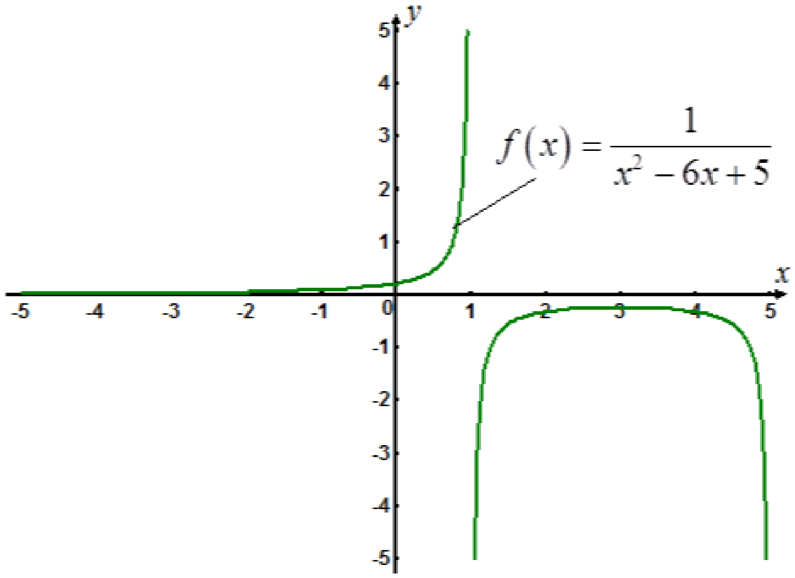
because $2-t\to 0^+$ as $t\to 2^-$. Thus $\int_0^2 \frac{w}{w-2}dw$ is divergent. This implies that $\int_0^5 \frac{w}{w-2}dw$ is divergent. [Do not need to evaluate $\int_2^5 \frac{w}{w-2}dw$.]

Answer 35E.

Consider the integral

$$\int_0^3 \frac{1}{x^2-6x+5}dx$$

Sketch the graph of the function $f(x)=\frac{1}{x^2-6x+5}$.



Observe the graph, the function has discontinuous at $x=1$.

So the integral $\int_0^3 \frac{1}{x^2-6x+5}dx$ changes as

$$\int_0^3 \frac{1}{x^2-6x+5}dx = \int_0^1 \frac{dx}{(x-1)(x-5)} + \int_1^3 \frac{dx}{(x-1)(x-5)}$$

$$= \lim_{t\to 1^-}\int_0^t \frac{dx}{(x-1)(x-5)} + \lim_{t\to 1^+}\int_t^3 \frac{dx}{(x-1)(x-5)}$$

$$= \lim_{t\to 1^-}\int_0^t \frac{dx}{(x-1)(x-5)} + \lim_{t\to 1^+}\int_t^3 \frac{dx}{(x-1)(x-5)}$$

$$= \lim_{t\to 1^-}\int_0^t \left[-\frac{1}{4(x-1)} + \frac{1}{4(x-5)}\right]dx + \lim_{t\to 1^+}\int_t^3 \left[-\frac{1}{4(x-1)} + \frac{1}{4(x-5)}\right]dx$$

$$= \lim_{t\to 1^-}\left[-\frac{1}{4}\ln(x-1) + \frac{1}{4}\ln(x-5)\right]_0^t + \lim_{t\to 1^+}\left[-\frac{1}{4}\ln(x-1) + \frac{1}{4}\ln(x-5)\right]_t^3$$

$$= \infty$$

Therefore $\int_0^3 \frac{1}{x^2-6x+5}dx$ is diverges.

Answer 36E.

Consider the following integral:

$$\int_{\pi/2}^{\pi} \csc x\,dx$$

Multiply and divide the integral by $\csc(x)-\cot(x)$ and simplify as shown below:

$$\int_{\pi/2}^{\pi} \csc x\,dx = \int_{\pi/2}^{\pi} \frac{\csc(x)(\csc x - \cot(x))}{\csc(x)-\cot(x)}dx$$

$$= \int_{\pi/2}^{\pi} \frac{\csc^2 x - \csc(x)\cot(x)}{\csc(x)-\cot(x)}dx$$

Substitute, $u=\csc(x)-\cot(x) \Rightarrow du = (\csc^2 x - \csc x \cot x)dx.$

$$= \int_{\pi/2}^{\pi} \frac{1}{u}du$$

$$= [\ln(u)]_{\pi/2}^{\pi}$$

$$= \ln[\csc(x)-\cot(x)]_{\pi/2}^{\pi}$$

$$= \ln[\csc(\pi)-\cot(\pi)] - \ln\left[\csc\left(\frac{\pi}{2}\right)-\cot\left(\frac{\pi}{2}\right)\right]$$

$$= \ln[0-\infty] - \ln[1-0]$$
$$= \ln[\infty] - 0$$
$$= \infty$$

Therefore, $\int_{\pi/2}^{\pi} \csc x\,dx$ is diverges.

Answer 37E.

Consider the integral,

$$\int_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^3}\,dx$$

The object is to determine whether the integral converges or diverges.

Rewrite the integral as,

$$\int_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^3}\,dx = \lim_{t\to 0^-}\int_{-1}^{t} \frac{e^{\frac{1}{x}}}{x^3}\,dx$$

$$= \lim_{t\to 0^-}\int_{-1}^{t} \frac{e^{\frac{1}{x}}}{x^2\cdot x}\,dx$$

Use substitution method: Let $\frac{1}{x}=u$

Differentiate $\frac{1}{x}=u$ on both sides.

$$\frac{-1}{x^2}dx = du$$

Upper limit: If $x=t$ then $u=\frac{1}{t}$.

Lower limit: If $x=-1$ then $u=-1$.

Now the integral is reduced as,

$$\int_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^3}\,dx = \lim_{t\to 0^-}\int_{-1}^{t} \frac{e^{\frac{1}{x}}}{x^3}\,dx$$

$$= \lim_{u\to -\infty}\int_{-1}^{u} -ue^{u}\,du$$

$$= \lim_{u\to -\infty}\int_{-1}^{u} -ue^{u}\,du$$

$$= -\lim_{u\to -\infty}\int_{-1}^{u} ue^{u}\,du$$

$$= -\lim_{u\to -\infty}\left[ue^{u}-e^{u}\cdot 1\right]_{-1}^{u}$$

$$= -\lim_{u\to -\infty}\left[\left(ue^{u}-e^{u}\right)-\left(-e^{-1}-e^{-1}\right)\right]$$

$$= -\lim_{u\to -\infty}\left[ue^{u}-e^{u}+\frac{2}{e}\right]$$

$$= -\left[(-\infty)e^{-\infty}-e^{-\infty}+\frac{2}{e}\right]$$

$$= \frac{-2}{e} \qquad \text{Use } e^{-\infty}=0$$

This is a finite number.

Therefore, the integral $\int_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^3}\,dx$ converges.

Answer 38E.

Consider the integral,

$$\int_0^1 \frac{e^{\frac{1}{x}}}{x^3}dx$$

The objective is to determine whether the following integral is convergent or divergent.

Rewrite the integral as,

$$\int_0^1 \frac{e^{\frac{1}{x}}}{x^3}dx = \lim_{R\to 0+}\int_R^1 \frac{e^{\frac{1}{x}}}{x^3}dx$$

$$= -\lim_{R\to 0-}\int_{-1}^{R} te^t dt \quad \begin{bmatrix} \frac{1}{x}=t \\ -\frac{1}{x^2}dx = dt \end{bmatrix}$$

$$= -\lim_{R\to 0-}\left[t\int_{-1}^{R} e^t dt - \int_{-1}^{R}\left(\frac{dt}{dt}\int e^t dt\right)dt\right] \quad [\text{Using By parts}]$$

$$= -\lim_{R\to 0-}\left[te^t\Big|_{-1}^{R} - e^t\Big|_{-1}^{R}\right]$$

$$= \boxed{2e^{-1}-1}$$

Since the integral has finite number as a result,

Therefore, the $\int_0^1 \frac{e^{\frac{1}{x}}}{x^3}dx$ integral is convergent.

Answer 39E.

Consider the integral:

$$\int_0^2 z^2 \ln z \, dz.$$

We need to determine, whether the integral, $\int_0^2 z^2 \ln z \, dz$ is convergent or divergent, and also evaluate this integral, whether it is convergent.

The function has an asymptote at $z=0$.

Since $z\to 0^+, \ln z \to -\infty$

Thus $\int_0^2 z^2 \ln z \, dz = \lim_{t\to 0^+}\int_t^2 z^2 \ln z \, dz$.

Using Integrate by parts, we have

$u = \ln z \qquad dv = z^2 dz$

$du = \frac{1}{z}dz \qquad v = \frac{z^3}{3}$

Therefore, $\int_0^2 z^2 \ln z \, dz = \lim_{t\to 0^+}\left\{\left[(\ln z).\frac{z^3}{3}\right]_t^2 - \int_t^2 \frac{1}{z}.\frac{z^3}{3}dz\right\}$

$$= \lim_{t\to 0^+}\left\{\left[\frac{z^3}{3}(\ln z)\right]_t^2 - \frac{1}{3}\int_t^2 z^2 \, dz\right\}$$

$$= \lim_{t\to 0^+}\left\{\left[\frac{z^3}{3}\ln z\right]_t^2 - \frac{1}{9}\left[z^3\right]_t^2\right\}$$

$$= \lim_{t\to 0^+}\left[\left(\frac{8}{3}\ln 2 - \frac{t^3}{3}\ln t\right) - \frac{1}{9}\left(8-t^3\right)\right]$$

$$= \frac{8}{3}\ln 2 - \frac{8}{9} - \frac{1}{9}\lim_{t\to 0^+}\left[t^3\left(3\ln t - 1\right)\right].$$

Now applying L-Hospital's rule on the limit, $\lim_{t\to 0^+}\left[t^3(3\ln t-1)\right]$ we have

$$\lim_{t\to 0^+}\left[t^3(3\ln t-1)\right]=\lim_{t\to 0^+}\frac{3\ln t-1}{t^{-3}}$$
$$=\lim_{t\to 0^+}\left(\frac{3/t}{-3/t^4}\right)$$
$$=\lim_{t\to 0^+}\left(-t^3\right)$$
$$=0$$

Now substitute, the value of the limit, $\lim_{t\to 0^+}\left[t^3(3\ln t-1)\right]=0$ in the integral,

$$\int_0^2 z^2\ln z\,dz=\frac{8}{3}\ln 2-\frac{8}{9}-\frac{1}{9}\lim_{t\to 0^+}\left[t^3(3\ln t-1)\right].$$

Then, the value of the integral is

$$\int_0^2 z^2\ln z\,dz=\frac{8}{3}\ln 2-\frac{8}{9}-\frac{1}{9}(0)$$
$$=\frac{8}{3}\ln 2-\frac{8}{9}-0$$

$=\frac{8}{3}\ln 2-\frac{8}{9}$, which is finite value.

Hence the integral, $\int_0^2 z^2\ln z\,dz$ is convergent, and also the value of the integral is

$$\boxed{\int_0^2 z^2\ln z\,dz=\frac{8}{3}\ln 2-\frac{8}{9}}.$$

Answer 40E.

We have to evaluate $\int_0^1 \frac{\ln x}{\sqrt{x}}dx$

The given function has vertical asymptote at $x=0$.

Thus $\int_0^1 \frac{\ln x}{\sqrt{x}}dx=\lim_{t\to 0^+}\int_t^1 \ln x\,\frac{1}{\sqrt{x}}dx$

Integrate by parts with

$$u=\ln x,\qquad dv=\frac{1}{\sqrt{x}}dx=x^{-1/2}dx$$
$$u=\frac{1}{x}dx,\qquad v=2x^{1/2}$$

Using L-Hospital rule

$$\int_0^1 \frac{\ln x}{\sqrt{x}}dx=2\lim_{t\to 0^+}\left(\frac{1/t}{-\frac{1}{2}t^{-3/2}}\right)-4 \qquad \text{[L-Hospital rule]}$$
$$=2\lim_{t\to 0^+}\left(-2t^{1/2}\right)-4$$
$$=-4$$

Thus $\boxed{\int_0^1 \ln x\,\frac{1}{\sqrt{x}}dx=-4}$ and is convergent

Answer 41E.

A definite integral that contains the limits that tends to infinite.

Either both the limits of the integration could be infinite, or one of the two could be infinite.

Also, the integrand can tend to infinite in the interval of integration provided.

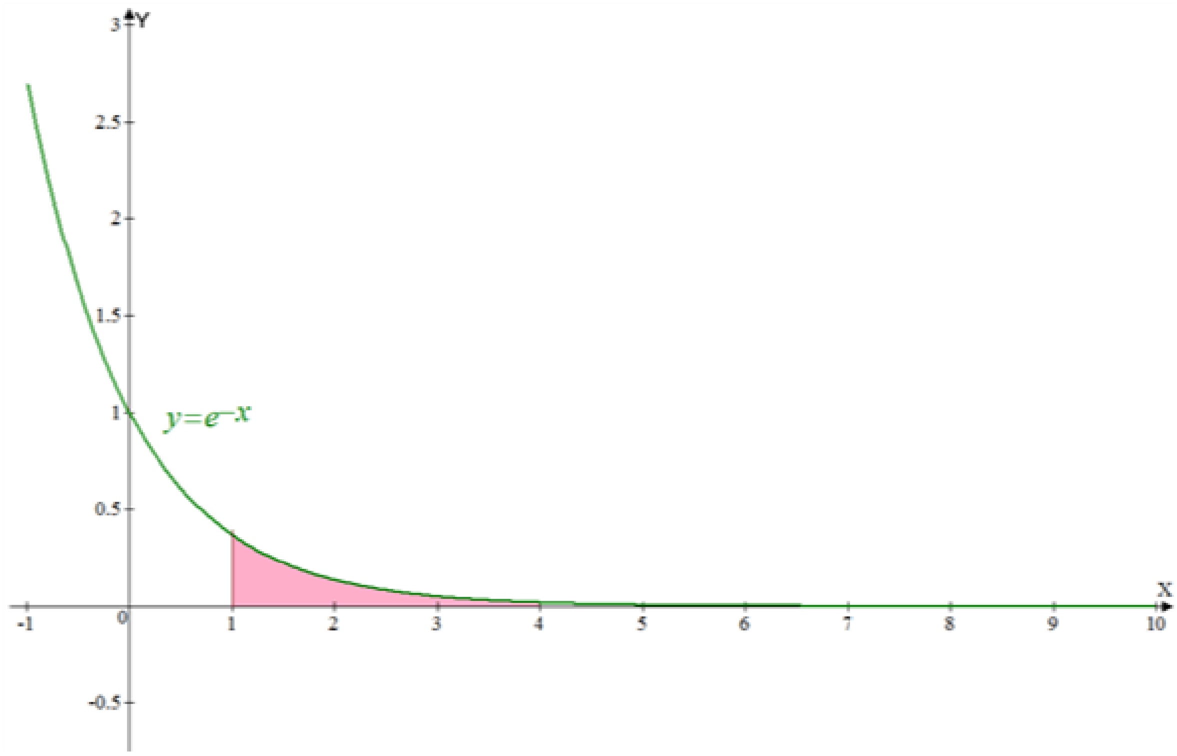
Consider the region S:

$$S=\{(x,y)/x\geq 1, 0\leq y\leq e^{-x}\}$$

Take the value of the curve y:

$$y=e^{-x}$$

Consider the graph of S as shown below:



Obtain the integral from the above region:

$$\text{Area}=\int_1^\infty e^{-x}dx$$

The integral so obtained represents an improper integral.

Evaluate the area of the above region as shown below:

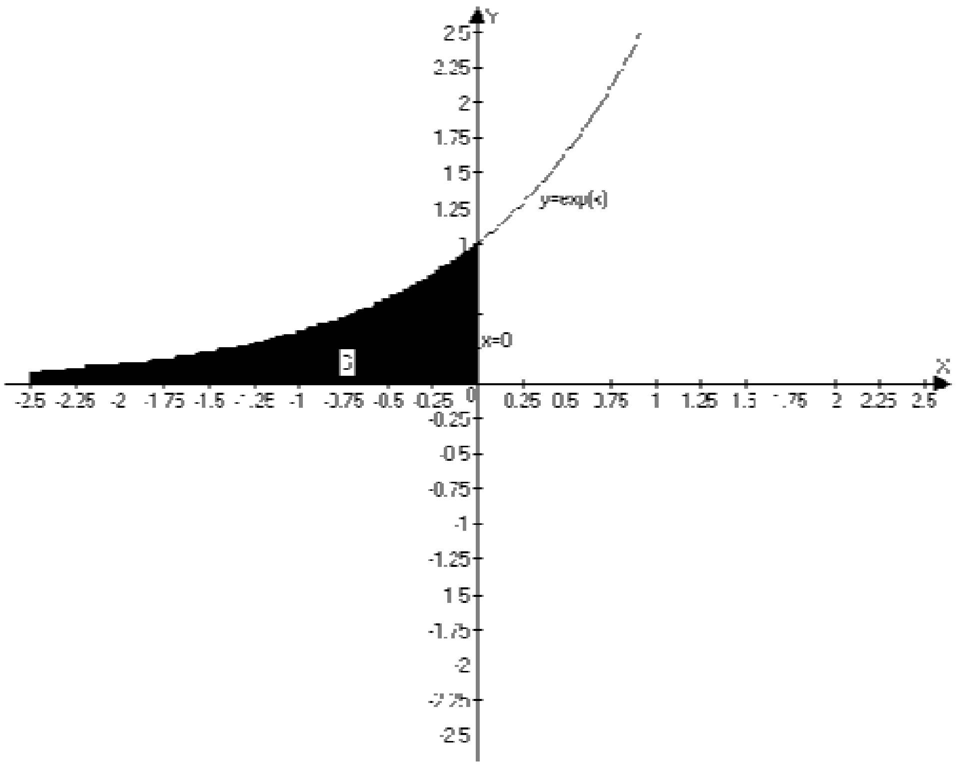
$$\begin{aligned}\int_1^\infty e^{-x}dx &= \left[\frac{e^{-x}}{-1}\right]_1^\infty \\ &= \left[\frac{-1}{e^x}\right]_1^\infty \\ &= \frac{1}{e} \\ &\approx 0.36788\end{aligned}$$

Hence, the area of the concerned region is $\boxed{0.36788}$.

Answer 42E.

Let $y = e^x$ and $s = \{(x,y)/x \le 0, 0 \le y \le e^x\}$

The graph of s is shown below



Therefore area $= \int_{-\infty}^{0} e^x dx$

$= \left[e^x\right]_{-\infty}^{0}$

$= 1 - 0$

$= 1$

$\boxed{\text{Therefore Area} = 1}$

Answer 43E.

A definite integral that contains the limits that tends to infinite.

Either both the limits of the integration could be infinite, or one of the two could be infinite.

Also, the integrand can tend to infinite in the interval of integration provided.

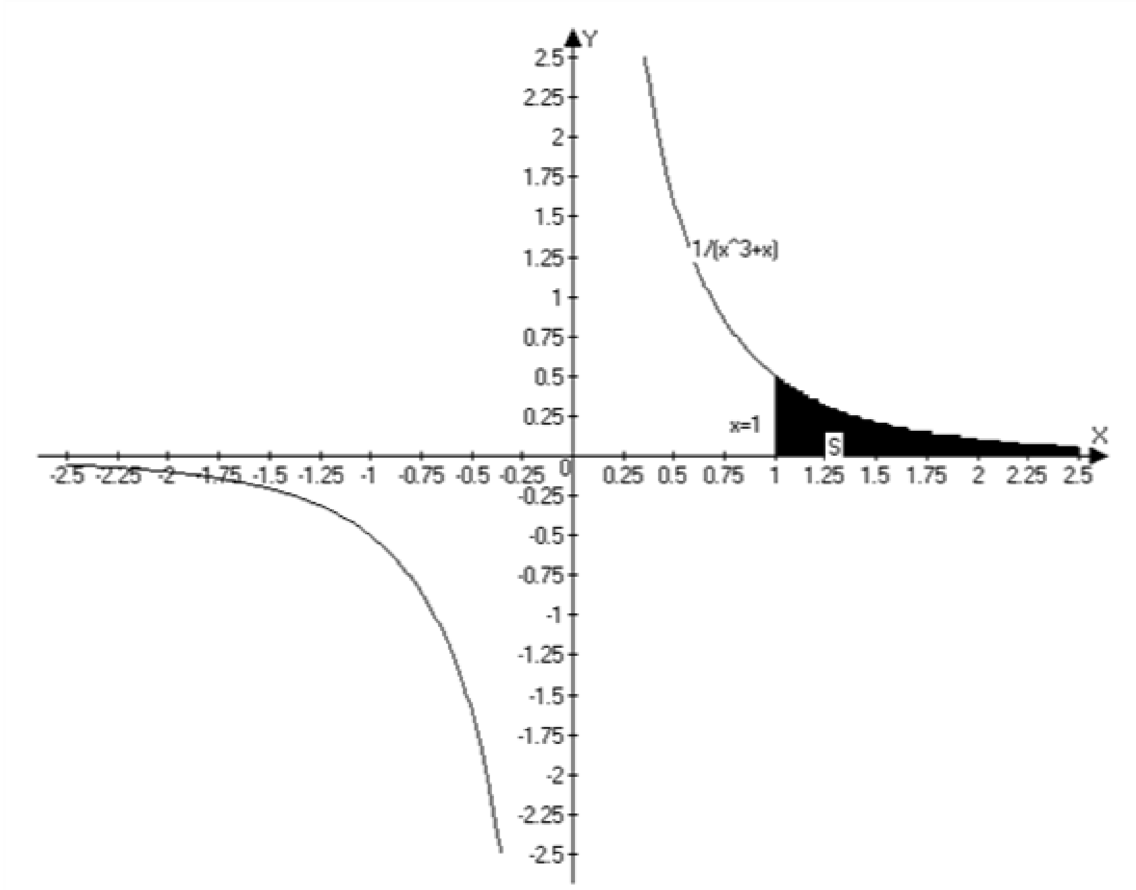
Consider the region S:

$$S = \left\{ (x, y) \middle/ x \geq 1, 0 \leq y \leq \frac{1}{x^3 + x} \right\}$$

Take the value of the curve y:

$$y = \frac{1}{x^3 + x}$$

Consider the graph of S as shown below:



Obtain the integral from the above region:

$$\text{Area} = \int_1^\infty \frac{1}{x^3 + x} dx$$

The integral so obtained represents an improper integral.

Evaluate the area of the above region as shown below:

$$\begin{aligned}
\int_1^\infty \frac{1}{x^3 + x} dx &= \int_1^\infty \frac{1}{x(x^2 + 1)} dx \\
&= \int_1^\infty \left(\frac{1}{x} - \frac{x}{x^2 + 1} \right) dx \\
&= \left[\ln x - \frac{1}{2} \ln (x^2 + 1) \right]_1^\infty \\
&= \left[\ln x - \ln \sqrt{x^2 + 1} \right]_1^\infty
\end{aligned}$$

Simplify the above value further and substitute the limits:

$$\int_1^\infty \frac{1}{x^3+x}dx = \left[\ln\left(\frac{x}{\sqrt{x^2+1}}\right)\right]_1^\infty$$

$$= \left[\ln\left(\frac{1}{\sqrt{1+\frac{1}{x^2}}}\right)\right]_1^\infty$$

$$= \ln 1 - \ln\frac{1}{\sqrt{2}}$$

$$= \ln\sqrt{2}$$

Solve the above value further to obtain the final result:

$$\int_1^\infty \frac{1}{x^3+x}dx = \frac{1}{2}\ln 2$$

$$\approx 0.34657$$

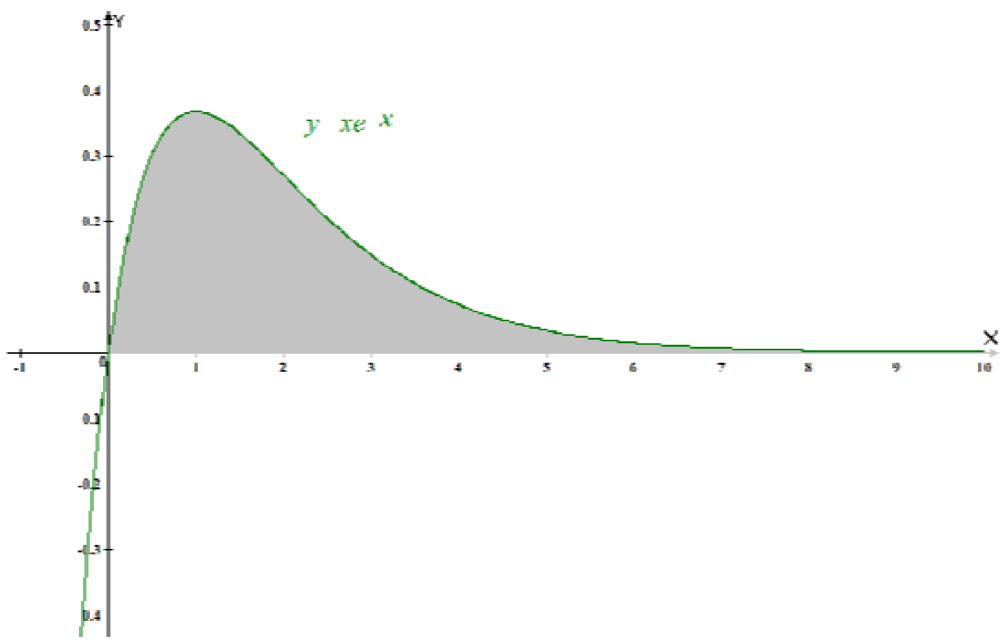
Hence, the final value is $\boxed{0.34657}$.

Answer 44E.

Let $s = \{(x,y)/x \ge 0, 0 \le y \le xe^{-x}\}$

Let $y = xe^{-x}$

The graph of s is shown below



Therefore area $= \int_0^\infty xe^{-x}dx$

$$= \left[\frac{xe^{-x}}{-1} - (1)\frac{e^{-x}}{1}\right]_0^\infty$$

$$= \left[-xe^{-x} - e^{-x}\right]_0^\infty$$

$$= 1$$

Answer 45E.

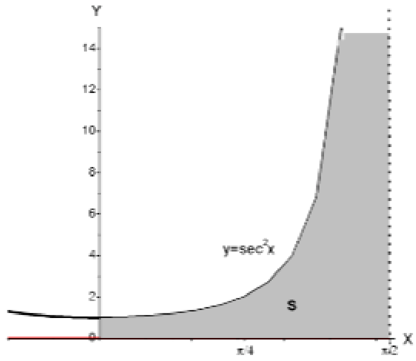


Fig.1

Shaded region is $S = \left\{(x,y) \mid 0 \le x \le \frac{\pi}{2},\ 0 \le y \le \sec^2 x\right\}$

Area is $A = \int_0^{\pi/2} \sec^2 x\, dx$

$= \lim_{t \to \frac{\pi}{2}} \int_0^t \sec^2 x\, dx$

$= \lim_{t \to \frac{\pi}{2}} \left[\tan x\right]_0^t$

$= \lim_{t \to \frac{\pi}{2}} \left[\tan t\right]$

$= \infty$

$\Rightarrow \boxed{A = \infty}$ Area is infinite

Answer 46E.

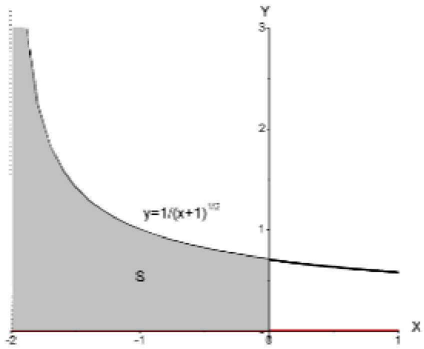


Fig.1

Shaded region is $S = \left\{(x,y) \mid -2 < x \le 0,\ 0 \le y \le 1/\sqrt{x+2}\right\}$

The area of the region $A = \int_{-2}^{0} \frac{1}{\sqrt{x+2}} dx$

Let $x + 2 = y \Rightarrow dx = dy$

When $x = -2, y = 0$ and when $x = 0, y = 2$

$$A = \int_0^2 y^{-1/2} dx$$

$$\Rightarrow A = \lim_{t \to 0} \int_t^2 y^{-1/2} dx$$

$$= \lim_{t \to 0} \left[2.y^{1/2}\right]_t^2$$

$$= \lim_{t \to 0} \left[2\sqrt{2} - 2\sqrt{t}\right]$$

$$= \lim_{t \to 0} 2\sqrt{2} - \lim_{t \to 0} 2\sqrt{t}$$

$$= 2\sqrt{2} - 0$$

$\Rightarrow \boxed{A = 2\sqrt{2}}$

Answer 47E.

(A) We have $g(x)=\frac{\sin^2 x}{x^2}$

Now with the help of computer we find the values of $\int_1^t g(x)dx$ for t = 2, 5, 10, 100, 1000 and 10000 and make a table

t	$\int_1^t \left[\frac{\sin^2 x}{x^2}\right]dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1000	0.672957
10000	0.673407

We see that the value of integral is approaching (0.675) when t is increasing. So it appears that $\int_1^t g(x)dx$ is convergent

(B) For $x \geq 1$ we have $\sin x \geq 1$

Then $\sin^2 x \geq 1$

And so $\frac{\sin^2 x}{x^2} \geq \frac{1}{x^2}$ for $x \geq 1$

Let $f(x)=\frac{1}{x^2}$ then $g(x) \leq f(x)$ for $x \geq 1$

We calculate

$$\int_1^\infty f(x)dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx$$
$$= \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t$$
$$= \lim_{t\to\infty}\left[-\frac{1}{t}+1\right]$$
$$= -0+1=1$$

So $f(x)$ is convergent

By the comparison theorem if $f(x) \geq g(x) \geq 0$ for $x \geq a$ and if $\int_a^\infty f(x)dx$ is convergent then $\int_a^\infty g(x)dx$ is also convergent

Here $f(x) \geq g(x) \geq 0$ for $x \geq 1$

And we saw that $\int_1^\infty f(x)dx$ is convergent

So $\boxed{\int_1^\infty g(x)dx \text{ is also convergent}}$

(C) We graph the functions $f(x)$ and $g(x)$ for $1 \leq x \leq 10$

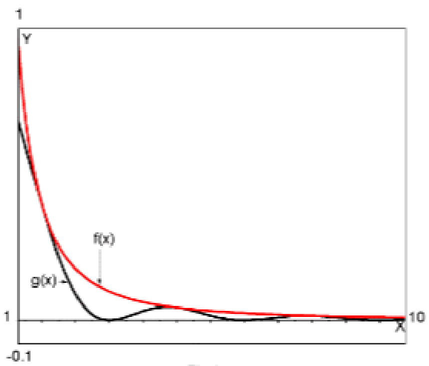


Fig.1

We see from the graph that the area under the graph of $f(x)$ for $x \geq 1$, is finite and since $f(x) \geq g(x)$, for $x \geq 1$, then the area under the graph of $g(x)$ must be finite on any interval [1, t]. Therefore $\int_1^\infty g(x)dx$ is convergent.

Answer 48E.

(A) We have $g(x) = \frac{1}{(\sqrt{x}-1)}$

Now we evaluate the values of $\int_2^t g(x)dx$ for $t = 5$, 10, 100, 1000 and 10000 with the help of computer and make a table

t	$\int_2^t g(x)dx$
5	3.830327
10	6.801200
100	23.32877
1000	69.02336
10000	208.1246

We see that the values are not approaching any fixed number so $\int_2^\infty g(x)\,dx$ is divergent

(B) For $x \geq 2$ $\quad \sqrt{x} \geq \sqrt{x}-1$ $\quad$ Or $\quad \frac{1}{\sqrt{x}} \leq \frac{1}{\sqrt{x}-1}$

Let $f(x) = \frac{1}{\sqrt{x}}$ then $0 \leq f(x) \leq g(x)$ for $x \geq 2$

Now $\int_2^\infty \frac{1}{\sqrt{x}}dx = \lim_{t\to\infty}\left[2.\sqrt{x}\right]_2^t$

$= \lim_{t\to\infty}\left[2\sqrt{t} - 2\sqrt{2}\right] = \infty$

So $\int_2^\infty f(x)dx$ is divergent

Thus we have $g(x) \geq f(x) \geq 0$ for $x \geq 2$ and $\int_2^\infty f(x)dx$ is divergent then by Comparison theorem

$\boxed{\int_2^\infty g(x)dx \text{ is also divergent}}$

(B) For $x \geq 2$ $\quad \sqrt{x} \geq \sqrt{x}-1$ $\quad$ Or $\quad \frac{1}{\sqrt{x}} \leq \frac{1}{\sqrt{x}-1}$

Let $f(x) = \frac{1}{\sqrt{x}}$ then $0 \leq f(x) \leq g(x)$ for $x \geq 2$

Now $\int_2^\infty \frac{1}{\sqrt{x}}dx = \lim_{t\to\infty}\left[2.\sqrt{x}\right]_2^t$

$= \lim_{t\to\infty}\left[2\sqrt{t} - 2\sqrt{2}\right] = \infty$

So $\int_2^\infty f(x)dx$ is divergent

Thus we have $g(x) \geq f(x) \geq 0$ for $x \geq 2$ and $\int_2^\infty f(x)dx$ is divergent then by Comparison theorem

$\boxed{\int_2^\infty g(x)dx \text{ is also divergent}}$

We see that from the area under the graph of $f(x)$ is infinite and $g(x) \geq f(x)$ for $x \geq 2$. So the area under the graph of $g(x)$ must be infinite.

Therefore $\int_2^{\infty} g(x)$ is divergent

Answer 49E.

Consider the integral,

$$\int_0^{\infty} \frac{x}{x^3+1} dx.$$

Use the Comparison Theorem to determine whether the integral is convergent or divergent.

Recalls that, suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^{\infty} f(x) dx$ converges then $\int_a^{\infty} g(x) dx$ converges.

(b) If $\int_a^{\infty} g(x) dx$ diverges then $\int_a^{\infty} f(x) dx$ diverges.

Rewrite the integral as,

$$\int_0^{\infty} \frac{x}{x^3+1} dx = \int_0^{1} \frac{x}{x^3+1} dx + \int_1^{\infty} \frac{x}{x^3+1} dx.$$

Observe that the first integral on the right – hand side is just an ordinary definite integral.

By *p*-series test, $\int_1^{\infty} \frac{1}{x^p} dx$ converges if p >1 and diverges if $p \leq 1$.

Compare the second integral on the interval $[1, \infty)$.

$$\frac{x}{(x^3+1)} < \frac{x}{x^3}$$
$$= \frac{1}{x^2}$$

Here, $p = 2 > 1$.

Observe that $\int_1^{\infty} \frac{1}{x^2} dx$ converges by *p*-series test.

By the comparison theorem, since $\int_1^{\infty} \frac{1}{x^2} dx$ converges.

Then, $\int_0^{\infty} \frac{x}{x^3+1} dx$ also converges.

Therefore, the integral $\int_0^{\infty} \frac{x}{x^3+1} dx$ converges.

Answer 50E.

For $x \geq 1$ $\quad 2+e^{-x} > 2$ $\quad$ since $e^{-x} > 0$ for all x

Then $\frac{2+e^{-x}}{x} > \frac{2}{x}$ $\quad$ on $[1, \infty)$

Let $f(x) = \frac{2+e^{-x}}{x}$ and $g(x) = \frac{2}{x}$

Now

$$\int_1^\infty g(x)\,dx = \int_1^\infty \frac{2}{x}\,dx$$
$$= \lim_{t\to\infty}\int_1^t \frac{2}{x}\,dx$$
$$= \lim_{t\to\infty}\left[2\ln x\right]_1^t$$
$$= \lim_{t\to\infty}(2\ln t) \qquad [\ln 1 = 0]$$
$$= \infty$$

So $\int_1^\infty g(x)\,dx$ is divergent for $x \geq 1$

Thus we have $f(x) \geq g(x) \geq 0$ and $\int_1^\infty g(x)\,dx$ is divergent for $x \geq 1$

Then by comparison theorem $\int_1^\infty f(x)\,dx = \int_1^\infty \frac{2+e^{-x}}{x}\,dx$ is $\boxed{\text{divergent}}$

Answer 51E.

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Consider the integral,

$$\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$$

The object is to use Comparison Theorem to determine whether the given integral is converges or not.

Recall that, If $f(x)$ and $g(x)$ are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ convergent.

If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Observe that $x+1 > x$ and $\sqrt{x^4-x} > \sqrt{x^4}$, $x > 0$

This implies that,

$$\frac{x+1}{\sqrt{x^4-x}} \geq \frac{x}{\sqrt{x^4}}$$
$$= \frac{1}{x}$$

Note that $f(x) \geq g(x)$ where $f(x) = \frac{x+1}{\sqrt{x^4-1}}$ and $g(x) = \frac{1}{x}$.

The object is to check whether the integral $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x}\,dx$ is convergent or not.

From the definition of an improper integral of type 1, then

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to+\infty}\int_1^t \frac{1}{x}\,dx$$
$$= \lim_{t\to+\infty}\left[\ln|x|\right]_1^t$$
$$= \lim_{t\to+\infty}(\ln t - \ln 1)$$
$$= \lim_{t\to+\infty}\ln t \qquad \text{Since } \ln 1 = 0$$
$$= \ln\infty$$
$$= \infty$$

The limit does not exist as a finite number and so the improper integral $\int_1^\infty \frac{1}{x}\,dx$ is divergent.

Hence, the integral $\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$ diverges by comparison theorem.

Answer 52E.

Consider the integral,

$$\int_0^\infty \frac{\arctan(x)}{2+e^x}dx$$

The objective is to determine whether the following integral is convergent or divergent by using comparison theorem.

Comparison theorem states that if *f* and *g* are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Consider the integrand,

$$\frac{\arctan(x)}{2+e^x}$$

Observe that,

$$-\frac{\pi}{2} < \arctan(x) < \frac{\pi}{2}$$
$$-\frac{\frac{\pi}{2}}{2+e^x} < \frac{\arctan(x)}{2+e^x} < \frac{\frac{\pi}{2}}{2+e^x}$$

Therefore, the integration $\int_0^\infty \frac{\arctan(x)}{2+e^x}dx$ can be written as,

$$\int_0^\infty \left(\frac{\frac{\pi}{2}}{2+e^x}\right)dx = \frac{\pi}{2}\int_0^\infty \frac{1}{2+e^x}dx$$
$$= \boxed{0.408}$$

Since integration $\int_0^\infty \left(\frac{\frac{\pi}{2}}{2+e^x}\right)dx$ is convergent, therefore the integral $\int_0^\infty \frac{\arctan(x)}{2+e^x}dx$ is also convergent.

Answer 53E.

Consider the integral,

$$\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}dx$$

The objective is to use comparison theorem to determine whether the given integral is convergent or divergent.

Comparison theorem states that, if *f* and *g* are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Consider the integrand,

$$\frac{\sec^2 x}{x\sqrt{x}}$$

We know that,

$$\sec^2 x > 1$$
$$\frac{\sec^2 x}{x\sqrt{x}} > \frac{1}{x\sqrt{x}}$$
$$f(x) > g(x)$$

Consider the integral,

$$\int_0^1 \left(\frac{1}{x\sqrt{x}}\right)dx = \lim_{R\to 0+} \int_R^\infty \frac{1}{x\sqrt{x}}dx$$
$$= \lim_{R\to 0+} \left.\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}\right|_R^1$$
$$= \boxed{\infty}$$

Since the integral $\int_0^1 \left(\frac{1}{x\sqrt{x}}\right)dx$ is divergent, therefore by Comparison Theorem the integral $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}dx$ is also divergent.

Answer 54E.

Consider the integral,

$$\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}dx$$

The objective is to use the comparison theorem to determine whether the given integral is convergent or divergent.

Comparison theorem states that if *f* and *g* are continuous functions with $f(x) \ge g(x) \ge 0$ for $x \ge a$.

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Consider the definite integral,

$$\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}} dx$$

We know that,

$$\sin^2 x \le 1$$
$$\frac{\sin^2 x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$$
$$g(x) \le f(x)$$

So, the integral

$$\int_0^1 \left(\frac{1}{\sqrt{x}}\right) dx = \lim_{R \to 0+} \int_R^1 \frac{1}{\sqrt{x}} dx$$
$$= \lim_{R \to 0+} \left. \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right|_R^1$$
$$= \boxed{2}$$

Since the integral $\int_0^1 \left(\frac{1}{\sqrt{x}}\right) dx$ is finite, therefore, the integral $\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}} dx$ is also convergent.

Answer 55E.

We have

$$\int_0^{\infty} \frac{1}{\sqrt{x}(1+x)} = \int_0^1 \frac{1}{\sqrt{x}(1+x)} dx + \int_1^{\infty} \frac{1}{\sqrt{x}(1+x)} dx$$

First we evaluate the integral $\int_0^1 \frac{1}{\sqrt{x}(1+x)} dx$

Let $x+1 = y \Rightarrow dx = dy$

When $x \to 0,\ y \to 1$ and when $x = 1, y = 2$

Then $\int_0^1 \frac{1}{\sqrt{x}(1+x)} dx = \lim_{t \to 1} \int_t^2 \frac{1}{y\sqrt{-1+y}} dy$

Using the formula $\int \frac{du}{u\sqrt{a+bu}} = \frac{2}{\sqrt{-a}} \tan^{-1} \sqrt{\frac{a+bu}{-a}} + C \quad$ if $\quad a < 0$

Then $\int_0^1 \frac{1}{\sqrt{x}(1+x)} dx = \lim_{t \to 1} \left[2\tan^{-1}\sqrt{y-1}\right]_t^2$

$$= \lim_{t \to 1} \left(2\tan^{-1} 1 - 2\tan^{-1}\sqrt{t-1}\right)$$
$$= \frac{2\pi}{4} - 0 = \frac{\pi}{2}$$

Now we evaluate the integral $\int_1^{\infty} \frac{1}{\sqrt{x}(1+x)} dx$

Using the same substitution as in first integral

When $x = 1, y = 2$ and $x \to \infty,\ \ y \to \infty$

Then $\int_1^{\infty} \frac{1}{\sqrt{x}(1+x)} dx = \lim_{u \to \infty} \int_2^u \frac{1}{y\sqrt{y-1}} dy$

$$= \lim_{u \to \infty} \left[2\tan^{-1}\sqrt{u-1} - 2\tan^{-1} 1\right]$$
$$= 2\lim_{u \to \infty} \tan^{-1}\sqrt{u-1} - \frac{\pi}{2}$$

Let $u-1=S^2$ so $u\to\infty$ then $S\to\infty$

So $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}(1+x)}dx = 2\lim_{S\to\infty}\tan^{-1}S - \frac{\pi}{2}$

$$= 2.\left(\frac{\pi}{2}\right) - \frac{\pi}{2}$$

$$= \frac{\pi}{2}$$

Then $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}(1+x)}dx = \frac{\pi}{2}+\frac{\pi}{2}$

$$= \pi$$

$$\Rightarrow \boxed{\int_1^\infty \frac{1}{\sqrt{x}(1+x)}dx = \pi}$$

Answer 56E.

We can write

$$\int_2^\infty \frac{1}{x\sqrt{x^2-4}}dx = \int_2^4 \frac{1}{x\sqrt{x^2-4}}dx + \int_4^\infty \frac{1}{x\sqrt{x^2-4}}dx$$

Now we evaluate the integral $\displaystyle\int_4^\infty \frac{1}{x\sqrt{x^2-4}}dx$

As first integral, we have

$$\int_4^\infty \frac{1}{x\sqrt{x^2-4}}dx = \lim_{S\to\infty}\left[\frac{1}{2}\sec^{-1}\frac{x}{2}\right]_4^S$$

$$= \frac{1}{2}\lim_{S\to\infty}\sec^{-1}\frac{S}{2} - \frac{1}{2}\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \int_4^\infty \frac{1}{x\sqrt{x^2-4}}dx = \frac{1}{2}\lim_{S\to\infty}\sec^{-1}\frac{S}{2} - \frac{\pi}{6}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{\pi}{6}$$

$$= \frac{\pi}{4} - \frac{\pi}{6}$$

Now we evaluate the integral $\displaystyle\int_4^\infty \frac{1}{x\sqrt{x^2-4}}dx$

As first integral, we have

$$\int_4^\infty \frac{1}{x\sqrt{x^2-4}}dx = \lim_{S\to\infty}\left[\frac{1}{2}\sec^{-1}\frac{x}{2}\right]_4^S$$

$$= \frac{1}{2}\lim_{S\to\infty}\sec^{-1}\frac{S}{2} - \frac{1}{2}\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \int_4^\infty \frac{1}{x\sqrt{x^2-4}}dx = \frac{1}{2}\lim_{S\to\infty}\sec^{-1}\frac{S}{2} - \frac{\pi}{6}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{\pi}{6}$$

$$= \frac{\pi}{4} - \frac{\pi}{6}$$

Then we have

$$\int_4^\infty \frac{1}{x\sqrt{x^2-4}}dx = \frac{\pi}{6}+\frac{\pi}{4}-\frac{\pi}{6}\boxed{=\frac{\pi}{4}}$$

Answer 57E.

If p = 1 then $\int_0^1 \frac{1}{x^p}dx$ is divergent

So let $p \neq 1$ then

$$\int_0^1 \frac{1}{x^p}dx = \lim_{t\to 0^+}\int_t^1 x^{-p}dx$$
$$= \lim_{t\to 0^+}\left[\frac{x^{-p+1}}{-p+1}\right]_t^1$$
$$= \frac{1}{(1-p)}\lim_{t\to 0^+}\left[1-\frac{1}{t^{p-1}}\right]$$

Now if p > 1 then p − 1 > 0 and $\frac{1}{t^{p-1}} \to \infty$ as $t \to 0^+$

Therefore $\int_0^1 \frac{1}{x^p}dx$ is divergent

If p < 1 then p − 1< 0 and $\frac{1}{t^{p-1}} = t^{1-p} \to 0$ as $t \to 0^+$

Then $\int_0^1 \frac{1}{x^p}dx = \frac{1}{1-p}(1-0)$

$$\Rightarrow \int_0^1 \frac{1}{x^p}dx = \frac{1}{1-p}$$

Thus $\int_0^1 \frac{1}{x^p}dx$ converges when p < 1 and the integral has the value $\frac{1}{1-p}$

Answer 58E.

If p = 1

$$\int_e^\infty \frac{1}{x\ln x}dx = \lim_{t\to\infty}\int_e^t \frac{1}{x\ln x}dx$$

Let $\ln x = y \Rightarrow \frac{1}{x}dx = dy$

When $x = e, y = 1$ and when $x = t, y = \ln t$

Then $\lim_{t\to\infty}\int_e^t \frac{1}{x\ln x}dx = \lim_{t\to\infty}\int_1^{\ln t}\frac{1}{y}dy$

$$= \lim_{t\to\infty}[\ln y]_1^{\ln t}$$
$$= \lim_{t\to\infty}[\ln(\ln t) - \ln(1)]$$
$$= \infty$$

Since $\ln t \to \infty$ as $t \to \infty$ so $\int_e^\infty \frac{1}{x(\ln x)^p}dx$ is divergent when $p = 1$

When p < 1

$$\lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)^p}dx = \lim_{t\to\infty}\int_1^{\ln t} y^{-p}dy$$
$$= \lim_{t\to\infty}\left[\frac{y^{-p+1}}{1-p}\right]_1^{\ln t}$$
$$= \lim_{t\to\infty}\left[\frac{1}{1-p}\left\{\frac{1}{(\ln t)^{p-1}} - 1\right\}\right]$$

Since p < 1 then p − 1 < 0 and then $\frac{1}{(\ln t)^{p-1}} \to \infty$ as $t \to \infty$ since $\ln t \to \infty$

as $t \to \infty$, So $\int_e^\infty \frac{1}{x(\ln x)^p}dx$ is divergent when $p < 1$

If $p > 1$ then $p - 1 > 0$ and then $\frac{1}{(\ln t)^{p-1}} \to 0$ as $t \to \infty$ since $\ln t \to \infty$ as $t \to \infty$

So $$\int_e^\infty \frac{1}{x(\ln x)^p} dx = \lim_{t\to\infty}\left[\frac{1}{1-p}\left\{\frac{1}{(\ln t)^{p-1}} - 1\right\}\right]$$
$$= \left[\frac{1}{1-p}\{0-1\}\right]$$
$$= \frac{1}{p-1}$$

Then $\int_e^\infty \frac{1}{x(\ln x)^p} dx$ is convergent when $\boxed{p>1}$ and $\boxed{\int_e^\infty \frac{1}{x(\ln x)^p} dx = \frac{1}{p-1}}$

Answer 59E.

If $p = -1$

Then $$\int_0^1 x^p \ln x\, dx = \int_0^1 \frac{\ln x}{x} dx$$
$$= \lim_{t\to 0^+}\int_t^1 \frac{\ln x}{x} dx$$

Let $\ln x = y \Rightarrow \frac{1}{x} dx = dy$

And when $x = t$, $y = \ln t$, when $x = 1$, $y = 0$

Then $$\lim_{t\to 0^+}\int_t^1 \frac{\ln x}{x} dx = \lim_{t\to 0^+}\int_{\ln t}^0 y\, dy$$
$$= \lim_{t\to 0^+}\left[\frac{y^2}{2}\right]_{\ln t}^0$$
$$= \lim_{t\to 0^+}\left[-\frac{(\ln t)^2}{2}\right]$$

Since $\ln t \to -\infty$ as $t \to 0^+$

So $\int_0^1 x^p \ln x\, dx$ is divergent when $p = -1$

If $p < -1$
$$\int_0^1 x^p \ln x\, dx = \lim_{t\to 0^+}\int_t^1 x^p \ln x\, dx$$

Using the formula $\int u^n \ln u\, du = \frac{u^{n+1}}{(n+1)^2}\left[(n+1)\ln u - 1\right] + C$

We have $$\int_0^1 x^p \ln x\, dx = \lim_{t\to 0^+}\left[\frac{x^{p+1}}{(p+1)^2}\{(p+1)\ln x - 1\}\right]_t^1$$
$$= \lim_{t\to 0^+}\left[-\frac{1}{(p+1)^2} - \frac{t^{p+1}}{(p+1)^2}\{(p+1)\ln t - 1\}\right]$$

When $p < -1 \Rightarrow p + 1 < 0$ then $\lim_{t\to 0^+} t^{p+1} = \infty$ and $\lim_{t\to 0^+}\ln t = -\infty$

So $\int_0^1 x^p \ln x\, dx$ is divergent when $p < -1$

If $p > -1$
Then $p + 1 > 0$
And so $\lim_{t\to 0^+} t^{p+1} = 0$

Then $\int_0^1 x^p \ln x\,dx = \lim_{t\to 0^+}\left[-\frac{1}{(p+1)^2} - \frac{t^{p+1}}{(p+1)^2}\{(p+1)\ln t - 1\}\right]$

$$= -\frac{1}{(p+1)^2} - 0$$

$$= -\frac{1}{(p+1)^2}$$

So $\int_0^1 x^p \ln x\,dx$ is convergent when $\boxed{p > -1}$ and $\boxed{\int_0^1 x^p \ln x\,dx = -\frac{1}{(p+1)^2}}$

Answer 60E.

(A) **For $n = 0$**

$$\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t e^{-x}dx$$

$$= \lim_{t\to\infty}\left[-e^{-x}\right]_0^t$$

$$= \lim_{t\to\infty}\left[-e^{-t} + 1\right]$$

Since $\lim_{t\to\infty} e^{-t} = 0$ then $\boxed{\int_0^\infty e^{-x}dx = 1}$ --- (1)

For $n = 1$

$$\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t x e^{-x}dx$$

$$= \lim_{t\to\infty}\left[-xe^{-x}\right]_0^t + \lim_{t\to\infty}\int_0^t e^{-x}dx \qquad \text{[Integrate by part]}$$

$$= \lim_{t\to\infty}\left[-t\,e^{-t} + 0\right] + \lim_{t\to\infty}\int_0^t e^{-x}dx$$

$$= 0 + 1 \qquad \text{From (1)} \qquad \left[\lim_{t\to\infty}\left(-te^{-t}\right) = 0\right]$$

$\boxed{\int_0^\infty xe^{-x}dx = 1!}$ --- (2)

For $n = 2$

$$\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t x^2 e^{-x}dx$$

$$= \lim_{t\to\infty}\left[-x^2 e^{-x}\right]_0^t + 2.\lim_{t\to\infty}\int_0^t xe^{-x}dx \qquad \text{[Integration by part]}$$

$$= 0 + 2.1 \qquad \text{[From (2)]}$$

$\Rightarrow \boxed{\int_0^\infty x^2 e^{-x}dx = 2!}$ --- (3)

For n = 3

$$\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t x^3 e^{-x}dx$$

$$= \lim_{t\to\infty}\left[-x^3 e^{-x}\right]_0^t + \lim_{t\to\infty}\int_0^t x^2 e^{-x}dx \qquad \text{[Integration by parts]}$$

$$= 0 + 3.2 \qquad \text{[From (3)]}$$
$$= 3.2.1 = 3!$$

Then $\boxed{\int_0^\infty x^3 e^{-x}dx = 3.2.1 = 3!}$ --- (4)

(B) From part (A) we can guess the value of $\int_0^\infty x^n e^{-x} dx$ when n is an arbitrary positive constant (integer) as

$$\boxed{\int_0^\infty x^n e^{-x} dx = n!}$$

(C) We see that for $n = 1$ [from part (A)]

$$\int_0^\infty xe^{-x} dx = 1!$$

So this statement is true for $n = 1$

We assume that it is true for $n = k$

Then $\int_0^\infty x^k e^{-x} dx = \lim_{t\to\infty} \int_0^t x^k e^{-x} dx = k!$ --- (5)

Now if $n = k + 1$

Then $\int_0^\infty x^{k+1} e^{-x} dx = \lim_{t\to\infty} \int_0^t x^{k+1} e^{-x} dx$

$$= \lim_{t\to\infty} \left[-x^{k+1} e^{-x}\right]_0^t + (k+1) \lim_{t\to\infty} \int_0^t x^k e^{-x} dx \quad \text{[Integration by part]}$$

$$= 0 + (k+1)\,k! \quad \text{[From (5)]}$$

$$\Rightarrow \int_0^\infty x^{k+1} e^{-x} dx = (k+1)!$$

Then by mathematical induction we have

$$\boxed{\int_0^\infty x^n e^{-x} dx = n!}$$ Proved

Answer 61E.

(A) We write $\int_{-\infty}^\infty x\,dx = \int_{-\infty}^0 x\,dx + \int_0^\infty x\,dx$

Now we evaluate the integrals on the right hand side

$$\int_{-\infty}^0 x\,dx = \lim_{t\to-\infty} \int_t^0 x\,dx$$

$$= \lim_{t\to-\infty} \left[\frac{x^2}{2}\right]_t^0$$

$$= \lim_{t\to-\infty} \left[-\frac{t^2}{2}\right] = -\infty$$

Since one of the integrals of right hand side is divergent then the integral $\int_{-\infty}^\infty x\,dx$ is divergent

(B) $\lim_{t\to\infty} \int_{-t}^t x\,dx = \lim_{t\to\infty} \left[\frac{x^2}{2}\right]_{-t}^t$

$$= \lim_{t\to\infty} \left[\frac{t^2}{2} - \frac{t^2}{2}\right]$$

$$= \lim_{t\to\infty} 0 = 0$$

[We can use the property of definite integrals of odd functions $\int_{-a}^a f(x)\,dx = 0$]

Thus $\lim_{t\to\infty} \int_{-t}^t x\,dx = 0$ so the integral $\int_{-\infty}^\infty x\,dx$ must be convergent but from part (A) we saw that $\int_{-\infty}^\infty x\,dx$ is divergent so this is a contradiction

So we can not define $\int_{-\infty}^\infty x\,dx = \lim_{t\to\infty} \int_{-t}^t x\,dx$

Answer 62E.

Consider the following average speed of molecules in an ideal gas:

$$\bar{v} = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2}\int_0^\infty v^3 e^{-M v^2/(2RT)} dv$$

Where M,R and T are the constants.

Apply some substitutions to simplify the constants.

$$\bar{v} = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2}\int_0^\infty v^3 e^{-Mv^2/(2RT)} dv$$

$$a = Mv^2/2RT$$

$$b = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2}$$

Therefore,

$$\bar{v} = b\int_0^\infty v^3 e^{-av^2} dv \qquad \text{...... (1)}$$

Use u-substitution method.

Let,

$$u = v^2$$

$$du = 2v\,dv$$

$$dv = \frac{1}{2v}du$$

Substitute these values in equation (1).

$$\bar{v} = \frac{1}{2}b\int_0^\infty ue^{-au} du$$

This can be written as,

$$\bar{v} = \frac{1}{2}b\lim_{t\to\infty}\int_0^t ue^{-au} du \qquad \text{......(2)}$$

Now, use integration by parts to solve the above improper integral.

Consider,

$$\int_0^t u\,e^{-au} du = \left[-\frac{ue^{-au}}{a}\bigg|_0^t + \int_0^t 1\cdot\frac{e^{-au}}{a}du\right]$$

$$= \left[-\frac{ue^{-au}}{a} - \frac{e^{-au}}{a^2}\right]_0^t$$

$$= \left[-\frac{v^2e^{-av^2}}{a} - \frac{e^{-av^2}}{a^2}\right]_0^t \qquad \left(\text{Since } u = v^2\right)$$

Substitute the above integral value in equation (2).

$$\bar{v} = \frac{1}{2}b\lim_{t\to\infty}\int_0^t ue^{au} du$$

$$= \frac{1}{2}b\lim_{t\to\infty}\left[-\frac{v^2e^{-av^2}}{a} - \frac{e^{-av^2}}{a^2}\right]_0^t$$

$$= \frac{1}{2}b\lim_{t\to\infty}\left[-\frac{t^2e^{-at^2}}{a} - \frac{e^{-at^2}}{a^2} - \left(-\frac{1}{a^2}\right)\right]$$

$$= \frac{1}{2}b\left(\frac{1}{a^2}\right) \qquad \left(\text{Since } \lim_{t\to\infty} e^{-t} = 0\right)$$

$$= \frac{b}{2a^2}$$

Now, plug in values for b and a in the above equation.

$$\bar{v} = \frac{b}{2a^2}$$

$$= \frac{\frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{\frac{3}{2}}}{2\left(\frac{M}{2RT}\right)^2}$$

$$= \frac{2}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{\frac{3}{2}-2}$$

$$= \frac{2}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{-\frac{1}{2}}$$

$$= \frac{2}{\sqrt{\pi}}\sqrt{\frac{2RT}{M}}$$

$$= \sqrt{\frac{8RT}{\pi M}}$$

Hence, $\boxed{\bar{v} = \sqrt{\frac{8RT}{\pi M}}}$.

Answer 63E.

In figure 1 the shaded region is $R = \left\{(x,y) \mid x \geq 1,\ 0 \leq y \leq \frac{1}{x}\right\}$

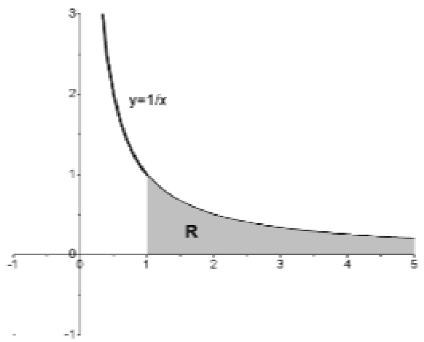


Fig.1

We consider a strip in this region vertically

If we rotate this region about x – axis, we get a typical disk with radius $\frac{1}{x}$

Then cross sectional area of disk $A(x) = \pi\left(\frac{1}{x}\right)^2 = \frac{\pi}{x^2}$

Then the volume of the solid

$$V = \int_1^\infty \frac{\pi}{x^2}\,dx$$

$$= \lim_{t\to\infty} \pi \int_1^t x^{-2}\,dx$$

$$= \pi \lim_{t\to\infty}\left[-x^{-1}\right]_1^t$$

$$= \pi \lim_{t\to\infty}\left[-\frac{1}{t}+1\right]$$

$$= \pi \lim_{t\to\infty}\left(-\frac{1}{t}\right) + \pi \lim_{t\to\infty} 1$$

$$= 0 + \pi$$

Thus volume of the solid is $\boxed{V = \pi}$ which is finite

Answer 64E.

The Newton's law of Gravitation states that two bodies with masses m_1 and m_2 attract each other with a force

$$F = G\frac{m_1 m_2}{r^2}$$

Where r is the distance between the two bodies and G is the gravitational constant.

Find the work needed to propel the space vehicle of weigh 1000kg from the Earth's gravitational field.

The required work is,

$$\begin{aligned} W &= \int_R^\infty F(r)dr \\ &= \int_R^\infty G\frac{m_1 m_2}{r^2}dr \\ &= Gm_1m_2\int_R^\infty \frac{1}{r^2}dr \\ &= Gm_1m_2\left[-\frac{1}{r}\right]_R^\infty \\ &= Gm_1m_2\left[\frac{1}{R}\right] \end{aligned}$$

Here,

$$\begin{aligned} &R = \text{Radius of the Earth} = 6.37\times10^6 E\,\text{m} \\ &m_1 = \text{Mass of the Earth} = 5.98\times10^{24}\ \text{kg} \\ &m_2 = \text{Mass of the Space vehicle} = 1000\ \text{kg} \\ &G = \text{Gravitational force} = 6.67\times10^{-11}\ \text{N.m}^2/\text{kg} \end{aligned}$$

Substitute these values in the above equation $W = Gm_1m_2\left[\frac{1}{R}\right]$.

$$\begin{aligned} W &= Gm_1m_2\left[\frac{1}{R}\right] \\ &= \left(6.67\times10^{-11}\ \text{N.m}^2/\text{kg}\right)\left(5.98\times10^{24}\ \text{kg}\right)\left(1000\ \text{kg}\right)\left[\frac{1}{6.37\times10^6\ \text{m}}\right] \\ &= 6.2481234\times10^{13}\ \text{Joules.} \qquad \text{Using a caluculator.} \end{aligned}$$

Therefore, the required work to propel the space vehicle of weigh 1000kg from the Earth's gravitational field is $\boxed{W = 6.2481234\times10^{13}\ \text{Joules.}}$

Answer 65E.

The Newton's law of Gravitation states that two bodies with masses m_1 and m_2 attract each other with a force

$$F = G\frac{m_1 m_2}{r^2}$$

Where r is the distance between the two bodies and G is the gravitational constant.

The required work done to propel a rocket of mass m out of the gravitational field of a planet with mass M and radius R is

$$\begin{aligned} W &= \int_R^\infty F(r)\,dr \\ &= \int_R^\infty G\frac{mM}{r^2}dr \\ &= \lim_{t\to\infty}\int_R^t G\frac{mM}{r^2}dr \\ &= \lim_{t\to\infty} GmM\int_R^t \frac{1}{r^2}dr \end{aligned}$$

Continue the above integration,

$$\begin{aligned}W &= GmM \lim_{t\to\infty}\left[-\frac{1}{r}\right]_R^t \\ &= GmM \lim_{t\to\infty}\left[-\frac{1}{t}+\frac{1}{R}\right] \\ &= \frac{GmM}{R} \qquad \lim_{t\to\infty}\frac{1}{t}=0.\end{aligned}$$

The fact that initial Kinetic energy of $\frac{1}{2}mv_0^2$ supplies the needed work.

So,

$$W=\frac{GmM}{R}=\frac{1}{2}mv_0^2$$

Do the little algebra, to get

$$\begin{aligned}\frac{GmM}{R}&=\frac{1}{2}mv_0^2 \\ v_0&=\sqrt{\frac{2GM}{R}}\end{aligned}$$

Therefore, the require escape velocity is $\boxed{v_0=\sqrt{\frac{2GM}{R}}}$.

Answer 66E.

Consider the function $x(r)=\frac{1}{2}(R-r)^2$

Substitute into the integral for the given function $x(r)=\frac{1}{2}(R-r)^2$ to begin:

$$\int_s^R \frac{2r}{\sqrt{r^2-s^2}}\left[\frac{1}{2}(R-r)^2\right]dr=\int_s^R\frac{r(R-r)^2}{\sqrt{r^2-s^2}}dr$$

Now, use a trigonometric substitution of $r=s\sec\theta$, which means $dr=s\sec\theta\tan\theta d\theta$.

$\int_s^R\frac{r(R-r)^2}{\sqrt{r^2-s^2}}dr=\int_s^R\frac{s\sec\theta(R-s\sec\theta)^2}{\sqrt{s^2\sec^2\theta-s^2}}(s\sec\theta\tan\theta)d\theta$ $=\int_s^R\frac{s\sec\theta(R-2Rs\sec\theta+s^2\sec^2\theta)}{\sqrt{s^2(\tan^2\theta)}}s\sec\theta\tan\theta d\theta$ $=\int_s^R\frac{s\sec\theta(R-2Rs\sec\theta+s^2\sec^2\theta)}{s\tan\theta}s\sec\theta\tan\theta d\theta$ $=s\int_s^R(R\sec^2\theta-2Rs\sec^3\theta+s^2\sec^4\theta)d\theta$	Square out the binomial. Factor out the common s^2 in the radical, and the trig id: $\tan^2\theta=\sec^2\theta-1$ Cancel the $s\tan\theta$ and distribute the $\sec\theta$.

Using a table of integrals at this point, we can get the integral of $\sec^4\theta$ reduced, and then fully integrate the other two.

$$\int\sec^4\theta d\theta=\frac{1}{3}\sec^2\theta\tan\theta+\frac{2}{3}\int\sec^2\theta d\theta=\frac{1}{3}\sec^2\theta\tan\theta+\frac{2}{3}\tan\theta$$

$\int\sec^3\theta d\theta=\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\ln|\sec\theta+\tan\theta|$ and

$$\int\sec^2\theta d\theta=\tan\theta$$

When $r=s$, we have $s=s\sec\theta$, which means $1=\sec\theta$ or $\theta=0$ for the lower limit.

When $r=R$, we have $R=s\sec\theta$, which means $\frac{R}{s}=\sec\theta$. Using the identity

$\tan^2\theta=\sec^2\theta-1$, we see that:

$\tan^2\theta=\left(\frac{R}{s}\right)^2-1=\frac{R^2-s^2}{s^2}$ and thus:

$\tan\theta=\frac{\sqrt{R^2-s^2}}{s}$ (we take the positive root because all values are positive in this case.

When $r=s$, we have $s=s\sec\theta$, which means $1=\sec\theta$ or $\theta=0$ for the lower limit.

When $r=R$, we have $R=s\sec\theta$, which means $\frac{R}{s}=\sec\theta$. Using the identity

$\tan^2\theta=\sec^2\theta-1$, we see that:

$\tan^2\theta=\left(\frac{R}{s}\right)^2-1=\frac{R^2-s^2}{s^2}$ and thus:

$\tan\theta=\frac{\sqrt{R^2-s^2}}{s}$ (we take the positive root because all values are positive in this case.

Thus, using the above substitutions for when $r=R$, we evaluate:

$$sR\tan\theta-2Rs^2\left(\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\ln|\sec\theta+\tan\theta|\right)+s^3\left(\frac{1}{3}\sec^2\theta\tan\theta+\frac{2}{3}\tan\theta\right)$$

$$=sR\left(\frac{\sqrt{R^2-s^2}}{s}\right)-2Rs^2\left(\frac{1}{2}\cdot\frac{R}{s}\cdot\frac{\sqrt{R^2-s^2}}{s}+\frac{1}{2}\ln\left(\frac{R+\sqrt{R^2-s^2}}{s}\right)\right)+$$

$$s^3\left(\frac{1}{3}\left(\frac{R}{s}\right)^2\left(\frac{\sqrt{R^2-s^2}}{s}\right)+\frac{2}{3}\frac{\sqrt{R^2-s^2}}{s}\right)$$

$$=R\sqrt{R^2-s^2}-R^2\sqrt{R^2-s^2}-Rs^2\ln\left(\frac{R+\sqrt{R^2-s^2}}{s}\right)+\frac{1}{3}R^2\sqrt{R^2-s^2}$$

$$+\frac{2}{3}s^2\sqrt{R^2-s^2}$$

Using the substitution of $\theta=0$ for when $r=s$, we have a much simpler evaluation:

$$R\sqrt{R^2-s^2}-R^2\sqrt{R^2-s^2}-Rs^2\ln\left(\frac{R+\sqrt{R^2-s^2}}{s}\right)+\frac{1}{3}R^2\sqrt{R^2-s^2}+\frac{2}{3}s^2\sqrt{R^2-s^2}$$

$$=R\sqrt{R^2-s^2}-\frac{2}{3}R^2\sqrt{R^2-s^2}-Rs^2\ln\left(\frac{R+\sqrt{R^2-s^2}}{s}\right)+\frac{2}{3}s^2\sqrt{R^2-s^2}$$

$$=R\sqrt{R^2-s^2}-Rs^2\ln\left(\frac{R+\sqrt{R^2-s^2}}{s}\right)-\frac{2}{3}\left(R^2-s^2\right)\sqrt{R^2-s^2}$$

Answer 67E.

Consider the data:

A manufacturer of light bulbs wants to produce bulbs that last about 700 hours but, of course, some bulbs burn out faster than others.

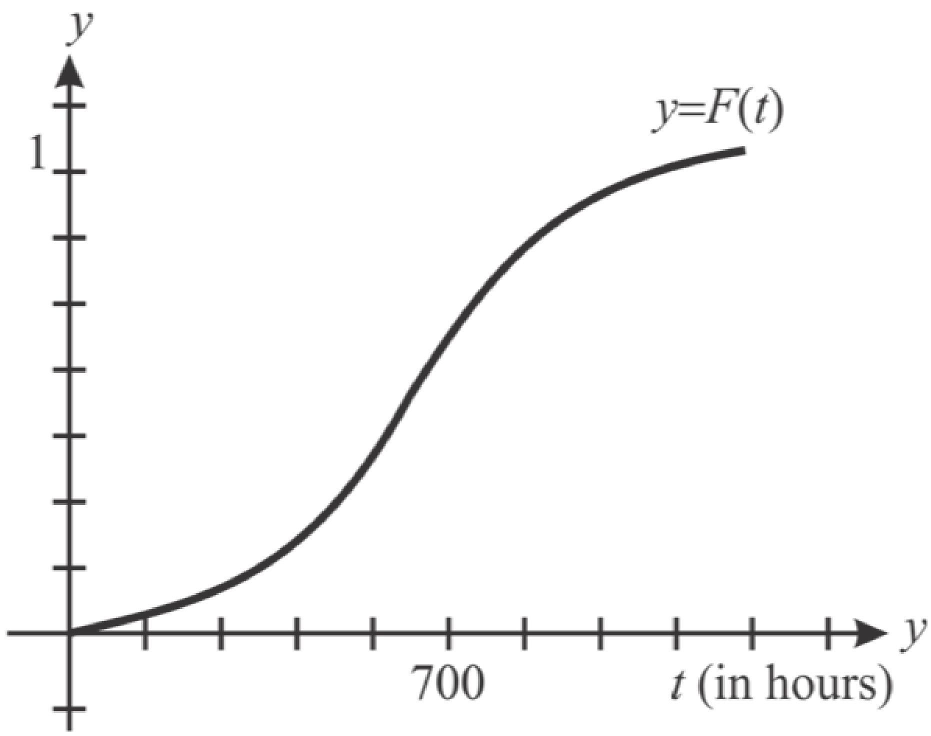
And $F(t)$ be the fraction of the company's bulbs that burn out before t hours, so $F(t)$ always lies between 0 and 1.

(a)

The objective is to draw the rough sketch of the function $F(t)$

Expect a small percentage of bulbs to burn out in the first few hundred hours, most of the bulbs to burn out after close to 700 hours, and a few awesome bulbs to burn on past 700 hours.

The rough sketch of the function, $F(t)$ is shown below:



(b)

The objective is to write the meaning of the derivative $r(t)=F'(t)$

The derivative $F'(t)$ is the rate at which the fraction of the burnt-out bulbs increases as t increases. So, it can be interpret $F'(t)$ as a fractional burnout rate.

(c)

From the data, $r(t)=F'(t)$

The objective is to find the value of the improper integral, $\int_0^\infty r(t)dt$

Now,

$$\int_0^\infty r(t)dt=\lim_{b\to\infty}\left(\int_0^b r(t)dt\right)$$
$$=\lim_{b\to\infty}\left(\int_0^b F'(t)dt\right)$$
$$=\lim_{b\to\infty}\left(\left[F(t)\right]_0^b\right)$$
$$=\lim_{b\to\infty}\left(\left[F(b)-F(0)\right]\right)$$
$$=\lim_{b\to\infty}F(b)-F(0)$$
$$=1-0$$
$$=1$$

Therefore, $\int_0^\infty r(t)dt=\boxed{1}$.

So, it can be interpreting this as the final limit is $\boxed{1}$, because all the bulbs will eventually burnout.

Answer 68E.

We have $M = -k\int_0^\infty t\, e^{kt}\, dt$

$$= -k \lim_{y\to\infty} \int_0^y t\, e^{kt}\, dt$$

$$= -k \lim_{y\to\infty} \left[\left\{ \frac{t e^{kt}}{k} \right\}_0^y + \frac{1}{k}\int_0^y e^{kt}\, dt \right]$$

$$= -k \left[\lim_{y\to\infty} \left\{ \frac{y e^{yk}}{k} - 0 \right\} - \frac{1}{k^2} \lim_{y\to\infty} \left[e^{kt} \right]_0^y \right]$$

$$= -k \lim_{y\to\infty} \left[\frac{y e^{yk}}{k} \right] + \frac{1}{k} \lim_{y\to\infty} \left[e^{yk} - e^0 \right]$$

$$= -k \lim_{y\to\infty} \left[\frac{y e^{yk}}{k} \right] + \frac{1}{k} \lim_{y\to\infty} \left(e^{yk} - 1 \right)$$

Since k = - 0.000121 < 0

Then $\lim_{y\to\infty} e^{yk} = 0$ for k < 0

Then $M = -k.0 + \frac{1}{k}(0-1)$

$$\Rightarrow M = -\frac{1}{k}$$

$$\Rightarrow M = -\frac{1}{-0.000121}$$

$$\Rightarrow \boxed{M \approx 8264.5} \text{ years}$$

Answer 69E.

Consider the integral inequality $\int_a^\infty \frac{1}{x^2+1}dx < 0.001$. ...... (1)

Determine how large the number a.

If $\int_a^t f(x)dx$ exists for every number $t \geq a$ then $\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$, provided this limit exists.

Rewrite the equation (1) as:

$$\int_a^\infty \frac{1}{x^2+1}dx = \lim_{t\to\infty}\int_a^t \frac{1}{x^2+1}dx$$
$$= \lim_{t\to\infty}\left[\tan^{-1}(x)\right]_a^t$$

$$\int_a^\infty \frac{1}{x^2+1}dx < 0.001 = \lim_{t\to\infty}\left[\tan^{-1} x\right]_a^t < 0.001$$

$$= \lim_{t\to\infty}\left[\tan^{-1} t - \tan^{-1} a\right] < 0.001$$

$$= \frac{\pi}{2} - \tan^{-1} a < 0.001$$

$$= \frac{\pi}{2} - 0.001 < \tan^{-1} a$$

$$= \tan\left(\frac{\pi}{2} - 0.001\right) < \tan\left(\tan^{-1} a\right)$$

$$= a > \tan\left(\frac{\pi}{2} - 0.001\right)$$

$$= a > 999.9996667$$

Therefore the number a is 1000.

Answer 70E.

Consider the integral $\int_0^{\infty} e^{-x^2}\,dx$.

Rewrite the given integral as $\int_0^{\infty} e^{-x^2}\,dx = \int_0^{4} e^{-x^2}\,dx + \int_4^{\infty} e^{-x^2}\,dx$.

Approximate the integral $\int_0^{4} e^{-x^2}\,dx$ using Simpson's rule.

Simpson's rule is,

$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$ Where

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{4-0}{8} \quad \text{Since } n=8$$

$$= \frac{1}{2}$$

Substitute the values in the Simpson's rule.

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$$

$$\int_0^4 e^{-x^2}dx \approx S_8 = \frac{1}{2\cdot 3}\begin{bmatrix} f(0)+4f(0.5)+2f(1)+4f(1.5)+2f(2)+4f(2.5) \\ +2f(3)+4f(3.5)+f(4) \end{bmatrix}$$

$$= \frac{1}{2\cdot 3}\left[e^{-0^2}+4e^{-0.5^2}+2e^{-1^2}+4e^{-1.5^2}+2e^{-2^2}+4e^{-2.5^2}+2e^{-3^2}+4e^{-3.5^2}+e^{-4^2}\right]$$

$$= \frac{1}{6}\begin{bmatrix} 1+4(0.778)+2(0.37)+4(0.105)+2(0.018)+4(0.001) \\ +2[0.0001]+4(4.79\times10^{-6})+f(1.125\times10^{-7}) \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 1+4(0.778)+2(0.37)+4(0.105)+2(0.018)+4(0.001) \\ +2[0.0001]+4(4.79\times10^{-6})+f(1.125\times10^{-7}) \end{bmatrix}$$

$$= 0.8862$$

The comparison theorem is,

Suppose the functions f and g are continuous functions with $f(x) \ge g(x) \ge 0$ for $x \ge a$.

(1) If $\int_a^{\infty} f(x)dx$ is convergent, then $\int_a^{\infty} g(x)dx$ is convergent.

(2) If $\int_a^{\infty} g(x)dx$ is divergent, then $\int_a^{\infty} f(x)dx$ is divergent.

Consider $x \ge 4$

$$x \ge 4$$

$$x^2 \ge 4x$$

$$-x^2 \le -4x$$

$$e^{-x^2} \le e^{-4x}$$

$$\int_4^{\infty} e^{-x^2}dx \le \int_4^{\infty} e^{-4x}$$

Find the integral $\int_4^\infty e^{-4x}$

$$\int_4^\infty e^{-4x} = \lim_{t\to\infty}\int_4^t e^{-4x}dx$$
$$= \lim_{t\to\infty}\left(\frac{e^{-4x}}{-4}\right)_4^t$$
$$= \lim_{t\to\infty}\left(-\left(\frac{e^{-4(t)}}{4}-\frac{e^{-4(4)}}{4}\right)\right)$$
$$= -\lim_{t\to\infty}\frac{e^{-4(t)}}{4}+\lim_{t\to\infty}\frac{e^{-4(4)}}{4}$$
$$= 0+\frac{e^{-16}}{4}$$
$$= \frac{e^{-16}}{4}$$
$$\approx 0.00000002$$

$f(x)=e^{-4x}$ and $g(x)=e^{-x^2}$ in the comparison theorem.

$$\int_4^\infty e^{-x^2}dx \le \int_4^\infty e^{-4x} \approx 0.00000002$$

Rewrite the above equation as

$$\int_4^\infty e^{-x^2}dx \le \int_4^\infty e^{-4x} < 0.0000001$$

Therefore, the second integral is smaller than $\int_4^\infty e^{-4x}$ which is less than 0.0000001.

Answer 71E.

Definition of the Laplace transform of the function $f(t)$ is,

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt$$

Generalisation of $L(f(t)) = F(s)$

Where, the function *F* is the domain of the set of all real numbers *s*.

The *s* is integral converges.

Here, $f(t)$ is continuous for $t \ge 0.$

(a)

Find the Laplace transform of the function $f(t)=1$

By definition of improper integrals, if $\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$.

$$L(1) = \lim_{t\to\infty}\int_0^t 1\cdot e^{-st}dt$$
$$= \lim_{t\to\infty}\int_0^t e^{-st}dt$$
$$= \lim_{t\to\infty}\left[\frac{e^{-st}}{-s}\right]_0^t$$
$$= -\frac{1}{s}\lim_{t\to\infty}\left[e^{-st}\right]_0^t$$
$$= -\frac{1}{s}\lim_{t\to\infty}\left[e^{-st}-e^{-0}\right]$$
$$= -\frac{1}{s}\left(\lim_{t\to\infty}e^{-st}-1\right)$$
$$= \frac{1}{s}$$

Therefore, the Laplace transform of the function $f(t)=1$ is $\boxed{\frac{1}{s}}$

(b)

Find the Laplace transform of the function $f(t)=e^t$

$$L(e^t)=\lim_{t\to\infty}\int_0^t e^t\cdot e^{-st}dt$$
$$=\lim_{t\to\infty}\int_0^t e^{-(s-1)t}dt$$
$$=\lim_{t\to\infty}\left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^t$$
$$=-\frac{1}{(s-1)}\lim_{t\to\infty}\left[e^{-(s-1)t}-e^{-(s-1)0}\right]$$
$$=-\frac{1}{(s-1)}\lim_{t\to\infty}\left[e^{-(s-1)t}-1\right]$$
$$=-\frac{1}{(s-1)}\left(\lim_{t\to\infty}e^{-(s-1)t}-1\right)$$
$$=-\frac{1}{(s-1)}(0-1)$$
$$=\frac{1}{s-1}$$

Therefore, the Laplace transform of the function $f(t)=e^t$ is $\boxed{\frac{1}{s-1}}$

(c)

Find the Laplace transform of the function $f(t)=t$

$$L(t)=\lim_{t\to\infty}\int_0^t t\cdot e^{-st}dt$$
$$=\lim_{t\to\infty}\left[\left[t\cdot\frac{e^{-st}}{-s}\right]_0^t-\int_0^t 1\cdot\frac{e^{-st}}{-s}dt\right]$$
$$=\lim_{t\to\infty}\left[\left[t\frac{e^{-st}}{-s}-0\right]-\left[\frac{e^{-st}}{s^2}\right]_0^t\right]$$
$$=\lim_{t\to\infty}\left[\left[t\frac{e^{-st}}{-s}-0\right]-\left[\frac{e^{-st}}{s^2}-\frac{1}{s^2}\right]\right]$$
$$=\frac{1}{s^2}$$

Therefore, the Laplace transform of the function $f(t)=t$ is $\boxed{\frac{1}{s^2}}$.

Answer 72E.

Consider the continuous function $f(t)$ such that $0\le f(t)\le Me^{at}$ where M and a are constants

To prove that the Laplace transform of $f(t)$ is exists for $s>a$

By the definition of Laplace transform

$$F(s)=\int_0^\infty f(t)e^{-st}dt$$

The function $F(s)$ is defined when integral convergent

The given inequality is

$$0\le f(t)\le Me^{at}$$
$$0\cdot e^{-st}\le f(t)\cdot e^{-st}\le Me^{at}\cdot e^{-st}$$
$$0\le f(t)e^{-st}\le Me^{-(s-a)t}$$

Consider the following integral for $s > a$

$$\int_0^\infty Me^{-(s-a)t}dt = \lim_{p\to\infty}\int_0^p Me^{-(s-a)t}dt$$

$$= \lim_{p\to\infty}\left[\frac{Me^{-(s-a)t}}{-(s-a)}\right]_0^p$$

$$= \lim_{p\to\infty}\left[\frac{Me^{-(s-a)p}}{-(s-a)} - \left(\frac{Me^{-(s-a)0}}{-(s-a)}\right)\right]$$

$$= \lim_{p\to\infty}\left[\frac{Me^{-(s-a)p}}{-(s-a)} + \frac{M}{(s-a)}\right]$$

As $p \to \infty$, $-(s-a)p \to -\infty$ then $e^{-(s-a)p} \to 0$

$$\lim_{p\to\infty}\left[\frac{Me^{-(s-a)p}}{-(s-a)} + \frac{M}{(s-a)}\right] = 0 + \frac{M}{(s-a)}$$

Consider the following integral for $s > a$

$$\int_0^\infty Me^{-(s-a)t}dt = \lim_{p\to\infty}\int_0^p Me^{-(s-a)t}dt$$

$$= \lim_{p\to\infty}\left[\frac{Me^{-(s-a)t}}{-(s-a)}\right]_0^p$$

$$= \lim_{p\to\infty}\left[\frac{Me^{-(s-a)p}}{-(s-a)} - \left(\frac{Me^{-(s-a)0}}{-(s-a)}\right)\right]$$

$$= \lim_{p\to\infty}\left[\frac{Me^{-(s-a)p}}{-(s-a)} + \frac{M}{(s-a)}\right]$$

As $p \to \infty$, $-(s-a)p \to -\infty$ then $e^{-(s-a)p} \to 0$

$$\lim_{p\to\infty}\left[\frac{Me^{-(s-a)p}}{-(s-a)} + \frac{M}{(s-a)}\right] = 0 + \frac{M}{(s-a)}$$

Answer 73E.

Consider the following function $f(t)$ such that

$0 \le f(t) \le Me^{at}, 0 \le f'(t) \le Ke^{at}$ for $t \ge 0$

And Laplace transform of $f(t)$ is $F(s)$ and Laplace transform of $f'(t)$ is $G(s)$

By the definition of Laplace transform

$$G(s) = sF(s) - f(0), s > a$$

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

$$G(s) = \int_0^\infty f'(t)e^{-st}dt$$

Recollect integral by parts

$$\int f(x)g(x)dx = \left[\int f(x)dx\right]g(x) - \int g'(x)\left[\int f(x)dx\right]dx$$

$$\int f'(t)e^{-st}dt = \left[\int f'(t)dt\right]e^{-st} - \int\left(e^{-st}\right)'\left[\int f'(t)dt\right]dt$$

$$\int f'(t)e^{-st}dt = \left[f(t)\right]e^{-st} - \int\left(-se^{-st}\right)\left[f(t)\right]dt$$

$$\int f'(t)e^{-st}dt = \left[f(t)\right]e^{-st} + s\int e^{-st}f(t)dt$$

$$\int_0^\infty f'(t)e^{-st}dt = \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty e^{-st}f(t)dt$$

But it is given that

$$0 \le f(t) \le Me^{at}$$
$$0 \cdot e^{-st} \le f(t)e^{-st} \le Me^{at} \cdot e^{-st}$$
$$0 \le f(t)e^{-st} \le Me^{at} \cdot e^{-st}$$
$$0 \le f(t)e^{-st} \le Me^{-(s-a)t}$$

if $s > a$ then as $t \to \infty$, $e^{-(s-a)t} \to 0$

$$\lim_{t\to\infty} 0 \le \lim_{t\to\infty} f(t)e^{-st} \le \lim_{t\to\infty} Me^{-(s-a)t}$$

$\lim_{t\to\infty} f(t)e^{-st} = 0$ and $\lim_{t\to 0} f(t)e^{-st} = f(0)e^{-s(0)} = f(0)$

$$\int_0^\infty f'(t)e^{-st}dt = \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

$$\int_0^\infty f'(t)e^{-st}dt = 0 - f(0) + sF(s)$$

$$\boxed{G(s) = sF(s) - f(0) \text{ if } s > a}$$

Answer 74E.

Consider $\int_{-\infty}^{\infty} f(x)dx$ is convergent and *a* and *b* are real numbers,

Prove the following equality.

$$\int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx = \int_{-\infty}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx \quad \text{...... (1)}$$

Case1:

Assume that $b > a$ without any loss of generality.

Rewrite the left hand side as:

$$\int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx \quad \text{...... (2)}$$

Rewrite the right hand side as:

$$\int_{-\infty}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx \quad \text{...... (3)}$$

Equations (2) and (3) are equal.

Therefore

$$\int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx = \int_{-\infty}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx.$$

Case2:

Assume that $a > b$ without any loss of generality.

Rewrite the left hand side as:

$$\int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx = \int_{-\infty}^{b} f(x)dx + \int_{b}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx \quad \text{...... (4)}$$

Rewrite the right hand side as:

$$\int_{-\infty}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx = \int_{-\infty}^{b} f(x)dx + \int_{b}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx \quad \text{...... (5)}$$

Equations (4) and (5) are equal.

Case3:

Assume that $a = b$ without any loss of generality.

Replace a with b in equation (1)

$$\int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx \text{ True}$$

Therefore, in any case $\int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx = \int_{-\infty}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx$ is true.

Answer 75E.

Consider the improper integral $\int_0^\infty x^2 e^{-x^2}\,dx$ ...... (1)

Rewrite the integral (1) as:

$$\int_0^\infty x^2 e^{-x^2}\,dx = -\frac{1}{2}\int_0^\infty x\cdot\left(-2xe^{-x^2}\right)dx \quad ...... (2)$$

Let $u = x$ and $dv = -2xe^{-x^2}\,dx$

Then $du = dx$ and $v = \int -2xe^{-x^2}\,dx$

$$= \int e^{-x^2}\,d\left(-x^2\right)$$

$$= e^{-x^2} \qquad \left(\text{Since } \int e^t\,dt = e^t\right)$$

Formula for integration by parts:

$$\int u\,dv = uv - \int v\,du,$$

The integral becomes

$$\int x\cdot\left(-2xe^{-x^2}\right)dx = xe^{-x^2} - \int e^{-x^2}\,dx$$

Substitute the limits of the integration.

$$\int_0^\infty x\cdot\left(-2xe^{-x^2}\right)dx = \left[xe^{-x^2}\right]_0^\infty - \int_0^\infty e^{-x^2}\,dx$$

$$= \left[\infty\cdot 0 - 0\cdot 1\right] - \int_0^\infty e^{-x^2}\,dx$$

$$= -\int_0^\infty e^{-x^2}\,dx$$

$$\int_0^\infty x\cdot\left(-2xe^{-x^2}\right)dx = -\int_0^\infty e^{-x^2}\,dx \quad ...... (3)$$

From the equation (2)

$$\int_0^\infty x^2 e^{-x^2}\,dx = -\frac{1}{2}\int_0^\infty x\cdot\left(-2xe^{-x^2}\right)dx$$

$$\int_0^\infty x^2 e^{-x^2}\,dx = -\frac{1}{2}\left(-\int_0^\infty e^{-x^2}\,dx\right) \text{ From (3)}$$

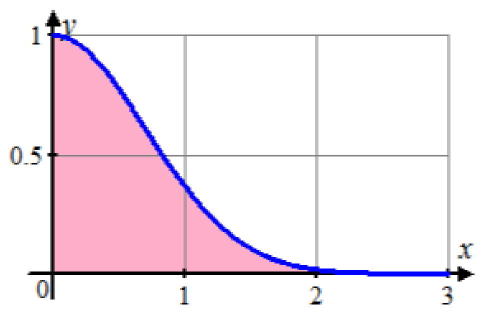
$$= \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

Therefore $\int_0^\infty x^2 e^{-x^2}\,dx = \boxed{\frac{1}{2}\int_0^\infty e^{-x^2}\,dx}$

Answer 76E.

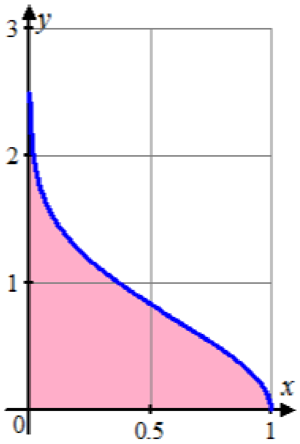
Give the graphical interpretation to show that $\int_0^\infty e^{-x^2}\,dx = \int_0^1 \sqrt{-\ln y}\,dy.$

First, sketch graph of the function, $y = e^{-x^2}$ and shade the enclosed region with *x*-axis and *y*-axis.



The area of the shaded region in the above diagram is represented by the integral, $\int_0^\infty e^{-x^2}\,dx.$

First, sketch graph of the function $y = \sqrt{-\ln x}$ in the interval, 0 to 1, and shade the enclosed region with *x*-axis and *y*-axis.



The area of the shaded region in the above diagram is represented by the integral,
$\int_0^1 \sqrt{-\ln y}\, dy$.

Observe these diagrams carefully. If we rotate the first diagram to 90 degrees angle and then take its mirror image, it will be the same as the second diagram. It follows that the area of the shaded regions in both the diagrams is equal.

This implies $\boxed{\int_0^\infty e^{-x^2} dx = \int_0^1 \sqrt{-\ln y}\, dy}$.

Answer 77E.

Consider the improper integral $\int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2} \right) dx.$

Find the value of the constant C for which this integral converges.

Using the definition $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$ with $a = 0$, we have

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2} \right) dx = \lim_{t\to\infty} \int_0^t \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2} \right) dx$$

Since $\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x + \sqrt{x^2+a^2}\right)$, the above integral becomes

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2} \right) dx = \lim_{t\to\infty} \left[\ln\left(x + \sqrt{4+x^2}\right) - C\ln|x+2| \right]_0^t$$

$$= \lim_{t\to\infty} \left[\ln\left(t + \sqrt{4+t^2}\right) - C\ln|t+2| - \ln 2 + C\ln 2 \right]$$

$$= \lim_{t\to\infty} \left[\ln\left(t + \sqrt{4+t^2}\right) - \ln|t+2|^C - \ln 2 + C\ln 2 \right]$$

$$= \lim_{t\to\infty} \left(\ln \frac{\left(t + \sqrt{4+t^2}\right)}{(t+2)^C} - \ln 2 + C\ln 2 \right)$$

$$= \lim_{t\to\infty} \left(\ln \frac{t\left(1 + \sqrt{\frac{4}{t^2}+1}\right)}{t^C\left(1+\frac{2}{t}\right)^C} \right) - \ln 2 + C\ln 2$$

$$= \lim_{t\to\infty} \left(\ln \frac{t^{1-C}\left(1 + \sqrt{\frac{4}{t^2}+1}\right)}{\left(1+\frac{2}{t}\right)^C} \right) - \ln 2 + C\ln 2 \qquad \text{......(1)}$$

In the above integral as $t \to \infty$, $\frac{1}{t} \to 0$. So in order to have the converge of the above integral we must have

$$1 - C = 0$$
$$\Rightarrow \boxed{C = 1}$$

If $C = 1$, then from the integral (1), we have

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2} \right) dx = \left(\ln \frac{t^0 \left(1+\sqrt{0+1}\right)}{(1+0)^1} \right) - \ln 2 + (1)\ln 2$$
$$= \left(\ln \frac{(1+1)}{1} \right) - \ln 2 + (1)\ln 2$$
$$= \ln 2 - \ln 2 + \ln 2$$
$$= \ln 2$$

Hence the value of the integral is $\boxed{\ln 2}$.

Answer 78E.

Consider the following improper integral:

$$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{c}{3x+1} \right) dx$$

The objective is to find the value of constant '*c*' for which the integral converges.

Use the definition $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$ with $a = 0$.

$$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1} \right) dx = \lim_{t\to\infty} \int_0^t \left(\frac{x}{x^2+1} - \frac{C}{3x+1} \right) dx$$
$$= \lim_{t\to\infty} \left(\frac{1}{2} \int_0^t \frac{2x}{x^2+1} dx - \frac{C}{3} \int_0^t \frac{3}{3x+1} dx \right)$$
$$= \lim_{t\to\infty} \left[\frac{1}{2} \ln\left(1+x^2\right) - \frac{C}{3} \ln|3x+1| \right]_0^t$$
$$= \lim_{t\to\infty} \left[\ln\left(1+x^2\right)^{\frac{1}{2}} - \ln|3x+1|^{\frac{C}{3}} \right]_0^t$$

Continuing of the above integration,

$$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{c}{3x+1} \right) dx = \lim_{t\to\infty} \left[\ln \frac{\sqrt{1+x^2}}{|3x+1|^{\frac{C}{3}}} \right]_0^t$$
$$= \lim_{t\to\infty} \left[\ln \frac{\sqrt{1+t^2}}{(3t+1)^{\frac{C}{3}}} - \ln \frac{1}{1} \right]$$
$$= \lim_{t\to\infty} \ln \frac{\sqrt{t^2+1}}{(3t+1)^{\frac{C}{3}}} \qquad (\text{since } \ln 1 = 0)$$
$$= \lim_{t\to\infty} \ln \frac{t\sqrt{1+\frac{1}{t^2}}}{t^{\frac{C}{3}}\left(3+\frac{1}{t}\right)^{\frac{C}{3}}}$$
$$= \lim_{t\to\infty} \ln \frac{t^{1-\frac{C}{3}}\sqrt{1+\frac{1}{t^2}}}{\left(3+\frac{1}{t}\right)^{\frac{C}{3}}} \quad \ldots\ldots (1)$$

The integral as $t \to \infty$, $\frac{1}{t} \to 0$.

When the integral is converges, the power of *t* value is,

$$1 - \frac{C}{3} = 0$$
$$C = 3$$

Substitute for $C = 3$ in equation (1).

$$\int_0^{\infty} \left(\frac{x}{x^2+1} - \frac{C}{3x+1} \right) dx = \ln \left[\frac{t^0 \sqrt{1+0}}{(3+0)^{\frac{3}{3}}} \right]$$
$$= \ln \left(\frac{1}{3} \right)$$

Therefore, the integral value is $\boxed{\ln \left(\frac{1}{3} \right)}$.

Answer 79E.

Consider a function $f(x) = \exp(\exp(-x))$.

Clearly this function is a continuous function for all values of *x* as it contains only exponential terms.

It means the function $f(x) = \exp(\exp(-x))$ continuous on $[0, \infty)$.

Consider the limit of the function.

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \exp(\exp(-x))$$
$$= \exp(\exp(-\infty))$$
$$= \exp(0) \quad \text{Since } \exp(-\infty) = 0$$
$$= 1$$

Hence $\lim_{x \to \infty} f(x) = 1$.

This follows that the function is satisfying all the given conditions.

Consider the improper integral $\int_0^{\infty} f(x) dx$.

$$\int_0^{\infty} f(x) dx = \int_0^{\infty} \exp(\exp(-x)) dx$$

Clearly this integral is not convergent as there is no function $g(x)$ in the interval $[0, \infty)$

such that $f(x) \le g(x)$ and $\int_0^{\infty} g(x) dx$ is convergent.

Therefore existence of the desired function is not possible.

Answer 80E.

Consider the following integral:

$$\int_0^\infty \frac{x^a}{1+x^b}dx,\ \text{if } a>-1 \text{ and b>a+1}$$

The objective is to find the integral $\int_0^\infty \frac{x^a}{1+x^b}dx$ is convergent.

Use the definition of improper integral,

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$

$$\int_0^\infty \frac{x^a}{1+x^b}dx = \lim_{t\to\infty}\int_0^t \frac{x^a}{1+x^b}dx$$

Apply the condition $a>-1$ on the right hand side integrand.

$$\frac{x^a}{1+x^b},\text{if } a>-1$$

$$\frac{x^a}{1+x^b}<\frac{x^a}{x^b}$$

$$\lim_{t\to\infty}\int_0^\infty \frac{x^a}{1+x^b}dx < \lim_{t\to\infty}\int_0^t \frac{x^a}{x^b}dx$$

$$< \lim_{t\to\infty}\int_0^t x^{a-b}dx$$

$$= \lim_{t\to\infty}\left[\frac{x^{a-b+1}}{a-b+1}\right]_0^t \quad \left(\text{use formula } \int x^n dx = \frac{x^{n+1}}{n+1}\right)$$

$$= \lim_{t\to\infty}\left[\frac{t^{a-b+1}}{a-b+1}-0\right]$$

$$= \lim_{t\to\infty}\left[\frac{t^{a-b+1}}{a-b+1}\right]$$

$$\lim_{t\to\infty}\int_0^t \frac{x^a}{1+x^b}dx < \lim_{t\to\infty}\left[\frac{t^{a-b+1}}{a-b+1}\right]$$

When the power of *t* is negative, use the condition.

$$a+1<b$$

$$a-b+1<0$$

Hence the $\lim_{t\to\infty}\left[\frac{1}{a-b+1}t^{a-b+1}\right]$ is exists.

Therefore, the integral is $\int_0^\infty \frac{x^a}{1+x^b}dx$ convergent to the condition $a>-1, a+1<b$.