# Mathematics & Statistics [Set 1]

Academic Year: 2018-2019 Marks: 80
Date & Time: 2nd March 2019, 11:00 am
Duration: 3h

## SECTION –A

**Question 1: Select and write the most appropriate answer from the given alternatives for each question:** **[1]**

$\frac{\pi}{6}, \frac{5\pi}{6}$
$\frac{5\pi}{6}, \frac{7\pi}{6}$
$\frac{5\pi}{6}, \frac{11\pi}{6}$
$\frac{\pi}{6}, \frac{11\pi}{6}$

**Solution:**
$\frac{5\pi}{6}, \frac{11\pi}{6}$
cot x = $\sqrt{3}$
cot x = $-\cot\left(\frac{\pi}{6}\right) = \cot\left(\pi - \frac{\pi}{6}\right) = \cot\left(2\pi - \frac{\pi}{6}\right)$
= $\cot\left(\frac{5\pi}{6}\right) = \cot\left(\frac{11\pi}{6}\right)$

**Question 2:** The acute angle between the two planes x+y+2z = 3 and 3x -2y +2z = 7 ________. **[1]**

$\sin^{-1}\left(\frac{5}{\sqrt{102}}\right)$
$\cos^{-1}\left(\frac{5}{\sqrt{102}}\right)$
$\sin^{-1}\left(\frac{15}{\sqrt{102}}\right)$
$\cos^{-1}\left(\frac{15}{\sqrt{102}}\right)$

**Solution:**

$$\cos^{-1}\left(\frac{5}{\sqrt{102}}\right)$$

$$\cos\theta = \left|\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right|$$

$$\cos\theta = \left|\frac{(1)(3) + (1)(-2) + (2)(2)}{\sqrt{1^2 + 1^2 + 2^2}\sqrt{3^2 + 2^2 + 2^2}}\right|$$

$$= \left|\frac{3 - 2 + 4}{\sqrt{6}\sqrt{17}}\right|$$

$$= \left|\frac{5}{\sqrt{102}}\right|$$

**Question 3:** The direction ratios of the line which is perpendicular to the lines with direction ratios –1, 2, 2 and 0, 2, 1 are _______. **[1]**

–2, –1, –2
2, 1, 2
2, –1, –2
–2, 1, –2

**Solution:**

–2, 1, –2

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= \hat{i}(2 - 4) - \hat{j}(-1) + \hat{k}(-2)$$

$$= -2\hat{i} + \hat{j} - 2\hat{k}$$

**Question 4:** **[1]**

If f(x) = $(1 + 2x)^{\frac{1}{x}}$ for x≠0 is continuous at x = 0 then f(0) = _____.

e
$e^2$
0
2

**Solution:**

$e^2$

$f(0) = \lim_{x \to 0} \left[(1+2x)^{\frac{1}{2x}}\right]^2 = e^2$

**Question 5:** **[1]**

$\int \frac{dx}{9x^2+1} =$ ___.

$\frac{1}{3}\tan^{-1}(2x) + c$

$\frac{1}{3}\tan^{-1}x + c$

$\frac{1}{3}\tan^{-1}(3x) + c$

$\frac{1}{3}\tan^{-1}(6x) + c$

**Solution:**

$\frac{1}{3}\tan^{-1}(3x) + c$

Let I = $\int \frac{dx}{9x^2+1}$

$= \frac{1}{9}\int \frac{dx}{(x^2) + \left(\frac{1}{3}\right)^2}$

$= \frac{1}{9}\frac{1}{\frac{1}{3}}\tan^{-1}\left(\frac{x}{\frac{1}{3}}\right) + C$

$= \frac{1}{3}\tan^{-1}(3x) + c$

**Question 6:** **[1]**

If $y = ae^{5x} + be^{-5x}$ then the differential equation is _______.

$\frac{d^2y}{dx^2} = 25y$

$\frac{d^2y}{dx^2} = -25y$

$\frac{d^2y}{dx^2} = -5y$

$\frac{d^2y}{dx^2} = 5y$

**Solution:**

$$\frac{d^2y}{dx^2} = 25y$$

## SECTION – B

**Question 7:** **[2]**

Write the truth values of the following statements:

(1) 2 is a rational number and $\sqrt{2}$ is an irrational number. .

(2) 2 + 3 = 5 or $\sqrt{2} + \sqrt{3} = \sqrt{5}$

**Solution:** (1) Let p : 2 is rational number, q : √2 is an irrational number. . So given compound statement is p ^ q. Truth value of p is T and that q is T. Hence, truth value p ^ q = T ^ T =T.

(2) Let p = 2 + 3 = 5 which is T.

q = $\sqrt{2} + \sqrt{3} = \sqrt{5}$ which is F

∴ Truth value of given statement is T ∨ F = T.

**Question 8.A:** Find the volume of the parallelepiped, if the coterminous edges are given by the vectors $2\hat{i} + 5\hat{j} - 4\hat{k}, 5\hat{i} + 7\hat{j} + 5\hat{k}, 4\hat{i} + 5\hat{j} - 2\hat{k}$. **[2]**

**Solution:**

Let the given vectors

$\bar{a} = 2\hat{i} + 5\hat{j} - 4\hat{k}, \bar{b} = 5\hat{i} + 7\hat{j} + 5\hat{k}, \bar{c} = 4\hat{i} + 5\hat{j} - 2\hat{k}$.

represent the co-terminous edges of the parallelopied

Consider $\bar{a}.\left(\bar{b} \times \bar{c}\right) = \begin{vmatrix} 2 & 5 & -4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{vmatrix}$

= 2 (-14-25) -5 (-10-20) -4 (25 - 28)

= - 78 +150 + 12 = 84

∴Volume of the parallelopiped = $\left|\bar{a}.\left(\bar{b} \times \bar{c}\right)\right|$

= 84 cubic units.

**OR**

**Question 8.B:** Find the value of p, if the vectors $\hat{i} - 2\hat{j} + \hat{k}, 2\hat{i} - 5\hat{j} + p\hat{k}, 5\hat{i} - 9\hat{j} + 4\hat{k}$ are coplanar. **[2]**

**Solution:**

$a = \hat{i} - 2\hat{j} + \hat{k},\ \ b = 2\hat{i} - 5\hat{j} + p\hat{k}\ , c = 5\hat{i} - 9\hat{j} + 4\hat{k}$

Given that $\bar{a}, \bar{b}, \bar{c}$ are coplanar.

**These vectors are coplanar if their scalar triple product is zero.**

Therefore, we have

$\therefore\ \bar{a}.\left(\bar{b} \times \bar{c}\right) = 0$

i.e. $\begin{vmatrix} 1 & -2 & 1 \\ 2 & -5 & p \\ 5 & -9 & 4 \end{vmatrix} = 0$

1(-20 +9p) + 2(8-5p) + 1 (-18 + 25) =0

= - 20 + 9p + 16 - 10p - 18 + 25 = 0

-p + 3 = 0

- p = - 3

p = 3

**Therefore, the value of p is 3.**

**Question 9:** Show that the points A (-7 , 4 , -2),B (-2 , 1 , 0)and C (3 ,-2 ,2) are collinear. **[2]**

**Solution:**

Let $\bar{a}, \bar{b}, \& \bar{c}$ be position vectors of A,B &C

$\overline{AB} = \bar{b} - \bar{a}$

= $\left(-2\hat{i} + \hat{j}\right) - \left(-7\hat{i} + 4\hat{j} - 2\hat{k}\right)$

=$5\hat{i} - 3\hat{j} + 2\hat{k}$

$\overline{AC}\ = \bar{c} - \bar{a}$

= $\left(3\hat{i} - 2\hat{j} + 2\hat{k}\right) - \left(-7\hat{i} + 4\hat{j} - 2\hat{k}\right)$

=$10\hat{i} - 6\hat{j} + 4\hat{k}$

= $2\left[5\hat{i} - 3\hat{j} + 2\hat{k}\right]$

$\Rightarrow \overline{AC} = 2\overline{(AB)}$

$\Rightarrow \overline{AC}$is a scalar multiple of $\overline{AB}$

$\Rightarrow \overline{AC}\ \&\ \overline{AB}$

$\because$ A is common

$\Rightarrow$ A B &C are collinear.

**Question 10:** Write the equation of the plane 3x + 4y -2z = 5 in the vector form. **[2]**

**Solution:** The equation of the given plane is 3x+ 4y-2z = 5

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be any vector in the plane and 3 , 4 ,-2 are direction ratios of $\bar{n}$ , which is normal to the plane

$\therefore \bar{n} = 3\hat{i} + 4\hat{j} - 2\hat{k}$

$\therefore \left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(3\hat{i} + 4\hat{j} - 2\hat{k}\right) = 5$

$\therefore \bar{r}.\left(3\hat{i} + 4\hat{j} - 2\hat{k}\right) = 5$ which is required form.

**Question 11:** **[2]**

If $y = x^x$, find $\frac{dy}{dx}$.

**Solution:**

$y = x^x$ ......(1)

Taking log on both sides,

log y = log $X^X$

logy =x log x

Differentiate both sides w.r.t x

$$\frac{1}{y}.\frac{dy}{dx} = \frac{x \times 1}{x} + \log x \times 1$$

$$\frac{dy}{dx} = y[1 + \log x]$$

$=X^X$ [ 1+ log x] from (1)

**Question 12:** Find the equation of tangent to the curve y = $x^2$ +4x + 1 at (-1, -2). **[2]**

**Solution:**

y = $x^2$+ 4x +1 at (-1 , -2)

Given, Equation of the curve is y = $x^2$ +4x + 1

Differentiating w.r.t. x

$$\frac{dy}{dx} = 2x + 4$$

$\left(\frac{dy}{dx}\right)_{at(-1,-2)}$ = 2(-1) + 4 = 2

∴ Slope of the tangent at the point (-1 , -2) = 2

∴ Equation of the tangent at point (-1 ,-2) is

y-(-2) = 2[x-(-1)]

2x - y = 0

**Question 13:** **[2]**

Evaluate : $\int \frac{e^x(1+x)}{\cos^2(xe^x)} dx$

**Solution:**

Let I = $\int \frac{e^x(1+x)}{\cos^2(xe^x)} dx$

Put $x.e^x = t$

Differentiate w.r.t. x,

$$x.e^x + e^x.1 = \frac{dt}{dx} \Rightarrow e^x(x+1)dx = dt$$

$$\therefore \int \frac{1}{\cos^2 t} dt$$

$= \int \sec^2 t\, dt$

= tan t + c

$$\therefore \int \frac{e^x(1+x)}{\cos^2(xe^x)} dx = \tan(x.e^x) + c$$

**Question 14:** **[2]**

Evaluate : $\int_0^{\frac{\pi}{2}} \sin^2 x\, dx$

**Solution:**

$$\int_0^{\frac{\pi}{2}} \sin^2 x\, dx$$

i = $\int_0^{\frac{\pi}{2}} \frac{1 - \cos 2x \;\; dx}{2}$ [∵ 1 - 2 $\cos^2 \theta$ = 2 $\sin^2 \theta$]

$$\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}}$$

$= \left(\frac{\pi}{4} - \frac{\sin\pi}{4}\right) - (0 - 0)$

$= \frac{\pi}{4}$

## SECTION – C

### Question 15: [3]

In , ΔABC prove that

$$\sin\left(\frac{B - C}{2}\right) = \left(\frac{b - c}{a}\right)\cos\left(\frac{A}{2}\right)$$

### Solution:

RHS = $\left(\frac{b - c}{a}\right)\cos\frac{A}{2}$ ...(by sine rule)

$= \left(\frac{k\sin B - k\sin C}{k\sin A}\right).\cos\frac{A}{2}$

$= \frac{k[\sin B - \sin C]}{k\sin A}.\cos\frac{A}{2}$

$= \frac{\left[2\cos\frac{B+C}{2}.\sin\frac{B-C}{2}\right]}{2\sin\frac{A}{2}\cos\frac{A}{2}}.\cos\frac{A}{2}$

$= \frac{2\cos\left(\frac{\pi}{2} - \frac{A}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}}$

$= \sin\left(\frac{B - C}{2}\right) = \text{LHS}$

Hence, the required result is proved.

**OR**

### Question 15.A: [3]

Show that $\sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{63}{16}\right)$

**Solution:**

Let $\sin^{-1}\left(\frac{5}{13}\right) = A \Rightarrow \sin A = \frac{5}{13}$

$\therefore \cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\left(\frac{5}{13}\right)^2}$

$\sqrt{1-\frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$

$\Rightarrow \tan A = \frac{5}{12}$

let $\cos^{-1}\left(\frac{3}{5}\right) = B \Rightarrow \cos B = \frac{3}{5}$

$\sin B = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\therefore \tan B = \frac{4}{3}$

Now , $\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}$

$\Rightarrow A + B = \tan^{-1}\left[\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}\right]$

$= \tan^{-1}\left[\frac{\frac{5+16}{12}}{\frac{36-20}{36}}\right]$

$A + B = \tan^{-1}\left[\frac{\frac{21}{16}}{3}\right]$

$\Rightarrow \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{63}{16}\right)$

= R.H.S

**Question 16:** If $A(\bar{a})$ and $B(\bar{b})$ are any two points in the space and R(r) be a point on the line segment AB dividing it internally in the ratio m : n, then prove that **[3]**

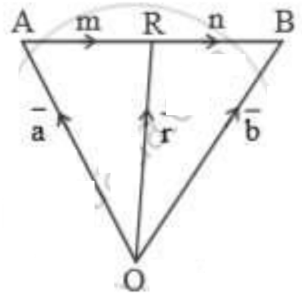
$r = \frac{m\bar{b}+n\bar{a}}{m+n}$

**Solution:** $\bar{a}, \bar{b}, \bar{r}$ are position vectors of points A,B , R respectively

$\overline{AR} = \bar{r} - \bar{a}, \overline{RB} = \bar{b} - \bar{r}$

$$\therefore \frac{AR}{RB} = \frac{m}{n}$$

R is a point of line segment AB and A–R–B which divides AB in the ratio m : n.




$$\therefore \frac{AR}{RB} = \frac{m}{n}$$

$$n\overline{AR} = m\overline{RB}$$

$$n(\bar{r} - \bar{a}) = m(\bar{b} - \bar{r})$$

$$\bar{r}(m + n) = m\bar{b} + n\bar{a}$$

$$\bar{r} = \frac{m\bar{b} + n\bar{a}}{m + n}$$

**Question 17:** The equation of a line is 2x -2 = 3y +1 = 6z -2 find the direction ratios and also find the vector equation of the line. **[3]**

**Solution:** The Cartesian equation of a line is

2x -2 =3y + 1 = 6z -2

$2(x-1) = 3\left(y + \frac{1}{3}\right) = 6\left(z - \frac{1}{3}\right)$ ....dividing by 6 all side

$$\frac{x-1}{3} = \frac{y + \frac{1}{3}}{2} = \frac{z - \frac{1}{3}}{1}$$

The direction ratios of the line are 3, 2, 1

Further, the line passes through the point $\left(1, -\frac{1}{3}, \frac{1}{3}\right)$

Let A = $\left(1, -\frac{1}{3}, \frac{1}{3}\right)$

Thus, the line passes through the point having the position vector.

$$\bar{a} = 1\hat{i} - \frac{1}{3}\hat{j} + \frac{1}{3}\hat{k}$$

Let $\hat{b} = 3\hat{i} + 2\hat{j} + \hat{k}$

Then, the line is parallel to the vector $\bar{b}$

Hence, the vector equation of the line is

$\bar{r} = \bar{a} + \lambda\bar{b}$ , where $\lambda \in R$

i.e. $\bar{r} = \left(1\hat{i} - \frac{1}{3}\hat{j} + \frac{1}{3}\hat{k}\right) + \lambda\left(3\hat{i} + 2\hat{j} + \hat{k}\right)$

**Question 18:** **[3]**

Discuss the continuity of the function

f(x) = $\frac{\log(2+x) - \log(2-x)}{\tan x}$ , for x ≠ 0

= 1 for x = 0 at the point x =0

**Solution:**

Here , f(0) = 1 ... (i) [Given]

Also , L = $\lim_{x \to 0} f(x)$

$$= \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\log(2+x) - \log(2-x)}{\tan x}$$

$$= \lim_{x \to 0} \frac{\log\frac{2+x}{2-x}}{\tan x} \quad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$= \lim_{x \to 0} \frac{\log\left(\frac{1+\frac{x}{2}}{1-\frac{x}{2}}\right)}{\tan x}$$

$$= \lim_{x \to 0} \left[\frac{\log\left(\frac{1+\frac{x}{2}}{1-\frac{x}{2}}\right)}{x}\right] \times \lim_{x \to 0} \frac{x}{\tan x}$$

$$= \left[\lim_{x \to 0} \frac{\log\left(1+\frac{x}{2}\right)}{x} - \lim_{x \to 0} \frac{\log\left(1-\frac{x}{2}\right)}{x}\right] \times \lim_{x \to 0} \frac{x}{\tan x}$$

$$= \left[\frac{1}{2} - \left(\frac{-1}{2}\right)\right] \times 1$$

= 1

From (i) and (ii), we have $\lim_{x \to 0} f(x) = f(0)$

∴ The given function is continuous at x = 0

**Question 19.A:** **[3]**

| x | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| P(X = x) | 0.45 | 0.35 | 0.15 | 0.03 | 0.02 |

Find the variance of X.

**Solution:**

| $x_i$ | $p_i$ | $p_i x_i$ | $p_i x_i^2$ |
|---|---|---|---|
| 0 | 0.45 | 0 | 0 |
| 1 | 0.35 | 0.35 | 0.35 |
| 2 | 0.15 | 0.30 | 0.60 |
| 3 | 0.03 | 0.09 | 0.27 |
| 4 | 0.02 | 0.08 | 0.32 |
| | Total | 0.82 | 1.54 |

$E(X) = \mu = \sum p_i x_i = 0.82$

To find Var (X) use the formula :

$$\mathrm{Var}(X) = \sum p_i x_i^2 - \mu^2$$

$= 1.54 - (0.82)^2$

= 1.54 -0.6724

= 0.8676

Thus , Var (X) = 0.8676

**OR**

**Question 19.B:** **[3]**

For the following probability density function (p. d. f) of X, find : (i) P ( X<1), (ii) P |x| < 1

if $f(x) = \frac{x^2}{18}, -3 < x < 3$

= 0, otherwise

**Solution:**

Given, f(x) is pdf

(i) P(X< 1) = $\int_{-3}^{1} f(x)dx$

$= \int_{-3}^{1} \left(\frac{x^2}{18}\right) dx$

$= \frac{1}{18}\left[\frac{x^3}{3}\right]_{-3}^{1} = \frac{14}{27} = 0.5185$

(ii) P(|X|<1) = P(-1<X<1) = $\int_{-1}^{1} f(x)dx$

$= \int_{-1}^{1} \frac{x^2}{18} dx$

$= \frac{1}{18}\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{27} = 0.03704$

**Question 20:** Given is X ~ B (n,p) **[3]**
If E(X) = 6, Var(X) = 4.2, find n and p.

**Solution:**

E(X) = 6 , Var(X) = 4.2 ,E(X) np, Var(X) = npq

$\therefore \frac{Var(X)}{E(X)} = \frac{npq}{np} = q = \frac{4.2}{6} = 0.7 \quad \therefore q = 0.7$

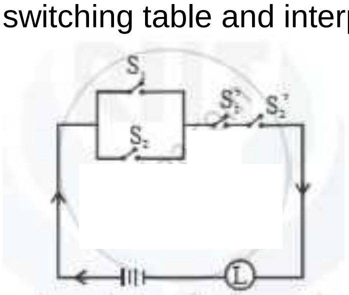
$\therefore p = 1 - q = 0.3$

Now , np = 6

$\therefore n = \frac{6}{p} = \frac{6}{0.3} = 20$

## SECTION – D

**Question 21:** Find the symbolic form of the given switching circuit. Construct its switching table and interpret your result. Diagram **[4]**



**Solution:** Let p,q denote the statements that the switches $S_1$, $S_2$ are closed respectively. Let ~p ,~ q denote the statements that switches ' $S_1$ and ' $S_2$ are closed respectively. .
The symbolic form of the given switching circuit is $(p \vee q) \wedge (\sim p \wedge \sim q) \equiv l$
The switching table for the given switching circuit is as follows:

| | | | | A | B | |
|---|---|---|---|---|---|---|
| p | q | ~p | ~q | p ∨q | ~p ∧ ~q | A∧B |
| 1 | 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 1 | 0 |

In the last column of the switching table, each entry is zero, i.e. it is a contradiction. Hence, no current will flow in the circuit irrespective of the positions of the switches.

**Question 22:** If three numbers are added, their sum is 2. If two times the second number is subtracted from the sum of first and third numbers we get 8 and if three times the first number is added to the sum of second and third numbers we get 4. Find the numbers using matrices. **[4]**

**Solution:** In, Δ ABC with usual notations prove that
$b^2 = c^2 + a^2 - 2\, ca \cos B$

**Solution:** Consider that for Δ ABC, ∠B is in a standard position i.e. vertex B is at the origin and the side BC is along positive x-axis. As ∠B is an angle of a triangle ∴ ∠B can be acute or ∠B can be obtuse.

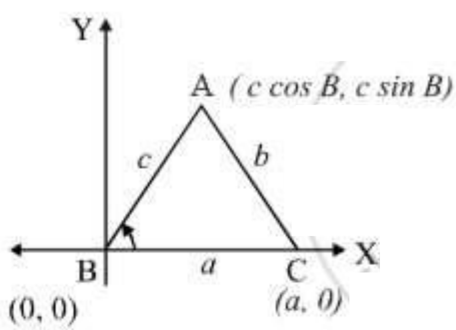


Fig. 1

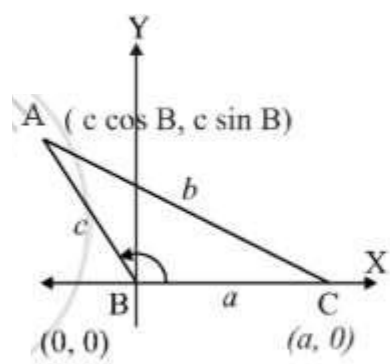


Fig. 2

Using the Cartesian co-ordinate system in above figure.

we get B ≡(0,0) A ≡(c cos B , c sin B) and C ≡ (a,0)

Now consider l(CA) = b

$\therefore b^2 = (a- c \cos B)^2 + (0-c \sin B)^2$ , by distance formula

$\therefore b^2 = a^2 - 2 ac \cos B + c^2 \cos^2 B + c^2 \sin^2 B$

$\therefore b^2 = a^2 - 2 ac \cos B + c^2 (\sin^2 B + \cos^2 B)$

$\therefore b^2 = a^2 + c^2 - 2 ac \cos B$

Hence proved.

**OR**

**Question 23.A:** **[4]**

In , $\Delta ABC$ with usual notations prove that

$$(a-b)^2 \cos^2 \left(\frac{C}{2}\right) + (a+b)^2 \sin^2 \left(\frac{C}{2}\right) = c^2$$

**Solution:**

Taking LHS

$$= (a-b)^2 \cos^2 \frac{C}{2} + (a+b)^2 \sin^2 \frac{C}{2}$$

$$= (a^2 + b^2 - 2ab)\cos^2 \frac{C}{2} + (a^2 + b^2 + 2ab).\sin \frac{C}{2}$$

$$= (a^2 + b^2)\cos \frac{C}{2} - 2\,ab \cos^2 \frac{C}{2} + (a^2 + b^2).\sin^2 \frac{C}{2} + 2\,ab \sin^2 \frac{C}{2}$$

$$= (a^2 + b^2)\left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2}\right) - 2ab\left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2}\right)$$

$$= (a^2 + b^2) - 2ab \cos C \qquad \{\text{By cosine Rule}\}$$

$$= c^2$$

**Question 24:** Find 'p' and 'q' if the equation $px^2 - 8xy + 3y^2 + 14x + 2y + q = 0$ represents a pair of perpendicular lines. **[4]**

**Solution:** Given general equation is $px^2 - 8xy + 3y^2 + 14x + 2y + q = 0$

a = p, h = -4, b = 3 , g = 7, f = 1 , c = q

Since the equation represents perpendicular lines

a +b =0

p + 3 = 0

∴ p = -3

∴ a = p = -3

Since the equation represents a pair of lines.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\begin{vmatrix} -3 & -4 & 7 \\ -4 & 3 & 1 \\ 7 & 1 & q \end{vmatrix} = 0$$

∴ -3(3q-1)+4(-4q-7)+7(-4-21)=0

∴ -9q +3 -16q - 28 - 28 - 147 = 0

∴ -25q + 3 - 28 - 175 = 0

∴ -25q + 3 - 203 = 0

∴ -25q - 200 = 0

∴ -25q = 200

$$\therefore q = \frac{200}{-25}$$

∴ q = -8

∴ p = -3 , q = -8

**Question 25:** **[4]**

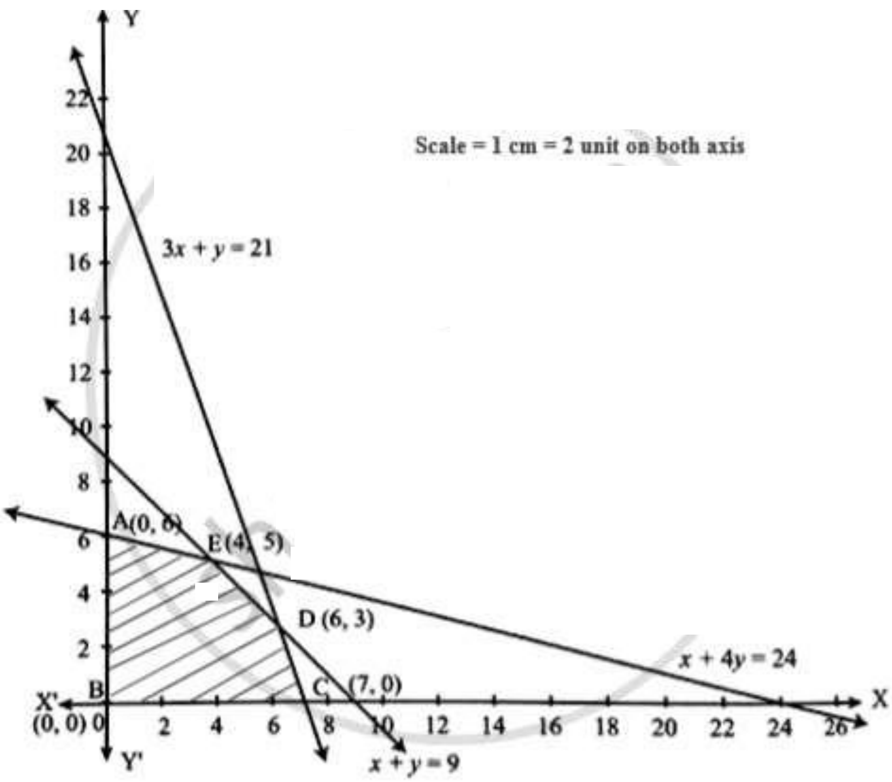
Maximize: z = 3x + 5y Subject to

x +4y ≤ 24 3x + y ≤ 21

x + y ≤ 9 x ≥ 0 , y ≥0

**Solution:**

| In equation | Point on x-axis | Point on y-axis | Feasible Region |
|---|---|---|---|
| x +4y ≤ 24 | (24,0) | (0,6) | Origin side |
| 3x +y ≤ 21 | (7,0) | (0,21) | Origin side |
| x + y ≤9 | (9,0) | (0,9) | Origin side |




From the figure common feasible region is ABCDEA
E is the point of intersection of x + y = 9 and x + 4y =24
Solving them we get E(4,5)
D is the point of intersection of x + y = 9 and 3x +y = 21
Solving them we get D(6,3)

| End Point | Value of z = 3x +5y |
|---|---|
| A(0,6) | 0 +30= 30 |
| B(0,0) | 0 + 0 = 0 |
| C(7,0) | 21 + 0 = 21 |
| D(6,3) | 18 +1 5 = 33 |
| E(4,5) | 12 +25 = 37 |

∴ z is maximum 37 at the point (4, 5)

**Question 26:** If X = f(t) and Y = g(t) Are Differentiable Functions of t , then prove that y is a differentiable function of x and **[4]**

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \text{where} \frac{dx}{dt} \neq 0$$

Hence find $\frac{dy}{dx}$ if $x = a\cos^2 t$ and $y = a\sin^2 t$.

**Solution:** $x = a\cos^2 t$ and $y = a\sin^2 t$

Differentiating x and y with respect to t, we get

$$\frac{dx}{dt} = a\cos^2 t, \frac{dy}{dt} = a\sin^2 t$$

$$\Rightarrow \frac{dx}{dt} = 2a\cos t \frac{d}{dt}\cos t, \frac{dy}{dt} = 2a\sin t \frac{d}{dt}\sin t$$

$$\Rightarrow \frac{dx}{dt} = 2a\cos t \times (-\sin t), \frac{dy}{dt} = a(2\sin t \times \cos t)\left(\frac{d}{dx}(at^2) = 2at\frac{d}{dx}t\right)$$

$$\Rightarrow \frac{dx}{dt} = -\ 2a\cos t\sin t, \frac{dy}{dt} = 2a\sin t\cos\left(\frac{d}{dx}(\sin\theta) = \cos\theta \text{and} \frac{d}{dx}(\cos\theta) = -\sin\theta\right)$$

$$\Rightarrow \frac{dx}{dt} = -a\sin 2t, \frac{dy}{dt} = a\sin 2t\ (\sin 2\theta = 2\sin\theta\cos\theta)$$

Therefore

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \text{ where } \frac{dx}{dt} \neq 0$$

$$\frac{dy}{dx} = \frac{a\sin 2t}{-a\sin 2t}$$

$$\Rightarrow \frac{dy}{dx} = -1$$

**Question 27:** f(x) = (x-1)(x-2)(x-3) , x ε[0,4], find if 'c' LMVT can be applied **[4]**

**Solution:** f(x) = (x-1)(x-2)(x-3) , x ∈[0,4],

∴ f(x) = $x^3$ -6$x^2$ + 11x -6

As f(x) is a polynomial in x
(1) f(x) is continuous on [0, 4]
(2) f(x) is differentiable on (0, 4)
Thus, all the conditions of LMVT are satisfied.

To verify LMVT we have to find c ∈ (0,4) such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0} \quad ........(1)$$

Now $f(4) = (4 - 1)(4 - 2)(4 - 3) = 6$

f(0) = (0-1)(0-2)(0-3) = -6 and

$f'(x) = 3x^2 - 12x + 11$

$\therefore f'(c) = 3c^2 - 12c + 11$

$\therefore$ from (1)

$$\frac{f(4) - f(0)}{4 - 0} = 3c^2 - 12c + 11$$

$$\frac{6 - (-6)}{4} = 3c^2 - 12c + 11$$

$3 = 3c^2 - 12c + 11$

$3c^2 - 12c + 8 = 0$

$$c = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$c = \frac{12 \pm \sqrt{144 - 96}}{6}$$

$$c = \frac{12 \pm \sqrt{48}}{6}$$

$$c = 12 \pm \frac{2}{3}\sqrt{3}$$

$$c = 4 \pm \frac{2}{\sqrt{3}}$$

Both the value of c lie between 0 and 4.

**OR**

**Question 27.A:** A rod of 108 meters long is bent to form a rectangle. Find its dimensions if the area is maximum. Let x be the length and y be the breadth of the rectangle. **[4]**

**Solution:** $\therefore 2x + 2y = 108$

$\therefore y = 54 - x$

Now, area of rectangle = xy =x (54 - x)

$\therefore$ f(x) = 54x -$x^2$

f ' (x) = 54 -2x

f '' (x) = -2

For extreme values f '(x) = 0

$\therefore$ 54 - 2x = 0

$\therefore$ x = 27

f '' (27) = -2 < 0 $\therefore$ Area is maximum when x =27 , y = 27

$\therefore$ The dimensions of rectangle are 27m × 27m

It is a square.

**Question 28:** **[4]**

Prove that: $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + c$

**Solution:**

Let I = $\int \frac{1}{\sqrt{x^2 + a^2}} dx$

Put x = a tan θ ⇒ tan θ = $\frac{x}{a}$

$\therefore$ dx = a $\sec^2$ θ dθ

$\therefore$ I = $\int \frac{1}{\sqrt{a^2\tan^2\theta + a^2}} a \sec^2 \theta \, d\theta$

$= \int \frac{a.\sec^2\theta}{a\sqrt{1 + \tan^2\theta}} \, d\theta$

$= \int \frac{\sec^2\theta}{\sec\theta} d\theta$

$= \int \sec\theta . d\theta$

$= \log\ |\sec\theta + \tan\theta| + c_1$

$= \log\left|\frac{x}{a} + \sqrt{\sec^2\theta}\right| + c_1$

$= \log\left|\frac{x}{a} + \sqrt{1} + \tan^2\theta\right| + c_1$

$= \log\left|\frac{x}{a} + \sqrt{1 + \frac{x^2}{a^2}}\right| + c_1$

$= \log\left|\frac{x}{a} + \frac{\sqrt{a^2 + x^2}}{a}\right| + c_1$

$= \log\left|x + \sqrt{x^2 + a^2}\right| - \log a + c_1$

$\therefore \int \frac{1}{\sqrt{x^2 + a^2}} dx = \log\left|x + \sqrt{x^2 + a^2}\right| - \log a + c,$

where $c = -\log a + c_1$

**Question 29:** **[4]**

Show that : $\int_0^{\frac{\pi}{4}} \log(1 + \tan x)dx = \frac{\pi}{8}\log 2$

**Solution:**

Let I = $\int_0^{\frac{\pi}{4}} \log(1 + \tan x)dx$

$= \int_0^{\frac{\pi}{4}} \log(1 + \tan x)dx$

$= \int_0^{\frac{\pi}{4}} \log\left\{1 + \tan\left(\frac{\pi}{4} - x\right)\right\}dx$

$\left(\because \int_0^{a} f(x)dx \int f(a - x)dx\right)$

$= \int_0^{\frac{\pi}{4}} \log\left\{1 + \frac{(\tan\frac{\pi}{4} - \tan x)}{1 + \tan\frac{\pi}{4}\tan x}\right\}dx$

$= \int_0^{\frac{\pi}{4}} \log\left\{1 + \frac{1 - \tan x}{1 + \tan x}\right\}dx$

$$= \int_0^{\frac{\pi}{4}} \log\left\{\frac{1+\tan x+1-\tan x}{1+\tan x}\right\}dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \{\log 2 - \log(1+\tan x)\}dx$$

$$= \int_0^{\frac{\pi}{4}} \log 2dx - \int_0^{\frac{\pi}{4}} \log(1+\tan x)dx$$

$$I = \log 2[x]\int_0^{\frac{\pi}{4}} - I$$

$$2I = \log 2\left[\frac{\pi}{4} - 0\right]$$

$$I = \frac{\pi}{8}.\log 2$$

$$\therefore \int_0^{\frac{\pi}{4}} \log(1+\tan x)dx = \frac{\pi}{8}\log 2$$

**Question 30:** **[4]**

Solve the differential equation:

$$\frac{dy}{dx} + y\sec x = \tan x$$

**Solution:**

$$\frac{dy}{dx} + y\sec x = \tan x$$

Comparing with $\frac{dy}{dx} + P.y = Q$, we get

P = sec x , Q = tan x

IF = $e^{\int P dx} = e^{\int \sec x dx} = e^{\ln|\sec x+\tan x}| = \sec x + \tan x$

therefore the general solution is

y(IF) = $\int Q(IF)dx + C$

y(secx + tan x) = $\int \tan x(\sec x + \tan x)dx + C$

= $\int (\sec x \tan x + \tan^2 x)dx + C$

$= \int (\sec x \tan x + \sec^2 x - 1)dx + C$

∴ y(sec x + tan x) = sec x + tan x - x +C

**Question 30.A:** **[4]**

Solve the differential equation:

$(x + y)\frac{dy}{dx} = 1$

**Solution:**

$(x + y)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{x + y} \Rightarrow \frac{dx}{dy} = x + y$

$\frac{dx}{dy} + (-1)x = y$

Comparing with $\frac{dx}{dy} + P.x = Q$we get

P = -1 and Q = y

$\therefore = e^{\int -1dy} = e^{-y}$

The general solution is

x (IF) = ∫ Q (IF) dy +C

$xe^{-y} = y \int e^{-y}\, dy + C$

$\Rightarrow xe^{-y} = -ye^{-y} - e^{-y} + C$

$\Rightarrow (x + y + 1) = ce^{y}$