CBSE Test Paper 03

CH-13 Limits and Derivatives

1. $\underset{n\to\infty}{Lt}\frac{\sum_{r=1}^{n}r^2}{n^3}$ is equal to
   a. $\frac{1}{3}$
   b. $\frac{1}{2}$
   c. $\frac{1}{4}$
   d. 0
2. If f (x) = $\sqrt{1-x^2}, x \in (0,1), then f'(x)$ is equal to
   a. $\sqrt{1-x^2}$
   b. $\sqrt{x^2-1}$
   c. $\frac{1}{\sqrt{1-x^2}}$
   d. $\frac{-x}{\sqrt{1-x^2}}$
3. Let f (x) = x sin$\frac{1}{x}$, x$\neq$0, then the value of the function at x = 0, so that f is continuous at x = 0, is
   a. -1
   b. $\sqrt{2}$
   c. 1
   d. 0
4. $\frac{d}{dx}\left(\sin^{-1}(1-x)\right)$ is equal to
   a. $\frac{-1}{\sqrt{2x-x^2}}$
   b. $\frac{1}{\sqrt{x^2-2x}}$
   c. $\frac{1-x}{(2x-x^2)^{\frac{3}{2}}}$
   d. $\frac{1}{\sqrt{2x-x^2}}$
5. $\frac{d}{dx}\left(\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)\right)$ is equal to
   a. $\frac{1}{\sqrt{x^2+a^2}}$
   b. $\sqrt{x^2+a^2}$
   c. $\frac{1}{x+\sqrt{x^2+a^2}}$
   d. $\sqrt{x^2-a^2}$

6. Fill in the blanks:

   The value of the limit $\lim_{x \to 4} \frac{4x+3}{x-2}$ is ________.

7. Fill in the blanks:

   The derivative of x at x = 1 is ________.

8. Find the derivative of the following functions: sec x

9. Evaluate $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x-3}{cosec x-2}$.

10. Evaluate $\lim_{x \to 1} \frac{ax^2+bx+c}{cx^2+bx+a}, \ a+b+c \neq 0$

11. Evaluate: $\lim_{x \to 1} \frac{x^3-1}{x-1}$.

12. Evaluate $\lim_{x \to 0} \frac{ax+x\cos x}{b\sin x}$.

13. Find the derivative of $\frac{(x-1)(x-2)}{(x-3)(x-4)}$.

14. Evaluate $\lim_{x \to \pi} \frac{1+\cos x}{\tan^2 x}$.

15. Solve: $\lim_{x \to 1} \frac{x^4-3x^3+2}{x^3-5x^2+3x+1}$

Solution

1. (a) $\frac{1}{3}$

   Explanation: $\underset{n\to\infty}{Lt}\frac{\frac{n(n+1)(2n+1)}{6}}{n^3}=\lim_{n\to\infty}\frac{\left(2n^2+3n+1\right)}{6n^2}$

   $\Rightarrow$ substitute $n=\frac{1}{t}$

   $\Rightarrow \underset{t\to0}{\mathrm{Lt}}\frac{\left(2+3t+t^2\right)}{6}=\frac{1}{3}$
2. (d) $\frac{-x}{\sqrt{1-x^2}}$

   Explanation:

   $f'(x)=\frac{1}{2\sqrt{1-x^2}}.-2x$
3. (d) 0

   Explanation: Here, if we directly put x= 0, f(0) = 0 * sin (1/0) = 0.

   At L.H.L, put x=0-h , $\underset{x\to0}{Lt}\ x.\sin\frac{1}{x}$ f(0-h) = = 0.

   At R.H.L, put x = 0+h , $\underset{x\to0}{Lt}\ x.\sin\frac{1}{x}$, f(0+h) = = 0.

   Hence, L.H.L = f(0) = R.H.L.

   f(x) is continuous at x=0.
4. (c) $\frac{1-x}{\left(2x-x^2\right)^{\frac{3}{2}}}$

   Explanation: $y=\sin^{-1}(1-x)$

   $\Rightarrow \sin y=1-x$

   $\Rightarrow (\cos y)y'=-1$

   $\Rightarrow y''=(y')^2(\tan y)=\frac{(\sin y)}{(\cos y)^3}$

   $\Rightarrow \frac{1-x}{\left(2x-x^2\right)^{\frac{3}{2}}}$
5. (b) $\sqrt{x^2+a^2}$

Explanation:

$$y' = \left(\frac{x}{2}\right)\cdot\left(\frac{2x}{2\sqrt{x^2+a^2}}\right) + \left(\sqrt{x^2+a^2}\right)\cdot\left(\frac{1}{2}\right) + \left(\frac{a^2}{2}\right)\left[\frac{1+\left(\frac{2x}{2\sqrt{x^2+a^2}}\right)}{x+\sqrt{x^2+a^2}}\right]$$

$$\Rightarrow \sqrt{x^2+a^2}$$

6. $\frac{19}{2}$

7. 1

8. Here f (x) = sec x
$\therefore$ f'$(x) = \frac{d}{dx}(\sec x)$
= sec x tan x

9. Given, $\lim\limits_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{cosec x - 2}$
$= \lim\limits_{x \to \frac{\pi}{6}} \frac{cosec^2 x - 1 - 3}{cosec\, x - 2}$ [$\because$ cosec^{2}x - cot^{2}x = 1]
$= \lim\limits_{x \to \frac{\pi}{6}} \frac{cosec^2 x - 4}{\cos ec x - 2} = \lim\limits_{x \to \frac{\pi}{6}} \frac{(cosec x - 2)(cosec x + 2)}{(cosec x - 2)}$ [by factorisation]
$= \lim\limits_{x \to \frac{\pi}{6}} (cosec x + 2) = cosec \frac{\pi}{6}$ + 2 = 2 + 2 = 4

10. Here $\lim\limits_{x \to 1} \frac{ax^2+bx+c}{cx^2+bx+a}$
$= \frac{a \times (1)^2 + b \times 1 + c}{c \times (1)^2 + b \times 1 + a} = \frac{a+b+c}{c+b+a} = 1$

11. When x = 1 the expression $\frac{x^3-1}{x-1}$ assumes the indeterminate form $\frac{0}{0}$. Therefore, (x -1) is a common factor in numerator and denominator. Factorising the numerator and denominator, we obtain
$\lim\limits_{x \to 1} \frac{x^3-1}{x-1} \left(\text{form } \frac{0}{0}\right)$
= $\lim\limits_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)}$ = $\lim\limits_{x \to 1}$ x^2 + x + 1 = 1^2 + 1 + 1 = 3.

12. We have, $\lim\limits_{x \to 0} \frac{ax + x\cos x}{b \sin x} = \lim\limits_{x \to 0}\left(\frac{ax}{b \sin x} + \frac{x \cos x}{b \sin x}\right)$
$= \frac{a}{b} \lim\limits_{x \to 0} \frac{x}{\sin x} + \frac{1}{b} \lim\limits_{x \to 0} \frac{x \cos x}{\sin x}$

$$= \frac{a}{b} \lim_{x \to 0} \frac{1}{\left(\frac{\sin x}{x}\right)} + \frac{1}{b} \lim_{bx \to 0} \frac{\cos x}{\left(\frac{\sin x}{x}\right)}$$

$$= \frac{a}{b} \frac{\lim\limits_{x \to 0} 1}{\lim\limits_{x \to 0}\left(\frac{\sin x}{x}\right)} + \frac{1}{b} \frac{\lim\limits_{x \to 0} \cos x}{\lim\limits_{x \to 0}\left(\frac{\sin x}{x}\right)} \left[\because \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}\right]$$

$$= \frac{a}{b} \times \frac{1}{1} + \frac{1}{b} \times \frac{1}{1} \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

$$= \frac{a+1}{b}$$

13. Let $y = \frac{(x-1)(x-2)}{(x-3)(x-4)}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\left[\begin{array}{r}(x-3)(x-4)\frac{d}{dx}[(x-1)(x-2)]-(x-1)\\ (x-2)\frac{d}{dx}[(x-3)(x-4)]\end{array}\right]}{[(x-3)(x-4)]^2}$$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}\right]$$

$$= \frac{(x-3)(x-4)\left[(x-1)\frac{d}{dx}(x-2)+(x-2)\frac{d}{dx}(x-1)\right]-(x-1)(x-2)\left[(x-3)\frac{d}{dx}(x-4)+(x-4)\frac{d}{dx}(x-3)\right]}{(x-3)^2(x-4)^2}$$

$$= \frac{(x-3)(x-4)[(x-1)\cdot 1+(x-2)\cdot 1]-(x-1)(x-2)[(x-3)\cdot 1+(x-4)\cdot 1)]}{(x-3)^2(x-4)^2}$$

$$= \frac{(x-3)(x-4)[2x-3]-(x-1)(x-2)[2x-7]}{(x-3)^2(x-4)^2}$$

$$= \frac{(x^2-7x+12)(2x-3)-(x^2-3x+2)(2x-7)}{(x-3)^2(x-4)^2}$$

$$= \frac{2x^3-14x^2+24x-3x^2+21x-36-2x^3+6x^2-4x+7x^2-21x+14}{(x-3)^{2(}x-4)^2}$$

$$= \frac{-4x^2+20x-22}{(x-3)^2(x-4)^2}$$

14. On putting $x = \pi + b$ and as $x \to \pi$, then $h \to 0$, we get

$$\lim_{x \to \pi} \frac{1+\cos x}{\tan^2 x} = \lim_{h \to 0} \frac{1+\cos(\pi+h)}{\tan^2(\pi+h)}$$

$$= \lim_{h \to 0} \frac{2\cos^2\left(\frac{\pi+h}{2}\right)}{\tan^2 h} \left[\begin{array}{l}\because \tan(\pi+b) = \tan b\\ 1+\cos 2\theta = 2\cos^2\theta\end{array}\right]$$

$$= \lim_{h \to 0} \frac{2\cos^2\left(\frac{\pi}{2}+\frac{h}{2}\right)}{\tan^2 h}$$

$$= \lim_{h \to 0} \frac{2\sin^2\frac{h}{2}}{\tan^2 h} \left[\begin{array}{l}\because \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta\\ \therefore \cos^2\left(\frac{\pi}{2}+\theta\right) = \sin^2\theta\end{array}\right]$$

$$= \lim_{h\to 0} \frac{2\sin^2 \frac{h}{2}}{\sin^2 h} \times \cos^2 h = \lim_{h\to 0} \frac{2\sin^2 \frac{h}{2}}{\left(2\sin \frac{h}{2} \cdot \cos \frac{h}{2}\right)^2} \cos^2 h$$

$$\left[\because \sin\theta = 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\right]$$

$$= \lim_{h\to 0} \frac{1}{2\cos^2 \frac{h}{2}} \times \cos^2 h = \frac{1}{2\times 1} \times (1)^2 [\because \cos 0^\circ = 1]$$

$$= \frac{1}{2}$$

15. Dividing $x^4 - 3x^3 + 2$ by $x^3 - 5x^2 + 3x + 1$

$$\begin{array}{r|l} & x + 2 \\ x^3 - 5x^3 + 3x + 1 & x^4 - 3x^3 + 2 \\ & \underline{\pm x^4 \mp 5x^3 \pm 3x^2 \pm x} \\ & 2x^3 - 3x^2 - x + 2 \\ & \underline{\pm 2x^3 \mp 10x^2 \pm 6x \pm 2} \\ & 7x^2 - 7x \end{array}$$

$$\Rightarrow \lim_{x\to 1} \frac{x^4 - 3x^3 + 2}{x^3 - 5x^2 + 3x + 1} = \lim_{x\to 1}(x+2) + \lim_{x\to 1} \frac{7x^2 - 7x}{x^3 - 5x^3 + 3x + 1}$$

$$= \lim_{x\to 1} x + 2 + \lim_{x\to 1} \frac{7x(x-1)}{x^3 - 5x^3 + 3x + 1}$$

$$= \lim_{x\to 1} x + 2 + \lim_{x\to 1} \frac{7x(x-1)}{(x-1)(x^2 - 4x - 1)}$$

$$= \lim_{x\to 1} x + 2 + \lim_{x\to 1} \frac{7x}{(x^2 - 4x - 1)}$$

$$= 1 + 2 + \frac{7}{(1-4-1)}$$

$$= 3 - \frac{7}{4}$$

$$= \frac{12-7}{4}$$

$$= \frac{5}{4}$$