Exercise 11.5

Answer 1

(a) An alternating series is a series whose terms are alternately positive and negative

(b) An alternating series converge under the following two conditions

(i) $0 \leq b_{n+1} \leq b_n$

(ii) $\lim_{n \to \infty} b_n = 0$

(c) Remainder $|R_n| \leq b_{n+1}$

Answer 2

Given alternating series is

$$\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \ \dots \ = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$$

This series satisfies

(i) $b_{n+1} \leq b_n$ for all n, because $\frac{1}{2n+3} < \frac{1}{2n+1}$ where $b_n = \frac{1}{2n+1}$

(ii) $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{2n+1}$

$= 0$

So the series is convergent by the Alternating Series test.

Answer 3

To determine the series converges or diverges, consider the series

$$-\frac{2}{5} + \frac{4}{6} - \frac{6}{7} + \frac{8}{8} - \frac{10}{9} + \dots$$

The series can be written as

$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n}{n+4} \right)$$

The above series contains terms with alternatively negative and posive signs.

So, it is alternating series.

To determine the convergence of it, apply alternating series test.

Alternating series test:

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \ldots \qquad b_n > 0$$

satisfies

(i) $b_{n+1} \le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Let $b_n = \dfrac{2n}{n+4}$

Now, apply the limit as $n \to \infty$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{2n}{n+4}$$

$$= \lim_{n\to\infty}\frac{2n}{n\left(1+\frac{4}{n}\right)}$$

$$= \lim_{n\to\infty}\frac{2}{1+\frac{4}{n}}$$

$$= \frac{2}{1+0}$$

$$= 2$$

Answer 4

Consider the following series:

$$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \cdots$$

The objective is to determine whether the series is convergent or divergent.

Observe that, the terms of the given series are alternatively positive and negative, so it is an alternating series.

Rewrite the given series, using the sigma notation as follows:

$$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \cdots = \frac{(-1)^{1-1}}{\sqrt{1+1}} + \frac{(-1)^{2-1}}{\sqrt{2+1}} + \frac{(-1)^{3-1}}{\sqrt{3+1}} + \frac{(-1)^{4-1}}{\sqrt{4+1}} + \cdots$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n+1}}$$

The Alternating Series Test is defined as follows:

Let the alternating series be represented as follows:

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \text{ here, } b_n > 0.$$

Now, if the following conditions are satisfied, then the series will be convergent:

(i) $b_{n+1} \le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

For the given series, the following variable can be obtained.

$$b_n = \frac{1}{\sqrt{n+1}}$$

Consider $b_{n+1} - b_n$ and solve as follows:

$$b_{n+1} - b_n = \frac{1}{\sqrt{(n+1)+1}} - \frac{1}{\sqrt{n+1}}$$

$$= \frac{\sqrt{n+1} - \sqrt{n+2}}{\left(\sqrt{n+2}\right)\left(\sqrt{n+1}\right)}$$

$$= \frac{\sqrt{n+1} - \sqrt{n+2}}{\left(\sqrt{n+2}\right)\left(\sqrt{n+1}\right)} \times \frac{\sqrt{n+1} + \sqrt{n+2}}{\sqrt{n+1} + \sqrt{n+2}}$$

$$= \frac{n+1-(n+2)}{\left(\sqrt{n+2}\right)\left(\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n+2}\right)}$$

$$= \frac{-1}{\left(\sqrt{n+2}\right)\left(\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n+2}\right)} < 0 \text{ for all } n$$

Thus, $b_{n+1} \le b_n$ for all n.

Now, calculate the limit as follows:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n\left(1+\frac{1}{n}\right)}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1/n}}{\sqrt{1+\frac{1}{n}}}$$

$$= \frac{0}{\sqrt{1+0}} \text{ As } n\to\infty,\ 1/n\to 0$$

$$= 0$$

Since $b_n = \frac{1}{\sqrt{n+1}}$, satisfies the conditions that $b_{n+1} \le b_n$ for all n and $\lim_{n\to\infty} b_n = 0$, so by Alternating Series Test, the following result is obtained:

Hence, $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n+1}}$ is $\boxed{\text{convergent}}$.

Answer 5

Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$$

Its need to determine whether the series convergence or divergence

On expanding the series notation, we have that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1} = \frac{(-1)^{1-1}}{2(1)+1} + \frac{(-1)^{2-1}}{2(2)+1} + \frac{(-1)^{3-1}}{2(3)+1} + \frac{(-1)^{4-1}}{2(4)+1} + \cdots$$

$$= \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$$

Observe that, terms of given series are alternatively positive and negative, so it is an alternating series.

The Alternating Series Test: if the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \quad b_n > 0$$

Satisfies

(i) $b_{n+1} \le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

For the given series, we have that

$$b_n = \frac{1}{2n+1}$$

Consider $b_{n+1} - b_n$

$$b_{n+1} - b_n = \frac{1}{2(n+1)+1} - \frac{1}{2n+1}$$

$$= \frac{1}{2n+3} - \frac{1}{2n+1}$$

$$= \frac{2n+1-2n-3}{(2n+3)(2n+1)}$$

$$= \frac{-2}{(2n+3)(2n+1)} < 0 \text{ for all } n$$

Thus $b_{n+1} \le b_n$ for all n

And also that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2n+1}$$

$$= \lim_{n\to\infty} \frac{1}{n\left(2+\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{1/n}{2+\frac{1}{n}}$$

$$= \frac{0}{2+0} \text{ As } n \to \infty,\ 1/n \to 0$$

$$= 0$$

Since $b_n = \frac{1}{2n+1}$, satisfies the conditions that $b_{n+1} \le b_n$ for all n and $\lim_{n\to\infty} b_n = 0$, by Alternating Series Test, the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$$

is $\boxed{\text{convergent}}$.

Answer 6

Consider the following series:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\ln(n+4)}$$

The objective is to determine whether the series is convergent or divergent

Expand the series notation as follows:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1}=\frac{(-1)^{1-1}}{\ln(1+4)}+\frac{(-1)^{2-1}}{\ln(2+4)}+\frac{(-1)^{3-1}}{\ln(3+4)}+\frac{(-1)^{4-1}}{\ln(4+4)}+\cdots$$

$$=\frac{1}{\ln 5}-\frac{1}{\ln 6}+\frac{1}{\ln 7}-\frac{1}{\ln 8}+\cdots$$

Observe that, the terms of given series are alternatively positive and negative, so it is an alternating series.

The Alternating Series Test is defined as follows:

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n=b_1-b_2+b_3-b_4+b_5-b_6+\cdots \text{ here, } b_n>0.$$

Now, if the following conditions are satisfied, then the series will be convergent:

(i) $b_{n+1}\le b_n$ for all n.

(ii) $\lim_{n\to\infty} b_n=0$.

For the given series, use the following variable and proceed as follows:

$$b_n=\frac{1}{\ln(n+4)}$$

Observe that, $n+5<n+4$ for all $n\in N.$

Since the logarithmic function is increasing on $(0,\infty)$, proceed as follows:

$\ln(n+5)<\ln(n+4)$ for all $n\in N.$

That is, $b_{n+1}<b_n$ for all $n\in N$.

Thus, $b_{n+1}\le b_n$ for all n.

As n becomes larger, $(n+4)$ increases similarly $\ln(n+4)$ also increases.

Proceeding further, using the reciprocal $1/\ln(n+4)$ approaches to zero.

Thus, $\lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{1}{\ln(n+4)}$

$=0$

So, $\lim_{n\to\infty} b_n=0$.

Since $b_n = \frac{1}{\ln(n+4)}$, satisfies the conditions that $b_{n+1} \le b_n$ for all n and $\lim_{n\to\infty} b_n = 0$, by Alternating Series Test, the following result is obtained.

Hence, $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)}$ is $\boxed{\text{convergent}}$.

Answer 7

Consider the series $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$.

First, see that the series is an alternating series: The factor $\frac{3n-1}{2n+1}$ is always positive for positive n, and $(-1)^n$ is negative on odd terms and positive on even terms. So the terms of the series alternate in sign.

Recollect the alternating series test:

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ satisfies the conditions $b_{n+1} \le b_n$ and $\lim_{n\to\infty} b_n = 0$ then the series is convergent, otherwise divergent.

Let $b_n = \frac{3n-1}{2n+1}$

$$\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{3n-1}{2n+1} \\
&= \lim_{n\to\infty} \frac{\cancel{n}\left(3-\frac{1}{n}\right)}{\cancel{n}\left(2+\frac{1}{n}\right)} \\
&= \lim_{n\to\infty} \frac{\left(3-\frac{1}{n}\right)}{\left(2+\frac{1}{n}\right)} \qquad \text{As } n\to\infty, \frac{1}{n}\to 0 \\
&= \lim_{n\to\infty} \frac{3}{2} \\
&= \frac{3}{2} \\
&\ne 0
\end{aligned}$$

Since $\lim_{n\to\infty} (-1)^n \frac{3n-1}{2n+1} \ne 0$.

Therefore from alternating series test the series $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$ is $\boxed{\text{divergent}}$

Answer 8

Consider the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{n^3+2}}$$

Its need to determine whether the given series converges or diverges

On expanding the series notation, we have that

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{n^3+2}}$$

$$=(-1)^1\frac{1}{\sqrt{1^3+2}}+(-1)^2\frac{2}{\sqrt{(2)^3+2}}+(-1)^3\frac{3}{\sqrt{(3)^3+2}}+(-1)^4\frac{4}{\sqrt{(4)^3+2}}+\cdots$$

$$=-\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{10}}-\frac{3}{\sqrt{29}}+\frac{4}{\sqrt{66}}+\cdots$$

Observe that, terms of given series are alternatively positive and negative, so it is an alternating series.

The Alternating Series Test: if the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1-b_2+b_3-b_4+b_5-b_6+\cdots \quad b_n>0$$

Satisfies

(i) $b_{n+1}\le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

For the given series, we have that

$$b_n=\frac{n}{\sqrt{n^3+2}}$$

Consider $b_{n+1}-b_n$

$$b_{n+1}-b_n=\frac{n+1}{\sqrt{(n+1)^3+2}}-\frac{n}{\sqrt{n^3+2}}$$

$$=\frac{(n+1)\sqrt{n^3+2}-n\sqrt{(n+1)^3+2}}{\sqrt{(n+1)^3+2}\sqrt{n^3+2}}$$

$$=\frac{(n+1)\sqrt{n^3+2}-n\sqrt{(n+1)^3+2}}{\sqrt{(n+1)^3+2}\sqrt{n^3+2}}\times\frac{(n+1)\sqrt{n^3+2}+n\sqrt{(n+1)^3+2}}{(n+1)\sqrt{n^3+2}+n\sqrt{(n+1)^3+2}}$$

$$=\frac{(n+1)^2(n^3+2)-n^2((n+1)^3+2)}{\sqrt{(n+1)^3+2}\sqrt{n^3+2}\left((n+1)\sqrt{n^3+2}+n\sqrt{(n+1)^3+2}\right)}$$

$$=\frac{-(n^4+2n^3+n^2-4n-2)}{\sqrt{(n+1)^3+2}\sqrt{n^3+2}\left((n+1)\sqrt{n^3+2}+n\sqrt{(n+1)^3+2}\right)}<0 \text{ for all}$$

$n(>1)\in N$

Thus $b_{n+1}\le b_n$ for all $n(>1)\in N$

And also that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^3+2}}$$

$$= \lim_{n\to\infty} \frac{n}{\sqrt{n^3\left(1+\frac{2}{n^3}\right)}}$$

$$= \lim_{n\to\infty} \frac{n}{n^{3/2}\sqrt{\left(1+\frac{2}{n^3}\right)}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}\sqrt{\left(1+\frac{2}{n^3}\right)}}$$

$$= \lim_{n\to\infty} \frac{1/\sqrt{n}}{\sqrt{\left(1+\frac{2}{n^3}\right)}}$$

$= \frac{0}{\sqrt{(1+0)}}$ As $n \to \infty$, $1/n \to 0$

$= 0$

Since $b_n = \frac{n}{\sqrt{n^3+2}}$, satisfies the conditions that $b_{n+1} \le b_n$ for all n and $\lim_{n\to\infty} b_n = 0$, by Alternating Series Test, the series

$$\sum_{n=1}^{\infty}(-1)^{n-1} b_n = \sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{n^3+2}}$$

is $\boxed{\text{convergent}}$.

Answer 9

Given series is $\sum_{n=1}^{\infty}(-1)^n e^{-n}$

This series satisfies

(i) $b_{n+1} \le b_n$ for all n, because $e^{-(n+1)} < e^{-n}$ where $b_n = e^{-n}$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{e^n}$

$= \frac{1}{e^{\infty}}$

$= \frac{1}{\infty}$

$= 0$

So the series is convergent by the Alternating Series test.

Answer 10

Given series $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{2n+3}$

Let $f(x)=\frac{\sqrt{x}}{2x+3}$

Then

$$f'(x)=\frac{(2x+3)\frac{1}{2\sqrt{x}}-(\sqrt{x})(2)}{(2x+3)^2}$$

$$=\frac{(2x+3)-4x}{2\sqrt{x}(2x+3)^2}$$

$$=\frac{3-2x}{2\sqrt{x}(2x+3)^2}$$

Since we are considering only positive x, we see that

$$f'(x)<0 \text{ if } 3-2x<0$$

$$i.e,\ x>\frac{3}{2}$$

Thus, f is decreasing on the interval $\left[\frac{3}{2},\infty\right]$

This means that $f(n+1)<f(n)$

and therefore $b_{n+1}<b_n$ when $n\geq 2$ where $b_n=\frac{\sqrt{n}}{2n+3}$.

Now

$$\lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{\sqrt{n}}{2n+3}$$

$$=\lim_{n\to\infty}\frac{1}{2\sqrt{n}+\frac{3}{\sqrt{n}}}$$

$$=0$$

Thus the given series is convergent by the Alternating Series test.

Answer 11

Consider the series,

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}$$

The objective is to determine whether the series converges or diverges.

Alternating Series Test:

The alternating series $\sum_{n=1}^{\infty}(-1)^n b_n$ is said to be convergent if it satisfies the following two conditions,

(i) $b_{n+1} \le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

Condition (i):

Show that, the sequence b_n is decreasing, that is, $b_{n+1} < b_n$.

For the given series, we have that

$$b_n = \frac{n^2}{n^3+4}$$

In order to show that b_n is decreasing, consider $f(x) = \frac{x^2}{x^3+4}$ and show that $f(x)$ is decreasing.

$$f'(x) = \frac{2x(x^3+4) - x^2(3x^2)}{(x^3+4)^2}$$

$$= \frac{2x(x^3+4) - 3x^4}{(x^3+4)^2}$$

$$= \frac{8x - x^4}{(x^3+4)^2}$$

$$< 0$$

Therefore, *f* is decreasing. That is, $f(n+1) < f(n)$

Hence, $b_{n+1} < b_n$.

So, the condition (i) satisfied.

Condition (ii):

Show that the limit of the sequence b_n tends to zero, as n tends to infinity.

$$\begin{aligned}\lim_{n\to\infty} b_n &= \lim_{n\to\infty}\frac{n^2}{n^3+4}\\ &= \lim_{n\to\infty}\frac{n^2}{n^3\left(1+\frac{4}{n^3}\right)}\\ &= \lim_{n\to\infty}\frac{1}{n\left(1+\frac{4}{n^3}\right)}\\ &= 0\end{aligned}$$

Since $\lim_{n\to\infty} b_n = 0$.

So, the given series satisfy the two conditions of the Alternating Series Test.

Thus, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}$ is convergent.

Answer 12

We have $\sum_{n=1}^{\infty}(-1)^{n+1} ne^{-n}$. Let $b_n = ne^{-n}$. In order to show that b_n is decreasing, consider

$f(x) = xe^{-x}$ or $f(x) = \frac{x}{e^x}$ and show that $f(x)$ is decreasing.

Find $f'(x)$.

$$\begin{aligned}f'(x) &= \frac{e^x - xe^x}{\left(e^x\right)^2}\\ &= \frac{1-x}{e^x}\end{aligned}$$

We note that $\frac{1-x}{e^x} < 0$, for $x \geq 1$. This means that $b_{n+1} \leq b_n$ for all n.

Apply the limits.

$$\begin{aligned}\lim_{n\to\infty} b_n &= \lim_{n\to\infty}\frac{n}{e^n}\\ &= \lim_{n\to\infty}\frac{1}{e^n}\\ &= 0\end{aligned}$$

Thus, by the alternating series test, the given series is convergent.

Answer 13

We have $\sum_{n=1}^{\infty}(-1)^{n-1}e^{\frac{2}{n}}$. Let $b_n = e^{\frac{2}{n}}$.

We know that $n+1 \geq n$. Then, $\frac{1}{n+1} \leq \frac{1}{n}$ and $\frac{2}{n+1} \leq \frac{2}{n}$.

This means that $e^{\frac{2}{n+1}} \leq e^{\frac{2}{n}}$. Also, $b_{n+1} \leq b_n$ for all n.

Apply the limits.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} e^{\frac{2}{n}}$$
$$= 1 \neq 0$$

We note that the condition (ii) of the alternating series test is not satisfied.

Also, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^{n-1}e^{\frac{2}{n}}$ does not exist. Thus, by the test for divergence the series diverges.

Answer 14

We have $\sum_{n=1}^{\infty}(-1)^{n-1}\arctan n$. Let $b_n = \arctan n$.

Apply the limits.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \arctan n$$
$$= \frac{\pi}{2}$$

So, the condition (ii) of the alternating series test is not satisfied. Also, $b_{n+1} > b_n$. But $\lim_{n\to\infty}(-1)^{n-1}$, arctan n does not exist.

Therefore, by the test for divergence, the series diverges.

Answer 15

Given series is $\sum_{n=0}^{\infty}\frac{\sin\left(n+\frac{1}{2}\right)\pi}{1+\sqrt{n}}$

We know that $\sin\left(n+\frac{1}{2}\right)\pi = (-1)^n, n = 0, 1, 2, \ldots$

Therefore given series can be written as

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$$

This series satisfies

(1) $b_{n+1} < b_n$, because $\frac{1}{1+\sqrt{n+1}} < \frac{1}{1+\sqrt{n}}$ where $b_n = \frac{1}{1+\sqrt{n}}$

(2) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{1+\sqrt{n}}$

$= 0$

So the series is converges by the Alternating Series test.

Answer 16

To determine the series converges or diverges, consider the series

$$\sum_{n=1}^{\infty} \frac{n\cos n\pi}{2^n}$$

Since $\cos n\pi = (-1)^n,\ n = 1,2,3,...$, so the series can be written as

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n}$$

On expanding the series with respect to n,

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n} = \frac{(-1)^1 1}{2^1} + \frac{(-1)^2 2}{2^2} + \frac{(-1)^3 3}{2^3} + \frac{(-1)^4 4}{2^4} + ...$$

$$= -\frac{1}{2} + \frac{2}{4} - \frac{3}{8} + \frac{4}{16} + ...$$

The above series contains tems with alternatively negative and positive signs.

So, it is alternating series.

To determine the convergence of it, apply alternating series test.

Alternating series test:

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + ... \qquad b_n > 0$$

Satisfies

(i) $b_{n+1} \le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Let $b_n = \frac{n}{2^n}$.

Then $b_n - b_{n+1} = \frac{n}{2^n} - \frac{n+1}{2^{n+1}}$

$= \frac{2n-n-1}{2^{2n+1}}$

$= \frac{n-1}{2^{2n+1}} > 0 \qquad \forall n > 1$

$b_n - b_{n+1} > 0$

$b_n > b_{n+1}$

So, b_n is decreasing sequence.

Now, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{2^n}$ $\left(\frac{\infty}{\infty} \text{ form, so use L'Hospital Rule to evaluate it}\right)$

$= \lim_{n\to\infty} \frac{1}{2^n \log(2)}$

$= \frac{1}{\infty} \qquad \text{Because as } n \to \infty,\ 2^n \to \infty$

$= 0$

So the series is $\boxed{\text{convergent}}$ by **Alternating Series Test**.

Answer 17

1679-11.5-17E RID: 1411| 17/02/2016

Consider the series,

$$\sum_{n=1}^{\infty}(-1)^n \sin\left(\frac{\pi}{n}\right).$$

Rewrite this series as,

$$\sum_{n=1}^{\infty}(-1)^n \sin\left(\frac{\pi}{n}\right)=(-1)^1\sin\left(\frac{\pi}{1}\right)+(-1)^2\sin\left(\frac{\pi}{2}\right)+(-1)^3\sin\left(\frac{\pi}{3}\right)+\cdots$$

$$=-\sin\left(\frac{\pi}{1}\right)+\sin\left(\frac{\pi}{2}\right)-\sin\left(\frac{\pi}{3}\right)+\cdots$$

This is an alternating series.

Recall the alternating series test, if $s=\sum(-1)^{n-1}b_n$ is the sum of the alternating series that satisfies the following conditions:

(a) $b_{n+1}\le b_n$ (b) $\lim_{n\to\infty} b_n=0$

Let $b_n=\sin\left(\frac{\pi}{n}\right)$.

Replace n by $n+1$ then the series becomes

$$b_{n+1}=\sin\left(\frac{\pi}{n+1}\right)$$

Observe that $\sin\left(\frac{\pi}{n+1}\right)\le\sin\left(\frac{\pi}{n}\right)$ for all $n\ge 2$.

Then,

$$\lim_{n\to\infty} b_n=\lim_{n\to\infty}\sin\left(\frac{\pi}{n}\right)$$

$$=\sin\left(\lim_{n\to\infty}\frac{\pi}{n}\right)\quad \text{As } n\to\infty \text{ then } \frac{1}{n}\to 0$$

$$=\sin 0$$

$$=0$$

So, the series satisfies all the conditions.

Therefore, the series $\sum_{n=1}^{\infty}(-1)^n \sin\left(\frac{\pi}{n}\right)$ converges.

Answer 18

The given series is $\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{\pi}{n}\right)$

n[th] term of the given series is

$$a_n = (-1)^n \cos\left(\frac{\pi}{n}\right)$$

And $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^n \cos\left(\frac{\pi}{n}\right)$

Now, $\lim_{n\to\infty}\cos\left(\frac{\pi}{n}\right) = \cos 0 = 1$

Therefore, $\lim_{n\to\infty}(-1)^n \cos\left(\frac{\pi}{n}\right)$ will have the value 1 or -1.

i.e. $\lim_{n\to\infty}(-1)^n \cos\left(\frac{\pi}{n}\right)$ does not exist.

Hence by test of divergence, the given series is $\boxed{\text{divergent}}$

Answer 19

The given series is $\sum_{n=1}^{\infty}(-1)^n \frac{n^n}{n!}$

Here n[th] term of the given series is

$$a_n = (-1)^n \frac{n^n}{n!}$$

Now, we have

$$\frac{n^n}{n!} = \frac{n\,n\,n\,n\,n \ldots\ldots\ldots\ldots n\,n}{n(n-1)(n-2) \qquad\qquad 2.1}$$

$$= \left(\frac{n}{n}\right)\left(\frac{n}{n-1}\right)\left(\frac{n}{n-2}\right)\left(\frac{n}{n-3}\right)\ldots\ldots\ldots\left(\frac{n}{2}\right)n$$

$$\geq n$$

Thus $\lim_{n\to\infty}\frac{n^n}{n!} = \infty$

Thus, $\lim_{n\to\infty}(-1)^n \frac{n^n}{n!} = \infty \neq 0$

Therefore, by test of divergence, the given series is divergent.

Answer 20

Given series $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n+1}-\sqrt{n}\right)$

Let

$$b_n = \left(\sqrt{n+1}-\sqrt{n}\right)$$

$$\Rightarrow b_n = \frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\left(\sqrt{n+1}+\sqrt{n}\right)}$$

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

This series satisfies

(1) $b_{n+1} < b_n$,because $\frac{1}{\sqrt{n+2}+\sqrt{n+1}} < \frac{1}{\sqrt{n+1}+\sqrt{n}}$

(2) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$

$= 0$

So the series is converges by the Alternating Series test.

Answer 21

Given series $\sum_{n=1}^{\infty} \frac{(-0.8)^n}{n!}$

$$Let \quad a_n = \frac{(-0.8)^n}{n!} x$$

$\therefore a_1 = -0.8, a_2 = 0.32, \ a_3 = -0.08534, \ a_4 = 0.01707,$

$a_5 = -0.0027307, \ a_6 = 0.00036409, \quad a_7 = -0.00004161$

and

$$S_1 = -0.8$$
$$S_2 = a_1 + a_2 = -0.48$$
$$S_3 = a_1 + a_2 + a_3 = -56534$$
$$S_4 = a_1 + a_2 + a_3 + a_4 = -0.54827$$
$$S_5 = a_1 + a_2 + a_3 + a_{4+} + a_5 = -0.55100$$
$$S_6 = a_1 + a_2 + a_3 + a_{4+} + a_5 + a_6 = -0.5506$$
$$S_7 = a_1 + a_2 + a + a_{4+} + a_5 + a_6 + a_7 = -0.5507$$

Notice that $a_8 = \frac{(-0.8)^8}{8!}$

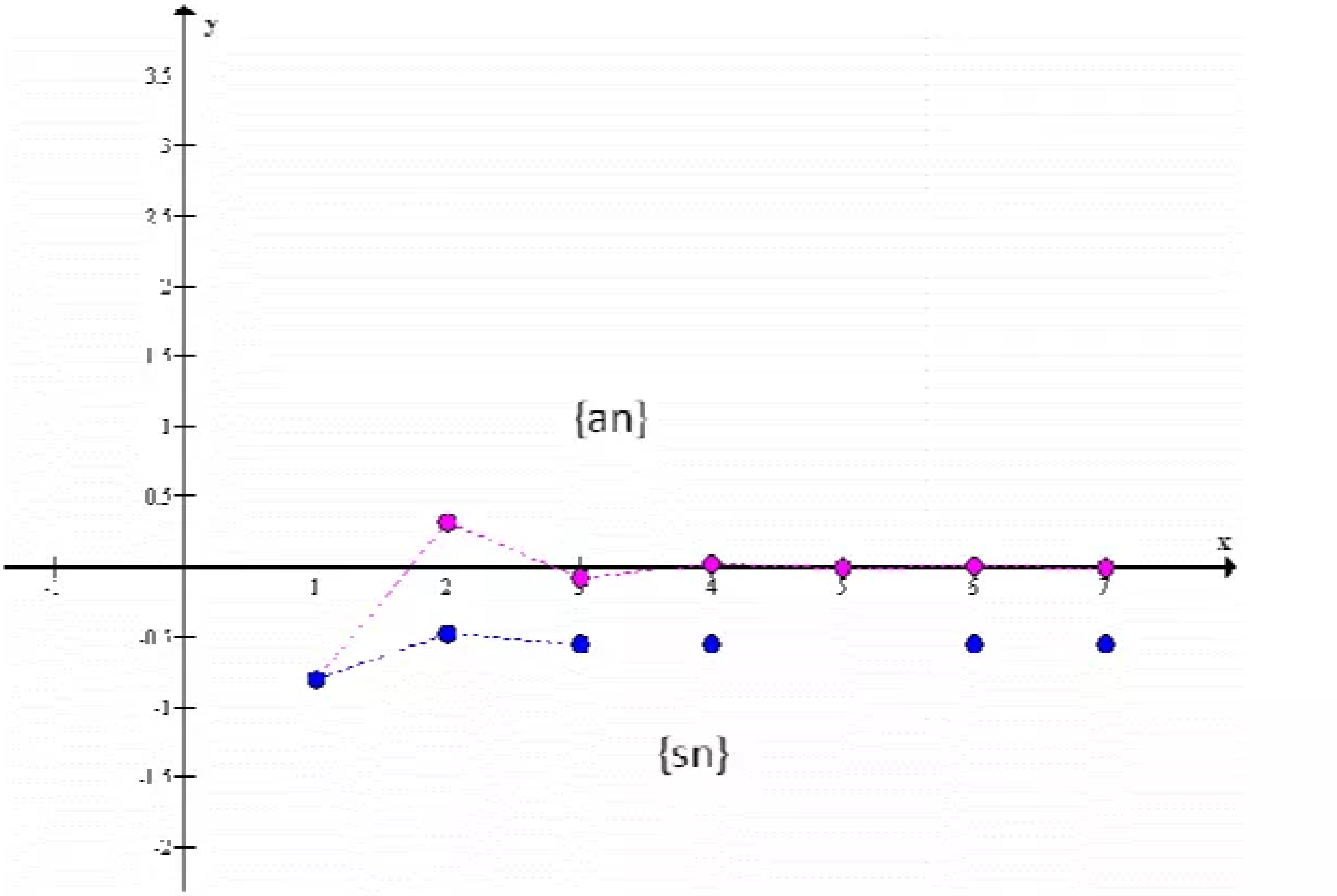
$= 0.0000416$

and $S_7 \cong -0.5507$

By the Alternating Series Estimation theorem, we know that

$$|s - s_7| \le a_8 < 0.0000416$$

This error of less than 0.0000416 does not effect the fourth decimal places, so we have $S \cong -0.5507$ correct to four decimal places.



Answer 22

Consider the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{8^n}$$

Recall that, for a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$, let s_n denote its nth partial sum, then

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

Let $a_n = (-1)^{n-1}\frac{n}{8^n}$ and s_n denote its nth partial sum.

On giving different values for n, we get the corresponding values for a_n, s_n that is we get both the sequence of terms and sequence of partial sums.

Consolidate the values in a table.

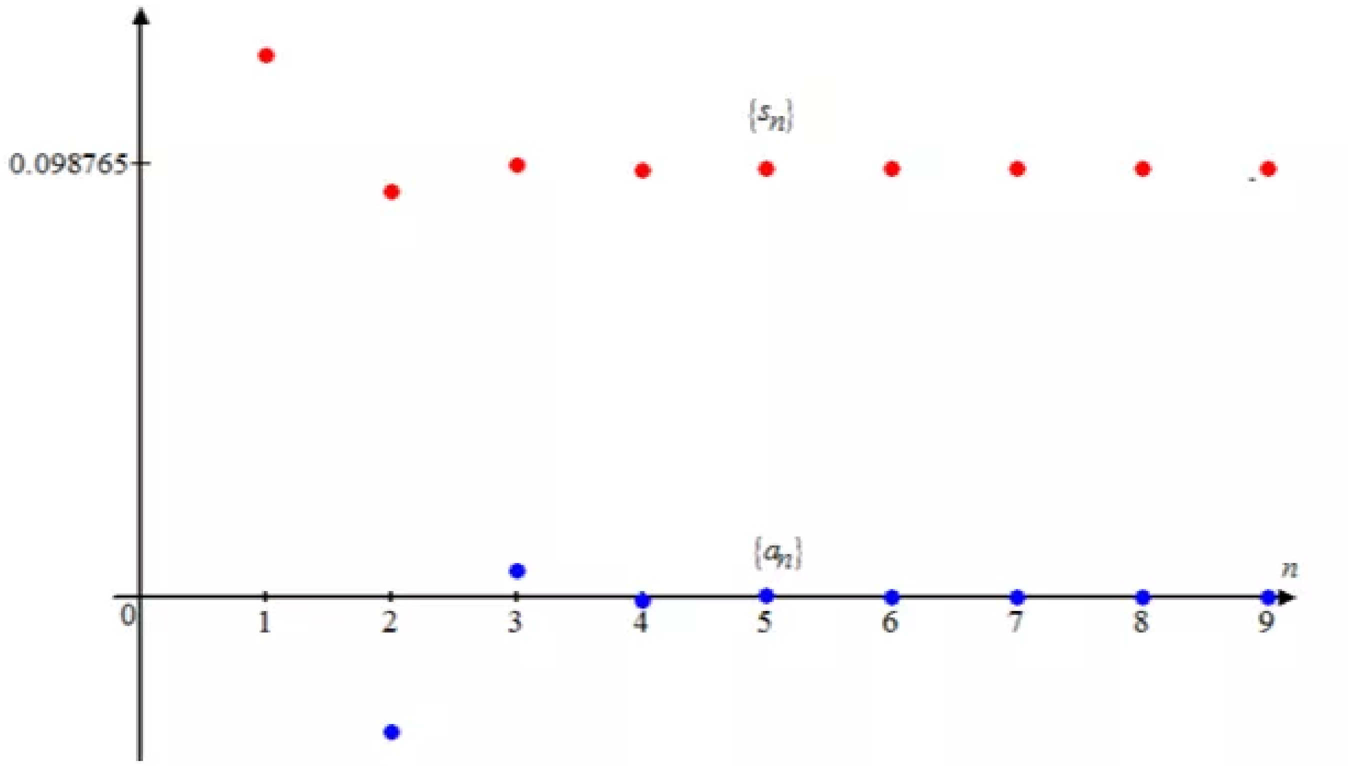
n	Sequence of terms $a_n = (-1)^{n-1}\frac{n}{8^n}$	Sequence of partial terms $s_n = \sum_{i=1}^{n} a_i$
1	$a_1 = (-1)^{1-1}\frac{1}{8^1}$ $= \frac{1}{8}$ $= 0.125$	$s_1 = \sum_{i=1}^{1} a_i$ $= a_1$ $= 0.125$
2	$a_2 = (-1)^{2-1}\frac{2}{8^2}$ $= -\frac{2}{64}$ $= -0.03125$	$s_2 = \sum_{i=1}^{2} a_i$ $= a_1 + a_2$ $= 0.125 - 0.03125$ $= 0.09375$

As in the above process we can evaluate the next terms.

Therefore, the first nine terms of the sequence a_n and nine partial sums s_n of the series are as follows.

n	a_n	s_n
1	0.125	0.125
2	−0.03125	0.09375
3	0.005859375	0.099609375
4	−0.0009765625	0.098632812
5	0.0001525878	0.09878539
6	−0.000022888	0.098762501
7	0.0000033378	0.098765838
8	−0.0000004768	0.098765361
9	0.00000006705	0.098765428

Below figure illustrates the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{8^n}$ by showing the graphs of the terms $a_n=(-1)^{n-1}\frac{n}{8^n}$ and the partial sums s_n on the same screen.



From the graph, it is observed that the values of s_n are zigzag across the point 0.098765.

So, the sum of the series is approximately 0.098765.

Let $s=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{8^n}$

First verify that, the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{8^n}$ is convergent or divergent.

To determine the convergence of it, apply alternating series test.

Alternating series test:

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n=b_1-b_2+b_3-b_4+b_5-b_6+\ldots \qquad b_n>0$$

Satisfies

(i) $b_{n+1}\le b_n$ for all n

(ii) $\lim_{n\to\infty}b_n=0$

then the series is convergent.

Suppose that $b_n = \frac{n}{8^n}$, then

$$b_{n+1} = \frac{n+1}{8^{n+1}}$$

Consider

$$\begin{aligned} b_n - b_{n+1} &= \frac{n}{8^n} - \frac{n+1}{8^{n+1}} \\ &= \frac{8n - n - 1}{8^{n+1}} \\ &= \frac{7n-1}{8^{n+1}} > 0 \quad \text{for all } n \in N \end{aligned}$$

So, $b_n - b_{n+1} > 0 \qquad \text{for all } n \in N$

And,

$$\begin{aligned} \lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{n}{8^n} \left(\begin{array}{l} \frac{\infty}{\infty}, \text{ so use} \\ \text{L'Hospital rule} \end{array} \right) \\ &= \lim_{n\to\infty} \frac{1}{8^n \log 8} \\ &= 0 \quad \text{Because as } n \to \infty, \frac{1}{8^n} \to 0 \end{aligned}$$

Therefore, by alternating series test

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{8^n}$ is convergent

To get a feel for how many terms we need to use in our approximation, let's write out first few terms of the series:

$$\begin{aligned} s &= \frac{(-1)^{1-1}1}{8^1} + \frac{(-1)^{2-1}2}{8^2} + \frac{(-1)^{3-1}3}{8^3} + \frac{(-1)^{4-1}4}{8^4} + \frac{(-1)^{5-1}5}{8^5} + \frac{(-1)^{6-1}6}{8^6} + \ldots \\ &= \frac{1}{8} - \frac{2}{8^2} + \frac{3}{8^3} - \frac{4}{8^4} + \frac{5}{8^5} - \frac{6}{8^6} \ldots \\ &= 0.125 - 0.03125 + 0.005859375 - 0.0009765625 \\ &\quad + 0.0001525878 - 0.000022888\ldots \end{aligned}$$

Notice that, $b_6 = \frac{6}{8^6}$

$$= \frac{3}{131072}$$
$$< \frac{3}{131000}$$
$$\approx 0.000022888$$

And,

$$s_5 = \frac{1}{8} - \frac{2}{8^2} + \frac{3}{8^3} - \frac{4}{8^4} + \frac{5}{8^5}$$
$$= 0.125 - 0.03125 + 0.005859375 - 0.0009765625 + 0.0001525878$$
$$= 0.09878539$$

Alternating Series Estimation Theorem states that, if $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

(i) $b_{n+1} \le b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$

then the erroe involved in using $s \approx s_n$ is the remainder R_n such that,

$$|R_n| = |s - s_n| \le b_{n+1}$$

By alternating series estimation theorem, we have that

$$|s - s_5| \le b_{5+1}$$
$$= b_6$$
$$< 0.000022888$$

This error of less than 0.000022888 does not affect the fourth decimal place, so we have $s \approx 0.09878$ correct to four decimal places.

Answer 23

Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} \quad (|\text{error}| < 0.00005)$$

Its need to show that the series is convergent and then determining the number of terms do we need to add in order to find the sum of the specified accuracy.

On expanding the series notation, we have that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} = \frac{(-1)^{1+1}}{1^6} + \frac{(-1)^{2+1}}{2^6} + \frac{(-1)^{3+1}}{3^6} + \frac{(-1)^{4+1}}{4^6} + \cdots$$
$$= 1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \cdots$$

Observe that, terms of given series are alternatively positive and negative, so it is an alternating series.

The Alternating Series Test: if the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \quad b_n > 0$$

Satisfies

(i) $b_{n+1} \le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

For the given series, we have that

$$b_n = \frac{1}{n^6}$$

Observe that, $n^6 < (n+1)^6$ for all $n \in N$

$$\Rightarrow \frac{1}{n^6} > \frac{1}{(n+1)^6} \text{ for all } n \in N$$

Thus $b_{n+1} \le b_n$ for all $n \in N$

And also that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^6}$$

$= 0$ As $n \to \infty$, $1/n^6 \to 0$

Since $b_n = \frac{1}{n^6}$, satisfies the conditions that $b_{n+1} \le b_n$ for all n and $\lim_{n\to\infty} b_n = 0$, by Alternating Series Test, the series

$$\sum_{n=1}^{\infty}(-1)^{n-1} b_n = \sum_{n=1}^{\infty}(-1)^{n} \frac{1}{n^6}$$

is $\boxed{\text{convergent}}$.

To get a feel for how many terms we need to use in our approximation, let,s write out first few terms of the series:

$$s = 1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \frac{1}{6^6} + \cdots$$

$$= 1 - \frac{1}{64} + \frac{1}{729} - \frac{1}{4096} + \frac{1}{15625} - \frac{1}{46656} + \cdots$$

Notice that $b_6 = \frac{1}{46656} < \frac{1}{20000} = 0.00005$ ...... (1)

And $s_5 = 1 - \frac{1}{64} + \frac{1}{729} - \frac{1}{4096} + \frac{1}{15625}$

$$\approx 1 - 0.015625 + 0.001372 - 0.000244 + 0.000064$$

$$\approx 0.985567$$

Alternating Series Estimation Theorem:

If $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

(i) $0 \le b_{n+1} \le b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$

then $|R_n| = |s - s_n| \le b_{n+1}$

Since the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ with $b_n = \frac{1}{n^6}$ satisfies the conditions that

(i) $0 \le b_{n+1} \le b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$

So, by **Alternating Series Estimation Theorem** we have that

$$|R_5| = |s - s_5| \le b_{5+1}$$

$= b_6 < 0.00005$ By (1)

Thus $|s - s_5| \le b_6 < 0.00005$

Hence, the series gives specified accuracy up to 5 terms.

Answer 24

Consider the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n\cdot 5^n}\quad (|\text{error}|<0.0001)$$

Its need to show that the series is convergent and then determining the number of terms do we need to add in order to find the sum of the specified accuracy.

On expanding the series notation, we have that

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n5^n}=\frac{(-1)^1}{1\cdot 5^1}+\frac{(-1)^2}{2\cdot 5^2}+\frac{(-1)^3}{3\cdot 5^3}+\frac{(-1)^4}{4\cdot 5^4}+\frac{(-1)^5}{5\cdot 5^5}+\cdots$$

$$=-\frac{1}{5}+\frac{1}{2\cdot 5^2}-\frac{1}{3\cdot 5^3}+\frac{1}{4\cdot 5^4}-\frac{1}{5\cdot 5^5}+\cdots$$

Observe that, terms of given series are alternatively positive and negative, so it is an alternating series.

The Alternating Series Test: if the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n=b_1-b_2+b_3-b_4+b_5-b_6+\cdots\quad b_n>0$$

Satisfies

(i) $b_{n+1}\le b_n$ for all n

(ii) $\lim\limits_{n\to\infty}b_n=0$

then the series is convergent.

For the given series, we have that

$$b_n=\frac{1}{n\cdot 5^n}$$

Consider $b_{n+1}-b_n$

$$b_{n+1}-b_n=\frac{1}{(n+1)\cdot 5^{n+1}}-\frac{1}{n\cdot 5^n}$$

$$=\frac{n-5(n+1)}{n(n+1)\cdot 5^{n+1}}$$

$$=\frac{-4n-5}{n(n+1)\cdot 5^{n+1}}<0 \text{ for all } n\in N$$

Thus $b_{n+1}-b_n<0$ for all $n\in N$

So, the condition $b_{n+1}\le b_n$ for all $n\in N$

And also that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n\cdot 5^n}$$

Recall the fact that, if a_n is a sequence of positive terms such that

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then $\lim_{n\to\infty} a_n = 0$

Suppose that $a_n = \frac{1}{n\cdot 5^n}$

Then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(n+1)\cdot 5^{n+1}} \cdot \frac{n\cdot 5^n}{1}$

$$= \lim_{n\to\infty} \frac{1}{(n+1)\cdot 5\cdot 5^n} \cdot \frac{n\cdot 5^n}{1}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)\cdot 5}$$

$= \frac{1}{(1+0)\cdot 5}$ As $n\to\infty$, and $1/n\to 0$

$$= \frac{1}{5} < 1$$

Thus $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{5} < 1$

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{5} < 1$, from the above fact we have that

$$\lim_{n\to\infty} \frac{1}{n\cdot 5^n} = 0$$

Since $b_n = \frac{1}{n\cdot 5^n}$, satisfies the conditions that $b_{n+1} \le b_n$ for all n and $\lim_{n\to\infty} b_n = 0$, by Alternating Series Test, the series

$$\sum_{n=1}^{\infty} (-1)^n b_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n\cdot 5^n}$$

is $\boxed{\text{convergent}}$.

To get a feel for how many terms we need to use in our approximation, let's write out first few terms of the series:

$$s=\frac{(-1)^1}{1\cdot 5^1}+\frac{(-1)^2}{2\cdot 5^2}+\frac{(-1)^3}{3\cdot 5^3}+\frac{(-1)^4}{4\cdot 5^4}+\frac{(-1)^5}{5\cdot 5^5}+\cdots$$

$$=-\frac{1}{5}+\frac{1}{2\cdot 5^2}-\frac{1}{3\cdot 5^3}+\frac{1}{4\cdot 5^4}-\frac{1}{5\cdot 5^5}+\cdots$$

$$=-\frac{1}{5}+\frac{1}{50}-\frac{1}{375}+\frac{1}{2500}-\frac{1}{15625}+\cdots$$

Notice that $b_5=\frac{1}{15625}<\frac{1}{10000}=0.0001$ ...... (1)

And $s_4=-\frac{1}{5}+\frac{1}{50}-\frac{1}{375}+\frac{1}{2500}$

$$=-0.2+0.02-0.002667+0.0004$$

$$=-0.182267$$

Alternating Series Estimation Theorem:

If $s=\sum(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies

(i) $0\le b_{n+1}\le b_n$ and (ii) $\lim_{n\to\infty} b_n=0$

then $|R_n|=|s-s_n|\le b_{n+1}$

Since the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n\cdot 5^n}$ with $b_n=\frac{1}{n\cdot 5^n}$ satisfies the conditions that

(i) $0\le b_{n+1}\le b_n$ and (ii) $\lim_{n\to\infty} b_n=0$

So, by **Alternating Series Estimation Theorem** we have that

$$|R_4|=|s-s_4|\le b_{4+1}$$

$=b_5<0.0001$ By (1)

Thus $|s-s_4|\le b_5<0.0001$

Hence, the series gives specified accuracy up to 4 terms.

Answer 25

Consider the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{10^n n!} \quad (|\text{error}|<0.000005)$$

Its need to show that the series is convergent and then determining the number of terms do we need to add in order to find the sum of the specified accuracy.

On expanding the series notation, we have that

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{10^n n!}=\frac{(-1)^0}{10^0\cdot 0!}+\frac{(-1)^1}{10^1\cdot 1!}+\frac{(-1)^2}{10^2\cdot 2!}+\frac{(-1)^3}{10^3\cdot 3!}+\frac{(-1)^4}{10^4\cdot 4!}+\frac{(-1)^5}{10^5\cdot 5!}+\cdots$$

$$=1-\frac{1}{10}+\frac{1}{10^2\cdot 2!}-\frac{1}{10^3\cdot 3!}+\frac{1}{10^4\cdot 4!}-\frac{1}{10^5\cdot 5!}+\cdots$$

Observe that, terms of given series are alternatively positive and negative, so it is an alternating series.

The Alternating Series Test: if the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n=b_1-b_2+b_3-b_4+b_5-b_6+\cdots \quad b_n>0$$

Satisfies

(i) $b_{n+1}\le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n=0$

then the series is convergent.

For the given series, we have that

$$b_n=\frac{1}{10^n n!}$$

Consider $b_{n+1}-b_n$

$$b_{n+1}-b_n=\frac{1}{10^{n+1}(n+1)!}-\frac{1}{10^n n!}$$

$$=\frac{1}{10^n\cdot 10\cdot(n+1)n!}-\frac{1}{10^n n!}$$

$$=\frac{1-10(n+1)}{10^{n+1}(n+1)!}$$

$$=\frac{9-10n}{10^{n+1}(n+1)!}<0 \text{ for all } n\in N$$

Thus $b_{n+1}-b_n<0$ for all $n\in N$

So, the condition $b_{n+1}\le b_n$ for all $n\in N$

And also that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{10^n n!}$$

Recall the fact that, if a_n is a sequence of positive terms such that

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} < 1$, then $\lim_{n\to\infty} a_n = 0$

Suppose that $a_n = \frac{1}{10^n n!}$

Then $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{1}{10^{n+1}(n+1)!}\cdot\frac{10^n\cdot n!}{1}$

$$= \lim_{n\to\infty}\frac{1}{10^n\cdot 10\cdot(n+1)n!}\cdot\frac{10^n\cdot n!}{1}$$

$$= \lim_{n\to\infty}\frac{1}{10(n+1)}$$

$= 0 < 1$ As $n\to\infty$, $(n+1)\to\infty$ and $1/10(n+1)\to 0$

Thus $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 0 < 1$

Since $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 0 < 1$, from the above fact we have that

$$\lim_{n\to\infty}\frac{1}{10^n n!} = 0$$

Since $b_n = \frac{1}{10^n n!}$, satisfies the conditions that $b_{n+1} \le b_n$ for all n and $\lim_{n\to\infty} b_n = 0$, by Alternating Series Test, the series

$$\sum_{n=0}^{\infty}(-1)^n b_n = \sum_{n=0}^{\infty}\frac{(-1)^n}{10^n\cdot n!}$$

is $\boxed{\text{convergent}}$.

Alternating Series Estimation Theorem:

If $s = \sum(-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

(i) $0 \le b_{n+1} \le b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$

then $|R_n| = |s - s_n| \le b_{n+1}$

Since the alternating series $\sum_{n=0}^{\infty}\frac{(-1)^n}{10^n \cdot n!}$ with $b_n=\frac{1}{10^n \cdot n!}$ satisfies the conditions that

(i) $0 \le b_{n+1} \le b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$

So, by **Alternating Series Estimation Theorem** we have that

$$|R_4| = |s - s_4| \le b_{4+1}$$

$= b_5 < 0.000005$ By (1)

Thus $|s - s_4| \le b_5 < 0.000005$

Hence, the series gives specified accuracy up to 4 terms.

Answer 26

Consider the following series;

$$\sum_{n=1}^{\infty}(-1)^{n-1} n e^{-n}$$

Show that this series is convergent and determine how many terms, need to add so that the partial sum is within 0.01 of the actual value of the series sum.

First, study the series, which is alternating or not.

Note that, ne^{-n} is always positive for positive n, and $(-1)^{n-1}$ is positive on odd terms and negative on even terms.

So, the terms of the series alternate in sign.

Also, the terms of the series decrease in size. For any k, consider the size difference between the $(k+1)^{th}$ term and the k^{th} term.

$$(n+1)e^{-(n+1)} - ne^{-n} = \frac{(n+1)}{e^{n+1}} - \frac{n}{e^n}$$

$$= \frac{e^n(n+1) - ne^{n+1}}{e^{2n+1}}$$

$$= \frac{e^n((n+1) - ne)}{e^{2n+1}}$$

$$= \frac{e^n(n(1-e)+1)}{e^{2n+1}}$$

We know that $1-e<-1$ and $n \ge 1$.

So $n(1-e)<-1$

$n(1-e)+1<0$

Since, the other terms e^n and e^{2n+1} are positive, the entire quantity $\frac{e^n(n(1-e)+1)}{e^{2n+1}}$ is negative.

This inequality becomes as follows.

$$(n+1)e^{-(n+1)} - ne^{-n} < 0$$
$$(n+1)e^{-(n+1)} < ne^{-n}$$

So, the terms of the series are decreasing in size.

Finally, the limit tends to zero as $n \to \infty$, which is shown below.

$$\lim_{n\to\infty} ne^{-n} = 0.$$

$$\lim_{n\to\infty} ne^{-n} = \lim_{n\to\infty} \frac{n}{e^n}$$

Consider, the function, $f(x)=\frac{x}{e^x}$.

If $\lim_{n\to\infty} \frac{n}{e^n}$ exists.

Let $x=n$, then $\lim_{x\to\infty} \frac{x}{e^x} = \lim_{n\to\infty} \frac{n}{e^n}$.

Now, compute, $\lim_{x\to\infty} \frac{x}{e^x}$. Since this limit has the indefinite form $\frac{\infty}{\infty}$ at ∞.

So, apply L'Hospital's rule.

Then, $\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$.

Therefore, the value of the limit, $\lim_{n\to\infty} \frac{n}{e^n} = 0$.

Since $\sum_{n=1}^{\infty} (-1)^{n-1} ne^{-n}$ is an alternating series of terms, whose terms decrease in size and satisfy the limit, $\lim_{n\to\infty} \frac{n}{e^n} = 0$, by the Alternating Series Test we know that $\sum_{n=1}^{\infty} (-1)^{n-1} ne^{-n}$ converges.

Find, how many terms we must add, so that the partial sum is within 0.01 of the actual sum of the series.

Use the following estimate, called the Alternating Series Estimation Theorem from the text.

Let b_n be the sequence, ne^{-n}, $s=\sum_{n=1}^{\infty}(-1)^{n-1}ne^{-n}$ the sum of the entire series, and s_n is the n^{th} partial sum.

Then, $|s-s_n|\le b_{n+1}$.

That is, the error $|s-s_n|$ from using the n^{th} partial sum as an estimate to s is no larger than b_{n+1}, the size of the $(n+1)^{th}$ term of the series.

Here, we want $|s-s_n|<0.01$.

So, if we find b_{n+1} such that $b_{n+1}<0.01$, we will have $|s-s_n|\le b_{n+1}<0.01$.

We want n with $b_{n+1}<0.01$.

So thee inequality becomes as follows.

$$(n+1)e^{-(n+1)}<0.01$$
$$\frac{n+1}{0.01}<e^{n+1}$$
$$100<\frac{e^{n+1}}{n+1}$$

Here, we do some estimating, since there is no simple direct way to solve for n.

Note that, $\frac{e^6}{6}=67.2\ldots<100$ while $\frac{e^7}{7}=156.6>100$.

So 6 is the least number that works for $n+1$.

Thus, the number, 6 is the least number that works for n.

That is, when $n\ge 6$, $|s-s_n|\le\frac{7}{e^7}<0.01$.

Hence, we need to add at least 6 terms.

Answer 27

We first observe that the series is convergent by the Alternating Series test because

(1) $\frac{1}{(2(n+1))!}<\frac{1}{(2n)!}$

(2) $\frac{1}{(2n)!}\to 0\, as\, n\to\infty$

To approximate the sum, first we write few terms of the series

$$S = \frac{-1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \frac{1}{10!} + \ldots$$
$$= \frac{-1}{2!} + \frac{1}{24} - \frac{1}{720} + \frac{1}{40,320} + \ldots$$

Notice that

$$b_4 = \frac{-1}{8!}$$
$$= \frac{1}{40,320}$$
$$< \frac{1}{40,000}$$
$$= 0.000025$$

and $S_3 = -\frac{1}{2} + \frac{1}{24} - \frac{1}{720}$
$$\cong -0.4597$$

By the Alternating Series Estimation theorem we know that

$$|S - S_3| \le b_4 < 0.000025$$

This error of less than 0.000025 does not affect the fourth decimal place.
So we have $S \cong -0.4597$ correct to five decimal places.

Answer 28

Let $s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$

First verify that, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ is convergent or divergent.

To determine the convergence of it, apply alternating series test.

Alternating series test:

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \ldots \qquad b_n > 0$$

Satisfies

(i) $b_{n+1} \le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Suppose that $b_n = \frac{1}{n^6}$, then

$$b_{n+1} = \frac{1}{(n+1)^6}$$

Since $n < n+1$ for all $n \in N$,

$$n^6 < (n+1)^6 \quad \text{for all } n \in N$$
$$\frac{1}{(n+1)^6} < \frac{1}{n^6} \quad \text{for all } n \in N$$

That is, $b_{n+1} \le b_n$ for all n

And,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^6}$$
$$= 0 \quad \text{Because as } n \to \infty, \frac{1}{n^6} \to 0$$

Therefore, by **alternating series** test

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ is convergent.

To get a feel for how many terms we need to use in our approximation, let's write out first few terms of the series:

$$s = \frac{(-1)^{1+1}}{1^6} + \frac{(-1)^{2+1}}{2^6} + \frac{(-1)^{3+1}}{3^6} + \frac{(-1)^{4+1}}{4^6} + \frac{(-1)^{5+1}}{5^6} + \frac{(-1)^{6+1}}{6^6} + \dots$$
$$= 1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \dots$$
$$= 1 - \frac{1}{64} + \frac{1}{729} - \frac{1}{4096} + \frac{1}{15625} - \dots$$

Notice that, $b_5 = \frac{1}{15625}$

$$< \frac{1}{15000}$$
$$\approx 0.0000666$$

And,

$$s_4 = 1 - \frac{1}{64} + \frac{1}{729} - \frac{1}{4096}$$
$$= 1 - 0.015625 + 0.001371 - 0.000244$$
$$= 0.985502$$

Alternating Series Estimation Theorem states that, if $s = \sum(-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

(i) $b_{n+1} \le b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$

then the erroe involved in using $s \approx s_n$ is the remainder R_n such that,

$$|R_n| = |s - s_n| \le b_{n+1}$$

By **alternating series estimation theorem**, we have that

$$\begin{aligned} |s - s_4| &\le b_{4+1} \\ &= b_5 \\ &< 0.0000666 \end{aligned}$$

This error of less than 0.0000666 does not affect the fourth decimal place, so we have $s \approx 0.9855$ correct to four decimal places.

Answer 29

We have the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{10^n}$

We write first few terms of the series

$$s = \frac{1}{10} - \frac{2^2}{10^2} + \frac{3^2}{10^3} - \frac{4^5}{10^4} + \frac{5^2}{10^5} - \frac{6^2}{10^6} + \ldots\ldots\ldots$$

$$= 0.1 - 0.04 + 0.009 - 0.0016 + 0.00025 - 0.000036 + 0.0000049 - \ldots$$

Since $b_7 = 0.0000049$, this will not affect the fourth decimal place of the sum of the series so we take the sum of first six terms only $s_6 = 0.067614$

We have $\boxed{s \approx 0.0676}$, correct to four decimal place.

Answer 30

We have $\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n n!}$

We write first few terms of the series

$$s = \frac{-1}{3\ \ 1!} + \frac{1}{3^2\ \ 2!} - \frac{1}{3^3\ \ 3!} + \frac{1}{3^4\ \ 4!} - \frac{1}{3^5\ \ 5!} + \ldots\ldots$$

$$= -\frac{1}{3} + \frac{1}{9(2)} - \frac{1}{27(6)} + \frac{1}{81(24)} - \frac{1}{243(120)} + \ldots\ldots$$

$$= -0.333333 + 0.055555 - 0.0061728 + 0.0005144 - 0.00003429 + 0.0000019$$

Since $b_6 \approx 0.0000019$, this will not affect the fourth decimal place of the sum of the series so we take the sum of first five terms only $s_5 \approx -0.283471$

So we have $\boxed{s \approx -0.2835}$, correct to four decimal place.

Answer 31

Consider the 50th partial sum s_{50} of the following series;

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

Determine, s_{50} is an overestimate or an underestimate to the sum of the entire series,

$$s=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

First, note that this is clearly a decreasing alternating series, whose terms approach

zero.

The expression, $(-1)^{n-1}$ alternates, and the factor $\frac{1}{n}$ decreases and approaches to zero.

A general pattern for the partial sums of decreasing alternating series, the first partial sum starts positive, the second partial sum goes down, the third partial sum rises back not as high as the first, the fourth partial sum falls but not as high as the second, and so on.

So, the sequence of odd terms is falling down to the limit, while the sequence of even terms is rising up to the limit.

For example, the n^{th} term of the sequence is $a_n=\frac{(-1)^{n-1}}{n}$.

Let $n=1$, then $s_1=a_1$.

So

$$a_1=\frac{(-1)^{1-1}}{1}$$
$$=1$$

Let $n=2$, then $s_2=s_1+a_2$.

So

$$s_1+a_2=1+\frac{(-1)^{2-1}}{2}$$
$$=1+\frac{(-1)^{1}}{2}$$
$$=1-\frac{1}{2}$$
$$=\frac{1}{2}$$

Therefore, $s_2=\frac{1}{2}$.

Similarly, the sums, s_3 and s_4 are as follows.

$s_3 = s_2 + a_3 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ rising back up but not as high as the first.

$s_4 = s_3 + a_4 = \frac{5}{6} - \frac{1}{4} = \frac{7}{12}$ falling, but not as low as the second.

The expression $(-1)^{n-1}$ is positive on odd terms and negative on even terms. The number 50 is even, so the last term added in s_{50} is negative, and by the above pattern, expect that s_{50} is an underestimate.

The proof is given below.

Note that, a_{51}, being an odd term, is positive, and a_{52}, an even term, is negative. Also, the terms of the series are decreasing in size, so $|a_{51}| \geq |a_{52}|$.

Then $a_{51} + a_{52} = |a_{51}| - |a_{52}|$, since, as noted, a_{51} is positive and a_{52} is negative, and $|a_{51}| - |a_{52}| \geq 0$ because $|a_{51}| \geq |a_{52}|$.

So, $a_{51} + a_{52} \geq 0$.

Clearly, there is nothing special about a_{51} and a_{52}: this holds for every pair of an odd term a_{2n+1} and the succeeding even term a_{2n+2} because of the combination of the alternating and decreasing property of the series.

Now, notice the following expression.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = s_{50} + a_{51} + a_{52} + a_{53} + a_{54} + \ldots a_{2n+1} + a_{2n+2} + \ldots$$
$$= s_{50} + (a_{51} + a_{52}) + (a_{53} + a_{54}) + \ldots (a_{2n+1} + a_{2n+2}) + \ldots$$

As we noted, each of the pairs in parenthesis is an odd term and the succeeding even term so is positive.

That means, the sum of the entire series is s_{50} plus a bunch of positive terms.

Therefore, s_{50} is an underestimate.

Answer 32

The given series is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$

When p > 0. Then it is clear that the given series is an alternating the terms in the given series are in decreasing order as $\frac{1}{n^p} > \frac{1}{(n+1)^p}$

Also, $\lim_{n \to \infty} \frac{1}{n^p} = 0$

Thus, by alternating series test, the given series is convergent.

When p = 0, then the given series is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^0} = \sum_{n=1}^{\infty} (-1)^{n-1}$

$= -1 + 1 - 1 + 1 - 1 + 1 - \ldots\ldots\ldots\ldots\ldots\ldots$

Here n^{th} term $a_n = (-1)^{n-1}$

And $\lim_{n \to \infty} a_n = \lim_{n \to \infty} (-1)^{n-1}$ which does not tend to a unique value.

i.e. $\lim_{n \to \infty} (-1)^{n-1}$ does not exist.

Therefore, by test for divergence, the given series is divergent.

When p<0, then $\lim_{n \to \infty} \frac{1}{n^p} = \infty$

n^{th} term of the given series is ,

$$a_n = \frac{(-1)^{n-1}}{n^p}$$

and $\lim_{n \to \infty} \frac{(-1)^{n-1}}{n^p}$ = does not exist Tends to ∞ or $-\infty$

Therefore, by test for divergence, the given series is divergent.
Hence,

The given series is convergent only for p > 0

Answer 33

The given series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+p)}$

It is clear that the given series is an alternating series.
The terms in the series are in decreasing order for all p.

And $\lim_{n \to \infty} \frac{1}{(n+p)} = 0$ for all p such that $n + p \neq 0$

Thus, by alternating series test, the given series is convergent if $n + p \neq 0$

i.e. p is not a negative integer.

Answer 34

Consider the series:

$$\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(\ln n)^p}{n}$$

To find the values of p for which the given series is convergent use the Cauchy's integral test to the series of absolute values:

$$\sum_{n=2}^{\infty}\left|(-1)^{n-1}\frac{(\ln n)^p}{n}\right| = \sum_{n=2}^{\infty}\frac{(\ln n)^p}{n}$$

Let,

$$f(x)=\frac{(\ln x)^p}{x}$$

Then,

$$f'(x)=\frac{xp(\ln x)^{p-1}\left(\frac{1}{x}\right)-(\ln x)^p}{x^2}$$
$$=\frac{(\ln x)^{p-1}}{x^2}(p-\ln x)$$

Now, consider the following facts:

When $x>e^p$,

Then

$$mx > me^p$$

So, $mx > p$

If $p>1$, then since both numerator and denominator are positive $\frac{mx}{x^2}>0$.

But

$$\begin{aligned} mx &> p \\ -mx &< -p \\ p-mx &< p-p \\ &< 0 \end{aligned}$$

Hence, $f'(x)<0 \;\; \forall\, x>e^p$

So, f is decreasing for all $x>e^p$.

Thus, the condition for the Cauchy integral test is satisfied then it follows that:

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty}\frac{(\ln x)}{x}dx.$$

Substitute

$\ln x = t$, then $\frac{1}{x}dx = dt$

And the corresponding limits are:

$x = 2$, $t = \ln 2$

$x = \infty$, $t = \infty$

Therefore,

$$\int_{2}^{\infty} f(x)\,dx = \int_{\ln 2}^{\infty} t^{p}\,dt$$

$$= \left[\frac{t^{p+1}}{p+1}\right]_{\ln 2}^{\infty}$$

$$= \infty$$

The series is $\sum_{n=1}^{\infty} \frac{(\ln x)^{p}}{n}$ divergent.

But, if $p < -1$, then the above integral looks like:

$$\int_{\ln 2}^{\infty} \frac{1}{t^{p}}\,dt .$$

Compare this with series:

$$\sum_{\ln 2}^{\infty} \frac{1}{x^{q}} .$$

This is a p-series that converges when $q > 1$.

Since $q = -p$, it follows that when $-p > 1$ or $p < -1$, the series converges.

Therefore, the given series will converge when $\boxed{p < -1}$.

Answer 35

Consider the series,

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n$$

Here $b_n = \frac{1}{n}$ if n is odd

And, $b_n = \frac{1}{n^2}$ if n is even

It is needed to prove that, this series is divergent.

With the given conditions on n, the series can be written as,

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = 1-\frac{1}{2^2}+\frac{1}{3}-\frac{1}{2^4}+\frac{1}{5}-\frac{1}{2^6}+\cdots$$

$$=\left(1+\frac{1}{3}+\frac{1}{5}+\cdots\right)-\left(\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\cdots\right)$$

$$=\sum_{n=1}^{\infty}\frac{1}{2n-1}-\sum_{n=1}^{\infty}\frac{1}{2^{2n}}$$

$$=\sum_{n=1}^{\infty}\frac{1}{2n-1}-\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n \quad \text{...... (1)}$$

We have, $\frac{1}{n} < \frac{1}{2n-1}, \forall n \in N$

And the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, because the auxiliary series, $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p>1$, and diverges if $p \le 1$

So, by comparison test, the series $\sum_{n=1}^{\infty}\frac{1}{2n-1}$ is convergent. ...... (2)

Now the geometric series $\sum_{n=1}^{\infty} r^n$ converges for $|r|<1$, and diverges if $|r|\ge 1$

So that, the series $\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$ is convergent ...... (3)

By (1), (2), and (3) we have the given series is the difference of the convergent and the divergent series.

Hence the given series is divergent.

Alternating Series Test:

If the Alternating Series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$, $b_n > 0$ satisfies

(i) $\{b_n\}$ is decreasing for all n

(ii) $\lim_{n\to\infty} b_n = 0$

Then the series is convergent.

Consider the series,

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = 1-\frac{1}{2^2}+\frac{1}{3}-\frac{1}{2^4}+\frac{1}{5}-\frac{1}{2^6}+\cdots$$

In this series, observe that $b_n > 0$, and

$$1>\frac{1}{2^2}<\frac{1}{3}>\frac{1}{2^4}<\frac{1}{5}>\frac{1}{2^6}\cdots$$

From this we can say that the sequence $\{b_n\}$ is not decreasing.

So the given series does not satisfy the first condition for the Alternating Series Test.

Hence the Alternating Series Test cannot apply for the convergence of the given series.

Answer 36

Use the following steps to show that $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = \ln 2$

Let h_n and s_n be the partial sums of the harmonic and alternating harmonic series.

(a) Show that $s_{2n} = h_{2n} - h_n$

By using induction to prove the above result

Let $P(n)$ be the proposition that $s_{2n} = h_{2n} - h_n$

Replace n by 1 in $P(n)$ then

$$s_{2(1)} = h_{2(1)} - h_1$$
$$s_2 = h_2 - h_1$$

$P(1)$ is the statement $s_2 = h_2 - h_1$.

Which is true since $1-\frac{1}{2} = \left(1+\frac{1}{2}\right)-1$

So suppose that $P(n)$ is true.

Next show that $P(n+1)$ must be true as a consequence.

Replace n with $n+1$ in $h_{2n}-h_n$

$$\begin{aligned} h_{2n}-h_n &= h_{2(n+1)}-h_{n+1} \\ &= h_{2n+2}-h_{n+1} \end{aligned}$$

By alternating harmonic series

$$=\left(h_{2n}+\frac{1}{2n+1}+\frac{1}{2n+2}\right)-\left(h_n+\frac{1}{n+1}\right)$$

$$=(h_{2n}-h_n)+\frac{1}{2n+1}-\frac{1}{2n+2}$$

$$=s_{2n}+\frac{1}{2n+1}-\frac{1}{2n+2}$$

Which is $P(n+1)$,and proves that $s_{2n}=h_{2n}-h_n$ for all n .

(b) We Know that

$h_{2n}-\ln(2n)\to\gamma$ and $h_n-\ln n\to\gamma$ as $n\to\infty$.

Now

$$\begin{aligned} s_{2n} &= h_{2n}-h_n \\ &= \left[h_{2n}-\ln(2n)\right]-(h_n-\ln n)+\left[\ln(2n)-\ln n\right] \end{aligned}$$

And

$$\begin{aligned} \lim_{n\to\infty} s_{2n} &= \lim_{n\to\infty}\left[\left[h_{2n}-\ln(2n)\right]-(h_n-\ln n)+\left[\ln(2n)-\ln n\right]\right] \\ &= \lim_{n\to\infty}\left[h_{2n}-\ln(2n)\right]-\lim_{n\to\infty}(h_n-\ln n)+\lim_{n\to\infty}\left[\ln(2n)-\ln n\right] \\ &= \gamma-\gamma+\lim_{n\to\infty}\left[\ln(2n)-\ln n\right] \\ &= \lim_{n\to\infty}\left[\ln 2+\ln n-\ln n\right] \end{aligned}$$

$$=\boxed{\ln 2}$$