Mathematics - 2014

सामान्य निर्देश : General Instructions :

इस प्रश्न-पत्र में 29 प्रश्न हैं, जो तीन खण्डों-अ, ब और स में बंटे हुए हैं। खण्ड-अ में 10 प्रश्न हैं, जिनमें प्रत्येक 1 अंक का है, खण्ड-ब में 12 प्रश्न हैं जिनमें प्रत्येक 4 अंक का है तथा खण्ड-स में 7 प्रश्न हैं जिनमें प्रत्येक 6 अंक का है।

कैलकुलेटर के उपयोग की अनुमति नहीं है। आवश्यकता हो तो परीक्षार्थी के माँग पर लघुगणकीय अथवा सांख्यिकीय सारणी उपलब्ध करायी जा सकती है।

Section-A
(Objective Questions)

Q.1. A binary composition * is defined on $R\times R$ by $(a,b)*(c,d)=(ac,bc+d)$, where a, b, c, d $\in R$. Find $(2,3)*(1,-2)$.

Soln.

$\because \quad (a,b)*(c,d)=(ac,bc+d)$

$\therefore \quad (2,3)*(1,-2)=\{2.1,3.1+(-2)\}$

$=(2,1)$ Ans.

Q.2. Evaluate: $\sin\left\{\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right\}$.

Soln. $\sin\left\{\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right\}$.

$=\sin\left\{\frac{\pi}{3}-\left(-\frac{\pi}{6}\right)\right\}$

$=\sin\left\{\frac{\pi}{3}+\frac{\pi}{6}\right\}$

$=\sin\left(\frac{3\pi}{6}\right)=\sin\frac{\pi}{2}$

$=1$ Ans.

Q.3. Find the values of a and b for which $\begin{bmatrix} a & b \\ -a & 2b\end{bmatrix}\begin{bmatrix} 2 \\ -1\end{bmatrix}=\begin{bmatrix} 5 \\ 4\end{bmatrix}$.

Soln. $\begin{bmatrix} a & b \\ -a & 2b\end{bmatrix}\begin{bmatrix} 2 \\ -1\end{bmatrix}=\begin{bmatrix} 5 \\ 4\end{bmatrix}$

or, $\begin{bmatrix} 2a & -b \\ -2a & -2b\end{bmatrix}=\begin{bmatrix} 5 \\ 4\end{bmatrix}$

$2a-b=5 \quad ...(1)$

$\therefore \quad -2a-2b=4 \quad ...(2)$

$-3b=9$

$b=-3$

putting the value of b in (1), we get

$a=1$

$\therefore \left.\begin{matrix} a=1 \\ b=-3\end{matrix}\right\} Ans.$

Q.4. Construct a 2×2 matrix whose $(i,j)th$ element is given by $a_{ij}=\frac{2i-j}{3}$.

Soln. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}=A$ (let)

$\because \quad a_{ij}=\frac{2i-j}{3}$

$\therefore \quad a_{11}=\frac{2.1-1}{3}=\frac{2-1}{3}=\frac{1}{3}$

$a_{12}=\frac{2.1-2}{3}=\frac{0}{3}=0$

$a_{21}=\frac{2.2-1}{3}=\frac{3}{3}=1$

$a_{22}=\frac{2.2-2}{3}=\frac{2}{3}$

putting the value in A

$A=\begin{bmatrix} 1/3 & 0 \\ 1 & 2/3\end{bmatrix}$ Ans.

Q.5. Show that $x=1$ is a root of the equation $\begin{vmatrix} x+1 & 2x & 11 \\ 2x & x+1 & -4 \\ -3 & 4x-7 & 6\end{vmatrix}=0$.

Soln. applying $C_2\to C_2-C_1$

$\Delta=\begin{vmatrix} x+1 & 2x-(x+1) & 11 \\ 2x & x+1-2x & -4 \\ -3 & 4x-7+3 & 6\end{vmatrix}=0$

$\Rightarrow \begin{vmatrix} x+1 & x-1 & 11 \\ 2x & -(x-1) & -4 \\ -3 & 4(x-1) & 6\end{vmatrix}=0$

$\Rightarrow (x-1)\begin{vmatrix} x+1 & 1 & 11 \\ 2x & -1 & -4 \\ -3 & 4 & 6\end{vmatrix}=0$

$\Rightarrow \quad x-1=0$

or, $\quad x=1$

Q.6. Find the slope of the tangent to the curve $y=x^3-2x+8$ at the point (1, 7).

Soln. $y=x^3-2x+8$

Differentiating y w.r.t. x

$\frac{dy}{dx}=3x^2-2.1+0$

or, $\frac{dy}{dx}=3x^2-2$

The slope of the tangent at (1, 7)

[illegible]

Q.8. If $z = 2 - i$ then find the value of [illegible]

Soln. [illegible]

Q.9. Find a vector perpendicular to both $\vec{a} = 3\hat{i} - 2\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 4\hat{j} - \hat{k}$.

Soln. [illegible]

Q.10. Find the value of λ if straight lines

$$\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-4}{\lambda} \quad \text{and} \quad \frac{x-2}{1} = \frac{y-5}{3} = \frac{z-1}{-7} \quad \text{are}$$

perpendicular to each other.

Soln. The direction ratios of 1st line are 2, 4, λ and the direction ratios of 2nd line are 1, 3, −7.

(पहली line का direction ratios 2, 4, λ है और दूसरी line का 1, 3, −7)

The both lines will be ⊥ to each other if

(दोनों एक दूसरे को ⊥ करते हैं)

$2 \cdot 1 + 4 \cdot 3 + \lambda(-7) = 0$

or. $2 + 12 - 7\lambda = 0$

or. $7\lambda = 14$

[illegible]

Q.12. Let $f(x) = 2x^2$ and $g(x) = 3x - 4$, $x \in R$. Find the following

(i) $fof(x)$ (ii) $gog(x)$

(iii) $fog(x)$ (iv) $gof(x)$

Soln.

(i) $fof(x) = f[f(x)] = f(2x^2) = 2(2x^2)^2$

$= 2 \cdot 4x^4 = 8x^4$ Ans.

(ii) $gog(x) = g[g(x)] = g(3x-4)$

$= 3(3x-4) - 4$

$= 9x - 12 - 4$

$= 9x - 16$ Ans.

(iii) $fog(x) = f[g(x)] = f(3x-4)$

$= 3(2x^2) - 4$

$= 6x^2 - 4$ Ans.

(iv) $gof(x) = g[f(x)] = g(2x^2)$

$= 2(3x-4)^2$

$= 2(9x^2 - 24x + 16)$

$= 18x^2 - 48x + 32$ Ans.

Q.13 Using the properties of a determinant, prove that

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$$

Soln. L.H.S.

$$\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

applying $R_1 \rightarrow R_1 + R_2 + R_3$ प्रयोग करने पर

$$\Delta = \begin{vmatrix} a-b-c+2b+2c & 2a+b-c-a+2c & 2a+2b+c-a-b \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

taking out $(a+b+c)$ from R_1 as common factor

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

applying $c_1 \to c_1 - c_2$ *and*

$c_2 \to c_2 - c_3$

$$= (a+b+c)\begin{vmatrix} 1-1 & 1-1 & 1 \\ 2b-(b-c-a) & b-c-a-2b & 2b \\ 2c-2c & 2c-(c-a-b) & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ a+b+c & -a-b-c & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix}$$

taking out $(a+b+c)$ from c_1 and c_2 as common factor

[c_1 और c_2 से $(a+b+c)$ *common* लेने पर]

$$= (a+b+c)^3\begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 2b \\ 0 & 1 & c-a-b \end{vmatrix}$$

now expanding it, we get

$= (a+b+c)^3.[1.\{1.1-0(-1)\}]$

$= (a+b+c)^3.1$

$= (a+b+c)^3$ R.H.S.

Q.14. If the following function $f(x)$ is continuous at $x=0$, find the value of a.

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}; & x<0 \\ a; & x=0 \\ \dfrac{a\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}; & x>0 \end{cases}$$

Soln. for $f(x)$ to be continuous at $x=0$

we must have

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\Rightarrow \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = a$

Now, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0} \dfrac{1-\cos 4x}{x^2}$

$\Rightarrow \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0} \dfrac{2\sin^2 2x}{x^2}$

$\Rightarrow \quad \lim_{x\to 0^-} f(x) = 2\lim_{x\to 0}\left(\dfrac{\sin 2x}{x}\right)^2$

$\Rightarrow \quad \lim_{x\to 0^-} f(x) = 2\times 4 \lim_{x\to 0}\left(\dfrac{\sin 2x}{2x}\right)^2 = 8.1^2 = 8$

and, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0} \dfrac{a\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}$

$\Rightarrow \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0} \dfrac{a\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} \times \dfrac{\sqrt{16+\sqrt{x}}+4}{\sqrt{16+\sqrt{x}}+4}$

$\Rightarrow \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0} \dfrac{a\sqrt{x}\sqrt{16+\sqrt{x}}+4}{16+\sqrt{x}-16}$

$\Rightarrow \quad \lim_{x\to 0^+} f(x) = a.8$

Since $f(x)$ is continuous at $x=0$

so, as $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

or, $\delta = \delta.a = a$

This is the possible for any value of a. So, this function is not continuous or $x=0$ for any value of a. Derived value of a does not exists.

Q.15. If $x = a\sin 2t(1+\cos 2t)$ and $y = b\cos 2t(1-\cos 2t)$, prove that at $t=\frac{\pi}{4}$, $\frac{dy}{dx} = \frac{b}{a}$.

Soln. $x = a\sin 2t(1+\cos 2t)$

$\dfrac{dx}{dt} = (a\sin 2t)[0+(-\sin 2t)]^2 + (1+\cos 2t).a\cos 2t.2$

$= (a\sin 2t)(-2\sin 2t) + (1+\cos 2t)(2a\cos 2t)$

$y = (b\cos 2t)(1-\cos 2t)$

$\dfrac{dy}{dt} = (b\cos 2t)[0-(-\sin 2t)].2 + (1-\cos 2t).b(-\sin 2t).2$

$\dfrac{dy}{dt} = (b\cos 2t)(2\sin 2t) + (1-\cos 2t)(-2b\sin 2t)$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(b\cos 2t)(2\sin 2t)+(1-\cos 2t)(-2b\sin 2t)}{(a\sin 2t)(-2\sin 2t)+(1+\cos 2t)(2a\cos 2t)}$$

when $t = \dfrac{\pi}{4}$

$$\frac{dy}{dx} = \frac{\left(b\cos 2.\frac{\pi}{4}\right)\left(2\sin 2.\frac{\pi}{4}\right)+\left(1-\cos 2.\frac{\pi}{4}\right)\left(-2b.\sin 2\frac{\pi}{4}\right)}{\left(a\sin 2.\frac{\pi}{4}\right)\left(-2\sin 2.\frac{\pi}{4}\right)+\left(1+\cos 2.\frac{\pi}{4}\right)\left(2a\cos 2.\frac{\pi}{4}\right)}$$

$$= \frac{0.1+(1-0)(-2b.1)}{(a.1)(-2.1)+(1+0)(2a.0)}$$

$$= \frac{0+1(-2b)}{-2a+0}$$

$$= \frac{-2b}{-2a} = \frac{b}{a} \quad \textit{proved.}$$

OR, (अथवा)

Q.15. If $(\cos x)^y = (\cos y)^x$, then find $\frac{dy}{dx}$.

Soln. $(\cos x)^y = (\cos y)^x$

taking log on both sides (दोनों तरफ log लेने पर)

$y.\log.\cos x = x.\log.\cos y$

$y.\dfrac{1}{\cos x}.(-\sin x) + \log\cos x.\dfrac{dy}{dx} = x.\dfrac{1}{\cos y}(-\sin y)\dfrac{dy}{dx} + \log.\cos y$

or, $-y.\tan x + \log.\cos x\dfrac{dy}{dx} = -x\tan y.\dfrac{dy}{dx} + \log.\cos y$

or, $\log.\cos x.\frac{dy}{dx}+x\tan y.\frac{dy}{dx}=y\tan x+\log.\cos y$

or, $\frac{dy}{dx}=\frac{y\tan x+\log.\cos y}{x\tan y+\log.\cos x}$ Ans.

Q.16. A stone is dropped into a quiet lake and the waves move in circles. If the radius of a circular wave increases at the rate 5 cm/sec, find the rate of increase in its area at the instant when its radius is 8 cm.

Soln. Let,

A = Area of circle

R = radius of circle

$\therefore$ $A=\pi R^2$

$$\frac{dA}{dt}=2\pi R\frac{dR}{dt} \quad ..(1)$$

Here, $\frac{dR}{dt}=5cm/\sec$

अब हमें $\frac{dA}{dt}$ निकालना है

we have to find out

$\frac{dA}{dt}$ when $R=8cm$

putting these values in (1), we get,

$$\frac{dA}{dt}=2.\pi.8.5$$

$$=80\pi$$

OR, (अथवा)

Q.16. Find the intervals on which the function $f(x)=(x-1)(x-2)^2$; $x\in R$ is (a) increasing (b) decreasin

Soln. $f(x)=(x-1)(x^2-4x+4)$

$=x^3-4x^2+4x-x^2+4x-4$

$=x^3-5x^2+8x-4$

$f'(x)=3x^2-5.2x+8.1$

$=3x^2-10x+8$

$=3x^2-6x-4x+8$

$=3x(x-2)-4(x-2)$

$=(3x-4)(x-2)$

(a) For $f(x)$ to be increasing, we must have

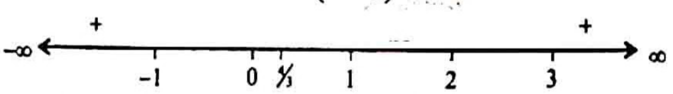
$f'(x)>0$

$\Rightarrow$ $(3x-4)(x-2)>0$

$\Rightarrow$ $x<\frac{4}{3}$ or $x>2$

$\Rightarrow$ $x\in\left(-\infty,\frac{4}{3}\right)\cup(2,\infty)$

So, $f(x)$ is increasing or $\left(-\infty,\frac{4}{3}\right)\cup(2,\infty)$



(b) For $f(x)$ to be decreasing, we must have

$(3x-4)(x-2)<0$

$\Rightarrow$ $\frac{4}{3}<x<2$

$\Rightarrow$ $x\in\left(\frac{4}{3},2\right)$.

so, $f(x)$ is decreasing on $\left(\frac{4}{3},2\right)$.

Q.17. Evaluate: $\int\frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$.

Soln. let $e^x=t$

$\therefore$ $e^x dx=dt$

putting the values

$$I=\int\frac{dt}{\sqrt{5-4t-t^2}}$$

$$=\int\frac{dt}{\sqrt{-(t^2+4t-5)}}$$

$$=\int\frac{dt}{\sqrt{-(t^2+2.t.2+4-4-5)}}$$

$$=\int\frac{dt}{\sqrt{-\{(t+2)^2-3^2\}}}$$

$$=\int\frac{dt}{\sqrt{3^2-(t+2)^2}}$$

$$=\sin^{-1}\frac{(t+2)}{3}+C$$

putting the value of t, we get

$$=\sin^{-1}\frac{(e^x+2)}{3}+C \text{ Ans.}$$

Q.18. Prove that $\int_0^\pi\frac{x}{1+\sin x}dx=\pi$.

Soln. we know that

$$\int_0^a xf(x)dx=\frac{a}{2}\int_0^a f(x)dx,\ if\ f(a-x)=f(x)$$

Here,

$$f(x)=\frac{1}{1+\sin x}$$

$\Rightarrow$ $f(\pi-x)=\frac{1}{1+\sin(\pi-x)}=\frac{1}{1+\sin x}$

$\Rightarrow$ $f(\pi-x)=f(x)$

L.H.S. = π

$\therefore$ $I=\int_0^\pi\frac{x}{1+\sin x}dx=\frac{\pi}{2}\int_0^\pi\frac{1}{1+\sin x}dx$

$$=\frac{\pi}{2}\int_0^\pi\frac{(1-\sin x)}{(1-\sin^2 x)}dx$$

$$=\frac{\pi}{2}\int_0^\pi\frac{1-\sin x}{\cos^2 x}dx$$

$$=\frac{\pi}{2}\int_0^\pi\left[\sec^2 x-\sec x.\tan x\right]dx$$

$$=\frac{\pi}{2}\left[\tan x-\sec x\right]_0^\pi$$

$$=\frac{\pi}{2}\left[(0+1)-(0-1)\right]$$

$$=\frac{\pi}{2}(2)$$

[illegible]

Q.19 Evaluate $\int_0^1 (3x^2+1)dx$ as a limit of a sum.

Soln. let $f(x) = (3x^2+1)$

$a = 0,\ b = 1$

$nh = (1-0) = 1$

$I = \int_0^1 (3x^2+1)$

$= \lim_{h\to 0} h\left[f(0)+f(0+h)+f(0+2h)+\ldots+f\{0+(n-1)h\}\right]$

$= \lim_{h\to 0} h\left[1+(3h^2+1)+(3\cdot 2^2h^2+1)+(3\cdot 3^2h^2+1)+\ldots\right.$

$\left.+3(n-1)^2h^2+1\right]$

$= \lim_{h\to 0} h\left[n+3h^2\left\{1^2+2^2+3^2+\ldots+(n-1)^2\right\}\right]$

$= \lim_{h\to 0} h\left[n+3h^2\left(\frac{1}{6}(n-1)n(2n-1)\right)\right]$

$= \lim_{h\to 0}\left[nh+\frac{1}{2}(nh-h)(nh)(2nh-h)\right]$

$= \lim_{h\to 0}\left[1+\frac{1}{2}(1-h).1.(2-h)\right] \because (nh=1)$

$= 1+\frac{1}{2}(1-0).1.(2-0)$

$= 1+\frac{1}{2}.1.1.2$

$= 1+1$

$= 2$ Ans.

Q.20. If $\vec{a} = \hat{i}-3\hat{j}+4\hat{k}$ and $\vec{b} = 2\hat{i}+\hat{j}+\hat{k}$, then find the following:

(i) $|\vec{a}|$ **(ii)** $\vec{a}.\vec{b}$

(iii) $\vec{a}\times\vec{b}$ **(iv)** Projection of $\vec{a}$ on $\vec{b}$.

Soln.(i) $|\vec{a}| = \sqrt{1^2+(-3)^2+4^2}$

$= \sqrt{1+9+16}$

$= \sqrt{26}$ Ans.

(ii) $\vec{a}.\vec{b} = \hat{i}.2\hat{i}+(-3\hat{j}).\hat{j}+4\hat{k}.\hat{k}$

$= 2-3+4$

$= 3$ Ans.

(iii) $\vec{a}.\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 4\\ 2 & 1 & 1\end{vmatrix}$

$= \hat{i}(-3-4)-\hat{j}(1-8)+\hat{k}[1-(-6)]$

$= -7\hat{i}+7\hat{j}+7\hat{k}$

$= 7(-\hat{i}+\hat{j}+\hat{k})$ Ans.

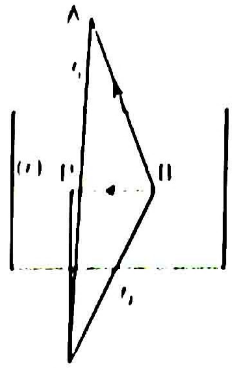
(iv) Projection of $\hat{a}$ on $\hat{b}$

$= |\hat{a}|\cos\theta$ · (*if θ be the angle between $\hat{a}$ & $\hat{b}$*)

[illegible]

Q.21 The position vectors of the two points A and B are $3\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}-2\hat{j}-4\hat{k}$ respectively. Find the vector equation of the plane passing through B and perpendicular to AB.

Soln.



we take any point P in the plane whose position vector is r, then as given $\vec{BP}$ is a vector in the plane and $\vec{BA}$ is ⊥ to the plane, so

$\vec{BP}.\vec{BA} = 0$

or, $(\vec{r}-\vec{r_2}).(\vec{r_1}-\vec{r_2}) = 0$

or, $\vec{r}.(\vec{r_1}-\vec{r_2}) = \vec{r_2}.(\vec{r_1}-\vec{r_2})$

or, $\vec{r}(2\hat{i}+3\hat{j}+6\hat{k}) = 1.2+(-2).3+(-4).6$

or, $\vec{r}(2\hat{i}+3\hat{j}+6\hat{k}) = -28$

This is required vector equation of the plane.

OR

Q.21. Find the equation of the straight line which passes through the point (0, −1, 4) and is parallel to the straight line

$$\frac{x-2}{1} = \frac{y+3}{7} = \frac{2z-6}{3}.$$

Soln. The d.r. of the line

$$\frac{x-2}{1} = \frac{y+3}{7} = \frac{2z-6}{3}$$

i.e. $$\frac{x-2}{1} = \frac{y+3}{7} = \frac{z-6}{3/2}$$

are $1, 7, 3/2$

so the required line which passes thr' (0, −1, 4) and ∥ to above line is

$$\frac{x-0}{1} = \frac{y+1}{7} = \frac{z-4}{3/2}$$

$$\frac{x}{1} = \frac{y+1}{7} = \frac{z-4}{3/2}$$

Q.22. If A and B are two independent events such that $P(A) = 0.35$ and $P(A\cup B) = 0.6$, find $P(B)$.

Soln.

$\because$ $P(A\cup B) = P(A)+p(B)-P(A\cap B)$

$= P(A)+P(B)-P(A).P(B)$

since A & B are independent

or, $\quad 0.6 = 0.35 + P(B) - 0.35P(B)$

or, $\quad 0.6 - 0.35 = P(B) - 0.35P(B)$

or, $\quad 0.25 = P(B)\{1 - 0.35\}$

$\therefore \quad P(B) = \dfrac{0.25}{0.65} = \dfrac{5}{13} = 0.38$ Ans

Section-C

Q.23. Using elementary row transformations, find the inverse of

$$A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}.$$

Soln. Writing $A = IA$

$$\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \leftrightarrow R_2$

or, $$\begin{bmatrix} -3 & 0 & -1 \\ 1 & 3 & -2 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Aplying $R_2 \leftrightarrow R_3$

or, $$\begin{bmatrix} -3 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_1 \leftrightarrow R_1 \times \left(-\dfrac{1}{3}\right)$

or, $$\begin{vmatrix} -3\times\left(-\frac{1}{3}\right) & 0 & -1\times\left(-\frac{1}{3}\right) \\ 2 & 1 & 0 \\ 1 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1\times\left(-\frac{1}{3}\right) & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} A$$

or, $$\begin{vmatrix} 1 & 0 & 1/3 \\ 2 & 1 & 0 \\ 1 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 0 & -1/3 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} A$$

Applying $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

or, $$\begin{bmatrix} 1 & 0 & 1/3 \\ 2-2.1 & 1 & 0-2/3 \\ 1-1 & 3-0 & -2-1/3 \end{bmatrix} = \begin{bmatrix} 0 & -1/3 & 0 \\ 0 & 0+2/3 & 1-2.0 \\ 1 & 0+1/3 & 0 \end{bmatrix} A$$

or, $$\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -2/3 \\ 0 & 3 & -7/3 \end{bmatrix} = \begin{bmatrix} 0 & -1/3 & 0 \\ 0 & +2/3 & 1 \\ 1 & +1/3 & 0 \end{bmatrix} A$$

Applying $R_3 \to R_3 - 3R_2$

or, $$\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -2/3 \\ 0 & 3-3.1 & -7/3-\left(3\times-\frac{2}{3}\right) \end{bmatrix} = \begin{bmatrix} 0 & -1/3 & 0 \\ 0 & +2/3 & 1 \\ 1 & +1/3-\left(3\times+2/3\right) & 0-2 \end{bmatrix} A$$

or, $$\begin{vmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & -1/3 \end{vmatrix} = \begin{vmatrix} 0 & -1/3 & 0 \\ 0 & +2/3 & 1 \\ 1 & -5/3 & -3 \end{vmatrix} A$$

Applying $R_3 \to R_3 \times -3$

or, $$\begin{vmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1/3 & 0 \\ 0 & +2/3 & 1 \\ -3 & +5 & 9 \end{vmatrix} A$$

Applying $R_1 \to R_3 \times \dfrac{1}{3}$

$R_2 \to R_2 + R_3 \times \dfrac{2}{3}$

or, $$\begin{bmatrix} 1 & 0 & \frac{1}{3}-1.\frac{1}{3} \\ 0 & 1 & -\frac{2}{3}+1\times\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & \frac{2}{3}+5.\frac{2}{3} & 1+9.\frac{2}{3} \\ -3 & 5 & 9 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix} A$$

Hence

$$A^{-1} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$$

OR

Q.23. Solve by matrix method, the following equations:

$$2x - y + z = 2$$
$$x + y - z = 1$$
$$4x + y + 2z = 7.$$

Soln. The given equations can be written as

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$$

or, $\quad AX = B$

or, $\quad X = A^{-1}B$

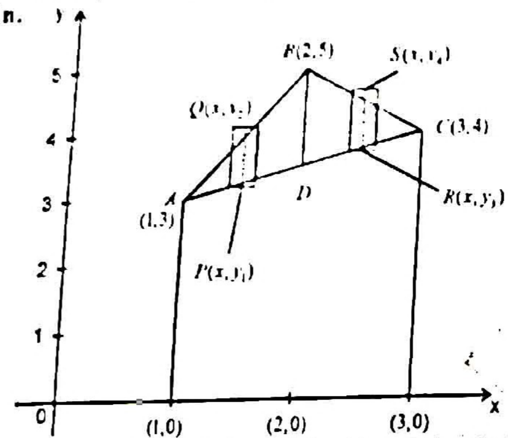
Now we have to find out A^{-1}

we will first find out co-factors of A

$y = \frac{3x}{2\sqrt{2}} - \frac{3}{4}$ Required Equation

Q.25. Using integration, find the area of the triangle whose vertices are (1, 3), (2, 5) and (3, 4).

Soln.



First we find the equations of the sides of triangle ABC by using

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

The equation of AB is

$$y - 3 = \frac{5-3}{2-1}(x-1)$$

$\Rightarrow \quad 2x - y + 1 = 0 \quad$ ...(1)

The equation of BC is

$$y - 5 = \frac{4-5}{3-2}(x-2)$$

$\Rightarrow \quad x + y - 7 = 0 \quad$ ...(2)

The equation of AC is

$$y - 3 = \frac{4-3}{3-1}(x-1)$$

$x - 2y + 5 = 0 \quad$ ...(3)

Clearly, Area of $\Delta ABC = Area\ ADB + Area\ BDC$

Area ADB: To find the area ADB we slice it into vertical strips. We observe that each vertical strip has its lower end on side AC and the upper and on AB. So, the approximating rectangle has length $= (y_2 - y_1)$, width $= \Delta x$ and

$$\int_1^2 (y_2 - y_1)\,dx$$

or, $y_1 = \frac{x+5}{2}$ and $y_2 = 2x + 1$

$$Area\ ADB = \frac{3}{2}\left[\frac{x^2}{2} - \frac{3}{2}x\right]_1^2 = \frac{3}{4}(2^2 - 1^2) - \frac{3}{2}(2-1)$$

$$= \frac{9}{4} - \frac{3}{2} = \frac{3}{4} \quad \text{...(A)}$$

Similarly, we have

$$Area\ BDC = \int_2^3 (y_4 - y_3)\,dx$$

$\because$ $R(x, y_3)$ and $S(x, y_4)$ lie on line (iii) and (ii) respectively

$\therefore$ $x - 2y_3 + 5 = 0$ & $x + y_4 - 7 = 0$

or, $\frac{x+5}{2} = y_3$ & $y_4 = -x + 7$

$$Area\ BDC = \int_2^3 -x + 7 - \left(\frac{x+5}{2}\right)dx$$

$$= \int_2^3 \left(\frac{-3x+9}{2}\right)$$

$$= \frac{1}{2}\int_2^3 (-3x+9)\,dx$$

$\therefore$ $Area\ ABC = A + B$

$$= \frac{3}{4} + \frac{3}{4}$$

$$= \frac{2.3}{4} = \frac{3}{2}\ sq.unit \text{ Ans.}$$

Q.26. Verify that $y = e^x(\sin x + \cos x)$ is a solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

Soln. $y = e^x(\sin x + \cos x) \quad$ ...(i)

Differentiating both sides

$$\frac{dy}{dx} = e^x(\cos x - \sin x) + e^x(\sin x + \cos x)$$

$\Rightarrow \quad \frac{dy}{dx} - y = e^x(\cos x - \sin x) \quad$ ..(ii)

$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x(-\sin x - \cos x) + e^x(\cos x - \sin x)$

$\Rightarrow \quad \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = -y + \left(\dfrac{dy}{dx} - y\right)$

using (i) & (ii)

$\Rightarrow \quad \dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$

So, y is the soln. of the above differential equation.

[दिया हुआ differential equation का हल है)

OR,

Q.26. Solve the differential equation $x\dfrac{dy}{dx} = y - x\tan\dfrac{y}{x}$.

$x\dfrac{dy}{dx} = y - x\tan\dfrac{y}{x}$

Dividing both sides by x

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{y}{x} - \tan\dfrac{y}{x}$

putting $y = vx$

$\therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

The given equation becomes

$v + x.\dfrac{dv}{dx} = \dfrac{v.x}{x} - \tan\dfrac{v.x}{x}$

$\Rightarrow \quad x\dfrac{dv}{dx} = -\tan v$

$\Rightarrow \quad \dfrac{dx}{x} = -\dfrac{dv}{\tan v} = -\cot v.dv$

Integrating both sides

$\Rightarrow \quad \int\dfrac{dx}{x} = \int -\cot v.dv$

$\Rightarrow \quad \log x = -\log\sin v + \log c$

$\Rightarrow \quad \log x = \log\left(\dfrac{c}{\sin v}\right)$

$\Rightarrow \quad x = \dfrac{c}{\sin\frac{y}{x}}$

$\Rightarrow \quad x.\sin\dfrac{y}{x} = c$ Solved

Q.27. Consider the equations of the straight lines given by

$L_1 : \vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} + \hat{j} + \hat{k})$

$L_2 : \vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$

If $\vec{a_1} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b_1} = \hat{i} - \hat{j} + \hat{k}$, $\vec{a_2} = 2\hat{i} - \hat{j} - \hat{k}$ **and**

$\vec{b_2} = 2\hat{i} + \hat{j} + 2\hat{k}$, **then find**

(i) $\vec{a_2} - \vec{a_1}$ (ii) $\vec{b_2} - \vec{b_1}$

(iii) $\vec{b_1} \times \vec{b_2}$ (iv) $\vec{a_1} \times \vec{a_2}$

(v) $\left(\vec{b_1} \times \vec{b_2}\right).\left(\vec{a_2} - \vec{a_1}\right)$

(vi) the shortest distance between L_1 and L_2.

Soln.

(i) $(\vec{a_2} - \vec{a_1}) = \hat{i} - 3\hat{j} - 2\hat{k}$ Ans.

(ii) $(\vec{b_2} - \vec{b_1}) = \hat{i} + 2\hat{j} + \hat{k}$ Ans.

(iii) $\vec{b_1} - \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$

$= -3\hat{i} - 0\hat{j} + 3\hat{k}$

$= -3\hat{i} + 3\hat{k}$ Ans.

(iv) $\vec{a_1} \times \vec{a_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 2 & -1 & -1 \end{vmatrix}$

$= -\hat{i} + 3\hat{j} - 5\hat{k}$ Ans.

(v) $\left(\vec{b_1} \times \vec{b_2}\right).\left(\vec{a_2} - \vec{a_1}\right)$

$= \left(-3\hat{i} - 0\hat{j} + 3\hat{k}\right).\left(\hat{i} - 3\hat{j} - 2\hat{k}\right)$

$= -3\hat{i}\hat{i} - 6\hat{k}\hat{k}$

$= -3 - 6$

$= -9$ Ans.

(vi) Shortest distance

$= \left|\dfrac{(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|}\right|$

$= \left|\dfrac{-9}{|3\hat{i} + 3\hat{k}|}\right| = \left|\dfrac{-9}{3\sqrt{2}}\right| = \left|\dfrac{-3}{\sqrt{2}}\right|$

$= \dfrac{3}{\sqrt{2}}$

Q.28. A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, whereas the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B on the job for 30% of the time and C on the job for 20% of the time.

Let E_1: be event that an item is produced by A, E_2: be event that an item is produced by B and E_3: be event that an item is produced by C. If E be the event that a defective item is produced, then find the following.

(i) $P(E_1)$ (ii) $P(E_2)$ (iii) $P(E_3)$

(iv) $P(E/E_1)$ (v) $P(E/E_2)$ (vi) $P(E_1/E)$.

Soln. (i) $P(E_1) = \dfrac{50}{100} = 0.5$ Ans

(ii) $P(E_2) = \dfrac{30}{100} = 0.3$ Ans

(iii) $P(E_3) = \dfrac{20}{100} = 0.2$ Ans.

(iv) $P\left(\dfrac{E}{E_1}\right) = \dfrac{1}{100} = 0.01$ Ans

$A_{11} = +\{2-(-1)\} = 3$

$A_{12} = -\{2-(-4)\} = -6$

$A_{13} = +\{1-4\} = -3$

$A_{21} = -\{-2-1\} = 3$

$A_{22} = +\{4-4\} = 0$

$A_{23} = -\{2-(-4)\} = -6$

$A_{31} = +\{1-1\} = 0$

$A_{32} = -\{-2-1\} = 3$

$A_{33} = +\{2-(-1)\} = 3$

$$|A| = 2\{2-(-1)\} - (-1)\{2-(-4)\} + 1\{1-4\}$$
$$= 2\{2+1\} + 1\{2+4\} + 1\{-3\}$$
$$= 2.3 + 6 - 3$$
$$= 6 + 6 - 3$$
$$= 12 - 3$$
$$= 9$$

$|A| \neq 0$, A is non singular matrix. So, its A^{-1} exists

[A एक non singular matrix है अत: इसका A^{-1} होगा)

Matrix made by its co-factors

$$\begin{bmatrix} 3 & -6 & -3 \\ 3 & 0 & -6 \\ 0 & 3 & 3 \end{bmatrix}$$

$$\text{Adjoint of } A = \begin{bmatrix} 3 & -6 & -3 \\ 3 & 0 & -6 \\ 0 & 3 & 3 \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & 3 & 0 \\ -6 & 0 & 3 \\ -3 & -6 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adjoint } A}{|A|} = \frac{\begin{bmatrix} 3 & 3 & 0 \\ -6 & 0 & 3 \\ -3 & -6 & 3 \end{bmatrix}}{9}$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{2}{3} & 0 & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{2}{3} & 0 & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} + \frac{1}{3} + 0 \\ -\frac{4}{3} + 0 + \frac{7}{3} \\ -\frac{2}{3} + \left(-\frac{2}{3}\right) + \frac{7}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\left.\begin{matrix} x = 1 \\ y = 1 \\ z = 1 \end{matrix}\right\} Ans.$$

Q.24. Find the equations of the tangent and normal to the curve $x = \sin 3t$, $y = \cos 2t$ at the point $t = \frac{\pi}{4}$.

Soln. $x = \sin 3t$

x at $t = \frac{\pi}{4}$

$$x_1 = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$$

$y = \cos 2t$

y at $t = \frac{\pi}{4}$

$$y_1 = \cos\frac{2.\pi}{4} = \cos\frac{\pi}{2} = 0$$

$$\frac{dx}{dt} = \cos 3t \times 3 = 3\cos 3t$$

$$\frac{dy}{dt} = -\sin 2t \times 2 = -2\sin 2t$$

$$\frac{dy}{dx} = \frac{-2\sin 2t}{3\cos 3t}$$

Now $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$

$$\left(\frac{dy}{dx}\right)_{t=\pi/4} = \frac{-2\sin 2.\frac{\pi}{4}}{3\cos 3.\frac{\pi}{4}} = \frac{-2.1}{3.\left(\frac{-1}{\sqrt{2}}\right)}$$

$$= \frac{2.\sqrt{2}}{3}$$

Equation of tangent at $t = \frac{\pi}{4}$

$$(y - y_1) = \left(\frac{dy}{dx}\right)_{t=\pi/4} (x - x_1)$$

$$y - 0 = \frac{2\sqrt{2}}{3}\left(x - \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad y = \frac{2\sqrt{2}}{3}x - \frac{2\sqrt{2}}{3\sqrt{2}}$$

(v) $P\left(\frac{E}{E_2}\right)=\frac{5}{100}=\frac{1}{20}=0.05$ Ans.

(vi) $P\left(\frac{E_1}{E}\right)=\frac{5}{34}$ Ans.

Q.29. Solve graphically the following L.P.P.

Minimise $Z=5x+10y$

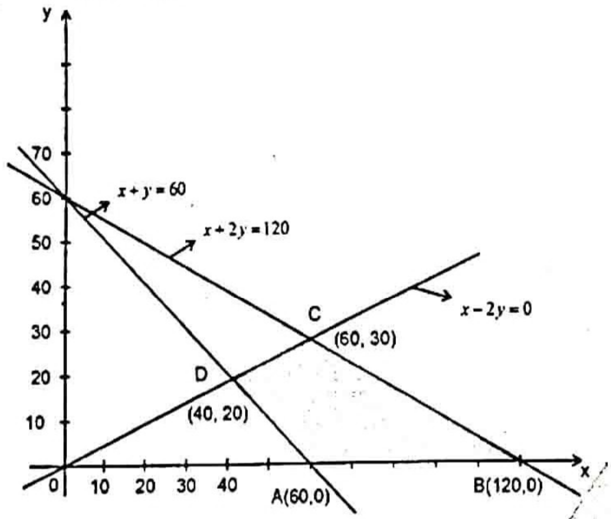
subject to $x+y\geq 60$

$x+2y\leq 120$

$x-2y\geq 0$

and $x,y\geq 0.$

Soln.



Since $x,y\geq 0$, so the feasible region will be in the 1st quadrant.

We change the inequality into equality, and draw the graph.

$x+y=60$

$\Rightarrow \quad \frac{x}{60}+\frac{y}{60}=1$

The line intercepts x axis at 60 and y axis at 60

$\because \quad x+y>60$, so the region will be upper side of the line.

Now consider the line

$x+2y=120$

$\Rightarrow \quad \frac{x}{120}+\frac{2y}{120}=1$

$\Rightarrow \quad \frac{x}{120}+\frac{y}{60}=1$

The intercepts x-axis at 120 and y axis at 60.

$\because \quad x+2y<120$

$\therefore$ feasible region will be lower side of the line.

Now, consider the line

$x-2y=0$

$\Rightarrow \quad x=2y$

when $x=0,\ y=0;\ x=2y=1$

$\because$ feasible region will be the lower side of the line

Thus the feasible region is Region ABCD

Now, we find the co-ordinate of point C and point D.

To find the co-ordinate of C

$$\begin{aligned} x+2y&=120\\ x-2y&=0\\ \hline 2x&=120 \end{aligned}$$

$x=60$

$y=30$

$C(60,30)$

Now, to find co-ordinage of D

$$\begin{aligned} x+y&=60\\ x-2y&=0\\ \hline 3y&=60 \end{aligned}$$

$y=20$

$x=40$

Now, at points A, B, C and D, we find the minimum value of the fraction $z=5x+10y$

$Z_A=5.60+10.0=300$

$Z_B=5.120+10.0=600$

$Z_C=5.60+10.30=600$

$Z_D=5.40+10.20=200+200=400$

The minimum value of the fraction is 300 at point A.

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