Chapter 2

Inverse Trigonometric Functions

Basics Concepts: Inverse Trigonometric Functions

What is an Inverse Function?

- A function accepts values, performs particular operations on these values, and generates an output. The inverse function agrees with the resultant, operates, and reaches back to the original function.
- If y=f(x) and x=g(y) are two functions such that f (g(y)) = y and g (f(y)) = x, then f and y are said to be inverse of each other.
  Example: If f(x) = 2x + 5 = y, then, g(y) = (y-5)/2 = x is the inverse of f(x).
- The inverse of f is denoted by f^{-1}

Inverse Trigonometric Functions

- Trigonometric functions are **many-one functions** but we know that inverse of function exists if the function is bijective.
- If we restrict the domain of trigonometric functions, then these functions become bijective and the inverse of trigonometric functions are defined within the restricted domain.
- **Example:** y = f(x) = sin x, then its inverse is x = sin-1 y.

Inverse Trigonometric Formulas

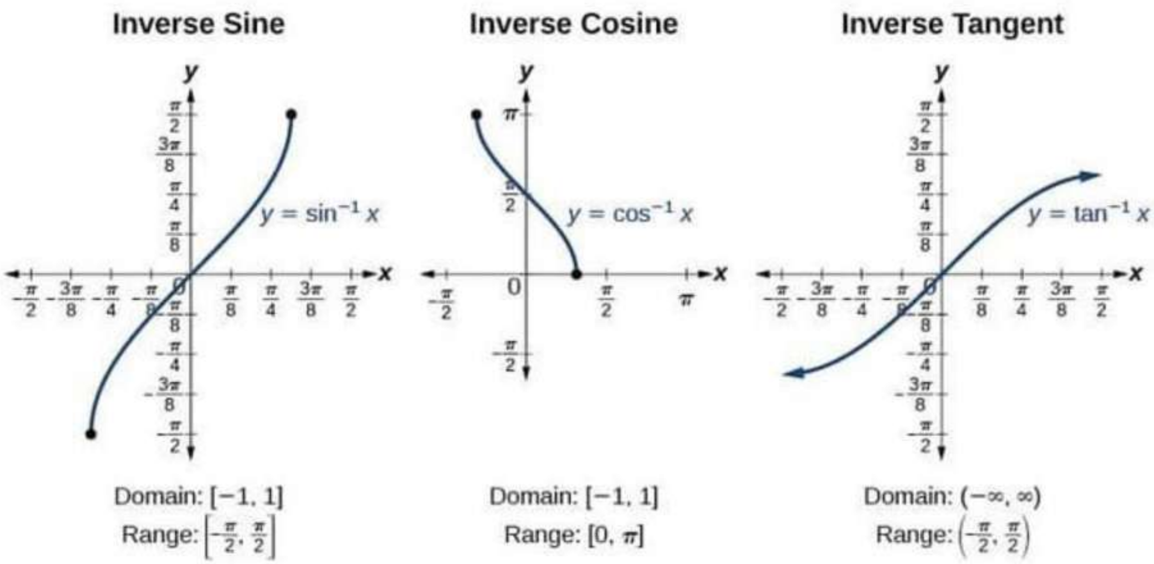


Fig: Inverse Trigonometric Functions

The inverse trigonometric functions are the inverse functions of the trigonometric functions written as $\cos^{-1} x$, $\sin^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\operatorname{cosec}^{-1} x$, $\sec^{-1} x$.

The inverse trigonometric functions are multivalued. For example, there are multiple values of ω such that $z = \sin\omega$, so $\sin^{-1}z$ is not uniquely defined unless a principal value is defined.

Such principal values are sometimes denoted with a capital letter so, for example, the principal value of the inverse sine may be variously denoted $\sin^{-1}z$ or arcsinz.

Let's say, if $y = \sin x$, then $x = \sin^{-1} y$, similarly for other trigonometric functions. This is one of the inverse trigonometric formulas. Now, $y = \sin^{-1} (x)$, $y \in [\pi/2 , \pi/2]$ and $x \in [-1,1]$.

- Thus, $\sin^{-1} x$ has infinitely many values for given $x \in [-1, 1]$.
- There is only one value among these values which lies in the interval $[\pi/2, \pi/2]$. This value is called the principal value.

Domain and Range of Inverse Trigonometric Formulas

Function	Domain	Range
$\sin^{-1}x$	[-1,1]	$[-\pi/2,\pi/2]$
$\cos^{-1}x$	[-1,1]	$[0,\pi]$

$\tan^{-1}x$	R	$(-\pi/2,\pi/2)$
$\cot^{-1}x$	R	$(0,\pi)$
$\sec^{-1}x$	R-(-1,1)	$[0,\pi]-\{\pi/2\}$
$\text{cosec}^{-1}x$	R-(-1,1)	$[-\pi/2,\pi/2]-\{0\}$

Solved Examples

Ques 1: Find the exact value of each expression without a calculator, in $[0,2\pi)$.

1. $\sin^{-1}(-3\sqrt{2})$
2. $\cos^{-1}(-2\sqrt{2})$
3. $\tan^{-1}\sqrt{3}$

Ans:

- Recall that $-3\sqrt{2}$ is from the $30-60-90$ triangle. The reference angle for sin and $3\sqrt{2}$ would be $60\circ$. Because this is sine and it is negative, it must be in the third or fourth quadrant. The answer is either $4\pi/3$ or $5\pi/3$.
- $-2\sqrt{2}$is from an isosceles right triangle. The reference angle is then $45\circ$. Because this is cosine and negative, the angle must be in either the second or third quadrant. The answer is either $3\pi/4$ or $5\pi/4$.
- $\sqrt{3}$is also from a $30-60-90$ triangle. Tangent is $\sqrt{3}$ for the reference angle $60\circ$. Tangent is positive in the first and third quadrants, so the answer would be $\pi/3$ or $4\pi/3$.

Notice how each one of these examples yields two answers. This poses a problem when finding a singular inverse for each of the trig functions. Therefore, we need to restrict the domain in which the inverses can be found.

Ques 2: Find the value of $\tan^{-1}(1.1106)$.

Ans: Let $A=\tan^{-1}(1.1106)$
Then, $\tan A = 1.1106$
$A = 48°$
$\tan 48 = 1.1106$

[Use calculator in degree mode]
$\tan^{-1} 1.1106 = 48°$

Properties of Inverse

Here are the properties of the inverse trigonometric functions with proof.

Property 1

1. $\sin^{-1}(1/x) = \text{cosec}^{-1}x$, $x \geq 1$ or $x \leq -1$
2. $\cos^{-1}(1/x) = \sec^{-1}x$, $x \geq 1$ or $x \leq -1$
3. $\tan^{-1}(1/x) = \cot^{-1}x$, $x > 0$

Proof : $\sin^{-1}(1/x) = \text{cosec}^{-1}x$, $x \geq 1$ or $x \leq -1$,

Let $\sin^{-1}x = y$

i.e. $x = \text{cosec } y$

$1/\pi = \sin y$

$$\sin^{-1} \frac{1}{x}) = y$$
$$\sin^{-1} \frac{1}{x}) = cosec^{-1}x$$
$$\sin^{-1}(\frac{1}{x}) = cosec^{-1}x$$
Hence, $\sin^{-1} \frac{1}{x} = cosec^{-1}x$ where, $x \geq 1$ or $x \leq -1$.

Property 2

1. $\sin^{-1}(-x) = -\sin^{-1}(x)$, $x \in [-1,1]$
2. $\tan^{-1}(-x) = -\tan^{-1}(x)$, $x \in R$
3. $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}(x)$, $|x| \geq 1$

Proof: $\sin^{-1}(-x) = -\sin^{-1}(x)$, $x \in [-1,1]$
Let, $\text{Sin}^{-1}(-x) = y$

Then $-x = \sin y$

$x = -\sin y$

$x = \sin(-y)$

$\sin^{-1}=\sin^{-1}(\sin(-y))$

$\sin^{-1}x=y$

$\sin^{-1}x=-\sin^{-1}(-x)$

Hence, $\sin^{-1}(-x)=-\sin^{-1} x \in [-1,1]$

Property 3

1. $\cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1,1]$
2. $\sec^{-1}(-x) = \pi - \sec^{-1}x, |x| \geq 1$
3. $\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in R$

Proof : $\cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1,1]$

Let $\cos^{-1}(-x)=y$

$\cos y=-x \quad x=-\cos y$

$x=\cos(\pi-y)$

Since, $\cos \pi-q=-\cos q$

$\cos^{-1} x=\pi-y$

$\cos^{-1} x=\pi-\cos-1-x$

Hence, $\cos^{-1}-x=\pi-\cos^{-1}x$

Solved Example

Ques 1: Prove that "$\sin^{-1}(-x) = -\sin^{-1}(x), \quad x \in [-1,1]$"

Ans: Let, $\sin^{-1}(-x)=y$

Then $-x=\sin y$

$x=-\sin y$

$x=\sin(-y)$

$\sin^{-1} x=\arcsin(\sin(-y))$

$\sin^{-1} x = y$

$\sin^{-1} x = -\sin^{-1}(-x)$

Hence, $\sin^{-1}(-x) = -\sin^{-1} x$, $x \in [-1,1]$

This concludes our discussion on the topic of trigonometric inverse functions.

Ques 2: $\sin^{-1}(\cos \pi/3)=?$

Ans: $\sin^{-1}\left(\cos \frac{\pi}{3}\right) = \sin^{-1} \frac{1}{2}$ [substitute $\cos(\pi/3)=1/2$]

$= \pi/6$ [substitute $\sin^{-1}(1/2) = \pi/6$]

Ques 3: Find the value of $\sin(\pi/4+\text{Cos}^{-1}(\sqrt{2}/2))$.

Ans:

Let y=sin $\left(\frac{\pi}{4} + cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$ and $A = cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$
Then, $\cos A = \sqrt{2}/2$
Multiplying the numerator as well as denominator by $\sqrt{2}$ we get:

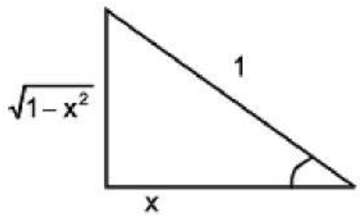
$\cos A = 1/\sqrt{2}$
$A = \pi/4$
Therefore
$y = \sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$
$y = \sin(\pi/2)$
hence, y=1.

Solved Examples: Inverse Trignometric Functions

Equations Involving Inverse Trigonometric Functions

Ex.1 Solve $\cos^{-1} x\sqrt{3} + \cos^{-1} x = \pi/2$.

Sol.



Given $\cos^{-1} x\sqrt{3} + \cos^{-1}x = \pi/2$. ...(1)

or $\cos^{-1} x\sqrt{3} = \pi/2 - \cos^{-1} x$

or $\cos \cos^{-1} x\sqrt{3} = \cos (\pi/2 - \cos^{-1}x)$

or $x\sqrt{3} = \sin \cos^{-1}x$ or

$x\sqrt{3} = \sin \sin^{-1} \sqrt{1-x^2}$

or $x\sqrt{3} = \sqrt{1-x^2}$

Squaring we get $3x^2 = 1 - x^2$ or $4x^2 = 1 = x = \pm 1/2$

Verification : When $x = 1/2$

L.H.S. of equation = $\cos1 (3 /2) + \cos^{-1} (1/2) = \pi/6 + \pi/3 + \pi/2$ = R.H.S. of equation

When $x = -1/2$

L.H.S. of equation = $\cos^{-1} (- 3 /2) + \cos^{-1} (-1/2) = \pi - \cos^{-1} (3 /2) + \pi - \cos^{-1} (1/2)$

$= \pi - \pi/6 + \pi - \pi/3 = 3p/2 \neq$ R.H.S. of equation

$\therefore x = 1/2$ is the only solution

Ex.2 Solve for x : $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$.

Sol.

We have $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$

$\Rightarrow \quad (\tan^{-1} x + \cot^{-1} x)^2 - 2 \tan^{-1} x . \cot^{-1} x = \frac{5\pi^2}{8}$

$\Rightarrow \quad \frac{\pi^2}{4} - 2. \frac{\pi}{2} . \tan^{-1} x + 2 (\tan^{-1} x)^2 = \frac{5\pi^2}{8}$

$\Rightarrow \left(\frac{\pi}{2}\right)^2 - 2 \tan^{-1} x . (\pi/2 - \tan^{-1} x) = \frac{5\pi^2}{8}$

$\Rightarrow 2 (\tan^{-1} x)^2 - \pi \tan^{-1} x - \frac{3\pi^2}{8} = 0$

$\tan^{-1} x = -\pi/4, 3\pi/4 = \tan^{-1} x = -\pi/4;\ x = -1,$ {neglecting $\tan^{-1} x = 3\pi/4$ as $\tan^{-1} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ }

Ex.3 Determine the integral values of ' k ' for which the system , $(\text{arc} \tan x)^2 + (\text{arc} \cos y)^2 = \pi^2 k$ and $\tan^{-1} x + \cos^{-1} y = \pi/2$ posses solution and find all the solutions.

Sol.

$\left.\begin{matrix} 0 \le (\tan^{-1} x)^2 \le \frac{\pi^2}{4} \\ 0 \le (\cos^{-1} y)^2 \le \pi^2 \end{matrix}\right] \quad \Rightarrow$

$\Rightarrow \quad (\tan^{-1} x)^2 + (\cos^{-1} x)^2 \le \frac{5\pi^2}{4}$

hence $\quad k\pi^2 \le \frac{5\pi^2}{4} \Rightarrow k \le \frac{5}{4} \qquad ...(1)$

Now put $\quad \tan^{-1} x = \frac{\pi}{2} - \cos^{-1} y$

$\Rightarrow \left(\frac{\pi}{2} - \cos^{-1} y\right)^2 + (\cos^{-1} y)^2 = \pi^2 k$

$\Rightarrow \quad 2t^2 - \pi t + \left(\frac{\pi^2}{4} - k\pi^2\right) = 0$

$\Rightarrow \quad \pi^2 - 8\left(\frac{\pi^2}{4} - k\pi^2\right) \ge 0 \quad = 1 - 2 + 8k \ge 0 = k \ge 1/2 \ ..(2)$

From (1) and (2) $\quad k = 1$

$\therefore\ t = \dfrac{\pi \pm \sqrt{8\pi^2 - \pi^2}}{4} = \dfrac{\pi \pm \sqrt{7}\,\pi}{4} = \left(1 \pm \sqrt{7}\right)\dfrac{\pi}{4}$ or $\cos^{-1} y$

$= \left(\sqrt{7} + 1\right)\dfrac{\pi}{4}$ (as $0 \le \cos^{-1} y \le \pi$)

$\therefore\ \tan^{-1} x = \dfrac{\pi}{2} - \left(\sqrt{7} + 1\right)\dfrac{\pi}{4}$

$= \dfrac{\pi}{4}\left[\left(1 - \sqrt{7}\right)\right]$

$\Rightarrow\ x = \tan\left(1 - \sqrt{7}\right)\dfrac{\pi}{4}$

G. Inequations involving inverse trigonometric functions

Ex.1 Find the interval in which $\cos^{-1} x > \sin^{-1} x$.

Sol.

We have, $\cos^{-1} x > \sin^{-1}$ {for cos–1 x to be real; x E [–1, 1]}

$2\cos^{-1} x > \pi/2 = \cos^{-1} x > \pi/4$ or $\cos(\cos^{-1} x) < \cos \pi/4$

$\Rightarrow\ x < \dfrac{1}{\sqrt{2}}$

$\therefore\ x \in \left(-1, \dfrac{1}{\sqrt{2}}\right)$

Ex.2 Find the solution set of the inequation $\sin^{-1}(\sin 5) > x^2 - 4x$

Sol.

$\sin^{-1}(\sin 5) > x^2 - 4x \Rightarrow \sin-1[\sin(5 - 2\pi)] > x^2 - 4x$

$\Rightarrow x^2 - 4x < 5 - 2\pi \Rightarrow x^2 - 4x + (2\pi - 5) < 0$

$\Rightarrow\ 2 - \sqrt{9 - 2\pi} < x < 2 + \sqrt{9 - 2\pi}$

$\Rightarrow\ x \in \left(2 - \sqrt{9 - 2\pi}, 2 + \sqrt{9 - 2\pi}\right)$

Summation of Series

Ex.1 Sum the series,

$$\tan^{-1}\frac{4}{1+3\cdot 4} + \tan^{-1}\frac{6}{1+8\cdot 9} + \tan^{-1}\frac{8}{1+15\cdot 16} + \ldots\ldots\text{ to 'n' terms.}$$

Sol.

$$T_n = \tan^{-1}\frac{2(n+1)}{1+\{(n+1)^2-1\}\{(n+1)^2\}}$$

$$= \tan^{-1}\frac{2n+2}{1+\left(n^2+2n\right)(n+1)^2}$$

$$= \tan^{-1}\left[\frac{2n+2}{1+n(n+2)(n+1)(n+1)}\right]$$

$$= \tan^{-1}\left[\frac{(n+1)(n+2)-n(n+1)}{1+\{n(n+1)\}\{(n+1)(n+2)\}}\right]$$

$= \tan^{-1}(n+1)(n+2) - \tan^{-1} n(n+1)$

Put n = 1, 2, 3, ..., n and add, we get $S_n = \tan^{-1}(n+1)(n+2) - \tan^{-1} 2$

Ex.2 Sum the series to 'n' terms, $\tan^{-1}\frac{2}{4} + \tan^{-1}\frac{2}{9} + \tan^{-1}\frac{2}{16} + \tan^{-1}\frac{2}{25} + \ldots\ldots$ **to 'n' terms. Also show that,** $S_\infty = \tan^{-1} 3$ **.**

Sol.

$$T_n = \tan^{-1}\frac{2}{n^2+2n+1}$$

$$= \tan^{-1}\frac{(n+2)-n}{1+n(n+2)}$$

$= \tan^{-1}(n+2) - \tan^{-1}(n)$

Hence, $S_n = \tan^{-1}(n+2) + \tan^{-1}(n+1) - (\tan^{-1} 1 + \tan^{-1} 2)$

$$S_\infty = \lim_{n\to\infty} S_n = \frac{\pi}{2}+\frac{\pi}{2}-\left(\pi+\tan^{-1}\left(\frac{1+2}{1-2}\right)\right) = \tan^{-1} 3$$

Ex.3 If the sum $\sum_{n=1}^{10}\sum_{m=1}^{10}\tan^{-1}\left(\frac{m}{n}\right) = k\pi$**, find the value of k.**

Sol.

$$S = \sum_{n=1}^{10}\left(\tan^{-1}\frac{1}{n}+\tan^{-1}\frac{2}{n}+\tan^{-1}\frac{3}{n}+\dots\dots+\tan^{-1}\frac{10}{n}\right) \text{ now consider}$$

$$\sum_{n=1}^{10}\tan^{-1}\frac{1}{n} = \underbrace{\tan^{-1}1}+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}+\dots\dots\dots\dots+\tan^{-1}\frac{1}{9}+\tan^{-1}\frac{1}{10}$$

$$\sum_{n=1}^{10}\tan^{-1}\frac{2}{n} = \tan^{-1}\frac{2}{1}+\underbrace{\tan^{-1}\frac{2}{2}}+\tan^{-1}\frac{2}{3}+\tan^{-1}\frac{2}{4}+\tan^{-1}\frac{2}{5}+\dots\dots+\tan^{-1}\frac{2}{10} \quad \dots\dots$$

$$\sum_{n=1}^{10}\tan^{-1}\frac{10}{n} = \tan^{-1}\frac{10}{1}+\tan^{-1}\frac{10}{2}+\tan^{-1}\frac{10}{3}+\tan^{-1}\frac{10}{4}+\dots\dots\dots\dots+\underbrace{\tan^{-1}\frac{10}{10}}$$

$$S = \underbrace{\left(10\cdot\frac{\pi}{4}\right)} + \underbrace{\left(\tan^{-1}\frac{1}{2}+\tan^{-1}2\right)} + \left(\tan^{-1}\frac{1}{3}+\tan^{-1}3\right) + \dots\dots$$

$$S = \frac{5\pi}{2}+\frac{45\pi}{2} = \frac{50\pi}{2} = 25\pi \qquad \Rightarrow \quad k = 25$$

Formulas: Inverse Trigonometric Functions

GENERAL DEFINITION(S): 1. $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ etc. denote angles or real numbers whose sine is x, whose cosine is x and whose tangent is x, provided that the answers given are numerically smallest available. These are also written as arc sinx, arc cosx etc .

If there are two angles one positive & the other negative having same numerical value, then positive angle should be taken.

2. PRINCIPAL VALUES AND DOMAINS OF INVERSE CIRCULAR FUNCTIONS :

(i) $y = \sin^{-1} x$ where $-1 \le x \le 1$; $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $\sin y = x$.

(ii) $y = \cos^{-1} x$ where $-1 \le x \le 1$; $0 \le y \le \pi$ and $\cos y = x$.

(iii) $y = \tan^{-1} x$ where $x \in R$; $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\tan y = x$.

(iv) $y = \text{cosec}^{-1} x$ where $x \le -1$ or $x \ge 1$; $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$ and $\text{cosec}\, y = x$

(v) $y = \sec^{-1} x$ where $x \le -1$ or $x \ge 1$; $0 \le y \le \pi$; $y \ne \pi/2$ and $\sec y = x$.

(vi) $y = \cot^{-1} x$ where $x \in R$, $0 < y < \pi$ and $\cot y = x$.

NOTE THAT : (a) 1st quadrant is common to all the inverse functions.

(b) 3rd quadrant is not used in inverse functions.

(c) 4th quadrant is used in the CLOCKWISE DIRECTION i.e. $-\pi/2 \le y \le 0$.

3. PROPERTIES OF INVERSE CIRCULAR FUNCTIONS:

P−1 (i) $\sin(\sin^{-1} x) = x, \ -1 \le x \le 1$

(ii) $\cos(\cos^{-1} x) = x, \ -1 \le x \le 1$

(iii) $\tan(\tan^{-1} x) = x, \ x \in R$

(iv) $\sin^{-1}(\sin x) = x, \ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(v) $\cos^{-1}(\cos x) = x \ ; \ 0 \le x \le \pi$

(vi) $\tan^{-1}(\tan x) = x \ ; \ -\frac{\pi}{2} < x < \frac{\pi}{2}$

P−2 (i) $\text{cosec}^{-1} x = \sin^{-1}(1/x)$; $x \le -1, \ x \ge 1$

(ii) $\sec^{-1} x = \cos^{-1}(1/x)$; $x \le -1, \ x \ge 1$

(iii) $\cot^{-1} x = \tan^{-1}(1/x)$; $x > 0 = \pi + \tan^{-1}(1/x)$; $x < 0$

P−3 (i) $\sin^{-1}(-x) = -\sin^{-1} x, -1 \le x \le 1$

(ii) $\tan^{-1}(-x) = -\tan^{-1} x, x \in R$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1} x, -1 \le x \le 1$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x, x \in R$

P−4 (i) $\sin^{-1} x + \cos^{-1} x = \pi/2 \ -1 \le x \le 1$

(ii) $\tan^{-1} x + \cot^{-1} x = \pi/2 \ x \in R$

(iii) $\text{cosec}^{-1} x + \sec^{-1} x = \pi/2 \ |x| \ge 1$

P−5 $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$ where $x > 0, y > 0 \ \& \ xy < 1$

$= \pi + \tan^{-1} \frac{x+y}{1-xy}$ where $x > 0, y > 0 \ \& \ xy > 1$

$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}$ where $x > 0, y > 0$

P−6 (i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$ where $x \ge 0, y \ge 0 \ \& \ (x^2 + y^2) \le 1$

Note that : $x^2 + y^2 \le 1 \Rightarrow 0 \le \sin^{-1} x + \sin^{-1} y \le \pi/2$

(ii) $\sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$ where $x \ge 0, y \ge 0 \ \& \ x^2 + y^2 > 1$

Note that : $x^2 + y^2 > 1 \Rightarrow \pi/2 < \sin^{-1} x + \sin^{-1} y < \pi$

(iii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$ where $x > 0, y > 0$

(iv) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right]$ where $x \ge 0, y \ge 0$

P−7 If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$ if, $x > 0, y > 0, z > 0 \ \& \ xy + yz + zx < 1$

Note : (i) If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$ then $x + y + z = xyz$

(ii) If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi/2$ then $xy + yz + zx = 1$

P−8 $2\tan^{-1} x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}$ Note very carefully that :

$$\sin^{-1}\frac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x < -1 \end{cases} \quad \cos^{-1}\frac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$$

$$\tan^{-1}\frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -(\pi - 2\tan^{-1}x) & \text{if } x > 1 \end{cases}$$

REMEMBER THAT : (i) $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = 3\pi/2 \Rightarrow x = y = z = 1$

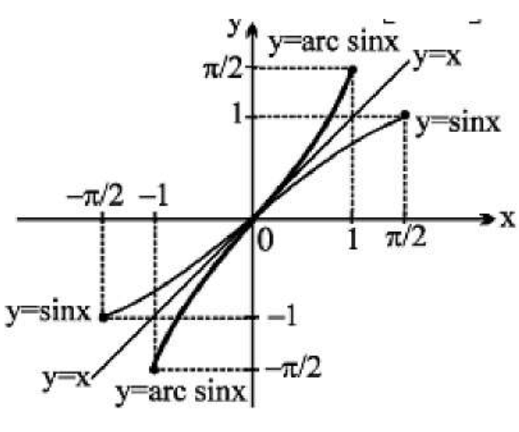
(ii) $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = 3\pi \Rightarrow x = y = z = -1$

(iii) $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$ and $\tan^{-1} 1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{2}$

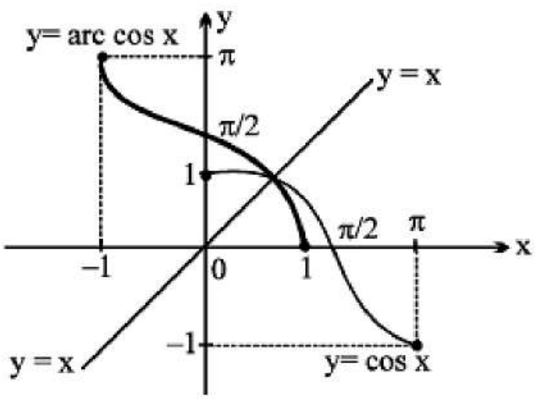
INVERSE TRIGONOMETRIC FUNCTIONS

SOME USEFUL GRAPHS

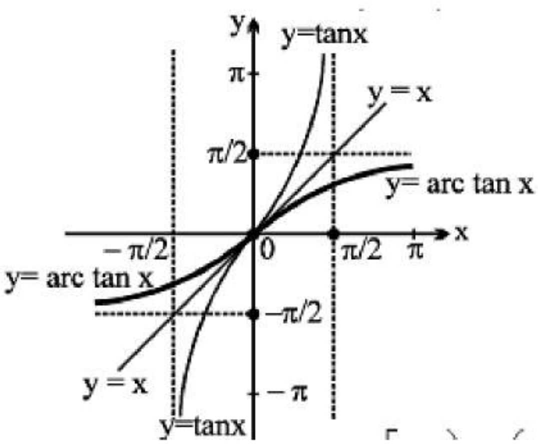
1. $y = \sin^{-1} x$, $|x| \le 1$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



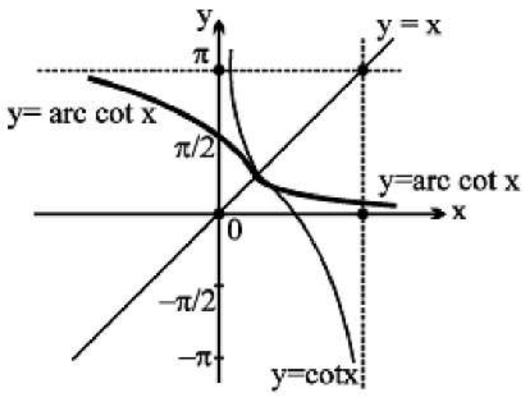
2. $y = \cos^{-1} x, |x| \le 1, y \in [0, \pi]$



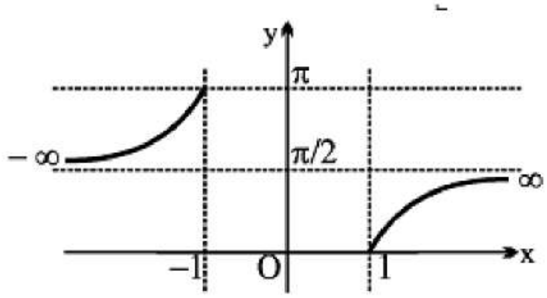
3. $y = \tan^{-1} x, x \in R, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



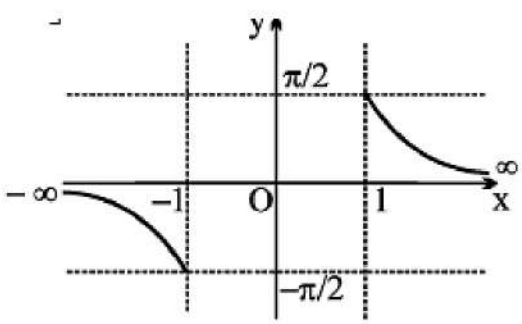
4. $y = \cot^{-1} x, x \in R, y \in (0, \pi)$



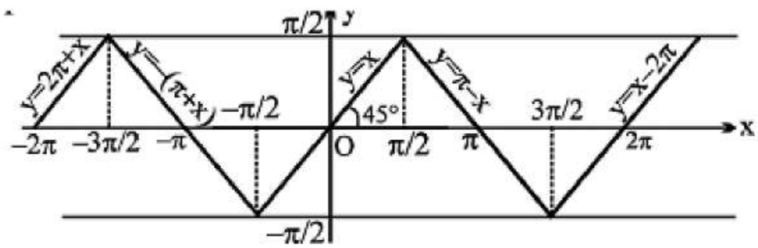
5. $y = \sec^{-1} x$, $|x| \geq 1$, $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



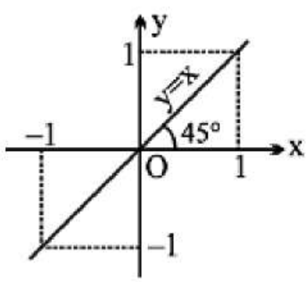
6. $y = \text{cosec}^{-1} x$, $|x| \geq 1$, $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



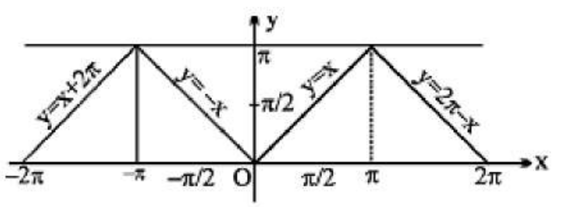
7. (a) $y = \sin^{-1}(\sin x)$, $x \in R$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, Periodic with period 2π



(b) $y = \sin(\sin^{-1} x), = x$ $x \in [-1, 1], y \in [-1, 1]$, y is aperiodic

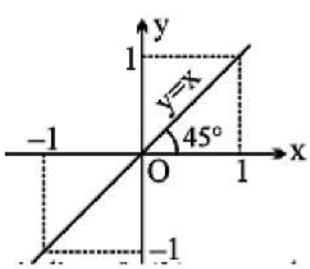


8. (a) $y = \cos^{-1}(\cos x), x \in R, y \in [0, \pi], = x$ periodic with period 2π



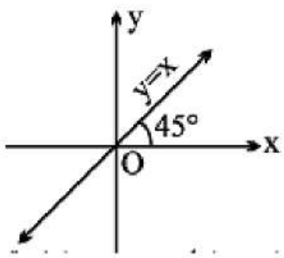
(b) $y = \cos(\cos^{-1} x), = x$

$x \in [-1, 1],\ y \in [-1, 1]$, y is aperiodic

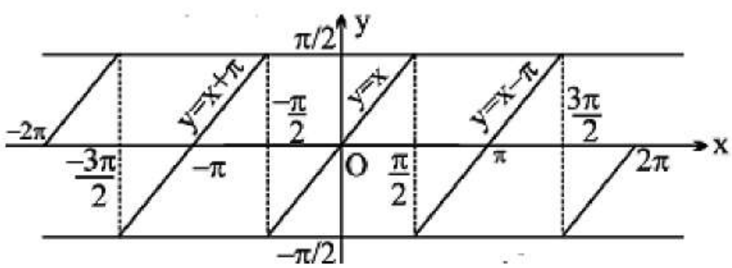


9. (a) $y = \tan(\tan^{-1} x),\ x \in R,\ y \in R,$ y is aperiodic $= x$

$x \in R - \left\{(2n-1)\frac{\pi}{2}\ n \in I\right\}$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, periodic with period π

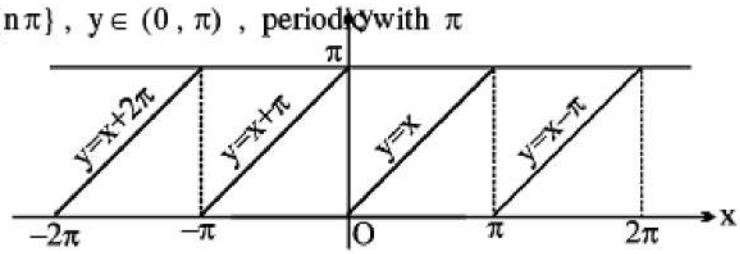


(b) $y = \tan^{-1}(\tan x), = x$



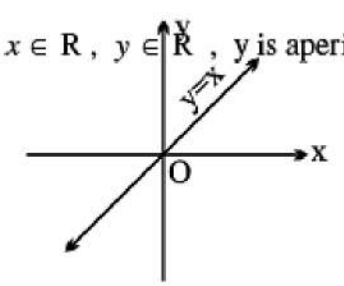
10. (a) $y = \cot^{-1}(\cot x), = x$

$x \in R - \{n\pi\}$, $y \in (0, \pi)$, periodic with π



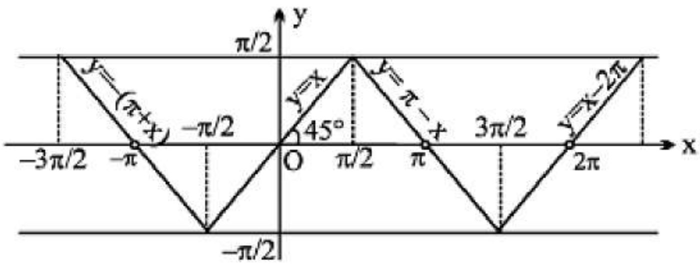
(b) $y = \cot(\cot -1\ x), = x$

$x \in R$, $y \in R$, y is aperiodic



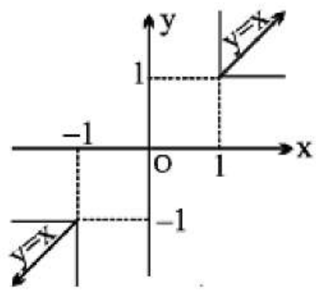
11. (a) $y = \text{cosec} -1\ (\text{cosec}\ x), = x$

$x \in R - \{ n\pi, n \in I \}, y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
y is periodic with period 2π



(b) $y = \text{cosec}(\text{cosec}^{-1} x), = x$

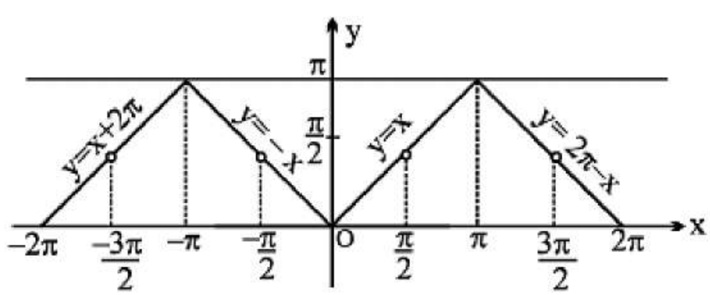
$|x| \geq 1$, $|y| \geq 1$, y is aperiodic



12. (a) $y = \sec -1 (\sec x), = x$

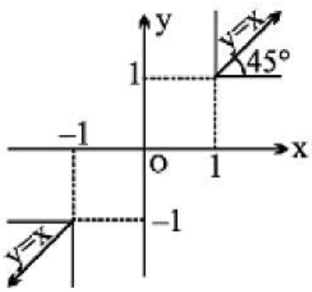
y is periodic with period 2π ;

$x \in R - \left\{(2n-1)\frac{\pi}{2} \; n \in I\right\} \; y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



(b) $y = \sec(\sec^{-1} x), = x$

$|x| \geq 1$; $|y| \geq 1]$, y is aperiodic


y
y=x
45°
1
–1
x
O
1
–1
y=x