DAY EIGHTEEN

Differential Equations

Learning & Revision for the Day

- Solution of Differential Equation
- Solutions of Differential Equations of First Order and First Degree
- Linear Differential Equation
- Inspection Method

An equation involving independent variables, a dependent variable and the derivatives of dependent variable w.r.t. independent variables, is called **differential equation**.

A differential equation which contains only one independent variable, is called an ordinary differential equation.

Order and Degree of a Differential Equation

- The **order** of a differential equation is the order of the highest derivative occurring in the differential equation.
- The **degree** of differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions.

Solution of Differential Equation

A solution of the differential equation is a relation between the dependent and independent variables of the equation not containing the derivatives, but satisfying the given differential equation.

General Solution

A general solution of a differential equation is a relation between the variables (not involving the derivatives) which contains the same number of the arbitrary constants as the order of the differential equation.

Particular Solution

Particular solution of the differential equation is obtained from the general solution by assigning particular values to the arbitrary constant in the general solution.

Solutions of Differential Equations of the First Order and First Degree

A differential equation of first order and first degree may be of the following types:

Differential Equation with Variable Separable

'Variable separable method' is used to solve such an equation in which variable can be separated completely, e.g. the term $f(x)$ with dx and the term $g(y)$ with dy

i.e. $g(y)dy = f(x)dx$

Reducible to Variable Separable Form

Differential equations of the form $\frac{dy}{dx} = f(ax + by + c)$ can be reduced to variable separable form by the substitution $ax + by + c = z$.

$\Rightarrow \quad a + b\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \left(\frac{dz}{dx} - a\right)\frac{1}{b} = f(z) \Rightarrow \frac{dz}{dx} = a + bf(z)$

Homogeneous Differential Equation

A function $f(x, y)$ is said to be a homogeneous function of degree n if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ for some non-zero constant λ.

- Let $f(x, y)$ and $g(x, y)$ be two homogeneous functions of same degree, then a differential equation expressible in the form

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$$

is called a **homogeneous differential equation**.

- To solve a homogeneous differential equation of the type $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)} = h\left(\frac{y}{x}\right)$, we put $y = vx$ and to solve a homogeneous differential equation of type $\frac{dx}{dy} = \frac{f(x, y)}{g(x, y)} = h\left(\frac{x}{y}\right)$, we put $x = vy$.

Linear Differential Equation

A differential equation of the form $\frac{dy}{dx} + Py = Q$, where, P and Q are the functions of x (or constants), is called a linear differential equation.
To solve such an equation, first find integrating Factor, IF $= e^{\int P dx}$. Then, the solution of the differential equation is given by $y(\text{IF}) = \int Q(\text{IF})\,dx + C$.

NOTE
- Sometimes given equation becomes a linear differential equation of the form $\frac{dx}{dy} + Rx = S$, where R and S are function of y (or constants).
- The integrating factor in this case is IF $= e^{\int R\,dy}$ and the solution is given by $x \cdot (\text{IF}) = \int (S \times \text{IF})dy + C$.

Bernoulli's Differential Equation (Reducible to Linear Form)

Let the differential equation be of the form

$$\frac{dy}{dx} + Py = Qy^n \quad \Rightarrow \quad y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q$$

Here, put $y^{-n+1} = z \Rightarrow (-n+1)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

$$\Rightarrow \quad \frac{dz}{dx} + (1 - n)\,Pz = (1 - n)\,Q$$

which is a linear differential equation in z.

The solution of this equation is given by

$$ze^{\int (1-n)P\,dx} = \int \left[(1-n) \cdot Q \cdot e^{\int (1-n)P\,dx}\right]dx + C$$

Inspection Method

If we can write the differential equation in the form $f\{f_1(x, y)\}\,d\{f_1(x, y)\} + \phi\{f_2(x, y)\}\,d\{f_2(x, y)\} + \ldots = 0$, then each term can be easily integrated separately.

For this use the following results.

(i) $d(x \pm y) = dx \pm dy$ (ii) $d(xy) = x\,dy + y\,dx$

(iii) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

(iv) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{x^2 + y^2}$

(v) $d[\log(xy)] = \frac{x\,dy + y\,dx}{xy}$

(vi) $d\left[\log\left(\frac{x}{y}\right)\right] = \frac{y\,dx - x\,dy}{xy}$

(vii) $d\left[\frac{1}{2}\log(x^2 + y^2)\right] = \frac{x\,dx + y\,dy}{x^2 + y^2}$

(viii) $d\left(-\frac{1}{xy}\right) = \frac{x\,dy + y\,dx}{x^2y^2}$

(ix) $d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x\,dy}{y^2}$

(x) $d(\sqrt{x^2 + y^2}) = \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}$

(xi) $d\left(\frac{1}{2}\log\frac{x + y}{x - y}\right) = \frac{x\,dy - y\,dx}{x^2 - y^2}$

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 The order of the differential equation whose general solution is given by

$$y = c_1 e^{2x + c_2} + c_3 e^x + c_4 \sin(x + c_5), \text{ is}$$

(a) 5 (b) 4 (c) 3 (d) 2

2 The degree of the differential equation

$$\frac{d^2y}{dx^2} + 3\left[\frac{dy}{dx}\right]^2 = x^2 \log\left[\frac{d^2y}{dx^2}\right], \text{ is}$$

(a) 1 (b) 2
(c) 3 (d) None of these

3 Order and degree of the differential equation, representing the family of curves $y^2 = 2c(x + \sqrt{c})$, where c is a positive parameter, are respectively equal to

(a) 1, 3 (b) 2, 3
(c) 2, 4 (d) 1, 2

4 The differential equation of all circles in the first quadrant which touch the coordinate axes is of order

(a) 1 (b) 2
(c) 3 (d) None of these

5 The differential equation of the family of curves $v = \frac{A}{r} + B$, where A and B are arbitrary constants, is

(a) $\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} = 0$ (b) $\frac{d^2v}{dr^2} - \frac{2}{r}\frac{dv}{dr} = 0$
(c) $\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0$ (d) None of these

6 The solution of the differential equation

$$x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2 + y^2} = 0, \text{ is}$$

(a) $y = x \tan\left(\frac{x^2 + y^2 + C}{2}\right)$

(b) $x = y \tan\left(\frac{x^2 + y^2 + C}{2}\right)$

(c) $y = x \tan\left(\frac{C - x^2 - y^2}{2}\right)$

(d) None of the above

7 The differential equation of the family of parabolas with focus at the origin and the X-axis as axis, is

(a) $y\left(\frac{dy}{dx}\right)^2 + 4x\frac{dy}{dx} = 4y$ (b) $y\left(\frac{dy}{dx}\right)^2 = 2x\frac{dy}{dx} - y$
(c) $y\left(\frac{dy}{dx}\right)^2 + y = 2xy\frac{dy}{dx}$ (d) $y\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} + y = 0$

8 The differential equation whose solution is $x^2 + y^2 + 2ax + 2by + c = 0$, where a, b, c are arbitrary constants, is

(a) $3y_1y_2 - (1 + y_1^2)y_3 = 0$ (b) $3y_1^2y_2 - (1 + y_1^2)y_3 = 0$
(c) $3y_1y_2^2 + (1 + y_1^2)y_3 = 0$ (d) $3y_1y_2^2 - (1 + y_1^2)y_3 = 0$

9 If $x\,dy = y\,(dx + y\,dy)$, $y(1) = 1$ and $y(x) > 0$. Then, $y(-3)$ is equal to

(a) 3 (b) 2
(c) 1 (d) 0

10 If $\frac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$, then $y(\log 2)$ is equal to

(a) 5 (b) 13 (c) -2 (d) 7

11 The differential equation $\frac{dy}{dx} = \frac{\sqrt{1 - y^2}}{y}$ determines family of circles with,

(a) variable radii and a fixed centre at (0, 1)
(b) variable radii and a fixed centre at (0, – 1)
(c) fixed radius 1 and variable centre along the X-axis
(d) fixed radius 1 and variables centre along the Y-axis

12 If the solution of the differential equation $\frac{dy}{dx} = \frac{ax + 3}{2y + f}$ represents a circle, then the value of a is

(a) 2 (b) -2 (c) 3 (d) -4

13 Solution of the differential equation $y - x\frac{dy}{dx} = y^2 + \frac{dy}{dx}$ is

(a) $Cy = (1 - x)(1 - y)$ (b) $Cx = (1 + x)(1 - y)$
(c) $Cy = (1 + x)(1 - y)$ (d) $Cx = (1 - x)(1 + y)$

14 Solution of the equation $ln\left(\frac{dy}{dx}\right) = ax + by$, is

(a) $ae^{-by} + be^{ax} + C = 0$
(b) $ae^{by} + be^{ax} + C = 0$
(c) $ae^{by} + be^{-ax} + C = 0$
(d) None of the above

15 The solution of the differential equation

$$\frac{dy}{dx} = e^{x-y} + x^2e^{-y} \text{ is}$$

(a) $e^y = e^x + \frac{x^3}{3} + C$ (b) $e^y = e^x + 2x + C$
(c) $e^y = e^x + x^3 + C$ (d) $y = e^x + C$

16 If $(2 + \sin x)\frac{dy}{dx} + (y + 1)\cos x = 0$ and $y(0) = 1$, then $y\left(\frac{\pi}{2}\right)$ is equal to → **JEE Mains 2017**

(a) $-\frac{1}{3}$ (b) $\frac{4}{3}$ (c) $\frac{1}{3}$ (d) $-2/3$

17 If a curve passes through the point $\left(2, \frac{7}{2}\right)$ and has slope $\left(1 - \frac{1}{x^2}\right)$ at any point (x, y) on it, then the ordinate of the point on the curve, whose abscissa is – 2, is

➞ **JEE Mains 2013**

(a) $-\frac{3}{2}$ (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) $-\frac{5}{2}$

18 Solution of the equation $(x+y)^2 \frac{dy}{dx} = 4, y(0) = 0$ is

(a) $y = 2\tan^{-1}\left(\frac{x+y}{2}\right)$ (b) $y = 4\tan^{-1}\left(\frac{x+y}{4}\right)$

(c) $y = 4\tan^{-1}\left(\frac{x+y}{2}\right)$ (d) None of these

19 The solution of $\frac{dy}{dx} + 1 = e^{x+y}$ is

(a) $e^{-(x+y)} + x + C = 0$ (b) $e^{-(x+y)} - x + C = 0$

(c) $e^{x+y} + x + C = 0$ (d) $e^{x+y} - x + C = 0$

20 Solution of the equation $xdy = (y + xf(y/x)/f'(y/x))\,dx$ is

(a) $|f(y/x)| = C|x|,\ C \in R$ (b) $|f(y/x)| = |x| + C, C > 0$

(c) $|f(y/x)| = C|x|, C > 0$ (d) None of these

21 If $x\frac{dy}{dx} = y(\log y - \log x + 1)$, then the solution of the equation is

(a) $\log\left(\frac{x}{y}\right) = Cy$ (b) $\log\left(\frac{y}{x}\right) = Cx$

(c) $x\log\left(\frac{y}{x}\right) = Cy$ (d) $y\log\left(\frac{x}{y}\right) = Cx$

22 The general solution of $y^2\ dx + (x^2 - xy + y^2)\ dy = 0$, is

(a) $\tan^{-1}\left(\frac{x}{y}\right) + \log y + C = 0$

(b) $2\tan^{-1}\left(\frac{x}{y}\right) + \log x + C = 0$

(c) $\log(y + \sqrt{x^2 + y^2}) + \log y + C = 0$

(d) $\sin^{-1}\left(\frac{x}{y}\right) + \log y + C = 0$

23 Solution of the differential equation $x^2dy + y(x+y)dx = 0$ is

(a) $y + 2x = c^2x^2y$ (b) $y - 2x = c^2x^2/y$

(c) $y + 2x = c^2x^2/y$ (d) None of these

24 If $(x^2 + y^2)dy = xydx$ and $y(x_0) = e, y(1) = 1$, then x_0 is

(a) $e\sqrt{3}$ (b) $\sqrt{2e^2 - 1}/\sqrt{2}$

(c) $\sqrt{e^2 - 1}/\sqrt{2}$ (d) $\sqrt{e^2 + 1}/\sqrt{2}$

25 If a curve $y = f(x)$ passes through the point (1,–1) and satisfies the differential equation $y(1 + xy)dx = xdy$, then $f\left(-\frac{1}{2}\right)$ is equal to

➞ **JEE Mains 2016**

(a) $-\frac{2}{5}$ (b) $-\frac{4}{5}$ (c) $\frac{2}{5}$ (d) $\frac{4}{5}$

26 The equation of the curve passing through the origin and satisfying the differential equation

$$(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2 \text{ is}$$

➞ **JEE Mains 2013**

(a) $(1 + x^2)y = x^3$ (b) $3(1 + x^2)y = 2x^3$

(c) $(1 + x^2)y = 3x^3$ (d) $3(1 + x^2)\ y = 4x^3$

27 The solution of the differential equation $\frac{dy}{dx} + \frac{3x^2}{1 + x^3}y = \frac{\sin^2 x}{1 + x^3}$, is

(a) $y(1 + x^3) = x + \frac{1}{2}\sin 2x + C$

(b) $y(1 + x^3) = Cx + \frac{1}{2}\sin 2x$

(c) $y(1 + x^3) = Cx - \frac{1}{2}\sin 2x$

(d) $y(1 + x^3) = \frac{x}{2} - \frac{1}{4}\sin 2x + C$

28 If $y(x)$ satisfies the differential equation $y' - y\tan x = 2x\sec x$ and $y(0) = 0$, then

➞ **IIT 2012**

(a) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$ (b) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$

(c) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$ (d) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{\pi^2}{3\sqrt{3}}$

29 The solution of the equation $\frac{dy}{dx} + y\tan x = x^m\cos x$, is

(a) $(m + 1)\ y = x^{m+1}\cos x + C(m + 1)\cos x$

(b) $my = (x^m + C)\cos x$

(c) $y = (x^{m+1} + C)\cos x$

(d) None of the above

30 Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dP(t)}{dt} = \frac{1}{2}P(t) - 200$. If $P(0) = 100$, then $P(t)$ is equal to

➞ **JEE Mains 2014**

(a) $400 - 300\,e^{\frac{t}{2}}$ (b) $300 - 200\,e^{-\frac{t}{2}}$

(c) $600 - 500\,e^{\frac{t}{2}}$ (d) $400 - 300\,e^{-\frac{t}{2}}$

31 The solution of differential equation

$$(xy^5 + 2y)\,dx - xdy = 0, \text{ is}$$

(a) $9x^8 + 4x^9y^4 = 9y^4C$ (b) $9x^8 - 4x^9y^4 - 9y^4C = 0$

(c) $x^8(9 + 4y^4) = 10y^4C$ (d) None of these

32 Let $Y = y(x)$ be the solution of the differential equation $\sin x\frac{dy}{dx} + y\cos x = 4x, x \in (0, \pi)$. If $y\left(\frac{\pi}{2}\right) = 0$, then $y(\pi/6)$ is equal to

➞ **JEE Mains 2018**

(a) $-\frac{8}{9\sqrt{3}}\pi^2$ (b) $-\frac{8}{9}\pi^2$ (c) $-\frac{4}{9}\pi^2$ (d) $\frac{4}{9\sqrt{3}}\pi^2$

33 **Statement I** The slope of the tangent at any point P on a parabola, whose axis is the axis of x and vertex is at the origin, is inversely proportional to the ordinate of the point P.

Statement II The system of parabolas $y^2 = 4ax$ satisfies a differential equation of degree 1 and order 1.

➞ **JEE Mains 2013**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

34 **Statement I** The elimination of four arbitrary constants in $y = (c_1 + c_2 + c_3 e^{c_4})x$ results into a differential equation of the first order $x\frac{dy}{dx} = y$.

Statement II Elimination of n independent arbitrary constants results in a differential equation of the nth order.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 The differential equation which represents the family of curves $y = c_1 e^{c_2 x}$ where c_1 and c_2 are arbitrary constant, is
(a) $y'' = y'y$ (b) $yy'' = y'$
(c) $yy'' = (y')^2$ (d) $y' = y^2$

2 The population $p(t)$ at time t of a certain mouse species satisfies the differential equation $\frac{dp(t)}{dt} = 0.5(t) - 450$. If $p(0) = 850$, then the time at which the population becomes zero is
(a) $2\log 18$ (b) $\log 9$
(c) $\frac{1}{2}\log 18$ (d) $\log 18$

3 A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right), x > 0$.
Then the equation of the curve is → **JEE Advanced 2013**
(a) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$
(b) $\text{cosec}\left(\frac{y}{x}\right) = \log x + 2$
(c) $\sec\left(\frac{2y}{x}\right) = \log x + 2$
(d) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

4 Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in R$, where $f'(x)$ denotes $\frac{df(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. Then the value of $y(2)$ is
(a) 2 (b) 4
(c) 5 (d) 0

5 Consider the differential equation $y^2 dx + \left(x - \frac{1}{y}\right) dy = 0$. If $y(1) = 1$, then x is given by
(a) $1 - \frac{1}{y} + \frac{e^{\frac{1}{y}}}{e}$ (b) $4 - \frac{2}{y} - \frac{e^{\frac{1}{y}}}{e}$
(c) $3 - \frac{1}{y} + \frac{e^{\frac{1}{y}}}{e}$ (d) $1 + \frac{1}{y} - \frac{e^{\frac{1}{y}}}{e}$

6 The equation of the curve $y = f(x)$ passing through the origin which satisfies the differential equation $\frac{dy}{dx} = \sin(10x + 6y)$ is
(a) $y = \frac{1}{3}\left\{\tan^{-1}\frac{5\tan 4x}{4 - 3\tan 4x} - 5x\right\}$
(b) $y = \frac{1}{3}\left\{\tan^{-1}\frac{5\tan 4x}{4 - 3\tan 4x} + 5x\right\}$
(c) $y = \frac{1}{3}\left\{\tan^{-1}\frac{5\tan 4x}{4 + 3\tan 4x} - 5x\right\}$
(d) None of the above

7 Let I be the purchase value of an equipment and $V(t)$ be the value after it has been used for t yr. The value $V(t)$ depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T - t)$, where $k > 0$ is a constant and T is the total life in years of the equipment. Then, the scrap value $V(T)$ of the equipment is
(a) $I - \frac{kT^2}{2}$ (b) $I - \frac{k(T - t)^2}{2}$
(c) e^{-kT} (d) $T^2 - \frac{1}{k}$

8 The degree of the differential equation satisfying the relation $\sqrt{1 + x^2} + \sqrt{1 + y^2} = \lambda\,(x\sqrt{1 + y^2} - y\sqrt{1 + x^2})$, is
(a) 1 (b) 2
(c) 3 (d) None of these

9 If length of tangent at any point on the curve $y = f(x)$ intercepted between the point of contact and X-axis is of length 1, the equation of the curve is

(a) $\sqrt{1-y^2} + \ln\left|(1-\sqrt{1-y^2})/y\right| = \pm x + C$

(b) $\sqrt{1-y^2} - \ln\left|(1-\sqrt{1-y^2})/y\right| = \pm x + C$

(c) $\sqrt{1-y^2} + \ln\left|(1+\sqrt{1-y^2})/y\right| = \pm x + C$

(d) None of the above

10 The solution of differential equation

$\cos x\, dy = y(\sin x - y)\, dx$, where, $0 < x < \dfrac{\pi}{2}$, is

(a) $\sec x = (\tan x + C)y$ (b) $y\sec x = \tan x + C$
(c) $y\tan x = \sec x + C$ (d) $\tan x = (\sec x + C)y$

11 The curves satisfying the differential equation $(1-x^2)y' + xy = ax$ are

(a) ellipse and parabola (b) ellipse and circles
(c) ellipse and hyperbola (d) None of these

12 A curve passes through (2, 0) and the slope of tangent at a point $P(x, y)$ is equal to $((x+1)^2 + y - 3)/(x+1)$. Then equation of the curve is → **IIT 2004**

(a) $y = x^2 + 2x$ (b) $y = x^2 - 2x$
(c) $y = 2x^2 - x$ (d) None of these

13 The solution of the differential equation

$$\frac{x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dots}{1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dots} = \frac{dx - dy}{dx + dy} \text{ is}$$

(a) $2ye^{2x} = Ce^{2x} + 1$ (b) $2ye^{2x} = Ce^{2x} - 1$
(c) $ye^{2x} = Ce^{2x} + 2$ (d) None of these

14 Let f be a real-valued differentiable function on R (the set of all real numbers) such that $f(1) = 1$. If the y-intercept of the tangent at any point $P(x, y)$ on the curve $y = f(x)$ is equal to the cube of the abscissa of P, then the value of $f(-3)$ is equal to

(a) 3 (b) 6 (c) 9 (d) 1

15 Let $f(x)$ be differentiable in the interval $(0, \pi)$ such that $f(1) = 1$ and $\lim_{t \to x} \dfrac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ for each $x > 0$. Then $f(x)$ is

(a) $\dfrac{1}{3x} + \dfrac{2x^2}{3}$ (b) $-\dfrac{1}{3x} + \dfrac{4x^2}{3}$
(c) $-\dfrac{1}{2x} + \dfrac{2}{x^2}$ (d) $\dfrac{1}{x}$

16 Let a solution $y = y(x)$ of the differential equation $x\sqrt{x^2-1}\, dy - y\sqrt{y^2-1}\, dx = 0$ satisfy $y(2) = \dfrac{2}{\sqrt{3}}$.

Statement I $y(x) = \sec\left(\sec^{-1} x - \dfrac{\pi}{6}\right)$.

Statement II $y(x)$ is given by $\dfrac{1}{y} = \dfrac{2\sqrt{3}}{x} - \sqrt{1 - \dfrac{1}{x^2}}$.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

ANSWERS

SESSION 1

1. (b)	**2.** (d)	**3.** (a)	**4.** (a)	**5.** (c)	**6.** (c)	**7.** (b)	**8.** (d)	**9.** (a)	**10.** (d)
11. (c)	**12.** (b)	**13.** (c)	**14.** (a)	**15.** (a)	**16.** (c)	**17.** (a)	**18.** (a)	**19.** (a)	**20.** (c)
21. (b)	**22.** (a)	**23.** (a)	**24.** (a)	**25.** (d)	**26.** (d)	**27.** (d)	**28.** (a)	**29.** (a)	**30.** (a)
31. (a)	**32.** (b)	**33.** (b)	**34.** (a)						

SESSION 2

1. (c)	**2.** (a)	**3.** (a)	**4.** (d)	**5.** (d)	**6.** (a)	**7.** (a)	**8.** (a)	**9.** (a)	**10.** (a)
11. (c)	**12.** (b)	**13.** (b)	**14.** (c)	**15.** (a)	**16.** (c)				

Hints and Explanations

SESSION 1

1 Given,
$y = c_1 \cdot e^{2x+c_2} + c_3 e^x + c_4 \sin(x + c_5)$
$= c_1 \cdot e^{c_2} e^{2x} + c_3 e^x$
$+ c_4(\sin x \cos c_5 + \cos x \sin c_5)$
$= Ae^{2x} + c_3 e^x + B\sin x + D\cos x$
Here, $A = c_1 e^{c_2}$, $B = c_4 \cos c_5$ and $D = c_4 \sin c_5$
Since, equation consists four arbitrary constants.
So, the order of differential equation is 4.

2 Since, the equation is not a polynomial in all differential coefficients, so its degree is not defined.

3 Given, $y^2 = 2c(x + \sqrt{c})$ ...(i)
On differentiating both side w.r.t. x, we get
$$2y\frac{dy}{dx} = 2c \Rightarrow c = y\frac{dy}{dx}$$
On putting in Eq. (i),
$$y^2 = 2xy\frac{dy}{dx} + 2y^{3/2}\left(\frac{dy}{dx}\right)^{3/2}$$
$$\Rightarrow 8y^3\left(\frac{dy}{dx}\right)^3 = \left(y^2 - 2xy\frac{dy}{dx}\right)^2$$
Which is the differential equation of order one and degree 3.

4 Clearly, the equation of family of circle which touch both the axes is $(x-a)^2 + (y-a)^2 = a^2$, where a is a parameter.
Since, there is only one parameter, therefore order of differential equation representing this family is 1.

5 We have, $v = \frac{A}{r} + B$, where A and B are parameters
On differentiating twice w.r.t. r, we get
$$\Rightarrow \frac{dv}{dr} = \frac{-A}{r^2} \quad \text{... (i)}$$
$$\text{and} \quad \frac{d^2v}{dr^2} = \frac{2A}{r^3} \quad \text{... (ii)}$$
Now, on substituting the value of A from Eq. (i) in Eq. (ii), we get
$$\frac{d^2v}{dr^2} = \frac{2}{r^3}\left(-r^2\frac{dv}{dr}\right) = \frac{-2}{r}\frac{dv}{dr}$$
$$\Rightarrow \frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0,$$ which is the required differential equation.

6 We have, $x\,dx + y\,dy + \frac{x dy - y dx}{x^2 + y^2} = 0$
$$\Rightarrow \frac{1}{2}d(x^2 + y^2) + d\left(\tan^{-1}\frac{y}{x}\right) = 0$$
On integrating, we get
$$\frac{1}{2}(x^2 + y^2) + \tan^{-1}\left(\frac{y}{x}\right) = \frac{C}{2}$$
$$\Rightarrow \frac{C - x^2 - y^2}{2} = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\therefore \quad y = x\tan\left(\frac{C - x^2 - y^2}{2}\right)$$

7 Equation of family of parabolas with focus at $(0, 0)$ and axis as X-axis is
$$y^2 = 4a(x - a) \quad \text{...(i)}$$
On differentiating Eq. (i) w.r.t. x, we get
$$2yy_1 = 4a$$
$$\therefore \quad y^2 = 2yy_1\left(x - \frac{yy_1}{2}\right)$$
$$\Rightarrow \quad y = 2xy_1 - yy_1^2$$
$$\Rightarrow \quad yy_1^2 = 2xy_1 - y$$

8 $x^2 + y^2 + 2ax + 2by + c = 0$ ...(i)
On differentiating Eq. (i) three times, we get
$$2x + 2yy_1 + 2a + 2by_1 = 0$$
$$\Rightarrow \quad x + yy_1 + a + by_1 = 0 \quad \text{...(ii)}$$
$$1 + y_1^2 + yy_2 + by_2 = 0 \quad \text{...(iii)}$$
$$3y_1y_2 + yy_3 + by_3 = 0 \quad \text{...(iv)}$$
On eliminating b from Eqs. (iii) and (iv), we get
$$3y_1y_2^2 - y_3 - y_1^2y_3 = 0$$

9 We have, $x\,dy = y\,(dx + y\,dy)$, $y > 0$
$$\therefore \quad \frac{x\,dy - y\,dx}{y^2} = dy$$
$$\Rightarrow \quad \frac{x}{y} = -y + C \quad \text{[integrating]}$$
Now, $y(1) = 1 \Rightarrow C = 2$
$$\therefore \quad \frac{x}{y} + y = 2$$
For $x = -3$, $-3 + y^2 = 2y$
$\Rightarrow y^2 - 2y - 3 = 0$
$\Rightarrow (y+1)(y-3) = 0$
$\therefore \quad y = 3 \quad [\because y > 0]$

10 Here, $\frac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$
$$\Rightarrow \quad \int\frac{dy}{y+3} = \int dx$$
$$\Rightarrow \quad \log_e|y + 3| = x + C$$
But $y(0) = 2$
$$\therefore \quad \log_e|2 + 3| = 0 + C$$
$$\Rightarrow \quad C = \log_e 5$$
$$\Rightarrow \quad \log_e|y + 3| = x + \log_e 5$$
When $x = \log_e 2$, then
$$\Rightarrow \log_e|y + 3| = \log_e 2 + \log_e 5 = \log_e 10$$
$$\therefore \quad y + 3 = 10 \Rightarrow y = 7$$

11 Clearly, $\int\frac{y}{\sqrt{1-y^2}}dy = \int dx$
$$\Rightarrow \quad -\sqrt{1-y^2} = x + C$$
$\Rightarrow (x + C)^2 + y^2 = 1$
Hence, the centre is $(-C, 0)$ and radius is 1.

12 We have, $\frac{dy}{dx} = \frac{ax + 3}{2y + f}$
$$\Rightarrow \quad (ax + 3)\,dx = (2y + f)\,dy$$
On integrating, we obtain
$$a\cdot\frac{x^2}{2} + 3x = y^2 + fy + C$$
$$\Rightarrow -\frac{a}{2}x^2 + y^2 - 3x + fy + C = 0$$
This will represent a circle, if $-\frac{a}{2} = 1$
[$\because$ coefficient of x^2 = coefficient of y^2]
$$\Rightarrow \quad a = -2$$

13 $(x+1)\frac{dy}{dx} = y - y^2$
$$\Rightarrow \quad \frac{dy}{y(1-y)} = \frac{dx}{x+1}$$
$$\Rightarrow \quad \left(\frac{1}{y} + \frac{1}{1-y}\right)dy = \frac{dx}{x+1}$$
$$\Rightarrow \quad \log y - \log(1-y) = \log(x+1) + \log C$$
$$\Rightarrow \quad \frac{y}{1-y} = C(x+1)$$
$$\Rightarrow \quad (x+1)(1-y) = Cy.$$

14 We have, $\frac{dy}{dx} = e^{ax}.e^{by}$
$$\Rightarrow \quad e^{-by}dy = e^{ax}dx$$
$$\Rightarrow \quad -\frac{1}{b}e^{-by} = \frac{1}{a}e^{ax} + C$$
$$\Rightarrow \quad be^{ax} + ae^{-by} + C' = 0 \quad [C' = abC]$$

15 We have,
$$\frac{dy}{dx} = e^{x-y} + x^2e^{-y} = e^{-y}(x^2 + e^x)$$
On separating the variables, we get
$$e^y dy = (x^2 + e^x)dx$$
On integrating both sides, we get
$$e^y = \frac{x^3}{3} + e^x + C$$

16 We have,
$$(2 + \sin x)\frac{dy}{dx} + (y+1)\cos x = 0$$
$$\Rightarrow \quad \frac{dy}{y+1} = -\frac{\cos x}{2 + \sin x}dx$$
$$\Rightarrow \quad \int\frac{dy}{y+1} = \int -\frac{\cos x}{2 + \sin x}dx$$
$$\Rightarrow \quad \ln(y+1) = -\ln(2 + \sin x) + \ln C$$
$$\Rightarrow \quad (y+1)(2 + \sin x) = C$$
Now, $y(0) = 1$

$\Rightarrow \quad (2)(2+0) = C$

$\Rightarrow \quad C = 4$

Thus, $(y+1)(2+\sin x) = 4$

Now, at $x = \frac{\pi}{2}$, $\quad (y+1)\left(2+\sin\frac{\pi}{2}\right) = 4$

$(y+1)(2+1) = 4$

$y = \frac{4}{3} - 1 = \frac{1}{3}$

17 Given, $\frac{dy}{dx} = 1 - \frac{1}{x^2}$

$\Rightarrow \quad dy = \left(1 - \frac{1}{x^2}\right)dx$

$\Rightarrow \quad y = x + \frac{1}{x} + C$ [integrating]

Since, the curve passing through the point $\left(2, \frac{7}{2}\right)$.

i.e. $\quad \frac{7}{2} = 2 + \frac{1}{2} + C \Rightarrow C = 1$

$\therefore \quad y = x + \frac{1}{x} + 1$ ...(i)

Now, at $x = -2$

Ordinate, $y = -2 - \frac{1}{2} + 1 = -3/2$

18 We have, $(x+y)^2 \frac{dy}{dx} = 4$

On putting $x + y = v$ and $1 + \frac{dy}{dx} = \frac{dv}{dx}$, we get

$\Rightarrow \quad v^2\left(\frac{dv}{dx} - 1\right) = 4$

$\Rightarrow \quad v^2 \frac{dv}{dx} = v^2 + 4.$

$\therefore \quad \frac{v^2 + 4 - 4}{v^2 + 4} dv = dx$

$\Rightarrow \quad v - 2\tan^{-1}(v/2) = x + C$

$\Rightarrow \quad x + y - 2\tan^{-1}\left(\frac{x+y}{2}\right) = x + C$

Now, $\quad y(0) = 0 \Rightarrow C = 0.$

$\therefore \quad y = 2\tan^{-1}\left(\frac{x+y}{2}\right)$

19 We have, $\frac{dy}{dx} + 1 = e^{x+y}$

On putting $x + y = z$ and $1 + \frac{dy}{dx} = \frac{dz}{dx}$, we get

$\therefore \quad \frac{dz}{dx} = e^z$

$\Rightarrow \quad e^{-z} dz = dx$

On integrating both sides, we get

$\frac{e^{-z}}{-1} = x + C$

$\Rightarrow \quad x + e^{-(x+y)} + C = 0$

20 $\frac{dy}{dx} = \frac{y}{x} + \frac{f(y/x)}{f'(y/x)}$

On putting $y/x = v$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$\frac{f'(v)}{f(v)} dv = \frac{dx}{x}$

$\Rightarrow \quad \log|f(v)| = \log|x| + \log C, C > 0$

$\Rightarrow \quad |f(y/x)| = C|x|, C > 0.$

21 Given, $\frac{dy}{dx} = \left(\frac{y}{x}\right)\left[\log\left(\frac{y}{x}\right) + 1\right]$

Put $y = tx$,

and $\quad \frac{dy}{dx} = t + x\frac{dt}{dx}$

Then, we get $t + x\frac{dt}{dx} = t\log t + t$

$\Rightarrow \quad \frac{dt}{t\log t} = \frac{dx}{x}$

$\Rightarrow \quad \log\log t = \log x + \log C$

$\therefore \quad \log\left(\frac{y}{x}\right) = Cx$

22 Given, $\frac{dx}{dy} + \frac{x^2 - xy + y^2}{y^2} = 0$

$\Rightarrow \quad \frac{dx}{dy} + \left(\frac{x}{y}\right)^2 - \left(\frac{x}{y}\right) + 1 = 0$

Put $\quad v = \frac{x}{y}$ or $x = vy$

and $\quad \frac{dx}{dy} = v + y\frac{dv}{dy}$

Then, we get $v + y\frac{dv}{dy} + v^2 - v + 1 = 0$

$\Rightarrow \quad \frac{dv}{v^2+1} + \frac{dy}{y} = 0$

$\Rightarrow \quad \tan^{-1}(v) + \log y + C = 0$ [integrating]

$\therefore \quad \tan^{-1}\left(\frac{x}{y}\right) + \log y + C = 0$

23 We have $\frac{dy}{dx} + \frac{y(x+y)}{x^2} = 0$

On putting $\quad y = vx$, we get

$v + x\frac{dv}{dx} + v(1+v) = 0$

or $\quad \frac{dx}{x} + \frac{dv}{v(v+2)} = 0$

or $\frac{dx}{x} + \frac{1}{2}\left(\frac{1}{v} - \frac{1}{v+2}\right)dv = 0$

On integrating, we get

$\log x + \frac{1}{2}[\log v - \log(v+2)] + \log C = 0$

or $\quad \log(v+2) = \log x^2 vC^2$

or $\quad y + 2x = C^2x^2y.$

24 Given, $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$\Rightarrow \quad v + \frac{xdv}{dx} = \frac{v}{1+v^2}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{1+v^2} - v = -\frac{v^3}{1+v^2}$

$\Rightarrow \quad \frac{1+v^2}{v^3} dv = -\frac{dx}{x}$

$\Rightarrow -\frac{1}{2v^2} + \log v = -\log x + C$

$\Rightarrow \frac{x^2}{2y^2} + C = \log\frac{y}{x} + \log x = \log y.$

Now, $\quad x = 1, y = 1$

$\Rightarrow \quad C = -1/2$

and $\quad x = x_0, y = e$

$\Rightarrow \quad \frac{x_0^2}{2e^2} - \frac{1}{2} = 1$

$\Rightarrow \quad x_0^2 = 3e^2.$

25 We have,

$y(1+xy)dx = xdy$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y(1+xy)}{x} = \frac{y}{x} + y^2$

On putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$v + x\frac{dv}{dx} = v + v^2x^2$

$\Rightarrow \quad \frac{dv}{v^2} = xdx$

$\Rightarrow \quad \int\frac{dv}{v^2} = \int xdx$

$\Rightarrow \quad -\frac{1}{v} = \frac{x^2}{2} + C$

$\Rightarrow \quad -\frac{x}{y} = \frac{x^2}{2} + C$

Put $(1, -1)$, then

$C = \frac{1}{2}$

$\therefore \quad -\frac{x}{y} = \frac{x^2}{2} + \frac{1}{2}$

Now, put $x = -\frac{1}{2}, y = \frac{4}{5}$

$\therefore \quad f\left(-\frac{1}{2}\right) = \frac{4}{5}$

26 Given, $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$

$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$

$\therefore \quad \text{IF} = e^{\int\frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$

and the solution is

$y \cdot (1+x^2) = \int 4x^2\,dx + C$

$y(1+x^2) = \frac{4x^3}{3} + C$

Now, $x = 0, y = 0$

$\Rightarrow \quad C = 0$

$\therefore \quad y(1+x^2) = \frac{4x^3}{3}$

$\Rightarrow \quad 3y(1+x^2) = 4x^3$

27 We have, $\frac{dy}{dx} + \frac{3x^2}{1+x^3}y = \frac{\sin^2 x}{1+x^3}$

Since, it is a linear equation with

$P = \frac{3x^2}{1+x^3}$

$\therefore \text{IF} = e^{\int P\,dx} = e^{\log(1+x^3)} = 1 + x^3$

and the solution is

$$y(1+x^3) = \int \frac{\sin^2 x}{1+x^3}(1+x^3)\,dx$$

$$= \int \frac{1-\cos 2x}{2}\,dx$$

$$\therefore\ y(1+x^3) = \frac{1}{2}x - \frac{\sin 2x}{4} + C$$

28 Given, $\frac{dy}{dx} - y\tan x = 2x\sec x, y(0) = 0$

$$\text{IF} = e^{-\int \tan x\,dx} = e^{-\log \sec x}$$

$$\text{IF} = \cos x$$

$$\therefore\ \cos x\cdot y = \int 2x\sec x\cdot\cos x\,dx$$

$$\Rightarrow\ \cos x\cdot y = x^2 + C$$

$$\Rightarrow\ y(0) = 0 \Rightarrow C = 0$$

$$\therefore\ y = x^2\sec x$$

and $y' = 2x\cdot\sec x + x^2\sec x\cdot\tan x$

$$y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16}\cdot\sqrt{2} = \frac{\pi^2}{8\sqrt{2}}$$

$$y'\left(\frac{\pi}{4}\right) = \frac{\pi}{2}\cdot\sqrt{2} + \frac{\pi^2}{16}\cdot\sqrt{2}$$

$$y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}\cdot 2 = \frac{2\pi^2}{9}$$

$$y'\left(\frac{\pi}{3}\right) = 2\frac{\pi}{3}\cdot 2 + \frac{\pi^2}{9}\cdot 2\cdot\sqrt{3}$$

$$= \frac{4\pi}{3} + \frac{2\pi^2\sqrt{3}}{9}$$

29 This is the linear equation of the form $\frac{dy}{dx} + Py = Q$.

where, $P = \tan x$ and $Q = x^m\cos x$

Now, integrating factor (IF)

$$= e^{\int P\,dx} = e^{\int \tan x\,dx}$$

$$= e^{\log\sec x} = \sec x$$

and the solution is given by,

$$y\cdot e^{\int P\,dx} = \int Q\cdot e^{\int P\,dx}\,dx + C$$

$$\Rightarrow y\cdot\sec x = \int x^m\cdot\cos x\cdot\sec x\,dx + C$$

$$\Rightarrow\ y\sec x = \frac{x^{m+1}}{m+1} + C$$

$$\therefore (m+1)y = x^{m+1}\cos x + C(m+1)\cos x$$

30 Given differential equation is $\frac{dP}{dt} - \frac{1}{2}P(t) = -200$, which is a linear differential equation.

Here, $P(t) = \frac{-1}{2}$ and $Q(t) = -200$

Now, $\text{IF} = e^{\int -\left(\frac{1}{2}\right)dt} = e^{-\frac{t}{2}}$

and the solution is

$$P(t).\text{IF} = \int Q(t)\ \text{IF}\ dt + K$$

$$P(t)\cdot e^{-\frac{t}{2}} = -\int 200\,e^{-\frac{t}{2}}\,dt + K$$

$$P(t)\cdot e^{-\frac{t}{2}} = 400\,e^{-\frac{t}{2}} + K$$

$$\Rightarrow P(t) = 400 + Ke^{\frac{t}{2}}$$

If $P(0) = 100$, then $K = -300$

$$\Rightarrow P(t) = 400 - 300\,e^{\frac{t}{2}}$$

31 We have, $(xy^5 + 2y)dx = x\,dy$

$$\Rightarrow\ x\frac{dy}{dx} - 2y = xy^5$$

$$\Rightarrow\ \frac{dy}{dx} - \frac{2y}{x} = y^5$$

$$\Rightarrow y^{-5}\frac{dy}{dx} - \frac{2y^{-4}}{x} = 1 \quad \ldots(\text{i})$$

Put, $y^{-4} = t$

$$\Rightarrow\ -4y^{-5}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow\ y^{-5}\frac{dy}{dx} = \frac{-1}{4}\frac{dt}{dx} \quad \ldots(\text{ii})$$

From Eqs. (i) and (ii), we get

$$-\frac{1}{4}\frac{dt}{dx} - \frac{2t}{x} = 1$$

$$\Rightarrow\ \frac{dt}{dx} + \frac{8t}{x} = -4$$

Now, $\text{IF} = e^{\int\frac{8}{x}dx} = e^{8\log x} = x^8$

and the solution is

$$t\cdot x^8 = \int (-4)x^8\,dx + C$$

$$\Rightarrow\ \frac{x^8}{y^4} = -\frac{4\cdot x^9}{9} + C$$

$$\Rightarrow 9x^8 + 4x^9\cdot y^4 = 9y^4C$$

32 We have, $\sin x\frac{dy}{dx} + y\cos x = 4x$

$$\Rightarrow\ \sin x\,dy + y\cos x\,dx = 4x\,dx$$

$$\Rightarrow\ d(y\sin x) = 4x\,dx$$

On integrating both sides, we get

$$y\sin x = 2x^2 + C$$

Since, it is passes through $\left(\frac{\pi}{2}, 0\right)$

$$\therefore\ 0 = \frac{\pi^2}{2} + C \Rightarrow C = -\pi^2/2$$

$$\Rightarrow\ y\sin x = 2x^2 - \pi^2/2$$

$$\Rightarrow\ y = 2x^2\text{cosec}\,x - \frac{\pi^2}{2}\text{cosec}\,x$$

$$\Rightarrow y(\pi/6) = 2\left(\frac{\pi^2}{36}\right)\text{cosec}\frac{\pi}{6} - \frac{\pi^2}{2}\text{cosec}\,\pi/6$$

$$= 2\left(\frac{\pi^2}{36}\right)2 - \frac{\pi^2}{2}\cdot 2$$

$$= -\frac{8\pi^2}{9}$$

33 **Statement I** Let the equation of parabola whose axis is the axis of x and vertex at the origin is

$$y^2 = 4ax$$

$$\Rightarrow\ 2y\frac{dy}{dx} = 4a$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{2a}{y}$$

$$\Rightarrow\ \frac{dy}{dx} \propto \frac{1}{y}$$

[where, $a \to$ parameter]

Statement II $y^2 = 4ax \quad \ldots(\text{i})$

$$\Rightarrow\ 2y\frac{dy}{dx} = 4a$$

$$\Rightarrow\ a = \frac{y}{2}\cdot\frac{dy}{dx} \quad [\text{from Eq. (i)}]$$

$$\Rightarrow\ y^2 = 4x\cdot\frac{y}{2}\cdot\frac{dy}{dx}$$

$$\Rightarrow\ y^2 = 2xy\frac{dy}{dx}$$

$$\Rightarrow\ y = 2x\cdot\frac{dy}{dx},$$

which has order = 1 and degree = 1

34 Let $c_1 + c_2 + c_3e^{c_4} = A$ [constant]

Then, $y = Ax$

$$\Rightarrow\ \frac{dy}{dx} = A$$

$$\Rightarrow\ y = x\frac{dy}{dx}$$

$$\therefore\ x\frac{dy}{dx} = y$$

SESSION 2

1 Given, $y = c_1e^{c_2x}$

$$\Rightarrow\ y' = c_1c_2e^{c_2x} \Rightarrow c_2 = \frac{y'}{y}$$

and $y'' = c_1c_2^2e^{c_2x} \Rightarrow y'' = y.\left(\frac{y'}{y}\right)^2$

$$\Rightarrow\ yy'' = (y')^2$$

2 Given,

$$p'(t) = \frac{dp(t)}{dt} = 0.5\ p(t) - 450$$

$$\Rightarrow\ \frac{2dp(t)}{p(t) - 900} = dt$$

$$\Rightarrow\ \int\frac{2dp(t)}{p(t) - 900} = \int dt$$

$$\Rightarrow\ 2\log|p(t) - 900| = t + C$$

To find the value of C, let's substitute $t = 0$ and $p(0) = 850$

$$\Rightarrow\ 2\log|p(0) - 900| = 0 + C$$

$$\Rightarrow\ C = 2\log|850 - 900|$$

$$\Rightarrow\ C = 2\log 50$$

Now, $2\log|p(t) - 900| = t + 2\log 50$

Now, put $p(t) = 0$, then

$$2\log|0 - 900| = t + 2\log 50$$

$$\Rightarrow\ t = 2\log\left|\frac{900}{50}\right| = 2\log 18$$

3 Given slope at (x, y) is

$$\frac{dy}{dx} = \frac{y}{x} + \sec(y/x)$$

Let $\frac{y}{x} = t \Rightarrow y = xt$

and $\frac{dy}{dx} = t + x\frac{dt}{dx}$

Now, $t + x\frac{dt}{dx} = t + \sec(t)$

$$\int \cos t\, dt = \int \frac{1}{x} dx$$

$$\sin t = \ln x + C$$

$$\sin(y/x) = \ln x + C$$

$\because$ This curve passes through $(1, \pi/6)$

$$\sin(\pi/6) = \ln(1) + C \Rightarrow C = \frac{1}{2}$$

Thus, $\sin\left(\frac{y}{x}\right) = \ln x + \frac{1}{2}$

4 We have, $y'(x) + y(x)g'(x) = g(x)g'(x)$

Linear differential equation with integrating factor $e^{g(x)}$

$\Rightarrow \quad y(x)\cdot e^{g(x)} = \int g(x)\cdot g'(x)\cdot e^{g(x)} dx$

$\Rightarrow \quad y(x)\cdot e^{g(x)} = e^{g(x)}(g(x)-1) + C$

Since, $y(0) = 0$ and $g(0) = 0$, therefore $C = 1$

$\Rightarrow \quad y(x) = (g(x)-1) + e^{-g(x)}$

$\Rightarrow \quad y(2) = (g(2)-1) + e^{-g(2)} = 0,$

as $g(2) = 0$.

5 Here, $\frac{dx}{dy} + \frac{1}{y^2}\cdot x = \frac{1}{y^3}$

[linear differential equation in x]

Clearly, $\quad \text{IF} = e^{\int \frac{1}{y^2} dy} = e^{-\frac{1}{y}}$

Now, complete solution is,

$$x\cdot e^{-\frac{1}{y}} = \int \frac{1}{y^3}\cdot e^{-\frac{1}{y}}\, dy$$

$\Rightarrow \quad x\cdot e^{-\frac{1}{y}} = \int \frac{1}{y}\cdot\frac{1}{y^2}\cdot e^{-\frac{1}{y}}\, dy$

Put $\quad -\frac{1}{y} = t \Rightarrow \frac{1}{y^2} dy = dt$

$\Rightarrow \quad xe^{-\frac{1}{y}} = \int -t\cdot e^t dt$

$\Rightarrow \quad xe^{-\frac{1}{y}} = -\{t\cdot e^t - \int 1\cdot e^t dt\} + C$

$\Rightarrow \quad xe^{-\frac{1}{y}} = -te^t + e^t + C$

$\Rightarrow \quad xe^{-\frac{1}{y}} = \frac{1}{y}\cdot e^{-\frac{1}{y}} + e^{-\frac{1}{y}} + C$

$\Rightarrow \quad e^{-1} = e^{-1} + e^{-1} + C \quad [\because y(1) = 1]$

$\Rightarrow \quad C = -\frac{1}{e}$

$\Rightarrow \quad xe^{-\frac{1}{y}} = \frac{1}{y}\cdot e^{-\frac{1}{y}} + e^{-\frac{1}{y}} - \frac{1}{e}$

$\therefore \quad x = \frac{1}{y} + 1 - \frac{1}{e}\cdot e^{\frac{1}{y}}$

6 On putting $10x + 6y = t$ and $10 + 6\frac{dy}{dx} = \frac{dt}{dx}$, we get

$$\frac{dt}{dx} = 6\sin t + 10 = 6\left(\frac{2\tan t/2}{1+\tan^2 t/2}\right) + 10$$

$\Rightarrow \frac{\sec^2 t/2}{10\tan^2 t/2 + 12\tan t/2 + 10} dt = dx$

$\Rightarrow \quad \frac{dz}{5z^2 + 6z + 5} = dx \qquad [z = \tan(t/2)]$

$\Rightarrow \quad \frac{1}{5}\int \frac{dz}{\left(z+\frac{3}{5}\right)^2 + (4/5)^2} = \int dx$

$\Rightarrow \quad \frac{1}{4}\tan^{-1}\frac{5z+3}{4} = x + C$

$\Rightarrow \quad 5\tan t/2 + 3 = 4\tan 4(x + C)$

$\Rightarrow \quad 5\tan(5x+3y) + 3 = 4\tan 4(x + C)$

Now, $x = 0, y = 0 \Rightarrow C = \frac{1}{4}\tan^{-1}(3/4)$

Hence, equation of the curve is

$$5\tan(5x+3y) = \frac{25\tan 4x}{4 - 3\tan 4x}$$

$\Rightarrow \quad y = \frac{1}{3}\left\{\tan^{-1}\frac{5\tan 4x}{4-3\tan 4x} - 5x\right\}.$

7 Given, $\frac{d\{V(t)\}}{dt} = -k(T-t)$

$\therefore \quad d\{V(t)\} = -k(T-t)\,dt \qquad \ldots(\text{i})$

$\Rightarrow \quad \int_0^T d\{V(t)\} = \int_0^T -k(T-t)dt$

$\Rightarrow \quad V(T) - V(0) = k\left[\frac{(t-T)^2}{2}\right]_0^T$

$\Rightarrow V(T) - I = \frac{k}{2}[(T-T)^2 - (0-T)^2]$

[$\because$ when $t = 0$, then $V(t) = I$]

$\therefore \quad V(T) = I - \frac{k}{2}T^2$

8 On putting $x = \tan A$, and $y = \tan B$ in the given relation, we get

$\cos A + \cos B = \lambda(\sin A - \sin B)$

$\Rightarrow \quad \tan\left(\frac{A-B}{2}\right) = \frac{1}{\lambda}$

$\Rightarrow \tan^{-1} x - \tan^{-1} y = 2\tan^{-1}\left(\frac{1}{\lambda}\right)$

On differentiating w.r.t. x, we get

$$\frac{1}{1+x^2} - \frac{1}{1+y^2}\cdot\frac{dy}{dx} = 0$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Clearly, it is a differential equation of degree 1.

9 $|y|\sqrt{1 + (dx/dy)^2} = 1$

$\Rightarrow \quad y^2\left(1 + \left(\frac{dx}{dy}\right)^2\right) = 1$

$\therefore \quad \frac{dy}{dx} = \pm\frac{y}{\sqrt{1-y^2}}$

$\Rightarrow \int \frac{\sqrt{1-y^2}}{y} dy = \pm x + C.$

On putting $y = \sin\theta$ and $dy = \cos\theta\, d\theta$, we get

$$C \pm x = \int \frac{\cos\theta\cdot\cos\theta\, d\theta}{\sin\theta}$$

$$= \int (\operatorname{cosec}\theta - \sin\theta)d\theta$$

$\Rightarrow \quad C \pm x = \log|\operatorname{cosec}\theta - \cot\theta| + \cos\theta$

$\Rightarrow \quad C \pm x = \log\left|\frac{1-\sqrt{1-y^2}}{y}\right| + \sqrt{1-y^2}.$

10 Given, $\quad \cos x\, dy = y\sin x\, dx - y^2 dx$

$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$

Put $\quad -\frac{1}{y} = z \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore \quad \frac{dz}{dx} + (\tan x)z = -\sec x$

This is a linear differential equation.

Now, $\text{IF} = e^{\int \tan x\, dx} = e^{\log\sec x} = \sec x$

Hence, the solution is

$z\cdot(\sec x) = \int -\sec x\cdot\sec x\, dx + C_1$

$\Rightarrow \quad -\frac{1}{y}\sec x = -\tan x + C_1$

$\therefore \quad \sec x = y(\tan x + C)$

where $C = -C_1$.

11 We have, $(1-x^2)\frac{dy}{dx} + xy = ax$

$\Rightarrow \quad \frac{dy}{dx} + \frac{x}{1-x^2}y = \frac{ax}{1-x^2} \qquad$ [L. D. E.]

$$\text{IF} = e^{\int (x/1-x^2)dx} = e^{-\frac{1}{2}\log|1-x^2|}$$

$$= \begin{cases} \frac{1}{\sqrt{1-x^2}}, & \text{if } -1 < x < 1 \\ \frac{1}{\sqrt{x^2-1}}, & \text{if } x < -1 \text{ or } x > 1. \end{cases}$$

If $-1 < x < 1$, then solution is

$$y\cdot\frac{1}{\sqrt{1-x^2}} = a\int \frac{x}{(1-x^2)^{3/2}} dx + C$$

$$y\cdot\frac{1}{\sqrt{1-x^2}} = \frac{a}{\sqrt{1-x^2}} + C$$

$\Rightarrow \quad y = a + C\sqrt{1-x^2}$

$\Rightarrow \quad (y-a)^2 = C^2(1-x^2)$

$\Rightarrow \quad C^2x^2 + (y-a)^2 = C^2$

which is an ellipse.

If $x < -1$ or $x > 1$, solution is

$$y\cdot\frac{1}{\sqrt{x^2-1}} = C + a\int \frac{x}{(1-x^2)\sqrt{x^2-1}} dx$$

$$= C - a\int \frac{x}{(x^2-1)^{3/2}} dx = C + \frac{a}{\sqrt{x^2-1}}$$

$\Rightarrow \quad y = a + C\sqrt{x^2-1}$

$\Rightarrow \quad (y-a)^2 = C^2x^2 - c^2$

$\Rightarrow \quad C^2x^2 - (y-a)^2 = C^2$

which represents a hyperbola.

12 Given,

$$\frac{dy}{dx} = \frac{(x+1)^2 + y - 3}{x+1} = (x+1) + \frac{y-3}{x+1}$$

Putting $x + 1 = X$, $y - 3 = Y$, $\frac{dy}{dx} = \frac{dY}{dX}$,

the equation becomes

$$\frac{dY}{dX} = X + \frac{Y}{X}$$

or $\quad \frac{dY}{dX} - \frac{1}{X} \cdot Y = X$ [L. D. E.]

$\text{IF} = e^{\int (-1/X) dX} = e^{-\log X} = X^{-1} = \frac{1}{X}$

$\therefore$ The solution is

$$Y \cdot \left(\frac{1}{X}\right) = C + \int X \cdot \left(\frac{1}{X}\right) dx = C + X$$

or $\quad \frac{(y-3)}{(x+1)} = C + x + 1$

Now, $\quad x = 2, y = 0$

$\Rightarrow \quad \frac{0-3}{2+1} = C + 2 + 1$

$\Rightarrow \quad C = -4$

$\therefore$ The equation of the curve is

$$\frac{y-3}{x+1} = x - 3 \text{ or } y = x^2 - 2x.$$

13 We have, $\dfrac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots} = \dfrac{dx - dy}{dx + dy}$

On applying componendo and dividendo, we get

$$\frac{\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) + \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)}{\left(x + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) - \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)}$$

$$= \frac{(dx - dy) + (dx + dy)}{(dx - dy) - (dx - dy)}$$

$$\Rightarrow \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)}{-\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)} = \frac{2dx}{-2dy} = -\frac{dx}{dy}$$

$\Rightarrow \quad \frac{e^x}{-e^{-x}} = -\frac{dx}{dy}$

$\Rightarrow \quad dy = e^{-2x} dx$

On integrating both sides, we get

$$y = \frac{e^{-2x}}{-2} + C_1$$

$\Rightarrow \quad 2ye^{2x} = -1 + 2C_1 e^{2x}$

$\Rightarrow \quad 2ye^{2x} = Ce^{2X} - 1$, where $C = 2C_1$

14 Equation of tangent is $Y - y = \frac{dy}{dx}(X - x)$

Y-intercept of tangent is $y - x\frac{dy}{dx}$

From given condition, $y - x\frac{dy}{dx} = x^3$,

$\Rightarrow \quad \frac{dy}{dx} - \frac{1}{x} \cdot y = -x^2$ ...(i)

Now, $\quad \text{IF} = e^{-\int \frac{1}{x} dx} = e^{-\log_e x} = \frac{1}{x}$

and solution is

$$\frac{1}{x} \cdot y = -\int x dx = -\frac{x^2}{2} + C$$

$\Rightarrow \quad f(x) = \frac{-x^3}{2} + Cx$

Now, $f(1) = 1 \Rightarrow C = 1 + \frac{1}{2} = \frac{3}{2}$

$\therefore \quad f(x) = \frac{-x^3}{2} + \frac{3}{2}x$

$\Rightarrow \quad f(-3) = \frac{27}{2} - \frac{9}{2} = 9$

15 $\lim\limits_{t \to x} \dfrac{t^2 f(x) - x^2 f(x) + x^2 f(x) - x^2 f(t)}{t - x} = 1$

$\lim\limits_{t \to x} \left[(t + x) f(x) + x^2 \left(\frac{f(x) - f(t)}{t - x}\right)\right] = 1$

$2x f(x) + x^2(-f'(x)) = 1$

$\Rightarrow \quad \dfrac{x^2 \frac{dy}{dx} - 2xy}{x^4} = -\frac{1}{x^4}$,

where $y = f(x)$

$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = -\frac{1}{x^4}$$

$\Rightarrow \quad x^{-2} y = \frac{1}{3x^3} + C$

$\Rightarrow \quad x^{-2} y = \frac{1}{3x^3} + \frac{2}{3}$ $\quad [\because f(1) = 1]$

$\Rightarrow \quad y = \frac{1}{3x} + \frac{2x^2}{3}$.

16 Given, $\dfrac{dy}{dx} = \dfrac{y\sqrt{y^2 - 1}}{x\sqrt{x^2 - 1}}$

$\Rightarrow \quad \int \frac{dy}{y\sqrt{y^2 - 1}} = \int \frac{dx}{x\sqrt{x^2 - 1}}$

$\Rightarrow \quad \sec^{-1} y = \sec^{-1} x + C$

Now, $x = 2$, $y = \frac{2}{\sqrt{3}}$

$\Rightarrow \quad \frac{\pi}{6} = \frac{\pi}{3} + C \Rightarrow C = -\frac{\pi}{6}$

$\therefore \quad y = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$

$\Rightarrow \quad \frac{1}{y} = \cos\left(\cos^{-1}\frac{1}{x} - \cos^{-1}\frac{\sqrt{3}}{2}\right)$

$$\Rightarrow \quad \frac{1}{y} = \cos\left[\cos^{-1}\left(\frac{\sqrt{3}}{2x} + \sqrt{1 - \frac{1}{x^2}} \cdot \sqrt{1 - \frac{3}{4}}\right)\right]$$

$\therefore \quad \frac{1}{y} = \frac{\sqrt{3}}{2x} + \frac{1}{2}\sqrt{1 - \frac{1}{x^2}}$