Chapter 13

# THE HYPERBOLA

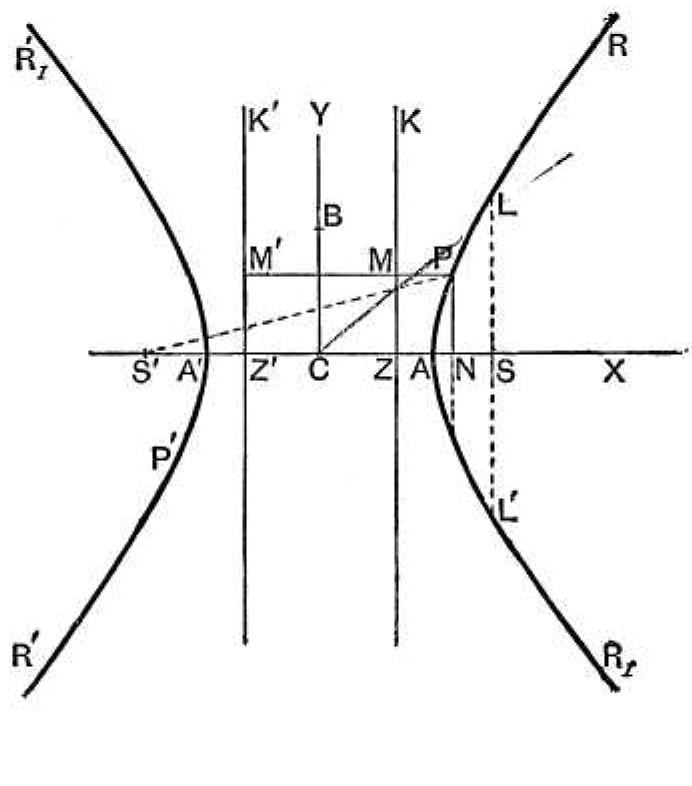
**295.** THE hyperbola is a Conic Section in which the eccentricity $e$ is greater than unity.

*To find the equation to a hyperbola.*

Let $ZK$ be the directrix, $S$ the focus, and let $SZ$ be perpendicular to the directrix.

There will be a point $A$ on $AZ$, such that

$$SA = e \,.\, AZ \qquad \text{.....................} \qquad (1).$$



Since $e > 1$, there will be another point $A'$, on $SZ$ produced, such that

$$SA' = e \,.\, A'Z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Let the length $AA'$ be called $2a$, and let $C$ be the middle point of $AA'$.

Subtracting (1) from (2), we have

$$2a = AA' = e \,.\, A'Z - e \,.\, AZ$$
$$= e\left[CA' + CZ\right] - e\left[CA - CZ\right] = e \,.\, 2CZ,$$

*i.e.*
$$CZ = \frac{a}{e} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Adding (1) and (2), we have

$$e\,(AZ + A'Z) = SA' + SA = 2CS,$$

*i.e.*
$$e \,.\, AA' = 2 \,.\, CS,$$

and hence
$$CS = ae \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Let $C$ be the origin, $CSX$ the axis of $x$, and a straight line $CY$, through $C$ perpendicular to $CX$, the axis of $y$.

Let $P$ be any point on the curve, whose coordinates are $x$ and $y$, and let $PM$ be the perpendicular upon the directrix, and $PN$ the perpendicular on $AA'$.

The focus $S$ is the point $(ae, 0)$.

The relation $SP^2 = e^2 \,.\, PM^2 = e^2 \,.\, ZN^2$ then gives

$$(x - ae)^2 + y^2 = e^2\left[x - \frac{a}{e}\right]^2,$$

*i.e.*
$$x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2.$$

Hence
$$x^2\,(e^2 - 1) - y^2 = a^2\,(e^2 - 1),$$

*i.e.*
$$\frac{x^2}{a^2} - \frac{y^2}{a^2\,(e^2 - 1)} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Since, in the case of the hyperbola, $e > 1$, the quantity $a^2\,(e^2 - 1)$ is positive. Let it be called $b^2$, so that the equation (5) becomes

$$\mathbf{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

where
$$b^2 = a^2e^2 - a^2 = CS^2 - CA^2 \quad \ldots\ldots\ldots\ldots\ldots (7),$$

and therefore
$$CS^2 = a^2 + b^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

**296.** The equation (6) may be written

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1 = \frac{x^2 - a^2}{a^2} = \frac{(x-a)(x+a)}{a^2},$$

*i.e.*
$$\frac{PN^2}{b^2} = \frac{AN \cdot NA'}{a^2},$$

so that
$$PN^2 : AN \cdot NA' :: b^2 : a^2.$$

If we put $x = 0$ in equation (6), we have $y^2 = -b^2$, shewing that the curve meets the axis $CY$ in imaginary points.

**Def.** The points $A$ and $A'$ are called the vertices of the hyperbola, $C$ is the centre, $AA'$ is the transverse axis of the curve, whilst the line $BB'$ is called the conjugate axis, where $B$ and $B'$ are two points on the axis of $y$ equidistant from $C$, as in the figure of Art. 315, and such that

$$B'C = CB = b.$$

**297.** Since $S$ is the point $(ae, 0)$, the equation referred to the focus as origin is, by Art. 128,

$$\frac{(x+ae)^2}{a^2} - \frac{y^2}{b^2} = 1,$$

*i.e.*
$$\frac{x^2}{a^2} + 2\frac{ex}{a} - \frac{y^2}{b^2} + e^2 - 1 = 0.$$

Similarly, the equations, referred to the vertex $A$ and foot of the directrix $Z$ respectively as origins, will be found to be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{2x}{a} = 0,$$

and
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{2x}{ae} = 1 - \frac{1}{e^2}.$$

The equation to the hyperbola, whose focus, directrix, and eccentricity are any given quantities, may be written down as in the case of the ellipse (Art. 249).

**298.** *There exist a second focus and a second directrix to the curve.*

On $SC$ produced take a point $S'$, such that

$$SC = CS' = ae,$$

and another point $Z'$, such that

$$ZC = CZ' = \frac{a}{e}.$$

Draw $Z'M'$ perpendicular to $AA'$, and let $PM$ be produced to meet it in $M'$.

The equation (5) of Art. 295 may be written in the form

$$x^2 + 2aex + a^2e^2 + y^2 = e^2x^2 + 2aex + a^2,$$

*i.e.* $$(x + ae)^2 + y^2 = e^2\left(\frac{a}{e} + x\right)^2,$$

*i.e.* $$S'P^2 = e^2(Z'C + CN)^2 = e^2 . PM'^2.$$

Hence any point $P$ of the curve is such that its distance from $S'$ is $e$ times its distance from $Z'K'$, so that we should have obtained the same curve if we had started with $S'$ as focus, $Z'K'$ as directrix, and the same eccentricity $e$.

**299.** *The difference of the focal distances of any point on the hyperbola is equal to the transverse axis.*

For (Fig., Art 295) we have

$$SP = e . PM, \text{ and } S'P = e . PM'.$$

Hence $$S'P - SP = e(PM' - PM) = e . MM'$$
$$= e . ZZ' = 2e . CZ = 2a$$
$$= \text{the transverse axis } AA'.$$

Also $SP = e . PM = e . ZN = e . CN - e . CZ = \mathbf{ex' - a},$

and $S'P = e . PM' = e . Z'N = e . CN + e . Z'C = \mathbf{ex' + a},$

where $x'$ is the abscissa of the point $P$ referred to the centre as origin.

**300.** *Latus-rectum of the Hyperbola.*

Let $LSL'$ be the latus-rectum, *i.e.* the double ordinate of the curve drawn through $S$.

By the definition of the curve, the semi-latus-rectum $SL$

$= e$ times the distance of $L$ from the directrix

$= e . SZ = e(CS - CZ)$

$$= e . CS - eCZ = ae^2 - a = \frac{b^2}{a},$$

by equations (3), (4), and (7) of Art. 295.

**301.** *To trace the curve*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ ..................... } (1).$$

The equation may be written in either of the forms

$$y = \pm b\sqrt{\frac{x^2}{a^2} - 1} \text{ ................. } (2),$$

or

$$x = \pm a\sqrt{\frac{y^2}{b^2} + 1} \text{ ................. } (3).$$

From (2), it follows that, if $x^2 < a^2$, *i.e.* if $x$ lie between $a$ and $-a$, then $y$ is impossible. There is therefore no part of the curve between $A$ and $A'$.

For all values of $x^2 > a^2$ the equation (2) shews that there are two equal and opposite values of $y$, so that the curve is symmetrical with respect to the axis of $x$. Also, as the value of $x$ increases, the corresponding values of $y$ increase, until, corresponding to an infinite value of $x$, we have an infinite value of $y$.

For all values of $y$, the equation (3) gives two equal and opposite values to $x$, so that the curve is symmetrical with respect to the axis of $y$.

If a number of values in succession be given to $x$, and the corresponding values of $y$ be determined, we shall obtain a series of points, which will all be found to lie on a curve of the shape given in the figure of Art. 295.

The curve consists of two portions, one of which extends in an infinite direction towards the positive direction of the axis of $x$, and the other in an infinite direction towards the negative end of this axis.

**302.** *The quantity* $\frac{x'^2}{a^2} - \frac{y'^2}{b^2} - 1$ *is positive, zero, or negative, according as the point* $(x', y')$ *lies within, upon, or without, the curve.*

Let $Q$ be the point $(x', y')$, and let the ordinate $QN$

through $Q$ meet the curve in $P$, so that, by equation (6) of Art. 295,

$$\frac{x'^2}{a^2} - \frac{PN^2}{b^2} = 1,$$

and hence
$$\frac{PN^2}{b^2} = \frac{x'^2}{a^2} - 1.$$

If $Q$ be within the curve then $y'$, *i.e.* $QN$, is less than $PN$, so that
$$\frac{y'^2}{b^2} < \frac{PN^2}{b^2}, \text{ i.e.} < \frac{x'^2}{a^2} - 1.$$

Hence, in this case, $\frac{x'^2}{a^2} - \frac{y'^2}{b^2} > 0$, *i.e.* is positive.

Similarly, if $Q$ be without the curve, then $y' > PN$, and we have
$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} - 1 \text{ negative.}$$

**303.** *To find the length of any central radius drawn in a given direction.*

The equation (6) of Art. 295, when transferred to polar coordinates, becomes

$$r^2\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) = 1,$$

*i.e.*
$$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2} = \frac{\cos^2\theta}{b^2}\left(\frac{b^2}{a^2} - \tan^2\theta\right)\ldots\ldots(1).$$

This is the equation giving the value of any central radius of the curve drawn at an inclination $\theta$ to the transverse axis.

So long as $\tan^2\theta < \frac{b^2}{a^2}$, the equation (1) gives two equal and opposite values of $r$ corresponding to any value of $\theta$.

For values of $\tan^2\theta > \frac{b^2}{a^2}$, the corresponding values of $\frac{1}{r^2}$ are negative, and the corresponding values of $r$ imaginary. Any radius drawn at a greater inclination than $\tan^{-1}\frac{b}{a}$

does not therefore meet the curve in any real points, so that all the curve is included within two straight lines drawn through $C$ and inclined at an angle $\pm \tan^{-1}\frac{b}{a}$ to $CX$.

Writing (1) in the form

$$r^2 = \frac{b^2}{\cos^2\theta\left(\frac{b^2}{a^2} - \tan^2\theta\right)},$$

we see that $r$ is least when the denominator is greatest, *i.e.* when $\theta = 0$. The radius vector $CA$ is therefore the least.

Also, when $\tan\theta = \pm\frac{b}{a}$, the value of $r$ is infinite.

For values of $\theta$ between 0 and $\tan^{-1}\frac{b}{a}$ the corresponding positive values of $r$ give the portion $AR$ of the curve (Fig., Art. 295) and the corresponding negative values give the portion $A'R'$.

For values of $\theta$ between 0 and $-\tan^{-1}\frac{b}{a}$, the positive values of $R$ give the portion $AR_1$, and the negative values give the portion $A'R_1'$.

The ellipse and the hyperbola since they both have a centre $C$, such that all chords of the conic passing through it are bisected at it, are together called **Central Conics.**

**304.** In the hyperbola any ordinate of the curve does not meet the circle on $AA'$ as diameter in real points. There is therefore no real eccentric angle as in the case of the ellipse.

When it is desirable to express the coordinates of any point of the curve in terms of one variable, the substitutions

$$\mathbf{x = a \sec\phi} \text{ and } \mathbf{y = b \tan\phi}$$

may be used; for these substitutions clearly satisfy the equation (6) of Art. 295.

The angle $\phi$ can be easily defined geometrically.

On $AA'$ describe the auxiliary circle, (Fig., Art. 306) and from the foot $N$ of any ordinate $NP$ of the curve draw a tangent $NU$ to this circle, and join $CU$. Then

$$CU = CN \cos NCU,$$

*i.e.* $$x = CN = a \sec NCU.$$

The angle $NCU$ is therefore the angle $\phi$.

Also $$NU = CU \tan \phi = a \tan \phi,$$

so that $$NP : NU :: b : a.$$

The ordinate of the hyperbola is therefore in a constant ratio to the length of the tangent drawn from its foot to the auxiliary circle.

This angle $\phi$ is not so important an angle for the hyperbola as the eccentric angle is for the ellipse.

**305.** Since the fundamental equation to the hyperbola only differs from that to the ellipse in having $-b^2$ instead of $b^2$, it will be found that many propositions for the hyperbola are derived from those for the ellipse by changing the sign of $b^2$.

Thus, as in Art. 260, the straight line $y = mx + c$ meets the hyperbola in points which are real, coincident, or imaginary, according as

$$c^2 >=< a^2m^2 - b^2.$$

As in Art. 262, the equation to the tangent at $(x', y')$ is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1.$$

As in Art. 263, the straight line

$$y = mx + \sqrt{a^2m^2 - b^2}$$

is always a tangent.

The straight line

$$x \cos \alpha + y \sin \alpha = p$$

is a tangent, if $$p^2 = a^2 \cos^2 \alpha - b^2 \sin^2 \alpha.$$

The straight line $lx + my = n$

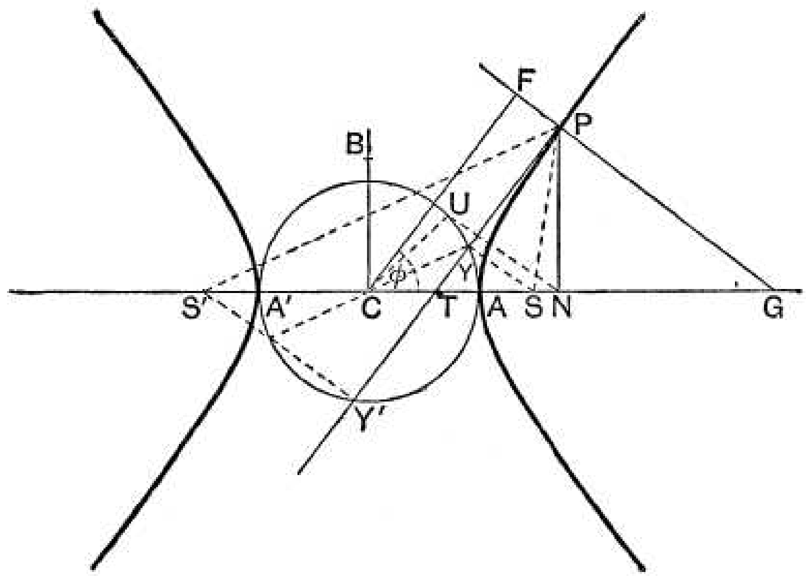
is a tangent, if $$n^2 = a^2l^2 - b^2m^2.$$ [Art. 264.]

The normal at the point $(x', y')$ is, as in Art. 266,

$$\frac{x-x'}{\frac{x'}{a^2}} = \frac{y-y'}{\frac{y'}{-b^2}}.$$

**306.** With some modifications the properties of Arts. 269 and 270 are true for the hyperbola also, if the corresponding figure be drawn.

In the case of the hyperbola the tangent bisects the interior, and the normal the exterior, angle between the focal distances $SP$ and $S'P$.




It follows that, if an ellipse and a hyperbola have the same foci $S$ and $S'$, they cut at right angles at any common point $P$. For the tangents in the two cases are respectively the internal and external bisectors of the angle $SPS'$, and are therefore at right angles.

**307.** The equation to the straight lines joining the points $(a \sec\phi,\ b\tan\phi)$ and $(a\sec\phi',\ b\tan\phi')$ can be shewn to be

$$\frac{x}{a}\cos\frac{\phi'-\phi}{2} - \frac{y}{b}\sin\frac{\phi+\phi'}{2} = \cos\frac{\phi+\phi'}{2}.$$

Hence, by putting $\phi' = \phi$, it follows that the tangent at the point $(a \sec \phi,\ b \tan \phi)$ is

$$\frac{x}{a} - \frac{y}{b} \sin \phi = \cos \phi.$$

It could easily be shewn that the equation to the normal is

$$ax \sin \phi + by = (a^2 + b^2) \tan \phi.$$

**308.** The proposition of Art. 272 is true also for the hyperbola.

As in Art. 273, the chord of contact of tangents from $(x_1, y_1)$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

As in Art. 274, the polar of any point $(x_1, y_1)$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

As in Arts. 279 and 281, the locus of the middle points of chords, which are parallel to the diameter $y = mx$, is the diameter $y = m_1 x$, where

$$mm_1 = \frac{b^2}{a^2}.$$

The proposition of Art. 278 is true for the hyperbola also, if we replace $b^2$ by $-b^2$.

**309. Director circle.** The locus of the intersection of tangents which are at right angles is, as in Art. 271, found to be the circle $x^2 + y^2 = a^2 - b^2$, *i.e.* a circle whose centre is the origin and whose radius is $\sqrt{a^2 - b^2}$.

If $b^2 < a^2$, this circle is real.

If $b^2 = a^2$, the radius of the circle is zero, and it reduces to a point circle at the origin. In this case the centre is the only point from which tangents at right angles can be drawn to the curve.

If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle, and so no tangents at right angles can be drawn to the curve.

## 310. Equilateral, or Rectangular, Hyperbola.

The particular kind of hyperbola in which the lengths of the transverse and conjugate axes are equal is called an equilateral, or rectangular, hyperbola. The reason for the name "rectangular" will be seen in Art. 318.

Since, in this case, $b = a$, the equation to the equilateral hyperbola, referred to its centre and axes, is $x^2 - y^2 = a^2$.

*The eccentricity of the rectangular hyperbola is* $\sqrt{2}$.

For, by Art. 295, we have, in this case,

$$e^2 = \frac{a^2 + b^2}{a^2} = \frac{2a^2}{a^2} = 2,$$

so that $$e = \sqrt{2}.$$

**311. Ex.** *The perpendiculars from the centre upon the tangent and normal at any point of the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *meet them in* $Q$ *and* $R$. *Find the loci of* $Q$ *and* $R$.

As in Art. 308, the straight line

$$x \cos \alpha + y \sin \alpha = p$$

is a tangent, if $$p^2 = a^2 \cos^2 \alpha - b^2 \sin^2 \alpha.$$

But $p$ and $\alpha$ are the polar coordinates of $Q$, the foot of the perpendicular on this straight line from $C$.

The polar equation to the locus of $Q$ is therefore

$$r^2 = a^2 \cos^2 \theta - b^2 \sin^2 \theta,$$

*i.e.*, in Cartesian coordinates,

$$(x^2 + y^2)^2 = a^2x^2 - b^2y^2.$$

If the hyperbola be rectangular, we have $a = b$, and the polar equation is

$$r^2 = a^2 (\cos^2 \theta - \sin^2 \theta) = a^2 \cos 2\theta.$$

Again, by Art. 307, any normal is

$$ax \sin \phi + by = (a^2 + b^2) \tan \phi \quad \text{.....................(1).}$$

The equation to the perpendicular on it from the origin is

$$bx - ay \sin \phi = 0 \quad \text{.......................... (2).}$$

If we eliminate $\phi$, we shall have the locus of $R$.

From (2), we have $$\sin \phi = \frac{bx}{ay},$$

and then $$\tan \phi = \frac{\sin \phi}{\sqrt{1 - \sin^2 \phi}} = \frac{bx}{\sqrt{a^2y^2 - b^2x^2}}.$$

Substituting in (1) the locus is

$$(x^2 + y^2)^2 (a^2y^2 - b^2x^2) = (a^2 + b^2)^2 x^2y^2.$$

## EXAMPLES XXXVI

Find the equation to the hyperbola, referred to its axes as axes of coordinates,

**1.** whose transverse and conjugate axes are respectively 3 and 4,

**2.** whose conjugate axis is 5 and the distance between whose foci is 13,

**3.** whose conjugate axis is 7 and which passes through the point $(3, -2)$,

**4.** the distance between whose foci is 16 and whose eccentricity is $\sqrt{2}$.

**5.** In the hyperbola $4x^2 - 9y^2 = 36$, find the axes, the coordinates of the foci, the eccentricity, and the latus rectum.

**6.** Find the equation to the hyperbola of given transverse axis whose vertex bisects the distance between the centre and the focus.

**7.** Find the equation to the hyperbola, whose eccentricity is $\frac{5}{4}$, whose focus is $(a, 0)$, and whose directrix is $4x - 3y = a$.

Find also the coordinates of the centre and the equation to the other directrix.

**8.** Find the points common to the hyperbola $25x^2 - 9y^2 = 225$ and the straight line $25x + 12y - 45 = 0$.

**9.** Find the equation of the tangent to the hyperbola $4x^2 - 9y^2 = 1$ which is parallel to the line $4y = 5x + 7$.

**10.** Prove that a circle can be drawn through the foci of a hyperbola and the points in which any tangent meets the tangents at the vertices.

**11.** An ellipse and a hyperbola have the same principal axes. Shew that the polar of any point on either curve with respect to the other touches the first curve.

**12.** In both an ellipse and a hyperbola, prove that the focal distance of any point and the perpendicular from the centre upon the tangent at it meet on a circle whose centre is the focus and whose radius is the semi-transverse axis.

**13.** Prove that the straight lines $\frac{x}{a} - \frac{y}{b} = m$ and $\frac{x}{a} + \frac{y}{b} = \frac{1}{m}$ always meet on the hyperbola.

**14.** Find the equation to, and the length of, the common tangent to the two hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

**15.** In the hyperbola $16x^2 - 9y^2 = 144$, find the equation to the diameter which is conjugate to the diameter whose equation is $x = 2y$.

**16.** Find the equation to the chord of the hyperbola

$$25x^2 - 16y^2 = 400$$

which is bisected at the point (5, 3).

**17.** In a rectangular hyperbola, prove that

$$SP \cdot S'P = CP^2.$$

**18.** the distance of any point from the centre varies inversely as the perpendicular from the centre upon its polar.

**19.** if the normal at $P$ meet the axes in $G$ and $g$, then $PG = Pg = PC$.

**20.** the angle subtended by any chord at the centre is the supplement of the angle between the tangents at the ends of the chord.

**21.** the angles subtended at its vertices by any chord which is parallel to its conjugate axis are supplementary.

**22.** The normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes in $M$ and $N$, and lines $MP$ and $NP$ are drawn at right angles to the axes; prove that the locus of $P$ is the hyperbola

$$a^2x^2 - b^2y^2 = (a^2 + b^2)^2.$$

**23.** If one axis of a varying central conic be fixed in magnitude and position, prove that the locus of the point of contact of a tangent drawn to it from a fixed point on the other axis is a parabola.

**24.** If the ordinate $MP$ of a hyperbola be produced to $Q$, so that $MQ$ is equal to either of the focal distances of $P$, prove that the locus of $Q$ is one or other of a pair of parallel straight lines.

**25.** Shew that the locus of the centre of a circle which touches externally two given circles is a hyperbola.

**26.** On a level plain the crack of the rifle and the thud of the ball striking the target are heard at the same instant; prove that the locus of the hearer is a hyperbola.

**27.** Given the base of a triangle and the ratio of the tangents of half the base angles, prove that the vertex moves on a hyperbola whose foci are the extremities of the base.

**28.** Prove that the locus of the poles of normal chords with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the curve

$$y^2a^6 - x^2b^6 = (a^2 + b^2)^2 x^2y^2.$$

**29.** Find the locus of the pole of a chord of the hyperbola which subtends a right angle at (1) the centre, (2) the vertex, and (3) the focus of the curve.

**30.** Shew that the locus of poles with respect to the parabola $y^2 = 4ax$ of tangents to the hyperbola $x^2 - y^2 = a^2$ is the ellipse $4x^2 + y^2 = 4a^2$.

**31.** Prove that the locus of the pole with respect to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ of any tangent to the circle, whose diameter is the line joining the foci, is the ellipse $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$.

**32.** Prove that the locus of the intersection of tangents to a hyperbola, which meet at a constant angle $\beta$, is the curve

$$(x^2+y^2+b^2-a^2)^2=4\cot^2\beta\,(a^2y^2-b^2x^2+a^2b^2).$$

**33.** From points on the circle $x^2+y^2=a^2$ tangents are drawn to the hyperbola $x^2-y^2=a^2$; prove that the locus of the middle points of the chords of contact is the curve

$$(x^2-y^2)^2=a^2(x^2+y^2).$$

**34.** Chords of a hyperbola are drawn, all passing through the fixed point $(h, k)$; prove that the locus of their middle points is a hyperbola whose centre is the point $\left(\frac{h}{2}, \frac{k}{2}\right)$, and which is similar to either the hyperbola or its conjugate.

## ANSWERS

**1.** $16x^2-9y^2=36$.

**2.** $25x^2-144y^2=900$.

**3.** $65x^2-36y^2=441$.

**4.** $x^2-y^2=32$.

**5.** 6, 4, $(\pm\sqrt{13}, 0)$, $2\frac{2}{3}$.

**6.** $3x^2-y^2=3a^2$.

**7.** $7y^2+24xy-24ax-6ay+15a^2=0$; $\left(-\frac{a}{3}, a\right)$; $12x-9y+29a=0$.

**8.** $(5, -\frac{20}{3})$.

**9.** $24y-30x=\pm\sqrt{161}$.

**14.** $y=\pm x\pm\sqrt{a^2-b^2}$; $(a^2+b^2)\sqrt{\frac{2}{a^2-b^2}}$.

**15.** $9y=32x$.

**16.** $125x-48y=481$.

**29.** (1) $b^4x^2+a^4y^2=a^2b^2(b^2-a^2)$; (2) $x=a\cdot\frac{a^2-b^2}{a^2+b^2}$;

(3) $x^2(a^2+2b^2)-a^2y^2-2a^3ex+a^2(a^2-b^2)=0$.

## SOLUTIONS/HINTS

**1.** $\frac{x^2}{(1\frac{1}{2})^2}-\frac{y^2}{2^2}=1$ or $16x^2-9y^2=36$.

**2.** $b=\frac{5}{2}$, and $a^2+b^2=\frac{169}{4}$; $\therefore\ a^2=36$ and $a=6$.

Hence the equation is $\frac{x^2}{36} - \frac{y^2}{(\frac{5}{2})^2} = 1$, etc.

**3.** If $\frac{x^2}{a^2} - \frac{y^2}{(3\frac{1}{2})^2} = 1$ passes through $(3, -2)$,

$$\text{then } \frac{9}{a^2} - \frac{4}{(3\frac{1}{2})^2} = 1, \text{ whence } a^2 = \frac{9 \times 49}{65},$$

and the equation becomes $65x^2 - 36y^2 = 441$.

**4.** $a^2e^2 = a^2 + b^2 = 8^2$; $\therefore a^2 = \frac{64}{2} = 32$.

$\therefore b^2 = 32$, and the equation is $x^2 - y^2 = 32$.

**5.** $\frac{x^2}{9} - \frac{y^2}{4} = 1$. Hence the transverse axis $= 6$, and the conjugate axis $= 4$.

$$\therefore e^2 = \frac{a^2 + b^2}{a^2} = \frac{9+4}{9} = \frac{13}{9}; \quad \therefore e = \tfrac{1}{3}\sqrt{13};$$

$\therefore$ the foci are $(\pm\sqrt{13}, 0)$, and the latus rectum $= 2\frac{b^2}{a} = \frac{8}{3}$.

**6.** If $2a = ae$, $e = 2$, and $b^2 = a^2e^2 - a^2 = 3a^2$.

Hence the required equation is $3x^2 - y^2 = 3a^2$.

**7.** $(x-a)^2 + y^2 = \frac{25}{16}\left(\frac{4x - 3y - a}{\sqrt{4^2 + 3^2}}\right)^2$, etc.

Let $S$ be the focus $(a, 0)$. Then the equation of the axis $SX$ (viz. the line through $(a, 0)$ perpendicular to the directrix) is $3x + 4y = 3a$.

Solving, the coordinates of $X$ are $\left(\frac{13a}{25}, \frac{9a}{25}\right)$. The coordinates of the centre $C$ (which divides $SX$ externally so that $CX : CS = 1 : e^2 = 16 . 25$) are [Art. 22]

$$\frac{16a - 13a}{16 - 25}, \text{ and } \frac{0 - 9a}{16 - 25}, \text{ or } \left(-\frac{a}{3}, a\right).$$

Also, if $(h, k)$ be the coordinates of $X'$,

$$h + \frac{13a}{25} = -\frac{2a}{3}, \text{ and } k + \frac{9a}{25} = 2a,$$

whence $$h = -\frac{89a}{75} \text{ and } k = \frac{41a}{25};$$

also the equation of the line through this point parallel to the directrix is

$$4\left(x + \frac{89a}{75}\right) - 3\left(y - \frac{41a}{25}\right) = 0, \text{ or } 12x - 9y + 29a = 0.$$

**8.** The abscissae are given by

$$25x^2 - \left(\frac{45 - 25x}{4}\right)^2 = 225, \text{ or } x^2 - \left(\frac{9 - 5x}{4}\right)^2 = 9,$$

*i.e.* $x^2 - 10x + 25 = 0$, which has equal roots, each 5, and

$$y = \frac{45 - 25x}{12} = -\frac{20}{3}.$$

**9.** The intersections of the line $4y = 5x + c$ with the hyperbola, are given by

$$4x^2 - 9\left(\frac{5x + c}{4}\right)^2 = 1, \text{ or } 161x^2 + 90cx + 9c^2 + 16 = 0,$$

which has equal roots if $(45c)^2 = 161(9c^2 + 16)$;

whence $$c = \pm\frac{\sqrt{161}}{6}, \text{ etc.}$$

**10.** Putting $x = a$ and $x = -a$ in the equation of a tangent in Art. 307, the points in which it cuts the tangents at the vertices are

$$\left(a,\ b\tan\frac{\phi}{2}\right) \text{ and } \left(-a,\ -b\cot\frac{\phi}{2}\right).$$

The locus of points equidistant from $(ae, 0)$ and

$$\left(a,\ b\tan\frac{\phi}{2}\right)$$

is $$(x - a)^2 + \left(y - b\tan\frac{\phi}{2}\right)^2 = (x - ae)^2 + y^2,$$

or $$(ae - a)(2x - a - ae) = b\tan\frac{\phi}{2}\left(2y - b\tan\frac{\phi}{2}\right),$$

and similarly the points equidistant from $(-ae, 0)$ and $\left(-a, -b\cot\frac{\phi}{2}\right)$ lie on the line

$$(ae-a)(2x+a+ae)=b\cot\frac{\phi}{2}\left(2y+b\cot\frac{\phi}{2}\right).$$

Each of these lines cuts the axis of $y$ in the same point, viz. where $y=\frac{b}{2}\left(\tan\frac{\phi}{2}-\cot\frac{\phi}{2}\right)$.

Also this point is equidistant from the two foci and thus from all four points. Hence, etc.

**11.** The polar of $(a\sec\phi, b\tan\phi)$ with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x}{a}\sec\phi+\frac{y}{b}\tan\phi=1$, which touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $-\phi$. [Art. 307.]

Similarly $\frac{x}{a}\cos\phi-\frac{y}{b}\sin\phi=1$, which is the polar of a point $(a\cos\phi, b\sin\phi)$ on the ellipse with respect to the hyperbola, touches the ellipse at the point whose eccentric angle is $-\phi$.

**12.** The equation of the line joining the points

$$(a\sec\phi, b\tan\phi) \text{ and } (ae, 0),$$

is

$$\frac{x-ae}{a\sec\phi-ae}=\frac{y}{b\tan\phi},$$

or

$$y=\frac{b}{a}\cdot\frac{\sin\phi}{1-e\cos\phi}(x-ae), \quad \text{..............(i)}$$

and the equation of the perpendicular from the centre upon $\frac{x}{a}-\frac{y}{b}\sin\phi=\cos\phi$, is

$$\sin\phi=-\frac{by}{ax}. \quad \text{....................(ii)}$$

Substituting for $\sin\phi$ in (1), we have $\cos\phi=\frac{aex-b^2}{ax}$.

$\therefore$ the required locus is

$$(aex-b^2)^2+b^2y^2=a^2x^2, \text{ or } (x-ae)^2+y^2=a^2.$$

Similarly for the ellipse.

**13.** Multiplying, we have $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

**14.** The line $bx - ay \sin\phi = ab\cos\phi$, which, by Art. 307, is any tangent to the first hyperbola, will touch the second if [Art. 305]

$$a^2b^2\cos^2\phi = a^4\sin^2\phi - b^4, \text{ whence } \tan^2\phi = \frac{b^2}{a^2 - b^2}.$$

$$\therefore \sin\phi = \pm\frac{b}{a}, \text{ and } \cos\phi = \pm\frac{\sqrt{a^2 - b^2}}{a}.$$

Substituting, the equation becomes $y = \pm x \mp \sqrt{a^2 - b^2}$.

If $\phi < \frac{\pi}{2}$, we take the upper sign for $\sin\phi$ and $\cos\phi$, and the equation to the common tangent is $y = x - \sqrt{a^2 - b^2}$.

Solving with the second hyperbola, the point of contact is

$$\left(-\frac{b^2}{\sqrt{a^2 - b^2}}, -\frac{a^2}{\sqrt{a^2 - b^2}}\right);$$

and the point of contact with the first hyperbola is

$$\left(\frac{a^2}{\sqrt{a^2 - b^2}}, \frac{b^2}{\sqrt{a^2 - b^2}}\right).$$

Hence the length of the common tangent

$$= (a^2 + b^2)\sqrt{\frac{2}{a^2 - b^2}}.$$

**15.** The line is $y = mx$, where $m \cdot \frac{1}{2} = \frac{16}{9}$, [Art. 308].

$$\therefore m = \tfrac{32}{9}.$$

**16.** The required equation is [Art. 280]

$$\frac{5}{\frac{1}{25}}(x - 5) - \frac{3}{\frac{1}{16}}(y - 3) = 0, \text{ or } 125x - 48y = 481.$$

**17.** If $P$ be the point $\phi$; then, since $e = \sqrt{2}$,

$$SP \cdot S'P = \{ea\sec\phi - a\}\{ea\sec\phi + a\} = 2a^2\sec^2\phi - a^2$$
$$= a^2(2\sec^2\phi - 1) = a^2\sec^2\phi + a^2\tan^2\phi = CP^2.$$

**18.** The perpendicular from (0, 0) upon the polar of $(h, k)$ with respect to the hyperbola $x^2 - y^2 = a^2$,

*i.e.* the line $xh - yk = a^2$, is $\dfrac{a^2}{\sqrt{h^2+k^2}}$. Hence, etc.

**19.** The equation of the normal (Art. 307) becomes (when $a = b$) $x \sin\phi + y = 2a \tan\phi$.

Hence $CG = 2a \sec\phi$, and $Cg = 2a \tan\phi$.

$$\therefore \ PG^2 = a^2 \sec^2\phi + b^2 \tan^2\phi = Pg^2 = PC^2.$$

**20.** Let the two points $P$ and $Q$ be $(a \sec\phi,\ a \tan\phi)$ and $(a \sec\theta,\ a \tan\theta)$.

The angle $P\hat{C}Q = \tan^{-1}(\sin\phi) - \tan^{-1}(\sin\theta)$

$$= \tan^{-1}\frac{\sin\phi - \sin\theta}{1 + \sin\phi \sin\theta}.$$

The angle between the tangents at $P$ and $Q$

$$= \tan^{-1}(\operatorname{cosec}\phi) - \tan^{-1}(\operatorname{cosec}\theta) \quad [\text{Art. } 307]$$

$$= \tan^{-1}\frac{\sin\theta - \sin\phi}{1 + \sin\theta \sin\phi} = \pi - P\hat{C}Q.$$

**21.** $$\operatorname{Tan} PAN = \frac{a\tan\phi}{a\sec\phi - a} = \frac{\sin\phi}{1 - \cos\phi},$$

and $$\tan PA'N = \frac{a\tan\phi}{a\sec\phi + a} = \frac{\sin\phi}{1 + \cos\phi}.$$

$$\therefore \ \tan PAN \,.\, \tan PA'N = \frac{\sin^2\phi}{1 - \cos^2\phi} = 1.$$

$\therefore \ \angle PAN$ and $\angle PA'N$ are complementary.

So $\angle P'AN$ and $\angle PA'N$ are complementary. Hence, etc.

**22.** The coordinates $(x', y')$ of $P$ are found by putting $x$ and $y = 0$ in the equation of the normal in Art. 307.

$$\therefore \ x' = \frac{a^2+b^2}{a} \sec\phi, \text{ and } y' = \frac{a^2+b^2}{b} \tan\phi.$$

Eliminating $\phi$, we have

$$\therefore \ a^2x'^2 - b^2y'^2 = (a^2+b^2)^2. \quad \therefore \text{ etc.}$$

**23.** Let the transverse axis be fixed. Putting $x = 0$ in the equation of the tangent in Art. 307,

$$b \cot \phi = \text{cons.} = c,$$

and $$x = a \sec \phi,\ y = b \tan \phi.$$

Eliminating $\phi$ and $b$,

$$\frac{x^2}{a^2} - \frac{y}{c} = \sec^2 \phi - \tan^2 \phi = 1,$$

which is a parabola. Similarly for the ellipse.

**24.** The coordinates of $Q$, if $P$ is the point

$$(a \sec \phi,\ b \tan \phi)$$

are $x = a \sec \phi$, $y = + ea \sec \phi \pm a$, whence $y = ex \pm a$.

**25.** Let $a$ and $b$ be the radii of the given circles whose centres are $S$ and $S'$; $P$ the centre of the variable circle, and $r$ its radius.

Then $PS \sim PS' = (a + r) \sim (b + r) = a \sim b$.

$\therefore$ the locus of $P$ is a hyperbola whose foci are $S$ and $S'$ [Art. 299].

**26.** Let $S$ be the target, $S'$ the firing-point, and $P$ the hearer. Let $v$ and $v'$ be the velocities of the bullet and sound respectively. Then since the time of sound from $S'$ to $P$

= time of bullet from $S'$ to $S$ + time of sound from $S$ to $P$,

$$\therefore\ \frac{S'P}{v'} = \frac{SS'}{v} + \frac{SP}{v'}.$$

$$\therefore\ S'P - SP = \frac{v'}{v} . SS' = \text{constant}.$$

$\therefore$ the locus of $P$ is a hyperbola [Art. 299].

**27.** Let $ABC$ be the $\triangle$; since

$$\tan \frac{B}{2} = \frac{r}{s-b}, \text{ and } \tan \frac{C}{2} = \frac{r}{s-c},$$

$$\therefore\ \frac{s-b}{s-c} = \text{constant}. \quad \therefore\ \frac{a+c-b}{a+b-c} = \text{constant}.$$

$$\therefore\ \frac{a}{b-c} = \text{constant}. \quad \therefore\ b - c = \text{constant}.$$

$\therefore$ the locus of $A$ is a hyperbola whose foci are $B$ and $C$.

**28.** The lines $ax \sin \phi + by = (a^2 + b^2) \tan \phi$, and

$$b^2xh - a^2yk = a^2b^2,$$

which is the polar of $(h, k)$, are identical if

$$\frac{a \sin \phi}{b^2h} = -\frac{b}{a^2k} = \frac{(a^2 + b^2) \tan \phi}{a^2b^2};$$

$$\therefore \ \operatorname{cosec} \phi = -\frac{a^3k}{b^3h}, \text{ and } \cot \phi = -\frac{(a^2 + b^2) kh}{b^3h}.$$

$$\therefore \ a^6k^2 - b^6h^2 = (a^2 + b^2)^2 h^2k^2, \text{ etc.}$$

**29.** (1) The lines joining the origin to the common points of the line $b^2xh - a^2yk = a^2b^2$ and the hyperbola are

$$(b^2xh - a^2yk)^2 = a^2b^2 (b^2x^2 - a^2y^2).$$

If they are at right angles, then

$$b^4h^2 + a^4k^2 = a^2b^2 (b^2 - a^2).$$

(2) Removing the origin to the point $(a, 0)$ the equation to the hyperbola becomes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -\frac{2x}{a}$, and the polar of $(h, k)$ becomes $b^2xh - a^2yk = a^2b^2 - ab^2h$.

The equation of the lines joining the origin to their common points is

$$(a^2b^2 - ab^2h)(b^2x^2 - a^2y^2) = -2ab^2x (b^2xh - a^2yk),$$

or

$$(a - h)(b^2x^2 - a^2y^2) = -2x (b^2xh - a^2yk).$$

They are at right angles if

$$(a - h)(b^2 - a^2) = -2b^2h, \text{ or } h (b^2 + a^2) = a (a^2 - b^2).$$

(3) On moving the origin to the point $(ae, 0)$, the equations to the ellipse and the polar of $(h, k)$ become

$$b^2x^2 - a^2y^2 + 2aeb^2x + b^4 = 0, \ \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

and

$$b^2xh - a^2yk = b^2a (a - eh), \ \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

and the equation of the lines joining the origin to their common points is

$$a^2 (b^2x^2 - a^2y^2)(a - eh)^2 + 2a^2ex (a - eh)(b^2xh - a^2yk) + (b^2xh - a^2yk)^2 = 0.$$

These are at right angles if

$$a^2 (b^2 - a^2)(a - eh)^2 + 2a^2b^2eh (a - eh) + b^4h^2 + a^4k^2 = 0,$$

which reduces to $h^2 (a^2 + 2b^2) - a^2k^2 - 2a^3eh + a^2 (a^2 - b^2) = 0$.

**30.** The lines $2ax - yk + 2ah = 0$, and

$$x - y\sin\phi - a\cos\phi = 0,$$

are coincident if $2a = \frac{k}{\sin\phi} = -\frac{2h}{\cos\phi}$.

Eliminating $\phi$, we have $k^2 + 4h^2 = 4a^2$.

**31.** The lines $x\cos\theta + y\sin\theta = ae$, and

$$b^2xh - a^2yk = a^2b^2$$

are coincident if $\frac{\cos\theta}{b^2h} = -\frac{\sin\theta}{a^2k} = \frac{ae}{a^2b^2}$.

Eliminating $\theta$, we have $\frac{h^2}{a^4} + \frac{k^2}{b^4} = \frac{1}{a^2e^2} = \frac{1}{a^2+b^2}$.

**32.** Take equation (2) of Art. 272 and change $b^2$ into $-b^2$. Then if $\tan\theta_1$ and $\tan\theta_2$ are the roots,

$$\tan\theta_1 + \tan\theta_2 = \frac{2x_1y_1}{x_1^2 - a^2}, \text{ and } \tan\theta_1\tan\theta_2 = \frac{y_1^2 + b^2}{x_1^2 - a^2};$$

$$\therefore\ (\tan\theta_1 - \tan\theta_2)^2 = \frac{4x_1^2y_1^2 - 4(y_1^2 + b^2)(x_1^2 - a^2)}{(x_1^2 - a^2)^2}$$

$$= \frac{4(a^2y_1^2 - b^2x_1^2 + a^2b^2)}{(x_1^2 - a^2)^2}.$$

$$\therefore\ \cot^2\beta = \frac{(1 + \tan\theta_1\tan\theta_2)^2}{(\tan\theta_1 - \tan\theta_2)^2} = \frac{(x_1^2 + y_1^2 + b^2 - a^2)^2}{4(a^2y_1^2 - b^2x_1^2 + a^2b^2)}, \text{ etc.}$$

**33.** The equation of Art. 280, on putting $b^2 = -a^2$, becomes $hx - ky = h^2 - k^2$.

If this is coincident with $x\cos\theta - y\sin\theta = a$, *i.e.* the polar of any point $(a\cos\theta, a\sin\theta)$ on the circle with respect to the hyperbola, then

$$\frac{\cos\theta}{h} = \frac{\sin\theta}{k} = \frac{a}{h^2 - k^2}.$$

Eliminating $\theta$, we have $(h^2 - k^2)^2 = a^2(h^2 + k^2)$.

**34.** The chord whose middle point is $(x_1, y_1)$ passes through the point $(h, k)$ if

$$\frac{y_1}{b^2}(k - y_1) = \frac{x_1}{a^2}(h - x_1) \ldots \text{[cf. Art. 280]}$$

which is equivalent to

$$\frac{\left(x_1 - \frac{h}{2}\right)^2}{a^2} - \frac{\left(y_1 - \frac{k}{2}\right)^2}{b^2} = \frac{1}{4}\left\{\frac{h^2}{a^2} - \frac{k^2}{b^2}\right\}.$$

This is a hyperbola, whose centre is $\left(\frac{h}{2}, \frac{k}{2}\right)$, and which is similar to the given hyperbola or its conjugate according as the right-hand member, $\frac{h^2}{a^2} - \frac{k^2}{b^2}$, is positive or negative.

**312. Asymptote. Def.** An asymptote is a straight line, which meets the conic in two points both of which are situated at an infinite distance, but which is itself not altogether at infinity.

**313.** *To find the asymptotes of the hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

As in Art. 260, the straight line

$$y = mx + c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

meets the hyperbola in points, whose abscissae are given by the equation

$$x^2(b^2 - a^2m^2) - 2a^2mcx - a^2(c^2 + b^2) = 0 \quad \ldots\ldots (2).$$

If the straight line (1) be an asymptote, both roots of (2) must be infinite.

Hence (C. Smith's Algebra, Art. 123), the coefficients of $x^2$ and $x$ in it must both be zero.

We therefore have

$$b^2 - a^2m^2 = 0, \text{ and } a^2mc = 0.$$

Hence $$m = \pm\frac{b}{a}, \text{ and } c = 0.$$

Substituting these values in (1), we have, as the required equation,

$$y = \pm\frac{b}{a}x.$$

There are therefore two asymptotes both passing through the centre and equally inclined to the axis of $x$, the inclination being

$$\tan^{-1}\frac{b}{a}.$$

The equation to the asymptotes, written as one equation, is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

**Cor.** For all values of $c$ one root of equation (2) is infinite if $m = \pm\frac{b}{a}$. Hence any straight line, which is parallel to an asymptote, meets the curve in one point at infinity and in one finite point.

**314.** That the asymptote passes through two coincident points at infinity, *i.e.* touches the curve at infinity, may be seen by finding the equations to the tangents to the curve which pass through any point $\left(x_1, \frac{b}{a}x_1\right)$ on the asymptote $y = \frac{b}{a}x$.

As in Art. 305 the equation to either tangent through this point is

$$y = mx + \sqrt{a^2m^2 - b^2},$$

where $$\frac{b}{a}x_1 = mx_1 + \sqrt{a^2m^2 - b^2},$$

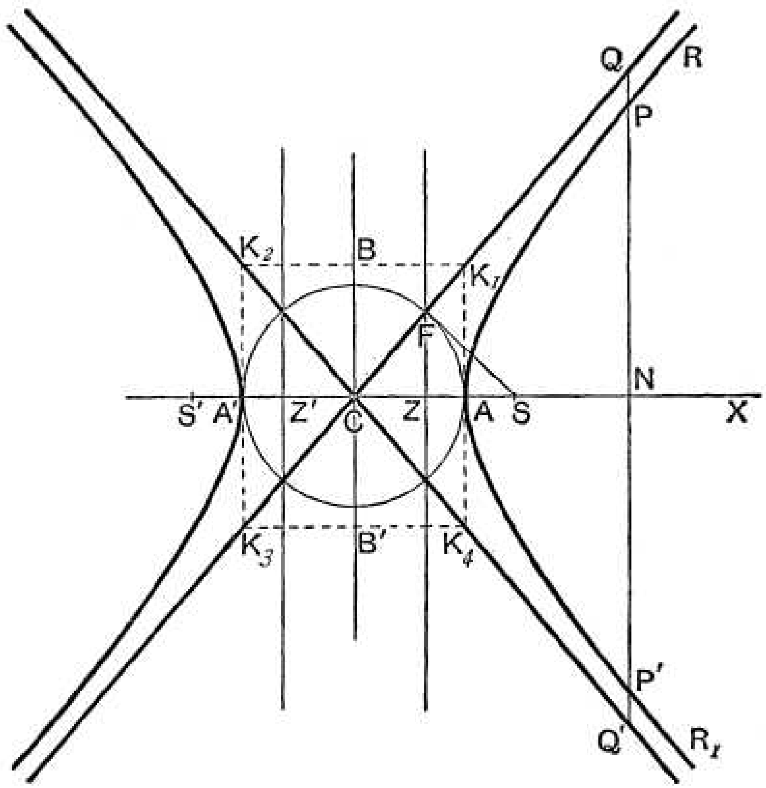
*i.e.* on clearing of surds,

$$m^2(x_1^2 - a^2) - 2m\frac{b}{a}x_1^2 + (x_1^2 + a^2)\frac{b^2}{a^2} = 0.$$

One root of this equation is $m = \frac{b}{a}$, so that one tangent through the given point is $y = \frac{b}{a}x$, *i.e.* the asymptote itself.

**315.** *Geometrical construction for the asymptotes.*

Let $A'A$ be the transverse axis, and along the conjugate axis measure off $CB$ and $CB'$, each equal to $b$. Through $B$ and $B'$ draw parallels to the transverse axis and through $A$ and $A'$ parallels to the conjugate axis, and let these meet respectively in $K_1$, $K_2$, $K_3$, and $K_4$, as in the figure.




Clearly the equations of $K_1CK_3$ and $K_2CK_4$ are

$$y = \frac{b}{a}x, \text{ and } y = -\frac{b}{a}x,$$

and these are therefore the equations of the asymptotes.

**316.** Let any double ordinate $PNP'$ of the hyperbola be produced both ways to meet the asymptotes in $Q$ and $Q'$, and let the abscissa $CN$ be $x'$.

Since $P$ lies on the curve, we have, by Art. 302,

$$NP = \frac{b}{a}\sqrt{x'^2 - a^2}.$$

Since $Q$ is on the asymptote whose equation is $y = \frac{b}{a}x$,

we have $$NQ = \frac{b}{a}x$$

Hence $$PQ = NQ - NP = \frac{b}{a}(x' - \sqrt{x'^2 - a^2}),$$

and $$P'Q = \frac{b}{a}(x' + \sqrt{x'^2 - a^2}).$$

Therefore $$PQ \cdot P'Q = \frac{b^2}{a^2}\{x'^2 - (x'^2 - a^2)\} = b^2.$$

Hence, if from any point on an asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point and the curve, is always equal to the square on the semi-conjugate axis.

Again,

$$PQ = \frac{b}{a}(x' - \sqrt{x'^2 - a^2}) = \frac{b}{a}\frac{a^2}{x' + \sqrt{x'^2 - a^2}}$$
$$= \frac{ab}{x' + \sqrt{x'^2 - a^2}}.$$

$PQ$ is therefore always positive, and therefore the part of the curve, for which the coordinates are positive, is altogether between the asymptote and the transverse axis.

Also as $x'$ increases, *i.e.* as the point $P$ is taken further and further from the centre $C$, it is clear that $PQ$ continually decreases; finally, when $x'$ is infinitely great, $PQ$ is infinitely small.

The curve therefore continually approaches the asymptote but never actually reaches it, although, at a very great distance, the curve would not be distinguishable from the asymptote.

This property is sometimes taken as the definition of an asymptote.

**317.** If $SF$ be the perpendicular from $S$ upon an asymptote, the point $F$ lies on the auxiliary circle. This follows from the fact that the asymptote is a tangent, whose point of contact happens to lie at infinity, or it may be proved directly.

For

$$CF = CS \cos FCS = CS \cdot \frac{CA}{CK} = \sqrt{a^2+b^2} \cdot \frac{a}{\sqrt{a^2+b^2}} = a.$$

Also $Z$ being the foot of the directrix, we have

$$CA^2 = CS \cdot CZ, \qquad \text{(Art. 295)}$$

and hence $CF^2 = CS \cdot CZ$, *i.e.* $CS : CF :: CF : CZ$.

By Euc. VI. 6, it follows that $\angle CZF = \angle CFS =$ a right angle, and hence that $F$ lies on the directrix.

Hence *the perpendiculars from the foci on either asymptote meet it in the same points as the corresponding directrix, and the common points of intersection lie on the auxiliary circle.*

**318. Equilateral or Rectangular Hyperbola.** In this curve (Art. 310) the quantities $a$ and $b$ are equal. The equations to the asymptotes are therefore $y = \pm x$, *i.e.* they are inclined at angles $\pm 45°$ to the axis of $x$, and hence they are at right angles. Hence the hyperbola is generally called a **rectangular** hyperbola.

**319. Conjugate Hyperbola.** The hyperbola which has $BB'$ as its transverse axis, and $AA'$ as its conjugate axis, is said to be the conjugate hyperbola of the hyperbola whose transverse and conjugate axes are respectively $AA'$ and $BB'$.

Thus the hyperbola

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

is conjugate to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Just as in Art. 313, the equation to the asymptotes of (1) is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 0,$$

which, by the same article, is the equation to the asymptotes of (2).

Thus a hyperbola and its conjugate have the same asymptotes.

The conjugate hyperbola is the dotted curve in the figure of Art. 323.

**320.** *Intersections of a hyperbola with a pair of conjugate diameters.*

The straight line $y = m_1 x$ intersects the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

in points whose abscissæ are given by

$$x^2 \left[\frac{1}{a^2} - \frac{m_1^2}{b^2}\right] = 1,$$

*i.e.* by the equation $\quad x^2 = \dfrac{a^2 b^2}{b^2 - a^2 m_1^2}$.

The points are therefore real or imaginary, according as

$$a^2 m_1^2 \text{ is } < \text{ or } > b^2,$$

*i.e.* according as

$$m_1 \text{ is numerically } < \text{ or } > \frac{b}{a} \ldots\ldots\ldots\ldots (1),$$

*i.e.* according as the inclination of the straight line to the axis of $x$ is less or greater than the inclination of the asymptotes.

Now, by Art. 308, the straight lines $y = m_1 x$ and $y = m_2 x$ are conjugate diameters if

$$m_1 m_2 = \frac{b^2}{a^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence one of the quantities $m_1$ and $m_2$ must be less than $\frac{b}{a}$ and the other greater than $\frac{b}{a}$.

Let $m_1$ be $< \frac{b}{a}$, so that, by (1), the straight line $y = m_1 x$ meets the hyperbola in real points.

Then, by (2), $m_2$ must be $> \frac{b}{a}$, so that, by (1), the straight line $y = m_2 x$ will meet the hyperbola in imaginary points.

It follows therefore that only one of a pair of conjugate diameters meets a hyperbola in real points.

**321.** *If a pair of diameters be conjugate with respect to a hyperbola, they will be conjugate with respect to its conjugate hyperbola.*

For the straight lines $y = m_1 x$ and $y = m_2 x$ are conjugate with respect to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (1),$$

if

$$m_1 m_2 = \frac{b^2}{a^2} \qquad (2).$$

Now the equation to the conjugate hyperbola only differs from (1) in having $-a^2$ instead of $a^2$ and $-b^2$ instead of $b^2$, so that the above pair of straight lines will be conjugate with respect to it, if

$$m_1 m_2 = \frac{-b^2}{-a^2} = \frac{b^2}{a^2} \qquad (3).$$

But the relation (3) is the same as (2).

Hence the proposition.

**322.** *If a pair of diameters be conjugate with respect to a hyperbola, one of them meets the hyperbola in real points and the other meets the conjugate hyperbola in real points.*

Let the diameters be $y = m_1 x$ and $y = m_2 x$, so that

$$m_1 m_2 = \frac{b^2}{a^2}.$$

As in Art. 320 let $m_1 < \frac{b}{a}$, and hence $m_2 > \frac{b}{a}$, so that the straight line $y = m_1 x$ meets the hyperbola in real points.

Also the straight line $y = m_2 x$ meets the conjugate hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ in points whose abscissæ are given by the equation $x^2 \left( \frac{m_2^2}{b^2} - \frac{1}{a^2} \right) = 1$, *i.e.* by $x^2 = \frac{a^2 b^2}{m_2^2 a^2 - b^2}$.

Since $m_2 > \frac{b}{a}$, these abscissæ are real.

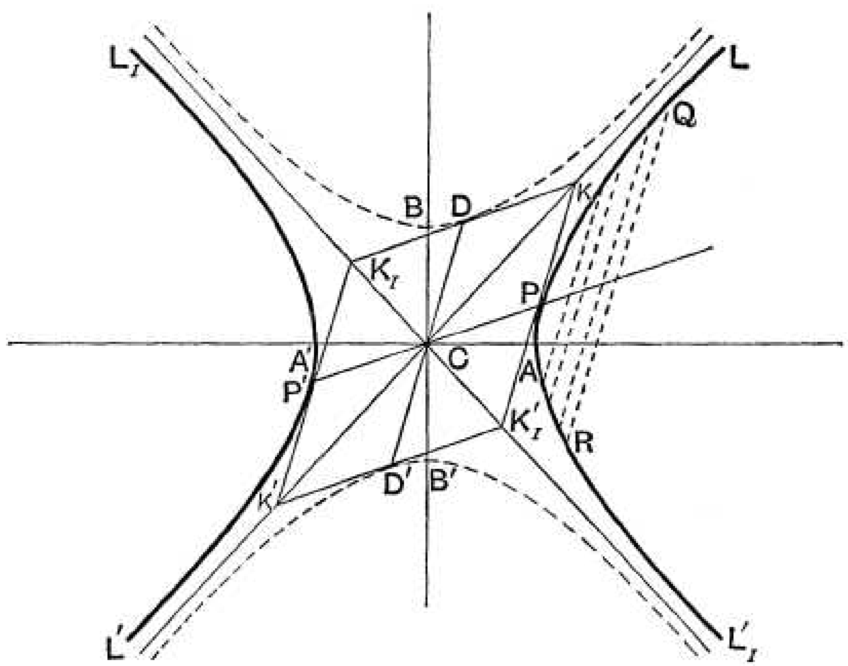
Hence the proposition.

**323.** *If a pair of conjugate diameters meet the hyperbola and its conjugate in P and D, then* (1) $CP^2 - CD^2 = a^2 - b^2$, *and* (2) *the tangents at P, D and the other ends of the diameters passing through them form a parallelogram whose vertices lie on the asymptotes and whose area is constant.*

Let $P$ be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose coordinates are $(a \sec \phi,\ b \tan \phi)$.

The equation to the diameter $CP$ is therefore

$$y = \frac{b \tan \phi}{a \sec \phi} x = x \cdot \frac{b}{a} \sin \phi.$$




By Art. 308, the equation to the straight line, which is conjugate to $CP$, is

$$y = x \frac{b}{a \sin \phi}.$$

This straight line meets the conjugate hyperbola

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1,$$

in the points $(a\tan\phi,\ b\sec\phi)$, and $(-a\tan\phi,\ -b\sec\phi)$ so that $D$ is the point $(a\tan\phi,\ b\sec\phi)$.

We therefore have

$$CP^2=a^2\sec^2\phi+b^2\tan^2\phi,$$

and

$$CD^2=a^2\tan^2\phi+b^2\sec^2\phi.$$

Hence

$$CP^2-CD^2=(a^2-b^2)(\sec^2\phi-\tan^2\phi)=a^2-b^2.$$

Again, the tangents at $P$ and $D$ to the hyperbola and the conjugate hyperbola are easily seen to be

$$\frac{x}{a}-\frac{y}{b}\sin\phi=\cos\phi,\ \ldots\ldots\ldots\ (1),$$

and

$$\frac{y}{b}-\frac{x}{a}\sin\phi=\cos\phi.\ \ldots\ldots\ldots\ (2).$$

These meet at the point

$$\frac{x}{a}=\frac{y}{b}=\frac{\cos\phi}{1-\sin\phi}.$$

This point lies on the asymptote $CL$.

Similarly, the intersection of the tangents at $P$ and $D'$ lies on $CL_1'$, that of tangents at $D'$ and $P'$ on $CL'$, and those at $D$ and $P'$ on $CL_1$.

If tangents be therefore drawn at the points where a pair of conjugate diameters meet a hyperbola and its conjugate, they form a parallelogram whose angular points are on the asymptotes.

Again, the perpendicular from $C$ on the straight line (1)

$$=\frac{\cos\phi}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}\sin^2\phi}}=\frac{ab\cos\phi}{\sqrt{b^2+a^2\sin^2\phi}}$$

$$=\frac{ab}{\sqrt{b^2\sec^2\phi+a^2\tan^2\phi}}=\frac{ab}{CD}=\frac{ab}{PK},$$

so that $PK \times$ perpendicular from $C$ on $PK = ab$,

*i.e.* area of the parallelogram $CPKD = ab$.

Also the areas of the parallelograms $CPKD$, $CDK_1P'$, $CP'K'D'$, and $CD'K_1'P$ are all equal.

The area $KK_1K'K_1'$ therefore $= 4ab$.

**Cor.** $PK = CD = D'C = K_1'P$, so that the portion of a tangent to a hyperbola intercepted between the asymptotes is bisected at the point of contact.

**324.** *Relation between the equation to the hyperbola, the equation to its asymptotes, and the equation to the conjugate hyperbola.*

The equations to the hyperbola, the asymptotes, and the conjugate hyperbola are respectively

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad\qquad (1),$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \qquad\qquad (2),$$

and $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \qquad\qquad (3).$$

We notice that the equation (2) differs from equation (1) by a constant, and that the equation (3) differs from (2) by exactly the same quantity that (2) differs from (1).

If now we transform the equations in any way we please—by changing the origin and directions of the axes—by the most general substitutions of Art. 132 and by multiplying the equations by any—the same—constant, we shall alter the left-hand members of (1), (2), and (3) in exactly the same way, and the right-hand constants in the equations will still be constants, and differ in the same way as before.

Hence, whatever be the form of the equation to a hyperbola, the equation to the asymptotes only differs from it by a constant, and the equation to the conjugate hyperbola differs from that to the asymptotes by the same constant.

**325.** As an example of the foregoing article, let it be required to find the asymptotes of the hyperbola

$$3x^2 - 5xy - 2y^2 + 5x + 11y - 8 = 0 \qquad \dots\dots\dots\dots(1).$$

Since the equation to the asymptotes only differs from it by a constant, it must be of the form

$$3x^2 - 5xy - 2y^2 + 5x + 11y + c = 0 \qquad \dots\dots\dots\dots(2).$$

Since (2) represents the asymptotes it must represent two straight lines. The condition for this is (Art. 116)

$$3(-2)c + 2 \cdot \tfrac{5}{2} \cdot \tfrac{11}{2}\left(-\tfrac{5}{2}\right) - 3\left(\tfrac{11}{2}\right)^2 - (-2)\left(\tfrac{5}{2}\right)^2 - c\left(-\tfrac{5}{2}\right)^2 = 0,$$

*i.e.*
$$c = -12.$$

The equation to the asymptotes is therefore

$$3x^2 - 5xy - 2y^2 + 5x + 11y - 12 = 0,$$

and the equation to the conjugate hyperbola is

$$3x^2 - 5xy - 2y^2 + 5x + 11y - 16 = 0.$$

**326.** As another example we see that the equation to any hyperbola whose asymptotes are the straight lines

$$Ax + By + C = 0 \text{ and } A_1x + B_1y + C_1 = 0,$$

is
$$(Ax + By + C)(A_1x + B_1y + C_1) = \lambda^2 \qquad \dots\dots\dots\dots (1),$$

where $\lambda$ is any constant.

For (1) only differs by a constant from the equation to the asymptotes, which is

$$(Ax + By + C)(A_1x + B_1y + C_1) = 0 \qquad \dots\dots\dots\dots (2).$$

If in (1) we substitute $-\lambda^2$ for $\lambda^2$ we shall have the equation to its conjugate hyperbola.

It follows that any equation of the form

$$(Ax + By + C)(A_1x + B_1y + C_1) = \lambda^2$$

represents a hyperbola whose asymptotes are

$$Ax + By + C = 0, \text{ and } A_1x + B_1y + C_1 = 0.$$

Thus the equation $x(x+y) = a^2$ represents a hyperbola whose asymptotes are $x = 0$ and $x + y = 0$.

Again, the equation $x^2 + 2xy \cot 2\alpha - y^2 = a^2$,

*i.e.*
$$(x \cot \alpha - y)(x \tan \alpha + y) = a^2,$$

represents a hyperbola whose asymptotes are

$$x \cot \alpha - y = 0, \text{ and } x \tan \alpha + y = 0.$$

**327.** It would follow from the preceding articles that the equation to any hyperbola whose asymptotes are $x = 0$ and $y = 0$ is $xy = \text{const.}$

The constant could be easily determined in terms of the semi-transverse and semi-conjugate axes.

In Art. 328 we shall obtain this equation by direct transformation from the equation referred to the principal axes.

## EXAMPLES XXXVII

**1.** Through the positive vertex of the hyperbola a tangent is drawn; where does it meet the conjugate hyperbola?

**2.** If $e$ and $e'$ be the eccentricities of a hyperbola and its conjugate, prove that

$$\frac{1}{e^2}+\frac{1}{e'^2}=1.$$

**3.** Prove that chords of a hyperbola, which touch the conjugate hyperbola, are bisected at the point of contact.

**4.** Shew that the chord, which joins the points in which a pair of conjugate diameters meets the hyperbola and its conjugate, is parallel to one asymptote and is bisected by the other.

**5.** Tangents are drawn to a hyperbola from any point on one of the branches of the conjugate hyperbola; shew that their chord of contact will touch the other branch of the conjugate hyperbola.

**6.** A straight line is drawn parallel to the conjugate axis of a hyperbola to meet it and the conjugate hyperbola in the points $P$ and $Q$; shew that the tangents at $P$ and $Q$ meet on the curve

$\frac{y^4}{b^4}\left(\frac{y^2}{b^2}-\frac{x^2}{a^2}\right)=\frac{4x^2}{a^2}$, and that the normals meet on the axis of $x$.

**7.** From a point $G$ on the transverse axis $GL$ is drawn perpendicular to the asymptote, and $GP$ a normal to the curve at $P$. Prove that $LP$ is parallel to the conjugate axis.

**8.** Find the asymptotes of the curve $2x^2+5xy+2y^2+4x+5y=0$, and find the general equation of all hyperbolas having the same asymptotes.

**9.** Find the equation to the hyperbola, whose asymptotes are the straight lines $x+2y+3=0$, and $3x+4y+5=0$, and which passes through the point $(1, -1)$.

Write down also the equation to the conjugate hyperbola.

**10.** In a rectangular hyperbola, prove that $CP$ and $CD$ are equal, and are inclined to the axis at angles which are complementary.

**11.** $C$ is the centre of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and the tangent at any point $P$ meets the asymptotes in the points $Q$ and $R$. Prove that the equation to the locus of the centre of the circle circumscribing the triangle $CQR$ is $4(a^2x^2-b^2y^2)=(a^2+b^2)^2$.

**12.** A series of hyperbolas is drawn having a common transverse axis of length $2a$. Prove that the locus of a point $P$ on each hyperbola, such that its distance from the transverse axis is equal to its distance from an asymptote, is the curve $(x^2-y^2)^2=4x^2(x^2-a^2)$.

## ANSWERS

**1.** At the points $(a, \pm b\sqrt{2})$.

**8.** $(2x+y+2)(x+2y+1)=0, \quad (2x+y+2)(x+2y+1)=\text{const.}$

**9.** $3x^2+10xy+8y^2+14x+22y+7=0$;

$3x^2+10xy+8y^2+14x+22y+23=0$.

## SOLUTIONS/HINTS

**1.** The line $x=a$ cuts the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1, \text{ where } y=\pm b\sqrt{2}.$$

**2.** $e^2=\dfrac{a^2+b^2}{a^2}$, and $e'^2=\dfrac{a^2+b^2}{b^2}$.

**3.** The tangent at the point $(a\tan\phi, b\sec\phi)$ to the conjugate hyperbola is

$$\frac{y}{b}.\sec\phi-\frac{x}{a}\tan\phi=1=\sec^2\phi-\tan^2\phi,$$

or $$\frac{\sec\phi}{b}(y-b\sec\phi)=\frac{\tan\phi}{a}(x-a\tan\phi),$$

which is the chord of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, which is bisected at the point $(a\tan\phi, b\sec\phi)$. [Cf. Art. 280.]

**4.** By Art. 323, the coordinates of $P$ and $D$ are

$$(a\sec\phi, b\tan\phi), \text{ and } (a\tan\phi, b\sec\phi),$$

$\therefore$ the "$m$" of $PD=-\dfrac{b}{a}$. Also the middle point of $PQ$

viz. $\{\frac{1}{2}a(\sec\phi+\tan\phi), \frac{1}{2}b(\sec\phi+\tan\phi)\}$

lies on the asymptote $bx=ay$.

**5.** The polar of the point $(a\tan\phi, b\sec\phi)$ with respect to the original hyperbola is $\dfrac{x}{a}\tan\phi-\dfrac{y}{b}\sec\phi=1$, which is a tangent to the conjugate hyperbola at the point

$$(-a\tan\phi, -b\sec\phi).$$

**6.** The line $x=a\sec\phi$ meets the conjugate hyperbola where $y = b\sqrt{1 + \sec^2\phi}$.

Hence the equations of the two tangents are

$\frac{x}{a} - \frac{y}{b}\sin\phi = \cos\phi$, and $\frac{x}{a} - \frac{y}{b}\sqrt{1 + \cos^2\phi} = -\cos\phi$.

Where these meet we therefore have

$$\frac{x}{a} - \cos\phi = \frac{y}{b}\sin\phi \quad \text{.................. (i)}$$

and

$$\frac{x}{a} + \cos\phi = \frac{y}{b}\sqrt{1 + \cos^2\phi}. \quad \text{...........(ii)}$$

Square and add. $\therefore\ \cos^2\phi = \frac{y^2}{b^2} - \frac{x^2}{a^2}$.

Substitute in (ii) $\therefore\ \frac{x}{a} + \sqrt{\frac{y^2}{b^2} - \frac{x^2}{a^2}} = \frac{y}{b}\sqrt{1 + \frac{y^2}{b^2} - \frac{x^2}{a^2}}$.

On rationalizing, this equation becomes

$$\frac{y^4}{b^4}\left\{\frac{y^2}{b^2} - \frac{x^2}{a^2}\right\} = \frac{4x^2}{a^2}.$$

The equations to the normals are

$$ax\sin\phi + by = (a^2 + b^2)\tan\phi,$$

and $\quad ax\sqrt{1 + \cos^2\phi} + by = (a^2 + b^2)(\sqrt{1 + \sec^2\phi})$,

which meet on the axis of $x$ where $x = \frac{a^2 + b^2}{a}\sec\phi$.

**7.** If $P$ be the point $(a\sec\phi,\ b\tan\phi)$, then, by the last example, $CG = \frac{a^2 + b^2}{a}\sec\phi$.

$\therefore$ the equation of the line through $G$ perpendicular to $bx - ay = 0$ is $ax + by = (a^2 + b^2)\sec\phi$.

Solving, $x = a\sec\phi$, giving the abscissa of $L$.

$\therefore$ $PL$ is perpendicular to the transverse axis.

**8.** The equation $2x^2 + 5xy + 2y^2 + 4x + 5y + c = 0$ represents two straight lines if

$$4c + 25 - \tfrac{25}{2} - 8 - c\tfrac{25}{4} = 0, \quad [\text{Art. } 116]$$

whence $c = 2$, etc. See Art. 326.

**9.** The hyperbola $(x+2y+3)(3x+4y+5)=\lambda^2$ passes through the point $(1, -1)$ if $\lambda^2 = 2.4 = 8$. Hence etc.

The equation of the conjugate hyperbola is

$$(x+2y+3)(3x+4y+5)=-8. \quad \text{[See Art. 324.]}$$

**10.** $CP^2 = CD^2 = a^2(\sec^2\phi + \tan^2\phi)$. [Art. 323.]

The "$m$" of $CP$ is $\sin\phi$, and the "$m$" of $CD = \dfrac{1}{\sin\phi}$.

Hence the inclinations of $CP$ and $CD$ to the transverse axis are complementary.

**11.** Solving $\dfrac{x}{a} - \dfrac{y}{b}\sin\phi = \cos\phi$, [Art. 307] with $\dfrac{x}{a} = \dfrac{y}{b}$,

we obtain $\left(\dfrac{a\cos\phi}{1-\sin\phi}, \dfrac{b\cos\phi}{1-\sin\phi}\right)$ for the coordinates of $Q$, and, similarly, $\left(\dfrac{a\cos\phi}{1+\sin\phi}, -\dfrac{b\cos\phi}{1+\sin\phi}\right)$ for the coordinates of $R$.

The required point is the intersection of the lines

$$x^2+y^2 = \left(x - \frac{a\cos\phi}{1-\sin\phi}\right)^2 + \left(y - \frac{b\cos\phi}{1-\sin\phi}\right)^2,$$

and $$x^2+y^2 = \left(x - \frac{a\cos\phi}{1+\sin\phi}\right)^2 + \left(y + \frac{b\cos\phi}{1+\sin\phi}\right)^2;$$

or $$2(ax+by)(1-\sin\phi) = (a^2+b^2)\cos\phi,$$

and $$2(ax-by)(1+\sin\phi) = (a^2+b^2)\cos\phi.$$

Multiplying, we have $4(a^2x^2 - b^2y^2) = (a^2+b^2)^2$.

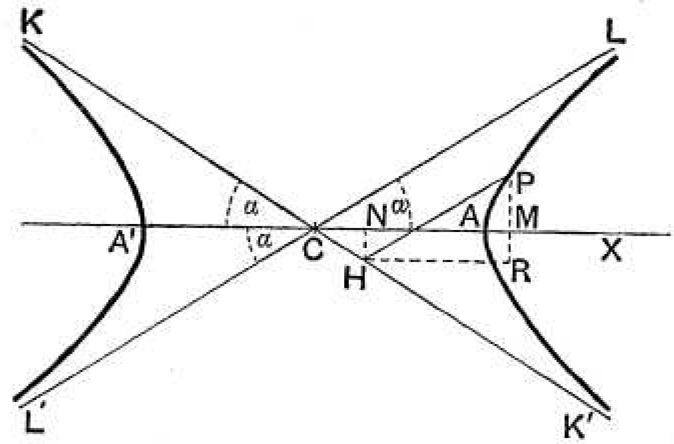
**12.** Let $b^2x^2 - a^2y^2 = a^2b^2$ be the equation of one of the hyperbolas. Then, if $(h, k)$ be the coordinates of $P$,

we have $$\frac{bh-ak}{\sqrt{a^2+b^2}} = k, \text{ whence } b = \frac{2ahk}{h^2-k^2}.$$

Also $$b^2h^2 - a^2k^2 = a^2b^2.$$

$$\therefore \frac{4a^2h^2k^2}{(h^2-k^2)^2}(h^2-a^2) = a^2k^2, \quad \text{or} \quad 4h^2(h^2-a^2) = (h^2-k^2)^2.$$

**328.** *To find the equation to a hyperbola referred to its asymptotes.*




Let $P$ be any point on the hyperbola, whose equation referred to its axes is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Draw $PH$ parallel to one asymptote $CL$ to meet the other $CK'$ in $H$, and let $CH$ and $HP$ be $h$ and $k$ respectively. Then $h$ and $k$ are the coordinates of $P$ referred to the asymptotes.

Let $\alpha$ be the semi-angle between the asymptotes, so that, by Art. 313, $\tan \alpha = \frac{b}{a}$,

and hence $$\frac{\sin \alpha}{b} = \frac{\cos \alpha}{a} = \frac{1}{\sqrt{a^2 + b^2}}.$$

Draw $HN$ perpendicular to the transverse axis, and $HR$ parallel to the transverse axis, to meet the ordinate $PM$ of the point $P$ in $R$.

Then, since $PH$ and $HR$ are parallel respectively to $CL$ and $CM$, we have $\angle PHR = \angle LCM = \alpha$.

Hence $CM = CN + HR = CH \cos \alpha + HP \cos \alpha$

$$= (h+k)\frac{a}{\sqrt{a^2+b^2}},$$

and $MP = RP - HN = HP \sin \alpha - CH \sin \alpha$

$$= (k-h)\frac{b}{\sqrt{a^2+b^2}}.$$

Therefore, since $CM$ and $MP$ satisfy the equation (1), we have

$$\frac{(h+k)^2}{a^2+b^2} - \frac{(k-h)^2}{a^2+b^2} = 1, \text{ i.e. } hk = \frac{a^2+b^2}{4}.$$

Hence, since $(h, k)$ is any point on the hyperbola, the required equation is

$$\mathbf{xy = \frac{a^2+b^2}{4}}.$$

This is often written in the form $xy = c^2$, where $4c^2$ equals the sum of the squares of the semiaxes of the hyperbola.

Similarly, the equation to the conjugate hyperbola is, when referred to the asymptotes,

$$xy = -\frac{a^2+b^2}{4}.$$

**329.** *To find the equation to the tangent at any point of the hyperbola* $xy = c^2$.

Let $(x', y')$ be any point $P$ on the hyperbola, and $(x'', y'')$ a point $Q$ on it, so that we have

$$x'y' = c^2 \qquad \text{.......................} (1),$$

and

$$x''y'' = c^2 \qquad \text{.......................} (2).$$

The equation to the line $PQ$ is then

$$y - y' = \frac{y''-y'}{x''-x'}(x-x') \qquad \text{..............} (3).$$

But, by (1) and (2), we have

$$\frac{y''-y'}{x''-x'}=\frac{\frac{c^2}{x''}-\frac{c^2}{x'}}{x''-x'}=\frac{c^2}{x'x''}\frac{x'-x''}{x''-x'}=-\frac{c^2}{x'x''}.$$

Hence the equation (3) becomes

$$y-y'=-\frac{c^2}{x'x''}(x-x') \quad \ldots\ldots\ldots\ldots\ldots (4).$$

Let now the point $Q$ be taken indefinitely near to $P$, so that $x''=x'$ ultimately, and therefore, by Art. 149, $PQ$ becomes the tangent at $P$.

Then (4) becomes

$$y-y'=-\frac{c^2}{x'^2}(x-x') = -\frac{y'}{x'}(x-x'), \text{ by (1)}.$$

The required equation is therefore

$$xy'+x'y=2x'y'=2c^2 \quad \ldots\ldots\ldots\ldots\ldots (5).$$

The equation (5) may also be written in the form

$$\frac{x}{x'}+\frac{y}{y'}=2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

**330.** *The tangent at any point of a hyperbola cuts off a triangle of constant area from the asymptotes, and the portion of it intercepted between the asymptotes is bisected at the point of contact.*

Take the asymptotes as axes and let the equation to the hyperbola be $xy=c^2$.

The tangent at any point $P$ is $\frac{x}{x'}+\frac{y}{y'}=2$.

This meets the axes in the points $(2x', 0)$ and $(0, 2y')$.

If these points be $L$ and $L'$, and the centre be $C$, we have

$$CL=2x', \text{ and } CL'=2y'.$$

If $2\alpha$ be the angle between the asymptotes, the area of the triangle

$$LCL'=\tfrac{1}{2}CL\,.\,CL'\sin 2\alpha=2x'y'\sin 2\alpha=\frac{a^2+b^2}{2}\,.\,2\sin\alpha\cos\alpha=ab.$$

(Art. 328.)

Also, since $L$ is the point $(2x', 0)$ and $L'$ is $(0, 2y')$, the middle point of $LL'$ is $(x', y')$, *i.e.* the point of contact $P$.

**331.** As in Art. 274, the polar of any point $(x_1, y_1)$ with respect to the curve can be shewn to be

$$xy_1 + x_1y = 2c^2.$$

Since, in general, the point $(x_1, y_1)$ does not lie on the curve the equation to the polar cannot be put into the form (6) of Art. 329.

**332.** The equation to the normal at the point $(x', y')$ is $y - y' = m(x - x')$, where $m$ is chosen so that this line is perpendicular to the tangent

$$y = -\frac{y'}{x'}x + \frac{2c^2}{x'}.$$

If $\omega$ be the angle between the asymptotes we then obtain, by Art. 93,

$$m = \frac{x' - y' \cos \omega}{y' - x' \cos \omega},$$

so that the required equation to the normal is

$$y(y' - x' \cos \omega) - x(x' - y' \cos \omega) = y'^2 - x'^2.$$

$$\left[\text{Also } \cos \omega = \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = \frac{a^2 - b^2}{a^2 + b^2}\right].$$

If the hyperbola be rectangular, then $\omega = 90°$, and the equation to the normal becomes $xx' - yy' = x'^2 - y'^2$.

## 333. Equation referred to the asymptotes. One Variable.

The equation $xy = c^2$ is clearly satisfied by the substitution $x = ct$ and $y = \frac{c}{t}$.

Hence, for all values of $t$, the point whose coordinates are $\left(ct, \frac{c}{t}\right)$ lies on the curve, and it may be called the point "$t$."

The tangent at the point "$t$" is by Art. 329,

$$\frac{x}{t} + yt = 2c.$$

Also the normal is, by the last article,

$$y(1-t^2\cos\omega)-x(t^2-\cos\omega)=\frac{c}{t}(1-t^4),$$

or, when the hyperbola is rectangular,

$$y-xt^2=\frac{c}{t}(1-t^4).$$

The equations to the tangents at the points "$t_1$" and "$t_2$" are

$$\frac{x}{t_1}+yt_1=2c, \text{ and } \frac{x}{t_2}+yt_2=2c,$$

and hence the tangents meet at the point

$$\left(\frac{2ct_1t_2}{t_1+t_2},\quad \frac{2c}{t_1+t_2}\right).$$

The line joining "$t_1$" and "$t_2$," which is the polar of this point, is therefore, by Art. 331,

$$x+yt_1t_2=c(t_1+t_2).$$

This form also follows by writing down the equation to the straight line joining the points

$$\left(ct_1, \frac{c}{t_1}\right) \text{ and } \left(ct_2, \frac{c}{t_2}\right).$$

**334. Ex. 1.** *If a rectangular hyperbola circumscribe a triangle, it also passes through the orthocentre of the triangle.*

Let the equation to the curve referred to its asymptotes be

$$xy=c^2 \qquad \text{.................................(1).}$$

Let the angular points of the triangle be $P$, $Q$, and $R$, and let their coordinates be

$$\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right), \text{ and } \left(ct_3, \frac{c}{t_3}\right)$$

respectively.

As in the last article, the equation to $QR$ is

$$x+yt_2t_3=c(t_2+t_3).$$

The equation to the straight line, through $P$ perpendicular to $QR$, is therefore

$$y-\frac{c}{t_1}=t_2t_3[x-ct_1],$$

*i.e.*

$$y+ct_1t_2t_3=t_2t_3\left[x+\frac{c}{t_1t_2t_3}\right] \qquad \text{....................(2).}$$

Similarly, the equation to the straight line through $Q$ perpendicular to $RP$ is

$$y+ct_1t_2t_3=t_3t_1\left[x+\frac{c}{t_1t_2t_3}\right] \qquad\qquad(3).$$

The common point of (2) and (3) is clearly

$$\left(-\frac{c}{t_1t_2t_3},\ -ct_1t_2t_3\right) \qquad\qquad(4),$$

and this is therefore the orthocentre.

But the coordinates (4) satisfy (1). Hence the proposition.

Also if $\left(ct_4, \frac{c}{t_4}\right)$ be the orthocentre of the points "$t_1$," "$t_2$," and "$t_3$," we have $t_1t_2t_3t_4=-1$.

**Ex. 2.** *If a circle and the rectangular hyperbola $xy=c^2$ meet in the four points "$t_1$," "$t_2$," "$t_3$," and "$t_4$," prove that*

(1) $t_1t_2t_3t_4=1$,

(2) *the centre of mean position of the four points bisects the distance between the centres of the two curves,*

*and* (3) *the centre of the circle through the points "$t_1$," "$t_2$," "$t_3$" is*

$$\left\{\frac{c}{2}\left(t_1+t_2+t_3+\frac{1}{t_1t_2t_3}\right),\ \frac{c}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+t_1t_2t_3\right)\right\}.$$

Let the equation to the circle be

$$x^2+y^2-2gx-2fy+k=0,$$

so that its centre is the point $(g, f)$.

Any point on the hyperbola is $\left(ct, \frac{c}{t}\right)$. If this lie on the circle,

we have
$$c^2t^2+\frac{c^2}{t^2}-2gct-2f\frac{c}{t}+k=0,$$

so that
$$t^4-2\frac{g}{c}t^3+\frac{k}{c^2}t^2-\frac{2f}{c}t+1=0 \qquad\qquad(1).$$

If $t_1$, $t_2$, $t_3$, and $t_4$ be the roots of this equation, we have, by Art. 2,

$$t_1t_2t_3t_4=1 \qquad\qquad(2),$$

$$t_1+t_2+t_3+t_4=\frac{2g}{c} \qquad\qquad(3),$$

and
$$t_2t_3t_4+t_3t_4t_1+t_4t_1t_2+t_1t_2t_3=\frac{2f}{c} \qquad\qquad(4).$$

Dividing (4) by (2), we have

$$\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{1}{t_4}=\frac{2f}{c} \qquad\qquad(5).$$

The centre of the mean position of the four points,

$$i.e. \text{ the point } \left\{\frac{c}{4}(t_1+t_2+t_3+t_4), \frac{c}{4}\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{1}{t_4}\right)\right\},$$

is therefore the point $\left(\frac{g}{2}, \frac{f}{2}\right)$, and this is the middle point of the line joining (0, 0) and $(g, f)$.

Also, since $t_4=\frac{1}{t_1t_2t_3}$, we have

$$g=\frac{c}{2}\left(t_1+t_2+t_3+\frac{1}{t_1t_2t_3}\right), \text{ and } f=\frac{c}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+t_1t_2t_3\right).$$

Again, since $t_1t_2t_3t_4=1$, we have product of the abscissae of the four points = product of their ordinates $=c^4$.

## EXAMPLES XXXVIII

**1.** Prove that the foci of the hyperbola $xy=\frac{a^2+b^2}{4}$ are given by

$$x=y=\pm\frac{a^2+b^2}{2a}.$$

**2.** Shew that two concentric rectangular hyperbolas, whose axes meet at an angle of 45°, cut orthogonally.

**3.** A straight line always passes through a fixed point; prove that the locus of the middle point of the portion of it, which is intercepted between two given straight lines, is a hyperbola whose asymptotes are parallel to the given lines.

**4.** If the ordinate $NP$ at any point $P$ of an ellipse be produced to $Q$, so that $NQ$ is equal to the subtangent at $P$, prove that the locus of $Q$ is a hyperbola.

**5.** From a point $P$ perpendiculars $PM$ and $PN$ are drawn to two straight lines $OM$ and $ON$. If the area $OMPN$ be constant, prove that the locus of $P$ is a hyperbola.

**6.** A variable line has its ends on two lines given in position and passes through a given point; prove that the locus of a point which divides it in any given ratio is a hyperbola.

**7.** The coordinates of a point are $a\tan(\theta+\alpha)$ and $b\tan(\theta+\beta)$, where $\theta$ is variable; prove that the locus of the point is a hyperbola.

**8.** A series of circles touch a given straight line at a given point. Prove that the locus of the pole of a given straight line with regard to these circles is a hyperbola whose asymptotes are respectively a parallel to the first given straight line and a perpendicular to the second.

**9.** If a right-angled triangle be inscribed in a rectangular hyperbola, prove that the tangent at the right angle is the perpendicular upon the hypothenuse.

**10.** In a rectangular hyperbola, prove that all straight lines, which subtend a right angle at a point $P$ on the curve, are parallel to the normal at $P$.

**11.** Chords of a rectangular hyperbola are at right angles, and they subtend a right angle at a fixed point $O$; prove that they intersect on the polar of $O$.

**12.** Prove that any chord of a rectangular hyperbola subtends angles which are equal or supplementary (1) at the ends of a perpendicular chord, and (2) at the ends of any diameter.

**13.** In a rectangular hyperbola, shew that the angle between a chord $PQ$ and the tangent at $P$ is equal to the angle which $PQ$ subtends at the other end of the diameter through $P$.

**14.** Show that the normal to the rectangular hyperbola $xy=c^2$ at the point "$t$" meets the curve again at a point "$t'$" such that

$$t^3t'=-1.$$

**15.** If $P_1$, $P_2$, and $P_3$ be three points on the rectangular hyperbola $xy=c^2$, whose abscissæ are $x_1$, $x_2$, and $x_3$, prove that the area of the triangle $P_1P_2P_3$ is

$$\frac{c^2}{2}\frac{(x_2-x_3)(x_3-x_1)(x_1-x_2)}{x_1x_2x_3},$$

and that the tangents at these points form a triangle whose area is

$$2c^2\frac{(x_2-x_3)(x_3-x_1)(x_1-x_2)}{(x_2+x_3)(x_3+x_1)(x_1+x_2)}.$$

**16.** Find the coordinates of the points of contact of common tangents to the two hyperbolas

$$x^2-y^2=3a^2 \text{ and } xy=2a^2.$$

**17.** The transverse axis of a rectangular hyperbola is $2c$ and the asymptotes are the axes of coordinates; shew that the equation of the chord which is bisected at the point $(2c, 3c)$ is $3x+2y=12c$.

**18.** Prove that the portions of any line which are intercepted between the asymptotes and the curve are equal.

**19.** Shew that the straight lines drawn from a variable point on the curve to any two fixed points on it intercept a constant distance on either asymptote.

**20.** Shew that the equation to the director circle of the conic $xy=c^2$ is $$x^2+2xy\cos\omega+y^2=4c^2\cos\omega.$$

**21.** Prove that the asymptotes of the hyperbola $xy=hx+ky$ are $x=k$ and $y=h$.

**22.** Shew that the straight line $y = mx + 2c\sqrt{-m}$ always touches the hyperbola $xy = c^2$, and that its point of contact is $\left(\frac{c}{\sqrt{-m}}, c\sqrt{-m}\right)$.

**23.** Prove that the locus of the foot of the perpendicular let fall from the centre upon chords of the rectangular hyperbola $xy = c^2$ which subtend half a right angle at the origin is the curve

$$r^4 - 2c^2r^2 \sin 2\theta = c^4.$$

**24.** A tangent to the parabola $x^2 = 4ay$ meets the hyperbola $xy = k^2$ in two points $P$ and $Q$. Prove that the middle point of $PQ$ lies on a parabola.

**25.** If a hyperbola be rectangular, and its equation be $xy = c^2$, prove that the locus of the middle points of chords of constant length $2d$ is

$$(x^2 + y^2)(xy - c^2) = d^2xy.$$

**26.** Shew that the pole of any tangent to the rectangular hyperbola $xy = c^2$, with respect to the circle $x^2 + y^2 = a^2$, lies on a concentric and similarly placed rectangular hyperbola.

**27.** Prove that the locus of the poles of all normal chords of the rectangular hyperbola $xy = c^2$ is the curve

$$(x^2 - y^2)^2 + 4c^2xy = 0.$$

**28.** Any tangent to the rectangular hyperbola $4xy = ab$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the points $P$ and $Q$; prove that the normals at $P$ and $Q$ to the ellipse meet on a fixed diameter of the ellipse.

**29.** Prove that triangles can be inscribed in the hyperbola $xy = c^2$, whose sides touch the parabola $y^2 = 4ax$.

**30.** A point moves on the given straight line $y = mx$; prove that the locus of the foot of the perpendicular let fall from the centre upon its polar with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a rectangular hyperbola, one of whose asymptotes is the diameter of the ellipse which is conjugate to the given straight line.

**31.** A quadrilateral circumscribes a hyperbola; prove that the straight line joining the middle points of its diagonals passes through the centre of the curve.

**32.** $A$, $B$, $C$, and $D$ are the points of intersection of a circle and a rectangular hyperbola. If $AB$ pass through the centre of the hyperbola, prove that $CD$ passes through the centre of the circle.

**33.** If a circle and a rectangular hyperbola meet in four points $P$, $Q$, $R$, and $S$, shew that the orthocentres of the triangles $QRS$, $RSP$, $SPQ$, and $PQR$ also lie on a circle.

Prove also that the tangents to the hyperbola at $R$ and $S$ meet in a point which lies on the diameter of the hyperbola which is at right angles to $PQ$.

**34.** A series of hyperbolas is drawn, having for asymptotes the principal axes of an ellipse; shew that the common chords of the hyperbolas and the ellipse are all parallel to one of the conjugate diameters of the ellipse.

**35.** A circle, passing through the centre of a rectangular hyperbola, cuts the curve in the points $A$, $B$, $C$, and $D$; prove that the circum-circle of the triangle formed by the tangents at $A$, $B$, and $C$ goes through the centre of the hyperbola, and has its centre at the point of the hyperbola which is diametrically opposite to $D$.

**36.** Given five points on a circle of radius $a$; prove that the centres of the rectangular hyperbolas, each passing through four of these points, all lie on a circle of radius $\frac{a}{2}$.

**37.** If a rectangular hyperbola circumscribe a triangle, shew that it meets the circle circumscribing the triangle in a fourth point, which is at the other end of the diameter of the hyperbola which passes through the orthocentre of the triangle.

Hence prove that the locus of the centre of a rectangular hyperbola which circumscribes a triangle is the nine-point circle of the triangle.

**38.** Two rectangular hyperbolas are such that the asymptotes of one are parallel to the axes of the other and the centre of each lies on the other. If any circle through the centre of one cut the other again in the points $P$, $Q$, and $R$, prove that $PQR$ is a triangle such that each side is the polar of the opposite vertex with respect to the first hyperbola.

## ANSWERS

**16.** $(\pm\frac{3}{4}\sqrt{6}a, \mp\frac{1}{4}\sqrt{6}a)$; $(\pm\frac{1}{3}\sqrt{6}a, \pm\sqrt{6}a)$.

## SOLUTIONS/HINTS

**1.** If $SH'$, parallel to $CL$ (Fig., Art. 328), meets $CK'$ in $H'$, and $(x, y)$ be the coordinates of $S$, we have by symmetry $CH' = H'S$, and

$$\therefore\ CH' \cos\alpha = H'S \cos\alpha = \tfrac{1}{2} CS.$$

$$\therefore\ 2x \cos\alpha = 2y \cos\alpha = \sqrt{a^2 + b^2}.$$

$$\therefore\ x = y = \frac{a^2 + b^2}{2a},$$ since, by Art. 328, $\cos\alpha = \frac{a}{\sqrt{a^2 + b^2}}$.

**2.** Since the asymptotes of each are the axes of the other, let $x^2 - y^2 = a^2$ ...(i) and $xy = b^2$ ...(ii) be the equations of the two rectangular hyperbolas.

The tangents at the point $(a \sec \phi, a \tan \phi)$ are

$$x - y \sin \phi = a \cos \phi, \text{ and } y + x \sin \phi = \frac{2b^2}{a} \cos \phi,$$

which are clearly at right angles. [Art. 69.]

**3.** See Ex. XI, No. 15.

**4.** Let $P$ be the point "$\phi$" of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Then if $(x, y)$ are the coordinates of $Q$,

$$x = a \cos \phi, \text{ and } y = \frac{a^2 - a^2 \cos^2 \phi}{a \cos \phi} \quad \text{[Art. 269]}$$

$$= \frac{a^2 \sin^2 \phi}{a \cos \phi}.$$

$$\therefore \; xy = a^2 \sin^2 \phi.$$

Eliminating $\phi$, $x^2 + xy = a^2$, which is a hyperbola. See Art. 326.

**5.** See Art. 103.

We have $PM \,.\, OM + PN \,.\, ON = \text{constant}$.

$\therefore \; (h + k \cos \omega) k \sin \omega + (k + h \cos \omega) h \sin \omega = \text{constant}$.

$\therefore \; 2hk + (h^2 + k^2) \cos \omega = \text{constant} = c^2$ say.

$\therefore$ the required equation to the locus of $P$ is

$$x^2 + y^2 + 2xy \sec \omega = c^2 \sec \omega,$$

which is a hyperbola. [Art. 326.]

**6.** Taking the two given lines as axes, let $\frac{x}{p} + \frac{y}{q} = 1$ be the equation of the variable line. Since it passes through the fixed point $(h, k)$, $\therefore \; \frac{h}{p} + \frac{k}{q} = 1$.

If $(x, y)$ be a point dividing it in the ratio $m : n$,

$$x = \frac{m}{m+n} \cdot p \text{ and } y = \frac{n}{m+n} \cdot q.$$

Eliminating $p$ and $q$, we have $\frac{mh}{x} + \frac{nk}{y} = m + n$,

or $(m+n)xy = mhy + nkx$, which is a hyperbola.

**7.** If $x = a\tan(\theta + \alpha)$ and $y = b\tan(\theta + \beta)$,

$$\therefore \tan^{-1}\frac{x}{a} = \theta + \alpha \text{ and } \tan^{-1}\frac{y}{b} = \theta + \beta.$$

$$\therefore \tan^{-1}\frac{x}{a} - \tan^{-1}\frac{y}{b} = \alpha - \beta.$$

$$\therefore \tan(\alpha - \beta) = \frac{\frac{x}{a} - \frac{y}{b}}{1 + \frac{x}{a}\frac{y}{b}}.$$

$$\therefore xy + ab = (bx - ay)\cot(\alpha - \beta),$$

which is a hyperbola.

**8.** Let $x^2 + y^2 = 2\lambda x$ for different values of $\lambda$ be the equation to the circles.

The equation to the polar of $(h, k)$ is

$$xh + yk = \lambda(x + h).$$

This is identical with a given straight line $ax + by = c$ if

$$\frac{h - \lambda}{a} = \frac{k}{b} = \frac{\lambda h}{c}.$$

$$\therefore \frac{k}{b} = \frac{h(h-\lambda) + \lambda h}{ah + c} = \frac{h^2}{ah + c}.$$

$\therefore$ the equation to the locus of $(h, k)$ is $bx^2 - axy = cy$, which is a hyperbola, whose asymptotes are parallel to

$$x = 0, \text{ and } bx - ay = 0.$$

**9.** Let "$t_1$," "$t_2$," and "$t_3$" be the angular points of the triangle, the angle at "$t_3$" being a right angle.

The "$m$'s" of the chords joining "$t_3$" and "$t_1$," and "$t_3$" and "$t_2$" are

$$-\frac{1}{t_1t_3} \text{ and } -\frac{1}{t_2t_3}, \quad [\text{Art. 333}].$$

$$\therefore \ t_1t_2t_3^2+1=0. \quad [\text{Art. 69.}]$$

The "$m$'s" of the tangent at "$t_3$" and the chord joining "$t_1$" and "$t_2$" are $-\dfrac{1}{t_3^2}$ and $-\dfrac{1}{t_1t_2}$.

Hence these lines are perpendicular by the above condition.

**10.** See the previous example.

**11.** By Art. 331, the equation to the polar of $(h, k)$ with respect to the curve $xy=c^2$ is $kx+hy=2c^2$.

If we remove the origin to the point $(h, k)$ we see that the polar of the origin with respect to the curve

$$(x+h)(y+k)=c^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

is
$$k(x+h)+h(y+k)=2c^2,$$

or
$$kx+hy=2(c^2-hk)=2C \text{ say.} \quad \ldots\ldots\ldots \text{(ii)}$$

Let $lx+my=p$ and $mx-ly=q$ be two perpendicular chords of (i). The equations of the two pairs of lines joining the origin to the extremities of these chords are [Art. 122]

$$xyp^2+(kx+hy)(lx+my)p=C(lx+my)^2,$$

and
$$xyq^2+(kx+hy)(mx-ly)q=C(mx-ly)^2.$$

Since each pair are at right angles,

$$\therefore \ p(lk+mh)=C(l^2+m^2), \text{ and } q(mk-lh)=C(l^2+m^2).$$

Adding, we have

$$k(pl+qm)+h(pm-ql)=2C(l^2+m^2),$$

which is the condition that the point of intersection of the chords, viz. $\left(\dfrac{pl+qm}{l^2+m^2}, \dfrac{pm-lq}{l^2+m^2}\right)$, should lie on the line (ii).

**12.** (1) Let $P$, $Q$, $R$, $S$, the extremities of the chords be the points $t_1$, $t_2$, $t_3$, $t_4$ respectively.

The $m$'s of $PQ$ and $SQ$ are $-\frac{1}{t_1 t_2}$ and $-\frac{1}{t_2 t_4}$.

$$\therefore\ P\hat{Q}S = \tan^{-1}\frac{-\frac{1}{t_1t_2}+\frac{1}{t_2t_4}}{1+\frac{1}{t_1t_2^2t_4}} = \tan^{-1}\frac{t_2(t_1-t_4)}{1+t_1t_2^2t_4}.$$

$$= \tan^{-1}\frac{\frac{1}{t_1}+t_1t_2t_3}{t_3-t_2},$$

since $$t_1t_2t_3t_4 = -1. \quad \text{[Art. 334.]}$$

Similarly $$P\hat{R}S = \tan^{-1}\left(\pm\frac{\frac{1}{t_1}+t_1t_2t_3}{t_2-t_3}\right).$$

$\therefore$ $P\hat{Q}S$ and $P\hat{R}S$ are either equal or supplementary.

(2) Let $t_1$ and $t_2$ be the extremities of the chord $PQ$, and $t$ and $-t$ the extremities of the diameter $RR'$.

Then $P\hat{R}Q = \tan^{-1}\frac{t(t_2 \sim t_1)}{1+t^2t_1t_2}$, as in (1) and

$$P\hat{R}'Q = \tan^{-1}\frac{-t(t_2\sim t_1)}{1+t^2t_1t_2}. \quad \text{Hence etc.}$$

**13.** Let $PQ$ be the points $t_1$ and $t_2$. Then the angle between $PQ$ and the tangent at $P$ [Art. 333]

$$= \tan^{-1}\frac{\frac{1}{t_1^2}\sim\frac{1}{t_1t_2}}{1+\frac{1}{t_1^3t_2}} = \tan^{-1}\frac{t_1(t_1\sim t_2)}{1+t_1^3t_2},$$

and the angle which $PQ$ subtends at the point $-t_1$ is found by putting $t = t_1$ in the expression for $P\hat{R}'Q$ in the previous Ex. Hence etc.

**14.** The normal at $t_1$ and the chord $t_1t_2$, viz.

$$y - xt_1^2 = \frac{c}{t_1}(1 - t_1^4),$$

and $yt_1t_2 + x = c(t_1 + t_2)$, [Art. 333], are identical if

$$\frac{1}{t_1t_2} = -\frac{t_1^2}{1}. \quad \therefore \ t_1^3t_2 = -1.$$

**15.** (1) See Ex. II, No. 9.

(2) The equations to the tangents are

$$y = -\frac{c^2}{x_1^2}x + \frac{2c^2}{x_1}, \text{ etc.}$$

Substitute in the result of Ex. X, No. 18.

**16.** The tangent at the point "$\alpha$" to the hyperbola $x^2 - y^2 = 3a^2$ viz. $x - y\sin\alpha = \sqrt{3}\,.\,a\cos\alpha$, [Art. 307], will be a tangent to $xy = 2a^2$, if the equation

$$y(y\sin\alpha + \sqrt{3}\,.\,a\cos\alpha) = 2a^2 \ \ldots\ldots\ldots\ldots \text{(i)}$$

has equal roots; the condition for which is

$$3\cos^2\alpha + 8\sin\alpha = 0, \text{ or } 3\sin^2\alpha - 8\sin\alpha - 3 = 0,$$

whence $\sin\alpha = -\frac{1}{3}$, and $\therefore\ \cos\alpha = \pm\frac{2\sqrt{2}}{3}$.

The equation (1) becomes $y^2 \pm 2\sqrt{6}\,.\,ay + 6a^2 = 0$.

$$\therefore\ y = \pm\sqrt{6}\,.\,a, \text{ and } \therefore\ x = \pm\tfrac{1}{3}\sqrt{6}a.$$

Also $\sqrt{3}a\sec\alpha = \pm\frac{3\sqrt{6}}{4}a$, and $\sqrt{3}a\tan\alpha = \mp\frac{\sqrt{6}}{4}a$.

**17.** The equation of the polar of the point $(2c, 3c)$ is, [Art. 331], $3x + 2y = 2c$, and the line parallel to this and passing through $(2c, 3c)$ is

$$3(x - 2c) + 2(y - 3c) = 0, \text{ or } 3x + 2y = 12c.$$

**18.** The chord which is bisected at $(h, k)$ is the line through this point parallel to the polar of $(h, k)$, and is therefore $k(x - h) + h(y - k) = 0$, or $kx + hy = 2hk$.

This cuts the asymptotes in the points $(2h, 0)$ and $(0, 2k)$. Hence $(h, k)$ is also the middle point of the portion of the line intercepted between the asymptotes. Hence etc.

**19.** Let "$t_1$" and "$t_2$" be the fixed points, and "$t$" the variable point. The equations of the chords joining "$t$" to "$t_1$" and "$t_2$" are

$$x + ytt_1 = c(t + t_1), \text{ and } x + ytt_2 = c(t + t_2). \quad [\text{Art. 333.}]$$

Therefore the intercept on the axis of $x = c(t_1 + t_2)$, and that on the axis of $y = c\left(\frac{1}{t_1} - \frac{1}{t_2}\right)$.

**20.** The tangents at the points "$t_1$" and "$t_2$" are

$$t_1^2 y + x = 2ct_1 \quad \text{.......................(1)}$$

and

$$t_2^2 y + x = 2ct_2. \quad \text{....................(2)}$$

These are at right angles, by Art. 93, Cor. 2, if

$$t_1^2 t_2^2 - (t_1^2 + t_2^2)\cos\omega + 1 = 0. \quad \text{...........(3)}$$

Between (1), (2) and (3) we must eliminate $t_1$ and $t_2$. On solving (1) and (2), we easily have

$$t_1 + t_2 = \frac{2c}{y} \text{ and } t_1 t_2 = \frac{x}{y}.$$

Substituting in (3), we have

$$\frac{x^2}{y^2} - \left[\frac{4c^2}{y^2} - 2\frac{x}{y}\right]\cos\omega + 1 = 0. \quad \text{Hence etc.}$$

**21.** The equation may be written $(x - k)(y - h) = hk$. Hence the asymptotes are

$$x - k = 0, \text{ and } y - h = 0 \quad [\text{Art. 326}].$$

**22.** The common points of $xy = c^2$ and the line $y = mx + a$ are given by $(mx + a)x = c^2$, or

$$mx^2 + ax - c^2 = 0, \quad \text{.................. (1)}$$

which has equal roots if $a^2 + 4c^2m = 0$, so that $a = 2c\sqrt{-m}$, and the equation (1) becomes

$$-mx^2 - 2c\sqrt{-m}x + c^2 = 0, \text{ or } \{\sqrt{-m}\,.\,x - c\}^2 = 0.$$

$$\therefore x = \frac{c}{\sqrt{-m}} \text{ and } \therefore y = \sqrt{-m}c.$$

**23.** The lines joining the origin to the common points of $xy = c^2$ and the line $x\cos\alpha + y\sin\alpha = p$ are (Art. 122),

$$p^2xy = c^2(x\cos\alpha + y\sin\alpha)^2.$$

If these are inclined at 45°,

$$\frac{2\sqrt{\left(c^2\cos\alpha\sin\alpha - \frac{p^2}{2}\right)^2 - c^4\cos^2\alpha\sin^2\alpha}}{c^2} = 1 \quad \text{[Art. 110]}.$$

$$\therefore\ p^4 - 2c^2p^2\sin 2\alpha = c^4. \quad \text{Hence etc.}$$

**24.** The chord of the hyperbola which is bisected at the point $(p, q)$ is by Ex. 18 $qx + py = 2pq$.

The points in which this cuts the parabola $x^2 = 4ay$ are given by

$$px^2 = 4a(2pq - qx), \text{ or } px^2 + 4aqx - 8apq = 0,$$

which has equal roots if $4a^2q^2 + 8ap^2q = 0$,

*i.e.* if $$aq + 2p^2 = 0.$$

Hence $(p, q)$ lies on the parabola $2x^2 + ay = 0$.

**25.** Let $\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r$ be the equation of any line through the point $(h, k)$. The points in which this line cuts the hyperbola $xy = c^2$ are given by

$$(h + r\cos\theta)(k + r\sin\theta) = c^2,$$

or $$r^2\cos\theta \,.\, \sin\theta + r(k\cos\theta + h\sin\theta) + hk - c^2 = 0.$$

If $(h, k)$ is the middle point of this chord,

$$k\cos\theta + h\sin\theta = 0, \quad \text{..................(i)}$$

and $$d^2 = \frac{c^2 - hk}{\cos\theta \,.\, \sin\theta} \quad \text{....................(ii)}$$

$$= \frac{(c^2 - hk)(h^2 + k^2)}{-hk} \text{. ........... by (i)}$$

$\therefore$ the equation to the required locus is

$$d^2xy = (xy - c^2)(x^2 + y^2).$$

**26.** The common points of the line

$$xh + yk = a^2 \quad \dots\dots\dots\dots\dots\dots \text{(1)}$$

and the hyperbola $xy = c^2$ are given by $ky^2 - a^2y + hc^2 = 0$.

Hence (1) is a tangent if this equation has equal roots, the condition for which is

$$hk = \frac{a^4}{c^2}.$$

Also $(h, k)$ is the pole of (1) with respect to the given circle. Hence etc.

**27.** The lines $xy_1 + x_1y = 2c^2$, and $y - xt^2 = \frac{c}{t}(1 - t^4)$

are identical if $\qquad -\frac{y_1}{t^2} = \frac{x_1}{1} = \frac{2ct}{1-t^4}$.

Eliminating $t$,

$$x_1\left(1 - \frac{y_1^2}{x_1^2}\right) = 2c\sqrt{-\frac{y_1}{x_1}},$$

or $\qquad (x_1^2 - y_1^2)^2 + 4c^2x_1y_1 = 0.$ Hence etc.

**28.** Let $y = m_1x + \frac{a}{m_1}$, etc., be the equations of any three tangents to the parabola. If two of their points of intersection lie on the hyperbola $xy = c^2$, then

$$a^2\left\{\frac{1}{m_1} + \frac{1}{m_2}\right\} = c^2m_1m_2, \quad \dots\dots\dots\dots\text{(1)}$$

and $$a^2\left\{\frac{1}{m_2} + \frac{1}{m_3}\right\} = c^2m_2m_3. \quad \text{[Art. 234.] (2)}$$

Whence, subtracting,

$$a^2 = -c^2m_1m_2m_3.$$

Substituting for $m_2$ in (1), we have

$$a^2\left\{\frac{1}{m_3} + \frac{1}{m_1}\right\} = c^2m_3m_1,$$

which is the condition that the third point of intersection of the tangents should also lie on the hyperbola.

**29.** The equation of the line through the point $(h, mh)$, and perpendicular to

$$\frac{xh}{a^2} + \frac{ymh}{b^2} = 1,$$

is

$$y - mh = \frac{a^2 m}{b^2}(x - h).$$

Eliminating $h$, we have

$$(b^2 y - a^2 mx)(b^2 x + a^2 my) + ma^2 b^2 (a^2 - b^2) = 0.$$

Hence etc.

**30.** Two diameters of the rectangular hyperbola $x^2 - y^2 = a^2$ are conjugate, by Art. 321, if the product of their "$m$'s" is unity.

The conjugate diameters are therefore

$$y = mx \text{ .........(1) and } y = \frac{1}{m}x. \text{ ............(2)}$$

Let the fixed point on the hyperbola be

$$(a \sec\phi,\ a \tan\phi).$$

The perpendicular from it on (1) is

$$y - a\tan\phi = -\frac{1}{m}(x - a\sec\phi).$$

This meets (1) at the point

$$\left[a\frac{m\tan\phi + \sec\phi}{1 + m^2},\ \frac{am(m\tan\phi + \sec\phi)}{1 + m^2}\right] \text{.... (4)}$$

Similarly the foot of the perpendicular upon (2) is

$$\left[\frac{am(\tan\phi + m\sec\phi)}{1 + m^2},\ \frac{a(\tan\phi + m\sec\phi)}{1 + m^2}\right] \text{.... (5)}$$

The "$m$" of the line joining (4) and (5)

$$= \frac{a(\tan\phi + m\sec\phi) - am(m\tan\phi + \sec\phi)}{am(\tan\phi + m\sec\phi) - a(m\tan\phi + \sec\phi)} = -\frac{\tan\phi}{\sec\phi} = -\sin\phi,$$

and is independent of the two conjugate diameters selected. Hence, etc.

**31.** Let $\frac{x}{t_1} + yt_1 = 2c$, etc. be the equations of the sides of the quadrilateral. Then if $(x_1, y_1)$ and $(x_2, y_2)$ be the coordinates of the middle points of the diagonals, we have

$$x_1 = c\left\{\frac{t_1 t_2}{t_1 + t_2} + \frac{t_3 t_4}{t_3 + t_4}\right\} = c\,\frac{\Sigma t_1 t_2 t_3}{(t_1 + t_2)(t_3 + t_4)} \quad \text{(Art. 333)},$$

and $$y_1 = c\left\{\frac{1}{t_1 + t_2} + \frac{1}{t_3 + t_4}\right\} = c\,\frac{\Sigma t_1}{(t_1 + t_2)(t_3 + t_4)},$$

$$\therefore\ \frac{x_1}{y_1} = \frac{\Sigma t_1 t_2 t_3}{\Sigma t_1} = \frac{x_2}{y_2}, \text{ similarly.}$$

Hence, etc.

**32.** See Art. 334, Ex. 2.

If the line joining "$t_1$" and "$t_2$" passes through the centre of the hyperbola, then $t_1 + t_2 = 0$; and equations (3) and (5) become $t_3 + t_4 = \frac{2g}{c}$ and $\frac{1}{t_3} + \frac{1}{t_4} = \frac{2f}{c}$, whence

$$g + ft_3 t_4 = c\,(t_3 + t_4),$$

which is the condition that the chord joining "$t_3$" and "$t_4$" should pass through the point $(g, f)$.

**33.** Let "$t_1$", "$t_2$", "$t_3$", and "$t_4$" be the four points $P$, $Q$, $R$, $S$ so that by Art. 334, Ex. 2, we have

$$t_1 t_2 t_3 t_4 = 1. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

By Art. 334, Ex. 1, the orthocentre of the triangle $PQR$ is

$$\left(-\frac{c}{t_1 t_2 t_3},\ -ct_1 t_2 t_3\right),\ \textit{i.e.}, \text{ by (1)}, \left(-ct_4,\ -\frac{c}{t_4}\right).$$

The four orthocentres are thus

$$\left(-ct_1,\ -\frac{c}{t_1}\right),\ \left(-ct_2,\ -\frac{c}{t_2}\right),\ \left(-ct_3,\ -\frac{c}{t_3}\right) \text{ and } \left(-ct_4,\ -\frac{c}{t_4}\right).$$

The product of their "$t$'s" $= (-t_1)(-t_2)(-t_3)(-t_4) = 1$, by (1).

Hence, by Art. 334, Ex. 2, the four orthocentres are the intersections of the hyperbola and a circle.

The point of intersection of the tangents at $R$ and $S$ is

$$\left(\frac{2ct_3t_4}{t_3+t_4},\ \frac{2c}{t_3+t_4}\right),$$

the diameter through which is $x = yt_3t_4$, *i.e.* $y = xt_1t_2$, by (1).

This is perpendicular to the straight line joining $t_1$ and $t_2$.

**34.** The equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+2\lambda(xy-c^2)=0$, represents any conic through the intersection of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the hyperbola $xy=c^2$.

It will represent two straight lines if $\lambda^2=\frac{1}{a^2b^2}$. (Art. 116.)

Hence the common chords are parallel to $\frac{x}{a}\pm\frac{y}{b}=0$, and these are the same for all values of $c$, *i.e.* whatever be the hyperbola chosen.

**35.** See Art. 334, Ex. 2.

If $k=0$, then $\Sigma t_1t_2=0$,

*i.e.* $$t_1t_2+t_2t_4+t_1t_4+\frac{1}{t_1t_2t_4}(t_1+t_2+t_4)=0. \qquad \ldots\ldots\ldots(1)$$

Now if $(x, y)$ be the coordinates of a vertex of the triangle formed by the tangents,

$$x=\frac{2ct_1t_2}{t_1+t_2},\ \text{and}\ y=\frac{2c}{t_1+t_2}. \quad \text{(Art. 333.)}$$

Substitute in (1); $\therefore\ x^2+y^2+2ct_4x+\frac{2c}{t_4}y=0$,

which is the equation of a circle passing through the origin and whose centre is the point $\left(-ct_4,\ -\frac{c}{t_4}\right)$.

**36.** Let $(a\cos\alpha_1, a\sin\alpha_1)$ etc., be the coordinates of the five points.

Now the centre of mean position $(x', y')$ of any four of the points is the middle point of the line joining (0, 0) to the centre of the corresponding hyperbola. [Art. 334, Ex. 2.] Hence, if $(x', y')$ be the centre of the hyperbola through the first four points,

$$x' = \frac{a}{2}(\Sigma\cos\alpha_1 - \cos\alpha_5), \text{ and } y' = \frac{a}{2}(\Sigma\sin\alpha_1 - \sin\alpha_5).$$

$$\therefore \left(x' - \frac{a}{2}\Sigma\cos\alpha_1\right)^2 + \left(y' - \frac{a}{2}\Sigma\sin\alpha_1\right)^2 = \frac{a^2}{4};$$

so for the other four centres.

Hence their locus is the circle

$$\left(x - \frac{a}{2}\Sigma\cos\alpha_1\right)^2 + \left(y - \frac{a}{2}\Sigma\sin\alpha_1\right)^2 = \frac{a^2}{4}.$$

**37.** If $t_1$, $t_2$, $t_3$ be the angular points of the triangle, then by Art. 334, Ex. 1, $-\frac{1}{t_1t_2t_3}$ is the orthocentre, and by Art. 334, Ex. 2, $\frac{1}{t_1t_2t_3}$ is the fourth point in which the hyperbola meets the circumcircle. The centre of the hyperbola is thus midway between these two points.

Also it is well known from Plane Geometry that the middle point of the straight line joining the orthocentre of a triangle to any point on its circumcircle lies on the nine-point circle of the triangle.