CBSE Test Paper 05

CH-13 Limits and Derivatives

1. If f be a function such that f (9) = 9 and f ‘ (9) = 3, then $\underset{x \to 9}{Lt} \frac{\sqrt{f(x)}-3}{\sqrt{x}-3}$ is equal to
   a. 3
   b. 9
   c. 1
   d. 0
2. $\underset{x \to \frac{\pi}{4}}{Lt} \frac{\tan x-1}{x-\frac{\pi}{4}}$ is equal to
   a. 1
   b. $\frac{1}{2}$
   c. 0
   d. 2
3. Let f (x) = x sin $\frac{1}{x}$, x$\neq$0, then the value of the function at x = 0, so that f is continuous at x = 0, is
   a. -1
   b. $\sqrt{2}$
   c. 1
   d. 0
4. $\underset{h \to 0}{Lt} \frac{\sqrt[3]{8+h}-2}{h}$ is equal to
   a. $\frac{1}{24}$

b. 2

c. $\frac{1}{12}$

d. $\frac{1}{3}$

5. $\underset{x \to 0}{Lt} \left(\frac{\tan x - x}{x}\right) \sin\left(\frac{1}{x}\right)$ is equal to

a. 1

b. a real number other than 0 and 1

c. -1

d. 0

6. Fill in the blanks: The value of limit $\lim\limits_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$ is ________.
7. Fill in the blanks: The derivative of cosecx is ________.
8. Find the derivative of 99 x at x = 0100
9. Evaluate $\lim\limits_{x \to \frac{1}{2}} \frac{4x^2 - 1}{2x - 1}$.
10. If $\lim\limits_{x \to (-a)} \frac{x^7 + a^7}{x + a} = 7$, then find the value of a.
11. Evaluate $\lim\limits_{x \to 0} \frac{\sin(2+x) - \sin(2-x)}{x}$.
12. Find the limit $\lim\limits_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$.
13. Evaluate $\lim\limits_{x \to \frac{\pi}{6}} \frac{2 - \sqrt{3}\cos x - \sin x}{(6x - \pi)^2}$.
14. Evaluate $\lim\limits_{x \to \sqrt{10}} \frac{\sqrt{7 - 2x} - (\sqrt{5} - \sqrt{2})}{x^2 - 10}$
15. f(x) = $\frac{ax^2 + b}{x^2 + 1}$, $\lim\limits_{x \to 0}$ f(x) = 1 and $\lim\limits_{x \to \infty}$ f(x) = 1, then prove that f(-2) = f (2) = 1.

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Solution

1. (a) 3

   Explanation: Using L'Hospital;

   $\Rightarrow \lim_{x \to 9} \frac{\frac{f'(x)}{2\sqrt{f(x)}}}{\frac{1}{2\sqrt{x}}}$

   $\Rightarrow \lim_{x \to 9} \frac{\sqrt{x} \cdot f'(x)}{\sqrt{f(x)}}$

   $\Rightarrow$ Substituting the given values; we get

   $\Rightarrow$ 3

2. (d) 2

   Explanation: $\text{let } x - \frac{\pi}{4} = t$

   $\Rightarrow \underline{Lt}_{t \to 0} \frac{\tan\left(\frac{\pi}{4} + t\right) - 1}{t}$

   $\Rightarrow \underset{t \to 0}{\text{Lt}} \frac{2 \tan t}{(1 - \tan t)(t)}$

   $\Rightarrow$ 2

3. (d) 0

   Explanation: Here, if we directly put x= 0, f(0) = 0 * sin (1/0) = 0.

   At L.H.L, put x=0-h , $\underset{x \to 0}{Lt}\ x . \sin \frac{1}{x}$ f(0-h) = = 0.

   At R.H.L, put x = 0+h , $\underset{x \to 0}{Lt}\ x . \sin \frac{1}{x}$, f(0+h) = = 0.

   Hence, L.H.L = f(0) = R.H.L.

   f(x) is continuous at x=0.

4. (c) $\frac{1}{12}$

   Explanation: using L'Hospital Rule;

$\Rightarrow \underset{h \to 0}{\mathrm{Lt}} \frac{\left(\frac{1}{3}\right)(8+h)^{\frac{-2}{3}}}{1}$

$\Rightarrow \frac{1}{12}$

5. (d) 0

 Explanation: $\underset{x \to 0}{\mathrm{Lt}} \left(\frac{\tan x}{x} - \frac{x}{x}\right) \sin\left(\frac{1}{x}\right)$

 $\Rightarrow$ 0.Finite number = 0

6. 2

7. -cosecx cotx

8. Here $\frac{d}{dx}(99x) = 99$

 $\therefore$ Derivative of 99 x at x = 100 = 99

9. $\lim_{x \to \frac{1}{2}} \frac{4x^2-1}{2x-1} = \lim_{x \to \frac{1}{2}} \frac{(2x+1)(2x-1)}{(2x-1)}$ [using factorisation method]

 $= \lim_{x \to \frac{1}{2}} (2x+1) = 2\left(\frac{1}{2}\right)$ + 1 = 2

10. We have, $\lim_{x \to (-a)} \frac{x^7+a^7}{x+a} = 7$

 $\Rightarrow \quad \lim_{x \to (-a)} \frac{x^7-(-a)^7}{x-(-a)} = 7$

 $\Rightarrow$ 7(-a)$^{7-1}$ = 7 $\left[\because \lim_{x \to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right]$

 $\Rightarrow$ 7(-a)^6 = 7 $\Rightarrow$ 7a^6 = 7

 $\Rightarrow$ a^6 = 1 $a = \pm 1$

11. We have, $\lim_{x \to 0} \frac{\sin(2+x)-\sin(2-x)}{x}$

 $= \lim_{x \to 0} \frac{2\cos\left(\frac{2+x+2-x}{2}\right)\sin\left(\frac{2+x-2+x}{2}\right)}{x}$

 $\left[\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right) \cdot \sin\left(\frac{C-D}{2}\right)\right]$

 $= \lim_{x \to 0} \frac{2\cos 2 \sin x}{x}$

 $= 2\cos 2 \lim_{x \to 0} \frac{\sin x}{x} = 2\cos 2 \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$

12. Put y = 1 + x, then y $\rightarrow$ 1 as x $\rightarrow$ 0.

$$\therefore \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{y\to 1}\frac{\sqrt{y}-1}{y-1}$$
$$=\lim_{y\to 1}\frac{y^{\frac{1}{2}}-1^{\frac{1}{2}}}{y-1}$$
$$=\frac{1}{2}(1)^{\frac{1}{2}-1}\left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$
$$=\frac{1}{2}$$

13. We have, $\lim_{x\to \frac{\pi}{6}}\frac{2-\sqrt{3}\cos x-\sin x}{(6x-\pi)^2}$
$$=\lim_{h\to 0}\frac{2-\sqrt{3}\cos\left(\frac{\pi}{6}+h\right)-\sin\left(\frac{\pi}{6}+h\right)}{\left[6\left(\frac{\pi}{6}+h\right)-\pi\right]^2}$$
[putting $x=\frac{\pi}{6}+h$,as $x\to\frac{\pi}{6}$, then h $\to$ 0]
$$=\lim_{h\to 0}\frac{\left[2-\sqrt{3}\left(\cos\frac{\pi}{6}\cdot\cos h-\sin\frac{\pi}{6}\sin h\right)-\left[\sin\frac{\pi}{6}\cdot\cos h+\cos\frac{\pi}{6}\sin h\right]\right]}{(\pi+6h-\pi)^2}$$
$$=\lim_{h\to 0}\frac{2-\sqrt{3}\cdot\left(\frac{\sqrt{3}}{2}\cos h-\frac{1}{2}\sin h\right)-\left(\frac{1}{2}\cos h+\frac{\sqrt{3}}{2}\sin h\right)}{36h^2}$$
$$=\lim_{h\to 0}\frac{\left(2-\frac{3}{2}\cos h+\frac{\sqrt{3}}{2}\sin h-\frac{1}{2}\cos h-\frac{\sqrt{3}}{2}\sin h\right)}{36h^2}$$
$$=\lim_{h\to 0}\frac{2-2\cos h}{36h^2}=\lim_{h\to 0}\frac{1-\cos h}{18h^2}$$
$$=\lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{18h^2}=\frac{1}{9}\lim_{h\to 0}\frac{\sin^2\frac{h}{2}}{h^2}=\frac{1}{9}\lim_{\frac{h}{2}\to 0}\left[\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]^2\times\frac{1}{4}$$
$$=\frac{1}{36}\times 1=\frac{1}{36}$$

14. Here $\lim_{x\to\sqrt{10}}\frac{\sqrt{7-2x}-(\sqrt{5}-\sqrt{2})}{x^2-10}$
$$=\lim_{x\to\sqrt{10}}\frac{\sqrt{7-2x}-\sqrt{(\sqrt{5}-\sqrt{2})^2}}{x^2-10}=\lim_{x\to\sqrt{10}}\frac{\sqrt{7-2x}-\sqrt{7-2\sqrt{10}}}{x^2-10}\left[\frac{0}{0} from\right]$$
$$=\lim_{x\to\sqrt{10}}\frac{\sqrt{7-2x}-\sqrt{7-2\sqrt{10}}}{x^2-10}\times\frac{\sqrt{7-2x}+\sqrt{7-2\sqrt{10}}}{\sqrt{7-2x}+\sqrt{7-2\sqrt{10}}}$$
$$=\lim_{x\to\sqrt{10}}\frac{(7-2x)-(7-2\sqrt{10})}{(x-\sqrt{10})(x+\sqrt{10})(\sqrt{7-2x}+\sqrt{7-2\sqrt{10}})}$$
$$=\lim_{x\to\sqrt{10}}\frac{-2(x-\sqrt{10})}{(x-\sqrt{10})(x+\sqrt{10})(\sqrt{7-2x}+\sqrt{7-2\sqrt{10}})}$$

$$= \lim_{x \to \sqrt{10}} \frac{-2}{(x+\sqrt{10})(\sqrt{7-2x}+\sqrt{7-2\sqrt{10}})}$$

$$= \frac{-2}{(2\sqrt{10})\left(\sqrt{7-2\sqrt{10}}+\sqrt{7-2\sqrt{10}}\right)} = \frac{-2}{(2\sqrt{10})\ \left(2\sqrt{7-2\sqrt{10}}\right)}$$

$$= \frac{-1}{2\sqrt{10}\ \sqrt{7\ -2\sqrt{10}}}$$

$$= \frac{-1}{2\sqrt{10}\ (\sqrt{5}-\sqrt{21})}\ [\because\ (7-2\sqrt{10}) = \sqrt{5}-\sqrt{21}$$

15. f(x) = $\frac{ax^2+b}{x^2+1}$

Also $\lim_{x\to 0} f(x)$ = 1 ....(i) [given]

$\Rightarrow \lim_{x\to 0} \frac{ax^2+b}{x^2+1}$ = 1

$\Rightarrow \frac{\lim_{x\to 0} ax^2+b}{\lim_{x\to 0} x^2+1} = 1$

$\Rightarrow$ b = 1

Also, it is given that $\lim_{x\to\infty} f(x) = 1$

$\therefore \lim_{x\to\infty} \frac{ax^2+b}{x^2+1}$ = 1 ....(ii)

$\Rightarrow \lim_{x\to\infty} \frac{ax^2+1}{x^2+1}$ = 1

$\Rightarrow \lim_{x\to\infty} \frac{a+\frac{1}{x^2}}{1+\frac{1}{x^2}} = 1$

$\Rightarrow$ a = 1

Thus, f(x) = $\frac{ax^2+b}{x^2+1} = \frac{x^2+1}{x^2+1}$

So, f(-2) = 1 and f(2) = 1

Hence, f(-2) = f(2) = 1