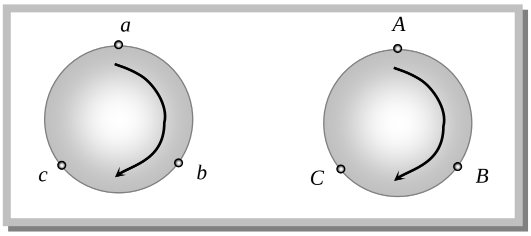
Properties of Triangles and Solutions of Triangles

3.1 Relation between Sides and Angles

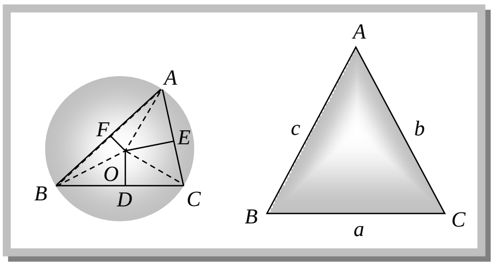
A triangle has six components, three sides and three angles. The three angles of a ΔABC are denoted by letters *A*, *B*, *C* and the sides opposite to these angles by letters *a*, *b* and *c* respectively. Following are some well known relations for a triangle (say ΔABC)



- $A + B + C = 180^o$ (or π)
- $a + b > c$, $b + c > a$, $c + a > b$
- $|a - b| < c, |b - c| < a, |c - a| < b$

Generally, the relations involving the sides and angles of a triangle are cyclic in nature, e.g. to obtain the second similar relation to $a + b > c$, we simply replace *a* by *b*, *b* by *c* and *c* by *a*. So, to write all the relations, follow the cycles given.

(1) **The law of sines or sine rule :** The sides of a triangle are proportional to the sines of the angles opposite to them *i.e.*, $\boxed{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k \text{ (say)}}$



More generally, if *R* be the radius of the circumcircle of the triangle *ABC*,

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R}$$

Note: ❑ The above rule may also be expressed as $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

❑ The sine rule is very useful tool to express sides of a triangle in terms of sines of angle and vice-versa in the following manner $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = K \text{ (Let)} \Rightarrow a = K\sin A, b = K \sin B, c = K \sin C$.

Similarly, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = \lambda$ (Let) $\Rightarrow \sin A = \lambda a, \; \sin B = \lambda b, \; \sin C = \lambda c$.

Example: 1 If the angles of a triangle are in the ratio 4 : 1 : 1, then the ratio of the longest side to the perimeter is **[IIT Screening 2003]**

(a) $\sqrt{3} : (2+\sqrt{3})$ (b) $1:6$ (c) $1:(2+\sqrt{3})$ (d) $2:3$

Solution: (a) $4x + x + x = 180 \Rightarrow 6x = 180 \Rightarrow x = 30^o$

$$\frac{\sin 120^o}{a} = \frac{\sin 30^o}{b} = \frac{\sin 30^o}{c}$$

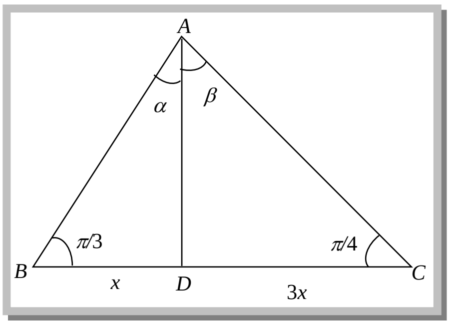
$$\therefore \; a:(a+b+c) = (\sin 120^o):(\sin 120^o + \sin 30^o + \sin 30^o) = \frac{\sqrt{3}}{2} : \frac{\sqrt{3}+2}{2} = \sqrt{3} : \sqrt{3}+2.$$

Example: 2 In a triangle ABC, $\angle B = \frac{\pi}{3}$ and $\angle C = \frac{\pi}{4}$ and D divides BC internally in the ratio 1 : 3. Then $\frac{\sin \angle BAD}{\sin \angle CAD}$ is equal to

[UPSEAT 2003, 2001; IIT 1995]

(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{6}}$ (d) $\sqrt{\frac{2}{3}}$

Solution: (c) Let $\angle BAD = \alpha, \angle CAD = \beta$



In ΔADB, applying sine formulae, we get $\frac{x}{\sin\alpha} = \frac{AD}{\sin\left(\frac{\pi}{3}\right)}$ ....(i)

In ΔADC, applying sine formulae, we get, $\frac{3x}{\sin\beta} = \frac{AD}{\sin\left(\frac{\pi}{4}\right)}$ ....(ii)

Dividing (i) by (ii), we get,

$$\Rightarrow \frac{x}{\sin\alpha} \times \frac{\sin\beta}{3x} = \frac{AD}{\sin\left(\frac{\pi}{3}\right)} \times \frac{\sin\left(\frac{\pi}{4}\right)}{AD} \Rightarrow \frac{\sin\beta}{3\sin\alpha} = \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{2}} = \sqrt{\frac{2}{3}} \Rightarrow \frac{\sin\beta}{\sin\alpha} = 3\sqrt{\frac{2}{3}} = \sqrt{6}$$

$$\therefore \frac{\sin \angle BAD}{\sin \angle CAD} = \frac{\sin\alpha}{\sin\beta} = \frac{1}{\sqrt{6}}.$$

Example: 3 In a $\Delta ABC, A : B : C = 3 : 5 : 4$. Then $[a + b + c\sqrt{2}]$ is equal to **[DCE 2001]**

(a) $2b$ (b) $2c$ (c) $3b$ (d) $3a$

Solution: (c) $A : B : C = 3 : 5 : 4 \Rightarrow A + B + C = 12x = 180^o \Rightarrow x = 15^o$

$\therefore A = 45^o$, $B = 75^o$, $C = 60^o$

$$\frac{a}{\sin 45^o} = \frac{b}{\sin 75^o} = \frac{c}{\sin 60^o} = k \text{ (say)}$$

i.e., $a = \frac{1}{\sqrt{2}}K$, $b = \frac{\sqrt{3}+1}{2\sqrt{2}}K$, $c = \frac{\sqrt{3}}{2}K$. Hence $[a + b + c\sqrt{2}] = 3b$.

Example: 4 In any triangle ABC if $2\cos B = \frac{a}{c}$, then the triangle is

(a) Right angled (b) Equilateral (c) Isosceles (d) None of these

Solution: (c) $2\cos B = \frac{a}{c} = \frac{k\sin A}{k\sin C} = \frac{\sin A}{\sin C} \Rightarrow 2\cos B\sin C = \sin A \Rightarrow \sin(B + C) - \sin(B - C) = \sin A$

$\Rightarrow \sin(180^o - A) - \sin(B - C) = \sin A \Rightarrow \sin A - \sin(B - C) = \sin A \Rightarrow \sin(B - C) = 0 \Rightarrow B - C = 0 \Rightarrow B = C$

$\therefore$ Triangle is isosceles.

3.2 The Law of Cosines or Cosine Rule

In any triangle ABC, the square of any side is equal to the sum of the squares of the other two sides diminished by twice the product of these sides and the cosine of their included angle, that is for a triangle ABC,

(1) $a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$

(2) $b^2 = c^2 + a^2 - 2ca\cos B \Rightarrow \cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$

(3) $c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow \cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Combining with $\sin A = \dfrac{a}{2R}, \sin B = \dfrac{b}{2R}, \sin C = \dfrac{c}{2R}$

We have by division, $\tan A = \dfrac{abc}{R(b^2 + c^2 - a^2)}$, $\tan B = \dfrac{abc}{R(c^2 + a^2 - b^2)}$, $\tan C = \dfrac{abc}{R(a^2 + b^2 - c^2)}$

Where *R*, be the radius of the circum-circle of the triangle *ABC*.

Example: 5 The smallest angle of the ΔABC, when $a = 7, b = 4\sqrt{3}$ and $c = \sqrt{13}$, is **[MP PET 2003]**

(a) 30^o (b) 15^o (c) 45^o (d) None of these

Solution: (a) Smallest angle is opposite to smaller side

$\therefore \cos C = \dfrac{b^2 + a^2 - c^2}{2ab} = \dfrac{49 + 48 - 13}{2 \times 7 \times 4\sqrt{3}} = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2} \Rightarrow \angle C = 30^o$.

Example: 6 In a ΔABC, if $\dfrac{b+c}{11} = \dfrac{c+a}{12} = \dfrac{a+b}{13}$, then $\cos C =$ **[Karnataka CET 2003]**

(a) $\dfrac{7}{5}$ (b) $\dfrac{5}{7}$ (c) $\dfrac{17}{36}$ (d) $\dfrac{16}{17}$

Solution: (b) $\dfrac{b+c}{11} = \dfrac{c+a}{12} = \dfrac{a+b}{13} = \lambda$ (Let)

$\therefore b + c = 11\lambda$ ......(i)

$c + a = 12\lambda$ ....(ii) and $a + b = 13\lambda$ ....(iii)

From (i) + (ii) + (iii), $2(a + b + c) = 36\lambda \therefore a + b + c = 18\lambda$ ....(iv)

Now subtract (i), (ii) and (iii) from (iv), $a = 7\lambda, b = 6\lambda, c = 5\lambda$.

Now $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab} = \dfrac{(7\lambda)^2 + (6\lambda)^2 - (5\lambda)^2}{2.7\lambda.6\lambda} = \dfrac{49\lambda^2 + 36\lambda^2 - 25\lambda^2}{84\lambda^2} = \dfrac{60\lambda^2}{84\lambda^2} = \dfrac{5}{7}$.

Example: 7 In a ΔABC, $2ac\sin\left(\dfrac{A - B + C}{2}\right)$ is equal to **[IIT Screening 2000]**

(a) $a^2 + b^2 - c^2$ (b) $c^2 + a^2 - b^2$ (c) $b^2 - c^2 - a^2$ (d) $c^2 - a^2 - b^2$

Solution: (b) $2ac\sin\dfrac{A - B + C}{2} = 2ac\sin\dfrac{\pi - 2B}{2} = 2ac\cos B = 2ac\dfrac{c^2 + a^2 - b^2}{2ca} = c^2 + a^2 - b^2$.

Example: 8 In ambiguous case if *a*, *b* and *A* are given and if there are two possible values of third side, are c_1 and c_2, then **[UPSEAT 1999]**

(a) $c_1 - c_2 = 2\sqrt{(a^2 + b^2\sin^2 A)}$ (b) $c_1 - c_2 = 2\sqrt{(a^2 - b^2\sin^2 A)}$

(c) $c_1 - c_2 = 4\sqrt{(a^2 + b^2\sin^2 A)}$ (d) $c_1 - c_2 = 3\sqrt{(a^2 - b^2\sin^2 A)}$

Solution: (b) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ or $c^2 - (2b\cos A)c + (b^2 - a^2) = 0$

Which is quadratic equation in *c*. Let there be two roots, c_1 and c_2 of above quadratic equation then $c_1 + c_2 = 2b\cos A$ and $c_1c_2 = b^2 - a^2$

$\therefore c_1 - c_2 = \sqrt{[(c_1 + c_2)^2 - 4c_1c_2]} = \sqrt{[(2b\cos A)^2 - 4(b^2 - a^2)} = \sqrt{[4a^2 - 4b^2(1 - \cos^2 A)]} = 2\sqrt{(a^2 - b^2\sin^2 A)}$.

Example: 9 In a ΔABC, if $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, then a^2, b^2, c^2 are in **[Karnataka CET 1999]**

(a) A.P. (b) G.P. (c) H.P. (d) None of these

Solution: (a) $\frac{\sin A}{\sin C} = \frac{\sin A\cos B - \cos A \sin B}{\sin B \cos C - \cos B \sin C} \Rightarrow \frac{a}{c} = \frac{a\cos B - b\cos A}{b\cos C - c\cos B}$ (Using sine formula)

$\Rightarrow ab\cos C - ac\cos B = ac\cos B - bc\cos A \Rightarrow ab\cos C + bc\cos A = 2ac\cos B$

$\Rightarrow \frac{a^2+b^2-c^2}{2} + \frac{b^2+c^2-a^2}{2} = \frac{c^2+a^2-b^2}{1} \Rightarrow b^2 = c^2+a^2-b^2 \Rightarrow b^2 = \frac{c^2+a^2}{2} \Rightarrow a^2, b^2, c^2$ are in A.P.

Example: 10 In a triangle ABC, AD is altitude from A. Given $b > c$, $\angle C = 23^o$ and $AD = \frac{abc}{b^2-c^2}$, then $\angle B$ equal to **[IIT 1994]**

(a) 67^o (b) 44^o (c) 113^o (d) None of these

Solution: (c) $\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{a^2-(b^2-c^2)}{2ac}$

Now, $AD = \frac{abc}{b^2-c^2}$; $\therefore \cos B = \frac{a^2 - \frac{abc}{AD}}{2ac}$

Also $AD = b\sin 23^o$; $\therefore \cos B = \frac{a - \frac{c}{\sin 23^o}}{2c}$

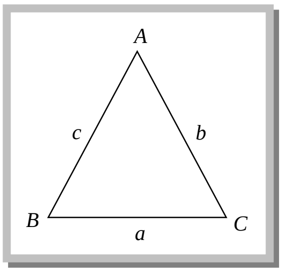
By sine formulae $\Rightarrow \frac{a}{c} = \frac{\sin(B+23^o)}{\sin 23^o}$; $\therefore \cos B = \frac{\left[\frac{\sin(B+23^o)}{\sin 23^o} - \frac{1}{\sin 23^o}\right]}{2}$

$\Rightarrow \sin(23^o - B) = -1 = \sin(-90^o)$; therefore $23^o - B = -90^o$ or $B = 113^o$.

3.3 Projection Formulae

In any triangle ABC, $b\cos C + c\cos B = k\sin B\cos C + k\sin C\cos B$ (from sine rule)

$$= k[\sin(B+C)] = k\sin(\pi - A) = k\sin A = a$$



Similarly, we can deduct other projection formulae from sine rule.

(i) $a = b\cos c + c\cos B$ (ii) $b = c\cos A + a\cos C$ (iii) $c = a\cos B + b\cos A$

i.e., any side of a triangle is equal to the sum of the projection of other two sides on it.

Example: 11 In a ΔABC, $\frac{\cos C + \cos A}{c+a} + \frac{\cos B}{b}$ is equal to **[EAMCET 2001]**

(a) $\frac{1}{a}$ (b) $\frac{1}{b}$ (c) $\frac{1}{c}$ (d) $\frac{c+a}{b}$

Solution: (b) $\frac{\cos C + \cos A}{c+a} + \frac{\cos B}{b} = \frac{(b\cos C + b\cos A) + (c\cos B + a\cos B)}{b(c+a)}$

$= \frac{(b\cos C + c\cos B) + (b\cos A + a\cos B)}{b(c+a)} = \frac{a+c}{b(c+a)}$ (Using projection formulae) $= \frac{1}{b}$.

Example: 12 If k be the perimeter of the ΔABC, then $b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2}$ is equal to

(a) k (b) $2k$ (c) $\frac{k}{2}$ (d) None of these

Solution: (c) $b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2} = \frac{b}{2}(1+\cos C) + \frac{c}{2}(1+\cos B) = \frac{b}{2} + \frac{c}{2} + \frac{1}{2}(b\cos C + c\cos B) = \frac{a+b+c}{2} = \frac{k}{2}$.

Example: 13 $(a+b+c)(\cos A+\cos B+\cos C)=$

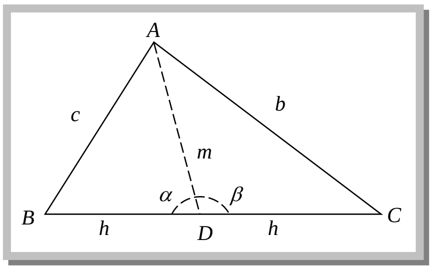
(a) $\sum a\sin^2\frac{A}{2}$ (b) $\sum a\cos^2\frac{A}{2}$ (c) $2\sum a\sin^2\frac{A}{2}$ (d) $2\sum a\cos^2\frac{A}{2}$

Solution: (d) $(a+b+c)(\cos A+\cos B+\cos C)=9$ terms.

$$=\underset{(3)}{\sum a\cos A}+\underset{(6)}{\sum(b\cos C+c\cos B)}=a\cos A+b\cos B+c\cos C+(a+b+c)=\sum a(1+\cos A)=2\sum a\cos^2\frac{A}{2}.$$

3.4 Theorem of the Medians: (Apollonius Theorem)

In every triangle the sum of the squares of any two sides is equal to twice the square on half the third side together with twice the square on the median that bisects the third side.



For any triangle ABC, $b^2+c^2=2(h^2+m^2)=2\{m^2+(a/2)^2\}$ by use of cosine rule.

If Δ be right angled, the mid point of hypotenuse is equidistant from the three vertices so that $DA=DB=DC$

$\therefore b^2+c^2=a^2$ which is pythagoras theorem. This theorem is very useful for solving problems of height and distance.

Example: 14 If AD, BE and CF are the medians of a ΔABC then $(AD^2+BE^2+CF^2):(BC^2+CA^2+AB^2)$ is equal to

(a) 4 : 2 (b) 3 : 2 (c) 3 : 4 (d) 2 : 3

Solution: (c) We have, $AB^2+AC^2=2(AD^2+BD^2)\Rightarrow\frac{c^2+b^2}{2}-\frac{a^2}{4}=AD^2$ ......(i)

$\frac{a^2+c^2}{2}-\frac{b^2}{4}=BE^2$ ......(ii) and $\frac{a^2+b^2}{2}-\frac{c^2}{4}=CF^2$ ....(iii)

$$a^2+b^2+c^2-\frac{a^2+b^2+c^2}{4}=AD^2+BE^2+CF^2$$

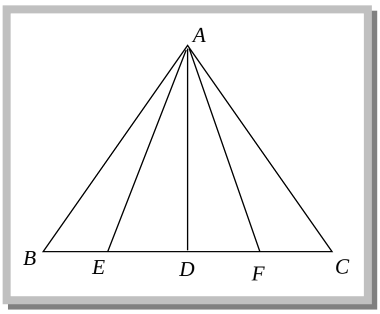
Adding (i), (ii) and (iii) we get, $(AD^2+BE^2+CF^2):(a^2+b^2+c^2)=3:4$.

Example: 15 AD is a median of the ΔABC, if AE and AF are medians of the triangles ABD and ADC respectively and $AD=m_1$, $AE=m_2$, $AF=m_3$, then $\frac{a^2}{8}$ is equal to

(a) $m_2^2+m_3^2-2m_1^2$ (b) $m_1^2+m_2^2-2m_3^2$ (c) $m_2^2+m_3^2-m_1^2$ (d) None of these

Solution: (a) In ΔABC, $AD^2=m_1^2=\frac{c^2+b^2}{2}-\frac{a^2}{4}$

In ΔABD, $AE^2=m_2^2=\frac{c^2+AD^2}{2}-\frac{\left(\frac{a}{2}\right)^2}{4}$



In ΔADC, $AF^2=m_3^2=\frac{AD^2+b^2}{2}-\frac{\left(\frac{a}{2}\right)^2}{4}$

$\therefore\ m_2^2+m_3^2=AD^2+\frac{b^2+c^2}{2}-\frac{a^2}{8}=m_1^2+m_1^2+\frac{a^2}{4}-\frac{a^2}{8}$

$m_2^2+m_3^2=2m_1^2+\frac{a^2}{8}\Rightarrow\frac{a^2}{8}=m_2^2+m_3^2-2m_1^2$.

3.5 Napier's Analogy (Law of Tangents)

For any triangle ABC,

(1) $\tan\left(\frac{A-B}{2}\right)=\left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$ (2) $\tan\left(\frac{B-C}{2}\right)=\left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$ (3) $\tan\left(\frac{C-A}{2}\right)=\left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$

Note : ❑ **Mollweide's formula:** For any triangle, $\frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}, \frac{a-b}{c}=\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}$.

Example: 16 If $\tan\frac{B-C}{2}=x\cot\frac{A}{2}$, then x equal to **[MP PET 1992, 2002]**

(a) $\frac{c-a}{c+a}$ (b) $\frac{a-b}{a+b}$ (c) $\frac{b-c}{b+c}$ (d) None of these

Solution: (c) We know, $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2} \Rightarrow x=\frac{b-c}{b+c}$.

Example: 17 If in a ΔABC $a=6, b=3$ and $\cos(A-B)=\frac{4}{5}$, then **[Roorkee 1997]**

(a) $C=\frac{\pi}{4}$ (b) $A=\sin^{-1}\frac{2}{\sqrt{5}}$ (c) $ar(\Delta ABC)=9$ (d) None of these

Solution:(b,c) $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}=\frac{1}{3}\cot\frac{C}{2}$

$\therefore \frac{4}{5}=\cos(A-B)=\frac{1-\tan^2\frac{A-B}{2}}{1+\tan^2\frac{A-B}{2}}=\frac{1-\frac{1}{9}\cot^2\frac{C}{2}}{1+\frac{1}{9}\cot^2\frac{C}{2}}$

$\therefore \tan^2\frac{C}{2}=1 \Rightarrow C=\frac{\pi}{2} \quad \therefore ar(\Delta ABC)=\frac{1}{2}ab=\frac{1}{2}.6.3=9$

Also, $\sin A=\frac{6}{\sqrt{3^2+6^2}}=\frac{2}{\sqrt{5}}$.

Example: 18 In a $\Delta ABC, A=\frac{\pi}{3}$ and $b:c=2:3$. If $\tan\alpha=\frac{\sqrt{3}}{5}, 0<\alpha<\frac{\pi}{2}$, then

(a) $B=60^o+\alpha$ (b) $C=60^o+\alpha$ (c) $B=60^o-\alpha$ (d) $C=60^o-\alpha$

Solution:(b,c) $\tan\frac{C-B}{2}=\frac{c-b}{c+b}\cot\frac{A}{2} \Rightarrow \tan\frac{C-B}{2}=\frac{1}{5}\cot 30^o=\frac{\sqrt{3}}{5}=\tan\alpha$

$\therefore C-B=2\alpha$ and $C+B=180^o-60^o=120^o$ *i.e.*, $B=60^o-\alpha$, $C=60^o+\alpha$.

Example: 19 In a ΔABC, $a=2b$ and $|A-B|=\frac{\pi}{3}$. The measure of $\angle C$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) None of these

Solution: (b) Clearly, $A>B$ $(\because a>b)$

Now $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2} \Rightarrow \tan 30^o=\frac{1}{3}\cot\frac{C}{2}$

$\therefore \sqrt{3}=\cot\frac{C}{2}$ or $\frac{C}{2}=\frac{\pi}{6} \Rightarrow C=\frac{\pi}{3}$.

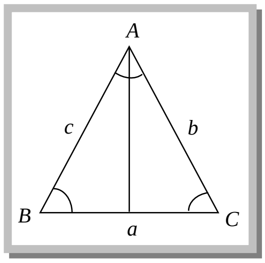
3.6 Area of Triangle

Let three angles of ΔABC are denoted by A, B, C and the sides opposite to these angles by letters a, b, c respectively.

(1) **When two sides and the included angle be given:** The area of triangle ABC is given by, $\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$ *i.e.*, $\Delta = \frac{1}{2}$ (Product of two sides) × sine of included angle

(2) **When three sides are given:** Area of $\Delta ABC = \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

where semiperimeter of triangle $s = \frac{a+b+c}{2}$



(3) **When three sides and the circum-radius be given:** Area of triangle $\Delta = \frac{abc}{4R}$,

where R be the circum-radius of the triangle.

(4) **When two angles and included side be given :**

$$\Delta = \frac{1}{2}a^2\frac{\sin B\sin C}{\sin(B+C)} = \frac{1}{2}b^2\frac{\sin A\sin C}{\sin(A+C)} = \frac{1}{2}c^2\frac{\sin A\sin B}{\sin(A+B)}$$

Example: 20 In a ΔABC if $a = 2x, b = 2y$ and $\angle C = 120^o$, then the area of the triangle is **[MP PET 1986, 2002]**

(a) xy (b) $xy\sqrt{3}$ (c) $3xy$ (d) $2xy$

Solution: (b) $\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}.2x.2y\sin 120^o = \sqrt{3}xy$.

Example: 21 In a ΔABC, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ and $a = 2$, then the area of a triangle is **[MP PET 2000; IIT 1993]**

(a) 1 (b) 2 (c) $\frac{\sqrt{3}}{2}$ (d) $\sqrt{3}$

Solution: (d) By sine rule, $\tan A = \tan B = \tan C$; $\therefore$ Triangle is equilateral .

Hence, $\Delta = \frac{1}{2}.a.a.\sin 60^o = \frac{1}{2}.2.2.\frac{\sqrt{3}}{2} = \sqrt{3}$.

Example: 22 In a triangle ABC, a, b, A are given and c_1, c_2 are two values of third side c. The sum of the areas of triangles with sides a, b, c_1 and a, b, c_2 is

(a) $\frac{1}{2}a^2\sin 2A$ (b) $\frac{1}{2}b^2\sin 2A$ (c) $b^2\sin 2A$ (d) $a^2\sin 2A$

Solution: (b) Let the triangles be $\Delta_1 = ABC_1$ and $\Delta_2 = ABC_2$ A, b, a are given and c has two values. Hence we apply cosine formulae $\cos A = \frac{b^2+c^2-a^2}{2bc}$ or $c^2 - 2bc\cos A + b^2 - a^2 = 0$.

Above is quadratic in c If c_1, c_2 be the two values of c, then $c_1 + c_2 = 2b\cos A, c_1c_2 = b^2 - a^2$

$\Delta_1 = \frac{1}{2}ab\sin C_1$, $\Delta_2 = \frac{1}{2}ab\sin C_2$

$\therefore$ $\Delta_1 + \Delta_2 = \frac{1}{2}ab(\sin C_1 + \sin C_2) = \frac{1}{2}abk(2b\cos A) = b^2ak\cos A = b^2\sin A\cos A = \frac{1}{2}b^2\sin 2A$.

Example: 23 If Δ stands for the area of a triangle ABC, then $a^2\sin 2B + b^2\sin 2A =$ **[WB JEE 1988]**

(a) 3Δ (b) 2Δ (c) 4Δ (d) -4Δ

Solution: (c) Use sine rule, $\Delta = \frac{1}{2}\ ab \sin C$

L.H.S. = $k^2(\sin^2 A.2\sin B\cos B + \sin^2 B.2\sin A\cos A) = k^2[2\sin A.\sin B.\sin(A+B)] = 2ab\sin C = 4\Delta$.

Example: 24 In a ΔABC, $A = \frac{2\pi}{3}, b - c = 3\sqrt{3}$ and $ar(\Delta ABC) = \frac{9\sqrt{3}}{2}cm^2$. Then a is

(a) $6\sqrt{3}cm.$ (b) $9cm.$ (c) $18\ cm.$ (d) None of these

Solution: (b) $\frac{1}{2}bc\sin\frac{2\pi}{3} = \frac{9\sqrt{3}}{2}$ or $\frac{1}{2}.\frac{\sqrt{3}}{2}.bc = \frac{9\sqrt{3}}{2} \Rightarrow bc = 18$

Also, $\cos\frac{2\pi}{3} = \frac{b^2+c^2-a^2}{2bc} \Rightarrow -\frac{1}{2} = \frac{(b-c)^2+2bc-a^2}{2bc}$ or $(b-c)^2+3bc-a^2 = 0$ or $27+54 = a^2 \Rightarrow a = 9$.

Example: 25 If p_1, p_2, p_3 are altitudes of a triangle ABC from the vertices A, B, C and Δ, the area of the triangle, then $p_1^{-2}+p_2^{-2}+p_3^{-2}$ is equal to

(a) $\frac{a+b+c}{\Delta}$ (b) $\frac{a^2+b^2+c^2}{4\Delta^2}$ (c) $\frac{a^2+b^2+c^2}{\Delta^2}$ (d) None of these

Solution: (b) We have $\frac{1}{2}ap_1 = \Delta, \frac{1}{2}bp_2 = \Delta, \frac{1}{2}cp_3 = \Delta \Rightarrow p_1 = \frac{2\Delta}{a}, p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$

$\therefore \frac{1}{p_1^2}+\frac{1}{p_2^2}+\frac{1}{p_3^2} = \frac{a^2+b^2+c^2}{4\Delta^2}$.

3.7 Half Angle Formulae

If $2s$ shows the perimeter of a triangle ABC then, *i.e.*, $2s = a+b+c$, then

(1) **Formulae for** $\sin\frac{A}{2}, \sin\frac{B}{2}, \sin\frac{C}{2}$ **:**

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ca}}$ (iii) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(2) **Formulae for** $\cos\frac{A}{2}, \cos\frac{B}{2}, \cos\frac{C}{2}$ **:**

(i) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$ (iii) $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(3) **Formulae for** $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ **:**

(i) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ (ii) $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$ (iii) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Note : ❑ $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sqrt{\frac{(s-b)(s-c)}{bc}}.\sqrt{\frac{s(s-a)}{bc}} = \left\{\frac{2}{bc}\right\}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$

Similarly $\sin B = \frac{2\Delta}{ca}, \sin C = \frac{2\Delta}{ab}$

❑ $\tan\frac{A}{2} = \frac{(s-b)(s-c)}{\Delta}, \tan\frac{B}{2} = \frac{(s-c)(s-a)}{\Delta}, \tan\frac{C}{2} = \frac{(s-a)(s-b)}{\Delta}$

Important Tips

☞ $\tan\frac{A}{2}\tan\frac{B}{2} = \left\{\frac{(s-b)(s-c)}{s(s-a)}.\frac{(s-c)(s-a)}{s(s-b)}\right\}^{1/2} = \frac{s-c}{s} \Rightarrow \cot\frac{A}{2}\cot\frac{B}{2} = \frac{s}{s-c}$

☞ $\tan\frac{A}{2}+\tan\frac{B}{2} = \sqrt{\left(\frac{s-c}{s}\right)}\left[\sqrt{\left(\frac{s-b}{s-a}\right)}+\sqrt{\left(\frac{s-a}{s-b}\right)}\right] = \sqrt{\left(\frac{s-c}{s}\right)}\left[\frac{s-b+s-a}{\sqrt{(s-a)(s-b)}}\right] = \frac{c}{s}\left[\frac{s(s-c)}{(s-a)(s-b)}\right]^{1/2} = \frac{c}{s}\cot\frac{C}{2}$

Another form: $\frac{c.(s-c)}{\{s(s-a)(s-b)(s-c)\}^{1/2}} = \frac{c}{s}(s-c)$

☞ $\tan\frac{A}{2}-\tan\frac{B}{2} = \frac{a-b}{s}(s-c)$

☞ $\cot\frac{A}{2}+\cot\frac{B}{2} = \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A}{2}\tan\frac{B}{2}} = \frac{c}{s-c}\cot\frac{C}{2}$.

Example: 26 If in any ΔABC ; $\cot\frac{A}{2},\cot\frac{B}{2},\cot\frac{C}{2}$ are in A.P., then **[MP PET 2003]**

(a) $\cot\frac{A}{2}\cot\frac{B}{2}=4$ (b) $\cot\frac{A}{2}\cot\frac{C}{2}=3$ (c) $\cot\frac{B}{2}\cot\frac{C}{2}=1$ (d) $\cot\frac{B}{2}\tan\frac{C}{2}=0$

Solution: (b) **Trick:** Take $A=B=C=60^o$, then $\cot\frac{A}{2},\cot\frac{B}{2}$ and $\cot\frac{C}{2}$ are in A.P. with common difference zero. Now option (b) satisfies.

Example: 27 In a ΔABC, if $3a=b+c$, then the value of $\cot\frac{B}{2}\cot\frac{C}{2}$ is **[Roorkee 1986; MP PET 1990, 97, 98]**

(a) 1 (b) 2 (c) $\sqrt{3}$ (d) $\sqrt{2}$

Solution: (b) $\cot\frac{B}{2}.\cot\frac{C}{2} = \sqrt{\frac{s(s-b)}{(s-a)(s-c)}}.\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \frac{s}{s-a}$

Given $3a=b+c \Rightarrow a+b+c=4a \Rightarrow \therefore \cot\frac{B}{2}.\cot\frac{C}{2} = \frac{2a}{a} = 2$.

Example: 28 In a triangle ABC, $\tan\frac{A}{2}=\frac{5}{6}$ and $\tan\frac{C}{2}=\frac{2}{5}$, then **[EAMCET 1994]**

(a) a,b,c are in A.P. (b) $\cos A,\cos B,\cos C$ are in A.P.

(c) $\sin A,\sin B,\sin C$ are in A.P. (d) Both (a) and (c)

Solution: (d) $\tan\frac{A}{2}.\tan\frac{C}{2} = \frac{(s-b)}{s} \Rightarrow \frac{5}{6}.\frac{2}{5} = \frac{s-b}{s} \Rightarrow 2s=3b \Rightarrow a+b+c=3b \Rightarrow a+c=2b$.

Example: 29 If the sides of triangle a, b, c be in A.P. then $\tan\frac{A}{2}+\tan\frac{C}{2}$ equal to **[Roorkee 1993]**

(a) $\frac{2}{3}\cot\frac{A}{2}$ (b) $\frac{2}{3}\cot\frac{B}{2}$ (c) $\frac{2}{3}\cot\frac{C}{2}$ (d) None of these

Solution: (b) $\tan\frac{A}{2}+\tan\frac{C}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}+\sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{b}{s}\cot\frac{B}{2} = \frac{2b}{2s}\cot\frac{B}{2}$

$\because$ a, b, c in A.P.

$\therefore$ $a+c=2b \Rightarrow 2s=3b = \frac{2b}{3b}\cot\frac{B}{2}$

Hence, $\tan\frac{A}{2}+\tan\frac{C}{2} = \frac{2}{3}\cot\frac{B}{2}$.

Example: 30 In ΔABC, $\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\left(a\sin^2\frac{B}{2}+b\sin^2\frac{A}{2}\right)$ equal to **[Roorkee 1988]**

(a) $\cot C$ (b) $c\cot C$ (c) $\cot\frac{C}{2}$ (d) $c\cot\frac{C}{2}$

Solution: (d) $\left\{\cot\frac{A}{2}+\cot\frac{B}{2}\right\}\left\{a\sin^2\frac{B}{2}+b\sin^2\frac{A}{2}\right\}=\left\{\frac{\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}\right\}\left\{a\sin^2\frac{B}{2}+b\sin^2\frac{A}{2}\right\}=\left\{\cos\frac{C}{2}\right\}\left\{a\frac{\sin\frac{B}{2}}{\sin\frac{A}{2}}+b\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}}\right\}$

$$=\sqrt{\frac{s(s-c)}{ab}}\left\{a\frac{\sqrt{\frac{(s-a)(s-c)}{ac}}}{\sqrt{\frac{(s-b)(s-c)}{bc}}}+b\frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{(s-a)(s-c)}{ac}}}\right\}=\sqrt{\frac{s(s-c)}{ab}}\left\{\sqrt{\left(\frac{s-a}{s-b}\right)ab}+\sqrt{\left(\frac{s-b}{s-a}\right)ab}\right\}$$

$$=\sqrt{s(s-c)}\left\{\frac{s-a+s-b}{\sqrt{(s-a)(s-b)}}\right\}=\sqrt{s(s-c)}\left\{\frac{2s-a-b}{\sqrt{(s-a)(s-b)}}\right\}=c\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}=c\cot\frac{C}{2}.$$

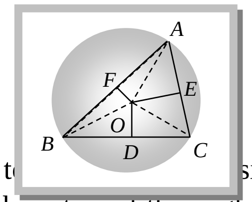
Trick : Such type of unconditional problems can be checked by putting the particular values for $a=1$, $b=\sqrt{3}$, $c=2$ and $A=30^o$, $B=60^o$, $C=90^o$, Here expression is equal to 2 which is given by (d).

3.8 Circle Connected with Triangle

(1) **Circumcircle of a triangle and its radius**

(i) **Circumcircle :** The circle which passes through the angular points of a triangle is called its circumcircle. The centre of this circle is the point of intersection of perpendicular bisectors of the sides and is called the circumcentre. Its radius is always denoted by R. The circumcentre may lie within, outside or upon one of the sides of the triangle.

(ii) **Circum-radius :** The circum-radius of a ΔABC is given by

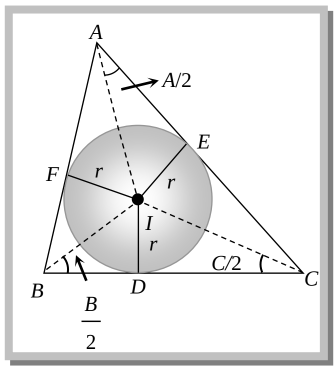


(a) $\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=R$ (b) $R=\frac{abc}{4\Delta}$ [$\Delta=$area of ΔABC]

(2) **Inscribed circle or incircle of a triangle and its radius**

(i) **In-circle or inscribed circle :** The circle which can be inscribed within a triangle so as t sides is called its inscribed circle or in circle. The centre of this circle is the point of intersection of the bisectors of the angles of the triangle. The radius of this circle is always denoted by r and is equal to the length of the perpendicular from its centre to any one of the sides of triangle.

(ii) **In-radius :** The radius r of the inscribed circle of a triangle ABC is given by



(a) $r=\frac{\Delta}{s}$ (b) $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(c) $r=(s-a)\tan\frac{A}{2}, r=(s-b)\tan\frac{B}{2},\ r=(s-c)\tan\frac{C}{2}$

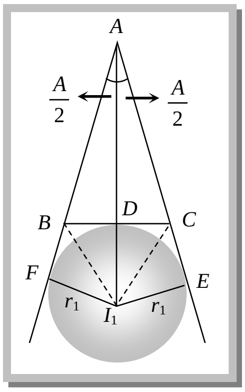
(d) $r=\frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}, r=\frac{b\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}}, r=\frac{c\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$

(e) $\cos A+\cos B+\cos C=1+\frac{r}{R}$

(3) **Escribed circles of a triangle and their radii**

(i) **Escribed circle :** The circle which touches the side *BC* and two sides *AB* and *AC* produced of a triangle *ABC* is called the escribed circle opposite to the angle A. Its radius is denoted by r_1. Similarly, r_2 and r_3 denote the radii of the escribed circles opposite to the angles *B* and *C* respectively.

The centres of the escribed circles are called the ex-centres. The centre of the escribed circle opposite to the angle *A* is the point of intersection of the external bisectors of angles *B* and *C*. The internal bisectors of angle *A* also passes through the same point. The centre is generally denoted by I_1



(ii) **Radii of ex-circles :** In any ΔABC, we have

(a) $r_1 = \dfrac{\Delta}{s-a}, r_2 = \dfrac{\Delta}{s-b}, r_3 = \dfrac{\Delta}{s-c}$ (b) $r_1 = s\tan\dfrac{A}{2},\ r_2 = s\tan\dfrac{B}{2}, r_3 = s\tan\dfrac{C}{2}$

(c) $r_1 = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}, r_2 = \dfrac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}, r_3 = \dfrac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

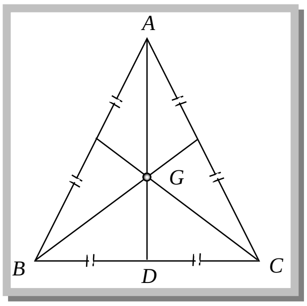
(d) $r_1 + r_2 + r_3 - r = 4R$ (e) $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r}$

(f) $\dfrac{1}{r^2} + \dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \dfrac{1}{r_3^2} = \dfrac{a^2+b^2+c^2}{\Delta^2}$ (g) $\dfrac{1}{bc} + \dfrac{1}{ca} + \dfrac{1}{ab} = \dfrac{1}{2Rr}$

(h) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$ (i) $\Delta = 2R^2\sin A.\sin B.\sin C = 4Rr\cos\dfrac{A}{2}.\cos\dfrac{B}{2}.\cos\dfrac{C}{2}$

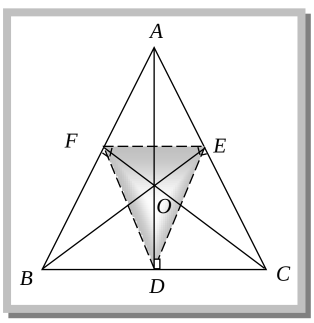
(j) $r_1 = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}; r_2 = \cos\dfrac{A}{2}.\sin\dfrac{B}{2}.\cos\dfrac{C}{2}; r_3 = 4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}$

(4) **Centroid (G) :** Common point of intersection of medians of a triangle. Divides every median in the ratio 2:1. Always lies inside the triangle.



(5) **Orthocentre of a triangle :** The point of intersection of perpendicular drawn from the vertices on the opposite sides of a triangle is called its orthocentre.

3.9 Pedal Triangle



Let the perpendiculars *AD*, *BE* and *CF* from the vertices *A*, *B* and *C* on the opposite sides *BC*, *CA* and *AB* of ΔABC respectively, meet at *O*. Then *O* is the orthocentre of the ΔABC. The triangle *DEF* is called the pedal triangle of the ΔABC.

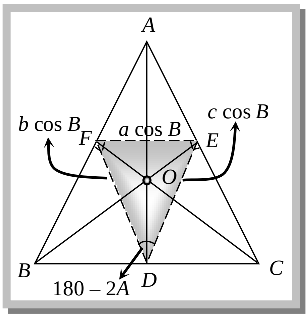
Othocentre of the triangle is the incentre of the pedal triangle.

If *O* is the orthocentre and *DEF* the pedal triangle of the ΔABC, where *AD*, *BE*, *CF* are the perpendiculars drawn from *A*, *B*, *C* on the opposite sides *BC*, *CA*, *AB* respectively, then

(i) $OA = 2R\cos A, OB = 2R\cos B$ and $OC = 2R\cos C$

(ii) $OD = 2R\cos B\cos C, OE = 2R\cos C\cos A$ and $OF = 2R\cos A\cos B$

(1) **Sides and angles of a pedal triangle:** The angles of pedal triangle DEF are: $180-2A, 180-2B, 180-2C$ and sides of pedal triangle are:



$EF = a\cos A$ or $R\sin 2A$; $FD = b\cos B$ or $R\sin 2B$; $DE = c\cos C$ or $R\sin 2C$

If given ΔABC is obtuse, then angles are been represented by $2A,\ 2B,\ 2C-180^o$ and the sides are $a\cos A,\ b\cos B,\ -c\cos C$.

(2) **Area and circum-radius and in-radius of pedal triangle**

Area of pedal triangle = $\frac{1}{2}$(Product of the sides) $\times$ (sine of included angle)

$$\Delta = \frac{1}{2}R^2.\sin 2A.\sin 2B.\sin 2C$$

Circum-radius of pedal triangle = $\dfrac{EF}{2\sin FDE} = \dfrac{R\sin 2A}{2\sin(180^o - 2A)} = \dfrac{R}{2}$.

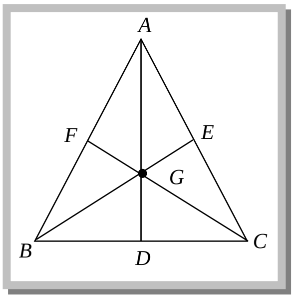
In-radius of pedal triangle = $\dfrac{\text{area of } \Delta DEF}{\text{semi - perimeter of } \Delta DEF} = \dfrac{\frac{1}{2}R^2\sin 2A.\sin 2B.\sin 2C}{2R\sin A.\sin B.\sin C} = 2R\cos A.\cos B.\cos C$.

Important Tips

- *Circum-centre, Centroid and Orthocentre are collinear.*
- *In any right angled triangle, the orthocentre coincides with the vertex containing the right angled.*
- *The mid-point of the hypotenuse of a right angled triangle is equidistant from the three vertices of the triangle.*
- *The mid-point of the hypotenuse of a right angled triangle is the circumcentre of the triangle.*
- *The length of the medians AD, BE, CF of* ΔABC *are given by*

  $AD = \frac{1}{2}\sqrt{2b^2+2c^2-a^2} = \frac{1}{2}\sqrt{b^2+c^2+2bc\cos A}$, $BE = \frac{1}{2}\sqrt{2c^2+2a^2-b^2} = \frac{1}{2}\sqrt{c^2+a^2+2ca.\cos B}$

  $CF = \frac{1}{2}\sqrt{2a^2+2b^2-c^2} = \frac{1}{2}\sqrt{a^2+b^2+2ab.\cos c}$



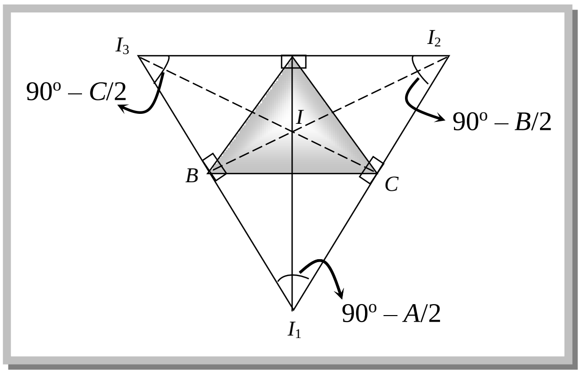
- *The distance between the circumcentre O and centroid G of* ΔABC *is given by*

  $OG = \frac{1}{3}OH = \frac{1}{3}R\sqrt{1-8\cos A.\cos B.\cos C}$, *Where H is the orthocentre of* ΔABC .
- *The distance between the orthocentre H and centroid G of* ΔABC *is given by* $HG = \frac{2}{3}R\sqrt{1-8\cos A.\cos B.\cos C}$.
- *The distance between the circumcentre O and the incentre I of* ΔABC *given by* $OI = R\sqrt{1-8\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}}$
- If I_1 is the centre of the escribed circle opposite to the angle *B*, then $OI_1 = R\sqrt{1+8\sin\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}}$

  Similarly, $OI_2 = R\sqrt{1+8\cos\frac{A}{2}.\sin\frac{B}{2}.\cos\frac{C}{2}}$, $OI_3 = R\sqrt{1+8\cos\frac{A}{2}.\cos\frac{B}{2}.\sin\frac{C}{2}}$
- *Circle circumscribing the pedal triangle of a given triangle bisects the sides of the given triangle and also the lines joining the vertices of the given triangle to the orthocentre of the given triangle. This circle is known as "Nine point circle".*
- *Circumcentre of the pedal triangle of a given triangle bisects the line joining the circum-centre of the triangle to the orthocentre.*

3.10 Ex-central Triangle

Let ABC be a triangle and I be the centre of incircle. Let I_1, I_2 and I_3 be the centres of the escribed circles which are opposite to A, B, C respectively then $I_1I_2I_3$ is called the Ex-central triangle of ΔABC.



$I_1I_2I_3$ is a triangle, thus the triangle ABC is the pedal triangle of its ex-central triangle $I_1I_2I_3$. The angles of ex-central triangle $I_1I_2I_3$ are $90^o - \frac{A}{2}, 90^o - \frac{B}{2}, 90^o - \frac{C}{2}$

and sides are $I_1I_3 = 4R\cos\frac{B}{2}$; $I_1I_2 = 4R\cos\frac{C}{2}$; $I_2I_3 = 4R\cos\frac{A}{2}$

Area and circum-radius of the ex-central triangle

Area of triangle = $\frac{1}{2}$ (Product of two sides) × (sine of included angles)

$$\Delta = \frac{1}{2}\left(4R\cos\frac{B}{2}\right).\left(4R\cos\frac{C}{2}\right)\times\sin\left(90^o - \frac{A}{2}\right)$$

$$\Delta = 8R^2\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$$

Circum-radius $= \frac{I_2I_3}{2\sin I_2I_1I_3} = \frac{4R\cos\frac{A}{2}}{2\sin\left(90^o - \frac{A}{2}\right)} = 2R$.

Example: 31 In a ΔABC $r_1 < r_2 < r_3$, then **[EAMCET 2003]**

(a) $a < b < c$ (b) $a > b > c$ (c) $b < a < c$ (d) $a < c < b$

Solution: (a) In a ΔABC, $r_1 < r_2 < r_3$

$$\Rightarrow \frac{1}{r_1} > \frac{1}{r_2} > \frac{1}{r_3} \Rightarrow \frac{s-a}{\Delta} > \frac{s-b}{\Delta} > \frac{s-c}{\Delta} \Rightarrow (s-a) > (s-b) > (s-c) \Rightarrow -a > -b > -c \Rightarrow a < b < c.$$

Example: 32 Which of the following pieces of data does NOT uniquely determine an acute-angled triangle ABC (R being the radius of the circum-circle) **[IIT Screening 2002]**

(a) $a, \sin A, \sin B$ (b) a, b, c (d) $a, \sin B, R$ (d) $a, \sin A, R$

Solution: (d) $\frac{a}{\sin A} = R$ and $b = 2R\sin B$. So two sides and two angles are known. So $\angle C$ is known. Therefore, two sides and included angle is known. So, Δ is uniquely known in case (a).

If a, b, c are known the Δ is uniquely known in case (b). $b = 2R\sin B, \sin A = \frac{a}{2R}$. So, sides a, b and angle A, B are known. So $\angle C$ is known. Therefore two sides and included angle is known. So, Δ is uniquely known in case (c).

$\frac{a}{\sin A} = R$. So, only a side and an angle is known. So, Δ is not uniquely known in case (d).

Example: 33 In a triangle ABC, let $\angle C = \frac{\pi}{2}$. If r is the in radius and R is the circum-radius of the triangle, then $2(r+R)$ is equal to

[IIT Screening 2000]

(a) $a+b$ (b) $b+c$ (c) $c+a$ (d) $a+b+c$

Solution: (a) $\frac{c}{\sin C}=2R$, $\therefore$ $c=2R\sin 90^o=2R$

Also $r=(s-c)\tan\frac{C}{2}=(s-c)$ $[\because \tan 45^o=1]$

$2r=2s-2c=a+b-c=a+b-2R$, $2(r+R)=a+b$.

Example: 34 If R is the radius of the circumcircle of the ΔABC and Δ is its area, then **[Karnataka CET 2000]**

(a) $R=\frac{a+b+c}{\Delta}$ (b) $R=\frac{a+b+c}{4\Delta}$ (c) $R=\frac{abc}{4\Delta}$ (d) $R=\frac{abc}{\Delta}$

Solution: (c) Area of the triangle $ABC(\Delta)=\frac{bc}{2}\sin A$. From the sine formula, $a=2R\sin A$ or $\sin A=\frac{a}{2R}$ $\Rightarrow$ $\Delta=\frac{1}{2}bc.\frac{a}{2R}=\frac{abc}{4R}$ or $R=\frac{abc}{4\Delta}$.

Example: 35 If the length of the sides of a triangle are 3, 4 and 5 units, then R (the circum-radius) is **[UPSEAT 2000]**

(a) 2.0 unit (b) 2.5 unit (c) 3.0 unit (d) 3.5 unit

Solution: (b) Given, Sides are 3, 4, 5.

Since $3^2+4^2=5^2$. So, triangle is a right angle triangle. $R=\frac{5}{2}=2.5$ unit.

Example: 36 If x,y,z are perpendicular drawn from the vertices of triangle having sides *a, b,* and *c*, then the value of $\frac{bx}{c}+\frac{cy}{a}+\frac{az}{b}$ will be

(a) $\frac{a^2+b^2+c^2}{2R}$ (b) $\frac{a^2+b^2+c^2}{R^2}$ (c) $\frac{a^2+b^2+c^2}{4R}$ (d) $\frac{2(a^2+b^2+c^2)}{R}$

Solution: (a) Let area of triangle be Δ, then according to question, $\Delta=\frac{1}{2}ax=\frac{1}{2}by=\frac{1}{2}cz$

$\therefore$ $\frac{bx}{c}+\frac{cy}{a}+\frac{az}{b}=\frac{b}{c}\left(\frac{2\Delta}{a}\right)+\frac{c}{a}\left(\frac{2\Delta}{b}\right)+\frac{a}{b}\left(\frac{2\Delta}{c}\right)=\frac{2\Delta(b^2+c^2+a^2)}{abc}=\frac{2(a^2+b^2+c^2)}{abc}.\frac{abc}{4R}=\frac{a^2+b^2+c^2}{2R}$.

Example: 37 If r_1,r_2,r_3 in a triangle be in H.P., then the sides are in **[EAMCET 1993]**

(a) A.P. (b) G.P. (c) H.P. (d) None of these

Solution: (a) $\frac{1}{r_2}-\frac{1}{r_1}=\frac{1}{r_3}-\frac{1}{r_2}$ $\Rightarrow$ $\frac{1}{\frac{\Delta}{s-b}}-\frac{1}{\frac{\Delta}{s-a}}=\frac{1}{\frac{\Delta}{s-c}}-\frac{1}{\frac{\Delta}{s-b}}$ $\Rightarrow$ $\frac{s-b-s+a}{\Delta}=\frac{s-c-s+b}{\Delta}$ $\Rightarrow$ $\frac{a-b}{\Delta}=\frac{b-c}{\Delta}$

$\Rightarrow$ $\frac{b-a}{\Delta}=\frac{c-b}{\Delta}$ $\Rightarrow$ $2b=a+c$.

Example: 38 If the sides of a triangle are 13,14 15, then the radius of its incircle is **[EAMCET 1987]**

(a) $\frac{67}{8}$ (b) $\frac{65}{4}$ (c) 4 (d) 24

Solution: (c) $r=\frac{\Delta}{s}$ and $s=\frac{a+b+c}{2}=\frac{13+14+15}{2}=21$

$\Rightarrow$ $\Delta=\sqrt{21\times 8\times 7\times 6}=84$ $\Rightarrow$ $r=\frac{84}{21}=4$.

Example: 39 In an equilateral triangle the inradius and the circumradius are connected by **[EAMCET 1983]**

(a) $r=4R$ (b) $r=\frac{R}{2}$ (c) $r=\frac{R}{3}$ (d) None of these

Solution: (b) $r = 4R\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$

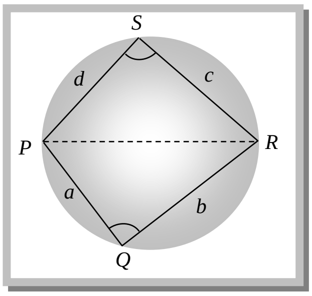
For an equilateral triangle, $A = B = C = 60^o$

$\therefore\ r = 4R\sin 30^o.\sin 30^o.\sin 30^o = 4R.\frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{R}{2}$.

3.11 Cyclic Quadrilateral

A quadrilateral *PQRS* is said to be cyclic quadrilateral if there exists a circle passing through all its four vertices *P*, *Q*, *R* and *S*.

Let a cyclic quadrilateral be such that $PQ = a, QR = b, RS = c$ and $SP = d$. Then $\angle Q + \angle S = 180^o$ and $\angle A + \angle C = 180^o$. Let $2s = a + b + c + d$



and $\square$= Area of cyclic quadrilateral *PQRS*

$\square$= Area of ΔPQR + Area of ΔPRS = $\frac{1}{2}ab\sin Q + \frac{1}{2}cd\sin S$

$\square = \frac{1}{2}ab\sin Q + \frac{1}{2}cd\sin(\pi - Q) = \frac{1}{2}(ab + cd)\sin Q$

$\square = \frac{1}{2}(ab + cd)\sin Q$ ......(i)

In ΔPQR and ΔPRS ,

From cosine rule, $PR^2 = PQ^2 + QR^2 - 2PQ.QR\cos Q = a^2 + b^2 - 2ab\cos Q$ ......(ii)

and $PR^2 = PS^2 + RS^2 - 2PS.RS\cos S$

$PR^2 = d^2 + c^2 - 2cd\cos(\pi - Q)$

$PR^2 = d^2 + c^2 + 2cd\cos Q$ ....(iii)

From (ii) and (iii) we have, $\square = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ ....(iv)

Therefore, (1) Area of cyclic quadrilateral = $\frac{1}{2}(ab + cd)\sin Q$

(2) Area of cyclic quadrilateral = $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, where $2s = a + b + c + d$

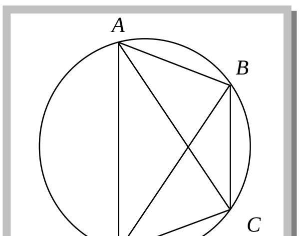
(3) $\cos Q = \dfrac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$

(4) **Circumradius of cyclic quadrilateral :** Circum circle of quadrilateral *PQRS* is also the circumcircle of ΔPQR. Hence circumradius of cyclic quadrilateral *PQRS* = *R* = circumradius of $\Delta PQR = \dfrac{PR}{2\sin B} = \dfrac{PR(ab+cd)}{4\Delta}$

But $PR = \sqrt{\dfrac{(ac+bd)(ad+bc)}{(ab+cd)}}$

Hence $R = \dfrac{1}{4\Delta}\sqrt{(ac+bd)(ad+bc)(ab+cd)} = \dfrac{1}{4}\sqrt{\dfrac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}}$

(5) **Ptolemy's theorem :** In a cyclic quadrilateral *PQRS*, the product of diagonals is equal to the sum of the products of the length of the opposite sides *i.e.*, According to Ptolemy's theorem, for a cyclic quadrilateral *PQRS* $PR.QS = PQ.RS + RQ.PS$.



Example: 40 A cyclic quadrilateral *ABCD* of area $\frac{3\sqrt{3}}{4}$ is inscribed in a unit circle. If one of its sides *AB* = 1 and the diagonal *BD* = $\sqrt{3}$ then the lengths of the other sides are **[Roorkee 1995]**

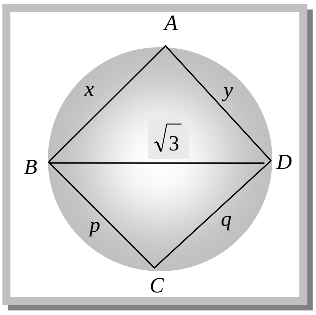
(a) 2, 1, 1 (b) 2, 1, 2 (c) 3, 1, 2 (d) None of these

Solution: (a) By sine formula in ΔABC, $\frac{\sqrt{3}}{\sin A} = 2R \Rightarrow \frac{\sqrt{3}}{\sin A} = 2 \Rightarrow \sin A = \frac{\sqrt{3}}{2} \Rightarrow A = \frac{\pi}{3}$

Now, $AB = x = 1$

By cosine formula in ΔABD $\cos\frac{\pi}{3} = \frac{x^2+y^2-3}{2xy} \Rightarrow \frac{1}{2} = \frac{1+y^2-3}{2y} \Rightarrow y = y^2 - 2$

$\Rightarrow y^2 - y - 2 = 0 \Rightarrow (y-2)(y+1) = 0 \Rightarrow y = 2$ $[\because y \neq -1]$



$\therefore AD = 2$

Since $\angle A = 60^o$ $\therefore \angle C = 120^o$

In $\Delta BDC, 3 = p^2 + q^2 - 2pq\cos 120^o \Rightarrow 3 = p^2 + q^2 + pq$ ....(i)

Also area of quadrilateral $ABCD = \frac{3\sqrt{3}}{4}$

$\therefore \frac{3\sqrt{3}}{4} = \Delta ABD + \Delta BCD = \frac{1}{2}.1.2\sin 60^o + \frac{1}{2}p.q.\sin 120^o = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}pq$

$\Rightarrow \frac{\sqrt{3}}{4}pq = \frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}-2\sqrt{3}}{4} = \frac{\sqrt{3}}{4} \Rightarrow pq = 1$

$\therefore$ (i) gives, $3 = p^2 + q^2 + 1 \Rightarrow p^2 + q^2 = 2$, $[p, q > 0]$

$\therefore p^2 + \frac{1}{p^2} = 2 \Rightarrow p^4 - 2p^2 + 1 = 0 \Rightarrow (p^2-1)^2 = 0 \Rightarrow p^2 = 1, \Rightarrow \therefore p = 1, q = 1$

$\therefore AB = 1, AD = 2, BC = CD = 1$.

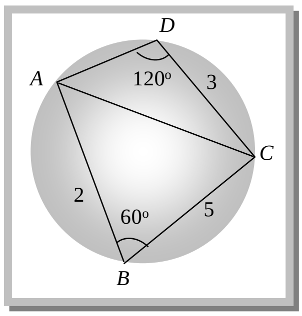
Example: 41 The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the third side is 3, the remaining fourth side is **[UPSEAT 1994]**

(a) 2 (b) 3 (c) 4 (d) 5

Solution: (a) Since *ABCD* is cyclic quadrilateral and $\angle ABC = 60^o$

$\therefore \angle ADC = 180^o - 60^o = 120^o$

Let $AB = 2$, $BC = 5$ and $CD = 3$



In ΔABC, $AB^2 + AC^2 + 2AB.BC\cos 60^o = AC^2$ or $4 + 25 - 2\times 2\times 5\times\frac{1}{2} = AC^2$

$\therefore AC^2 = 19$; In ΔABC $AD^2 + CD^2 + 2AD.CD\cos 60^o = AC^2$

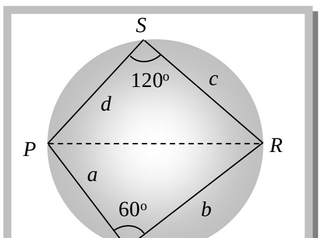
or $AD^2 + 9 + 2AD.3.\frac{1}{2} = 19$ or $AD^2 + 3AD - 10 = 0$ or $AD^2 + 5AD - 2AD - 10 = 0$

or $AD(AD+5) - 2(AD+5) = 0$ or $(AD-2)(AD+5) = 0$. Therefore, fourth side is $AD = 2$.

Example: 42 Two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the area of the quadrilateral is $4\sqrt{3}$, then the remaining two sides are **[Roorkee 1991]**

(a) 2, 3 (b) 3, 4 (c) 4, 5 (d) 5, 6

Solution: (a) Let $a = PQ = 2$, $b = QR = 5$, $RS = c$, $SP = d$



Given, Area of quadrilateral $PQRS = 4\sqrt{3}$

$\Rightarrow$ Area of $(\Delta PQR + \Delta PRS) = \frac{1}{2}[2 \times 5 \sin 60^o + c \times d \sin 120^o]$

$\Rightarrow$ $4\sqrt{3} = \frac{5\sqrt{3}}{2} + \frac{\sqrt{3}}{4}cd \Rightarrow cd = 6$ ....(i)

Now by cosine formula $RS^2 + SP^2 - 2RS.SP.\cos 120^o = PR^2 = PQ^2 + QR^2 - 2PQ.QR \cos 60^o$

$\Rightarrow$ $c^2 + d^2 - 2cd\left(-\frac{1}{2}\right) = 4 + 25 - 2(2)(5)\left(\frac{1}{2}\right) \Rightarrow c^2 + d^2 + cd = 19 \Rightarrow c^2 + d^2 = 19 - 6 = 13$ ....(ii)

Solving (i) and (ii) we get $c = 2$ and $d = 3$.

3.12 Regular Polygon

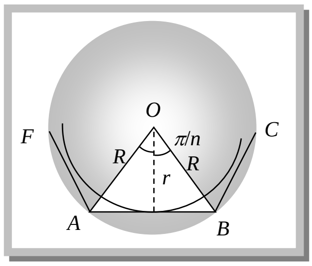
A regular polygon is a polygon which has all its sides equal and all its angles equal.

(1) Each interior angle of a regular polygon of n sides is $\left(\frac{2n-4}{n}\right) \times$ right angles $= \boxed{\left[\frac{2n-4}{n}\right] \times \frac{\pi}{2}}$ radians.

(2) The circle passing through all the vertices of a regular polygon is called its *circumscribed* circle.

If a is the length of each side of a regular polygon of n sides, then the radius R of the circumscribed circle, is given by

$$\boxed{R = \frac{a}{2}.\text{cosec}\left(\frac{\pi}{n}\right)}$$



(3) The circle which can be inscribed within the regular polygon so as to touch all its sides is called its inscribed circle.

Again if a is the length of each side of a regular polygon of n sides, then the radius r of the inscribed circle is given by $\boxed{r = \frac{a}{2}.\cot\left(\frac{\pi}{n}\right)}$

(4) The area of a regular polygon is given by $\Delta = n \times$ area of triangle OAB

$\boxed{= \frac{1}{4}na^2\cot\left(\frac{\pi}{n}\right)}$ (in terms of side)

$\boxed{= nr^2.\tan\left(\frac{\pi}{n}\right)}$ (in terms of in-radius)

$\boxed{= \frac{n}{2}.R^2\sin\left(\frac{2\pi}{n}\right)}$ (in terms of circum-radius)

Example: 43 The sum of the radii of inscribed and circumscribed circles for an n sided regular polygon of side a, is **[AIEEE 2003]**

(a) $a\cot\left(\frac{\pi}{n}\right)$ (b) $\frac{a}{2}\cot\left(\frac{\pi}{2n}\right)$ (c) $a\cot\left(\frac{\pi}{2n}\right)$ (d) $\frac{a}{4}\cot\left(\frac{\pi}{2n}\right)$

Solution: (c) $\tan\left(\frac{\pi}{n}\right) = \frac{a}{2r}$ and $\sin\left(\frac{\pi}{n}\right) = \frac{a}{2R} \Rightarrow r + R = \frac{a}{2}\left[\cot\frac{\pi}{n} + \text{cosec}\frac{\pi}{n}\right] = \frac{a}{2}\cot\left(\frac{\pi}{2n}\right)$.

Example: 44 The area of the circle and the area of a regular polygon of n sides and its perimeter equal to that of the circle are in the ratio of

(a) $\tan\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$ (b) $\cos\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$ (c) $\sin\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$ (d) $\cot\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$

Solution: (a) Let r be the radius of the circle and A_1 be its area $\therefore$ $A_1 = \pi r^2$

Since the perimeter of the circle is the same as the perimeter of a regular polygon of n sides $\therefore\ 2\pi r = na$, where 'a' is the length of one side of the regular polygon, $\therefore\ a = \frac{2\pi r}{n}$

Let A_2 be the area of the polygon, then

$$A_2 = \frac{1}{4} na^2 . \cot\frac{\pi}{n} = \frac{1}{4} n . \frac{4\pi^2 r^2}{n^2} \cot\frac{\pi}{n} = \pi r^2 . \frac{\pi}{n} . \cot\frac{\pi}{n}$$

$$\therefore\ A_1 : A_2 = \pi r^2 : \pi r^2 . \frac{\pi}{n} . \cot\frac{\pi}{n} = 1 : \frac{\pi}{n}\cot\frac{\pi}{n} = \tan\frac{\pi}{n} : \frac{\pi}{n}.$$

Example: 45 A regular polygon of nine sides, each of length 2 is inscribed in a circle. The radius of the circle is **[IIT 1994]**

(a) $\operatorname{cosec}\frac{\pi}{9}$ (b) $\operatorname{cosec}\frac{\pi}{3}$ (c) $\cot\frac{\pi}{9}$ (d) $\tan\frac{\pi}{9}$

Solution: (a) We know that radius of the circumcircle is given by $R = \frac{a}{2}\operatorname{cosec}\left(\frac{\pi}{n}\right)$; Here, $a = 2, n = 9$

$\therefore$ Required radius = $\frac{2}{2}\operatorname{cosec}\frac{\pi}{9} = \operatorname{cosec}\frac{\pi}{9}$.

Example: 46 If the number of sides of two regular polygons having the same perimeter be n and $2n$ respectively, their areas are in the ratio

(a) $\dfrac{2\cos\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{2n}\right)}$ (b) $\dfrac{2\cos\left(\frac{\pi}{n}\right)}{1+\cos\left(\frac{\pi}{n}\right)}$ (c) $\dfrac{\cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}$ (d) None of these

Solution: (b) Let s be the perimeter of both the polygons. Then the length of each side of the first polygon is $\frac{s}{n}$ and that of second polygon is $\frac{s}{2n}$.

If A_1, A_2 denote their areas, then $A_1 = \frac{n}{4}\left[\frac{s}{n}\right]^2 \cot\frac{\pi}{n}$ and $A_2 = \frac{1}{4}.(2n)\left(\frac{s}{2n}\right)^2 . \cot\left(\frac{\pi}{2n}\right)$

$$\frac{A_1}{A_2} = \frac{2\cot\left(\frac{\pi}{n}\right)}{\cot\left(\frac{\pi}{2n}\right)} = \frac{2\cos\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{2n}\right)}{\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{2n}\right)} = \frac{2\cos\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{2n}\right)}{2\sin\left(\frac{\pi}{2n}\right)\cos\left(\frac{\pi}{2n}\right)\cos\left(\frac{\pi}{2n}\right)} \Rightarrow \frac{A_1}{A_2} = \frac{2\cos\left(\frac{\pi}{n}\right)}{1+\cos\left(\frac{\pi}{n}\right)}.$$

3.13 Solutions of Triangles

Different formulae will be used in different cases and sometimes the same problem may be solved in different ways by different formulae. We should, therefore, look for that formula which will suit the problem best.

(1) Solution of a right angled triangle

(2) Solution of a triangle in general

(1) Solution of a right angled triangle

(i) **When two sides are given:** Let the triangle be right angled at C. Then we can determine the remaining elements as given in the following table

Given	Required

a,b	$\tan A = \frac{a}{b}$, $B = 90^o - A$, $c = \frac{a}{\sin A}$
a,c	$\sin A = \frac{a}{c}$, $b = c\cos A, B = 90^o - A$

(ii) **When a side and an acute angle are given :** In this case, we can determine the remaining elements as given in the following table

Given	Required
a, A	$B = 90^o - A, b = a\cot A, c = \frac{a}{\sin A}$
c, A	$B = 90^o - A,\ a = c\sin A, b = c\cos A$

(2) Solution of a triangle in general

(i) When three sides a, b and c are given in this case, the remaining elements are determined by using the following formulae, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where $2s = a+b+c =$ perimeter of triangle

$$\sin A = \frac{2\Delta}{bc}, \qquad \sin B = \frac{2\Delta}{ac}, \qquad \sin C = \frac{2\Delta}{ab}$$

$$\tan\frac{A}{2} = \frac{\Delta}{s(s-a)}, \qquad \tan\frac{B}{2} = \frac{\Delta}{s(s-b)}, \qquad \tan\frac{C}{2} = \frac{\Delta}{s(s-c)}$$

(ii) **When two sides a, b and the included angle C are given :** In this case, we use the following formulae $\Delta = \frac{1}{2}ab\sin C$; $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$; $\frac{A+B}{2} = 90^o - \frac{C}{2}$ and $c = \frac{a\sin C}{\sin A}$

(iii) **When one sides a and two angles A and B are given :** In this case, we use the following formulae to determine the remaining elements $A + B + C = 180^o \Rightarrow C = 180^o - A - B$

$$b = \frac{a\sin B}{\sin A} \text{ and } c = \frac{a\sin C}{\sin A} \Rightarrow \Delta = \frac{1}{2}ca\sin B$$

(iv) **When two sides a, b and the angle A opposite to one side is given :** In this case, we use the following formulae

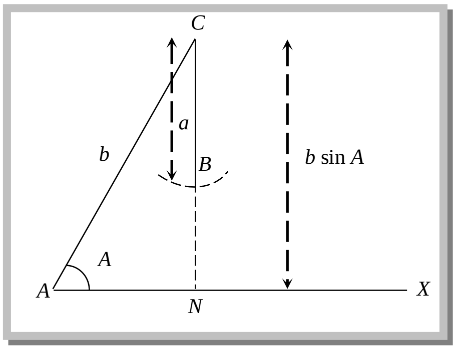
$$\sin B = \frac{b}{a}\sin A \qquad \text{..........(i)}$$

$$C = 180^o - (A+B), c = \frac{a\sin C}{\sin A}$$

Special Cases

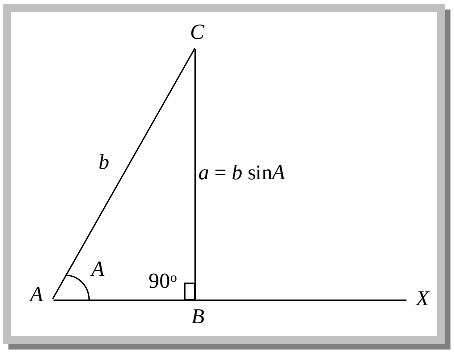
Case I : When a is an acute angle

(a) If $a < b\sin A$, there is no triangle. When $a < b\sin A$, from (i), $\sin B > 1$, which is impossible.



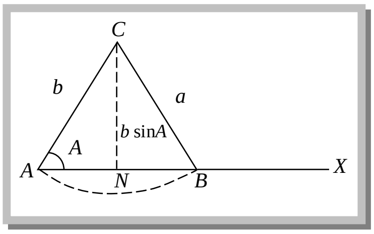
From the following figure, If $AC = b, \angle CAX = A$, then perpendicular $CN = b\sin A$. Now taking C as centre, If we draw an arc of radius a then it will never intersect the line AX and hence no triangle ABC can be constructed in this case.

(b) If $a = b\sin A$, then only one triangle is possible which is right angled at B. When $a = b\sin A$, then from sine rule. $\sin B = 1$, $\therefore \angle B = 90^o$



from fig. It is clear that $CB = a = b\sin A$

Thus, in this case, only one triangle is possible which is right angled at B.



(c) If $a > b\sin A$, then three possibilities will arise:

(i) $a = b$ In this case, from sine rule $\sin B = \sin A$; $\therefore B = A$ or $B = 180^o - A$

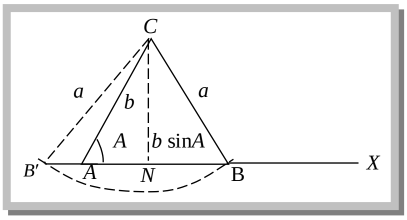
But $B = 180^o - A \Rightarrow A + B = 180^o$, which is not possible in a triangle.

$\therefore$ In this case, we get $\angle A = \angle B$.

Hence, if $b = a > b\sin A$ then only one isosceles triangle ABC is possible in which $\angle A = \angle B$.

(ii) $a > b$ In the following figure, Let $AC = b, \angle CAX = A$, and $a > b$, also $a > b\sin A$.

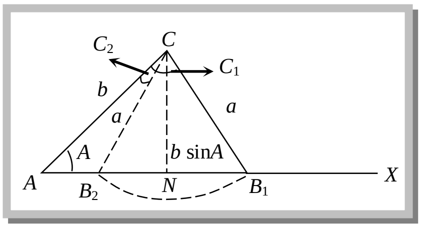
Now taking C as centre, if we draw an arc of radius a, it will intersect AX at one point B and hence only one ΔABC is constructed. Also this arc will intersect XA produced at B' and $\Delta AB'C$ is also formed but this Δ is inadmissible (because $\angle CAB'$ is an obtuse angle in this triangle)



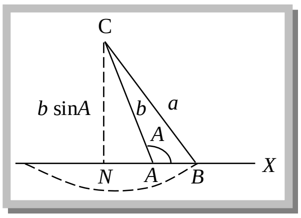
Hence, if $a > b$ and $a > b\sin A$, then only one triangle is possible.

(iii) $b > a$ ($i.e.,$ $b > a > b\sin A$)

In fig. let $AC = b, \angle CAX = A$. Now taking C as centre, if we draw an arc of radius a, then it will intersect AX at two points B_1 and B_2. Hence if $b > a > \sin A$, then there are two triangles.

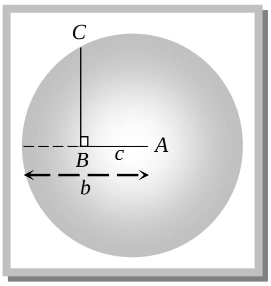


Case II : When *A* is an obtuse angle: In this case, there is only one triangle, if $a > b$



Case III: $b > c$ and $B = 90^o$

Again the circle with A as centre and b as radius will cut the line only in one point. So, only one triangle is possible.

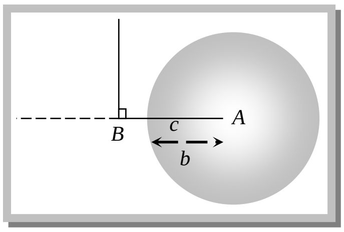


Case IV: $b \le c$ and $B = 90^o$

The circle with A as centre and b as radius will not cut the line in any point. So, no triangle is possible.

This is, sometimes called an ambiguous case.

Alternative method: By applying cosine rule, we have $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$



$\Rightarrow a^2 - (2c\cos B)a + (c^2 - b^2) = 0$

$\Rightarrow a = c\cos B \pm \sqrt{(c\cos B)^2 - (c^2 - b^2)}$

$\Rightarrow a = c\cos B \pm \sqrt{b^2 - (c\sin B)^2}$

This equation leads to following cases:

Case I : If $b < c\sin B$,no such triangle is possible.

Case II : Let $b = c\sin B,$ there are further following case.

(a) B is an obtuse angle $\Rightarrow$ $\cos B$ is negative. There exists no such triangle.

(b) B is an acute angle $\Rightarrow$ $\cos B$ is positive. There exists only one such triangle.

Case III : Let $b > c\sin B$. There are further following cases :

(a) B is an acute angle $\Rightarrow$ cos$B$ is positive.

In this case two values of a will exists if and only if $c\cos B > \sqrt{b^2 - (c\sin B)^2}$ or $c > b$

Two such triangle is possible. If $c < b,$ only one such triangle is possible.

(b) B is an obtuse angle $\Rightarrow$ cos$B$ is negative. In this case triangle will exist if and only if $\sqrt{b^2 - (c\sin B)^2} \not> |c\cos B|$ $\Rightarrow$ $b > c$. So, in this case only one such triangle is possible. If $b < c$ there exists no such triangle.

Note : ❑ If one side a and two angles B and C are given, then $A = 180^o - (B + C)$ and $b = \dfrac{a\sin B}{\sin A}, c = \dfrac{a\sin C}{\sin A}$

❑ If the three angles A, B, C are given. We can only find the ratios of the sides a, b, c by using sine rule (since there are infinite similar triangle possible).

Important Tips

☞ *A triangle which does not contain a right angle is called an oblique triangle.*

☞ *If triangle is an acute angled triangle means every angle is less than 90^o.*

☞ *If triangle is an obtuse angled triangle means one of its angle is greater than 90^o.*

☞ *If A and B are complementry angles then $A + B = 90^o$.*

☞ *If A and B are supplementary angle, then $A + B = 180^o$.*

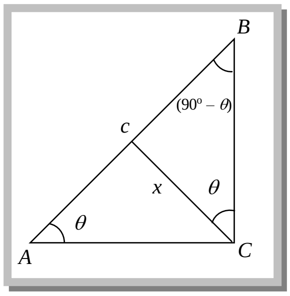
☞ $\Sigma(p - q) = (p - q) + (q - r) + (r - p) = 0$, $\Sigma p(q - r) = p(q - r) + q(r - p) + r(p - q) = 0$.

$\Sigma(p + a)(q - r) = \Sigma p(q - r) + a\Sigma(q - r) = 0$.

Example: 47 If in a right angled triangle the hypotenuse is four times as long as the perpendicular drawn to it from opposite vertex, then one of its acute angle is **[MP PET 1998]**

(a) 15^o (b) 30^o (c) 45^o (d) None of these

Solution: (a) If x is length of perpendicular drawn to it from opposite vertex of a right angled triangle, so, length of diagonal $AB = y_1 + y_2$ ....(i)



From $\Delta OCB, y_2 = x\cot\theta$ and from $\Delta OCA, y_1 = x\tan\theta$

Put the values in equation (i), then $AB = x(\tan\theta + \cot\theta)$ ....(ii)

Since, length of hypotenuse = 4 (Length of perpendicular)

$\therefore\ x(\tan\theta + \cot\theta) = 4x \Rightarrow \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta . \cos\theta} = 4$

$\Rightarrow \sin 2\theta = \dfrac{1}{2} \Rightarrow 2\theta = 30^o \Rightarrow \theta = 15^o$.

Trick : $\dfrac{\text{Length of hypotenuse}}{\text{Length of perpendicular drawn from opposite vertex to hypotenuse}} = \dfrac{2}{\sin 2\theta}$

$\Rightarrow 4 = \dfrac{2}{\sin 2\theta} \Rightarrow \sin 2\theta = \dfrac{1}{2} \Rightarrow \sin 2\theta = \sin 30^o \Rightarrow \theta = 15^o$.

Example: 48 The number of triangles ABC that can be formed with $a = 3, b = 8$ and $\sin A = \dfrac{5}{3}$ is **[Roorkee 1998]**

(a) 0 (b) 1 (c) 2 (d) 3

Solution: (a) Given, $a = 3, b = 8$ and $\sin A = \dfrac{5}{13}$.

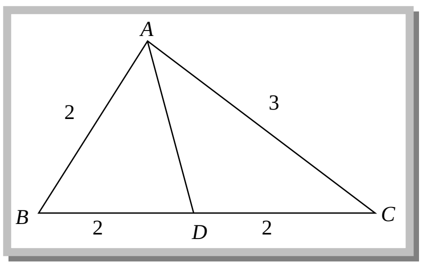
$\therefore b\sin A = 8 \times \left(\dfrac{5}{13}\right) = \dfrac{40}{13} > a(= 3)$

Thus in this case no triangle is possible.

Example: 49 In any triangle $AB = 2, BC = 4, CA = 3$ and D is mid point of BC, then **[Roorkee 1995]**

(a) $\cos B = \dfrac{11}{6}$ (b) $\cos B = \dfrac{7}{8}$ (c) $AD = 2.4$ (d) $AD^2 = 2.5$

Solution: (d) From ΔABC , $\cos B = \dfrac{2^2 + 4^2 - 3^2}{2\times 2\times 4} = \dfrac{11}{16}$



From ΔABD , $\cos B = \dfrac{2^2 + 2^2 - AD^2}{2\times 2\times 2} = \dfrac{11}{16}$

$\therefore\ \dfrac{11}{16} = \dfrac{2^2 + 2^2 - AD^2}{2\times 2\times 2}$

$\Rightarrow AD^2 = 2.5$.

Example: 50 If $b = 3, c = 4$ and $B = \dfrac{\pi}{3}$, then the number of triangles that can be constructed is **[Roorkee 1992]**

(a) Infinite (b) Two (c) One (d) Nil

Solution: (d) Here, $c\sin B = 4\sin\left(\frac{\pi}{3}\right) = 2\sqrt{3} > b(=3)$

Thus, we have $b < c\sin B$

Hence no triangle is possible, *i.e.*, the number of triangles that can be constructed is nil.

Example: 51 If two sides of a triangle are $2\sqrt{3}$ and $2\sqrt{2}$ the angle opposite the shorter side is 45^o. The maximum value of the third side is

(a) $2+\sqrt{6}$ (b) $\sqrt{2}+\sqrt{6}$ (c) $\sqrt{6}-2$ (d) None of these

Solution: (b) Let $a = 2\sqrt{3}, b = 2\sqrt{2}$ $\therefore B = 45^o$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{2\sqrt{3}}{\sin A} = \frac{2\sqrt{2}}{\sin 45^o} = \frac{c}{\sin C} \qquad ....(i)$$

$$\therefore \quad \sin A = \frac{\sqrt{3}}{2} \Rightarrow A = 60^o.$$

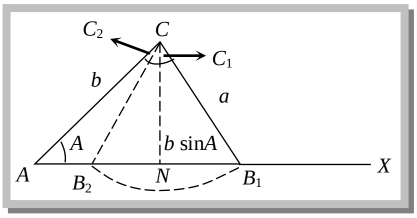
$$\therefore \quad C = 180^o - A - B = 75^o$$

From (i) $c = 4\sin C = 4\sin(45^o + 30^o) = \sqrt{2}+\sqrt{6}$.

Example: 52 In an ambiguous case, If the remaining angles of the triangles formed with *a, b* and *A* be B_1, C_1 and B_2, C_2 then $\frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_2} =$

(a) $2\cos A$ (b) $\cos A$ (c) $2\sin A$ (d) $\sin A$

Solution: (a) In $\Delta's\, ACB_1$ and ACB_2



$$\frac{\sin C_1}{\sin B_1} = \frac{AB_1}{AC} = \frac{c_1}{b} \text{ and } \frac{\sin C_2}{\sin B_2} = \frac{c_2}{b}$$

$$\therefore \frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_2} = \frac{c_1 + c_2}{b}$$

$$\left[\begin{array}{l} \because \cos A = \dfrac{b^2 + c^2 - a^2}{2bc} \text{ or } c^2 - (2b\cos A)c + (b^2 - a^2) = 0 \\ \therefore c_1 + c_2 = 2b\cos A \end{array}\right]$$

$$\frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_2} = \frac{2b\cos A}{b} = 2\cos A.$$

Assignment

Relation between Sides and Angles

Basic Level

1. The triangle PQR of which the angles P, Q, R satisfy $\cos P = \frac{\sin Q}{2\sin R}$ is **[Orissa JEE 2002]**

(a) Equilateral (b) Right angled (c) Any triangle (d) Isosceles

2. If angles of a triangle are in the ratio of 2 : 3 : 7, then the sides are in the ratio of **[MP PET 1996, 1993; BIT Ranchi 1992]**

(a) $\sqrt{2} : 2 : (\sqrt{3}+1)$ (b) $2 : \sqrt{2} : (\sqrt{3}+1)$ (c) $\sqrt{2} : (\sqrt{3}+1) : 2$ (d) $2 : (\sqrt{3}+1) : \sqrt{2}$

3. The smallest angle of the triangle whose sides are $6+\sqrt{12}, \sqrt{48}, \sqrt{24}$ is **[EAMCET 1985]**

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) None of these

4. In a ΔABC, if $c^2+a^2-b^2=ac$, then $\angle B =$ **[MP PET 1983, 1989, 1990]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) None of these

5. In a triangle ABC, $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C =$ **[MP PET 1985]**

(a) 0 (b) 1 (c) $a+b+c$ (d) $2(a+b+c)$

6. In a ΔABC, $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=$ **[ISM Dhanbad 1973]**

(a) 0 (b) $a+b+c$ (c) $a^2+b^2+c^2$ (d) $2(a^2+b^2+c^2)$

7. In any triangle ABC, the value of $a(b^2+c^2)\cos A+b(c^2+a^2)\cos B+c(a^2+b^2)\cos C$ is **[MP PET 1994]**

(a) $3abc^2$ (b) $3a^2bc$ (c) $3abc$ (d) $3ab^2c$

8. If in a triangle, $a\cos^2\frac{C}{2}+c\cos^2\frac{A}{2}=\frac{3b}{2}$, then its sides will be in **[MP PET 1982; AIEEE 2003]**

(a) A.P. (b) G.P. (c) H.P. (d) None of these

9. If a, b, c are the sides and A, B, C are the angles of a triangle ABC, then $\tan\left(\frac{A}{2}\right)$ is equal to **[MP PET 1994]**

(a) $\frac{\sqrt{(s-c)(s-a)}}{s(s-b)}$ (b) $\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ (c) $\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$ (d) $\sqrt{\frac{(s-a)s}{(s-b)(s-c)}}$

10. In a ΔABC $(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2}$ is equal to **[WB JEE 1989]**

(a) 0 (b) 1 (c) ± 1 (d) 2

11. In a triangle $ABC, a=2cm, b=3cm, c=4cm$ then angle A is **[MP PET 2002]**

(a) $\cos^{-1}\left(\frac{1}{24}\right)$ (b) $\cos^{-1}\left(\frac{11}{16}\right)$ (c) $\cos^{-1}\left(\frac{7}{8}\right)$ (d) $\cos^{-1}\left(-\frac{1}{4}\right)$

12. In a ΔABC, if $(\sin A+\sin B+\sin C)(\sin A+\sin B-\sin C)=3\sin A\sin B$, then the angle C is equal to **[AMU 1999]**

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

13. In a triangle ABC, right angled at C, the value of $\tan A+\tan B$ is **[Pb. CET 1990; Karnataka CET 1999; MP PET 2001]**

(a) $a+b$ (b) $\frac{a^2}{bc}$ (c) $\frac{b^2}{ac}$ (d) $\frac{c^2}{ab}$

14. In a ΔABC, if $b = 20, c = 21$ and $\sin A = \frac{3}{5}$, then $a =$ **[EAMCET 2003]**

(a) 12 (b) 13 (c) 14 (d) 15

15. In ΔABC, if $a = 16, b = 24$ and $c = 20$, then $\cos\frac{B}{2}$ equal to **[MP PET 1988]**

(a) $\frac{3}{4}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

16. The angles of a triangle are in the ratio 1 : 3 : 5 then the greatest angle is **[Kerala (Engg.) 2002]**

(a) $\frac{5\pi}{9}$ (b) $\frac{2\pi}{9}$ (c) $\frac{7\pi}{9}$ (d) $\frac{11\pi}{9}$

17. If in a triangle ABC, $2\cos A = \sin B \operatorname{cosec} C$, then **[MP PET 1996]**

(a) $a = b$ (b) $b = c$ (c) $c = a$ (d) $2a = bc$

18. If in a triangle the angles are in A.P. and $b : c = \sqrt{3} : \sqrt{2}$, then $\angle A$ is equal to **[IIT 1981; Haryana CEE 1998]**

(a) 30^o (b) 60^o (c) 15^o (d) 75^o

19. In ΔABC, $\frac{\sin(A-B)}{\sin(A+B)} =$ **[MP PET 1986]**

(a) $\frac{a^2-b^2}{c^2}$ (b) $\frac{a^2+b^2}{c^2}$ (c) $\frac{c^2}{a^2-b^2}$ (d) $\frac{c^2}{a^2+b^2}$

20. If the lengths of the sides of a triangle are 3, 5, 7 then the largest angle of the triangle is **[IIT 1994]**

(a) $\frac{\pi}{2}$ (b) $\frac{5\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{3\pi}{4}$

21. The lengths of the sides of a triangle are $\alpha - \beta$, $\alpha + \beta$ and $\sqrt{3\alpha^2 + \beta^2}, (\alpha > \beta > 0)$. Its largest angle is **[Roorkee 1999]**

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

22. In ΔABC, $\operatorname{cosec} A(\sin B\cos C + \cos B\sin C) =$ **[MP PET 1986, 1995]**

(a) $\frac{c}{a}$ (b) $\frac{a}{c}$ (c) 1 (d) $\frac{c}{ab}$

23. If $a = 9, b = 8$ and $c = x$ satisfies $3\cos C = 2$, then **[MP PET 1984]**

(a) $x = 5$ (b) $x = 6$ (c) $x = 4$ (d) $x = 7$

24. If in a triangle ABC side $a = (\sqrt{3}+1)\ cms$ and $\angle B = 30^o, \angle C = 45^o$, then the area of the triangle is **[MP PET 1997]**

(a) $\frac{\sqrt{3}+1}{3}cm^2$ (b) $\frac{\sqrt{3}+1}{2}cm^2$ (c) $\frac{\sqrt{3}+1}{2\sqrt{2}}cm^2$ (d) $\frac{\sqrt{3}+1}{3\sqrt{2}}cm^2$

25. In ΔABC, if $\cos\frac{A}{2} = \sqrt{\frac{b+c}{2c}}$, then **[MP PET 1990]**

(a) $a^2 + b^2 = c^2$ (b) $b^2 + c^2 = a^2$ (c) $c^2 + a^2 = b^2$ (d) $b - c = c - a$

26. In ΔABC, $\frac{\sin B}{\sin(A+B)} =$ **[MP PET 1989]**

(a) $\frac{b}{a+b}$ (b) $\frac{b}{c}$ (c) $\frac{c}{b}$ (d) None of these

27. If $\angle B = 60^o$, then **[Karnataka CET 1992]**

(a) $(a-b)^2 + ab = c^2$ (b) $(b-c)^2 + bc = a^2$ (c) $(c-a)^2 + ca = b^2$ (d) $a^2 + b^2 + c^2 = 2b^2 + ac$

28. If $b^2 + c^2 = 3a^2$, then $\cot B + \cot C - \cot A =$ **[MP PET 1991]**

(a) 1 (b) $\frac{ab}{4\Delta}$ (c) 0 (d) $\frac{ac}{4\Delta}$

29. In triangle ABC, $A = 30^o$, $b = 8$, $a = 6$, then $B = \sin^{-1} x$, where $x =$ **[Karnataka CET 1990]**

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) 1

30. In ΔABC, $1 - \tan\frac{A}{2}\tan\frac{B}{2} =$ **[Roorkee 1973]**

(a) $\frac{2c}{a+b+c}$ (b) $\frac{a}{a+b+c}$ (c) $\frac{2}{a+b+c}$ (d) $\frac{4a}{a+b+c}$

31. If $\cos^2 A + \cos^2 C = \sin^2 B$, then ΔABC is **[MP PET 1991]**

(a) Equilateral (b) Right angled (c) Isosceles (d) None of these

32. If in a triangle ABC, $\angle C = 60^o$, then $\frac{1}{a+c} + \frac{1}{b+c} =$ **[IIT 1975]**

(a) $\frac{1}{a+b+c}$ (b) $\frac{2}{a+b+c}$ (c) $\frac{3}{a+b+c}$ (d) None of these

33. In ΔABC, if $a = 3, b = 4, c = 5$, then $\sin 2B =$ **[MP PET 1983]**

(a) 4/5 (b) 3/20 (c) 24/25 (d) 1/50

34. If in a triangle $ABC, b = \sqrt{3}, c = 1$ and $B - C = 90^o$, then $\angle A$ is **[MP PET 1983]**

(a) 30^o (b) 45^o (c) 75^o (d) 15^o

35. If in a triangle ABC, $\cos A + \cos B + \cos C = \frac{3}{2}$, then the triangle is **[IIT 1984]**

(a) Isosceles (b) Equilateral (c) Right angled (d) None of these

36. If a^2, b^2, c^2 are in A.P., then which of the following are also in A.P. **[ISM Dhanbad 1989]**

(a) $\sin A, \sin B, \sin C$ (b) $\tan A, \tan B, \tan C$ (c) $\cot A, \cot B, \cot C$ (d) None of these

37. $\cot\frac{A+B}{2}.\tan\frac{A-B}{2} =$ **[Roorkee 1975]**

(a) $\frac{a+b}{a-b}$ (b) $\frac{a-b}{a+b}$ (c) $\frac{a}{a+b}$ (d) None of these

38. If the sides of a triangle are in A.P., then the cotangent of its half the angles will be in **[MP PET 1993]**

(a) H.P. (b) G.P. (c) A.P. (d) No particular order

39. In ΔABC, if $\cot A, \cot B, \cot C$ be in A.P., then a^2, b^2, c^2 are in **[MP PET 1997]**

(a) H.P. (b) G.P. (c) A.P. (d) None of these

Advance Level

40. If the perpendicular AD divides the base of the triangle ABC such that BD, CD and AD are in the ratio 2, 3 and 6, then angle A is equal to

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

41. In a triangle ABC, $\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$, then the value of angle A is **[IIT 1993]**

(a) 45^o (b) 30^o (c) 90^o (d) 60^o

42. If the two angles on the base of a triangle are $\left(22\frac{1}{2}\right)^o$ and $\left(112\frac{1}{2}\right)^o$, then the ratio of the height of the triangle to the length of the base is

(a) 1 : 2 (b) 2 : 1 (c) 2 : 3 (d) 1 : 1

43. If $\Delta = a^2 - (b-c)^2$, where Δ is the area of triangle *ABC*, then tan *A* is equal to **[Karnataka CET 1990]**

(a) $\frac{15}{16}$ (b) $\frac{8}{15}$ (c) $\frac{8}{17}$ (d) $\frac{1}{2}$

44. The perimeter of a ΔABC is 6 times the arithmetic mean of the sines of its angles. If the side *a* is 1, then the angle A is **[IIT 1992]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

45. If $A_1A_2A_3.....A_n$ be a regular polygon of *n* sides and $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$, then **[IIT 1994]**

(a) $n = 5$ (b) $n = 6$ (c) $n = 7$ (d) None of these

46. If an triangle *PQR*, sin *P*, sin *Q*, sin *R* are in A.P., then **[IIT 1998]**

(a) The altitudes are in A.P. (b) The altitudes are in H.P.

(c) The medians are in G.P. (d) The medians are in A.P.

47. Points *D*, *E* are taken on the side *BC* of a triangle *ABC* such that *BD*=*DE*=*EC*. If $\angle BAD = x$, $\angle DAE = y$, $\angle EAC = z$, then the values of $\frac{\sin(x+y)\sin(y+z)}{\sin x \sin z} =$

(a) 1 (b) 2 (c) 4 (d) None of these

48. In a ΔABC, $2\cos\left(\frac{A-C}{2}\right) = \frac{a+c}{\sqrt{a^2+c^2-ac}}$, then

(a) $B = \frac{\pi}{3}$ (b) $B = C$ (c) A, B, C are in A.P. (d) Both (a) and (c)

49. If P is the product of the sines of angles of a triangle, and *q* the product of their cosines, then the tangents of the angle are roots of the equation

(a) $qx^3 - px^2 + (1+q)x - p = 0$ (b) $px^3 - qx^2 + (1+p)x - q = 0$

(c) $(1+q)x^3 - px^2 + qx - p = 0$ (d) None of these

Circle Connected with Triangle

Basic Level

50. Which is true in the following **[UPSEAT 1999]**

(a) $a\cos A + b\cos B + c\cos C = R\sin A\sin B\sin C$ (b) $a\cos A + b\cos B + c\cos C = 2R\sin A\sin B\sin C$

(c) $a\cos A + b\cos B + c\cos C = 4R\sin A\sin B\sin C$ (d) $a\cos A + b\cos B + c\cos C = 8R\sin A\sin B\sin C$

51. In $\Delta ABC, a\cos A + b\cos B + c\cos C =$ **[WB JEE 1971]**

(a) $4R\sin A\sin B\sin C$ (b) $3R\sin A\sin B\sin C$ (c) $\sin A\sin B\sin C$ (d) $4R\cos A\cos B\cos C$

52. The in-radius of the triangle whose sides are 3, 5, 6 is **[EAMCET 1982]**

(a) $\sqrt{\frac{8}{7}}$ (b) $\sqrt{8}$ (c) $\sqrt{7}$ (d) $\sqrt{\frac{7}{8}}$

53. In an equilateral triangle of side $2\sqrt{3}$ *cm*, the circum-radius is **[EAMCET 1978]**

(a) 1 *cm* (b) $\sqrt{3}$ *cm* (c) 2 *cm* (d) $2\sqrt{3}$ *cm*

54. If the lengths of the sides of a triangle are 3, 4 and 5 units, then *R* the circum-radius is **[MNR 1990]**

(a) 2.0 (b) 2.5 (c) 3.0 (d) 3.5

55. In a triangle *ABC*, *a* : *b* : *c* = 4 : 5 : 6. The ratio of the radius of the circumcircle to that of the incircle is **[IIT 1996]**

(a) $\frac{16}{9}$ (b) $\frac{16}{7}$ (c) $\frac{11}{7}$ (d) $\frac{7}{16}$

56. If R is the radius of the circumcircle of the ΔABC and Δ is its area, then **[Karnataka CET 2000]**

(a) $R=\frac{a+b+c}{\Delta}$ (b) $R=\frac{a+b+c}{4\Delta}$ (c) $R=\frac{abc}{4\Delta}$ (d) $R=\frac{abc}{\Delta}$

57. A circle is inscribed in an equilateral triangle of side a. The area of any square inscribed in this circle is **[IIT 1994]**

(a) $\frac{a^2}{12}$ (b) $\frac{a^2}{6}$ (c) $\frac{a^2}{3}$ (d) $2a^2$

58. The diameter of the circum-circle whose sides are 61, 60, 11 **[EAMCET 1988]**

(a) 60 (b) 61 (c) 62 (d) 63

59. In a equilateral triangle, circum-radius : in-radius : ex-radius *i.e.,* $R:r:r_1 =$ **[Kurukshetra CEE 1999]**

(a) $1:1:1$ (b) $1:2:3$ (c) $2:1:3$ (d) $3:2:4$

60. In a ΔABC, I is the in-centre. The ratio $IA:IB:IC$ is equal to

(a) $\sin\frac{A}{2}:\sin\frac{B}{2}:\sin\frac{C}{2}$ (b) $\cos\frac{A}{2}:\cos\frac{B}{2}:\cos\frac{C}{2}$ (c) $\text{cosec}\frac{A}{2}:\text{cosec}\frac{B}{2}:\text{cosec}\frac{C}{2}$ (d) $\sec\frac{A}{2}:\sec\frac{B}{2}:\sec\frac{C}{2}$

61. In a triangle ABC if $r_1=2r_2=3r_3$, then

(a) $2a=b+c$ (b) $a+c=2b$ (c) $a+b-2c=0$ (d) None of these

62. If in a triangle R and r are the circumradius and in-radius respectively, then the H.M. of the ex-radii of the triangle is

(a) $3r$ (b) $2R$ (c) $R+r$ (d) None of these

Advance Level

63. If the radius of the circumcircle of an isosceles triangle PQR is equal to PQ (= PR), then the angle P is **[IIT 1992]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

64. If p_1,p_2,p_3 are respectively the perpendiculars from the vertices of a triangle to the opposite sides, then $p_1.p_2.p_3=$ **[DCE 1997; EAMCET 1994]**

(a) $\frac{a^2b^2c^2}{R^2}$ (b) $\frac{a^2b^2c^2}{4R^2}$ (c) $\frac{4a^2b^2c^2}{R^2}$ (d) $\frac{a^2b^2c^2}{8R^2}$

65. If r_1,r_2,r_3 are the radii of the escribed circles of a triangle ABC and if r is the radius of its incircle, then $r_1r_2r_3-r(r_1r_2+r_2r_3+r_3r_1)$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

66. In a triangle, the line joining the circum centre to the incentre is parallel to BC, then $\cos B+\cos C=$ **[EAMCET 1994]**

(a) $\frac{3}{2}$ (b) 1 (c) $\frac{3}{4}$ (d) $\frac{1}{2}$

67. The sides of a triangle inscribed in a given circle subtend angles α,β,γ at the centre. The minimum value of the A.M. of $\cos\left(\alpha+\frac{\pi}{2}\right),\cos\left(\beta+\frac{\pi}{2}\right)$ and $\cos\left(\gamma+\frac{\pi}{2}\right)$ is equal to **[UPSEAT 1996; IIT 1994]**

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{-\sqrt{3}}{2}$ (c) $\frac{-2}{\sqrt{3}}$ (d) $\sqrt{2}$

Solution of Triangle

Basic Level

68. In a ΔABC, if $\angle C = 30^o, a = 47\,cm$ and $b = 94\,cm$, then the triangle is **[MP PET 1986]**

(a) Right angled (b) Right angled isosceles (c) Isosceles (d) Obtuse angled

69. In a triangle *ABC*, if $a\sin A = b\sin B$, then the nature of the triangle **[MP PET 1983]**

(a) $a > b$ (b) $a < b$ (c) $a = b$ (d) $a + b = c$

70. If the sides of a triangle be 6, 10 and 14 then the triangle is **[MP PET 1982]**

(a) Obtuse angled (b) Acute angled (c) Right angled (d) Equilateral

71. If in a triangle *ABC*, *a*, *b*, *c* and angle *A* is given and $c\sin A < a < c$, then **[UPSEAT 1999]**

(a) $b_1 + b_2 = 2c\cos A$ (b) $b_1 + b_2 = c\cos A$ (c) $b_1 + b_2 = 3c\cos A$ (d) $b_1 + b_2 = 4c\sin A$

72. In a triangle *ABC* if $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, then the triangle is **[Karnataka CET 1991]**

(a) Right angled (b) Obtuse angled (c) Equilateral (d) Isosceles

73. If $\cot\frac{A}{2} = \frac{b+c}{2}$, then the ΔABC is **[EAMCET 1994]**

(a) Isosceles (b) Equilateral (c) Right angled (d) None of these

74. In any ΔABC if $a\cos B = b\cos A$, then the triangle is **[MP PET 1984]**

(a) Equilateral Triangle (b) Isosceles Triangle

(c) Scalene Triangle (d) Right angled Triangle

75. If one side of a triangle is twice the other side and the angles opposite to these sides differ by 60^o, then the triangle is

(a) Equilateral (b) Isosceles (c) Right angled (d) None of these

76. The sides of a triangle are $3x + 4y, 4x + 3y$ and $5x + 5y$ units where $x > 0, y > 0$. The triangle is **[AIEEE 2002]**

(a) Right angled (b) Equilateral (c) Obtuse angled (d) None of these

77. If $A = 60^o, a = 5, b = 4\sqrt{3}$ in ΔABC, then $B =$

(a) 30^o (b) 60^o (c) 90^o (d) None of these

78. If $A = 30^o, a = 7, b = 8$ in ΔABC, then *B* has

(a) One solution (b) Two solutions (c) No solution (d) None of these

79. If one angle of a triangle is 30^o and the lengths of the sides adjacent to it are 40 and $40\sqrt{3}$, the triangle is

(a) Right angled (b) Isosceles (c) Both (a) and (b) (d) None of these

80. If in ΔABC, $a = 5; b = 4; A = \frac{\pi}{2} + B$, then the value of *C*

(a) Cannot be evaluated (b) Is equal to $\tan^{-1}\frac{1}{9}$ (c) Is equal to $\tan^{-1}\frac{1}{40}$ (d) Is equal to $2\tan^{-1}\frac{1}{9}$

81. We are given *b*, *c* and sin*B* such that *B* is acute and $b < c\sin B$. Then **[Karnataka CET 1993]**

(a) No triangle is possible (b) One triangle is possible

(c) Two triangles are possible (d) A right angled triangle is possible

Advance Level

82. In a right angled triangle the hypotenuse is $2\sqrt{2}$ times the length of perpendicular drawn from the opposite vertex on the hypotenuse, then the other two angles are **[UPSEAT 1999, 94; MP PET 1998]**

(a) $\frac{\pi}{3}, \frac{\pi}{6}$ (b) $\frac{\pi}{4}, \frac{\pi}{4}$ (c) $\frac{\pi}{8}, \frac{3\pi}{8}$ (d) $\frac{\pi}{12}, \frac{5\pi}{12}$

83. In a triangle *ABC*, $a = 4, b = 3, \angle A = 60^o$. Then *c* is the root of the equation **[Roorkee 1993]**

(a) $c^2 - 3c - 7 = 0$ (b) $c^2 + 3c + 7 = 0$ (c) $c^2 - 3c + 7 = 0$ (d) $c^2 + 3c - 7 = 0$

84. In a triangle ABC, angle A is greater than angle B. If the measure of angles A and B satisfy the equation $3\sin x - 4\sin^3 x - k = 0,\ 0 < k < 1$, then the measure of angle C is **[IIT 1990; DCE 2001]**

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

85. In a ΔABC a, b, A are given and b_1, b_2 are two values of the third side b such that $b_2 = 2b_1$. Then $\sin A =$

(a) $\sqrt{\frac{9a^2 - c^2}{8a^2}}$ (b) $\sqrt{\frac{9a^2 - c^2}{8c^2}}$ (c) $\sqrt{\frac{9a^2 + c^2}{8a^2}}$ (d) None of these

86. The exists a triangle ABC satisfying

(a) $\tan A + \tan B + \tan C = 0$ (b) $\frac{\sin A}{2} = \frac{\sin B}{3} = \frac{\sin C}{7}$

(c) $(a+b)^2 = c^2 + ab$ and $\sqrt{2}(\sin A + \cos A) = \sqrt{3}$ (d) $\sin A + \sin B = \frac{\sqrt{3}+1}{4}, \cos A \cos B = \frac{\sqrt{3}}{2} = \sin A \sin B$

87. In the ambiguous case, given a, b and A. Then the difference between the two values of c is **[AMU 1998]**

(a) $2\sqrt{a^2 - b^2}$ (b) $\sqrt{a^2 - b^2 \sin^2 A}$ (c) $2\sqrt{a^2 - b^2 \sin^2 A}$ (d) $\sqrt{a^2 - b^2}$

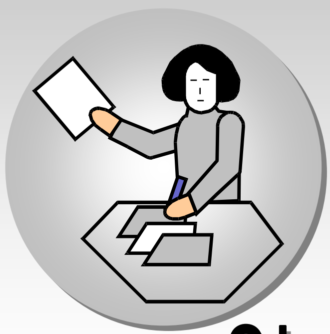
88. The sides of a triangle are in A.P. and its area is $\frac{3}{5}$ × (Area of an equilateral triangle of the same perimeter). Then the ratio of the sides is

(a) $1 : 2 : 3$ (b) $3 : 5 : 7$ (c) $1 : 3 : 5$ (d) None of these

89. There exists a triangle ABC satisfying the conditions **[IIT 1996]**

(a) $b \sin A = a, A < \frac{\pi}{2}$ or $b \sin A < a, A < \frac{\pi}{2}, b < a$ (b) $b \sin A > a, A > \frac{\pi}{2}$

(c) $b \sin A > a, A < \frac{\pi}{2}$ (d) None of these



Answer Sheet

Properties of Triangles and Solutions of Triangles ***Assignment (Basic & Advance Level)***

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**

d	**a**	**c**	**c**	**c**	**a**	**c**	**a**	**b**	**a**	**c**	**b**	**d**	**b**	**a**	**a**	**c**	**d**	**a**	**c**
21	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
c	**c**	**d**	**b**	**a**	**b**	**c**	**c**	**c**	**a**	**b**	**c**	**c**	**a**	**b**	**c**	**b**	**c**	**c**	**c**
41	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**
c	**a**	**b**	**a**	**c**	**b**	**c**	**d**	**a**	**c**	**a**	**a**	**c**	**b**	**b**	**c**	**b**	**b**	**c**	**c**
61	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**	**70**	**71**	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**	**80**
b	**a**	**d**	**d**	**a**	**b**	**b**	**d**	**c**	**a**	**a**	**c**	**c**	**b**	**c**	**c**	**d**	**b**	**b**	**d**
81	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**											
a	**c**	**a**	**c**	**b**	**c**	**c**	**b**	**a**											