$\therefore \quad d^2 = 9$

$\therefore \quad d = 3$ or $d = -3$

If $a = 6$ and $d = 3$, then the required terms are $-3, 3, 9, 15$ and if $a = 6$ and $d = -3$, then they are $15, 9, 3, -3$.

Example 17 : The income of a person is ₹ 3,50,000 in the first year. He receives an increment of ₹ 15,000 to his income per year. What will be his income at the end of 15th year ? How much amount he will receive in 15 years ?

Solution : Here we have an A.P. with $a = 3,50,000$ and $d = 15,000$

Now, $t_n = a + (n - 1)d$

$\therefore \quad t_{15} = 3,50,000 + 14(15,000) = 5,60,000$

and $\quad S_n = \frac{n}{2}(a + l)$

$= \frac{15}{2}(3,50,000 + 5,60,000) = 68,25,000$

$\therefore$ At the end of 15th year his income will be ₹ 5,60,000 and total amount he will receive in 15 years is ₹ 68,25,000.

Exercise 7.2

1. Find the desired terms in the following A.P.s :

   (1) 16th term in $-17, -13, -9, ...$

   (2) 31st term in $101, 96, 91, ...$

   (3) 10th term in $3, \frac{9}{2}, 6, \frac{15}{2}, ...$

2. If the nineth term of an A.P. is 30, find the sum of its first seventeen terms.
3. Find the sum of all natural numbers lying between 100 and 500, which are divisible by 5.
4. The first term of an A.P. is 4 and the sum of first five terms is one-sixth of the sum of the next five terms. Find the 8th term.
5. The sum of first n terms of an A.P. is $3n^2 + 5n$. Which term of it is 164 ?
6. If the sum of the first m terms of an A.P. is n and the sum of the first n terms is m, obtain the sum of the first $(m + n)$ terms.
7. If pth, qth and rth terms of an A.P. are l, m, n respectrively, then find the value of $l(q - r) + m(r - p) + n(p - q)$.
8. The ratio of the sum of first n terms of two A.P.s is $(3n - 13) : (5n - 1)$ for all $n \in N$. Find the ratio of their 13th terms.
9. The ratio of the nth terms of two A.P.s is $(2n - 1) : (4n + 3)$ for all $n \in N$. Find the ratio of the sum of the first 25 terms.
10. Find the sum of all integers from 100 to 200 which are divisible by 2 but not by 5.
11. If the 10th term of an A.P. is $\frac{1}{20}$ and the 20th term is $\frac{1}{10}$, then find the 200th term.
12. The sum of three consecutive terms of an A.P. is 9 and the sum of their squares is 59, find these terms.

13. If the sum of four consecutive terms of an A.P. is 32 and the product of whose 2nd and 3rd term is 60, find these terms.

14. A man starts repaying his loan with the initial instalment of ₹ 200. If he increases the instalment by ₹ 20 every month, how much total amount he will pay at the end of 20th instalment ?

15. Bhargav saves ₹ 50 in the first week of a year and then increases his weekly savings by ₹ 17.50. In the nth week, his weekly savings become ₹ 207.50. Find n and the amount he had saved.

16. A spiral is made up of successive semicircles, with centres alternately at P and Q. Semicircles start with centre P with radii 1 *cm*, 3 *cm*, 5 *cm*, ...; and centre Q with radii 2 *cm*, 4 *cm*, 6 *cm* ... What is the length of the spiral if it is made up of 20 such semicircles ? (See figure 7.1)

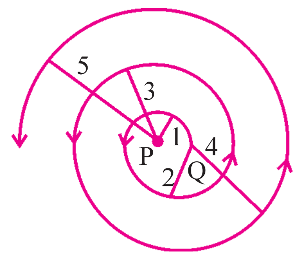

Figure 7.1

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7.3 Geometric Progression (G.P.)

Let us observe some sequences like :

(1) 3, 6, 12, 24,... (2) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$ (3) 0.1, 0.01, 0.001, 0.0001, ...

We note that each term (except the first) progresses in a definite order.

We observe that in (1) the second term onwards each term is double than the preceding term; and in (3) each term is 0.1 times the preceding.

So, the ratio of any term to the preceding term is a constant, i.e. same for all terms (except the first). A Sequence with this property is called a **Geometric Progression (G.P.). The constant ratio is called the common ratio of the G.P. Thus, if the ratio of each term to the preceding is a non-zero constant, then the sequence is a G.P.** We define G.P. as :

Geometric Progression : A sequence $f : N \to R$, $f(n) = Ar^n$, $A \in R - \{0\}$, $r \in R - \{0\}$ is called a Geometric Progression. A G.P. is an exponential function.

Putting $n = 1, 2, 3,...$ we get the terms of a G.P. as $Ar, Ar^2, Ar^3, ...$

Theorem 4 : The ratio of any two consecutive terms of a G.P. is a non-zero constant.

Proof : Suppose $f : N \to R$ is a G.P., then for some $A \neq 0$ and some $r \neq 0$, we have $f(n) = Ar^n$, $\forall n \in N$.

The ratio of two consecutive terms $f(k + 1)$ and $f(k)$ is $\frac{f(k+1)}{f(k)} = \frac{Ar^{k+1}}{Ar^k} = r$, a non-zero constant.

Converse of this theorem is also true.

Suppose the first term of a sequence is $a \neq 0$ and the ratio of two consecutive terms is r, where $r \neq 0$, then the terms of the sequence are $a, ar, ar^2, ..., ar^{n-1}$. Thus, nth term $t_n = ar^{n-1}$.

We will prove this by the principle of mathematical induction.

Theorem 5 : If the ratio of two consecutive terms of a sequence is a non-zero constant, then the sequence is a G.P.

Proof : Consider P(n) : $f(n) = ar^{n-1}$; $a, r \in R - \{0\}$

(1) For $n = 1$, $f(1) = ar^0 = a$, the first term

$\therefore$ P(1) is true.

(2) Let P(k) : $f(k) = ar^{k-1}$ be true for some $k \in$ N.

Now we shall prove that P($k + 1$) is also true.

$\therefore \quad \dfrac{f(k+1)}{f(k)} = r$ **(given)**

$\therefore \quad f(k+1) = r \cdot f(k) = r \cdot (ar^{k-1}) = ar^k = ar^{(k+1)-1}$

Thus P(k) is true. $\Rightarrow$ P($k + 1$) is true.

$\therefore$ By the principle of mathematical induction P(n) is true for all $n \in$ N.

Here, $\{f(n)\}$ is a G.P.

We note that the G.P., whose first term is a and common ratio is r, is $a, ar, ar^2, ..., ar^{n-1}, ...$

Here also we write nth term of a G.P. as t_n.

So, $t_n = ar^{n-1}$, $a \neq 0$, $r \neq 0$.

Note : (1) Now onwards common ratio will be termed as the 'ratio'.

(2) If a, b, c are consecutive terms in G.P. then $\dfrac{b}{a} = \dfrac{c}{b} \Leftrightarrow b^2 = ac$.

Example 18 : Find nth and the 8th term of the G.P. 54, 36, 24, 16, ...

Solution : Here $a = 54$, $r = \dfrac{t_2}{t_1} = \dfrac{36}{54} = \dfrac{2}{3}$

Now $t_n = ar^{n-1} = 54\left(\dfrac{2}{3}\right)^{n-1} = \dfrac{2 \times 3^3 \times 2^{n-1}}{3^{n-1}}$

$\therefore \quad t_n = 2^n \cdot 3^{4-n}$

Taking $n = 8$ in t_n, we have

$t_8 = 2^8 \cdot 3^{-4}$

$= \dfrac{256}{81}$

$\therefore$ The nth term of the G.P. is $2^n \cdot 3^{4-n}$ and the 8th term is $\dfrac{256}{81}$.

Example 19 : If the third term of a G.P. is 18 and its sixth term is 486, find the 9th term.

Solution : Here $t_3 = 18$ and $t_6 = 486$

Now $t_n = ar^{n-1}$

$\therefore \quad t_3 = ar^2 = 18$ and $t_6 = ar^5 = 486$

$\therefore \quad \dfrac{t_6}{t_3} = \dfrac{ar^5}{ar^2} = \dfrac{486}{18}$

$\therefore \quad r^3 = 27$

$\therefore \quad r = 3$

Also $ar^2 = 18$. So $9a = 18$

$\therefore \quad a = 2$

$\therefore \quad t_9 = ar^8 = 2(3)^8 = 13122$

$\therefore$ The 9th term of the G.P. is 13122.

Geometric Series :

The series corresponding to a G.P. is called a geometric series.

If first term of a G.P. is 'a' and ratio is 'r', then the nth term of the geometric series is $S_n = a + ar + ar^2 + ... + ar^{n-1}$.

Now we shall prove the formula for S_n by the principle of mathematical induction.

Theorem 6 : If first term of a G.P. is a and the ratio is r, then the sum of first n term is $S_n = \frac{a(r^n - 1)}{r - 1}$, $a \neq 0$, $r \neq 0$, $r \neq 1$, $n \in N$ and $S_n = na$ if $r = 1$.

Proof : Let the statement P(n) : $S_n = \frac{a(r^n - 1)}{r - 1}$, $a \neq 0$, $r \neq 0$, $r \neq 1$, $n \in N$.

(1) For $n = 1$, $S_1 = \frac{a(r - 1)}{r - 1} = a$, i.e. the sum of the first term is the first term 'a' itself. Thus, P(1) is true.

(2) Let P(k) : $S_k = \frac{a(r^k - 1)}{r - 1}$ be true for some $k \in N$.

Let $n = k + 1$

$$S_{k+1} = \frac{a(r^k - 1)}{r - 1} + ar^{(k+1)-1}$$

$$= \frac{a(r^k - 1)}{r - 1} + ar^k$$

$$= \frac{a}{r - 1}[r^k - 1 + r^{k+1} - r^k]$$

$$= \frac{a(r^{k+1} - 1)}{r - 1}$$

Thus, P(k) is true. $\Rightarrow$ P($k + 1$) is true.

Also we have seen that P(1) is true.

$\therefore$ By the principle of mathematical induction P(n) is true for all $n \in N$.

If $r = 1$, it is clear that $S_n = a + a + ... + a(n \text{ times}) = na$

Note : The formula for S_n can also be written as $S_n = \frac{a(1 - r^n)}{1 - r}$, usually we use this form when $r < 1$.

Example 20 : For a G.P. $t_2 = 6$ and $t_5 = 48$, find S_6.

Solution : $t_2 = 6$ and $t_5 = 48$

$\therefore \quad ar = 6$ and $ar^4 = 48$

$\therefore \quad \frac{ar^4}{ar} = \frac{48}{6}$

$\therefore \quad r^3 = 8 = 2^3$

$\therefore \quad r = 2$ and $ar = 6$

$\therefore \quad a = 3$

$$\text{Now } S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\therefore \quad S_6 = \frac{3(2^6 - 1)}{2 - 1} = 189$$

Example 21 : The third term of a G.P. is $\frac{3}{4}$ and the sum of the first five terms is $\frac{32}{33}$ times the sum of the first ten terms. Find the sum of the first four terms.

Solution : Here $t_3 = \frac{3}{4}$ and $S_5 = \frac{32}{33} \cdot S_{10}$

$$\therefore \quad ar^2 = \frac{3}{4} \text{ and } \frac{a(r^5 - 1)}{r - 1} = \frac{32}{33} \cdot \frac{a(r^{10} - 1)}{r - 1}$$

$$\therefore \quad \frac{33}{32} = r^5 + 1$$

$$\therefore \quad r^5 = \frac{33}{32} - 1$$

$$\therefore \quad r^5 = \frac{1}{32} = \left(\frac{1}{2}\right)^5$$

$$\therefore \quad r = \frac{1}{2}$$

As $ar^2 = \frac{3}{4}$, $a\left(\frac{1}{2}\right)^2 = \frac{3}{4}$

$\therefore \quad a = 3$

$$\text{Now, } S_4 = \frac{a(1 - r^4)}{1 - r} = \frac{3\left(1 - \frac{1}{16}\right)}{1 - \frac{1}{2}} = 6\left(\frac{15}{16}\right)$$

$$\therefore \quad S_4 = \frac{45}{8}$$

$\therefore$ The sum of the first four terms is $\frac{45}{8}$.

Note : Sometimes we need to assume some consecutive terms of a G.P.

If **three** or **five** or **seven** consecutive terms of a G.P. are given, then we assume the middle term as 'a' and preceding terms are obtained dividing a by r, r^2, r^3, ... and succeeding terms are obtained, multiplying a by r, r^2, r^3, ...

So we assume,

The 3 consecutive terms for G.P. : $\frac{a}{r}, a, ar$

The 5 consecutive terms for G.P. : $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

The 7 consecutive terms for G.P. : $\frac{a}{r^3}, \frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2, ar^3$

If **four** or **six** consecutive terms of a G.P. are given, then there are two middle terms, so we assume them as $\frac{a}{r}$ and ar. The terms preceding to $\frac{a}{r}$ are obtained by dividing $\frac{a}{r}$ by r^2, r^4, r^6, ... and the terms succeeding to ar are obtained by multiplying r^2, r^4, r^6, ... So we assume, The 4 consecutive terms for G.P. : $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

The 6 consecutive terms for G.P. : $\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$

Example 22 : Three numbers are consecutive term of a G.P. Their sum and product are $\frac{31}{5}$ and 1 respectively, find the numbers.

Solution : Let the three numbers in a G.P. be $\frac{a}{r}, a, ar$

Their product $\left(\frac{a}{r}\right)(a)(ar) = 1$ and sum $\left(\frac{a}{r}\right) + (a) + (ar) = \frac{31}{5}$

$\therefore \quad a^3 = 1$ and $\frac{1}{r} + 1 + r = \frac{31}{5}$

$\therefore \quad a = 1$

$\therefore \quad 5r^2 - 26r + 5 = 0$

$\therefore \quad (5r - 1)(r - 5) = 0$

$\therefore \quad r = \frac{1}{5}$ or $r = 5$

Taking $a = 1$ and $r = \frac{1}{5}$, the numbers are $5, 1, \frac{1}{5}$

(If we take $r = 5$, then the same numbers can be obtained.)

Example 23 : Find the sum of the sequence 5, 55, 555, ... upto first n terms.

Solution : $S_n = 5 + 55 + 555 + \dots n$ terms

$= \frac{5}{9}[9 + 99 + 999 + \dots n \text{ terms}]$

$= \frac{5}{9}[(10 - 1) + (10^2 - 1) + (10^3 - 1) + \dots n \text{ terms}]$

$= \frac{5}{9}[(10 + 10^2 + 10^3 + \dots n \text{ terms}) - (1 + 1 + 1 + \dots n \text{ terms})]$

$= \frac{5}{9}\left[\frac{10(10^n - 1)}{10 - 1} - n\right]$ **(here $a = 10$, $r = 10$)**

$= \frac{5}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$

$= \frac{50}{81}(10^n) - \frac{50}{81} - \frac{5n}{9}$

Example 24 : The number of bacteria in a certain culture increase at the rate 4 % every hour. If initially there are 40 bacteria present, then how many bacteria will be present at the end of the 4th hour ? How many bacteria have increased during the 4th hour ?

Solution : Initially number of bacteria is 40. Bacteria increase at the rate of 4 % at the end of each hour.

So at the end of the first hour, the number of bacteria will be

$40 + 40\left(\frac{4}{100}\right) = 40(1 + 0.04) = 40(1.04)$

At the end of the second hour, they are $40(1.04)^2$. At the end of the third hour they are $40(1.04)^3$.

Thus, number of bacteria present in the successive hours form a G.P. with $T_1 = a = 40$ and $r = 1.04$

At the end of the fourth hour they are $T_5 = 40(1.04)^4 = 46.7943$

i.e. at the end of the fourth hour number of bacteria present is approximately 47.

During the fourth hour the increase in number of bacteria

= Number of bacteria present at the end of the fourth hour − Number of bacteria present at the end of the third hour

$= 40[(1.04)^4 - (1.04)^3]$

$= 40(1.04)^3 \ (1.04 - 1)$

$= 40(1.04)^3 \ (0.04)$

$= 1.7987$

$\therefore$ During the fourth hour approximately 2 bacteria have increased.

Exercise 7.3

1. Find the indicated terms of the following G.P.s :

(1) The 12th term of $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, \ldots$

(2) The 11th term of $7, \frac{-7}{2}, \frac{7}{4}, \frac{-7}{8}, \ldots$

(3) The 8th term of $-2, -2\sqrt{2}, -4, -4\sqrt{2}, \ldots$

2. Do as directed for the following G.P.s :

(1) $t_7 = 96$, $r = 2$, find t_{10}.

(2) $a = 2$, $r = \sqrt{2}$, $t_n = 128$, find n.

(3) $a = 3$, $r = 3$, $S_n = 363$, find n.

(4) $r = \frac{1}{3}$, $S_3 = \frac{585}{4}$, find a.

3. The sum of the first three terms of a G.P. is 21 and the sum of the next three terms is 168, find the sum of the first five terms.

4. If the sum of the first two terms of a G.P. is $\frac{9}{2}$ and the sixth term is 8 times the third term, find G.P.

5. Sum to first n terms of a sequence :

(1) 7, 77, 777, 7777, ... (2) 3, 33, 303, 3003, ...

6. Find the sum of : $a(a + b) + a^2(a^2 + b^2) + a^3(a^3 + b^3) + \ldots$ n terms $(a, b \neq 0, \pm 1)$

7. The product of five positive numbers in G.P. is 32 and ratio of the greatest number to the smallest number is 81 : 1, find the numbers.

8. In a G.P., the $(p + q)$th term is m and the $(p - q)$th term is n. Find its pth term in terms of m and n.

9. If 1, a, b, c, 2 are consecutive terms in a G.P., then find the value of abc.

10. If pth, qth and rth terms of a G.P. are themselves consecutive terms of a G.P., prove that p, q, r are in A.P.

11. If x, y, z are three consecutive terms in a G.P., prove that $\frac{1}{x+y} + \frac{1}{y+z} = \frac{1}{y}$.

12. Find four positive consecutive terms in a G.P. such that their product is 16 and having sum of second and third terms equal to 5.

13. A motorcycle was purchased for ₹ 60,000. If its price goes down by 10 % each year, what would be its price at the end of the fourth year.

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7.4 Means

Arithmetic Mean (A.M.) : If three distinct numbers a, A, b are consecutive terms of an A.P., then A is called an arithmetic mean of the two numbers a and b.

a, A and b are in A.P.

$\therefore \quad A - a = b - A$

$\therefore \quad 2A = a + b$

$\therefore \quad A = \frac{a+b}{2}$

Thus, A.M. of a and b is $A = \frac{a+b}{2}$, i.e. it is the average of a and b.

For instance, the A.M. of 4 and 12 is $A = \frac{4+12}{2} = 8$.

Arithmetic Means : For given two distinct real numbers a and b, if numbers $a, A_1, A_2, A_3, \dots A_n, b$ are consecutive terms of an A.P., we say that $A_1, A_2, A_3, \dots, A_n$ are n arithmetic means between a and b.

Suppose $A_1, A_2, A_3, \dots, A_n$ are n arithmetic means between a and b.

Here we have $n + 2$ terms in an A.P.. The first term is a and $(n + 2)$nd term is b.

$\therefore \quad t_{n+2} = b = a + [(n + 2) - 1]\, d$

$\therefore \quad b - a = (n + 1)d$

$\therefore \quad \frac{b-a}{n+1} = d$

Here, arithmetic mean $A_1 = a + d$

$$= a + \left(\frac{b-a}{n+1}\right)$$

Thus $A_1 = a + \left(\frac{b-a}{n+1}\right)$, $A_2 = a + 2\left(\frac{b-a}{n+1}\right)$, $A_3 = a + 3\left(\frac{b-a}{n+1}\right)$, ...

n arithmetic means between a and b are $A_k = a + k\left(\frac{b-a}{n+1}\right)$, where $k = 1, 2, 3, \dots n$.

Here, A_k denotes kth A.M. out of n means between a and b.

For $n = 1$, $A_1 = a + \frac{b-a}{n+1} = \frac{a+b}{2}$, the A.M. of a and b.

Thus, A.M. of two distinct real numbers a and b is $A = \frac{a+b}{2}$.

Example 25 : Find four arithmetic means between 8 and 23.

Solution : Here, $a = 8$, $b = 23$ and $n = 4$

So $d = \frac{b-a}{n+1} = \frac{23-8}{4+1} = \frac{15}{5} = 3$

$\therefore$ The four arithmetic means between 8 and 23 are :

$8 + 3$, $8 + 2(3)$, $8 + 3(3)$, $8 + 4(3)$. They are 11, 14, 17 and 20.

Example 26 : If n arithmetic means are inserted between 1 and 31 such that the ratio of $(n-1)$th mean to the 7th mean is 9 : 5, find n.

Solution : Here, $a = 1$ and $b = 31$

Common difference $d = \frac{b-a}{n+1} = \frac{31-1}{n+1} = \frac{30}{n+1}$

$\frac{(n-1)\text{th A.M.}}{7\text{th A.M.}} = \frac{9}{5}$ **(given)**

$\therefore \quad \frac{1+(n-1)\left(\frac{30}{n+1}\right)}{1+7\left(\frac{30}{n+1}\right)} = \frac{9}{5}$

$\therefore \quad \frac{n+1+30n-30}{n+1+210} = \frac{9}{5}$

$\therefore \quad 5(31n - 29) = 9(n + 211)$

$\therefore \quad 155n - 145 = 9n + 1899$

$\therefore \quad 146n = 2044$

$\therefore \quad n = 14$

Geometric Mean (G.M.) : Given distinct positive real numbers a and b, if G is a positive number such that a, G, b are consecutive terms of G.P., then G is called a geometric mean of a and b.

a, G, b are in G.P.

$\therefore \quad \frac{G}{a} = \frac{b}{G}$

$\therefore \quad G^2 = ab$

$\therefore \quad G = \sqrt{ab}$

For instance, the G.M. of 2 and 18 is $G = \sqrt{2 \times 18} = 6$.

Geometric Means : For given distinct positive real numbers a and b, if the positive numbers $a, G_1, G_2, G_3, \dots G_n, b$ are consecutive terms of a G.P., then $G_1, G_2, G_3, \dots G_n$ are called the geometric means between a and b.

Now, we shall find the formula for n geometric means between a and b.

Suppose $G_1, G_2, G_3, \dots G_n$ are n geometric means between a and b, then $a, G_1, G_2, G_3, \dots G_n, b$ are consecutive terms in a G.P. in which the first term is a and the $(n + 2)$nd term is b.

$\therefore \quad t_{n+2} = b = ar^{n+1}$ where r is the common ratio of G.P.

$\therefore \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Thus, $G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, G_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, G_3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}}, \dots$

$\therefore \quad n$ geometric means between a and b are

$G_k = a\left(\frac{b}{a}\right)^{\frac{k}{n+1}}$, where $k = 1, 2, 3, \dots n$

Here, G_k denotes kth G.M. out of n G.M.s between a and b.

For $n = 1$, $G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{1+1}} = \sqrt{ab}$, the G.M. of a and b.

Thus, G.M. of two distinct positive real numbers a and b is $G = \sqrt{ab}$.

Example 27 : Find three G.M.s between 2 and $\frac{2}{81}$.

Solution : Here, $a = 2$, $b = \frac{2}{81}$ and $n = 3$

$\therefore \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} = \left(\frac{2}{81} \times \frac{1}{2}\right)^{\frac{1}{3+1}} = \left(\frac{1}{81}\right)^{\frac{1}{4}} = \frac{1}{3}$

Now, $G_1 = ar = 2 \cdot \frac{1}{3} = \frac{2}{3}$, $G_2 = ar^2 = 2 \cdot \left(\frac{1}{3}\right)^2 = \frac{2}{9}$, $G_3 = ar^3 = 2 \cdot \left(\frac{1}{3}\right)^3 = \frac{2}{27}$

$\therefore$ The three geometric means between 2 and $\frac{2}{81}$ are $\frac{2}{3}$, $\frac{2}{9}$ and $\frac{2}{27}$.

Example 28 : If the A.M. and G.M. of two positive real numbers are 7 and $2\sqrt{6}$ respectively, find the numbers.

Solution : Let a and b be two numbers whose A.M., A = 7 and G.M., G = $2\sqrt{6}$

$\therefore \quad A = \frac{a+b}{2} = 7$ and $G = \sqrt{ab} = 2\sqrt{6}$

$\therefore \quad a + b = 14$ and $ab = 24$

$\therefore \quad b = \frac{24}{a}$

$\therefore \quad a + \frac{24}{a} = 14$

$\therefore \quad a^2 - 14a + 24 = 0$

$\therefore \quad (a - 12)(a - 2) = 0$

$\therefore \quad a = 12 \quad$ or $\quad a = 2$

Now if $a = 12$, then $b = 2$ and if $a = 2$ then $b = 12$

$\therefore$ The required numbers are 2 and 12.

Example 29 : If G is the G.M. of a and b and A_1, A_2 are the two A.M.s between a and b, then prove that $G^2 = (2A_1 - A_2)(2A_2 - A_1)$.

Solution : A_1, A_2 are two A.M.s between a and b.

$\therefore \quad a, A_1, A_2, b$ are consecutive terms of A.P.

$\therefore \quad A_1 = \frac{a + A_2}{2}$ and $A_2 = \frac{A_1 + b}{2}$

$\therefore \quad 2A_1 - A_2 = a$ and $2A_2 - A_1 = b$

Now $G^2 = ab = (2A_1 - A_2)(2A_2 - A_1)$

Theorem 7 : If A and G are respectively the A.M. and G.M. of two distinct positive numbers a and b, then prove that $A > G$.

Proof : a and b are distinct and positive.

So, $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

$\therefore \quad A - G = \frac{a+b}{2} - \sqrt{ab}$

$= \frac{1}{2}(a + b - 2\sqrt{ab})$

$= \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 > 0$ $\quad (\because a \neq b;\ a, b > 0)$

$\therefore \quad A > G$

Example 30 : The difference between A.M. and G.M. of two positive real numbers is 12 and the ratio of these numbers is 1 : 9, find the numbers.

Solution : Suppose required numbers are a and b, $a, b \in R^+$.

$\therefore \quad A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

As $A > G$, so $A - G = 12$ and $\frac{a}{b} = \frac{1}{9}$ (given)

$\therefore \quad \frac{a+b}{2} - \sqrt{ab} = 12$ and $b = 9a$

$\therefore \quad \frac{a+9a}{2} - \sqrt{a \cdot 9a} = 12$

$\therefore \quad 5a - 3a = 12$

$\therefore \quad a = 6$ and $b = 54$

$\therefore$ The numbers are 6 and 54.

Example 31 : For positive real numbers a, b, c prove that $(a + b)(b + c)(c + a) \geq 8abc$.

Solution : We know that $\frac{a+b}{2} \geq \sqrt{ab}$.

Similarly $\frac{b+c}{2} \geq \sqrt{bc}$ and $\frac{c+a}{2} \geq \sqrt{ca}$.

Multiplying the respective sides of the above results, we get

$$\left(\frac{a+b}{2}\right)\left(\frac{b+c}{2}\right)\left(\frac{c+a}{2}\right) \geq abc$$

$\therefore \quad (a + b)(b + c)(c + a) \geq 8abc$

Exercise 7.4

1. Place 5 A.M.s between 3 and 4.

2. Place 3 A.M.s between -3 and 29.

3. Insert 5 G.M.s between $\frac{1}{8}$ and 8.

4. Insert 3 G.M.s between 2 and $\frac{1}{2}$.

5. Find two positive numbers whose A.M. and G.M. are 25 and 15 respectively.

6. If A.M. and G.M. of the roots of a quadratic equation are 10 and 8 respectively, then obtain the quadratic equation.

7. If $sec(x + y)$, $secx$, $sec(x - y)$ are in A.P. then prove that $cosx = \pm\sqrt{2}\ cos\frac{y}{2}$, where $cosx \neq 1$; $cosy \neq 1$.

8. If $\frac{1}{q}$ is A.M. of $\frac{1}{p}$ and $\frac{1}{r}$, then prove that $\frac{r+p}{q}$ is the A.M. of $\frac{p+q}{r}$ and $\frac{q+r}{p}$, where $p, q, r \neq 0$.

*

7.5 Sums of Some Special Series

Sequence of powers of Natural Numbers : We have obtained the formula for the sum of first n terms of an A.P. and a G.P., but it is not possible to find such formula for every sequence. There are some important sequences which are neither A.P. nor G.P. and we are able to calculate the sum of their n terms. We shall consider some such special sequences. We shall find formula for the sum of first n natural numbers, their squares and their cubes.

We shall introduce the notation 'Σ' pronounced as 'sigma', which is to be specially used for such series. It is the enlarged form of the upright capital Greek letter sigma, Σ means summation.

$\sum\limits_{n=2}^{n=6} t_n$ reads as 'sigma' t_n where n runs from 2 to 6 and it denotes sum of t_n for n = 2, 3, 4, 5 and 6. i.e. $\sum\limits_{n=2}^{n=6} t_n = t_2 + t_3 + t_4 + t_5 + t_6$.

For example,

$$\sum_{n=2}^{n=6} (2n+3) = \{2(2)+3\} + \{2(3)+3\} + \{2(4)+3\} + \{2(5)+3\} + \{2(6)+3\}$$

$$= 7 + 9 + 11 + 13 + 15$$

$$= 55$$

i.e. we have to substitute n = 2, 3, 4, 5 and 6 in $(2n + 3)$ and we shall add the resulting numbers.

It is customary to write $\sum\limits_{n=2}^{6} t_n$ instead of writing $\sum\limits_{n=2}^{n=6} t_n$.

The symbol Σ has the following properties which can easily be proved. (Try yourself !)

(1) $\sum\limits_{i=1}^{n} (a_i + b_i) = \sum\limits_{i=1}^{n} a_i + \sum\limits_{i=1}^{n} b_i$

(2) $\sum\limits_{i=1}^{n} ma_i = m \sum\limits_{i=1}^{n} a_i$, where m is a constant not depending on i.

(3) $\sum\limits_{i=1}^{n} 1 = \sum\limits_{i=1}^{n} (i)^0 = 1^0 + 2^0 + 3^0 + ... + n^0$

$$= 1 + 1 + 1 + ... + 1 \ (n \text{ times})$$

$$= n$$

(4) $\sum\limits_{i=1}^{n} m = m \sum\limits_{i=1}^{n} 1 = mn$, where m is constant.

Note : (1) $\sum_{i=1}^{n} (a_i \cdot b_i) \neq \sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i$

(2) $\sum_{i=1}^{n} \left(\frac{a_i}{b_i}\right) \neq \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$, $b_i \neq 0\ \forall i \in \mathrm{N}$

Now we shall find the sum of first n terms of some special series, namely

(1) $\sum_{r=1}^{n} r$ (2) $\sum_{r=1}^{n} r^2$ (3) $\sum_{r=1}^{n} r^3$.

Now, $\sum_{i=1}^{n} r = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$, we can prove it by the formula for sum of first n terms in A.P. (Try it !).

Note : It is believed that the great mathematician Gauss obtained the sum $1 + 2 + 3 + ... + 100$ at the age of 5 to 6 years. When his teacher had asked to sum the numbers 1 to 100, he had answered the question in no time. He had paired the numbers and added numbers in pairs as $1 + 100 = 101$, $2 + 99 = 101$, $3 + 98 = 101$, ... $50 + 51 = 101$, each pair giving the same sum 101 and there are such fifty pairs, so the sum of 1 to 100 is $50 \times 101 = 5050$.

From this also, we can have $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$.

Theorem 8 : $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$, $n \in \mathrm{N}$

Proof : Here $S_n = 1^2 + 2^2 + 3^2 + ... + n^2$.

We consider the identity : $x^3 - (x - 1)^3 = 3x^2 - 3x + 1$

Putting $x = 1, 2, 3, ..., n$, we have

$1^3 - 0^3 = 3(1)^2 - 3(1) + 1$

$2^3 - 1^3 = 3(2)^2 - 3(2) + 1$

$3^3 - 2^3 = 3(3)^2 - 3(3) + 1$

$\vdots$

$n^3 - (n - 1)^3 = 3(n)^2 - 3(n) + 1$

Adding all the above results, we have

$n^3 - 0^3 = 3[1^2 + 2^2 + 3^2 + ... + n^2] - 3[1 + 2 + 3 + ... + n] + [1 + 1 + 1 + ... + 1, n \text{ times}]$

$\therefore \quad n^3 = 3 \cdot S_n - 3 \sum_{r=1}^{n} r + n$

$\therefore \quad 3S_n = n^3 + \frac{3n(n+1)}{2} - n$

$\therefore \quad S_n = \frac{1}{6}\left(2n^3 + 3n^2 + 3n - 2n\right)$

$= \frac{1}{6}\left(2n^3 + 3n^2 + n\right)$

$= \frac{1}{6} \cdot n\left(2n^2 + 3n + 1\right)$

$= \frac{1}{6} \cdot n(n + 1)(2n + 1)$

$\therefore \quad \sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{1}{6} \cdot n(n + 1)(2n + 1)$

Theorem 9 : $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$, $n \in N$

Proof : Here $S_n = 1^3 + 2^3 + 3^3 + ... + n^3$

We consider the identity : $x^4 - (x - 1)^4 = 4x^3 - 6x^2 + 4x - 1$

Putting $x = 1, 2, 3, ..., n$, we have

$1^4 - 0^4 = 4(1)^3 - 6(1)^2 + 4(1) - 1$

$2^4 - 1^4 = 4(2)^3 - 6(2)^2 + 4(2) - 1$

$3^4 - 2^4 = 4(3)^3 - 6(3)^2 + 4(3) - 1$

$\vdots$

$n^4 - (n - 1)^4 = 4n^3 - 6n^2 + 4n - 1$

Adding all the above results, we have

$n^4 - 0^4 = 4[1^3 + 2^3 + 3^3 + ... + n^3] - 6[1^2 + 2^2 + 3^2 + ... + n^2]$

$+ 4[1 + 2 + 3 + ... + n] - [1 + 1 + 1 + ... + 1, n \text{ times}]$

$\therefore \quad n^4 = 4 \cdot S_n - 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - n$

$\therefore \quad 4S_n = n^4 + 6\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + n$

$\therefore \quad S_n = \frac{1}{4}\left[n^4 + n(n + 1)(2n + 1) - 2n(n + 1) + n\right]$

$= \frac{1}{4} \cdot n\left[n^3 + (n + 1)(2n + 1) - 2(n + 1) + 1\right]$

$= \frac{1}{4} \cdot n\left(n^3 + 2n^2 + 3n + 1 - 2n - 2 + 1\right)$

$= \frac{1}{4} \cdot n\left(n^3 + 2n^2 + n\right)$

$= \frac{1}{4}n \cdot n\left(n^2 + 2n + 1\right)$

$= \frac{1}{4}n^2(n + 1)^2$

$\therefore \quad S_n = \frac{1}{4} \cdot n^2(n + 1)^2$ or $S_n = \left[\frac{1}{2}n(n + 1)\right]^2$

$\therefore \quad \sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \frac{1}{4}n^2(n+1)^2$

$\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$, $\sum_{r=1}^{n} r^3$ are also denoted by Σn, Σn^2, Σn^3 respectively.

Example 32 : Obtain the following sums :

(1) $\sum_{r=7}^{16} 2r^3$, (2) $\sum_{r=10}^{20} (3r - r^2)$

Solution : (1) $\sum_{r=7}^{16} 2r^3 = 2 \sum_{r=7}^{16} r^3$

$$= 2\left[\sum_{r=1}^{16} r^3 - \sum_{r=1}^{6} r^3\right]$$

$$= 2\left[\frac{(16)^2 \cdot (17)^2}{4} - \frac{(6)^2 \cdot (7)^2}{4}\right]$$

$$= 2\ [18496 - 441]$$

$$= 2\ [18055] = 36110$$

(2) $\sum_{r=10}^{20} (3r - r^2) = 3\left[\sum_{r=1}^{20} r - \sum_{r=1}^{9} r\right] - \left[\sum_{r=1}^{20} r^2 - \sum_{r=1}^{9} r^2\right]$

$$= 3\left[\frac{20(20+1)}{2} - \frac{9(9+1)}{2}\right] - \left[\frac{20(20+1)\ (20(2)+1)}{6} - \frac{9(9+1)\ (2(9)+1)}{6}\right]$$

$$= 3\left[\frac{(20)\ (21)}{2} - \frac{9(10)}{2}\right] - \left[\frac{20(21)\ (41)}{6} - \frac{9(10)\ (19)}{6}\right]$$

$$= 3\ (210 - 45) - (2870 - 285)$$

$$= 495 - 2585 = -2090$$

Example 33 : Sum to n terms $1^3 + 3^3 + 5^3 + ...$

Solution : Let us think of the nth term (general term) of the series 1, 3, 5, ... It is an A.P. with the first term $a = 1$ and $d = 2$.

$\therefore \quad t_n = a + (n-1)d = 1 + (n-1)2 = 2n - 1$

$\therefore \quad$ the nth term of the given sequence is $(2n-1)^3$.

Note : To get general term, it is not necessary to show the method how we get it. For example in this question it is clear that 1, 3, 5,... is a sequence of odd natural numbers, so the nth odd natural number is obvious $2n - 1$.

Now, $S_n = 1^3 + 3^3 + 5^3 + ... + (2n-1)^3$

$$= \sum_{r=1}^{n} (2r-1)^3$$

$$= \sum_{r=1}^{n} (8r^3 - 12r^2 + 6r - 1)$$

$$= 8\sum_{r=1}^{n} r^3 - 12\sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$$

$$= 8 \cdot \frac{n^2(n+1)^2}{4} - 12 \cdot \frac{n(n+1)(2n+1)}{6} + 6 \cdot \frac{n(n+1)}{2} - n$$

$$= n(n+1)\,[(2n(n+1) - 2(2n+1) + 3)] - n$$

$$= n(n+1)\,(2n^2 + 2n - 4n - 2 + 3) - n$$

$$= n(n+1)\,(2n^2 - 2n + 1) - n$$

$$= n[(n+1)(2n^2 - 2n + 1) - 1]$$

$$= n(2n^3 + 2n^2 - 2n^2 - 2n + n + 1 - 1)$$

$$= n(2n^3 - n)$$

$$= n^2(2n^2 - 1)$$

Example 34 : Sum to n terms the series $1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + \ldots$ and hence obtain the sum of first 50 terms.

Solution : Here, to begin with we see that the nth term of 1, 2, 3, ... is n and that of 4, 5, 6, ... is $(n + 3)$.

$$\therefore \quad t_n = n(n+3)$$

$$\therefore \quad S_n = \sum_{r=1}^{n} r(r+3)$$

$$= \sum_{r=1}^{n} (r^2 + 3r)$$

$$= \sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2}$$

$$= \frac{n(n+1)(2n+1) + 9n(n+1)}{6}$$

$$= \frac{n(n+1)(2n+1+9)}{6}$$

$$= \frac{n(n+1)\cdot(2n+10)}{6}$$

$$= \frac{n(n+1)(n+5)}{3}$$

Substituting $n = 50$, we have

$$S_{50} = \frac{50(51)(55)}{3} = 46750$$

Thus, sum of the first n terms, $S_n = \frac{n(n+1)(n+5)}{3}$ and the sum of the first fifty terms is 46750.

Example 35 : Sum the series :

$(1 + x) + (1 + x + x^2) + (1 + x + x^2 + x^3) + \dots n$ terms. $(x \neq 1)$

Solution : Here, $S_n = \frac{1-x^2}{1-x} + \frac{1-x^3}{1-x} + \frac{1-x^4}{1-x} + \dots n$ terms

$$= \frac{1}{1-x}[(1 + 1 + 1 + \dots n \text{ terms}) - (x^2 + x^3 + x^4 + \dots n \text{ terms})]$$

$$= \frac{1}{1-x}\left[n - \frac{x^2(1-x^n)}{1-x}\right]$$

($x^2 + x^3 + x^4 + \dots$ *n* terms is a geometric series; $a = x^2$, $r = x$)

Example 36 : If the sum of the first n terms of the series,

$1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + \dots$ is $\frac{n(n+1)^2}{2}$ when n is even, then what is the sum of the series when n is odd ?

Solution : When n is odd, then last term will be n^2.

$\therefore$ Series is $1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + \dots + 2(n-1)^2 + n^2$

$$= \frac{(n-1)((n-1)+1)^2}{2} + n^2 \qquad \textbf{((n - 1) is even)}$$

$$= \frac{(n-1) \cdot n^2 + 2n^2}{2}$$

$$= \frac{n^2(n-1+2)}{2}$$

$$= \frac{n^2(n+1)}{2}$$

(Verify your answer !)

Example 37 : Sum the series, $1 + \frac{1}{2}(1 + 2) + \frac{1}{3}(1 + 2 + 3) + \dots 16$ terms.

Solution : Here, $t_n = \frac{1}{n}(1 + 2 + 3 + \dots + n) = \frac{1}{n}\Sigma n$

$$= \frac{1}{n} \cdot \frac{n(n+1)}{2}$$

$$= \frac{n+1}{2}$$

$$\therefore \quad S_n = \sum_{r=1}^{16}\left(\frac{r+1}{2}\right) = \frac{1}{2}\left(\sum_{r=1}^{16} r + \sum_{r=1}^{16} 1\right)$$

$$= \frac{1}{2}\left[\frac{16(17)}{2} + 16\right]$$

$$= \frac{1}{2}(136 + 16)$$

$$= \frac{1}{2}(152) = 76$$

Exercise 7.5

1. Obtain the following sums :

(1) $\sum_{r=1}^{10} (2r^2 + 3)$ (2) $\sum_{r=2}^{10} (4r^2 - 28r + 49)$ (3) $\sum_{r=6}^{15} (r^2 - r - 1)$ (4) $\sum_{r=8}^{20} (2 - r^2)$

2. Find the sum of the first n terms of each series below :

(1) $3^2 + 7^2 + 11^2 + ...$

(2) $1^3 + 4^3 + 7^3 + ...$

(3) $2 \cdot 1 + 5 \cdot 3 + 8 \cdot 5 + ...$

(4) $3 \cdot 4 \cdot 5 + 4 \cdot 6 \cdot 5 + 5 \cdot 8 \cdot 5 + ...$

(5) $(5^4 - 1^4) + (8^4 - 4^4) + (11^4 - 7^4) + ...$

(6) $1^2 + \left(\frac{1^2 + 2^2}{2}\right) + \left(\frac{1^2 + 2^2 + 3^2}{3}\right) + ...$

(7) $(2^2 - 1^2) + (4^2 - 3^2) + (6^2 - 5^2) + ...$

(8) $1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + ...$

(9) $(n^2 - 1^2) + 2(n^2 - 2^2) + 3(n^2 - 3^2) + ...$

3. Sum the following series :

(1) $2^3 - 3^3 + 4^3 - 5^3 + ... + 22^3 - 23^3$

(2) $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + ... + 29^2 - 30^2$

*

Miscellaneous Examples :

Example 38 : If α_1 is the sum of the first n natural numbers, α_2 denotes the sum of their squares and α_3 is the sum of their cubes, prove that $9\alpha_2^2 = \alpha_3 (1 + 8\alpha_1)$.

Solution : $\alpha_1 = \frac{n(n+1)}{2}$, $\alpha_2 = \frac{n(n+1)(2n+1)}{6}$, $\alpha_3 = \frac{n^2(n+1)^2}{4}$

$$\text{Now, } \alpha_3 [1 + 8\alpha_1] = \frac{n^2(n+1)^2}{4} \left[1 + 8 \cdot \frac{n(n+1)}{2}\right]$$

$$= \frac{n^2(n+1)^2}{4} [4n^2 + 4n + 1]$$

$$= \frac{n^2(n+1)^2(2n+1)^2}{4} \times \frac{9}{9}$$

$$= \left(\frac{n(n+1)(2n+1)}{6}\right)^2 \cdot 9 = 9\alpha_2^2$$

Example 39 : Obtain the sum of the series,

$$\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \left(x^3 + \frac{1}{x^3}\right)^2 + ... \ n \text{ terms } (x \neq 0, x \neq \pm 1)$$

Solution : $\left(x^2 + 2 + \frac{1}{x^2}\right) + \left(x^4 + 2 + \frac{1}{x^4}\right) + \left(x^6 + 2 + \frac{1}{x^6}\right) + ... \ n$ terms

$$= (x^2 + x^4 + x^6 + ... + n \text{ terms}) + \left(\frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} + ... \ n \text{ terms}\right) + (2 + 2 + 2 + ... \ n \text{ terms})$$

$$= \frac{x^2(x^{2n} - 1)}{x^2 - 1} + \frac{\frac{1}{x^2}\left(1 - \frac{1}{x^{2n}}\right)}{1 - \frac{1}{x^2}} + 2n$$

$$= \frac{x^2(x^{2n} - 1)}{x^2 - 1} + \frac{x^{2n} - 1}{(x^2 - 1) \cdot x^{2n}} + 2n$$

Exercise 7

1. Find the 30th term of the sequence 5, 0, −5, −10, ... Also find which term would be −200, if any.
2. For an A.P., the 12th term is 64 and the 20th term is 112, find the A.P.
3. Ramu travels at the speed of 40 *km/hr*. He reduces his speed every hour by 4 *km*. How much time would he take to travel 216 *km* ?
4. 200 wooden blocks are stacked in such a way that 20 blocks are in the bottom row, 19 are in the next upper row, 18 are in the upper row next to it and the process is continued. How many rows will be formed ? How many blocks are there in the upper most row ?
5. 1, 5, 25 are the pth, the qth and the rth terms respectively of a G.P. Prove that p, q, r are in A.P.
6. If a, b, c are consecutive terms of A.P. and $a, c - b, b - a$ are consecutive terms of G.P., then find $a : b : c$.
7. Sum to n terms of the series : $6 + 6.6 + 6.66 + 6.666 + ...$
8. If the sums of first $n, 2n, 3n$ terms of A.P. are α, β, γ respectively, prove that $\gamma = 3(\beta - \alpha)$.
9. Find S_{20} for an A.P. having t_3 as 7 and t_7 is 2 more than three times its t_3.
10. If a, b, c are consecutive terms of A.P., a^2, b^2, c^2 are consecutive terms of G.P., $a + b + c = \frac{3}{2}$ and $a < b < c$, find a.
11. If $a_n = 3 - 5n$, find S_n.
12. If S_{30} of an A.P. is 1635 and t_{30} is 98, find A.P.
13. If A.M. is three times G.M. of two positive numbers a and b, find $a : b$.
14. Sum the series : $1 + \frac{1^3 + 2^3}{2} + \frac{1^3 + 2^3 + 3^3}{3} + ... + \frac{1^3 + 2^3 + 3^3 + ... + 20^3}{20}$.
15. Sum of six consecutive terms of an A.P. is 48 and the product of the first and the last numbers is 39. Find these numbers.
16. Product of five consecutive terms of G.P. is 243. If the sum of the second and the fourth number is $\frac{51}{4}$, find the numbers.
17. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) In a sequence $\left\{\frac{n + (-1)^n}{2}\right\}$, the difference of the 12th and the 21st term is ... ☐

(a) 0 (b) $\frac{-1}{2}$ (c) $\frac{7}{2}$ (d) $\frac{33}{2}$

(2) If the 5th term of an A.P. is 7, then the sum of the first 9 terms is ... ☐

(a) 36 (b) 49 (c) 45 (d) 63

(3) If the third term of an A.P. is 9 and its tenth term is 21, then the sum of its first 12 terms is ... ☐

(a) 180 (b) 360 (c) 150 (d) 210

(4) A.M. of two positive numbers is 2. If a larger number is increased by 1, then their G.M. is also 2, then the numbers are ... ☐

(a) 1, 3 (b) $\frac{1}{2}, \frac{7}{2}$ (c) $\frac{2}{3}, \frac{10}{3}$ (d) 0.7, 3.3

(5) If for a G.P., $r = \frac{1}{3}$ and $S_4 = \frac{80}{27}$, then $a =$ ......

(a) $\frac{2}{3}$ (b) 3 (c) 2 (d) $\frac{3}{2}$

(6) If 25, $x - 6$ and $x - 12$ are consecutive terms of G.P., then $x =$ ......

(a) 8 (b) 12 (c) 16 (d) 20

(7) $\sum_{r=1}^{n} \left(\sum_{m=1}^{r} m \right) =$ ......

(a) $\frac{n(n+1)(2n+1)}{6}$ (b) $\frac{n(n+1)(n+2)}{6}$ (c) $\frac{n^2(n+1)^2}{4}$ (d) $\frac{n(n+1)(2n+1)}{12}$

(8) If S_1, S_2 and S_3 are the sums of the first n_1, n_2, n_3 terms of an A.P. respectively then

$\frac{2S_1}{n_1}(n_2 - n_3) + \frac{2S_2}{n_2}(n_3 - n_1) + \frac{2S_3}{n_3}(n_1 - n_2) =$ ......

(a) 0 (b) 1 (c) $S_1 \ S_2 \ S_3$ (d) $n_1 \ n_2 \ n_3$

(9) If the first term of a G.P. is 3 and the common ratio is 2, then the sum of first five to ten terms is ......

(a) 2976 (b) 3024 (c) 1488 (d) 3114

(10) 3 + 4 + 8 + 9 + 13 + 14 + 18 + 19 + ... 20 terms is ...

(a) 511 (b) 536 (c) 549 (d) 520

(11) If third term of a G.P. is 3, then the product of first five terms is ...

(a) 3^5 (b) 5^3 (c) 3^3 (d) 5^5

(12) A_1 and A_2 are two A.M.s inserted between a and b, while G_1 and G_2 are two G.M.s inserted between a and b, then $\frac{G_1G_2}{A_1 + A_2} =$ ....

(a) $\frac{a+b}{2ab}$ (b) $\frac{a+b}{ab}$ (c) $\frac{2ab}{a+b}$ (d) $\frac{ab}{a+b}$

(13) If the length of the sides of a right triangle are in A.P. then the *cosines* of their acute angles are ...

(a) $\frac{\sqrt{3}}{2}, \frac{1}{2}$ (b) $\frac{5}{13}, \frac{12}{13}$ (c) $\frac{3}{5}, \frac{4}{5}$ (d) $\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}$

(14) Let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2^n - 1}$, $n \in N$, then...

(a) $S_{100} < 100$ (b) $S_{100} > 100$ (c) $S_{200} = 100$ (d) $S_{200} > 200$

(15) If $a_1, a_2, a_3, ... a_n$ are in A.P. with common difference d, then

$\sin d\,[\operatorname{cosec} a_1 \cdot \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \cdot \operatorname{cosec} a_3 + ... + \operatorname{cosec} a_{n-1} \cdot \operatorname{cosec} a_n] =$ ......

(a) $\operatorname{cosec} a_1 - \operatorname{cosec} a_n$ (b) $\sec a_1 - \sec a_n$

(c) $\cot a_1 - \cot a_n$ (d) $\tan a_1 - \tan a_n$

(16) For an A.P., if $4t_4 = 7t_7$, then $t_{11} =$ ......

(a) -1 (b) 0 (c) 11 (d) 44

(17) $0 < \theta \le \frac{\pi}{2}$, then the minimum value of $sin^3\theta + cosec^3\theta$ is ... ☐

(a) 2 (b) 1 (c) 0 (d) not possible

(18) a, b, c, d, e, f are in A.P., then $d - b$ = ... ☐

(a) $2(c - a)$ (b) $2(f - c)$ (c) $2(d - c)$ (d) $2(f - b)$

*

Summary

We studied following points in this chapter :

1. If S_n is given then to find a_n by $a_1 = S_1$; $a_n = S_n - S_{n-1}$, $n > 1$.

2. The nth term of A.P. is $t_n = a + (n - 1)d$ where, a is the first term and d is the common difference.

3. The sum of the first n terms of an A.P. is $S_n = \frac{n}{2}\{2a + (n - 1)d\}$.

4. The nth term of a G.P. is $t_n = ar^{n-1}$, $a \neq 0$, $r \neq 0$, where a is the first term and r is the common ratio.

The sum of the first n terms of G.P. is $S_n = \begin{cases} \frac{a(r^n - 1)}{r - 1} & r \neq 1 \\ na & r = 1 \end{cases}$

5. The arithmetic mean of two numbers a and b is $A = \frac{a+b}{2}$. If there are n arithmetic means inserted between a and b, then $d = \frac{b-a}{n+1}$ and the kth mean is given by $A_k = a + k\left(\frac{b-a}{n+1}\right)$, where $k = 1, 2, 3, \ldots n$.

6. The geometric mean of two positive numbers a and b is $G = \sqrt{ab}$. If there are n geometric means inserted between a and b, then $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$ and the kth mean is given by $G_k = a\left(\frac{b}{a}\right)^{\frac{k}{n+1}}$, where $k = 1, 2, 3, \ldots n$ ($G > 0$, $G_i > 0$ $\forall i \in N$)

7. $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$, $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$.

Bhaskara (1114–1185), also known as Bhaskara II and Bhaskaracharya ("Bhaskara the teacher"), was an Indian mathematician and astronomer. He was born near Vijayvada. Bhaskara is said to have been the head of an astronomical observatory at Ujjain, the leading mathematical center of ancient India.

Bhaskara and his works represent a significant contribution to mathematical and astronomical knowledge in the 12th century. He has been called the greatest mathematician of medieval India. His main work the *Siddhanta Shiromani*, Sanskrit for "Crown of treatises," is divided into four parts called *Lilavati*, *Bijaganita*, *Grahaganita* and *Goladhyaya*. These four sections deal with arithmetic, algebra, mathematics of the planets and spheres respectively.

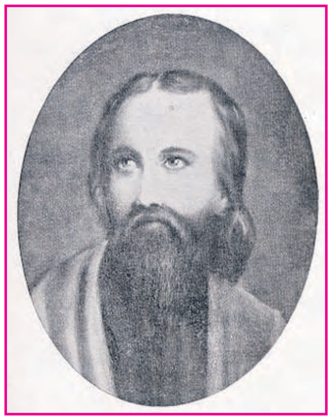
Chapter 8

CONICS

Proof is an idol before whom the pure mathematician tortures himself.
– Arthur Stanely Eddington

In most sciences one generation tears down what another has built and what one has established another undoes. In mathematics alone each generation adds a new story to the old structure.
– Hermann Hankel

8.1 Introduction


Apollonius (262 BC - 190 BC)

We shall study about some special curves, viz., a circle, an ellipse, a parabola and a hyperbola in this chapter. The curves mentioned above can be obtained by taking intersection of a plane with a double napped right circular cone. These curves are called **conic sections** or more commonly **conics**. The names parabola and hyperbola are given by **Apollonius**. He is in fact considered as pioneer of studying such curves. These curves have a very wide range of applications in many fields of physics, optics etc. In the sixteenth century, **Galileo** observed that the path of a projectile is a parabola. This fact is now used in the design of an artillery. In the seventeenth century after prolonged observations **Keplar** gave laws of planetary motion in which it is said that the orbits of the earth and other planets around the sun are ellipses. Afterwards, **Newton** gave theoretical proof of **Keplar's** laws in a more general situation. Now a days a dish antenna for television and for other communication is also designed using the concept of conics. Thus, the study of conics is very important; and it has got applications in mechanics, space science, communication, optics etc. In this chapter we will discuss these curves, their equations and their properties.

8.2 Circle

We know that the set of all points in a plane at the same distance from a fixed point is called a **circle**. The fixed point is called the **centre** and the fixed distance is called the **radius** of the circle.

Cartesian Equation of a Circle Centered at (h, k) and Radius r :

Let the point $C(h, k)$ be the centre of a circle and $P(x, y)$ be any point on the circle. Now since the radius of the circle is given to be r, we get

$$CP = r \Leftrightarrow CP^2 = r^2$$

$$\Leftrightarrow (x - h)^2 + (y - k)^2 = r^2$$

Thus the cartesian equation of a circle with centre $C(h, k)$ and radius r is given by

$$(x - h)^2 + (y - k)^2 = r^2$$

This form of the equation is also called the **centre-radius form of the equation of a circle**.

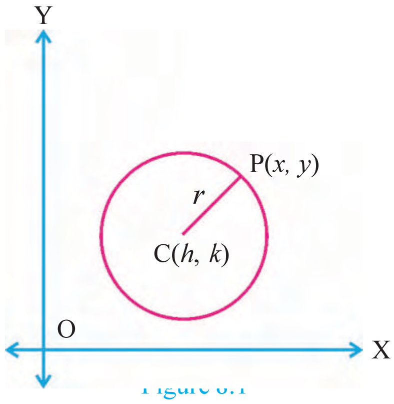


Figure 8.1

8.3 Standard Form of the Equation of a Circle

The **standard equation** of a circle is obtained by taking origin as the centre. Thus in this form of the equation of the circle, centre is the origin and radius is, say r. Hence we put $h = 0$, $k = 0$ in above equation of a circle and get $x^2 + y^2 = r^2$. This form of the equation is called the **standard form of the equation of a circle**.

Further, if radius $r = 1$, the standard form reduces to $x^2 + y^2 = 1$. This is called the **equation of the unit circle**.

Example 1 : Obtain the equation of the circle with centre $(1, -1)$ and radius 2.

Solution : Here the centre is $(1, -1)$ and the radius is 2. So, the equation of the circle is

$$(x - 1)^2 + (y + 1)^2 = 2^2 = 4$$

That is, $x^2 + y^2 - 2x + 2y - 2 = 0$.

Example 2 : Show that the point, $(2\,sin\alpha, 2\,cos\alpha)$; $\alpha \in R$, lies on the circle, $x^2 + y^2 = 4$.

Solution : We know that a point lies on the circle, if the coordinates of the point satisfy the equation of the circle. Substituting $x = 2\,sin\alpha$, $y = 2\,cos\alpha$ in the given equation, we get

L.H.S. $= (2\,sin\alpha)^2 + (2\,cos\alpha)^2$

$= 4\,sin^2\alpha + 4\,cos^2\alpha = 4 =$ R.H.S.

$\therefore$ $(2\,sin\alpha, 2\,cos\alpha)$, $\alpha \in R$ is on the circle $x^2 + y^2 = 4$.

Example 3 : Find the equation of the circle whose radius is 5 and centre is point of intersection of the lines $x + y = 1$ and $4x + 3y = 0$.

Solution : The point of intersection of the lines is the point satisfying both the equations $x + y = 1$ and $4x + 3y = 0$. Solving them $(-3, 4)$ is the centre of the circle.

Also radius is 5. So the equation of the required circle is,

$(x + 3)^2 + (y - 4)^2 = 5^2$, that is $x^2 + y^2 + 6x - 8y = 0$.

> **Note :** If the centre is the point of the intersection of two lines, then the lines contain diameters of the circle.

Example 4 : Find k, if the circle $x^2 + y^2 - 2x + 448y + k = 0$ passes through the origin.

Solution : The circle passes through $(0, 0)$. Substituting $x = 0 = y$ in the equation of the circle $0 + 0 - 0 + 0 + k = 0$. Thus we get $k = 0$.

> **Note :** A circle passes through the origin if and only if the constant term in the equation is equal to zero.

Example 5 : Find the equation of the set of complex numbers $z = x + iy$, so that $|z - z_1| = 5$, where $z_1 = 1 - 2i$.

Solution : We have $|z - z_1| = 5$

$\therefore \quad |z - z_1|^2 = 5^2$

$\therefore \quad |(x + iy) - (1 - 2i)|^2 = 25$

$\therefore \quad |(x - 1) + i(y + 2)|^2 = 25$

$\therefore \quad (x - 1)^2 + (y + 2)^2 = 25 \qquad$ **(i)**

$\therefore \quad x^2 + y^2 - 2x + 4y - 20 = 0$

From (i) it is clear that the set is a circle with centre (1, −2) and radius 5.

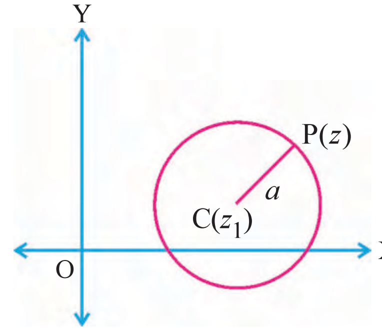


Figure 8.2

Note : In general the set of all complex numbers z satisfying $|z - z_1| = a$, $a \in R^+$ represents a circle with radius a, centered at z_1. The Argand diagram of the given circle is shown in figure 8.2. In fact if C and P represent z_1 and z respectively in the Argand plane and if $CP = |z - z_1| = a$ then P is on the circle with centre C and radius a.

Example 6 : Find the equation of the circles which touch X-axis.

Solution : If the radius of the circle is a, then the coordinates of the centre C are $(h, \pm a)$ or $(-h, \pm a)$ (figure 8.3). The equations of the circles are,

$(x - h)^2 + (y \pm a)^2 = a^2 \qquad$ or

$(x + h)^2 + (y \pm a)^2 = a^2$

$x^2 + y^2 - 2hx \pm 2ay + h^2 = 0 \qquad$ or

$x^2 + y^2 + 2hx \pm 2ay + h^2 = 0$

Thus, these four equations represent required circles.

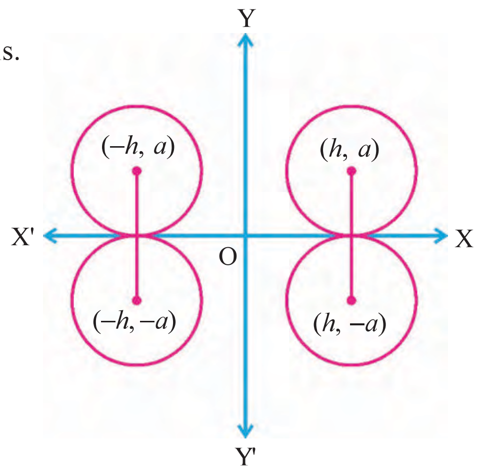


Figure 8.3

Note : If a circle of radius a touches Y-axis (figure 8.4) then its centre will be $(\pm a, k)$ or $(\pm a, -k)$, and hence the equations of such circles are of the form,

$$x^2 + y^2 \pm 2ax + 2ky + k^2 = 0$$

or

$$x^2 + y^2 \pm 2ax - 2ky + k^2 = 0$$

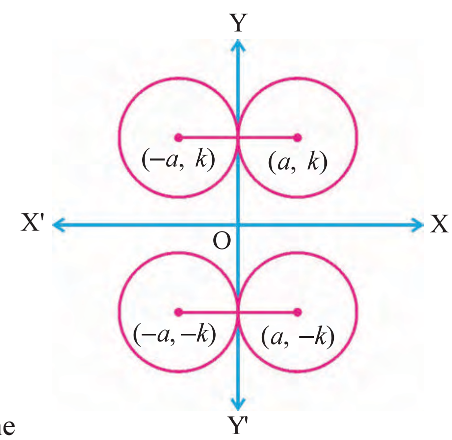


Figure 8.4

Example 7 : Find the equation of the circle with radius a in the first quadrant, if it touches both the axes.

Solution : If the circle touches both the axes in first quadrant, then its centre will be $C(a, a)$ (figure 8.5) and radius a. Hence its equation is $(x - a)^2 + (y - a)^2 = a^2$.

$\therefore \quad x^2 + y^2 - 2ax - 2ay + a^2 = 0$ is the equation.

Note : For the circle of radius a, touching both the axes in the other quadrants, the centre has coordinates as given in the following table. (figure 8.5)

Quadrant	Centre
I	(a, a)
II	$(-a, a)$
III	$(-a, -a)$
IV	$(a, -a)$

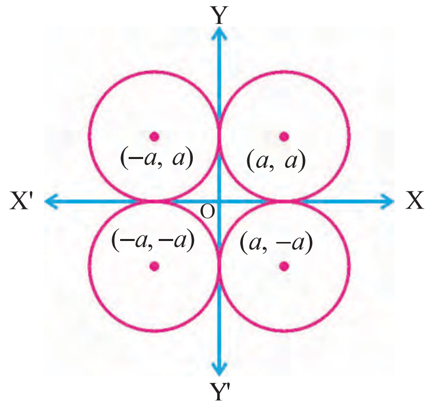


Figure 8.5

Exercise 8.1

1. Find the equation of the circle which radius and centre given below :

No.	Centre	Radius
1.	$(-2, 3)$	5
2.	$(-1, 1)$	$\sqrt{2}$
3.	$(-4\cos\alpha, 4\sin\alpha)$	5
4.	$(-\sqrt{2}, -\sqrt{5})$	$\sqrt{5}$
5.	$(1, 0)$	1

2. Find the equation of the circle for which lines containing the diameters are $x - y = 5$, $2x + y = 4$ and radius is 5.

3. Find the equation of the circle which touches Y-axis and has centre $(-2, -5)$.

4. Find the equation of the circle in the third quadrant having radius 3 and touching both the axes.

5. Find the equation of the circle passing through the origin, having radius $\sqrt{5}$ and having centre on $\overrightarrow{OX}$.

*

8.4 General Form of the Equation of a Circle

As discussed above each circle has unique centre and radius. Let for a circle the centre be (h, k) and radius be r. So, any circle has an equation of the form $(x - h)^2 + (y - k)^2 = r^2$ or $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$. Here h, k are real numbers and r is a positive number. From this equation, we observe the following :

(1) The equation of any circle is a quadratic equation in two variables.

(2) In the equation of a circle, coefficients of x^2 and y^2 are non-zero and equal (We will take these coefficients equal to 1)

(3) There is no xy-term, i.e. the coefficient of the xy-term is 0.

Thus, we take the general form of the equation of a circle in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Now if the equation of a circle is given in the above form, then we wish to determine the centre and radius of the circle. For this we will rearrange the terms, so that the equation reduces to the centre-radius form. Thus,

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{(i)}$$

$$\Leftrightarrow \; x^2 + 2gx + g^2 + y^2 + 2fy + f^2 - g^2 - f^2 + c = 0$$

$$\Leftrightarrow \; (x + g)^2 + (y + f)^2 = g^2 + f^2 - c$$

If $g^2 + f^2 - c > 0$, then above equation can be written as,

$$(x + g)^2 + (y + f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

In fact, above relation tells us that the distance of the point P(x, y) from the point C$(-g, -f)$ is $\sqrt{g^2 + f^2 - c}$.

Thus the equation (i) represents a circle if the constants g, f and c satisfy $g^2 + f^2 - c > 0$; and in this case the centre of the circle is C$(-g, -f)$ and the radius is $\sqrt{g^2 + f^2 - c}$. Equation (i) is called the general form of the equation of a circle.

Note : If $g^2 + f^2 - c = 0$, only $(-g, -f)$ satisfies equation (i).

Example 8 : Does the equation $x^2 + y^2 + 6x - 8y + 20 = 0$ represent a circle ? If yes, find its centre and radius.

Solution : Comparing the equation with the general form, we find that $g = 3, f = -4$ and $c = 20$. Thus, $g^2 + f^2 - c = 3^2 + (-4)^2 - 20 = 5 > 0$.

Hence the given equation represents a circle.

The centre of the circle is $(-g, -f) = (-3, 4)$ and the radius is $\sqrt{g^2 + f^2 - c} = \sqrt{5}$.

Another Method :

Adjusting the terms in the equation to get sum of squares, we write

$$x^2 + y^2 + 6x - 8y + 20 = 0$$

$$\therefore \; x^2 + 6x + 9 + y^2 - 8y + 16 - 5 = 0$$

$$\therefore \; (x + 3)^2 + (y - 4)^2 = 5$$

This is the equation of the circle centered at C$(-3, 4)$ and radius as $r = \sqrt{5}$.

Example 9 : Determine which of the following equations represents a circle. Find the centre and radius for those which represent a circle.

(1) $x^2 + 2y^2 - 2x + 6y - 8 = 0$

(2) $2x^2 + 2y^2 - 2x + 6y - 8 = 0$

(3) $x^2 + y^2 - 2\sqrt{2}x + y - \frac{91}{4} = 0$

(4) $x^2 + y^2 - 2x\cos\beta + 2y\sin\beta = 0; \; \beta \in R$

(5) $2x^2 + 2y^2 - 2xy + 6y + 22x - 1008 = 0$

(6) $x^2 + y^2 - 4x - 6y + 13 = 0$

Solution : (1) In this equation coefficients of x^2 and y^2 are not equal and hence it is not an equation of a circle.

(2) Dividing the equation by 2 gives $x^2 + y^2 - x + 3y - 4 = 0$. This gives $g = -\frac{1}{2}$, $f = \frac{3}{2}$ and $c = -4$. Now, $g^2 + f^2 - c = \left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 - (-4) = \frac{13}{2} > 0$, hence the equation represents a circle with centre $\left(\frac{1}{2}, -\frac{3}{2}\right)$ and radius $\sqrt{\frac{13}{2}}$.

(3) Here, $g = -\sqrt{2}$, $f = \frac{1}{2}$ and $c = -\frac{91}{4}$. Now, $g^2 + f^2 - c = 2 + \frac{1}{4} + \frac{91}{4} = 25 > 0$. Hence the equation represents a circle with centre $\left(\sqrt{2}, -\frac{1}{2}\right)$ and radius 5.

(4) Here, $g = -\cos\beta$, $f = \sin\beta$ and $c = 0$.

Now, $g^2 + f^2 - c = \cos^2\beta + \sin^2\beta = 1 > 0$. Hence the equation represents a circle with centre $(\cos\beta, -\sin\beta)$ and radius 1.

(5) This equation contains a term with xy and hence it does not represent a circle.

(6) Here, $g = -2$, $f = -3$ and $c = 13$. Now, $g^2 + f^2 - c = (-2)^2 + (-3)^2 - 13 = 0$, hence this equation does not represent a circle.

Note : In above examples for equations given in (2), (3), (4) and (6), we can use the method of sum of squares. Also see that if $c < 0$, then $g^2 + f^2 - c$ is always positive. Hence $x^2 + y^2 + 2gx + 2fy + c = 0$ always represents a circle, if $c < 0$.

Example 10 : Find the equation of the circle passing through the points (1, 1) and (−5, 1) and having centre on the line $x + 3y - 1 = 0$.

Solution : Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ **(i)**

We need to determine the values of constants g, f and c using given conditions. The centre of the circle given in (i) is $(-g, -f)$. Now given that the centre of the required circle lies on the line $x + 3y - 1 = 0$, $(-g, -f)$ satisfies the equation of the line and hence, we get

$$-g - 3f - 1 = 0 \quad \text{or} \quad g + 3f + 1 = 0 \qquad \textbf{(ii)}$$

Also using coordinates of given points in (i), we get

$$2g + 2f + c + 2 = 0 \qquad \textbf{(iii)}$$

$$-10g + 2f + c + 26 = 0$$

i.e., $10g - 2f - c - 26 = 0$ **(iv)**

In (ii), (iii) and (iv), we have a system of three linear equations in three unknowns g, f and c.

Using (iii) + (iv), we get $12g - 24 = 0$

$\therefore \quad g = 2$

$\therefore \quad g + 3f + 1 = 0$ gives $f = -1$

Further, $2g + 2f + c + 2 = 0$

$\therefore \quad 4 - 2 + c + 2 = 0$ **(by taking $g = 2$ and $f = -1$)**

$\therefore \quad c = -4$

$\therefore \quad x^2 + y^2 + 4x - 2y - 4 = 0$ is the equation of the required circle.

Example 11 : $A(x_1, y_1)$ and $B(x_2, y_2)$ are given end-points of a diameter of a circle. Find the equation of the circle.

Solution : As shown in the figure 8.6, let $A(x_1, y_1)$ and $B(x_2, y_2)$ be given end-points of a diameter of a circle and $P(x, y)$ be any point on the circle other than A or B. Now as we have studied in standard 10, the angle inscribed in a semi-circle is a right angle. Thus, ΔPAB is a right angled triangle with right angle at P (figure 8.6). According to Pythagoras theorem we have,

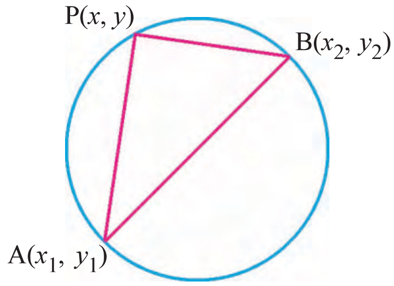


Figure 8.6

$PA^2 + PB^2 = AB^2$

Also we have, $PA^2 = (x - x_1)^2 + (y - y_1)^2$

$PB^2 = (x - x_2)^2 + (y - y_2)^2$

$AB^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$\therefore \quad (x_1 - x_2)^2 + (y_1 - y_2)^2 = (x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2$

$\Leftrightarrow \quad x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2$

$\quad = x^2 - 2xx_1 + x_1^2 + y^2 - 2yy_1 + y_1^2 + x^2 - 2xx_2 + x_2^2 + y^2 - 2yy_2 + y_2^2$

$\Leftrightarrow \quad -2x_1x_2 - 2y_1y_2 = x^2 - 2xx_1 + y^2 - 2yy_1 + x^2 - 2xx_2 + y^2 - 2yy_2$

$\Leftrightarrow \quad 2x^2 + 2y^2 - 2xx_1 - 2yy_1 - 2xx_2 - 2yy_2 + 2x_1x_2 + 2y_1y_2 = 0$

$\Leftrightarrow \quad x^2 + y^2 - xx_1 - yy_1 - xx_2 - yy_2 + x_1x_2 + y_1y_2 = 0$

This equation can also be written as $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ **(i)**

$A(x_1, y_1)$ and $B(x_2, y_2)$ also satisfy equation (i).

Hence equation (i) represents the circle having diameter $\overline{AB}$.

Other Method :

$\overline{AB}$ is diameter. Hence the centre is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Radius is $\sqrt{\left(\frac{x_1 + x_2}{2} - x_1\right)^2 + \left(\frac{y_1 + y_2}{2} - y_1\right)^2} = \frac{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}{2}$

$\therefore$ The equation of the circle is

$$\left(x - \frac{x_1 + x_2}{2}\right)^2 + \left(y - \frac{y_1 + y_2}{2}\right)^2 = \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{4}$$

$\Leftrightarrow \quad x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + \frac{(x_1 + x_2)^2 + (y_1 + y_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2}{4} = 0$

$\Leftrightarrow \quad x_1^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + x_1x_2 + y_1y_2 = 0$

$\Leftrightarrow \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

This is called the diameter form of the equation of a circle.

Note : $\overline{AP}$ and $\overline{PB}$ are perpendicular, hence product of their slopes is equal to -1. Using this also the equation of circle having diameter $\overline{AB}$ can be derived.

Exercise 8.2

1. Which of the following equations represents a circle ? Find the centre and the radius if the equation represents a circle :

(1) $x - y + 4 = 0$ (2) $x^2 + y^2 = 1$

(3) $x^2 + y^2 - 2x - 2y + 1 = 0$ (4) $x^2 - y^2 - 2x + 2y = 1008$

(5) $x^2 + 3y^2 - 6x + 8y = 0$ (6) $3x^2 + 3y^2 - 5x + 6y + 8 = 0$

(7) $x^2 + y^2 - x + y = 0$ (8) $9x^2 - 6x + 9y - 35 = 0$

(9) $x^2 + y^2 - 2x\,tan\alpha + 2y\,sec\alpha + 2\,tan^2\alpha = 0; \left(\alpha \in R, \alpha \neq \frac{(2n+1)\pi}{2}; n \in Z\right)$

(10) $x^2 + y^2 - 2xy\,tan\alpha + 2y\,sec\alpha + 2\,tan^2\alpha = 0; \alpha \in \left[0, \frac{\pi}{2}\right)$

2. Find the equation of the circle with centre (3, 4) and passing through the origin.

3. Find the equation of the circle which passes through the point (2, −1) and whose centre lies on both the lines $x + y = 5$ and $4x + y = 5$.

4. Obtain the equation of the circle that touches both the axes and passes through the point (−6, 3).

5. Show that the centres of the circles $x^2 + y^2 - 4x - 2y + 4 = 0$, $x^2 + y^2 - 2x - 4y + 1 = 0$ and $x^2 + y^2 + 2x - 8y + 1 = 0$ are collinear. Also show that their radii are in G.P.

6. Obtain the equation of the circle for given extremities of diameter using slopes of line segments.

*

8.5 Eccentricity

Geometric Definition of Conics : The set of points whose distance from a fixed point and whose perpendicular distance from a fixed line not passing through the given point are in a constant ratio is called a conic. The fixed point is called a focus of the conic and the fixed line is called a directrix of the conic. This constant ratio is called eccentricity of the conic and it is represented by symbol *e*.

8.6 Parabola

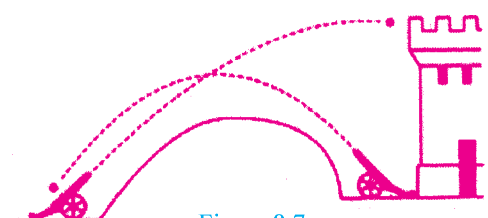


Figure 8.7

In 17th century, Galieo discovered that when an object, say a stone, is thrown in the air, then it follows parabolic path. Here 'para' means 'for' and bola means 'throwing'. Hence the name parabola. This discovery by Galileo made it possible for cannoneers to work out the kind of path a cannonball would travel if it were hurtled through the air at a specific angle. A formal definition of a parabola is as follows :

Definition : A parabola is the set of all points in a plane which are equidistant from a fixed line and a fixed point (not on the line) in the plane (figure 8.8). Here by the distance of a point from the line, we mean its perpendicular distance.

Points P_1, P_2 and P_3 are shown on the parabola.

By definition, $B_1P_1 = SP_1$, $B_2P_2 = SP_2$, $B_3P_3 = SP_3$. Similarly for all points on the parabola.

Let S be a fixed point and let l be a fixed line not passing through S. S is called the focus and l the directrix of the parabola.

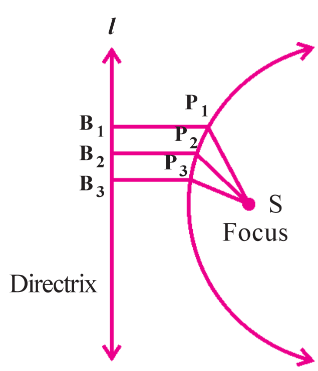


Figure 8.8

If P is a point on a parabola and perpendicular distance of P from the directrix is PM, then according to definition of parabola SP = PM.

$\therefore \quad \frac{SP}{PM} = 1$

$\therefore$ According to definition of eccentricity of a conic, parabola is a conic with eccentricity 1.

Let Z be the foot of the perpendicular from S to the line l. Let A be mid-point of $\overline{ZS}$. Thus SA = AZ and $\overline{ZS}$ is perpendicular to the directrix l. Hence A lies on parabola.

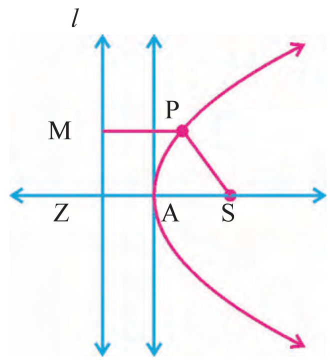


Figure 8.9

Choose A as origin, $\overleftrightarrow{AS}$ as X-axis and direction of $\overrightarrow{AS}$ as positive direction of the X-axis. Further, we take distance ZS = $2a$. Then S would be $(a, 0)$ and $Z(-a, 0)$. The equation of the directrix l would be $x = -a$.

Let $P(x, y)$ be any point on the parabola and let M be the foot of the perpendicular drawn from P to the directrix l. Then the coordinates of M are $(-a, y)$. Now as the point P is on the parabola,

SP = PM

$\therefore \quad SP^2 = PM^2$

$\therefore \quad (x - a)^2 + y^2 = (x + a)^2$

$\therefore \quad y^2 = (x + a)^2 - (x - a)^2$

$\therefore \quad y^2 = 4ax$ **(i)**

If a point $P(x, y)$ satisfies the equation $y^2 = 4ax$, then by taking above steps in the reverse order, we get SP = PM, i.e. the point P is on the parabola.

$\therefore$ The equation of the parabola is $y^2 = 4ax$.

This equation is called the standard form of the equation of a parabola.

Example 12 : Find the standard equation of the parabola whose focus is (4, 0) and equation of the directrix is $x + 4 = 0$.

Solution : As the focus is at (4, 0) and the directrix is $x + 4 = 0$, $a = 4$. The equation of the parabola is, $y^2 = 4(4)x$.

$\therefore \quad y^2 = 16x$ is the equation of the parabola.

8.7 Some definitions and results related to parabola

(1) **The axis of a parabola is defined as the line passing through the focus and perpendicular to the directrix. Accordingly for the parabola $y^2 = 4ax$, X-axis is axis of the parabola.**

(2) **The point of intersection of the parabola with its axis is called the vertex of the parabola. For $y^2 = 4ax$, the origin is the vertex.**

(3) **If the axis of the parabola is chosen as Y-axis and vertex at the origin, then the equation of the parabola comes out to be $x^2 = 4by$ (figure 8.10(ii)).** In this case focus is $(0, b)$ and the equation of the directrix is $y = -b$. Here $| b |$ is the distance between the vertex of the parabola and its focus.

(4) If the point P(x, y) is on the parabola $y^2 = 4ax$, then P(x, $-y$) is also a point on the parabola. Thus the parabola $y^2 = 4ax$ is symmetric about X-axis. (i.e. replacing y by $-y$ there is no change in the equation.) Similarly, the parabola $x^2 = 4by$ is symmetric about Y-axis. (i.e. replacing x by $-x$, there is no change in the equation.)

(5) Parabolas $y^2 = 4ax$ and $x^2 = 4by$ are shown in the figures ($a \neq 0$, $b \neq 0$).

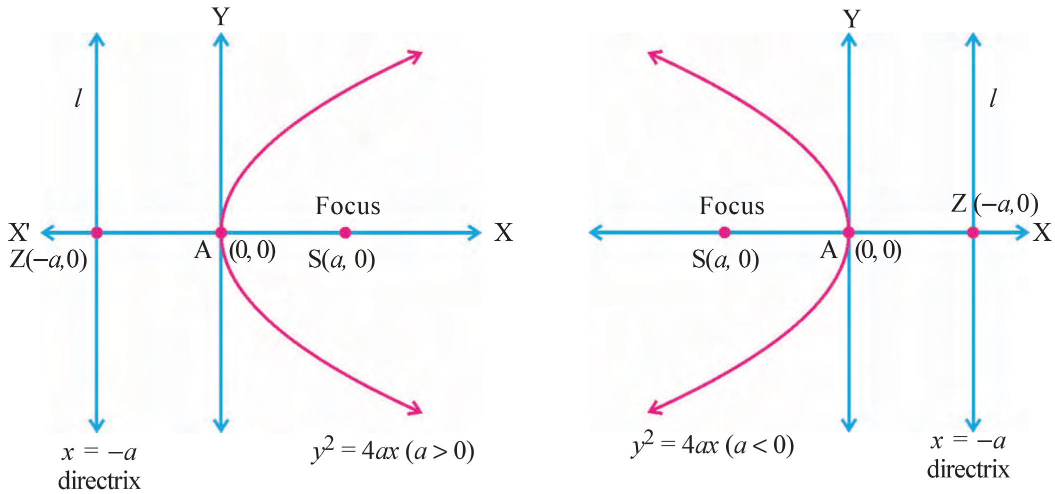


Figure 8.10 (i)

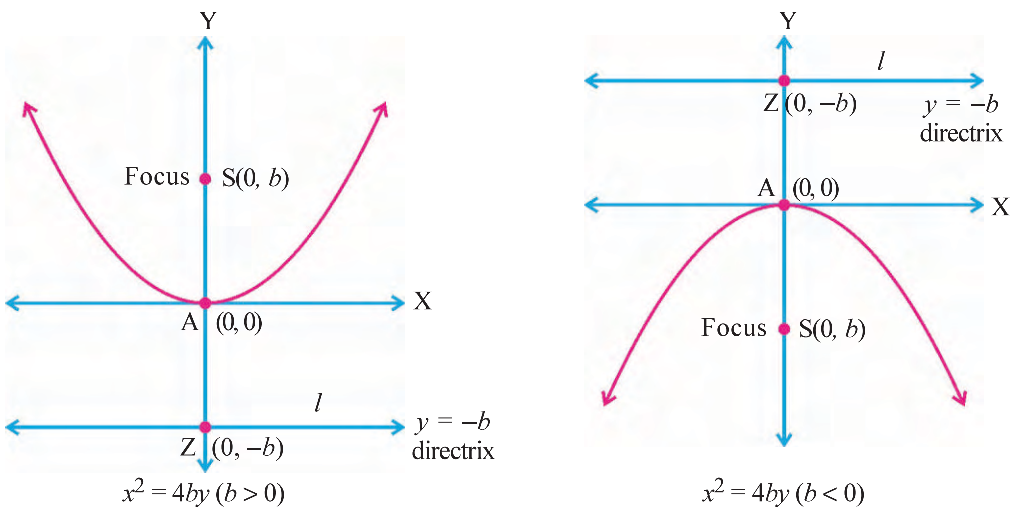


Figure 8.10(ii)

(6) **The line segment joining any two points of a parabola is called a chord of parabola. A chord passing through the focus is called a focal-chord. A focal-chord perpendicular to the axis, (or equivalently, parallel to the directrix), is called the latus-rectum of the parabola.**

8.8 Latus-rectum of a parabola

Let the end-points of the latus-rectum of a parabola $y^2 = 4ax$ be L and L'. Hence the line $\overleftrightarrow{LL'}$ is a vertical line. Since it passes through the focus (a, 0), its equation is $x = a$. Now as the points L and L' are on parabola also, using equation of the parabola we get, $y^2 = 4ax = 4a \cdot a = 4a^2$, which means $y = \pm 2a$. Thus, the coordinates of the end-points of the latus-rectum are L(a, $2|a|$) and L'(a, $-2|a|$). The length of the latus-rectum is the distance LL'.

Length of the latus-rectum $= LL'$

$$= \sqrt{(a-a)^2 + (2|a| + 2|a|)^2}$$

$$= 4\,|\,a\,|$$

Note : For the parabola $x^2 = 4by$, the end-points of the latus-rectum would be $L(2|b|, b)$ and $L'(-2|b|, b)$; and hence the length of the latus-rectum would be $4\,|\,b\,|$.

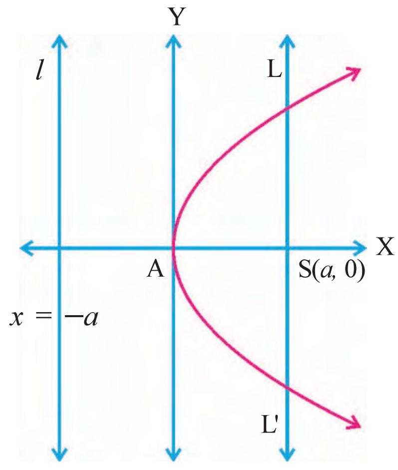


Figure 8.11

8.9 Parametric equations of a parabola

For any real parameter t, $x = at^2$ and $y = 2at$ satisfy the equation $y^2 = 4ax$. Conversely, suppose a point (x_1, y_1) lies on the parabola $y^2 = 4ax$. Let $t = \frac{y_1}{2a}$, then $x_1 = at^2$. In other words corresponding to any point on the parabola $y^2 = 4ax$, there exists a real number t such that $x = at^2$ and $y = 2at$.

Thus, $(at^2, 2at)$ is a point on the parabola $y^2 = 4ax$ and any point on $y^2 = 4ax$ is of the form $(at^2, 2at)$ for some $t \in R$.

$x = at^2$, $y = 2at$ are called parametric equations of the parabola $y^2 = 4ax$. The point $P(at^2, 2at)$ is called a t-point of the parabola and it is denoted by $P(t)$.

Example 13 : Find the equation of the parabola whose focus is $(2, 3)$ and directrix is $3x + 4y - 10 = 0$.

Solution : Let $P(x, y)$ be a point on the parabola. Now by the definition of a parabola, if focus is S and PM is perpendicular distance of P from the directrix, then

$SP = PM$, i.e. $SP^2 = PM^2$

$$\therefore \quad (x-2)^2 + (y-3)^2 = \left(\frac{3x+4y-10}{\sqrt{9+16}}\right)^2 = \frac{(3x+4y-10)^2}{25}$$

$$\therefore \quad 25(x^2 - 4x + 4 + y^2 - 6y + 9) = 9x^2 + 16y^2 + 24xy - 60x - 80y + 100$$

$\therefore \quad 16x^2 - 24xy + 9y^2 - 40x - 70y + 225 = 0$ is the equation of the required parabola.

Example 14 : By shifting the origin to $(4, 3)$ find the coordinates of the focus and the equation of the directrix for the parabola $(y-3)^2 = 16(x-4)$.

Solution : Let (x', y') be coordinates of $P(x, y)$ with respect to the new origin, then

$x = x' + h = x' + 4, \quad y = y' + k = y' + 3$

$\therefore$ The equation of parabola becomes $(y')^2 = 16x'$.

$\therefore \quad 4a = 16$, i.e. $a = 4$

New coordinates of the focus (x', y') are $(a, 0)$ i.e. $(4, 0)$.

Now, $x = x' + 4, \quad y = y' + 3$

$\therefore$ Original coordinates of the focus are $(8, 3)$.

∴ The equation of the directrix in new coordinate system : $x' + a = 0$

$x' + 4 = 0$

∴ Its equation is $x - 4 + 4 = 0$.

∴ The equation of the directrix is $x = 0$.

Verification : From SP = PM, $(x - 8)^2 + (y - 3)^2 = x^2$,

∴ $(y - 3)^2 = x^2 - (x^2 - 16x + 64) = 16(x - 4)$.

Example 15 : For each equation given below, find the coordinates of the focus, the equation of the directrix, length of the latus-rectum and coordinates of the end-points of the latus-rectum for the parabola :

(1) $x^2 = -8y$ (2) $y^2 = 8x$ (3) $x^2 = 3y$ (4) $y^2 = -10x$

Solution : (1) Comparing $x^2 = -8y$ with the standard equation $x^2 = 4by$ we get, $4b = -8$. So $b = -2$. Here the axis of the parabola is Y-axis. Hence the focus is $(0, b) = (0, -2)$.

The equation of the directrix is $y = -b$, so $y = 2$ is the equation of the directrix.

The length of the latus-rectum is $4\,|\,b\,| = 8$.

The end-points of the latus-rectum are $L(2|\,b\,|, b) = L(4, -2)$ and $L'(-2|\,b\,|, b) = L'(-4, -2)$.

(2) Comparing $y^2 = 8x$ with the equation $y^2 = 4ax$, we get $a = 2$. Here the axis of the parabola is X-axis.

Focus is $(a, 0) = (2, 0)$.

The equation of the directrix is $x = -a$, i.e. $x = -2$ or $x + 2 = 0$ is the equation of the directrix.

The length of the latus-rectum is $4\,|\,a\,| = 8$.

The end-points of the latus-rectum are $L(a, 2|\,a\,|) = L(2, 4)$ and $L'(a, -2|\,a\,|) = L'(2, -4)$.

(3) Comparing $x^2 = 3y$ with the standard equation $x^2 = 4by$ we get, $4b = 3$, i.e. $b = \frac{3}{4}$. Here the axis of the parabola is Y-axis.

Focus is $(0, b) = \left(0, \frac{3}{4}\right)$.

The equation of the directrix is $y = -b$, so $y = -\frac{3}{4}$ or $4y + 3 = 0$ is the equation of the directrix.

The length of the latus-rectum is $4\,|\,b\,| = 3$.

The end-points of the latus-rectum are $L(2|\,b\,|, b) = L\left(\frac{3}{2}, \frac{3}{4}\right)$ and $L'(-2|\,b\,|, b) = L'\left(-\frac{3}{2}, \frac{3}{4}\right)$.

(4) Comparing $y^2 = -10x$ with the equation $y^2 = 4ax$, we get $a = -\frac{5}{2}$. Here the axis of the parabola is X-axis.

Focus is $(a, 0) = \left(-\frac{5}{2}, 0\right)$.

The equation of the directrix is $x = -a$. So $x = \frac{5}{2}$ or $2x - 5 = 0$ is the equation of the dirextrix.

The length of the latus-rectum is $4\,|\,a\,| = 10$.

The end-points of the latus-rectum are $L(a, 2|\,a\,|) = L\left(-\frac{5}{2}, 5\right)$ and

$L'(a, -2|\,a\,|) = L'\left(-\frac{5}{2}, -5\right)$.

Example 16 : Find the standard equation of the parabola having vertex at origin, focus at $(0, -3)$ and directrix $y = 3$.

Solution : Here focus $(0, -3)$ is on Y-axis and the directrix $y = 3$ is parallel to X-axis. Thus the equation of the parabola is of the form $x^2 = 4by$, with $b = -3$. Thus the equation of the required parabola is $x^2 = -12y$.

Example 17 : Find the standard equation of the parabola symmetric about X-axis, vertex at origin and passing through $(5, -5)$.

Solution : Given that the parabola is symmetric about X-axis and vertex is at origin. Hence the standard form of the equation is $y^2 = 4ax$. Further since the parabola passes through the point $(5, -5)$, we get $(-5)^2 = 4a(5)$

$\therefore \quad 25 = 20a$

$\therefore \quad a = \frac{5}{4}$

Hence the equation of the parabola is $y^2 = 5x$.

8.10 Properties of a Parabola

Property 1 : Let $P(t_1)$ and $Q(t_2)$ be two points on a parabola $y^2 = 4ax$. If $\overline{PQ}$ is a focal chord, then $t_1t_2 = -1$.

Proof : If $\overline{PQ}$ is the latus-rectum, then $P(a, 2a)$.

$\therefore \quad at_1^2 = a, \quad 2at_1 = 2a$

$\therefore \quad t_1 = 1$

Similarly for $Q(a, -2a)$, $t_2 = -1$

$\therefore \quad t_1t_2 = -1$

Now, suppose $\overline{PQ}$ is not the latus-rectum.

$\therefore \quad at_1^2 \neq a, \quad at_2^2 \neq a.$

Now, slope of $\overleftrightarrow{SP}$ = slope of $\overleftrightarrow{SQ}$

$$\therefore \quad \frac{2at_1}{at_1^2 - a} = \frac{2at_2}{at_2^2 - a}$$

$$\therefore \quad \frac{t_1}{t_1^2 - 1} = \frac{t_2}{t_2^2 - 1}$$

$\therefore \quad t_1(t_2^2 - 1) = t_2(t_1^2 - 1)$

$\therefore \quad t_1t_2^2 - t_1 = t_2t_1^2 - t_2$

$\therefore \quad t_1t_2^2 - t_2t_1^2 = t_1 - t_2$

$\therefore \quad t_1t_2\,(t_2 - t_1) = -(t_2 - t_1)$

$\therefore \quad t_1t_2 = -1 \qquad (\because t_1 \neq t_2)$

Property 2 : Let S be the focus of the parabola $y^2 = 4ax$, $(a > 0)$ and $\overline{PQ}$ be a focal chord. Then $\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$.

Proof : Let $P(t_1)$ and $Q(t_2)$ be the end-points of a focal chord. The coordinates of the focus are $(a, 0)$. The coordinates of the points P and Q are $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ respectively.

$$SP^2 = (at_1^2 - a)^2 + (2at_1)^2$$
$$= (at_1^2 - a)^2 + 4a^2t_1^2$$
$$= (at_1^2 + a)^2$$

$\therefore$ $SP = a(t_1^2 + 1)$ Similarly, $SQ = a(t_2^2 + 1)$ **$(a > 0)$**

Now, $$\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a(t_1^2+1)} + \frac{1}{a(t_2^2+1)}$$
$$= \frac{1+t_1^2+t_2^2+1}{a(t_1^2+1)(t_2^2+1)}$$
$$= \frac{1+t_1^2+t_2^2+t_1^2t_2^2}{a(t_1^2+1)(t_2^2+1)} \qquad (t_1t_2 = -1)$$
$$= \frac{(1+t_1^2)(1+t_2^2)}{a(t_1^2+1)(t_2^2+1)} = \frac{1}{a}$$

Property 3 : Let P be any point on a parabola and S be the focus of the parabola $y^2 = 4ax$. Let $\overleftrightarrow{PQ}$ be a line parallel to the axis of the parabola. Let the bisector of the angle $\angle SPQ$ intersect axis of the parabola in point G. Then $\overline{SP} \cong \overline{SG}$.

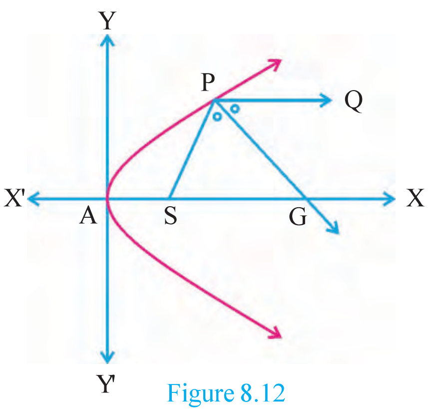


Figure 8.12

Proof : Here $\overleftrightarrow{PQ}$ is parallel to the axis of the parabola, that is to X-axis. Also $\overrightarrow{PG}$ is the bisector of $\angle SPQ$. Thus as shown in the figure 8.12, $\overleftrightarrow{PG}$ is a transversal to parallel lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{SG}$. Hence $m\angle SGP = m\angle QPG$. Also, $m\angle SPG = m\angle QPG$. Thus, $m\angle SGP = m\angle SPG$. Hence ΔSPG is an isosceles triangle with $\overline{SP} \cong \overline{SG}$.

Note : This property has some applications in optics, for designing the mirrors. If a light source is placed at the focus of a parabolic mirror, then the light will travel parallel to the axis of the mirror. This fact is used in head-light of a vehicle, whereas any light ray which is parallel to axis of a parabolic mirror is reflected in to the focus. This is used in a dish antenna of television.

Exercise 8.3

1. Obtain the coordinates of foci, the equations of directrices and draw a rough sketch for following parabola :

(1) $2y^2 = x$ (2) $x^2 = -4y$ (3) $4x^2 = -y$ (4) $y^2 = 12x$

2. Find the standard equation of the parabola satisfying conditions given below :

(1) Vertex (0, 0), focus (0, −2).

(2) Vertex (0, 0), X-axis as axis of the parabola and pasing through (1, −4).

3. (1) Find the equation of the parabola whose focus is (−1, 2) and directrix is $x - y + 1 = 0$.

(2) Find the equation of the parabola whose focus is (−3, −4) and directrix is $3x - 4y - 5 = 0$.

4. Find the length of the latus-rectum and the equation of the directrix of the parabola $(x + 1)^2 = 4(y + 2)$ by shifting the origin to (−1, −2).

5. Find the area of the triangle formed by the end-points of the latus-rectum and the vertex of the parabola $x^2 = 12y$.

6. One end-point of a focal-chord of the parabola $y^2 = 4ax$ is $(at_1^2, 2at_1)$, find its other end-point. From this show that the length of the focal-chord is $\left(t_1 + \frac{1}{t_1}\right)^2$.

7. Distance SP of a point P on the parabola $y^2 = 12x$ from its focus S is 6 units. Find the coordinates of the point P.

*

8.11 Ellipse

Any cylinder sliced at an angle will reveal an ellipse in cross-section. To demonstrate this tilt a glass of water and the surface of the liquid acquires an elliptical outline (figure 8.14). Also in salads, cucumber is often cut obliquely to obtain elliptical slices.

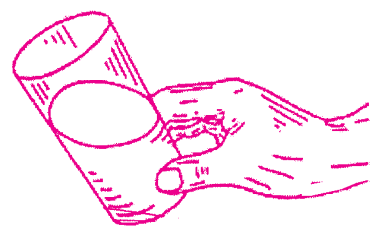


Figure 8.13

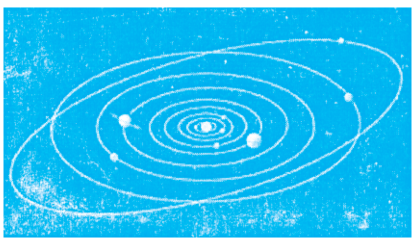


Figure 8.14

The early Greek astronomers thought that the planets moved in circular orbits about the unmoving earth, since the circle is the simplest mathematical curve. In the 17th century, Johannes Kepler eventually discovered that each planet travels around the sun in an elliptical orbit with the sun at one of its foci.

We have discussed eccentricity of a conic section. A conic section with $e < 1$ is called an ellipse.

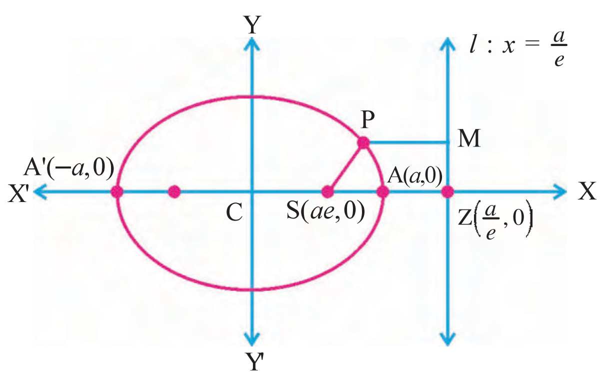


Figure 8.15

Standard equation of an ellipse :

Suppose S is the focus, l is the directrix and e is the eccentricity of an ellipse. Let P be a point on the ellipse. Let M be the foot of perpendicular from P to l.