Chapter 1

INTRODUCTION TO MATHEMATICAL ANALYSIS

§ 1.1. Real Numbers. The Absolute Value of a Real Number

Any decimal fraction, terminating or nonterminating, is called a *real* number.

Periodic decimal fractions are called *rational* numbers. Every rational number may be written in the form of a ratio, $\frac{p}{q}$, of two integers p and q, and vice versa.

Nonperiodic decimal fractions are called *irrational* numbers.

If X is a certain set of real numbers, then the notation $x \in X$ means that the number x belongs to X, and the notation $x \notin X$ means that the number x does not belong to X.

A set of real numbers x satisfying the inequalities $a < x < b$, where a and b are fixed numbers, is called an *open interval* (a, b). A set of real numbers x satisfying the inequalities $a \leqslant x \leqslant b$ is called a *closed interval* $[a, b]$. A set of real numbers x, satisfying the inequalities $a \leqslant x < b$ or $a < x \leqslant b$, is called a *half-open interval* $[a, b)$ or $(a, b]$. Open, closed, and half-open intervals are covered by a single term *interval*.

Any real number may be depicted as a certain point on the coordinate axis which is called a *proper point*. We may also introduce two more, so-called *improper points*, $+\infty$ and $-\infty$ infinitely removed from the origin of coordinates in the positive and negative directions, respectively. By definition, the inequalities $-\infty < x < +\infty$ hold true for any real number x.

The interval $(a-\varepsilon, a+\varepsilon)$ is called the *ε-neighbourhood* of the number a.

The set of real numbers $x > M$ is called the *M-neighbourhood* of the improper point $+\infty$.

The set of real numbers $x < M$ is called the *M-neighbourhood* of the improper point $-\infty$.

The *absolute value* of a number x (denoted $|x|$) is a number that satisfies the conditions

$$|x| = -x \text{ if } x < 0;$$
$$|x| = x \quad \text{if } x \geqslant 0.$$

The properties of absolute values are:

(1) the inequality $|x| \leqslant \alpha$ means that $-\alpha \leqslant x \leqslant \alpha$;

(2) the inequality $|x| \geqslant \alpha$ means that $x \geqslant \alpha$ or $x \leqslant -\alpha$;

(3) $|x \pm y| \leqslant |x| + |y|$;

(4) $|x \pm y| \geqslant ||x| - |y||$;

(5) $|xy| = |x||y|$;

(6) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ $(y \neq 0)$.

1.1.1. Prove that the number

$$0.1010010001\ldots\underbrace{1000\ldots0}_{n}1\ldots$$

is irrational.

Solution. To prove this, it is necessary to ascertain that the given decimal fraction is not a periodic one. Indeed, there are n zeros between the nth and $(n+1)$th unities, which cannot occur in a periodic fraction.

1.1.2. Prove that any number, with zeros standing in all decimal places numbered 10^n and only in these places, is irrational.

1.1.3. Prove that the sum of, or the difference between, a rational number α and an irrational number β is an irrational number.

Solution. Consider the sum of α and β. Suppose $\alpha + \beta = \gamma$ is a rational number, then $\beta = \gamma - \alpha$ is also a rational number, since it is the difference between two rational numbers, which contradicts the condition. Hence, the supposition is wrong and the number $\alpha + \beta$ is irrational.

1.1.4. Prove that the product $\alpha\beta$ and the quotient α/β of a rational number $\alpha \neq 0$ and an irrational number β is an irrational number.

1.1.5. (a) Find all rational values of x at which $y = \sqrt{x^2 + x + 3}$ is a rational number.

Solution. (a) Suppose x and $y = \sqrt{x^2 + x + 3}$ are rational numbers. Then the difference $y - x = q$ is also a rational number. Let us now express x through q

$$y - x = \sqrt{x^2 + x + 3} - x = q,$$

$$\sqrt{x^2 + x + 3} = q + x,$$

$$x^2 + x + 3 = q^2 + 2qx + x^2,$$

$$x = \frac{q^2 - 3}{1 - 2q}.$$

By a direct check it is easy to ascertain that $q \neq 1/2$.

Prove the reverse, namely, $y=\sqrt{x^2+x+3}$ is a rational number if $x=\frac{q^2-3}{1-2q}$, where q is any rational number not equal to $1/2$.

Indeed,

$$y=\sqrt{x^2+x+3}=\sqrt{\frac{(q^2-3)^2}{(1-2q)^2}+\frac{q^2-3}{1-2q}+3}=$$

$$=\sqrt{\frac{q^4-2q^3+7q^2-6q+9}{(1-2q)^2}}=\sqrt{\frac{(q^2-q+3)^2}{(1-2q)^2}}=\frac{q^2-q+3}{|1-2q|} \quad \left(q\neq\frac{1}{2}\right).$$

The latter expression is rational at any rational q not equal to $1/2$.

(b) Prove that $\sqrt{2}$ is an irrational number.

1.1.6. Prove that the sum $\sqrt{3}+\sqrt{2}$ is an irrational number.

Solution. Assume the contrary, i.e. that the number $\sqrt{3}+\sqrt{2}$ is rational. Then the number

$$\sqrt{3}-\sqrt{2}=\frac{1}{\sqrt{3}+\sqrt{2}}$$

is also rational, since it is the quotient of two rational numbers. Whence the number

$$\sqrt{2}=\frac{1}{2}[(\sqrt{3}+\sqrt{2})-(\sqrt{3}-\sqrt{2})]$$

is rational, which contradicts the irrational nature of the number $\sqrt{2}$ (see Problem **1.1.5**). Hence, the supposition is wrong, and the number $\sqrt{3}+\sqrt{2}$ is irrational.

1.1.7. Prove that for every positive rational number r satisfying the condition $r^2<2$ one can always find a larger rational number $r+h\,(h>0)$ for which $(r+h)^2<2$.

Solution. We may assume $h<1$. Then $h^2<h$ and $(r+h)^2<$ $<r^2+2rh+h$. That is why it is sufficient to put $r^2+2rh+h=2$, i.e. $h=(2-r^2)/(2r+1)$.

1.1.8. Prove that for every positive rational number s satisfying the condition $s^2>2$ one can always find a smaller rational number $s-k\,(k>0)$ for which $(s-k)^2>2$.

1.1.9. Solve the following inequalities:

(a) $|2x-3|<1$;

(b) $(x-2)^2\geqslant 4$;

(c) $x^2+2x-8\leqslant 0$;

(d) $|x^2-7x+12|>x^2-7x+12$.

Solution. (a) The inequality $|2x-3|<1$ is eqivalent to the inequalities

$$-1<2x-3<1,$$

whence

$$2<2x<4 \text{ and } 1<x<2.$$

(d) The given inequality is valid for those values of x at which $x^2-7x+12<0$, whence $3<x<4$.

1.1.10. Find out whether the following equations have any solutions:

(a) $|x|=x+5$; (b) $|x|=x-5$?

Solution. (a) At $x\geqslant 0$ we have $x=x+5$. Hence, there are no solutions. At $x<0$ we have $-x=x+5$, whence $x=-5/2$. This value satisfies the initial equation.

(b) At $x\geqslant 0$ we have $x=x-5$. Hence, there are no solutions. At $x<0$ we have $-x=x-5$, whence $x=5/2$, which contradicts our supposition ($x<0$). Thus, the equation has no solution.

1.1.11. Determine the values of x satisfying the following equalities:

(a) $\left|\frac{x-1}{x+1}\right|=\frac{x-1}{x+1}$;

(b) $|x^2-5x+6|=-(x^2-5x+6)$.

1.1.12. Determine the values of x satisfying the following equalities:

(a) $|(x^2+4x+9)+(2x-3)|=|x^2+4x+9|+|2x-3|$;

(b) $|(x^4-4)-(x^2+2)|=|x^4-4|-|x^2+2|$.

Solution. (a) The equality $|a+b|=|a|+|b|$ is valid if and only if both summands have the same sign. Since

$$x^2+4x+9=(x+2)^2+5>0$$

at any values of x, the equality is satisfied at those values of x at which $2x-3\geqslant 0$, i.e. at $x\geqslant 3/2$.

(b) The equality $|a-b|=|a|-|b|$ holds true if and only if a and b have the same sign and $|a|\geqslant|b|$.

In our case the equality will hold true for the values of x at which

$$x^4-4\geqslant x^2+2.$$

Whence

$$x^2-2\geqslant 1;\quad |x|\geqslant\sqrt{3}.$$

1.1.13. Solve the inequalities:

(a) $|3x-5|-|2x+3|>0$;

(b) $|x^2-5x|>|x^2|-|5x|$.

1.1.14. Find the roots of the following equations.

(a) $|\sin x|=\sin x+1$;

(b) $x^2-2|x|-3=0$.

Solution. (a) This equation will hold true only for those values of x at which $\sin x<0$, that is why we may rewrite it in the following way:

$$-\sin x = \sin x + 1, \text{ or } \sin x = -{}^1/_2;$$

whence $x = \pi k - (-1)^k \pi/6$ $(k = 0, \pm 1, \pm 2, \ldots)$.

(b) This equation can be solved in a regular way by considering the cases $x \geqslant 0$ and $x \leqslant 0$. We may also solve this equation rewriting it in the form

$$|x|^2 - 2|x| - 3 = 0.$$

Substituting y for $|x|$, we obtain

$$y^2 - 2y - 3 = 0,$$

whence $y_1 = 3$, $y_2 = -1$. Since $y = |x| \geqslant 0$, the value $y_2 = -1$ does not fit in. Hence

$$y = |x| = 3,$$

i.e. $x_1 = -3$, $x_2 = 3$.

§ 1.2. Function. Domain of Definition

The independent variable x is defined by a set X of its values. If to each value of the independent variable $x \in X$ there corresponds one definite value of another variable y, then y is called the *function* of x with a *domain of definition* (or *domain*) X or, in functional notation, $y = y(x)$, or $y = f(x)$, or $y = \varphi(x)$, and so forth. The set of values of the function $y(x)$ is called the *range* of the given function.

In particular, the functions defined by the set of natural numbers 1, 2, 3, ..., are called *numerical sequences*. They are written in the following way: $x_1, x_2, \ldots, x_n, \ldots$ or $\{x_n\}$.

1.2.1. Given the function $f(x) = (x+1)/(x-1)$. Find $f(2x)$, $2f(x)$, $f(x^2)$, $[f(x)]^2$.

Solution.

$$f(2x) = \frac{2x+1}{2x-1}; \quad 2f(x) = 2\frac{x+1}{x-1};$$

$$f(x^2) = \frac{x^2+1}{x^2-1}; \quad [f(x)]^2 = \left(\frac{x+1}{x-1}\right)^2.$$

1.2.2. (a) Given the function

$$f(x) = \log \frac{1-x}{1+x}.$$

Show that at $x_1, x_2 \in (-1, 1)$ the following identity holds true:

$$f(x_1) + f(x_2) = f\left(\frac{x_1 + x_2}{1 + x_1 x_2}\right).$$

Solution. At $x \in (-1, 1)$ we have $(1-x)/(1+x) > 0$ and hence

$$f(x_1)+f(x_2)=\log\frac{1-x_1}{1+x_1}+\log\frac{1-x_2}{1+x_2}=\log\frac{(1-x_1)(1-x_2)}{(1+x_1)(1+x_2)}. \qquad (1)$$

On the other hand,

$$f\left(\frac{x_1+x_2}{1+x_1x_2}\right)=\log\frac{1-\frac{x_1+x_2}{1+x_1x_2}}{1+\frac{x_1+x_2}{1+x_1x_2}}=\log\frac{1+x_1x_2-x_1-x_2}{1+x_1x_2+x_1+x_2}=$$

$$=\log\frac{(1-x_1)(1-x_2)}{(1+x_1)(1+x_2)},$$

which coincides with the right-hand member of expression (1).

(b) Given the function $f(x)=(a^x+a^{-x})/2$ $(a>0)$. Show that

$$f(x+y)+f(x-y)=2f(x)f(y).$$

1.2.3. Given the function $f(x)=(x+1)/(x^3-1)$. Find $f(-1)$; $f(a+1)$; $f(a)+1$.

1.2.4. Given the function $f(x)=x^3-1$. Find

$$\frac{f(b)-f(a)}{b-a}\ (b\neq a) \text{ and } f\left(\frac{a+h}{2}\right).$$

1.2.5. Given the function

$$f(x)=\begin{cases} 3^{-x}-1, & -1\leqslant x<0,\\ \tan(x/2), & 0\leqslant x<\pi,\\ x/(x^2-2), & \pi\leqslant x\leqslant 6.\end{cases}$$

Find $f(-1)$, $f(\pi/2)$, $f(2\pi/3)$, $f(4)$, $f(6)$.

Solution. The point $x=-1$ lies within the interval $[-1, 0)$. Hence

$$f(-1)=3^{-(-1)}-1=2.$$

The points $x=\pi/2$, $x=2\pi/3$ belong to the interval $[0, \pi)$. Hence

$$f(\pi/2)=\tan(\pi/4)=1;\quad f(2\pi/3)=\tan(\pi/3)=\sqrt{3}.$$

The points $x=4$, $x=6$ belong to the interval $[\pi, 6]$. Hence

$$f(4)=\frac{4}{16-2}=\frac{2}{7};\quad f(6)=\frac{6}{36-2}=\frac{3}{17}.$$

1.2.6. The function $f(x)$ is defined over the whole number scale by the following law:

$$f(x)=\begin{cases} 2x^3+1, & \text{if } x\leqslant 2,\\ 1/(x-2), & \text{if } 2<x\leqslant 3,\\ 2x-5, & \text{if } x>3.\end{cases}$$

Find: $f(\sqrt{2})$, $f(\sqrt{8})$, $f(\sqrt{\log_2 1024})$.

1.2.7. In the square $ABCD$ with side $AB=2$ a straight line MN is drawn perpendicularly to AC. Denoting the distance from the vertex A to the line MN as x, express through x the area S of the triangle AMN cut off from the square by the straight line MN. Find this area at $x=\sqrt{2}/2$ and at $x=2$ (Fig. 1).

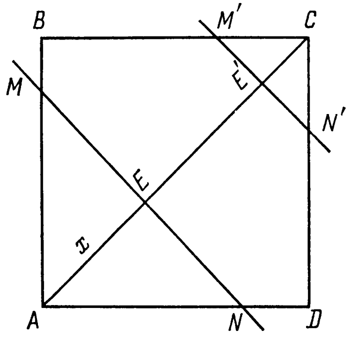

Fig. 1

Solution. Note that $AC=2\sqrt{2}$; hence $0 \leqslant x \leqslant 2\sqrt{2}$. If $x \leqslant \sqrt{2}$, then

$$S(x) = S_{\triangle AMN} = x^2.$$

If $x > \sqrt{2}$, then

$$S(x) = 4-(2\sqrt{2}-x)^2 = -x^2 + 4x\sqrt{2}-4.$$

Thus,

$$S(x) = \begin{cases} x^2, & 0 \leqslant x \leqslant \sqrt{2}, \\ -x^2+4x\sqrt{2}-4, & \sqrt{2} < x \leqslant 2\sqrt{2}. \end{cases}$$

Since $\sqrt{2}/2 < \sqrt{2}$, $S(\sqrt{2}/2) = (\sqrt{2}/2)^2 = 1/2$. Since $2 > \sqrt{2}$,

$$S(2) = -4+8\sqrt{2}-4 = 8(\sqrt{2}-1).$$

1.2.8. Bring the number α_n, which is equal to the nth decimal place in the expansion of $\sqrt{2}$ into a decimal fraction, into correspondence with each natural number n. This gives us a certain function $\alpha_n = \varphi(n)$. Calculate $\varphi(1)$, $\varphi(2)$, $\varphi(3)$, $\varphi(4)$.

Solution. Extracting a square root, we find $\sqrt{2} = 1.4142\ldots$. Hence

$$\varphi(1) = 4; \quad \varphi(2) = 1; \quad \varphi(3) = 4; \quad \varphi(4) = 2.$$

1.2.9. Calculate $f(x) = 49/x^2 + x^2$ at the points for which $7/x + x = 3$.

Solution. $f(x) = 49/x^2 + x^2 = (7/x+x)^2 - 14$, but $7/x+x=3$, hence $f(x) = 9-14 = -5$.

1.2.10. Find a function of the form $f(x) = ax^2+bx+c$, if it is known that $f(0)=5$; $f(-1)=10$; $f(1)=6$.

Solution.

$$f(0) = 5 = a\cdot 0^2 + b\cdot 0 + c,$$
$$f(-1) = 10 = a-b+c,$$
$$f(1) = 6 = a+b+c.$$

Determine the coefficients a, b, c from the above system. We have: $a=3$; $b=-2$; $c=5$; hence $f(x)=3x^2-2x+5$.

1.2.11. Find a function of the form

$$f(x)=a+bc^x \quad (c>0),$$

if $f(0)=15$; $f(2)=30$; $f(4)=90$.

1.2.12. Find $\varphi[\psi(x)]$ and $\psi[\varphi(x)]$ if

$$\varphi(x)=x^2 \text{ and } \psi(x)=2^x.$$

Solution.

$$\varphi[\psi(x)]=[\psi(x)]^2=(2^x)^2=2^{2x},$$
$$\psi[\varphi(x)]=2^{\varphi(x)}=2^{x^2}.$$

1.2.13. Given the function

$$f(x)=\frac{5x^2+1}{2-x}.$$

Find $f(3x)$; $f(x^3)$; $3f(x)$; $[f(x)]^3$.

1.2.14. Let

$$f(x)=\begin{cases} 3^x & \text{at } -1<x<0, \\ 4 & \text{at } 0\leqslant x<1, \\ 3x-1 & \text{at } 1\leqslant x\leqslant 3. \end{cases}$$

Find $f(2)$, $f(0)$, $f(0.5)$, $f(-0.5)$, $f(3)$.

1.2.15. Prove that if for an exponential function $y=a^x$ $(a>0;\ a\neq 1)$ the values of the argument $x=x_n$ $(n=1, 2, \ldots)$ form an arithmetic progression, then the corresponding values of the function $y_n=a^{x_n}$ $(n=1, 2, \ldots)$ form a geometric progression.

1.2.16. $f(x)=x^2+6$, $\varphi(x)=5x$. Solve the equation $f(x)=|\varphi(x)|$.

1.2.17. Find $f(x)$ if

$$f(x+1)=x^2-3x+2.$$

1.2.18. Evaluate the functions

$$f(x)=x^2+1/x^2 \text{ and } \varphi(x)=x^4+1/x^4$$

for the points at which $1/x+x=5$.

1.2.19. $f(x)=x+1$; $\varphi(x)=x-2$; solve the equation

$$|f(x)+\varphi(x)|=|f(x)|+|\varphi(x)|.$$

1.2.20. A rectangle with altitude x is inscribed in a triangle ABC with the base b and altitude h. Express the perimeter P and area S of the rectangle as functions of x.

1.2.21. Find the domains of definition of the following functions:

(a) $f(x)=\sqrt{x-1}+\sqrt{6-x}$;

(b) $f(x)=\sqrt{x^2-x-2}+\dfrac{1}{\sqrt{3+2x-x^2}}$;

(c) $f(x)=\dfrac{x}{\sqrt{x^2-x-2}}$;

(d) $f(x)=\sqrt{\sin x-1}$;

(e) $f(x)=\sqrt{\log\dfrac{5x-x^2}{4}}$;

(f) $f(x)=\log_x 5$;

(g) $f(x)=\log\dfrac{x^2-5x+6}{x^2+4x+6}$;

(h) $f(x)=\arcsin\dfrac{x-3}{2}-\log(4-x)$;

(i) $f(x)=\dfrac{1}{\log(1-x)}+\sqrt{x+2}$;

(j) $f(x)=\log\cos x$;

(k) $f(x)=\arccos\dfrac{3}{4+2\sin x}$;

(l) $y=\dfrac{1}{\sqrt{|x|-x}}$.

Solution. (a) The domain of definition of the given function consists of those values of x at which both items take on real values. To ensure this the following two conditions must be satisfied:

$$\begin{cases} x-1\geqslant 0, \\ 6-x\geqslant 0. \end{cases}$$

By solving the inequalities we obtain $x\geqslant 1$; $x\leqslant 6$. Hence, the domain of definition of the function will be the segment $[1,6]$.

(e) The function is defined for the values of x for which

$$\log\frac{5x-x^2}{4}\geqslant 0.$$

This inequality will be satisfied if

$$\frac{5x-x^2}{4}\geqslant 1, \text{ or } x^2-5x+4\leqslant 0.$$

Solving the latter inequality, we find $1\leqslant x\leqslant 4$. Thus, the segment $[1,4]$ is the domain of definition of the function.

(f) The function is defined for all positive x different from unity, which means that the domain of definition of the function consists of the intervals $(0, 1)$ and $(1, +\infty)$.

(k) The function is defined for the values of x for which

$$-1 \leqslant \frac{3}{4+2\sin x} \leqslant 1.$$

Since $4+2\sin x > 0$ at any x, the problem is reduced to solving the inequality

$$\frac{3}{4+2\sin x} \leqslant 1.$$

Whence

$$3 \leqslant 4+2\sin x, \text{ i.e. } \sin x \geqslant -{}^1/_2.$$

By solving the latter inequality we obtain

$$-\frac{\pi}{6}+2k\pi \leqslant x \leqslant \frac{7\pi}{6}+2k\pi \quad (k=0, \pm 1, \pm 2, \ldots).$$

(l) The function is defined for the values of x for which $|x|-x>0$, whence $|x|>x$. This inequality is satisfied at $x<0$. Hence, the function is defined in the interval $(-\infty, 0)$.

1.2.22. Find the domains of definition of the following functions:

(a) $f(x)=\sqrt{\arcsin(\log_2 x)}$;

(b) $f(x)=\log_2\log_3\log_4 x$;

(c) $f(x)=\frac{1}{x}+2^{\arcsin x}+\frac{1}{\sqrt{x-2}}$;

(d) $f(x)=\log|4-x^2|$;

(e) $f(x)=\sqrt{\cos(\sin x)}+\arcsin\frac{1+x^2}{2x}$.

Find the ranges of the following functions:

(f) $y=\frac{1}{2-\cos 3x}$;

(g) $y=\frac{x}{1+x^2}$.

Solution. (a) For the function $f(x)$ to be defined the following inequality must be satisfied

$$\arcsin(\log_2 x) \geqslant 0,$$

whence $0 \leqslant \log_2 x \leqslant 1$ and $1 \leqslant x \leqslant 2$.

(b) The function $\log_2\log_3\log_4 x$ is defined for $\log_3\log_4 x>0$, whence $\log_4 x>1$ and $x>4$. Hence, the domain of definition is the interval $4<x<+\infty$.

(c) The given function is defined if the following inequalities are satisfied simultaneously:

$$x \neq 0; \ -1 \leqslant x \leqslant 1 \text{ and } x>2,$$

but the inequalities $-1 \leqslant x \leqslant 1$ and $x>2$ are incompatible, that is why the function is not defined for any value of x.

(e) The following inequalities must be satisfied simultaneously:

$$\cos(\sin x) \geqslant 0 \quad \text{and} \quad \left|\frac{1+x^2}{2x}\right| \leqslant 1.$$

The first inequality is satisfied for all values of x, the second, for $|x| = 1$. Hence, the domain of definition of the given function consists only of two points $x = \pm 1$.

(f) We have

$$\cos 3x = \frac{2y-1}{y}.$$

Since

$$-1 \leqslant \cos 3x \leqslant 1, \quad \text{we have} \quad -1 \leqslant \frac{2y-1}{y} \leqslant 1,$$

whence, taking into account that $y > 0$, we obtain

$$-y \leqslant 2y - 1 \leqslant y \text{ or } \frac{1}{3} \leqslant y \leqslant 1.$$

(g) Solving with respect to x, we obtain

$$x = \frac{1 \pm \sqrt{1-4y^2}}{2y}.$$

The range of the function y will be determined from the relation

$$1 - 4y^2 \geqslant 0.$$

Whence

$$-\frac{1}{2} \leqslant y \leqslant \frac{1}{2}.$$

1.2.23. Solve the equation

$$\text{arc } \tan \sqrt{x(x+1)} + \text{arc } \sin \sqrt{x^2+x+1} = \pi/2.$$

Solution. Let us investigate the domain of definition of the function on the left side of the equation. This function will be defined for

$$x^2 + x \geqslant 0, \ 0 \leqslant x^2 + x + 1 \leqslant 1,$$

whence $x^2 + x = 0$.

Thus, the left member of the equation attains real values only at $x_1 = 0$ and $x_2 = -1$. By a direct check we ascertain that they are the roots of the given equation.

This problem shows that a study of domains of definition of a function facilitates the solution of equations, inequalities, etc.

1.2.24. Find the domains of definition of the following functions:

(a) $y = \dfrac{2x-3}{\sqrt{x^2+2x+3}}$;

(b) $y = \log \sin(x-3) + \sqrt{16-x^2}$;

(c) $y=\sqrt{3-x}+\arccos\frac{x-2}{3}$;

(d) $y=\frac{x}{\log(1+x)}$.

1.2.25. The function $f(x)$ is defined on the interval $[0, 1]$. What are the domains of definition of the following functions:

$$\text{(a) } f(3x^2);\quad \text{(b) } f(x-5);\quad \text{(c) } f(\tan x)?$$

Solution. The given functions are functions of functions, or *superpositions* of functions, i. e. *composite* functions.

a) Let us introduce an intermediate argument $u=3x^2$. Then the function $f(3x^2)=f(u)$ is defined if $0\leqslant u\leqslant 1$, i. e. $0\leqslant 3x^2\leqslant 1$, whence $-1/\sqrt{3}\leqslant x\leqslant 1/\sqrt{3}$.

(c) Similarly: $0\leqslant \tan x\leqslant 1$, whence

$$k\pi\leqslant x\leqslant \pi/4+k\pi \quad (k=0, \pm 1, \pm 2, \ldots).$$

1.2.26. The function $f(x)$ is defined on the interval $[0, 1]$. What are the domains of definition of the functions

$$\text{(a) } f(\sin x);\quad \text{(b) } f(2x+3)?$$

§ 1.3. Investigation of Functions

A function $f(x)$ defined on the set X is said to be *non-decreasing* on this set (respectively, *increasing*, *non-increasing*, *decreasing*), if for any numbers $x_1, x_2\in X$, $x_1<x_2$, the inequality $f(x_1)\leqslant f(x_2)$ (respectively, $f(x_1)<f(x_2)$, $f(x_1)\geqslant f(x_2)$, $f(x_1)>f(x_2)$) is satisfied. The function $f(x)$ is said to be *monotonic* on the set X if it possesses one of the four indicated properties. The function $f(x)$ is said to be *bounded above* (or *below*) on the set X if there exists a number M (or m) such that $f(x)\leqslant M$ for all $x\in X$ (or $m\leqslant f(x)$ for all $x\in X$). The function $f(x)$ is said to be *bounded on the set X* if it is bounded above and below.

The function $f(x)$ is called *periodic* if there exists a number $T>0$ such that $f(x+T)=f(x)$ for all x belonging to the domain of definition of the function (together with any point x the point $x+T$ must belong to the domain of definition). The least number T possessing this property (if such a number exists) is called the *period of the function* $f(x)$. The function $f(x)$ takes on the *maximum value* at the point $x_0\in X$ if $f(x_0)\geqslant f(x)$ for all $x\in X$, and the *minimum value* if $f(x_0)\leqslant f(x)$ for all $x\in X$. A function $f(x)$ defined on a set X which is symmetric with respect to the origin of coordinates is called *even* if $f(-x)=f(x)$, and *odd* if $f(-x)=-f(x)$.

In analysing the behaviour of a function it is advisable to determine the following:

1. The domain of definition of the function.
2. Is the function even, odd, periodic?
3. The zeros of the function.
4. The sign of the function in the intervals between the zeros.
5. Is the function bounded and what are its minimum and maximum values?

The above items do not exhaust the analysis of a function, and later on their scope will be increased.

1.3.1. Find the intervals of increase and decrease of the function $f(x)=ax^2+bx+c$, and its minimum and maximum values.

Solution. Isolating a perfect square from the square trinomial, we have

$$f(x)=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}.$$

If $a>0$, then the function $f(x)$ will increase at those values of x satisfying the inequality $x+b/(2a)>0$, i. e. at $x>-b/(2a)$, and decrease when $x+b/(2a)<0$, i. e. at $x<-b/(2a)$. Thus, if $a>0$, the function $f(x)$ decreases in the interval $\left(-\infty, -\frac{b}{2a}\right)$ and increases in the interval $(-b/(2a), +\infty)$. Obviously, at $x=-b/(2a)$ the function $f(x)$ assumes the minimum value

$$f_{\min}=f\left(-\frac{b}{2a}\right)=\frac{4ac-b^2}{4a}.$$

At $a>0$ the function has no maximum value.

Similarly, at $a<0$ the function $f(x)$ will increase in the interval $\left(-\infty, -\frac{b}{2a}\right)$ and decrease in the interval $(-b/(2a), \infty)$; at $x=-b/(2a)$ the function $f(x)$ takes on the maximum value

$$f_{\max}=f\left(-\frac{b}{2a}\right)=\frac{4ac-b^2}{4a},$$

whereas it has no minimum value.

1.3.2. (a) Find the minimum value of the function

$$y=3x^2+5x-1.$$

(b) Find the rectangle with the maximum area from among all rectangles of a given perimeter.

Solution. (a) Apply the results of Problem **1.3.1**: $a=3>0$, $b=5$, $c=-1$. The minimum value is attained by the function at the point $x=-5/6$

$$y_{\min}=\frac{4ac-b^2}{4a}=-\frac{37}{12}.$$

(b) We denote by $2p$ the length of the perimeter of the required rectangle, and by x the length of one of its sides; then the area S

of the rectangle will be expressed as

$$S = x(p-x) \text{ or } S = px - x^2.$$

Thus, the problem is reduced to the determination of the maximum value of the function $S(x) = -x^2 + px$. Apply the results of Problem **1.3.1**: $a = -1 < 0$, $b = p$, $c = 0$. The maximum value is attained by the function at the point $x = -b/(2a) = p/2$. Hence, one of the sides of the desired rectangle is $p/2$, the other side being equal to $p - x = p/2$, i. e. the required rectangle is a square.

1.3.3. Show that

(a) the function $f(x) = x^3 + 3x + 5$ increases in the entire domain of its definition;

(b) the function $g(x) = x/(1+x^2)$ decreases in the interval $(1, +\infty)$.

Solution. The function is defined for all points of the number scale. Let us take arbitrary points x_1 and x_2, $x_1 < x_2$ on the number scale and write the following difference:

$$f(x_2) - f(x_1) = (x_2^3 + 3x_2 + 5) - (x_1^3 + 3x_1 + 5) =$$
$$= (x_2 - x_1)(x_2^2 + x_1x_2 + x_1^2 + 3) =$$
$$= (x_2 - x_1)\left[\left(x_1 + \frac{1}{2}x_2\right)^2 + \frac{3}{4}x_2^2 + 3\right].$$

Since $x_2 - x_1 > 0$ and the expression in the brackets is positive at all x_1 and x_2, then $f(x_2) - f(x_1) > 0$, i. e. $f(x_2) > f(x_1)$, which means that the function $f(x)$ increases for all values of x.

1.3.4. Find the intervals of increase and decrease for the following functions:

(a) $f(x) = \sin x + \cos x$;

(b) $\tan(x + \pi/3)$.

Solution. (a) Using the familiar trigonometric formulas, we find

$$f(x) = \sqrt{2}\cos(x - \pi/4).$$

It is known that the function $\cos x$ decreases in the intervals

$$2n\pi \leqslant x \leqslant (2n+1)\pi$$

and increases in the intervals

$$(2n-1)\pi \leqslant x \leqslant 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Hence, the intervals of decrease of the function $f(x)$ are:

$$\pi/4 + 2n\pi \leqslant x \leqslant \pi/4 + (2n+1)\pi \quad (n = 0, \pm 1, \ldots),$$

and the intervals of increase of the same function are:

$$\pi/4 + (2n-1)\pi \leqslant x \leqslant \pi/4 + 2n\pi \quad (n = 0, \pm 1, \ldots).$$

1.3.5. Find the minimum and maximum values of the function

$$f(x) = a\cos x + b\sin x \quad (a^2 + b^2 > 0).$$

Solution. The given function can be represented as:

$$f(x)=\sqrt{a^2+b^2}\cos(x-\alpha),$$

where $\cos\alpha=a/\sqrt{a^2+b^2}$, $\sin\alpha=b/\sqrt{a^2+b^2}$. Since $|\cos(x-\alpha)|\leqslant 1$, the maximum value of $f(x)$ equals $+\sqrt{a^2+b^2}$ (at $\cos(x-\alpha)=1$), the minimum value of $f(x)$ being equal to $-\sqrt{a^2+b^2}$ (at $\cos(x-\alpha)=-1$).

1.3.6. Find the minimum value of the function

$$f(x)=3^{(x^2-2)^3+8}.$$

Solution. We denote by $\varphi(x)$ the exponent, i. e.

$$\varphi(x)=(x^2-2)^3+8.$$

The function $f(x)=3^{\varphi(x)}$ takes on the minimum value at the same point as the function $\varphi(x)$.

Hence

$$\varphi(x)=x^6-6x^4+12x^2=x^2[(x^2-3)^2+3].$$

Whence it is clear that the function $\varphi(x)$ attains the minimum value (equal to zero) at $x=0$. That is why the minimum value of the function $f(x)$ is equal to $3^0=1$.

1.3.7. Test the function

$$f(x)=\tan x+\cot x, \text{ where } 0<x<\pi/2,$$

for increase and decrease.

1.3.8. Given: n numbers $a_1, a_2, \ldots, a_n$. Determine the value of x at which the function

$$f(x)=(x-a_1)^2+(x-a_2)^2+\ldots+(x-a_n)^2$$

takes on the minimum value.

Solution. Rewrite the function $f(x)$ in the following way:

$$f(x)=nx^2-2(a_1+a_2+\ldots+a_n)x+(a_1^2+a_2^2+\ldots+a_n^2),$$

wherefrom it is clear that $f(x)$ is a quadratic trinomial ax^2+bx+c, where $a=n>0$. Using the results of Problem **1.3.1**, we find that the function assumes the minimum value at $x=-b/(2a)$, i. e. at $x=(a_1+a_2+\ldots+a_n)/n$.

Thus, the sum of the squares of deviations of the value of x from n given numbers attains the minimum value when x is the mean arithmetic value for these numbers.

1.3.9. Which of the given functions is (are) even, odd; and which of them is (are) neither even, nor odd?

(a) $f(x)=\log(x+\sqrt{1+x^2})$;
(b) $f(x)=\log\frac{1-x}{1+x}$;
(c) $f(x)=2x^3-x+1$;
(d) $f(x)=x\frac{a^x+1}{a^x-1}$.

Solution. (a) It can be seen that $f(+x)+f(-x)=0$. Indeed,

$$f(+x)+f(-x)=\log(x+\sqrt{1+x^2})+\log(-x+\sqrt{1+x^2})=$$
$$=\log(1+x^2-x^2)=0,$$

hence, $f(x)=-f(-x)$ for all x, which means that the function is odd.

(b) $f(-x)=\log\frac{1+x}{1-x}=\log\left(\frac{1-x}{1+x}\right)^{-1}=-\log\frac{1-x}{1+x}$.

Thus, $f(-x)=-f(x)$ for all x from the domain of definition $(-1, 1)$. Hence, the function is odd.

1.3.10. Which of the following functions is (are) even and which is (are) odd?
(a) $f(x)=4-2x^4+\sin^2 x$;
(b) $f(x)=\sqrt{1+x+x^2}-\sqrt{1-x+x^2}$;
(c) $f(x)=\frac{1+a^{kx}}{1-a^{kx}}$;
(d) $f(x)=\sin x+\cos x$;
(e) $f(x)=\text{const}$.

1.3.11. Prove that if $f(x)$ is a periodic function with period T, then the function $f(ax+b)$, where $a>0$, is periodic with period T/a.

Solution. Firstly,

$$f[a(x+T/a)+b]=f[(ax+b)+T]=f(ax+b),$$

since T is the period of the function $f(x)$. Secondly, let T_1 be a positive number such that

$$f[a(x+T_1)+b]=f(ax+b).$$

Let us take an arbitrary point x from the domain of definition of the function $f(x)$ and put $x'=(x-b)/a$. Then

$$f(ax'+b)=f\left(a\frac{x-b}{a}+b\right)=f(x)=f[a(x'+T_1)+b]=$$
$$=f(ax'+b+aT_1)=f(x+aT_1).$$

Whence it follows that the period $T\leqslant aT_1$, i. e. $T_1\geqslant T/a$ and T/a is the period of the function $f(ax+b)$.

Note. The periodic function $f(x)=A\sin(\omega x+\varphi)$, where A, ω, φ are constants, is called a *harmonic* with amplitude $|A|$, frequency ω

and initial phase φ. Since the function $\sin x$ has a period 2π, the function $A\sin(\omega x+\varphi)$ has a period $T=2\pi/\omega$.

1.3.12. Indicate the amplitude $|A|$, frequency ω, initial phase φ and period T of the following harmonics:

(a) $f(x)=5\sin 4x$;
(b) $f(x)=4\sin(3x+\pi/4)$;
(c) $f(x)=3\sin(x/2)+4\cos(x/2)$.

1.3.13. Find the period for each of the following functions:

(a) $f(x)=\tan 2x$;
(b) $f(x)=\cot(x/2)$;
(c) $f(x)=\sin 2\pi x$.

Solution. (a) Since the function $\tan x$ has a period π, the function $\tan 2x$ has a period $\pi/2$.

1.3.14. Find the period for each of the following functions:

(a) $f(x)=\sin^4 x+\cos^4 x$;
(b) $f(x)=|\cos x|$.

Solution. (a) $\sin^4 x+\cos^4 x=(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x=$

$$=1-\frac{1}{2}\sin^2 2x=1-\frac{1}{4}(1-\cos 4x)=\frac{3}{4}+\frac{1}{4}\sin\left(4x+\frac{\pi}{2}\right);$$

whence $T=2\pi/\omega=2\pi/4=\pi/2$.

(b) $f(x)=|\cos x|=\sqrt{\cos^2 x}=\sqrt{(1+\cos 2x)/2}$; but the function $\cos 2x$ has a period $T=\pi$; hence, the given function has the same period.

1.3.15. Prove that the function $f(x)=\cos x^2$ is not a periodic one.

Solution. Let us prove the contrary. Suppose the function has a period T; then the identity $\cos(x+T)^2\equiv\cos x^2$ is valid.

By the conditions of equality of cosines for a certain integer k we have

$$x^2+2Tx+T^2\pm x^2\equiv 2\pi k.$$

But this identity is impossible, since k may attain only integral values, and the left member contains a linear or quadratic function of the continuous argument x.

1.3.16. Find the greatest value of the function

$$f(x)=\frac{2}{\sqrt{2x^2-4x+3}}.$$

1.3.17. Which of the following functions are even, and which are odd:

(a) $f(x)=\sqrt[3]{(1-x)^2}+\sqrt[3]{(1+x)^2}$;

(b) $f(x) = x^2 - |x|$;
(c) $f(x) = x\sin^2 x - x^3$;
(d) $f(x) = (1+2^x)^2/2^x$?

1.3.18. Find the period for each of the following functions:
(a) $f(x) = \arctan(\tan x)$;

(b) $f(x) = 2\cos\frac{x-\pi}{3}$.

1.3.19. Prove that the functions
(a) $f(x) = x + \sin x$; (b) $f(x) = \cos\sqrt{x}$
are non-periodic.

§ 1.4. Inverse Functions

Let the function $y = f(x)$ be defined on the set X and have a range Y. If for each $y \in Y$ there exists a single value of x such that $f(x) = y$, then this correspondence defines a certain function $x = g(y)$ called *inverse* with respect to the given function $y = f(x)$. The sufficient condition for the existence of an inverse function is a strict monotony of the original function $y = f(x)$. If the function increases (decreases), then the inverse function also increases (decreases).

The graph of the inverse function $x = g(y)$ coincides with that of the function $y = f(x)$ if the independent variable is marked off along the y-axis. If the independent variable is laid off along the x-axis, i. e. if the inverse function is written in the form $y = g(x)$, then the graph of the inverse function will be symmetric to that of the function $y = f(x)$ with respect to the bisector of the first and third quadrants.

1.4.1. Find the inverse to the function $y = 3x + 5$.

Solution. The function $y = 3x + 5$ is defined and increases throughout the number scale. Hence, an inverse function exists and increases. Solving the equation $y = 3x + 5$ with respect to x we obtain $x = (y-5)/3$.

1.4.2. Show that the function $y = k/x$ $(k \neq 0)$ is inverse to itself.

Solution. The function is defined and monotonic throughout the entire number scale except $x = 0$. Hence, an inverse function exists. The range of the function is the entire number scale, except $y = 0$. Solving the equation $y = k/x$ with respect to x, we get $x = k/y$.

1.4.3. Find the inverse of the function

$$y = \log_a\left(x + \sqrt{x^2+1}\right), \quad (a > 0,\ a \neq 1).$$

Solution. The function $y = \log_a\left(x + \sqrt{x^2+1}\right)$ is defined for all x, since $\sqrt{x^2+1} > |x|$, and is odd [see Problem **1.3.9** (a)]. It increases for positive values of x, hence, it increases everywhere and has an inverse function. Solving the equation

$$y = \log_a (x + \sqrt{x^2+1})$$

with respect to x, we find

$$a^y = x + \sqrt{x^2+1}; \quad a^{-y} = -x + \sqrt{x^2+1},$$

whence

$$x = \frac{1}{2}(a^y - a^{-y}) = \sinh (y \ln a).$$

1.4.4. Show that the functions

$$f(x) = x^2 - x + 1, \ x \geqslant 1/2 \text{ and } \varphi(x) = 1/2 + \sqrt{x - 3/4}$$

are mutually inverse, and solve the equation

$$x^2 - x + 1 = 1/2 + \sqrt{x - 3/4}.$$

Solution. The function $y = x^2 - x + 1 = (x - 1/2)^2 + 3/4$ increases in the interval $1/2 \leqslant x < \infty$, and with x varying in the indicated interval we have $3/4 \leqslant y < \infty$. Hence, defined in the interval $3/4 \leqslant y < \infty$ is the inverse function $x = g(y)$, $x \geqslant 1/2$, which is found from the equation

$$x^2 - x + (1 - y) = 0.$$

Solving the equation with respect to x, we obtain

$$x = g(y) = 1/2 + \sqrt{y - 3/4} = \varphi(y).$$

Let us now solve the equation

$$x^2 - x + 1 = 1/2 + \sqrt{x - 3/4}.$$

Since the graphs of the original and inverse functions can intersect only on the straight line $y = x$, solving the equation $x^2 - x + 1 = x$ we find $x = 1$.

1.4.5. Find the inverse of $y = \sin x$.

Solution. The domain of definition of the function $y = \sin x$ is the entire number scale, the range of the function is the interval $[-1, 1]$. But the condition of existence of an inverse function is not fulfilled.

Divide the x-axis into intervals $n\pi - \pi/2 \leqslant x \leqslant n\pi + \pi/2$. If n is even, then the function increases on the intervals $n\pi - \pi/2 \leqslant x \leqslant n\pi + \pi/2$; if n is odd, the function decreases on the intervals $n\pi - \pi/2 \leqslant x \leqslant n\pi + \pi/2$. Hence, on each of the indicated intervals there exists an inverse function defined on the interval $[-1, 1]$.

In particular, for an interval $-\pi/2 \leqslant x \leqslant \pi/2$ there exists an inverse function $x = \arcsin y$.

The inverse of the function $y = \sin x$ on the interval $n\pi - \pi/2 \leqslant \leqslant x \leqslant n\pi + \pi/2$ is expressed through $\arcsin y$ in the following way:

$$x = (-1)^n \arcsin y + n\pi \quad (n = 0, \pm 1, \pm 2, \ldots).$$

1.4.6. Find the inverse of the given functions:

(a) $y = \sin(3x-1)$ at $-(\pi/6 + 1/3) \leqslant x \leqslant (\pi/6 + 1/3)$;
(b) $y = \arcsin(x/3)$ at $-3 \leqslant x \leqslant 3$;
(c) $y = 5^{\log x}$;
(d) $y = 2^{x(x-1)}$.

1.4.7. Prove that the function $y = (1-x)/(1+x)$ is inverse to itself.

§ 1.5. Graphical Representation of Functions

1.5.1. Sketch the graph of each of the following functions:

(a) $f(x) = x^4 - 2x^2 + 3$;
(b) $f(x) = \dfrac{2x}{1+x^2}$;
(c) $f(x) = \sin^2 x - 2\sin x$;
(d) $f(x) = \arccos(\cos x)$;
(e) $f(x) = \sqrt{\sin x}$;
(f) $f(x) = x^{1/\log x}$.

Solution. (a) The domain of definition of the function $f(x)$ is the entire number scale. The function $f(x)$ is even, hence its graph is symmetrical about the ordinate axis and it is sufficient to investigate the function at $x \geqslant 0$.

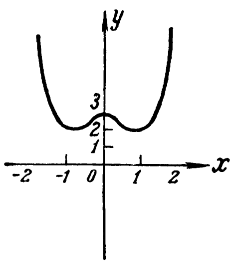

Fig. 2

Let us single out a perfect square $f(x) = (x^2-1)^2 + 2$. Since the first summand $(x^2-1)^2 \geqslant 0$, the minimum value of the function, equal to 2, is attained at the points $x = \pm 1$ (see Fig. 2).

The function $f(x)$ decreases from 3 to 2 on the closed interval $0 \leqslant x \leqslant 1$ and increases unboundedly on the open interval $1 < x < \infty$.

(b) The domain of definition of the function $f(x)$ is the entire number scale. The function $f(x)$ is odd, therefore its graph is symmetrical about the origin of coordinates and it is sufficient to investigate the function at $x \geqslant 0$.

Since $f(0) = 0$, the graph passes through the origin. It is obvious that there are no other points of intersection with the coordinate axes. Note that $|f(x)| \leqslant 1$. Indeed, $(1-|x|)^2 \geqslant 0$ or $1+x^2 \geqslant 2|x|$, whence

$$1 \geqslant \frac{2|x|}{1+x^2} = |f(x)|.$$

Since $f(x) \geqslant 0$ at $x \geqslant 0$ and $f(1)=1$, in the interval $[0, \infty)$ the maximum value of the function $f(x)$ equals 1, the minimum value being zero (see Fig. 3).

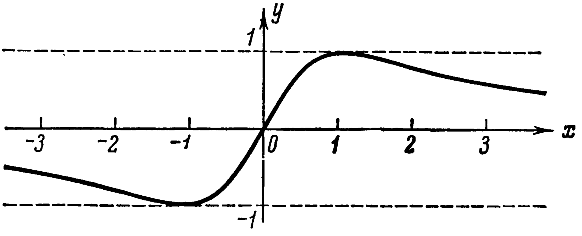

Fig. 3

Let us prove that the function increases on the closed interval $0 \leqslant x \leqslant 1$. Let $0 \leqslant x_1 < x_2 \leqslant 1$. Then

$$f(x_2)-f(x_1) = \frac{2x_2}{1+x_2^2} - \frac{2x_1}{1+x_1^2} = \frac{2x_2+2x_2x_1^2-2x_1-2x_1x_2^2}{(1+x_2^2)(1+x_1^2)} =$$

$$= \frac{2(x_2-x_1)(1-x_1x_2)}{(1+x_2^2)(1+x_1^2)} > 0$$

and $f(x_2) > f(x_1)$.

Similarly, we can show that on the interval $(1, \infty)$ the function decreases. Finally,

$$f(x) = 2x/(1+x^2) < 2x/x^2 = 2/x,$$

whence it is clear that $f(x)$ tends to zero with an increase in x.

(c) The domain of definition of the function $f(x)$ is the entire number scale. The function has a period 2π, that is why it is quite sufficient to investigate it on the interval $[0, 2\pi]$, where it becomes zero at the points $x=0$; $x=\pi$; $x=2\pi$.

Writing the given function in the form

$$f(x) = (1-\sin x)^2 - 1,$$

we note that it increases with a decrease in the function $\sin x$ and decreases as $\sin x$ increases. Hence, the function $f(x)$ decreases on the intervals $[0, \pi/2]$ and $[3\pi/2, 2\pi]$, and increases on the interval $[\pi/2, 3\pi/2]$. Since $f(\pi/2) = -1$, and $f(3\pi/2) = 3$, the range of the function is $-1 \leqslant f(x) \leqslant 3$ (Fig. 4).

(d) The domain of definition of the function is the entire number scale. Indeed, $|\cos x| \leqslant 1$ at any x, hence, $\operatorname{arc\,cos}(\cos x)$ has a meaning. The function $f(x)$ is a periodic one with the period 2π, hence, it is sufficient to sketch its graph on the interval $[0, 2\pi]$. But on this interval the following equality is true:

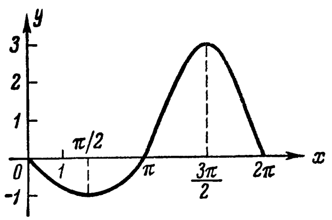

Fig. 4

$$f(x) = \begin{cases} x, & 0 \leqslant x \leqslant \pi, \\ 2\pi - x, & \pi \leqslant x \leqslant 2\pi. \end{cases}$$

Indeed, the first assertion follows from the definition of the function $\operatorname{arc\,cos} x$, while the second one can be proved in the following way. Let us put $x' = 2\pi - x$, $\pi \leqslant x \leqslant 2\pi$; then $0 \leqslant x' \leqslant \pi$ and

$$f(x) = \operatorname{arc\,cos}[\cos(2\pi - x')] = \operatorname{arc\,cos}(\cos x') = x' = 2\pi - x.$$

Taking all this into consideration, we draw the graph (see Fig. 5).

(e) The function $y = \sqrt{\sin x}$ is a periodic one with period 2π; that is why we may confine ourselves to the interval $[0, 2\pi]$. But

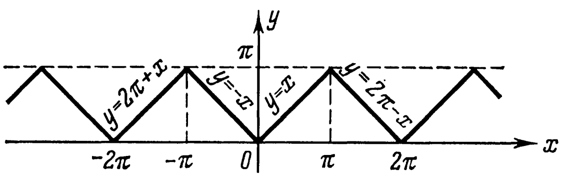

Fig. 5

the function is not defined in the whole interval $[0, 2\pi]$, it is defined only in the interval $[0, \pi]$, as in the interval $(\pi, 2\pi)$ the radicand is negative. The graph is symmetrical about the straight

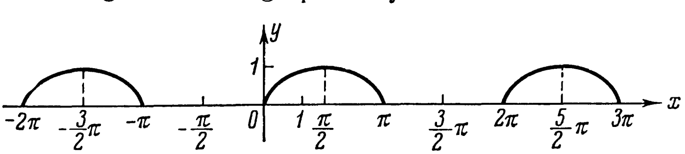

Fig. 6

line $x = \pi/2$, as well as the graph $y = \sin x$ (see Fig. 6). Here we have an example of a periodic function which does not exist in the infinite set of intervals.

(f) The domain of definition of the function is

$$0 < x < 1 \text{ and } 1 < x < \infty.$$

Reduce the formula to the form

$$f(x) = x^{1/\log x} = x^{\log_x 10} = 10.$$

Hence, the graph of the given function is the half-line $y=10$ in the right-hand halfplane with the point $x=1$ removed (see Fig. 7).

1.5.2. Sketch the graphs of functions defined by different formulas in different intervals (and in those reducible to them):

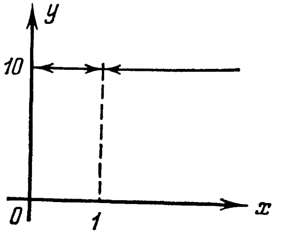

Fig. 7

(a) $y=\begin{cases} \sin x & \text{at } -\pi \leqslant x \leqslant 0, \\ 2 & \text{at } 0 < x \leqslant 1, \\ 1/(x-1) & \text{at } 1 < x \leqslant 4; \end{cases}$

(b) $y=\begin{cases} -2 & \text{at } x > 0, \\ 1/2 & \text{at } x = 0, \\ -x^3 & \text{at } x < 0; \end{cases}$

(c) $y=x+\sqrt{x^2}$;

(d) $y=2/(x+\sqrt{x^2})$.

Solution. (a) The domain of definition of the function is the interval $[-\pi, 4]$. The graph of the function consists of a portion of the sinusoid $y=\sin x$ on the interval $-\pi \leqslant x \leqslant 0$, straight line

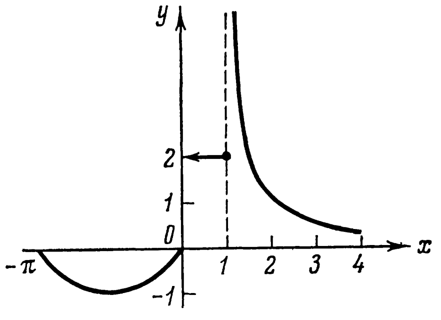

Fig. 8

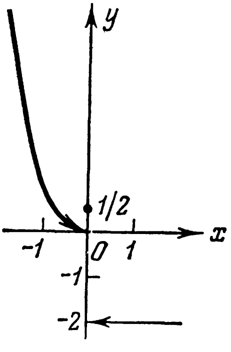

Fig. 9

$y=2$ on the interval $(0, 1]$ and a part of the branch of the hyperbola $y=1/(x-1)$ on the interval $(1,4]$ (see Fig. 8).

(b) The graph of the function consists of a portion of a cubic parabola, an isolated point and a half-line (see Fig. 9).

(c) The function may be given by two formulas:

$$y=\begin{cases} 2x, & \text{if } x \geqslant 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Thus, the graph of our function is a polygonal line (see Fig. 10).

(d) From (c) it follows that the function is defined only in the interval $(0, +\infty)$, y being equal to $1/x$ $(x > 0)$. Thus, the graph of our function is the right-hand part of an equilateral hyperbola (see Fig. 11).

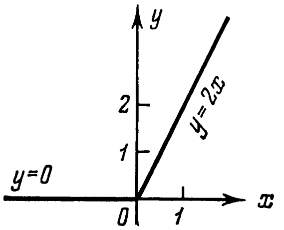

Fig. 10

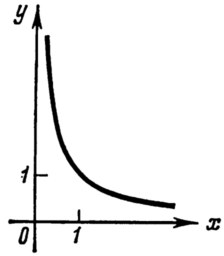

Fig. 11

1.5.3. Sketch the graphs of the following functions:

(a) $y = \cos x + |\cos x|$;

(b) $y = |x+2|\,x$.

Solution. (a) $\cos x + |\cos x| = \begin{cases} 2\cos x & \text{at } \cos x \geqslant 0, \\ 0 & \text{at } \cos x < 0. \end{cases}$

Doubling the non-negative ordinates of the graph for the function $y = \cos x$ (the broken line in Fig. 12) and assuming $y = 0$ at

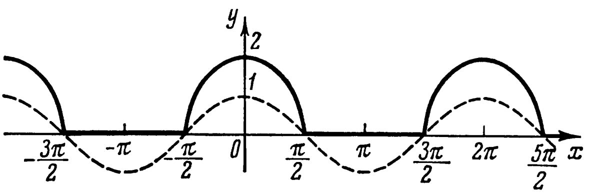

Fig. 12

the points where $\cos x < 0$, we can sketch the desired graph (the solid line in the same figure).

(b) The function $|x+2|\,x$ may be given by two formulas:

$$y = \begin{cases} (x+2)\,x & \text{at } x \geqslant -2, \\ -(x+2)\,x & \text{at } x \leqslant -2. \end{cases}$$

Plotting separately both parabolas: $y = (x+2)\,x = (x+1)^2 - 1$, and $y = -[(x+1)^2 - 1]$, retain only the parts corresponding to the above indicated intervals. Drawn in a solid line in Fig. 13 is the graph of the given function, the broken line showing the deleted parts of the constructed parabolas.

1.5.4. Sketch the graph of the function

$$y = 2|x-2| - |x+1| + x.$$

Solution. At $x \geqslant 2$

$$y = 2(x-2) - (x+1) + x = 2x - 5.$$

At $-1 \leqslant x \leqslant 2$

$$y = -2(x-2) - (x+1) + x = -2x + 3.$$

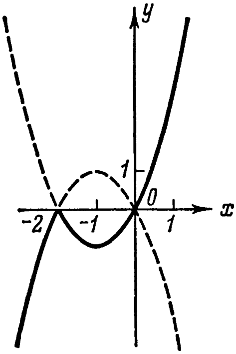

Fig. 13

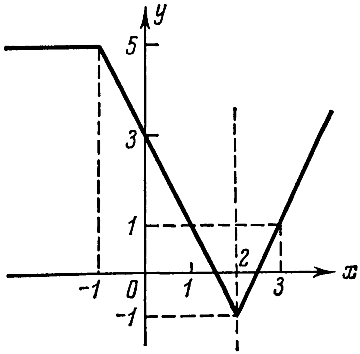

Fig. 14

Finally, at $x \leqslant -1$

$$y = -2(x-2) + (x+1) + x = 5.$$

Hence, the given function can be rewritten in the following way:

$$y = \begin{cases} 5, & x \leqslant -1, \\ -2x+3, & -1 \leqslant x \leqslant 2, \\ 2x-5, & x \geqslant 2. \end{cases}$$

Therefore the graph is a polygonal line (see Fig. 14).

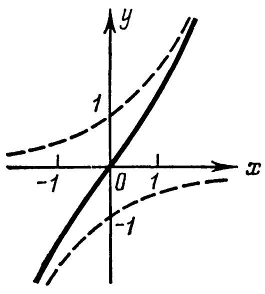

Fig. 15

1.5.5. Sketch the graph of the function

$$y = 2^x - 2^{-x}.$$

Solution. Draw graphs for the functions $y_1 = 2^x$ and $y_2 = -2^{-x}$ (broken lines in Fig. 15), and add graphically their ordinates. In doing so bear in mind that $y_2 < y < y_1$, and that y_2 tends to zero with an increase in x, whereas y_1 tends to zero with a decrease in x (the solid line in Fig. 15).

1.5.6. Sketch the graph of the function

$$y = x \sin x.$$

Solution. Being the product of two odd functions $y_1 = x$ and $y_2 = \sin x$, the function y is an even one, that is why we shall analyse it for $x \geqslant 0$.

We draw graphs for $y_1 = x$ and $y_2 = \sin x$ (the broken lines in Fig. 16).

At the points where $y_2 = \sin x = 0$, $y = y_1 \cdot y_2 = 0$, and at the points where $y_2 = \sin x = \pm 1$, $y = \pm y_1 = \pm x$. The latter equality

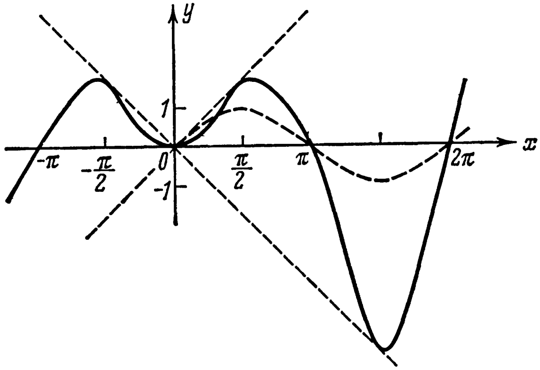

Fig. 16

indicates the expedience of graphing the auxiliary function $y_3 = -x$.

Marking the indicated points and joining them into a smooth curve, we obtain the required graph (the solid line in Fig. 16).

1.5.7. Sketch the graph of the function $y = x(x^2 - 1)$ by multiplying the ordinates of the graphs $y_1 = x$ and $y_2 = x^2 - 1$.

1.5.8. Graph the following functions:

(a) $y = x/(x^2 - 4)$, (b) $y = 1/\arccos x$.

Solution. (a) Since the function is odd, it is sufficient to investigate it for $x \geqslant 0$.

Let us consider it as the quotient of the two functions:

$$y_1 = x \text{ and } y_2 = x^2 - 4.$$

Since at $x = 2$ the denominator $y_2 = 0$, the function is not defined at the point 2. In the interval $[0, 2)$ the function y_1 increases from 0 to 2, the function y_2 is negative and $|y_2| = 4 - x^2$ decreases from 4 to 0; hence, the quotient $f(x) = y_1/y_2$ is negative and increases in absolute value, i.e. $f(x)$ decreases in the interval $[0,2)$ from 0 to $-\infty$.

In the interval $(2, \infty)$ both functions are positive and increasing. Their quotient decreases since from $2 \leqslant x_1 < x_2$ it follows that

$$y_2 - y_1 = \frac{x_2}{x_2^2 - 4} - \frac{x_1}{x_1^2 - 4} = \frac{(x_1 - x_2)(x_1 x_2 + 4)}{(x_2^2 - 4)(x_1^2 - 4)} < 0.$$

The indicated quotient tends to zero as $x \to \infty$, since $y = \frac{1/x}{1-4/x^2} \to 0$. The general outline of the graph is presented in Fig. 17 (three solid lines).

(b) Denote $y_1 = \arccos x$. The domain of definition of this function $|x| \leqslant 1$. At $x = 1$ we have $y_1 = 0$, hence, $y = 1/y_1 \to \infty$ at

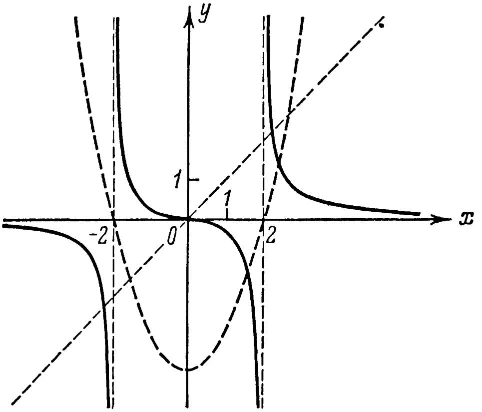

Fig. 17

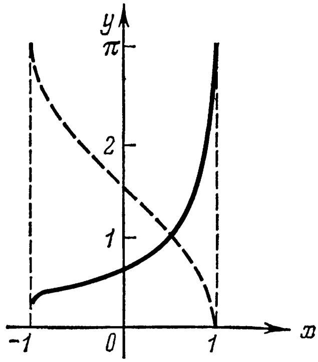

Fig. 18

$x \to 1$, i. e. $x = 1$ is a vertical asymptote. The function y_1 decreases on the entire interval of definition $[-1, 1)$, hence $y = 1/y_1$ increases. The maximum value $y_1 = \pi$ is attained at $x = -1$. Accordingly, the minimum value of the function is $1/\pi$. The solid line in Fig. 18 represents the general outline of the graph.

Simple Transformations of Graphs

I. The graph of the function $y = f(x + a)$ is obtained from the graph of the function $y = f(x)$ by translating the latter graph along the x-axis by $|a|$ scale units in the direction opposite to the sign of a (see Fig. 19).

II. The graph $y = f(x) + b$ is obtained from the graph of the function $y = f(x)$ by translating the latter graph along the y-axis by $|b|$ scale units in the direction opposite to the sign of b (see Fig. 20).

III. The graph of the function $y = f(kx)$ $(k > 0)$ is obtained from the graph of the function $y = f(x)$ by "compressing" the latter graph against the y-axis in the horizontal direction k times at $k > 1$ and by "stretching" it in the horizontal direction from the y-axis $1/k$ times at $k < 1$ (see Fig. 21).

IV. The graph of the function $y = kf(x)$ $(k > 0)$ is obtained from the graph of function $y = f(x)$ by "stretching" it in the horizontal direction k times at $k > 1$ and "compressing" it against the x-axis (i. e. vertically) $1/k$ times at $k < 1$ (see Fig. 21).

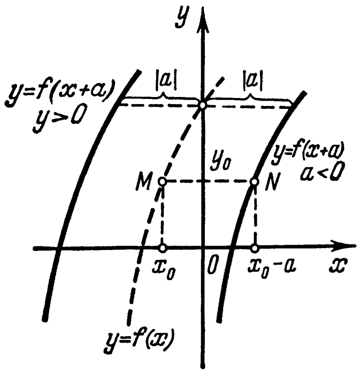

Fig. 19

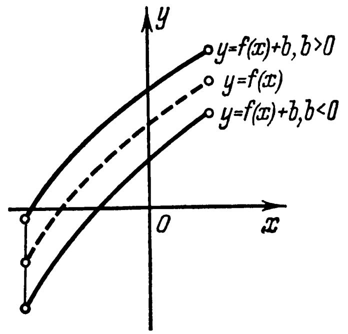

Fig. 20

V. The graph of the function $y = -f(x)$ is symmetrical to that of the function $y = f(x)$ about the x-axis, while the graph of the function $y = f(-x)$ is symmetrical to that of the function $y = f(x)$ about the y-axis.

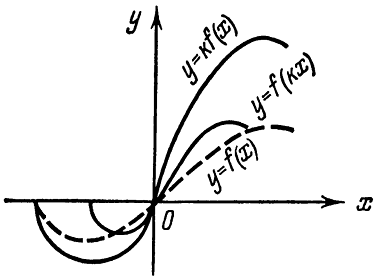

Fig. 21

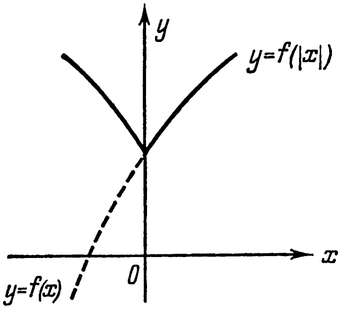

Fig. 22

VI. The graph of the function $y = f(|x|)$ is obtained from the graph of the function $y = f(x)$ in the following way: for $x \geqslant 0$ the graph of the function $y = f(x)$ is retained, then this retained part of the graph is reflected symmetrically about the y-axis, thus determining the graph of the function for $x \leqslant 0$ (see Fig. 22).

VII. The graph of the function $y = |f(x)|$ is constructed from the graph $y = f(x)$ in the following way: the portion of the graph of the function $y=f(x)$ lying above the x-axis remains unchanged, its other portion located below the x-axis being transformed symmetrically about the x-axis (see Fig. 23).

VIII. The graphs of the more complicated functions

$$y=\lambda f(kx+a)+b$$

are drawn from the graph of $y=f(x)$ applying consecutively transformations I to V.

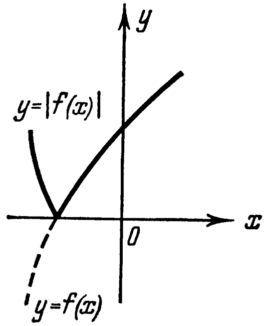

Fig. 23

1.5.9. Graph the function

$$y=3\sqrt{-2(x+2.5)}-0.8$$

by transforming the graph $y=\sqrt{x}$.

Solution. Sketch the graph of the function $y=\sqrt{x}$ (which is the upper branch of the parabola $y^2=x$) (Fig. 24, *a*), and transform it in the following sequence.

Sketch the graph of the function $y=3\sqrt{2x}$ by enlarging $3\sqrt{2}$ times the ordinates of the points on the graph of the function $y=\sqrt{x}$ and leaving their abscissas unchanged (see Fig. 24, *b*).

Then sketch the graph of the function $y=3\sqrt{-2x}$ which will be the mirror image of the preceding graph about the y-axis (see Fig. 24, *c*).

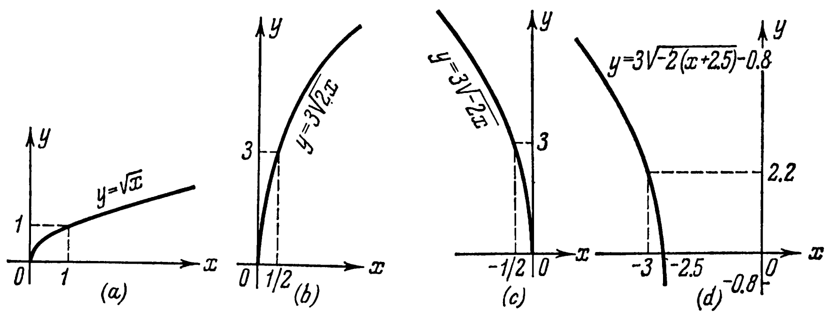

Fig. 24

By shifting the obtained graph 2.5 scale units leftward and then 0.8 unit downward draw the desired graph of the function $y=3\sqrt{-2(x+2.5)}-0.8$ (see Fig. 24, *d*).

1.5.10. Graph the function $y=3\cos x-\sqrt{3}\sin x$ by transforming the cosine curve.

Solution. Transform the given function

$$y = 3\cos x - \sqrt{3}\sin x = 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x\right) = 2\sqrt{3}\cos\left(x + \frac{\pi}{6}\right).$$

Thus, we have to sketch the graph of the function

$$y = 2\sqrt{3}\cos(x + \pi/6),$$

which is the graph of the function $y = 2\sqrt{3}\cos x$ translated by $\pi/6$ leftward. The function has a period of 2π, hence it is sufficient to draw its graph for $-\pi \leqslant x \leqslant \pi$ (see Fig. 25).

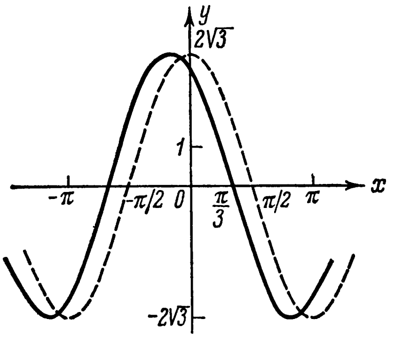

Fig. 25

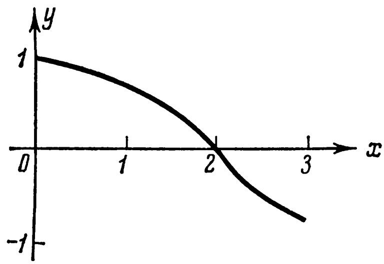

Fig. 26

The graph of any function of the form $y = a\cos x + b\sin x$, where a and b are constants, is sketched in a similar way.

1.5.11. Graph the following functions:

(a) $y = \dfrac{x+3}{x+1}$;

(b) $y = \dfrac{1}{x^2 - 9}$;

(c) $$y = \begin{cases} x^2 + x + 1, & \text{if } -1 \leqslant x < 0, \\ \sin^2 x, & \text{if } 0 \leqslant x \leqslant \pi, \\ (x-1)/(x+1), & \text{if } \pi < x \leqslant 5; \end{cases}$$

(d) $y = x + 1/x$;

(e) $y = x^2 - x^3$;

(f) $y = x + \sin x$;

(g) $y = 1/\cos x$;

(h) $y = 3\sin(2x - 4)$;

(i) $y = 2\sqrt{-3(x + 1.5)} - 1.2$;

(j) $y = |x^2 - 2x - 1|$;
(k) $y = ||x| - 1|$;
(l) $y = \cos(\sin x)$;
(m) $y = |\sin x| + \sin x$ on the interval $[0, 3\pi]$;
(n) $y = x^2 \operatorname{sign} x$, where $\operatorname{sign} x = \begin{cases} 1 \text{ at } x > 0, \\ 0 \text{ at } x = 0, \\ -1 \text{ at } x < 0. \end{cases}$

1.5.12. The function $y = f(x)$ is given graphically (Fig. 26). Sketch the graphs of the following functions:

(a) $y = f(x+1)$;
(b) $y = f(x/2)$;
(c) $y = |f(x)|$;
(d) $y = (|f(x)| \pm f(x))/2$;
(e) $y = |f(x)|/f(x)$.

§ 1.6. Number Sequences. Limit of a Sequence

The number a is called the *limit of a sequence* $x_1, x_2, \ldots, x_n, \ldots$ as $n \to \infty$, $a = \lim\limits_{n \to \infty} x_n$ if for any $\varepsilon > 0$ there exists a number $N(\varepsilon) > 0$ such that the inequality $|x_n - a| < \varepsilon$ holds true for all $n > N(\varepsilon)$.

A sequence which has a finite limit is said to be *convergent*.

A sequence $\{x_n\}$ is called *infinitely small* if $\lim x_n = 0$, and *infinitely large* if $\lim x_n = \infty$.

1.6.1. Given the general term of the sequence $\{x_n\}$:

$$x_n = \frac{\sin(n\pi/2)}{n}.$$

Write the first five terms of this sequence.

Solution. Putting consecutively $n = 1, 2, 3, 4, 5$ in the general term x_n, we obtain

$$x_1 = \frac{\sin(\pi/2)}{1} = 1;$$

$$x_2 = \frac{\sin(2\pi/2)}{2} = 0;$$

$$x_3 = \frac{\sin(3\pi/2)}{3} = -\frac{1}{3};$$

$$x_4 = \frac{\sin(4\pi/2)}{4} = 0;$$

$$x_5 = \frac{\sin(5\pi/2)}{5} = \frac{1}{5}.$$

1.6.2. Knowing the first several terms of the sequence, write one of the possible expressions for the general term:

(a) $\frac{2}{3}, \frac{5}{8}, \frac{10}{13}, \frac{17}{18}, \frac{26}{23};$

(b) $1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \frac{1}{5}.$

Note. A knowledge of the first several terms of a sequence is not sufficient to define this sequence. That is why this problem should be understood as one of finding a certain simple inductive regularity compatible with the given terms.

Solution. (a) Note that the numerator of each of the given terms of the sequence equals the square of the number of this term plus unity, i.e. n^2+1, while the denominators form the arithmetic progression 3, 8, 13, 18, ... with the first term $a_1=3$ and the common difference $d=5$. Hence,

$$a_n = a_1 + d(n-1) = 3+5(n-1) = 5n-2,$$

thus we have

$$x_n = \frac{n^2+1}{5n-2}.$$

(b) Here the general term of the sequence can be written with the aid of two formulas: one for the terms standing in odd places, the other for those in even places:

$$x_n = \begin{cases} k & \text{at } n = 2k-1, \\ 1/(k+1) & \text{at } n = 2k. \end{cases}$$

It is also possible to express the general term by one formula, which will be more complicated, for instance,

$$x_n = \frac{n+1}{4}[1-(-1)^n] + \frac{1}{n+2}[1+(-1)^n].$$

1.6.3. Find the first several terms of the sequence if the general term is given by one of the following formulas:

(a) $x_n = \sin(n\pi/3)$;
(b) $x_n = 2^{-n}\cos n\pi$;
(c) $x_n = (1+1/n)^n$.

1.6.4. Using the definition of the limit of a sequence, prove that

(a) $\lim x_n = 1$ if $x_n = (2n-1)/(2n+1)$,
(b) $\lim x_n = 3/5$ if $x_n = (3n^2+1)/(5n^2-1)$. Beginning with which n is the inequality $|x_n - 3/5| < 0.01$ fulfilled?

Solution. (a) For any $\varepsilon > 0$ let us try to find a natural number $N(\varepsilon)$ such that for any natural number $n > N(\varepsilon)$ the inequality $|x_n - 1| < \varepsilon$ is fulfilled.

For this purpose let us find the absolute value of the difference

$$\left|\frac{2n-1}{2n+1}-1\right|=\left|\frac{-2}{2n+1}\right|=\frac{2}{2n+1}.$$

Thus, the inequality $|x_n-1|<\varepsilon$ is satisfied if $\frac{2}{2n+1}<\varepsilon$, whence $n>1/\varepsilon-{}^1/_2$. Hence the integral part of the number $1/\varepsilon-{}^1/_2$ may be taken as $N(\varepsilon)$, i.e. $N=E(1/\varepsilon-{}^1/_2)$.

So, for each $\varepsilon>0$ we can find a number N such that from the inequality $n>N$ it will follow that $|x_n-1|<\varepsilon$, which means that

$$\lim_{n\to\infty}\frac{2n-1}{2n+1}=1.$$

(b) Let us find the absolute value of the difference $|x_n-3/5|$:

$$\left|\frac{3n^2+1}{5n^2-1}-\frac{3}{5}\right|=\frac{8}{5(5n^2-1)}.$$

Let $\varepsilon>0$ be given. Choose n so that the inequality

$$\frac{8}{5(5n^2-1)}<\varepsilon.$$

is fulfilled.

Solving this inequality, we find

$$n^2>\frac{8}{25\varepsilon}+\frac{1}{5};\qquad n>\frac{1}{5}\sqrt{\frac{8+5\varepsilon}{\varepsilon}}.$$

Putting

$$N=E\left(\frac{1}{5}\sqrt{\frac{8+5\varepsilon}{\varepsilon}}\right),$$

we conclude that at $n>N$

$$|x_n-3/5|<\varepsilon,$$

which completes the proof.

If $\varepsilon=0.01$, then

$$N=E\left(\frac{1}{5}\sqrt{\frac{8+5\varepsilon}{\varepsilon}}\right)=E\left(\frac{1}{5}\sqrt{805}\right)=5,$$

and all terms of the sequence, beginning with the 6th, are contained in the interval $(3/5-0.01;\ 3/5+0.01)$.

1.6.5. Given a sequence with the general term $x_n=\frac{3n-5}{9n+4}$. It is known that $\lim\limits_{n\to\infty}x_n=1/3$. Find the number of points x_n lying outside the open interval

$$L=\left(\frac{1}{3}-\frac{1}{1\,000};\ \frac{1}{3}+\frac{1}{1\,000}\right).$$

Solution. The distance from the point x_n to the point 1/3 is equal to

$$\left|x_n-\frac{1}{3}\right|=\left|-\frac{19}{3(9n+4)}\right|=\frac{19}{3(9n+4)}.$$

Outside the interval L there will appear those terms of the sequence for which this distance exceeds 0.001, i.e.

$$\frac{19}{3(9n+4)}>\frac{1}{1\,000},$$

whence

$$1\leqslant n<\frac{18\,988}{27}=703\frac{7}{27}.$$

Hence, 703 points ($x_1, x_2, \ldots, x_{703}$) are found outside the interval L.

1.6.6. Prove that the number $l=0$ is not the limit of a sequence with the general term $x_n=(n^2-2)/(2n^2-9)$.

Solution. Estimate from below the absolute value of the difference

$$\left|\frac{n^2-2}{2n^2-9}-0\right|=\frac{|n^2-2|}{|2n^2-9|}.$$

At $n\geqslant 3$ the absolute value of the difference remains greater than the constant number $^1/_2$; hence, there exists such $\varepsilon>0$, say, $\varepsilon={}^1/_2$, that the inequality

$$\left|\frac{n^2-2}{2n^2-9}-0\right|>\frac{1}{2}$$

holds true for any $n\geqslant 3$.

The obtained inequality proves that $l=0$ is not the limit of the given sequence.

1.6.7. Prove that the sequence

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{5},\ \frac{3}{4},\ \frac{1}{7},\ \frac{4}{5},\ \ldots$$

with the general term

$$x_n=\begin{cases}1/n, & \text{if } n=2k-1,\\ n/(n+2), & \text{if } n=2k,\end{cases}$$

has no limit.

Solution. It is easy to show that the points x_n with odd numbers concentrate about the point 0, and the points x_n with even numbers, about the point 1. Hence, any neighbourhood of the point 0, as well as any neighbourhood of the point 1, contains an infinite set of points x_n. Let a be an arbitrary real number. We can always choose such a small $\varepsilon>0$ that the ε-neighbourhood of the point a will not contain at least a certain neighbourhood of either point 0 or point 1. Then an infinite set of numbers x_n will be found outside this neighbourhood, and that is why one cannot assert that all the numbers x_n, beginning with a certain one, will enter the ε-neighbourhood of the number a. This means, by definition, that the number a is not the limit of the given sequence. But a is an arbitrary number, hence no number is the limit of this sequence.

1.6.8. Prove that $\lim x_n = 1$ if $x_n = (3^n + 1)/3^n$.

1.6.9. Prove that $\lim x_n = 2$ if $x_n = (2n+3)/(n+1)$. Find the number of the term beginning with which the inequality $|(2n+3)/(n+1) - 2| < \varepsilon$, where $\varepsilon = 0.1;\ 0.01;\ 0.001$, is fulfilled.

1.6.10. Prove that the sequence

$$\frac{1}{2},\ \frac{1}{2},\ \frac{3}{4},\ \frac{1}{4},\ \frac{7}{8},\ \frac{1}{8},\ \ldots,$$

with the general term

$$x_n = \begin{cases} 1 - \dfrac{1}{2^{(n+1)/2}} & \text{if } n \text{ is odd,} \\ \dfrac{1}{2^{n/2}} & \text{if } n \text{ is even,} \end{cases}$$

has no limit.

1.6.11. Prove that at any arbitrarily large $a > 0$ $\lim x_n = 0$ if $x_n = a^n/n!$

Solution. Let a natural number $k > 2a$. Then at $n > k$

$$\frac{a^n}{n!} = \frac{a}{1} \cdot \frac{a}{2} \cdot \ldots \cdot \frac{a}{n} = \left(\frac{a}{1} \cdot \frac{a}{2} \cdot \ldots \cdot \frac{a}{k}\right)\left(\frac{a}{k+1} \cdot \frac{a}{k+2} \cdot \ldots \cdot \frac{a}{n}\right) <$$

$$< a^k \left(\frac{1}{2}\right)^{n-k} = (2a)^k \left(\frac{1}{2}\right)^n.$$

Since $\lim (1/2)^n = 0$ (prove it!), then at a sufficiently large n we have: $\left(\frac{1}{2}\right)^n < \frac{\varepsilon}{(2a)^k}$ and, hence, $a^n/n! < \varepsilon$, which means that $\lim (a^n/n!) = 0$.

1.6.12. Test the following sequences for limits:

(a) $x_n = 1/(2\pi)$;

(b) $x_n = \begin{cases} 1 & \text{for an even } n, \\ 1/n & \text{for an odd } n; \end{cases}$

(c) $x_n = \frac{1}{n} \cos \frac{n\pi}{2}$;

(d) $x_n = n\,[1 - (-1)^n]$.

1.6.13. Prove that the sequence with the general term

$$x_n = 1/n^k \quad (k > 0)$$

is an infinitely small sequence.

Solution. To prove that the sequence x_n is infinitely small is to prove that $\lim\limits_{n \to \infty} x_n = 0$.

Take an arbitrary $\varepsilon > 0$. Since $|x_n| = 1/n^k$, we have to solve the inequality

$$1/n^k < \varepsilon,$$

whence $n > \sqrt[k]{1/\varepsilon}$. Hence N may be expressed as the integral part of $\sqrt[k]{1/\varepsilon}$, i. e. $N = E(\sqrt[k]{1/\varepsilon})$.

1.6.14. Prove that the sequences with the general terms

$$\text{(a)} \quad x_n = \frac{1-(-1)^n}{n}, \qquad \text{(b)} \quad x_n = \frac{1}{n} \sin\left[(2n-1)\frac{\pi}{2}\right]$$

are infinitely small as $n \to \infty$.

1.6.15. Show that the sequence with the general term $x_n = (-1)^n 2/(5\sqrt[3]{n}+1)$ is infinitely small as $n \to \infty$. Find a number N beginning with which the points x_n belong to the interval $(-1/10, 1/10)$.

Solution. Take an arbitrary $\varepsilon > 0$ and estimate $|x_n|$:

$$|x_n| = \frac{2}{5\sqrt[3]{n}+1} < \frac{2}{5\sqrt[3]{n}} < \frac{2}{2\sqrt[3]{n}} = \frac{1}{\sqrt[3]{n}}.$$

That is why $|x_n| < \varepsilon$ as soon as $n > 1/\varepsilon^3$. Hence $\lim\limits_{n \to \infty} x_n = 0$, i. e. the sequence is infinitely small.

We take now $\varepsilon = 1/10$. Since $|x_n| < 1/\sqrt[3]{n}$, x_n will necessarily be smaller than 1/10 if $1/\sqrt[3]{n} < 1/10$ or $n > 1\,000$. Hence N may be taken equal to 1 000. But we can obtain a more accurate result by solving the inequality

$$|x_n| = \frac{2}{5\sqrt[3]{n}+1} < \frac{1}{10}.$$

It holds true at $n > (19/5)^3 = 3.8^3 = 54.872$. Hence N may be taken equal to $54 \ll 1\,000$.

1.6.16. It is known that if $x_n = a + \alpha_n$, where α_n is an infinitesimal as $n \to \infty$, then $\lim\limits_{n \to \infty} x_n = a$. Taking advantage of this rule, find the limits:

$$\text{(a)} \quad x_n = \frac{3^{n+1} + \sin(n\pi/4)}{3^n}; \qquad \text{b)} \quad x_n = \frac{2^n + (-1)^n}{2^n}.$$

Solution. (a) $x_n = \frac{3^{n+1} + \sin(n\pi/4)}{3^n} = 3 + \alpha_n$, where $\alpha_n = \frac{\sin(n\pi/4)}{3^n}$ is an infinitesimal as $n \to \infty$, hence $\lim_{n \to \infty} x_n = 3$.

1.6.17. Prove that $\lim_{n \to \infty} \sqrt[n]{n} = 1$.

Solution. Let us prove that the variable $\sqrt[n]{n}$ can be represented as the sum $1 + \alpha_n$, where α_n is an infinitesimal as $n \to \infty$.

Let us put $\sqrt[n]{n} = 1 + \alpha_n$. Raising to the nth power we obtain

$$n = (1 + \alpha_n)^n = 1 + n\alpha_n + \frac{n(n-1)}{2!}\alpha_n^2 + \ldots + \alpha_n^n,$$

wherefrom we arrive at the conclusion that for any $n > 1$ the following inequality holds true:

$$n > 1 + \frac{n(n-1)}{2!}\alpha_n^2$$

(since all the terms on the right are non-negative). Transposing the unity to the left and reducing the inequality by $n - 1$ we obtain

$$1 > \frac{n}{2}\alpha_n^2,$$

whence it follows that $2/n > \alpha_n^2$ or $\sqrt{2/n} > \alpha_n > 0$. Since $\lim_{n \to \infty} \sqrt{2/n} = 0$, $\lim_{n \to \infty} \alpha_n$ also equals zero, i. e. α_n is an infinitesimal. Hence it follows that

$$\lim_{n \to \infty} \sqrt[n]{n} = 1.$$

1.6.18. Prove that the sequence with the general term

$$x_n = 3^{\sqrt[3]{n}}$$

is infinitely large as $n \to \infty$.

Solution. Let us take an arbitrary positive number M and solve the inequality

$$3^{\sqrt[3]{n}} > M.$$

Taking the logarithm, we obtain

$$\sqrt[3]{n} > \log_3 M, \quad n > (\log_3 M)^3.$$

If we now take $N = E(\log_3 M)^3$, then for all $n > N$ the inequality $|x_n| > M$ will be fulfilled, which means that the sequence is infinitely large.

1.6.19. Prove that

$$\lim_{n \to \infty} \sqrt[n]{a} = 1 \quad (a > 0).$$

§ 1.7. Evaluation of Limits of Sequences

If the sequences $\{x_n\}$ and $\{y_n\}$ are convergent, then

(1) $\lim (x_n \pm y_n) = \lim x_n \pm \lim y_n$;

(2) $\lim (x_n y_n) = \lim x_n \lim y_n$;

(3) $\lim \frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n}$ $(\lim y_n \neq 0)$.

If $x_n \leqslant y_n$ then $\lim x_n \leqslant \lim y_n$.

1.7.1. Find $\lim\limits_{n\to\infty} x_n$ if

(a) $x_n = \frac{3n^2+5n+4}{2+n^2}$; (b) $x_n = \frac{5n^3+2n^2-3n+7}{4n^3-2n+11}$;

(c) $x_n = \frac{4n^2-4n+3}{2n^3+3n+4}$; (d) $x_n = \frac{1^2+2^2+\ldots+n^2}{5n^3+n+1}$;

(e) $x_n = \frac{1+2+\ldots+n}{n^2}$.

Solution. (a) $x_n = \dfrac{3+\frac{5}{n}+\frac{4}{n^2}}{\frac{2}{n^2}+1}$,

$$\lim_{n\to\infty} x_n = \frac{\lim\limits_{n\to\infty} (3+5/n+4/n^2)}{\lim\limits_{n\to\infty}\left(\frac{2}{n^2}+1\right)} = 3.$$

(d) Recall that

$$1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Hence

$$x_n = \frac{n(n+1)(2n+1)}{6(5n^3+n+1)} = \frac{2n^3+3n^2+n}{6(5n^3+n+1)} = \frac{2+\frac{3}{n}+\frac{1}{n^2}}{30+\frac{6}{n^2}+\frac{6}{n^3}},$$

$$\lim_{n\to\infty} x_n = 1/15.$$

1.7.2. Find $\lim\limits_{n\to\infty} x_n$, if

(a) $x_n = \left(\frac{3n^2+n-2}{4n^2+2n+7}\right)^3$; (b) $x_n = \left(\frac{2n^3+2n^2+1}{4n^3+7n^2+3n+4}\right)^4$;

(c) $x_n = \sqrt[n]{5n}$; (d) $x_n = \sqrt[n]{n^8}$;

(e) $x_n = \sqrt[n]{n^5}$; (f) $x_n = \sqrt[n]{6n+3}$.

Solution. (a) $\lim\limits_{n\to\infty}\left(\frac{3n^2+n-2}{4n^2+2n+7}\right)^3 =$

$$= \lim_{n\to\infty}\left(\frac{3n^2+n-2}{4n^2+2n+7}\right)\left(\frac{3n^2+n-2}{4n^2+2n+7}\right)\left(\frac{3n^2+n-2}{4n^2+2n+7}\right) =$$

$$= \left(\lim_{n\to\infty}\frac{3+1/n-2/n^2}{4+2/n+7/n^2}\right)^3 = \left(\frac{3}{4}\right)^3 = \frac{27}{64}.$$

(c) In solving this example, and also the rest of the examples of Problem **1.7.2**, take advantage of the following equalities (see Problems **1.6.17** and **1.6.19**):

$$\lim_{n \to \infty} \sqrt[n]{n} = 1 \quad \text{and} \quad \lim_{n \to \infty} \sqrt[n]{a} = 1. \tag{1}$$

We have

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \sqrt[n]{5n} = \lim_{n \to \infty} \sqrt[n]{5} \lim_{n \to \infty} \sqrt[n]{n},$$

but from (1) it follows that $\lim \sqrt[n]{5} = 1$ and $\lim \sqrt[n]{n} = 1$; hence $\lim x_n = 1 \cdot 1 = 1$.

1.7.3. Find

$$\lim_{n \to \infty} \left(\frac{2n^3}{2n^2+3} + \frac{1-5n^2}{5n+1} \right).$$

Solution. Summing the fractions, we obtain

$$x_n = \frac{2n^3 - 13n^2 + 3}{10n^3 + 2n^2 + 15n + 3}.$$

Whence

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{2n^3 - 13n^2 + 3}{10n^3 + 2n^2 + 15n + 3} = \frac{1}{5}.$$

Note. If we put

$$y_n = \frac{2n^3}{2n^2+3}; \quad z_n = \frac{1-5n^2}{5n+1},$$

then the limit of their sum $\lim (y_n + z_n) = 1/5$, though each of the summands is an infinitely large quantity. Thus, from the convergence of a sum of sequences it does not, generally speaking, follow that the summands converge too.

1.7.4. Find $\lim\limits_{n \to \infty} x_n$ if

(a) $x_n = \sqrt{2n+3} - \sqrt{n-1}$;

(b) $x_n = \sqrt{n^2+n+1} - \sqrt{n^2-n+1}$;

(c) $x_n = n^2 (n - \sqrt{n^2+1})$;

(d) $x_n = \sqrt[3]{n^2 - n^3} + n$;

(e) $x_n = \dfrac{\sqrt{n^2+1} + \sqrt{n}}{\sqrt[4]{n^3+n} - \sqrt{n}}$;

(f) $x_n = \sqrt[3]{(n+1)^2} - \sqrt[3]{(n-1)^2}$;

(g) $x_n = \dfrac{1-2+3-4+5-6+\ldots-2n}{\sqrt{n^2+1} + \sqrt{4n^2-1}}$;

(h) $x_n = \dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \ldots + \dfrac{1}{n(n+1)}$.

Solution. (a) $x_n=\sqrt{n}(\sqrt{2+3/n}-\sqrt{1-1/n})\to+\infty$ as $n\to\infty$, since the second multiplier has a positive limit.

(c) $$x_n=\frac{n^2(n-\sqrt{n^2+1})}{1}=\frac{-n^2}{n+\sqrt{n^2+1}}=$$
$$=-n\cdot\frac{1}{1+\sqrt{1+\frac{1}{n^2}}}\to-\infty \text{ as } n\to\infty.$$

(d) $$x_n=\frac{n^2}{(n^2-n^3)^{2/3}-n\sqrt[3]{n^2-n^3}+n^2}=$$
$$=\frac{1}{\left(\frac{1}{n}-1\right)^{2/3}-\left(\frac{1}{n}-1\right)^{1/3}+1}.$$

It means, $x_n\to 1/3$.

(e) Factoring out the terms of the highest power in the numerator and denominator, we have:

$$x_n=\frac{\sqrt{n^2+1}+\sqrt{n}}{\sqrt[4]{n^3+n}-\sqrt{n}}=\frac{n\left(\sqrt{1+\frac{1}{n^2}}+\sqrt{\frac{1}{n}}\right)}{n^{3/4}\left(\sqrt[4]{1+\frac{1}{n^2}}-\sqrt[4]{\frac{1}{n}}\right)}=$$
$$=n^{1/4}\frac{\sqrt{1+\frac{1}{n^2}}+\sqrt{\frac{1}{n}}}{\sqrt[4]{1+\frac{1}{n^2}}-\sqrt[4]{\frac{1}{n}}}\to+\infty \text{ as } n\to\infty.$$

1.7.5. Find $\lim\limits_{n\to\infty} x_n$ if

(a) $x_n=\dfrac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$; (b) $x_n=\dfrac{\sqrt{n^2+4n}}{\sqrt[3]{n^3-3n^2}}$;

(c) $x_n=\sqrt[3]{1-n^3}+n$; (d) $x_n=\dfrac{1}{2n}\cos n^3-\dfrac{3n}{6n+1}$;

(e) $x_n=\dfrac{2n}{2n^2-1}\cos\dfrac{n+1}{2n-1}-\dfrac{n}{1-2n}\,\dfrac{n(-1)^n}{n^2+1}$;

(f) $x_n=\dfrac{1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{9}+\ldots+\frac{1}{3^n}}$.

§ 1.8. *Testing Sequences for Convergence*

Bolzano-Weierstrass' theorem. A monotonic bounded sequence has a finite limit.

Theorem on passing to the limit in inequalities. If $x_n\leqslant y_n\leqslant z_n$ and $\lim\limits_{n\to\infty} x_n=\lim\limits_{n\to\infty} z_n=c$, then $\lim\limits_{n\to\infty} y_n=c$ too (c is a number, $+\infty$ or $-\infty$ but not ∞).

1.8.1. Prove that the sequence with the general term $x_n = (2n-1)/(3n+1)$ is an increasing one.

Solution. We have to prove that $x_{n+1} > x_n$ for any n, i.e. to prove that

$$\frac{2n+1}{3n+4} > \frac{2n-1}{3n+1}.$$

The latter inequality is equivalent to the obvious inequality

$$6n^2+5n+1 > 6n^2+5n-4.$$

Hence, $x_{n+1} > x_n$.

1.8.2. Given a sequence with the general term

$$x_n = \frac{10^n}{n!}.$$

Prove that this sequence decreases at $n \geqslant 10$.

Solution.

$$x_{n+1} = \frac{10^{n+1}}{(n+1)!} = \frac{10^n}{n!} \cdot \frac{10}{n+1} = x_n \frac{10}{n+1}.$$

Since $\frac{10}{n+1} < 1$ at $n \geqslant 10$, then $x_{n+1} < x_n$ beginning with this number, which means that the sequence decreases at $n \geqslant 10$.

1.8.3. Test the following sequences for boundedness:

(a) $x_n = \frac{5n^2}{n^2+3}$;

(b) $y_n = (-1)^n \frac{2n}{n+1} \sin n$;

(c) $z_n = n \cos \pi n$.

Solution. (a) The sequence $\{x_n\}$ is bounded, since it is obvious that $0 < \frac{5n^2}{n^2+3} < 5$ for all n.

(b) The sequence $\{y_n\}$ is bounded:

$$|y_n| = |(-1)^n| \cdot \frac{2n}{n+1} |\sin n| < \frac{2n}{n+1} < 2.$$

(c) The sequence $\{z_n\}$ is not bounded, since

$$|z_n| = |n \cos \pi n| = n.$$

1.8.4. Prove that the sequence

$$x_1 = \frac{x_0}{a+x_0}; \quad x_2 = \frac{x_1}{a+x_1}; \quad x_3 = \frac{x_2}{a+x_2}; \quad \ldots; \quad x_n = \frac{x_{n-1}}{a+x_{n-1}}; \quad \ldots$$

$(a > 1, \ x_0 > 0)$ converges.

Solution. Let us prove that this sequence is monotonic and bounded. Firstly, $x_n < x_{n-1}$ as

$$x_n = \frac{x_{n-1}}{a+x_{n-1}} < x_{n-1}.$$

Hence, the given sequence is a decreasing one. Secondly, all its terms are positive (by condition $a > 0$ and $x_0 > 0$), which means that the sequence is bounded below. Thus, the given sequence is monotonic and bounded, hence it has a limit.

1.8.5. Prove that the sequence with the general term

$$x_n = \frac{1}{5+1} + \frac{1}{5^2+1} + \frac{1}{5^3+1} + \ldots + \frac{1}{5^n+1}$$

$$\left(\text{i.e. } x_1 = \frac{1}{5+1};\ x_2 = \frac{1}{5+1} + \frac{1}{5^2+1};\ x_3 = \frac{1}{5+1} + \frac{1}{5^2+1} + \frac{1}{5^3+1};\ \ldots\right)$$

converges.

Solution. The sequence $\{x_n\}$ increases, since $x_{n+1} = x_n + 1/(5^{n+1}+1)$ and, hence, $x_{n+1} > x_n$. Besides, it is bounded above, since $1/(5^n+1) < 1/5^n$ at any n and

$$x_n = \frac{1}{5+1} + \frac{1}{5^2+1} + \frac{1}{5^3+1} + \ldots + \frac{1}{5^n+1} <$$
$$< \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \ldots + \frac{1}{5^n} = \frac{1/5 - 1/5^{n+1}}{1 - 1/5} = \frac{1}{4}\left(1 - \frac{1}{5^n}\right) < \frac{1}{4}.$$

Hence, the sequence converges.

1.8.6. Taking advantage of the theorem on the existence of a limit of a monotonic bounded sequence, prove that the following sequences are convergent:

(a) $x_n = \dfrac{n^2-1}{n^2}$;

(b) $x_n = 2 + \dfrac{1}{2!} + \dfrac{1}{3!} + \ldots + \dfrac{1}{n!}$.

1.8.7. Prove that the following sequences converge and find their limits:

(a) $x_1 = \sqrt{2}$; $x_2 = \sqrt{2+\sqrt{2}}$;

$x_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$; ...; $x_n = \sqrt{\underbrace{2+\sqrt{2+\ldots+\sqrt{2}}}_{n \text{ radicals}}}$; ...;

(b) $x_n = \dfrac{2^n}{(n+2)!}$;

(c) $x_n = \dfrac{E(ny)}{n}$;

(d) the sequence of successive decimal approximations 1; 1.4; 1.41; 1.414; ... of the irrational number $\sqrt{2}$;

(e) $x_n = n!/n^n$.

Solution. (a) It is obvious that $x_1 < x_2 < x_3 < \ldots < x_n < x_{n+1} < \ldots$, i.e. the sequence is *increasing*. It now remains to prove that it is bounded.

We have $x_n=\sqrt{2+x_{n-1}}$, $n=2, 3, \ldots$ Since $x_1=\sqrt{2}<2$ $x_2=\sqrt{2+x_1}<\sqrt{2+2}=2$, $x_3=\sqrt{2+x_2}<\sqrt{2+2}=2$, ... Let it be proved that $x_{n-1}<2$. Then $x_n=\sqrt{2+x_{n-1}}<\sqrt{2+2}=2$. Thus, with the aid of mathematical induction we have proved that $x_n<2$, i.e. the sequence is *bounded*. Hence, it has a finite limit. Let us find it. Denote

$$\lim_{n\to\infty} x_n=y.$$

Then, $x_n=\sqrt{2+x_{n-1}}$; raising to the second power, we obtain

$$x_n^2=2+x_{n-1}.$$

Passing to the limit, we can rewrite this equality as follows

$$\lim_{n\to\infty} x_n^2=\lim_{n\to\infty}(2+x_{n-1}), \quad \text{or} \quad y^2=2+y.$$

The roots of the obtained quadratic equation are:

$$y_1=2; \quad y_2=-1.$$

The negative root does not suit here, since $x_n>0$. Hence, $\lim\limits_{n\to\infty} x_n=y_1=2$.

(c) We have $ny-1<E(ny)\leqslant ny$ or $y-\frac{1}{n}<\frac{E(ny)}{n}\leqslant y$. But the sequences $\left\{y-\frac{1}{n}\right\}$ and $\{y\}$ converge, their limit being y, that is why $\lim\limits_{n\to\infty} x_n=y$.

(d) This sequence is non-decreasing, since each following term x_{n+1} is obtained from the preceding one x_n by adding one more significant digit to the decimal fraction. The sequence is bounded above, say, by the number 1.5. Hence, the sequence converges, its limit being $\sqrt{2}$.

(e) The sequence decreases monotonically. Indeed,

$$x_{n+1}=\frac{(n+1)!}{(n+1)^{n+1}}=\frac{n!}{(n+1)^n}=\frac{n!}{n^n}\cdot\frac{n^n}{(n+1)^n}=\frac{n^n}{(n+1)^n}x_n.$$

Since $\frac{n^n}{(n+1)^n}<1$, $x_{n+1}<x_n$.

Then, since $x_n>0$, the sequence is bounded below, hence $\lim\limits_{n\to\infty} x_n$ exists. Let us denote it l. Obviously, $l=\lim\limits_{n\to\infty} x_n\geqslant 0$. Now let us show that $l=0$. Indeed,

$$\frac{(n+1)^n}{n^n}=\left(\frac{n+1}{n}\right)^n=\left(1+\frac{1}{n}\right)^n\geqslant 1+n\frac{1}{n}=2.$$

Hence, $\frac{n^n}{(n+1)^n} < \frac{1}{2}$ and $x_{n+1} < \frac{1}{2} x_n$. Passing over to the limit, we obtain

$$l \leqslant \frac{1}{2} l,$$

which, together with $l \geqslant 0$, brings us to the conclusion:

$$l = 0.$$

1.8.8. Find the limits of the sequences with the following general terms:

$$x_n = \frac{n}{\sqrt{n^2+n}}; \quad z_n = \frac{n}{\sqrt{n^2+1}};$$

$$y_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+n}}.$$

Solution. Let us prove that $\lim\limits_{n \to \infty} x_n = 1$. Indeed,

$$|x_n - 1| = \left| \frac{n}{\sqrt{n^2+n}} - 1 \right| = \left| \frac{n - \sqrt{n^2+n}}{\sqrt{n^2+n}} \right| =$$

$$= \frac{n}{\sqrt{n^2+n}\,(n+\sqrt{n^2+n})} < \frac{1}{2n}.$$

We can prove similarly that

$$\lim_{n \to \infty} z_n = 1.$$

Then,

$$y_n < \underbrace{\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \ldots + \frac{1}{\sqrt{n^2+1}}}_{n} = \frac{n}{\sqrt{n^2+1}} = z_n.$$

On the other hand,

$$y_n > \underbrace{\frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + \ldots + \frac{1}{\sqrt{n^2+n}}}_{n} = \frac{n}{\sqrt{n^2+n}} = x_n.$$

Thus,

$$x_n < y_n < z_n, \quad \lim_{n \to \infty} x_n = \lim_{n \to \infty} z_n = 1$$

and according to the theorem on passing to the limit in inequalities

$$\lim_{n \to \infty} y_n = 1.$$

1.8.9. Using the theorem on passing to the limit in inequalities prove

$$\lim_{n \to \infty} \sqrt[n]{a} = 1 \quad (a > 0).$$

1.8.10. Prove the existence of the limit of the sequence $y_n = a^{1/2^n}$ $(a > 1)$ and calculate it.

1.8.11. Taking advantage of the theorem on the limit of a monotonic sequence, prove the existence of a finite limit of the sequence

$$x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}.$$

1.8.12. Taking advantage of the theorem on passing to the limit in inequalities, prove that

$$\lim_{n \to \infty} x_n = 1 \text{ if } x_n = 2n\left(\sqrt{n^2+1} - n\right).$$

1.8.13. Prove that the sequence

$$x_1 = \sqrt{a};\ x_2 = \sqrt{a + \sqrt{a}};$$

$$x_3 = \sqrt{a + \sqrt{a + \sqrt{a}}};\ \ldots;\quad x_n = \underbrace{\sqrt{a + \sqrt{a + \ldots + \sqrt{a}}}}_{n \text{ radicals}}$$

$$(a > 0)$$

has the limit $b = (\sqrt{4a+1} + 1)/2$.

1.8.14. Prove that the sequence with the general term

$$x_n = \frac{1}{3+1} + \frac{1}{3^2+2} + \ldots + \frac{1}{3^n+n}$$

has a finite limit.

1.8.15. Prove that a sequence of lengths of perimeters of regular 2^n-gons inscribed in a circle tends to a limit (called the length of circumference).

§ 1.9. The Limit of a Function

A point a on the real axis is called the *limit point* of a set X if any neighbourhood of the point a contains points belonging to X which are different from a (a may be either a proper or an improper point).

Let the point a be the limit point of the domain of definition X of the function $f(x)$. The number A is called the *limit of the function* $f(x)$ as $x \to a$, $A = \lim\limits_{x \to a} f(x)$, if for any neighbourhood V of the number A there exists a neighbourhood u of the number a such that for all $x \in X$ lying in u, $f(x) \in V$ (the definition of the limit of a function after Cauchy). The number A may be either finite or infinite. In particular, if the numbers A and a are finite we obtain the following definition.

A number A is called the *limit of a function* $f(x)$ as $x \to a$, $A = \lim_{x \to a} f(x)$, if for any $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that for all x satisfying the inequality $0 < |x - a| < \delta$ and belonging to the domain of definition of the function $f(x)$ the inequality $|f(x) - A| < \varepsilon$ holds true (the "ε-δ definition").

If $a = +\infty$, the definition is as follows. A number A is called the *limit of a function* $f(x)$ as $x \to +\infty$, $A = \lim_{x \to +\infty} f(x)$, if for any $\varepsilon > 0$ there exists a number $M(\varepsilon) > 0$ such that for all x satisfying the inequality $x > M(\varepsilon)$ and belonging to the domain of definition of the function $f(x)$ the inequality $|f(x) - A| < \varepsilon$ holds true (the "ε-M definition").

The notation $\lim_{x \to a} f(x) = \infty$ means that $\lim_{x \to a} |f(x)| = +\infty$. The rest of the cases are considered similarly.

The definition of the limit of a function after Heine. The notation $\lim_{x \to a} f(x) = A$ means that for any sequence of values of x converging to the number a

$$x_1, x_2, \ldots, x_n, \ldots$$

(belonging to the domain of definition of the function and differing from a) the corresponding sequence of values of y

$$y_1 = f(x_1);\ y_2 = f(x_2);\ \ldots;\ y_n = f(x_n), \ldots$$

has a limit, which is the number A.

1.9.1. Taking advantage of the definition of the limit after Heine (i.e. in terms of sequences) and of the theorems on the limits of sequences, prove that

$$\lim_{x \to 2} \frac{3x+1}{5x+4} = \frac{1}{2}.$$

Solution. Let us consider any sequence $x_1, x_2, \ldots$ satisfying the following two conditions: (1) the numbers $x_1, x_2, \ldots$ belong to the domain of definition of the function $f(x) = (3x+1)/(5x+4)$ (i.e. $x_n \neq -4/5$); (2) the sequence $\{x_n\}$ converges to the number 2, i.e. $\lim_{n \to \infty} x_n = 2$.

To the sequence $\{x_n\}$ there corresponds the sequence of values of the function

$$\frac{3x_1+1}{5x_1+4};\ \frac{3x_2+1}{5x_2+4};\ \ldots;$$

proceeding from the theorem on the limits (§ 1.7),

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} \frac{3x_n+1}{5x_n+4} = \frac{\lim (3x_n+1)}{\lim (5x_n+4)} = \frac{6+1}{10+4} = \frac{1}{2}.$$

Thus, independently of the choice of a sequence $\{x_n\}$ which converges to the number $2\ (x_n \neq -4/5)$, the corresponding sequences of values of the function $f(x_n)$ converge to the number $1/2$, which, according to the definition of the limit of a function, means that

$$\lim_{x \to 2} \frac{3x+1}{5x+4} = \frac{1}{2}.$$

Note. The definition of the limit after Heine is conveniently applied when we have to prove that a function $f(x)$ *has no limit*. For this it is sufficient to show that there exist two sequences $\{x'_n\}$ and $\{x''_n\}$ such that $\lim\limits_{n \to \infty} x'_n = \lim\limits_{n \to \infty} x''_n = a$, but the corresponding sequences $\{f(x'_n)\}$ and $\{f(x''_n)\}$ do not have identical limits.

1.9.2. Prove that the following limits do not exist:

(a) $\lim\limits_{x \to 1} \sin \frac{1}{x-1}$; (b) $\lim\limits_{x \to 0} 2^{1/x}$; (c) $\lim\limits_{x \to \infty} \sin x$.

Solution. (a) Choose two sequences

$$x_n = 1 + \frac{1}{n\pi} \quad \text{and} \quad x'_n = 1 + \frac{2}{(4n+1)\pi} \quad (n = 1, 2, \ldots),$$

for which

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} x'_n = 1.$$

The corresponding sequences of values of the function are:

$$f(x_n) = \sin \frac{1}{1 + 1/(n\pi) - 1} = \sin n\pi = 0$$

and

$$f(x'_n) = \sin \frac{1}{1 + 2/[(4n+1)\pi] - 1} = \sin \frac{4n+1}{2}\pi = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1.$$

Hence,

$$\lim_{x_n \to 1} f(x_n) = 0 \quad \text{and} \quad \lim_{x'_n \to 1} f(x'_n) = 1,$$

i.e. the sequences $\{f(x_n)\}$ and $\{f(x'_n)\}$ have different limits, whence it follows that $\lim\limits_{x \to 1} \sin \frac{1}{x-1}$ does not exist.

(c) Choose two sequences, $x_n = \pi n$ and $x'_n = 2\pi n + \pi/2$ $(n = 1, 2, \ldots)$, for which $\lim\limits_{n \to \infty} x_n = \lim\limits_{n \to \infty} x'_n = \infty$. Since

$$\lim_{n \to \infty} \sin x_n = \lim_{n \to \infty} \sin \pi n = 0,$$

and

$$\lim_{n \to \infty} \sin x'_n = \lim_{n \to \infty} \sin(2\pi n + \pi/2) = 1,$$

$\lim\limits_{x \to \infty} \sin x$ does not exist.

Note. The above examples show that one cannot draw the conclusion about the existence of the limit of a function proceeding from the sequence of values of x of a *particular form* (for example, proceeding from $x_n = 1 + 2/((4n+1)\pi)$ in the item (a) of this problem), but it is necessary to consider an arbitrary sequence x_1, x_2, ..., x_n, ... having a given limit.

1.9.3. Proceeding from the definition of the limit of a function after Cauchy (i.e. in the terms of "ε-δ"; "ε-M", etc.), prove that

(a) $\lim\limits_{x \to 1} (3x-8) = -5$;

(b) $\lim\limits_{x \to +\infty} \frac{5x+1}{3x+9} = \frac{5}{3}$;

(c) $\lim\limits_{x \to 1} \frac{1}{(1-x)^2} = +\infty$;

(d) $\lim\limits_{x \to \infty} \log_a x = \infty \quad (a > 1)$;

(e) $\lim\limits_{x \to \infty} \arctan x = \pi/2$;

(f) $\lim\limits_{x \to \pi/6} \sin x = 1/2$.

Solution. (a) According to the "ε-δ" definition we are to prove that for any $\varepsilon > 0$ there exists $\delta > 0$ such that from the inequality $|x-1| < \delta$ it follows that $|f(x)-(-5)| = |f(x)+5| < \varepsilon$.

In other words, it is necessary to solve the inequality

$$|3x-8+5| = 3|x-1| < \varepsilon.$$

The latter inequality shows that the required inequality $|f(x)+5| < \varepsilon$ is fulfilled as soon as $|x-1| < \varepsilon/3 = \delta$. Hence, $\lim\limits_{x \to 1} (3x-8) = -5$.

(b) According to the "ε-M" definition of the limit one has to show that for any $\varepsilon > 0$ it is possible to find a number $M > 0$ such that for all $x > M$ the inequality

$$\left| \frac{5x+1}{3x+9} - \frac{5}{3} \right| < \varepsilon \qquad (*)$$

will be fulfilled.

Transforming this inequality, we obtain

$$\left| \frac{5x+1}{3x+9} - \frac{5}{3} \right| = \frac{14}{|3x+9|} < \varepsilon.$$

Since $x > 0$, it remains to solve the inequality

$$\frac{14}{3x+9} < \varepsilon,$$

whence

$$x > \frac{14-9\varepsilon}{3\varepsilon};$$

hence $M = \frac{14-9\varepsilon}{3\varepsilon}$.

Thus, for $\varepsilon > 0$ we have found $M = \frac{14-9\varepsilon}{3\varepsilon}$ such that for all values of $x > M$ the inequality (*) is fulfilled, and this means that

$$\lim_{x \to +\infty} \frac{5x+1}{3x+9} = \frac{5}{3}.$$

Let, for example, $\varepsilon = 0.01$; then $M = \frac{14-0.09}{0.03} = 463\frac{2}{3}$.

(c) We have to prove that for any $K > 0$ there exists $\delta > 0$ such that from the inequality

$$|x-1| < \delta$$

there always follows the inequality

$$\left|\frac{1}{(1-x)^2}\right| = \frac{1}{(1-x)^2} > K.$$

Let us choose an arbitrary number $K > 0$ and solve the inequality

$$\frac{1}{(1-x)^2} > K, \qquad (**)$$

whence

$$|1-x| < \frac{1}{\sqrt{K}} \quad (K > 0).$$

Thus, if we put $\delta = \frac{1}{\sqrt{K}}$, then the inequality (**) holds true as soon as $|x-1| < \delta$, which means that $\lim\limits_{x \to 1} \frac{1}{(1-x)^2} = +\infty$.

(d) We have to prove that for any $K > 0$ there exists $M > 0$ such that from the inequality $x > M$ there always follows the inequality $\log_a x > K$. Let us choose an arbitrary number $K > 0$ and consider the inequality $\log_a x > K$. If we put $a^K = M$, then at $x > M$ the inequality $\log_a x > K$ holds true. Hence,

$$\lim_{x \to +\infty} \log_a x = +\infty.$$

1.9.4. Prove that $\lim\limits_{x \to \infty} \cos x$ does not exist.

1.9.5. Using the sequences of the roots of the equations $\sin(1/x) = 1$ and $\sin(1/x) = -1$, show that the function $f(x) = \sin(1/x)$ has no limit as $x \to 0$.

1.9.6. Proceeding from Cauchy's definition of the limit of a function prove that:

(a) $\lim\limits_{x \to 1} (3x-2) = 1$; (b) $\lim\limits_{x \to 1} \dfrac{x-1}{\sqrt{x}-1} = 2$;

(c) $\lim\limits_{x \to 0} \sin x = 0$; (d) $\lim\limits_{x \to 0} \cos x = 1$;

(e) $\lim\limits_{x \to +\infty} \dfrac{2x-1}{3x+2} = \dfrac{2}{3}$;

(f) $\lim\limits_{x \to +\infty} a^x = +\infty$ $(a > 1)$;

(g) $\lim\limits_{x \to \infty} \dfrac{\sin x}{x} = 0$.

§ 1.10. *Calculation of Limits of Functions*

I. If the limits $\lim\limits_{x \to a} u(x)$ and $\lim\limits_{x \to a} v(x)$ exist, then the following theorems hold true:

(1) $\lim\limits_{x \to a} [u(x) \pm v(x)] = \lim\limits_{x \to a} u(x) \pm \lim\limits_{x \to a} v(x)$;

(2) $\lim\limits_{x \to a} [u(x) \cdot v(x)] = \lim\limits_{x \to a} u(x) \cdot \lim\limits_{x \to a} v(x)$;

(3) $\lim\limits_{x \to a} \dfrac{u(x)}{v(x)} = \dfrac{\lim\limits_{x \to a} u(x)}{\lim\limits_{x \to a} v(x)}$ $\quad (\lim\limits_{x \to a} v(x) \neq 0)$.

II. For all main elementary functions at any point of their domain of definition the equality $\lim\limits_{x \to a} f(x) = f(\lim\limits_{x \to a} x) = f(a)$ holds true.

III. If for all values of x in a certain neighbourhood of a point a (except for, perhaps, $x = a$) the functions $f(x)$ and $\varphi(x)$ are equal and one of them has a limit as x approaches a, then the other one has the same limit.

IV. The following limits are frequently used:

(1) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$;

(2) $\lim\limits_{x \to \infty} (1 + 1/x)^x = \lim\limits_{\alpha \to 0} (1+\alpha)^{1/\alpha} = e = 2.71828\ldots$;

(3) $\lim\limits_{x \to 0} \dfrac{\log_a (1+x)}{x} = \log_a e$ $\quad (a > 0;\ a \neq 1)$;

(4) $\lim\limits_{x \to 0} \dfrac{\ln (1+x)}{x} = 1$;

(5) $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \ln a$ $\quad (a > 0)$.

1.10.1. Find the limits:

(a) $\lim\limits_{x\to 1} \frac{4x^5+9x+7}{3x^6+x^3+1}$;

(b) $\lim\limits_{x\to 2} \frac{x^3+3x^2-9x-2}{x^3-x-6}$;

(c) $\lim\limits_{x\to -1} \frac{x+1}{\sqrt{6x^2+3}+3x}$;

(d) $\lim\limits_{x\to 1} \frac{x^p-1}{x^q-1}$ (p and q integers);

(e) $\lim\limits_{x\to 0} \frac{\sqrt{9+5x+4x^2}-3}{x}$;

(f) $\lim\limits_{x\to 2} \frac{\sqrt[3]{10-x}-2}{x-2}$;

(g) $\lim\limits_{x\to 2} \frac{\sqrt{x+7}-3\sqrt{2x-3}}{\sqrt[3]{x+6}-2\sqrt[3]{3x-5}}$;

(h) $\lim\limits_{x\to 3} \left[\log_a \frac{x-3}{\sqrt{x+6}-3}\right]$;

(i) $\lim\limits_{x\to 1} \frac{x^3-x^2-x+1}{x^3-3x+2}$;

(j) $\lim\limits_{x\to 1} \frac{\sqrt{x+8}-\sqrt{8x+1}}{\sqrt{5-x}-\sqrt{7x-3}}$.

Solution. (a) Since there exist limits of the numerator and denominator and the limit of the denominator is different from zero, we can use the theorem on the limit of a quotient:

$$\lim_{x\to 1} \frac{4x^5+9x+7}{3x^6+x^3+1} = \frac{\lim\limits_{x\to 1}(4x^5+9x+7)}{\lim\limits_{x\to 1}(3x^6+x^3+1)} = \frac{4+9+7}{3+1+1} = 4.$$

(b) The above theorem cannot be directly used here, since the limit of the denominator equals zero as $x \to 2$. Here the limit of the numerator also equals zero as $x \to 2$. Hence, we have the indeterminate form $\frac{0}{0}$. For $x \neq 2$ we have

$$\frac{x^3+3x^2-9x-2}{x^3-x-6} = \frac{(x-2)(x^2+5x+1)}{(x-2)(x^2+2x+3)} = \frac{x^2+5x+1}{x^2+2x+3}.$$

Thus, in any domain which does not contain the point $x=2$ the functions

$$f(x) = \frac{x^3+3x^2-9x-2}{x^3-x-6} \quad \text{and} \quad \varphi(x) = \frac{x^2+5x+1}{x^2+2x+3}$$

are equal; hence, their limits are also equal. The limit of the function $\varphi(x)$ is found directly:

$$\lim_{x\to 2} \varphi(x) = \lim_{x\to 2} \frac{x^2+5x+1}{x^2+2x+3} = \frac{15}{11};$$

hence,

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^3+3x^2-9x-2}{x^3-x-6} = \frac{15}{11}.$$

(c) Just as in (b), we remove the indeterminate form $\frac{0}{0}$ by transforming

$$\lim_{x\to -1} \frac{x+1}{\sqrt{6x^2+3}+3x} = \lim_{x\to -1} \frac{(x+1)(\sqrt{6x^2+3}-3x)}{3-3x^2} =$$
$$=\lim_{x\to -1} \frac{\sqrt{6x^2+3}-3x}{3(1-x)} = 1.$$

1.10.2. Find the limits:

(a) $\lim_{x\to\infty}\left(\frac{x^3}{3x^2-4}-\frac{x^2}{3x+2}\right)$;

(b) $\lim_{x\to+\infty}(\sqrt{9x^2+1}-3x)$;

(c) $\lim_{x\to+\infty}\frac{2\sqrt{x}+3\sqrt[3]{x}+5\sqrt[5]{x}}{\sqrt{3x-2}+\sqrt[3]{2x-3}}$;

(d) $\lim_{x\to-\infty}(\sqrt{2x^2-3}-5x)$;

(e) $\lim_{x\to+\infty}x(\sqrt{x^2+1}-x)$;

(f) $\lim_{x\to+\infty}\frac{\sqrt{2x^2+3}}{4x+2}$ and $\lim_{x\to-\infty}\frac{\sqrt{2x^2+3}}{4x+2}$;

(g) $\lim_{x\to\infty}5^{2x/(x+3)}$.

Solution. (a) $\lim_{x\to\infty}\left(\frac{x^3}{3x^2-4}-\frac{x^2}{3x+2}\right)$.

Here we have the indeterminate form $\infty-\infty$; let us subtract the fractions

$$\lim_{x\to\infty}\left(\frac{x^3}{3x^2-4}-\frac{x^2}{3x+2}\right)=\lim_{x\to\infty}\frac{2x^3+4x^2}{9x^3+6x^2-12x-8}=$$
$$=\lim_{x\to\infty}\frac{2+4/x}{9+6/x-12/x^2-8/x^3}=\frac{2}{9}.$$

Note. We see that in such examples the limit is equal to the ratio of the coefficients at the superior power of x (provided the polynomials are of the same degree).

(b) $\lim_{x\to+\infty}\frac{(\sqrt{9x^2+1}-3x)}{1}=\lim_{x\to+\infty}\frac{1}{\sqrt{9x^2+1}+3x}=0.$

(c) In handling such examples bear in mind that the function $f(x)=\sqrt[m]{p_n(x)}$, where $p_n(x)$ is a polynomial of degree n, tending to infinity in the same way as the function $\sqrt[m]{x^n}$. This allows us to single out the superior power of x and divide both the numerator and denominator by this power of x. In the given example the divisor is $\sqrt{x}$; then we obtain:

$$\lim_{x\to+\infty}\frac{2\sqrt{x}+3\sqrt[3]{x}+5\sqrt[5]{x}}{\sqrt{3x-2}+\sqrt[3]{2x-3}}=\lim_{x\to+\infty}\frac{2+3/\sqrt[6]{x}+5/\sqrt[10]{x^3}}{\sqrt{3-2/x}+\sqrt[6]{4/x-12/x^2+9/x^3}}=$$
$$=\frac{2}{\sqrt{3}}.$$

(d) Since the sum of two positive infinitely large quantities is also an infinitely large quantity, then

$$\lim_{x\to-\infty}(\sqrt{2x^2-3}-5x)=\lim_{x\to-\infty}[\sqrt{2x^2-3}+(-5x)]=+\infty.$$

(f) At $x>0$ we have $\sqrt{x^2}=x$, therefore

$$\lim_{x\to+\infty}\frac{\sqrt{x^2(2+3/x^2)}}{x(4+2/x)}=\lim_{x\to+\infty}\frac{x\sqrt{2+3/x^2}}{x(4+2/x)}=\frac{\sqrt{2}}{4}.$$

At $x<0$ we have $\sqrt{x^2}=-x$ and, hence,

$$\lim_{x\to-\infty}\frac{\sqrt{x^2(2+3/x^2)}}{x(4+2/x)}=\lim_{x\to-\infty}\frac{-x\sqrt{2+3/x^2}}{x(4+2/x)}=-\frac{\sqrt{2}}{4}.$$

Note. From this it follows, incidentally, that $\lim\limits_{x\to\infty}\frac{\sqrt{2x^2+3}}{4x+2}$ does not exist.

(g) $\lim\limits_{x\to\infty}5^{2x/(x+3)}=5^{\lim\limits_{x\to\infty}2x/(x+3)}=5^2=25.$

1.10.3. Find the limits:

(a) $\lim\limits_{x\to1}\frac{2x-2}{\sqrt[3]{26+x}-3}$;

(b) $\lim\limits_{x\to-1}\frac{x+1}{\sqrt[4]{x+17}-2}$;

(c) $\lim\limits_{x\to-1}\frac{1+\sqrt[3]{x}}{1+\sqrt[5]{x}}$;

(d) $\lim\limits_{x\to0}\frac{\sqrt[k]{1+x}-1}{x}$ (k positive integer);

(e) $\lim\limits_{x\to\pi/6}\frac{\sin(x-\pi/6)}{\sqrt{3}-2\cos x}$;

(f) $\lim\limits_{x\to\pi/2}\frac{\cos x}{\sqrt[3]{(1-\sin x)^2}}$;

(g) $\lim\limits_{x\to\pi/6}\frac{2\sin^2x+\sin x-1}{2\sin^2x-3\sin x+1}$.

Solution (method of substitution). (a) Let us put $26+x=z^3$. Then $x=z^3-26$ and $z\to3$ as $x\to1$; hence

$$\lim_{x\to1}\frac{2x-2}{\sqrt[3]{26+x}-3}=\lim_{z\to3}\frac{2z^3-54}{z-3}=\lim_{z\to3}\frac{2(z-3)(z^2+3z+9)}{z-3}=$$
$$=\lim_{z\to3}2(z^2+3z+9)=54.$$

(d) Let us put $1+x=z^k$; then $x=z^k-1$ and $z\to 1$ as $x\to 0$. Hence,

$$\lim_{x\to 0}\frac{\sqrt[k]{1+x}-1}{x}=\lim_{z\to 1}\frac{z-1}{z^k-1}=\frac{1}{k}$$ (see Problem **1.10.1** (d)).

(e) Let us put $x-\pi/6=z$; then $x=z+\pi/6$ and $z\to 0$ as $x\to\pi/6$. On substituting we obtain

$$\lim_{x\to\pi/6}\frac{\sin(x-\pi/6)}{\sqrt{3}-2\cos x}=\lim_{z\to 0}\frac{\sin z}{\sqrt{3}-2\cos(z+\pi/6)}=$$

$$=\lim_{z\to 0}\frac{\sin z}{\sqrt{3}-\sqrt{3}\cos z+\sin z}=\lim_{z\to 0}\frac{2\sin(z/2)\cos(z/2)}{2\sqrt{3}\sin^2(z/2)+2\sin(z/2)\cos(z/2)}=$$

$$=\lim_{z\to 0}\frac{\cos(z/2)}{\sqrt{3}\sin(z/2)+\cos(z/2)}=1.$$

1.10.4. Find the limits:

(a) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$; (b) $\lim\limits_{x\to 0}\frac{\tan x-\sin x}{x^3}$;

(c) $\lim\limits_{x\to 1}\frac{\cos(\pi x/2)}{1-x}$.

Solution. (a) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=\lim\limits_{x\to 0}\frac{2\sin^2(x/2)}{x^2}=\frac{1}{2}\lim\limits_{x\to 0}\left(\frac{\sin(x/2)}{x/2}\right)^2=\frac{1}{2}$;

(b) $$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{\cos x\cdot x^3}=$$

$$=\lim_{x\to 0}\frac{1}{\cos x}\cdot\frac{\sin x}{x}\cdot\frac{1-\cos x}{x^2}=\frac{1}{2};$$

(c) Let us put $1-x=z$. Then $x=1-z$ and $z\to 0$ as $x\to 1$. Hence,

$$\lim_{x\to 1}\frac{\cos\frac{\pi}{2}x}{1-x}=\lim_{z\to 0}\frac{\cos\left(\frac{\pi}{2}-\frac{\pi}{2}z\right)}{z}=\lim_{z\to 0}\frac{\sin\frac{\pi}{2}z}{z}=\frac{\pi}{2}.$$

Note. For a simpler method of solving similar problems see § **1.12.**

1.10.5. Find the limits:

(a) $\lim\limits_{x\to\infty}(1+1/x)^{7x}$; (b) $\lim\limits_{x\to 0}(1+x)^{1/(3x)}$;

(c) $\lim\limits_{x\to\infty}\left(\frac{x}{1+x}\right)^x$; (d) $\lim\limits_{x\to\infty}(1+k/x)^{mx}$;

(e) $\lim\limits_{x\to 0}\frac{\ln(1+x)}{3^x-1}$; (f) $\lim\limits_{x\to 0}\frac{e^{4x}-1}{\tan x}$;

(g) $\lim\limits_{x\to 0}\frac{\ln(a+x)-\ln a}{x}$; (h) $\lim\limits_{x\to 0}\frac{e^x-e^{-x}}{\sin x}$;

(i) $\lim\limits_{x\to e}\frac{\ln x-1}{x-e}$.

Solution. (a) $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{7x}=\lim\limits_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{x}\right]^{7}=$

$$=\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{7}=e^{7};$$

(e) $\lim\limits_{x\to 0}\frac{\ln(1+x)}{3^x-1}=\lim\limits_{x\to 0}\left[\frac{\ln(1+x)}{x}\cdot\frac{x}{3^x-1}\right]=\frac{1}{\ln 3}$.

(i) Put $x/e-1=z$; then $x=e(z+1)$; $z\to 0$ as $x\to e$. On substituting we obtain

$$\lim_{x\to e}\frac{\ln x-1}{x-e}=\lim_{x\to e}\frac{\ln(x/e)}{e(x/e-1)}=\frac{1}{e}\lim_{z\to 0}\frac{\ln(1+z)}{z}=\frac{1}{e}.$$

1.10.6. Find

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^{x}.$$

Solution. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x^2}\right)^{x}=\lim\limits_{x\to\infty}\left[\left(1+\frac{1}{x^2}\right)^{x^2}\right]^{1/x}=e^0=1.$

1.10.7. Find the limits:

(a) $\lim\limits_{x\to 1}\left(\frac{1+x}{2+x}\right)^{(1-\sqrt{x})/(1-x)}$;

(b) $\lim\limits_{x\to\infty}\left(\frac{x^2+2x-1}{2x^2-3x-2}\right)^{(2x+1)/(x-1)}$.

Solution. (a) Denote:

$$f(x)=(1+x)/(2+x);$$

$$\varphi(x)=\frac{1-\sqrt{x}}{1-x};$$

$$\lim_{x\to 1}f(x)=\lim_{x\to 1}\frac{1+x}{2+x}=\frac{2}{3};$$

$$\lim_{x\to 1}\varphi(x)=\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x}=\frac{1}{2}.$$

But at finite limits $\lim\limits_{x\to a}f(x)=A>0$, $\lim\limits_{x\to a}\varphi(x)=B$ the following relation holds true:

$$\lim_{x\to a}[f(x)]^{\varphi(x)}=e^{\lim\limits_{x\to a}\varphi(x)\ln f(x)}=e^{B\ln A}=A^B.$$

Hence,

$$\lim_{x\to 1}\left(\frac{1+x}{2+x}\right)^{(1-\sqrt{x})/(1-x)}=\left(\frac{2}{3}\right)^{1/2}=\sqrt{\frac{2}{3}}.$$

Note. If in handling examples of the form $\lim\limits_{x\to a}[f(x)]^{\varphi(x)}$ it turns out that $\lim\limits_{x\to a}f(x)=1$ and $\lim\limits_{x\to a}\varphi(x)=\infty$, then the following transformation may be recommended:

$$\lim_{x\to a}[f(x)]^{\varphi(x)} = \lim_{x\to a}\{1+[f(x)-1]\}^{\varphi(x)} =$$

$$= \lim_{x\to a}\{[1+(f(x)-1)]^{1/(f(x)-1)}\}^{\varphi(x)[f(x)-1]} = e^{\lim\limits_{x\to a}\varphi(x)[f(x)-1]}. \quad (*)$$

1.10.8. Find the limits:

(a) $\lim\limits_{x\to\infty}\left(\dfrac{2x^2+3}{2x^2+5}\right)^{8x^2+3}$; (b) $\lim\limits_{x\to 0}\left(\dfrac{1+\tan x}{1+\sin x}\right)^{1/\sin x}$;

(c) $\lim\limits_{x\to 1}(1+\sin\pi x)^{\cot\pi x}$;

(d) $\lim\limits_{x\to a}\left(\dfrac{\sin x}{\sin a}\right)^{1/(x-a)}$ ($a\neq k\pi$, with k an integer).

Solution. (a) Let us denote:

$$f(x)=\frac{2x^2+3}{2x^2+5};\quad \varphi(x)=8x^2+3;$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{2x^2+3}{2x^2+5}=1;$$

$$\lim_{x\to\infty}\varphi(x)=\lim_{x\to\infty}(8x^2+3)=\infty.$$

Use the formula (*):

$$\lim_{x\to\infty}\left(\frac{2x^2+3}{2x^2+5}\right)^{8x^2+3} = e^{\lim\limits_{x\to\infty}\varphi(x)[f(x)-1]};$$

$$f(x)-1=\frac{2x^2+3}{2x^2+5}-1=-\frac{2}{2x^2+5};$$

$$\lim_{x\to\infty}\varphi(x)[f(x)-1] = -\lim_{x\to\infty}\frac{2(8x^2+3)}{2x^2+5}=-8.$$

Therefore

$$\lim_{x\to\infty}\left(\frac{2x^2+3}{2x^2+5}\right)^{8x^2+3}=e^{-8}.$$

1.10.9. The function $f(x)$ is given with the aid of the limit

$$f(x)=\lim_{n\to\infty}\frac{x^{2n}-1}{x^{2n}+1}.$$

Investigate this function and graph it.

Solution. Consider three cases:

(1) $|x|>1$. Since in this case $\lim\limits_{n\to\infty}x^{2n}=\infty$, then

$$f(x)=\lim_{n\to\infty}\frac{1-1/x^{2n}}{1+1/x^{2n}}=1.$$

(2) $|x|<1$. In this case $\lim\limits_{n\to\infty}x^{2n}=0$; therefore $f(x)=-1$.

(3) $x=\pm 1$. In this case $x^{2n}=1$ at any n, and therefore $f(x)=0$.

Thus, the function under consideration can be written in the following way:

$$f(x)=\begin{cases} 1 & \text{if } |x|>1 \\ -1 & \text{if } |x|<1 \\ 0 & \text{if } x=\pm 1 \end{cases}$$

or, briefly, $f(x)=\operatorname{sign}(|x|-1)$ (see Problem **1.5.11** (n)).

The graph of this function is shown in Fig. 27.

1.10.10. The population of a country increases by 2% per year. By how many times does it increase in a century?

Solution. If we denote the initial number of inhabitants of a given country as A, then after a year the total population will amount to

$$A+\frac{A}{100}\cdot 2=\left(1+\frac{1}{50}\right)A.$$

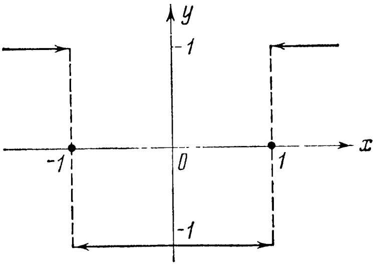

Fig. 27

After two years the population will amount to $A\left(1+\frac{1}{50}\right)^2$. After 100 years it will reach the total of $A\left(1+\frac{1}{50}\right)^{100}$, i.e. it will have increased $\left[\left(1+\frac{1}{50}\right)^{50}\right]^2$ times. Taking into account that $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$, we can approximately consider that $\left(1+\frac{1}{50}\right)^{50}\approx e$.

Hence, after 100 years the population of the country will have increased $e^2\approx 7.39$ times.

Of course, this estimation is very approximate, but it gives an idea as to the order of the increase in the population; $\left(\text{the quantity } \left(1+\frac{1}{50}\right)^{100}=7.245 \text{ to within three decimal places}\right)$.

1.10.11. Find the limits:

(a) $\lim\limits_{x\to 0}\dfrac{\cos x+4\tan x}{2-x-2x^4}$;

(b) $\lim\limits_{x\to -2}\dfrac{2x^2+5x-7}{3x^2-x-2}$;

(c) $\lim\limits_{x\to 1}\dfrac{\sqrt{5-x}-2}{\sqrt{2-x}-1}$;

(d) $\lim\limits_{x\to\infty}\dfrac{2x^2-5x+4}{5x^2-2x-3}$;

(e) $\lim\limits_{x\to\infty}(\sqrt{x^2+1}-\sqrt{x^2-1})$;

(f) $\lim\limits_{x\to\infty}\left(\dfrac{1-2x}{\sqrt[3]{1+8x^3}}+2^{-x^2}\right)$.

1.10.12. Find the limits:

(a) $\lim\limits_{x\to 0}\dfrac{\sqrt{x+4}-2}{\sin 5x}$; (b) $\lim\limits_{x\to 1}\dfrac{\sin(1-x)}{\sqrt{x}-1}$;

(c) $\lim\limits_{\alpha\to\pi}\dfrac{\sin\alpha}{1-\alpha^2/\pi^2}$; (d) $\lim\limits_{x\to\pi/4}\tan 2x\tan(\pi/4-x)$;

(e) $\lim\limits_{x\to\pi/3}\dfrac{\tan^3 x-3\tan x}{\cos(x+\pi/6)}$.

1.10.13. Find the limits:

(a) $\lim\limits_{x\to\infty}(1+4/x)^{x+3}$; (b) $\lim\limits_{x\to 0}\dfrac{e^{-x}-1}{x}$;

(c) $\lim\limits_{x\to 0}\dfrac{a^{2x}-1}{x}$; (d) $\lim\limits_{x\to 0}(1+3\tan^2 x)^{\cot^2 x}$;

(e) $\lim\limits_{x\to\pi/4}(\sin 2x)^{\tan^2 2x}$; (f) $\lim\limits_{x\to\infty}\left(\dfrac{2x-1}{2x+1}\right)^x$;

(g) $\lim\limits_{x\to\pi/2}(\tan x)^{\tan 2x}$; (h) $\lim\limits_{x\to\pi/2}(\sin x)^{\tan x}$;

(i) $\lim\limits_{x\to\infty}\left(\dfrac{3x^2+2x+1}{x^2+x+2}\right)^{(6x+1)/(3x+2)}$;

(j) $\lim\limits_{x\to\infty}\left(\dfrac{1+3x}{2+3x}\right)^{(1-\sqrt{x})/(1-x)}$;

(k) $\lim\limits_{x\to 0}\dfrac{e^{\alpha x}-e^{\beta x}}{x}$.

1.10.14. Find the limits:

(a) $\lim\limits_{x\to 0}\dfrac{\arccos(1-x)}{\sqrt{x}}$; (b) $\lim\limits_{x\to\pi/4}\dfrac{\ln\tan x}{1-\cot x}$;

(c) $\lim\limits_{x\to 0}\dfrac{1}{\sin x}\ln(1+a\sin x)$.

§ 1.11. *Infinitesimal and Infinite Functions. Their Definition and Comparison*

The function $\alpha(x)$ is called *infinitesimal* as $x\to a$ or as $x\to\infty$ if $\lim\limits_{x\to a}\alpha(x)=0$ or $\lim\limits_{x\to\infty}\alpha(x)=0$.

The function $f(x)$ is called *infinite* as $x\to a$ or as $x\to\infty$ if $\lim\limits_{x\to a}f(x)=\infty$ or $\lim\limits_{x\to\infty}f(x)=\infty$.

A quantity inverse to an infinite quantity is called an *infinitesimal*.

Infinitesimal functions possess the following properties:

(1) The sum and the product of any definite number of infinitesimal functions as $x \to a$ are also infinitesimals as $x \to a$.

(2) The product of an infinitesimal function by a bounded function is an infinitesimal.

Comparison of Infinitesimals. Let the functions $\alpha(x)$ and $\beta(x)$ be infinitesimal as $x \to a$. If

$$\lim_{x \to a} \frac{\alpha(x)}{\beta(x)} = c,$$

where c is a certain finite number different from zero, then the functions $\alpha(x)$ and $\beta(x)$ are called infinitesimals of the *same order*. If $c=1$, then the functions $\alpha(x)$ and $\beta(x)$ are called *equivalent*; notation: $\alpha(x) \sim \beta(x)$.

If $c=0$, then the function $\alpha(x)$ is called an infinitesimal of a *higher order* relative to $\beta(x)$, which is written thus: $\alpha(x) = o(\beta(x))$, and $\beta(x)$ is called an infinitesimal of a *lower order* with respect to $\alpha(x)$.

If $\lim\limits_{x \to a} \frac{\alpha(x)}{[\beta(x)]^n} = c$, where $0 < |c| < +\infty$, then the function $\alpha(x)$ is called an infinitesimal of the nth *order* as compared with the function $\beta(x)$. The concept of infinite functions of various orders is introduced similarly.

1.11.1. Prove that the functions

(a) $f(x) = \frac{2x-4}{x^2+5}$ as $x \to 2$,

(b) $f(x) = (x-1)^2 \sin^3 \frac{1}{x-1}$ as $x \to 1$ are infinitesimals.

Solution. (a) It is sufficient to find the limit

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{2x-4}{x^2+5} = 0.$$

(b) Firstly, the function $\varphi(x) = (x-1)^2$ is infinitesimal as $x \to 1$; indeed, $\lim\limits_{x \to 1} (x-1)^2 = 0$. Secondly, the function

$$\psi(x) = \sin^3 \frac{1}{x-1}; \quad x \neq 1,$$

is bounded:

$$\left| \sin^3 \frac{1}{x-1} \right| \leqslant 1.$$

Hence, the given function $f(x)$ represents the product of the bounded function $\psi(x)$ by the infinitesimal $\varphi(x)$, which means that $f(x)$ is an infinitesimal function as $x \to 1$.

1.11.2. Prove that the functions

(a) $f(x)=\frac{3x-12}{2x^2+7}$ as $x \to 4$;

(b) $f(x)=\frac{\sin x}{x}$ as $x \to \infty$

are infinitesimal.

1.11.3. Find

$$\lim_{x \to 0} x \sin(1/x).$$

Solution. Since x is an infinitesimal as $x \to 0$ and the function $\sin(1/x)$ is bounded, the product $x\sin(1/x)$ is an infinitesimal, which means that $\lim_{x \to 0} x\sin(1/x)=0$.

1.11.4. Compare the following infinitesimal functions (as $x \to 0$) with the infinitesimal $\varphi(x)=x$:

(a) $f_1(x)=\tan x^3$; (b) $f_2(x)=\sqrt[3]{\sin^2 x}$;

(c) $f_3(x)=\sqrt{9+x}-3$.

Solution. (a) We have

$$\lim_{x \to 0} \frac{\tan x^3}{x} = \lim_{x \to 0} \left[\frac{\tan x^3}{x^3} x^2\right] = \lim_{x \to 0} \frac{\tan x^3}{x^3} \lim_{x \to 0} x^2 = 0.$$

Hence, $\tan x^3$ is an infinitesimal of a higher order relative to x.

(b) We have

$$\lim_{x \to 0} \frac{\sqrt[3]{\sin^2 x}}{x} = \lim_{x \to 0} \left[\sqrt[3]{\frac{\sin^2 x}{x^2}} \frac{1}{\sqrt[3]{x}}\right] = \infty.$$

Hence, $\sqrt[3]{\sin^2 x}$ is an infinitesimal of a lower order as compared with x.

(c) We have

$$\lim_{x \to 0} \frac{\sqrt{9+x}-3}{x} = \lim_{x \to 0} \frac{1}{\sqrt{9+x}+3} = \frac{1}{6}.$$

Hence, the infinitesimals $\sqrt{9+x}-3$ and x are of the same order.

1.11.5. Determine the order of smallness of the quantity β with respect to the infinitesimal α.

(a) $\beta=\cos\alpha-\cos 2\alpha$; (b) $\beta=\tan\alpha-\sin\alpha$.

Solution. (a) $\beta=\cos\alpha-\cos 2\alpha=2\sin\frac{3}{2}\alpha\sin\frac{\alpha}{2}$.

Whence

$$\lim_{\alpha \to 0} \frac{\beta}{\alpha^2} = \lim_{\alpha \to 0} \frac{2\sin(3\alpha/2)\sin(\alpha/2)}{\alpha^2} = \frac{3}{2}.$$

Hence, β is an infinitesimal of the same order as α^2, i. e. of the second one with respect to α.

1.11.6. Assuming $x \to \infty$, compare the following infinitely large quantities:

(a) $f(x) = 3x^2 + 2x + 5$ and $\varphi(x) = 2x^3 + 2x - 1$;
(b) $f(x) = 2x^2 + 3x$ and $\varphi(x) = (x+2)^2$;
(c) $f(x) = \sqrt[3]{x+a}$ and $\varphi(x) = \sqrt[3]{x}$.

Solution. (a) The infinite function $3x^2 + 2x + 5$ is of a lower order as compared with the infinite function $2x^3 + 2x - 1$, since

$$\lim_{x \to \infty} \frac{3x^2+2x+5}{2x^3+2x-1} = \lim_{x \to \infty} \frac{3/x+2/x^2+5/x^3}{2+2/x^2-1/x^3} = 0.$$

1.11.7. Prove that the infinitesimals $\alpha = x$ and $\beta = x\cos(1/x)$ (as $x \to 0$) are not comparable, i. e. their ratio has no limit.

Solution. Indeed, $\lim\limits_{x \to 0} \frac{x\cos(1/x)}{x} = \lim\limits_{x \to 0} \cos(1/x)$ does not exist (prove it!), which means that these infinitesimal functions are not comparable.

1.11.8. If $x \to 0$, then which of the following infinitesimals is (are) of a higher order than x; of a lower order than x; of the same order as x?

(a) $100x$; (b) x^2; (c) $6\sin x$; (d) $\sin^3 x$; (e) $\sqrt[3]{\tan^3 x}$.

1.11.9. Let $x \to 0$. Determine the orders of the following infinitesimal functions with respect to x:

(a) $2\sin^4 x - x^5$; (b) $\sqrt{\sin^2 x + x^4}$;
(c) $\sqrt{1+x^3} - 1$; (d) $\sin 2x - 2\sin x$;
(e) $1 - 2\cos\left(x + \frac{\pi}{3}\right)$; (f) $2\sqrt{\sin x}$;
(g) $\frac{x}{x-1}$; (h) $\tan x + x^2$;
(i) $\cos x - \sqrt[3]{\cos x}$; (j) $e^x - \cos x$.

1.11.10. Assuming the side of a cube to be an infinitesimal, determine the order of smallness of the diagonal of the cube (d); of the area of its surface (S); of its volume (V).

§ 1.12. Equivalent Infinitesimals. Application to Finding Limits

If the functions $\alpha(x)$ and $\beta(x)$ are infinitesimal as $x \to a$ and if $\alpha(x) \sim \gamma(x)$, $\beta(x) \sim \delta(x)$, then

$\lim\limits_{x \to a} \frac{\alpha(x)}{\beta(x)} = \lim\limits_{x \to a} \frac{\gamma(x)}{\delta(x)}$ *(replacing an infinitesimal by an equivalent one).*

If

$$\lim_{x \to a} f(x) = k, \quad 0 < |k| < \infty,$$

then

$$f(x)\alpha(x) \sim k\alpha(x).$$

If

$$\alpha(x) \sim \gamma(x),$$
$$\beta(x) \sim \gamma(x),$$

then

$$\alpha(x) \sim \beta(x).$$

For two infinitesimal functions to be equivalent it is necessary and sufficient that their difference be an infinitesimal of a higher order as compared with each of the two.

Listed below are infinitesimal functions:

($\alpha(x)$ is an infinitesimal as $x \to 0$)

(1) $\sin\alpha(x) \sim \alpha(x)$; (2) $\tan\alpha(x) \sim \alpha(x)$;
(3) $1-\cos\alpha(x) \sim [\alpha(x)]^2/2$;
(4) $\arcsin\alpha(x) \sim \alpha(x)$; (5) $\arctan\alpha(x) \sim \alpha(x)$;
(6) $\ln[1+\alpha(x)] \sim \alpha(x)$; (7) $a^{\alpha(x)}-1 \sim \alpha(x)\ln a$ ($a > 0$), in particular, $e^{\alpha(x)}-1 \sim \alpha(x)$;
(8) $[1+\alpha(x)]^P-1 \sim P\alpha(x)$, in particular, $\sqrt[n]{1+\alpha(x)}-1 \sim \frac{\alpha(x)}{n}$.

1.12.1. Prove that as $x \to 0$

(a) $1-\frac{1}{\sqrt{1+x}} \sim \frac{1}{2}x$; (b) $1-\frac{1}{1+x} \sim x$;

(c) $\sin\sqrt{x\sqrt{x}} \sim \sqrt{x^2+\sqrt{x^3}}$.

Solution. (a) By formula (8) at $P = 1/2$ we have

$$1-\frac{1}{\sqrt{1+x}} = \frac{1}{\sqrt{1+x}}(\sqrt{1+x}-1) \sim 1\cdot\frac{1}{2}x.$$

(c) By formula (1) we have

$$\sin\sqrt{x\sqrt{x}} \sim \sqrt{x\sqrt{x}} = x^{3/4},$$
$$\sqrt{x^2+\sqrt{x^3}} = x^{3/4}\sqrt{1+x^{1/2}} \sim x^{3/4},$$

whence $\sin\sqrt{x\sqrt{x}} \sim \sqrt{x^2+\sqrt{x^3}}$.

1.12.2. Replace each of the following infinitesimals with an equivalent one:

(a) $3\sin\alpha-5\alpha^3$; (b) $(1-\cos\alpha)^2+16\alpha^3+5\alpha^4+6\alpha^5$.

Solution. (*a*) Note that the sum of two infinitesimals α and β of different orders is equivalent to the summand of the lower order, since the replacement of an infinitesimal with one equivalent to it is tantamount to the rejection of an infinitesimal of a higher order.

In our example the quantity $3\sin\alpha$ has the order of smallness 1, $(-5\alpha^3)$ — the order of smallness 3, hence

$$3\sin\alpha+(-5\alpha^3)\sim 3\sin\alpha\sim 3\alpha.$$

(b) $(1-\cos\alpha)^2+16\alpha^3+5\alpha^4+6\alpha^5=4\sin^4\frac{\alpha}{2}+16\alpha^3+5\alpha^4+6\alpha^5.$

The summand $16\alpha^3$ is of the lower order, therefore

$$(1-\cos\alpha)^2+16\alpha^3+5\alpha^4+6\alpha^5\sim 16\alpha^3.$$

1.12.3. With the aid of the principle of substitution of equivalent quantities find the limits:

(a) $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\ln(1+4x)}$; (b) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{1-\cos\frac{x}{2}}$;

(c) $\lim\limits_{x\to 0}\dfrac{\ln\cos x}{\sqrt[4]{1+x^2}-1}$; (d) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x+x^2}-1}{\sin 4x}$;

(e) $\lim\limits_{x\to 0}\dfrac{\sin 2x+\arcsin^2 x-\arctan^2 x}{3x}$;

(f) $\lim\limits_{x\to 0}\dfrac{3\sin x-x^2+x^3}{\tan x+2\sin^2 x+5x^4}$;

(g) $\lim\limits_{x\to 0}\dfrac{(\sin x-\tan x)^2+(1-\cos 2x)^4+x^5}{7\tan^7 x+\sin^6 x+2\sin^5 x}$;

(h) $\lim\limits_{x\to 0}\dfrac{\sin\sqrt[3]{x}\,\ln(1+3x)}{(\arctan\sqrt{x})^2\left(e^{5\sqrt[3]{x}}-1\right)}$;

(i) $\lim\limits_{x\to 0}\dfrac{1-\cos x+2\sin x-\sin^3 x-x^2+3x^4}{\tan^3 x-6\sin^2 x+x-5x^3}$.

Solution. (a) We have $\sin 5x\sim 5x$; $\ln(1+4x)\sim 4x$ (see the list of equivalent infinitesimals on page 72). Therefore

$$\lim_{x\to 0}\frac{\sin 5x}{\ln(1+4x)}=\lim_{x\to 0}\frac{5x}{4x}=\frac{5}{4}.$$

(c) $$\lim_{x\to 0}\frac{\ln\cos x}{\sqrt[4]{1+x^2}-1}=\lim_{x\to 0}\frac{\ln[1+(\cos x-1)]}{x^2/4}=$$

$$=4\lim_{x\to 0}\frac{\cos x-1}{x^2}=-4\lim_{x\to 0}\frac{x^2/2}{x^2}=-2.$$

(d) From the list of equivalent infinitesimals we find:

$$\sqrt{1+x+x^2}-1\sim(x+x^2)/2\sim x/2,\ \sin 4x\sim 4x.$$

Therefore

$$\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1}{\sin 4x}=\lim_{x\to 0}\frac{x/2}{4x}=\frac{1}{8}.$$

(e) Using the list of equivalent infinitesimal functions given on page 72 we obtain

$$\sin 2x+\arcsin^2 x-\arctan^2 x\sim\sin 2x\sim 2x.$$

Hence,

$$\lim_{x\to 0}\frac{\sin 2x+\arcsin^2 x-\arctan^2 x}{3x}=\lim_{x\to 0}\frac{2x}{3x}=\frac{2}{3}.$$

(h) $\sin\sqrt[3]{x}\sim\sqrt[3]{x}$; $\ln(1+3x)\sim 3x$;

$$\arctan\sqrt{x}\sim\sqrt{x};\quad e^{5\sqrt[3]{x}}-1\sim 5\sqrt[3]{x};$$

$$\lim_{x\to 0}\frac{\sin\sqrt[3]{x}\ln(1+3x)}{(\arctan\sqrt{x})^2\left(e^{5\sqrt[3]{x}}-1\right)}=\lim_{x\to 0}\frac{\sqrt[3]{x}\cdot 3x}{x\cdot 5\sqrt[3]{x}}=\frac{3}{5}.$$

1.12.4. Find the approximate values of the roots $\sqrt{1.02}$ and $\sqrt{0.994}$. Estimate the absolute error.

Solution. Use the approximate formula

$$\sqrt{1+x}\sim 1+x/2 \qquad (*)$$

(for x sufficiently close to zero). In our case

$$\sqrt{1+0.02}\sim 1+\frac{0.02}{2}=1.01;$$

$$\sqrt{1-0.006}\sim 1-\frac{0.006}{2}=0.997.$$

To estimate the error we note that

$$\frac{x}{2}-(\sqrt{1+x}-1)=\frac{1}{2}(x-2\sqrt{1+x}+2)=$$

$$=\frac{1}{2}(x+1-2\sqrt{x+1}+1)=\frac{1}{2}(\sqrt{x+1}-1)^2\sim\frac{1}{2}\left(\frac{x}{2}\right)^2=\frac{x^2}{8}.$$

Hence, the absolute error of the approximate formula (*) is estimated by the quantity $\frac{x^2}{8}$.

Using this estimate we find that the absolute error of the root $\sqrt{1.02}\approx 1.01$ is $\approx\frac{(0.02)^2}{8}=0.00005$, and the absolute error of $\sqrt{0.994}\approx 0.997$ amounts to $\approx\frac{(0.006)^2}{8}\approx 0.000005$.

1.12.5. Prove that, as $x\to 0$,

(a) $\sqrt[3]{1+x}-1\sim\frac{1}{3}x$;

(b) $\arctan mx \sim mx$;

(c) $1-\cos^3 x \sim \frac{3}{2}\sin^2 x$.

1.12.6. For $x \to 0$ determine the order of smallness, relative to the infinitesimal $\beta(x)=x$, of the following infinitesimals:

$$\text{(a) } \sqrt{\sin^2 x + x^4}; \quad \text{(b) } \frac{x^2(1+x)}{1+\sqrt[3]{x}}.$$

1.12.7. For $x \to 2$ determine the order of smallness, relative to the infinitesimal $\beta(x)=x-2$, of the following infinitesimals:

$$\text{(a) } 3(x-2)^2+2(x^2-4); \quad \text{(b) } \sqrt[3]{\sin \pi x}.$$

1.12.8. Making use of the method of replacing an infinitesimal with an equivalent one, find the following limits:

(a) $\lim\limits_{x\to 0}\frac{\sin 3x}{\ln(1+5x)}$;

(b) $\lim\limits_{x\to 0}\frac{\ln(1+\sin 4x)}{e^{\sin 5x}-1}$;

(c) $\lim\limits_{x\to 0}\frac{e^{\sin 3x}-1}{\ln(1+\tan 2x)}$;

(d) $\lim\limits_{x\to 0}\frac{\arctan 3x}{\arcsin 2x}$;

(e) $\lim\limits_{x\to 0}\frac{\ln(2-\cos 2x)}{\ln^2(\sin 3x+1)}$;

(f) $\lim\limits_{x\to 0}\frac{\sqrt{1+\sin 3x}-1}{\ln(1+\tan 2x)}$;

(g) $\lim\limits_{x\to 0}\frac{\ln(1+2x-3x^2+4x^3)}{\ln(1-x+2x^2-7x^3)}$;

(h) $\lim\limits_{x\to 0}\frac{\sqrt{1+x^2}-1}{1-\cos x}$.

1.12.9. Find an approximate value of the root $\sqrt[3]{1042}$.

§ 1.13. One-Sided Limits

A number A is called the *limit to the right of the function* $f(x)$ as $x \to x_0$ ($A = \lim\limits_{x\to x_0+0} f(x) = f(x_0+0)$) if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all x satisfying the inequality $0 < x-x_0 < \delta(\varepsilon)$ and belonging to the domain of definition of the function $f(x)$ the inequality $|f(x)-A| < \varepsilon$ holds true. The limit to the left of the function $f(x_0-0)$ as $x \to x_0-0$ is defined in a similar way. If $x_0=0$, then we write simply $x \to +0$ or $x \to -0$ and, respectively, $f(+0)$ and $f(-0)$.

1.13.1. Find the one-sided limits of the functions:

(a) $f(x)=\begin{cases} -2x+3 & \text{if } x \leqslant 1, \\ 3x-5 & \text{if } x > 1 \end{cases}$ as $x \to 1$;

(b) $f(x)=\frac{x^2-1}{|x-1|}$ as $x \to 1$;

(c) $f(x) = \frac{\sqrt{1-\cos 2x}}{x}$ as $x \to 0$;

(d) $f(x) = 3 + \frac{1}{1+7^{1/(1-x)}}$ as $x \to 1$;

(e) $f(x) = \cos(\pi/x)$ as $x \to 0$;

(f) $f(x) = 5/(x-2)^3$ as $x \to 2$.

Solution. (a) Let $x \leqslant 1$. Then $f(x) = -2x+3$. Hence, $f(1-0) = \lim\limits_{x \to 1-0} f(x) = 1$ is the limit to the left.

If $x > 1$, then $f(x) = 3x-5$; hence, $f(1+0) = \lim\limits_{x \to 1+0} f(x) = -2$ is the limit to the right (see Fig. 28).

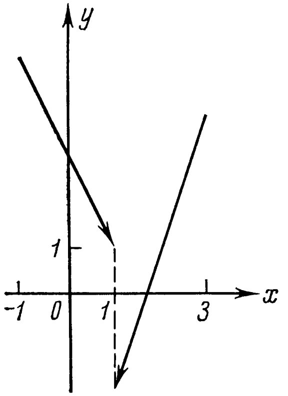

Fig. 28

(c) $f(x) = \frac{\sqrt{1-\cos 2x}}{x} = \frac{\sqrt{2\sin^2 x}}{x} = \frac{\sqrt{2}\,|\sin x|}{x}$,

but

$$|\sin x| = \begin{cases} \sin x, & \text{if } 0 < x < \pi/2, \\ -\sin x, & \text{if } -\pi/2 < x < 0. \end{cases}$$

Hence,

$$f(-0) = \lim_{x \to -0} f(x) = \lim_{x \to -0}\left(-\sqrt{2}\,\frac{\sin x}{x}\right) = -\sqrt{2},$$

$$f(+0) = \lim_{x \to +0} f(x) = \lim_{x \to +0}\left(\sqrt{2}\,\frac{\sin x}{x}\right) = \sqrt{2}.$$

(d) The expression $1/(1-x)$ tends to $+\infty$, when x tends to 1, remaining less than 1, therefore

$$\lim_{x \to 1-0} 7^{1/(1-x)} = +\infty, \quad \lim_{x \to 1-0} \frac{1}{1+7^{1/(1-x)}} = 0, \quad f(1-0) = 3.$$

Further, as $x \to 1+0$ we have $1/(1-x) \to -\infty$. Therefore $\lim\limits_{x \to 1+0} 7^{1/(1-x)} = 0$,

$$f(1+0) = \lim_{x \to 1+0}\left(3 + \frac{1}{1+7^{1/(1-x)}}\right) = 3+1 = 4.$$

(e) Let us choose two sequences, $\{x_n\}$ and $\{x_n'\}$, with the general terms

$$x_n = \frac{1}{2n} \text{ and } x_n' = \frac{2}{2n+1} \quad (n = 1, 2, \ldots)$$

respectively.

Then $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x'_n = 0$ and

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \cos 2\pi n = 1;$$

$$\lim_{n\to\infty} f(x'_n) = \lim_{n\to\infty} \cos (2n+1)\frac{\pi}{2} = 0.$$

Hence, the function $f(x)$ has no limit to the right at the point 0; taking into account that $f(x)$ is an even function, we conclude that it has no limit to the left either (see Fig. 29).

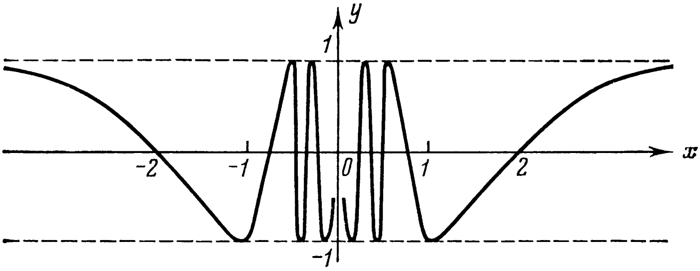

Fig. 29

1.13.2. Prove that, as $x \to 1$, the function

$$f(x) = \begin{cases} x+1 & \text{at } 0 \leqslant x < 1, \\ 3x+2 & \text{at } 1 < x < 3 \end{cases}$$

has a limit to the left equal to 2 and a limit to the right equal to 5.

1.13.3. Find the one-sided limits of the following functions as $x \to 0$:

(a) $f(x) = \dfrac{1}{2-2^{1/x}}$;

(b) $f(x) = e^{1/x}$;

(c) $f(x) = \dfrac{|\sin x|}{x}$.

§ 1.14. Continuity of a Function. Points of Discontinuity and Their Classification

Let the function $y = f(x)$ be defined on the set X and let the point $x_0 \in X$ be the limit point of this set. The function $f(x)$ is said to be *continuous at the point* x_0 if $\lim\limits_{x\to x_0} f(x) = f(x_0)$. The latter condition is equivalent to the condition $\lim\limits_{\Delta x\to 0} \Delta y(x_0) = \lim\limits_{\Delta x\to 0} [f(x_0+\Delta x) - f(x_0)] = 0$.

The function $f(x)$ is continuous at the point x_0 if and only if $f(x_0-0)=f(x_0+0)=f(x_0)$.

The function $f(x)$ is *continuous on the set* X if it is continuous at every point of this set.

Points of Discontinuity of the First Kind. Let the point x_0 be the limit point of the domain of definition X of the function $f(x)$. The point x_0 is called a *discontinuity of the first kind* of the function $f(x)$ if there exist the limits to the right and to the left and they are finite. If $f(x_0-0)=f(x_0+0)\neq f(x_0)$, then x_0 is called a *removable discontinuity*. Further, if $f(x_0-0)\neq f(x_0+0)$, then x_0 is a non-removable discontinuity of the first kind, and the difference $f(x_0+0)-f(x_0-0)$ is called a *jump discontinuity* of the function $f(x)$ at the point x_0.

Points of Discontinuity of the Second Kind. If at least one of the limits of $f(x_0-0)$ and $f(x_0+0)$ is non-existent and infinite, then point x_0 is called a *discontinuity of the second kind* of the function $f(x)$.

1.14.1. Using only the definition prove discontinuity of the function $f(x)=3x^4+5x^3+2x^2+3x+4$ at any x.

Solution. Let x_0 be an arbitrary point on the number scale. First find $\lim\limits_{x\to x_0} f(x)$:

$$\lim_{x\to x_0} f(x)=\lim_{x\to x_0}(3x^4+5x^3+2x^2+3x+4)=3x_0^4+5x_0+2x_0^2+3x_0+4.$$

Then compute the value of the function at the point x_0:

$$f(x_0)=3x_0^4+5x_0^3+2x_0^2+3x_0+4.$$

Comparing the results thus obtained, we see that

$$\lim_{x\to x_0} f(x_0)=f(x_0).$$

Hence, the function $f(x)$ is continuous at the point x_0 by definition. Since x_0 is an arbitrary point on the number scale, we have proved continuity of the function for all values of x.

1.14.2. Given the functions:

$$\text{(a) } f(x)=\begin{cases}\frac{1}{5}(2x^2+3) & \text{for } -\infty<x\leqslant 1,\\ 6-5x & \text{for } 1<x<3,\\ x-3 & \text{for } 3\leqslant x<\infty;\end{cases}$$

$$\text{(b) } f(x)=\begin{cases}-2x^2 & \text{for } x\leqslant 3,\\ 3x & \text{for } x>3;\end{cases}$$

$$\text{(c) } f(x)=\frac{|2x-3|}{2x-3}.$$

Find the points of discontinuity (if any). Determine the jump discontinuities of the functions at the points of discontinuity of the first kind.

Solution. (a) The domain of definition of the function is the entire number scale $(-\infty, \infty)$. In the open intervals $(-\infty, 1)$, $(1, 3)$, $(3, \infty)$ the function is continuous. Therefore discontinuities are possible only at the points $x=1$, $x=3$, at which analytic representation of the function is changed.

Let us find the one-sided limits of the function at the point $x=1$:

$$f(1-0)=\lim_{x\to 1-0}\frac{1}{5}(2x^2+3)=1;$$
$$f(1+0)=\lim_{x\to 1+0}(6-5x)=1.$$

The value of the function at the point $x=1$ is determined by the first analytic representation, i. e. $f(1)=(2+3)/5=1$. Since

$$f(1-0)=f(1+0)=f(1),$$

the function is continuous at the point $x=1$.

Consider the point $x=3$:

$$f(3-0)=\lim_{x\to 3-0}(6-5x)=-9;$$
$$f(3+0)=\lim_{x\to 3+0}(x-3)=0.$$

We see that the right-hand and the left-hand limits, though finite, are not equal to each other, therefore the function has a discontinuity of the first kind at the point $x=3$.

The jump of the function at the point of discontinuity is $f(3+0)-f(3-0)=0-(-9)=9$.

(c) The function is defined and continuous throughout the entire number scale, except at the point $x=3/2$. Since $2x-3>0$ for $x>3/2$ and $2x-3<0$ for $x<3/2$,

$$f(x)=\begin{cases} 1 \text{ at } x>3/2, \\ -1 \text{ at } x<3/2. \end{cases}$$

Hence,

$$f(3/2+0)=1,\quad f(3/2-0)=-1.$$

Therefore, at the point $x=3/2$ the function has a finite discontinuity of the first kind. The jump of the function at this point $f(3/2+0)-f(3/2-0)$ is equal to $1-(-1)=2$.

1.14.3. Test the following functions for continuity:

(a) $f(x)=\begin{cases} \dfrac{\sin x}{x} \text{ for } x\neq 0, \\ 1 \quad \text{for } x=0; \end{cases}$

(b) $f(x) = \sin(1/x)$;

(c) $f(x) = \begin{cases} x\sin(1/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0; \end{cases}$

(d) $f(x) = \begin{cases} 4\cdot 3^x & \text{for } x < 0, \\ 2a + x & \text{for } x \geqslant 0; \end{cases}$

(e) $f(x) = \arctan(1/x)$; (f) $f(x) = (x^3+1)/(x+1)$.

Solution. (a) The function is continuous at all points $x \neq 0$. At the point $x = 0$ we have

$$f(0) = 1; \quad \lim_{x \to -0} \frac{\sin x}{x} = \lim_{x \to +0} \frac{\sin x}{x} = 1.$$

Hence, at this point the function is continuous as well, which means that it is continuous for all values of x.

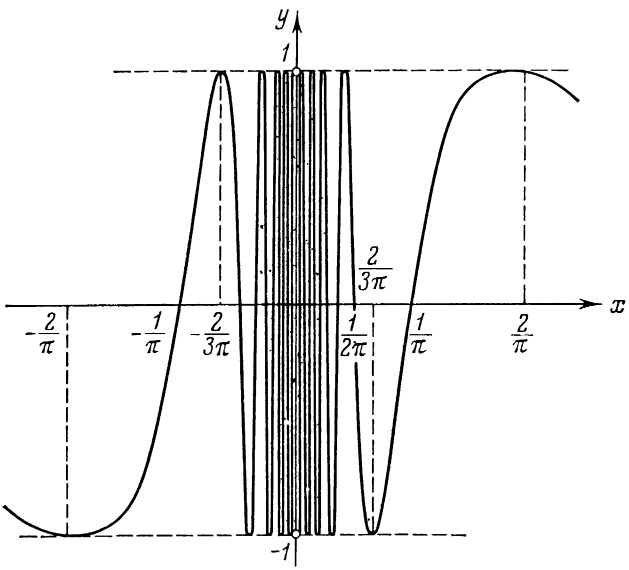

Fig. 30

(b) The function is defined and continuous for all $x \neq 0$. There are no one-sided limits at the point $x = 0$ (cf. Problem **1.13.1** (e)). Therefore, at the point $x = 0$ the function suffers a discontinuity of the second kind (see Fig. 30).

(d) $f(-0) = 4$, and $f(+0) = 2a$; the equality $f(-0) = f(+0) = f(0)$ will be fulfilled, i. e. the function $f(x)$ will be continuous at the point $x = 0$ if we put $2a = 4$, $a = 2$.

(f) $f(-1-0) = f(-1+0) = \lim\limits_{x \to -1} (x^2 - x + 1) = 3$, i.e. both one-sided limits are finite and coincide. But at the point $x = -1$ the function is not defined and, therefore, is not continuous. The graph of the function is the parabola $y=x^2-x+1$ with the point $M(-1, 3)$ removed. If we redefine the function putting $f(-1)=3$, then it will become continuous. Thus, at $x=-1$ the function has a removable discontinuity.

1.14.4. Test the following functions for continuity:

(a) $f(x)=E(x)$. It should be borne in mind that the function $E(x)$ is defined as the maximum integer n contained in the number x, i. e. as a number satisfying the inequality $n \leqslant x$.

(b)

$$\lambda(x)=\begin{cases} 1 \text{ if } x \text{ is rational,} \\ 0 \text{ if } x \text{ is irrational.} \end{cases}$$

$\lambda(x)$ is called the *Dirichlet function*. For instance, $\lambda(0)=1$; $\lambda(-1/2)=1$; $\lambda(\sqrt{2})=0$; $\lambda(\pi)=0$, etc.

Solution. (a) The function $E(x)$ is defined throughout the entire number scale and takes on only integral values. This function is discontinuous at every integral value n of the independent variable, since $E(n-0)=n-1$; $E(n+0)=n$ (see Fig. 31).

(b) Let us choose an arbitrary point x_0 on the x-axis; two cases are possible: (1) the number x_0 is rational; (2) the number x_0 is irrational.

In the first case $\lambda(x_0)=1$. In any vicinity of a rational point there are irrational points, where $\lambda(x)=0$. Hence, in any vicinity of x_0 there are points x for which

$$|\Delta y| = |\lambda(x_0)-\lambda(x)| = 1.$$

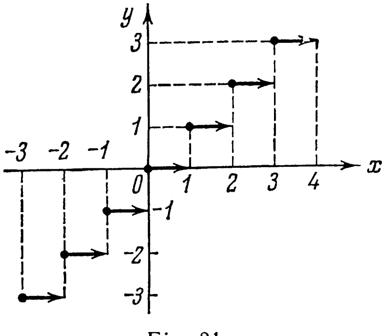

Fig. 31

In the second case $\lambda(x_0)=0$. In any vicinity of an irrational point there are rational points at which $\lambda(x)=1$. Hence, it is possible to find the values of x for which

$$|\Delta y| = |\lambda(x_0)-\lambda(x)| = 1.$$

Thus, in both cases the difference Δy does not tend to zero as $\Delta x \to 0$. Therefore, x_0 is a discontinuity. Since x_0 is an arbitrary point, the Dirichlet function $\lambda(x)$ is *discontinuous at each point*. The graph of this function consists of a set of points with irrational abscissas on the x-axis and of a set of points with rational abscissas on the straight line $y=1$, that is why it is impossible to sketch it.

1.14.5. Using the definition of continuity of a function in terms of "$\varepsilon-\delta$", test the following functions for continuity:

(a) $f(x)=ax+b \quad (a\neq 0)$;

(b) $f(x)=\begin{cases} x^2 & \text{if } x \text{ is rational,} \\ -x^2 & \text{if } x \text{ is irrational.} \end{cases}$

Solution. (a) Choose an arbitrary point x_0. According to the "$\varepsilon-\delta$" definition it is necessary to show that for any preassigned, arbitrarily small number $\varepsilon>0$ it is possible to find a number $\delta>0$ such that at $|x-x_0|<\delta$ the inequality $|f(x)-f(x_0)|<\varepsilon$ holds true.

Consider the absolute value of the difference

$$|f(x)-f(x_0)|=|(ax+b)-(ax_0+b)|=|ax+b-ax_0-b|=|a||x-x_0|.$$

Let us require that $|f(x)-f(x_0)|<\varepsilon$. This requirement will be fulfilled for all x satisfying the inequality

$$|a||x-x_0|<\varepsilon \text{ or } |x-x_0|<\varepsilon/|a| \quad (a\neq 0).$$

Hence, if we take $\delta\leqslant\varepsilon/|a|$, then at $|x-x_0|<\delta$ the inequality $|f(x)-f(x_0)|<\varepsilon$ is fulfilled. Continuity is thus proved for any point $x=x_0$.

(b) Choose an arbitrary point x_0. If $\{x_n\}$ is a sequence of rational numbers tending to x_0, then $\lim\limits_{x_n\to x_0} f(x_n)=x_0^2$. If $\{x_n'\}$ is a sequence of irrational numbers tending to x_0, then $\lim\limits_{x_n'\to x_0} f(x_n')=-x_0^2$. At $x_0\neq 0$ the indicated limits are different and hence the function is discontinuous at all points $x\neq 0$.

On the other hand, let now $x=0$. Find the absolute value of the difference $|f(x)-f(0)|$:

$$|f(x)-f(0)|=|\pm x^2-0|=x^2.$$

It is obvious that $x^2<\varepsilon$ at $|x|<\sqrt{\varepsilon}$. If $\varepsilon>0$ is given, then, putting $\delta\leqslant\sqrt{\varepsilon}$ and $|x-0|=|x|<\delta$, we obtain $|\Delta f(0)|=x^2<\varepsilon$. Hence, at the point $x=0$ the function is continuous. And so, the point $x=0$ is the only point at which the function is continuous. Note that the function under consideration can be expressed through the Dirichlet function (see Problem **1.14.4** (b)): $f(x)=x^2[2\lambda(x)-1]$.

1.14.6. Determine which kind of discontinuity the following functions have at the point $x=x_0$:

(a) $f(x)=\begin{cases} x+2 & \text{for } x<2, \\ x^2-1 & \text{for } x\geqslant 2; \end{cases} \quad x_0=2;$

(b) $f(x)=\arctan\dfrac{1}{x-5}$; $x_0=5$; (c) $f(x)=\dfrac{1}{1+2^{1/x}}$; $x_0=0$;

(d) $f(x)=\tan x$; $x_0=\pi/2$;

(e) $f(x)=\sqrt{x}-E(\sqrt{x})$; $x_0=n^2$, where n is a natural number.

Solution. (a) Find the one-sided limits at the point $x_0=2$;

$$f(2-0)=\lim_{x\to 2-0}(x+2)=4;$$

$$f(2+0)=\lim_{x\to 2+0}(x^2-1)=3.$$

Here the limits to the right and to the left exist, are finite but do not coincide, therefore the function has a discontinuity of the first kind at the point $x_0=2$.

(e) The function $E(\sqrt{x})$ has discontinuities of the first kind at every point $x=n^2$, where n is a natural number (see Problem **1.14.4** (a)), whereas the function $\sqrt{x}$ is continuous at all $x\geqslant 0$. Therefore the function $f(x)=\sqrt{x}-E(\sqrt{x})$ has discontinuities of the first kind at the points $1, 4, 9, \ldots, n^2, \ldots$

1.14.7. Test the following functions for continuity

(a) $f(x)=\dfrac{e^x-1}{x}$;

(b) $f(x)=\begin{cases}\dfrac{e^x-1}{x} & \text{for } x\neq 0,\\ 3 & \text{for } x=0;\end{cases}$

(c) $f(x)=\begin{cases}e^{1/x} & \text{for } x\neq 0,\\ 0 & \text{for } x=0;\end{cases}$

(d) $f(x)=\lim\limits_{n\to\infty}(\sin x)^{2n}$; (e) $f(x)=\dfrac{|\sin x|}{\sin x}$;

(f) $f(x)=E(x)+E(-x)$.

1.14.8. For each of the following functions find the points of discontinuity and determine the jumps of the function at these points:

(a) $f(x)=\dfrac{4}{x^2-2x+1}$;

(b) $f(x)=x+\dfrac{x+2}{|x+2|}$;

(c) $f(x)=\dfrac{2|x-1|}{x^2-x^3}$;

(d) $f(x)=\begin{cases}-x & \text{for } x\leqslant 1,\\ \dfrac{2}{x-1} & \text{for } x>1.\end{cases}$

1.14.9. Redefine the following functions at the point $x=0$ so as to make them continuous:

(a) $f(x)=\dfrac{\tan x}{x}$;

(b) $f(x)=\frac{5x^2-3x}{2x}$;

(c) $f(x)=\frac{\sqrt{1+x}-1}{x}$;

(d) $f(x)=\frac{\sin^2 x}{1-\cos x}$.

§ 1.15. Arithmetical Operations on Continuous Functions. Continuity of a Composite Function

If the functions $f(x)$ and $g(x)$ are continuous at the point $x=x_0$, then the functions

$$(1)\ f(x)\pm g(x);\quad (2)\ f(x)\cdot g(x);\quad (3)\ \frac{f(x)}{g(x)}\ (g(x_0)\neq 0)$$

are also continuous at this point.

If the function $u=\varphi(x)$ is continuous at the point $x=x_0$ and the function $y=f(u)$ is continuous at the point $u_0=\varphi(x_0)$, then the composite function $y=f[\varphi(x)]$ is continuous at the point $x=x_0$.

1.15.1. Test the following functions for continuity:

$$\text{(a) } f(x)=\frac{2x^5-8x^2+11}{x^4+4x^3+8x^2+8x+4};$$

$$\text{(b) } f(x)=\frac{3\sin^3 x+\cos^2 x+1}{4\cos x-2};$$

$$\text{(c) } f(x)=\frac{x^3\cos x+x^2\sin x}{\cos(1/\sin x)}.$$

Solution. (a) A function representing a ratio of two continuous functions (polynomials in this case) is discontinuous only at points for which the denominator becomes zero. But in our case

$$x^4+4x^3+8x^2+8x+4=(x^2+2x+2)^2,$$

and since $x^2+2x+2=(x+1)^2+1>0$ at any x, the denominator never becomes zero. Hence, the function $f(x)$ is continuous throughout the entire number scale.

(b) The function $f(x)$ suffers discontinuities only at points for which the denominator equals zero, i.e. at points which are the roots of the equation

$$4\cos x-2=0 \text{ or } \cos x=1/2,$$

whence

$$x=x_n=\pm\pi/3+2\pi n\ (n=0,\ \pm 1,\ \pm 2,\ \ldots).$$

Thus, the function $f(x)$ is continuous everywhere, except at the point x_n.

(c) Just as in the preceding example, the numerator is continuous throughout the entire number scale. As far as the denominator is concerned, according to the theorem on continuity of a composite function, it is continuous at points where the function $u = 1/\sin x$ is continuous, since the function $\cos u$ is continuous everywhere. Hence, the denominator is continuous everywhere, except at the points $x = k\pi$ (k an integer). Besides, we must exclude the points at which $\cos(1/\sin x) = 0$, i.e. the points at which $1/\sin x = (2p+1)\pi/2$ (p an integer), or $\sin x = 2/[(2p+1)\pi]$. Thus, the function $f(x)$ is continuous everywhere except at the points $x = k\pi$ and $x = (-1)^n \arcsin \frac{2}{(2p+1)\pi} + n\pi$ ($k, p, n = 0, \pm 1, \pm 2, \ldots$).

1.15.2. Test the following composite functions for continuity:

(a) $y = \cos x^n$, where n is a natural number;

(b) $y = \cos \log x$;

(c) $y = \sqrt{1/2 - \cos^2 x}$.

Solution. (a) We have a composite function $y = \cos u$, where $u = x^n$. The function $y = \cos u$ is continuous at any point u, and the function $u = x^n$ is continuous at any value of x. Therefore, the function $y = \cos x^n$ is continuous throughout the entire number scale.

(c) Here $y = \sqrt{1/2 - u^2}$, where $u = \cos x$. The function $\sqrt{1/2 - u^2}$ is defined and continuous on the interval $[-\sqrt{2}/2, \sqrt{2}/2]$, the function $u = \cos x$ is continuous throughout the entire number scale. Therefore, the function $y = \sqrt{1/2 - \cos^2 x}$ is continuous at all values of x for which

$$|\cos x| \leqslant \sqrt{2}/2, \quad \text{i.e.} \quad \begin{cases} \pi/4 + 2\pi n \leqslant x \leqslant 3\pi/4 + 2\pi n, \\ 5\pi/4 + 2\pi n \leqslant x \leqslant 7\pi/4 + 2\pi n. \end{cases}$$

1.15.3. For each of the following functions find the points of discontinuity and determine their character:

(a) $y = \frac{1}{u^2 + u - 2}$, where $u = \frac{1}{x-1}$;

(b) $y = u^2$, where $u = \begin{cases} x - 1 & \text{for } x \geqslant 0, \\ x + 1 & \text{for } x < 0; \end{cases}$

(c) $y = \frac{1 - u^2}{1 + u^2}$, where $u = \tan x$.

Solution. (a) The function

$$u = \varphi(x) = \frac{1}{x-1}$$

suffers a discontinuity at the point $x=1$. The function

$$y=f(u)=\frac{1}{u^2+u-2}$$

suffers a discontinuity at points where $u^2+u-2=0$, i.e. $u_1=-2$ and $u_2=1$. Using these values of u, find the corresponding values of x by solving the equations:

$$-2=\frac{1}{x-1}, \quad 1=\frac{1}{x-1};$$

whence $x=1/2$ and $x=2$.

Hence, the composite function is discontinuous at three points: $x_1=1/2$, $x_2=1$, $x_3=2$. Let us find out the character of discontinuities at these points.

$$\lim_{x\to 1} y=\lim_{u\to\infty} y=0,$$

therefore $x_2=1$ is a removable discontinuity.

$$\lim_{x\to 1/2} y=\lim_{u\to -2} y=\infty; \qquad \lim_{x\to 2} y=\lim_{u\to 1} y=\infty;$$

hence, the points $x_1=1/2$, $x_3=2$ are discontinuities of the second kind.

1.15.4. Given the function $f(x)=1/(1-x)$. Find the points of discontinuity of the composite function

$$y=f\{f[f(x)]\}.$$

Solution. The point $x=1$ is a discontinuity of the function

$$v=f(x)=\frac{1}{1-x}.$$

If $x\neq 1$, then

$$u=f[f(x)]=\frac{1}{1-1/(1-x)}=\frac{x-1}{x}.$$

Hence, the point $x=0$ is a discontinuity of the function $u=f[f(x)]$.

If $x\neq 0$, $x\neq 1$, then

$$y=f\{f[f(x)]\}=\frac{1}{1-(x-1)/x}=x$$

is continuous everywhere.

Thus, the points of discontinuity of this composite function are $x=0$, $x=1$, both of them being removable.

§ 1.16. The Properties of a Function Continuous on a Closed Interval. Continuity of an Inverse Function

I. The function $f(x)$, continuous on the interval $[a, b]$, possesses the following properties:

(1) $f(x)$ is bounded on $[a, b]$;

(2) $f(x)$ has the minimum and maximum values on $[a, b]$;

(3) If $m = \min\limits_{a \leqslant x \leqslant b} f(x)$, $M = \max\limits_{a \leqslant x \leqslant b} f(x)$, then for any A satisfying the inequalities $m \leqslant A \leqslant M$ there exists a point $x_0 \in [a, b]$ for which $f(x_0) = A$.

In particular, if $f(a) \cdot f(b) < 0$, then we can find a point c $(a < c < b)$ such that $f(c) = 0$.

II. **Continuity of an Inverse Function.** If the function $y = f(x)$ is defined, continuous and strictly monotonic on the interval X, then there exists a single-valued inverse function $x = \varphi(y)$ defined, continuous and also strictly monotonic in the range of the function $y = f(x)$.

1.16.1. Does the equation $\sin x - x + 1 = 0$ have a root?

Solution. The function

$$f(x) = \sin x - x + 1$$

is continuous over the entire number scale. Besides, this function changes sign, since $f(0) = 1$, and $f(3\pi/2) = -3\pi/2$. Hence, by property (3) within the interval $[0, 3\pi/2]$ there is at least one root of the given equation.

1.16.2. Has the equation $x^5 - 18x + 2 = 0$ roots belonging to the interval $[-1, 1]$?

1.16.3. Prove that any algebraic equation of an odd power with real coefficients

$$a_0x^{2n+1} + a_1x^{2n} + \ldots + a_{2n}x + a_{2n+1} = 0 \qquad (*)$$

has at least one real root.

Solution. Consider the function

$$f(x) = a_0x^{2n+1} + a_1x^{2n} + \ldots + a_{2n}x + a_{2n+1},$$

which is continuous throughout the number scale.

Let, for determinacy sake, $a_0 > 0$. Then

$$\lim_{x \to +\infty} f(x) = +\infty, \text{ and } \lim_{x \to -\infty} f(x) = -\infty.$$

Hence, we can find numbers a, b, $a < b$ such that $f(a) < 0$; $f(b) > 0$. By property (3), between a and b there exists a number c such that $f(c) = 0$, which proves that the equation (*) has at least one real root.

1.16.4. Let the function $f(x)$ be continuous on $[a, b]$ and let the equation $f(x)=0$ have a finite number of roots on the interval $[a, b]$. Arrange them in the ascending order:

$$a < x_1 < x_2 < x_3 < \ldots < x_n < b.$$

Prove that in each of the intervals

$$(a, x_1),\ (x_1, x_2),\ (x_2, x_3),\ \ldots,\ (x_n, b)$$

the function $f(x)$ retains the same sign.

Solution. If the function changed its sign on a certain interval, then we could find one more root of the function, which contradicts the condition. To determine the sign of the function on any of the indicated intervals it is sufficient to compute the value of the function at an arbitrary point of the appropriate interval.

1.16.5. Given a function on the interval $[-2, +2]$:

$$f(x)=\begin{cases} x^2+2 & \text{if } -2 \leqslant x < 0, \\ -(x^2+2) & \text{if } \quad 0 \leqslant x \leqslant 2. \end{cases}$$

Is there a point on this closed interval at which $f(x)=0$?

Solution. At the end-points of the interval $[-2, +2]$ the given function has different signs:

$$f(-2)=+6;\ f(+2)=-6.$$

But it is easy to notice that it does not become zero at any point of the interval $[-2, +2]$. Indeed, $x^2+2>0$ and $-(x^2+2)<0$ at any x; this is due to the fact that $f(x)$ has a discontinuity at the point $x=0$.

1.16.6. Does the function

$$f(x)=x^3/4-\sin \pi x+3$$

take on the value $2\frac{1}{3}$ within the interval $[-2, 2]$?

Solution. The function $f(x)=x^3/4-\sin \pi x+3$ is continuous within the interval $[-2, 2]$. Furthermore, at the end-points of this interval it attains the values

$$f(-2)=1;\ f(2)=5.$$

Since $1<2\frac{1}{3}<5$, then, by property (3), within the interval $[-2, 2]$ there exists at least one point x such that $f(x)=2\frac{1}{3}$.

1.16.7. Show that the function

$$f(x)=\begin{cases} 2^x+1 & \text{for } -1 \leqslant x < 0, \\ 2^x & \text{for } x=0, \\ 2^x-1 & \text{for } 0 < x \leqslant 1, \end{cases}$$

defined and bounded on the interval $[-1, 1]$, has neither maximum, nor minimum values.

Solution. In the interval $[-1, 0)$ the function increases from $3/2$ to 2 and in $(0, 1]$ it increases from 0 to 1, it does not attain either the value 2 or 0. Therefore the function is bounded but never reaches its upper and lower bounds. This is because there is a discontinuity at the point $x=0$.

1.16.8. Show that on any interval $[a, b]$ of length greater than unity the function $f(x)=x-E(x)$ attains its minimum value but never reaches its maximum.

Solution. In any interval $[n, n+1)$, where n is an integer, the given function $f(x)$ increases from 0 to 1, never attaining the maximum. Hence, $0 \leqslant f(x) < 1$ for any x. Since on the interval $[a, b]$ we can find at least one internal integral point n, then $f(n)=0$ and $\lim\limits_{x \to n-0} f(x)=1$, but $f(x) \neq 1$ for any x. It means that the function reaches its minimum value but never reaches its maximum. This is because there is a discontinuity at the point $x=n$ (see Fig. 32).

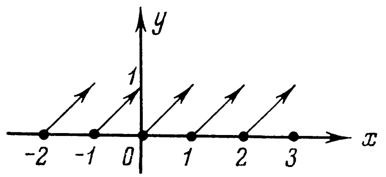

Fig. 32

1.16.9. Prove that the function $y=\sqrt[2n+1]{x}$ (n a natural number) is continuous throughout the number scale, considering it as a function inverse to $y=x^{2n+1}$.

Solution. The function $y=x^{2n+1}$ is continuous and increases from $-\infty$ to ∞ over the entire number scale. Hence, the inverse function $x=\sqrt[2n+1]{y}$ is defined for all y, continuous and increasing. Denoting the independent variable again as x, we find that the function $y=\sqrt[2n+1]{x}$ possesses the required properties.

1.16.10. Prove that for any function of the form

$$y=a_0x^{2n+1}+a_1x^{2n-1}+a_2x^{2n-3}+\ldots+a_nx+a_{n+1}, \qquad (*)$$

where $a_0, a_1, a_2, \ldots, a_n, a_{n+1}$ are positive numbers, there exists an inverse function increasing and continuous throughout the number scale.

Solution. As is known, the functions $x, x^3, x^5, \ldots, x^{2n+1}$ increase throughout the entire number scale. Then, since the coefficients a_i $(i=0, 1, \ldots, n+1)$ are positive, the function $f(x)=a_0x^{2n+1}+$ $+a_1x^{2n-1}+\ldots+a_nx+a_{n+1}$ also increases. Furthermore, it is continuous. Therefore, for a function of the form $(*)$ there exists an inverse function increasing and continuous over the entire number scale.

Note. This example establishes only the existence of an inverse function $x=g(y)$, but gives no analytic expression for it. It is not always possible to express it in radicals. The problems of the existence of an inverse function and of expressing it analytically should not be confused.

1.16.11. Prove that there exists only one continuous function $x=x(y)$ $(-\infty<y<\infty)$ which satisfies the *Kepler equation*:

$$x-\varepsilon\sin x=y \quad (0<\varepsilon<1).$$

Solution. Let us show that $y(x)$ is an increasing function. Let $x_1<x_2$ be arbitrary points on the number scale. Then

$$y(x_2)-y(x_1)=(x_2-\varepsilon\sin x_2)-(x_1-\varepsilon\sin x_1)=$$
$$=(x_2-x_1)-\varepsilon(\sin x_2-\sin x_1).$$

Estimate the absolute value of the difference $|\sin x_2-\sin x_1|$:

$$|\sin x_2-\sin x_1|=2\left|\sin\frac{x_2-x_1}{2}\right|\left|\cos\frac{x_2+x_1}{2}\right|\leqslant$$
$$\leqslant 2\left|\sin\frac{x_2-x_1}{2}\right|\leqslant 2\frac{|x_2-x_1|}{2}=|x_2-x_1|=(x_2-x_1).$$

Since $0<\varepsilon<1$,

$$\varepsilon|\sin x_2-\sin x_1|<(x_2-x_1),$$

whence

$$(x_2-x_1)-\varepsilon(\sin x_2-\sin x_1)=y(x_2)-y(x_1)>0.$$

Since $y(x)$ is a continuous function in the interval $(-\infty, \infty)$, the inverse function x is a single-valued and continuous function of y.

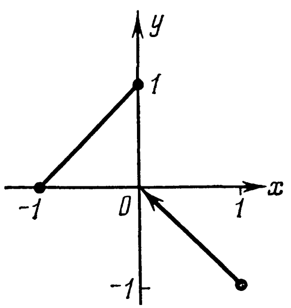

Fig. 33

1.16.12. Show that the equation

$$x^3-3x+1=0$$

has one root on the interval $[1, 2]$. Calculate this root approximately to within two decimal places.

1.16.13. The function $f(x)$ is defined on the interval $[a, b]$ and has values of the same sign on its end-points. Can one assert that there is no point on $[a, b]$ at which the function becomes zero?

1.16.14. Prove that the function

$$f(x)=\begin{cases} x+1 & \text{at } -1\leqslant x\leqslant 0, \\ -x & \text{at } \quad 0<x\leqslant 1 \end{cases}$$

is discontinuous at the point $x=0$ and still has the maximum and the minimum value on $[-1, 1]$ (see Fig. 33).

§ 1.17. Additional Problems

1.17.1. Prove the inequalities:
(a) $n! < \left(\frac{n+1}{2}\right)^n$ for a natural $n > 1$;
(b) $\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\ldots\frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$.

1.17.2. Prove the inequalities:
(a) $202^{303} > 303^{202}$;
(b) $200! < 100^{200}$.

1.17.3. Solve the inequalities:
(a) $||x|-2| \leqslant 1$;
(b) $||2-3x|-1| > 2$;
(c) $(x-2)\sqrt{x^2+1} > x^2+2$.

1.17.4. Can a sum, difference, product or quotient of irrational numbers be a rational number?

1.17.5. Do the equations
(a) $|\sin x| = \sin x + 3$, (b) $|\tan x| = \tan x + 3$
have any roots?

1.17.6. Prove the identity $\left(\frac{x+|x|}{2}\right)^2 + \left(\frac{x-|x|}{2}\right)^2 = x^2$.

1.17.7. Prove the Bernoulli inequality

$$(1+x_1)(1+x_2)\ldots(1+x_n) \geqslant 1+x_1+x_2+\ldots+x_n,$$

where $x_1, x_2, \ldots, x_n$ are numbers of like sign, and $1+x_i > 0$ $(i = 1, 2, \ldots, n)$.

1.17.8. Find the domains of definition of the following functions:
(a) $f(x) = \sqrt{x^3-x^2}$;
(b) $f(x) = \sqrt{\sin\sqrt{x}}$;
(c) $f(x) = \sqrt{-\sin^2 \pi x}$;
(d) $f(x) = \frac{1}{\sqrt{|x|-x}}$ and $g(x) = \frac{1}{\sqrt{x-|x|}}$;
(e) $f(x) = \arcsin(|x|-3)$;
(f) $f(x) = \arccos\frac{1}{\sin x}$.

1.17.9. Are the following functions identical?
(a) $f(x) = \frac{x}{x}$ and $\varphi(x) \equiv 1$;
(b) $f(x) = \log x^2$ and $\varphi(x) = 2\log x$;

(c) $f(x)=x$ and $\varphi(x)=(\sqrt{x})^2$;
(d) $f(x)\equiv 1$ and $\varphi(x)=\sin^2 x+\cos^2 x$;
(e) $f(x)=\log(x-1)+\log(x-2)$ and $\varphi(x)=\log(x-1)(x-2)$.

1.17.10. In what interval are the following functions identical?
(a) $f(x)=x$ and $\varphi(x)=10^{\log x}$;
(b) $f(x)=\sqrt{x}\sqrt{x-1}$ and $\varphi(x)=\sqrt{x(x-1)}$.

1.17.11. An isosceles triangle of a given perimeter $2p=12$ revolves about its base. Write the function $V(x)$, where V is the volume of the solid of revolution thus obtained and x is the length of the lateral side of the triangle.

1.17.12. Investigating the domain of definition of functions,
(a) solve the inequality

$$\sqrt{x+2}+\sqrt{x-5}\geqslant\sqrt{5-x};$$

(b) prove that the inequality

$$\log_{2-x}(x-3)\geqslant -5$$

has no solutions.

1.17.13. The function $y=\operatorname{sign} x$ was defined in Problem **1.5.11** (n). Show that

(a) $|x|=x\operatorname{sign} x$;
(b) $x=|x|\operatorname{sign} x$;
(c) $\operatorname{sign}(\operatorname{sign} x)=\operatorname{sign} x$.

1.17.14. Prove that if for a linear function

$$f(x)=ax+b$$

the values of the argument $x=x_n$ $(n=1, 2, \ldots)$ form an arithmetic progression, then the corresponding values of the function

$$y_n=f(x_n)\quad (n=1, 2, \ldots)$$

also form an arithmetic progression.

1.17.15. Prove that the product of two even or two odd functions is an even function, whereas the product of an even and an odd function is an odd function.

1.17.16. Prove that if the domain of definition of the function $f(x)$ is symmetrical with respect to $x=0$, then $f(x)+f(-x)$ is an even function and $f(x)-f(-x)$ is an odd one.

1.17.17. Prove that any function $f(x)$ defined in a symmetrical interval $(-l, l)$ can be presented as a sum of an even and an odd function. Rewrite the following functions in the form of a sum of an even and an odd function:

(a) $f(x)=\frac{x+2}{1+x^2}$; (b) $y=a^x$.

1.17.18. Extend the function $f(x)=x^2+x$ defined on the interval $[0, 3]$ onto the interval $[-3, 3]$ in an even and an odd way.

1.17.19. The function $\{x\}=x-E(x)$ is a fractional part of a number x. Prove that it is a periodic function with period 1.

1.17.20. Sketch the graph of a periodic function with period $T=1$ defined on the half-open interval $(0, 1]$ by the formula $y=x^2$.

1.17.21. Let us have two periodic functions $f(x)$ and $\varphi(x)$ defined on a common set. Prove that if the periods of these functions are commensurate, then their sum and product are also periodic functions.

1.17.22. Prove that the Dirichlet function $\lambda(x)$ (see Problem **1.14.4** (b)) is a periodic one but has no period.

1.17.23. Prove that if the function

$$f(x)=\sin x+\cos ax$$

is periodic, then a is a rational number.

1.17.24. Test the following functions for monotony:

(a) $f(x)=|x|$; (b) $f(x)=|x|-x$.

1.17.25. Prove that the sum of two functions increasing on a certain open interval is a function monotonically increasing on this interval. Will the difference of increasing functions be a monotonic function?

1.17.26. Give an example of a non-monotonic function that has an inverse.

1.17.27. Determine the inverse function and its domain of definition if

(a) $y=\tanh x$; (b) $y=\begin{cases} x & \text{if } -\infty<x<1, \\ x^2 & \text{if } 1\leqslant x\leqslant 4, \\ 2^x & \text{if } 4<x<\infty. \end{cases}$

1.17.28. Show that the equation $x^2+2x+1=-1+\sqrt{x}$ has no real roots.

1.17.29. Construct the graph of the function

$$y=f(x-l)+f(x+l),$$

where

$$f(x)=\begin{cases} k(1-|x|/l) & \text{at } |x|\leqslant l \\ 0 & \text{at } |x|>l. \end{cases}$$

1.17.30. Knowing the graph of the function $y=f(x)$, sketch the graphs of the following functions:

(a) $y=f^2(x)$; (b) $y=\sqrt{f(x)}$; (c) $y=f[f(x)]$.

1.17.31. Prove that the graphs of the functions $y=\log_a x$ and $y=\log_{a^n} x$ can be derived from each other by changing all ordinates in the ratio $1:1/n$.

1.17.32. Prove that if the graph of the function $y=f(x)$, defined throughout the number scale, is symmetrical about two vertical axes $x=a$ and $x=b$ $(a<b)$, then this function is a periodic one.

1.17.33. Let the sequence x_n converge and the sequence y_n diverge. What can be said about convergence of the sequences

(a) x_n+y_n; (b) x_ny_n?

1.17.34. Let the sequences x_n and y_n diverge. Can one assert that the sequences x_n+y_n, x_ny_n diverge too?

1.17.35. Let α_n be an interior angle of a regular n-gon ($n=3$, $4, \ldots$). Write the first several terms of the sequence α_n. Prove that $\lim \alpha_n=\pi$.

1.17.36. Prove that from $\lim\limits_{n\to\infty} x_n=a$ it follows that $\lim\limits_{n\to\infty}|x_n|=|a|$. Is the converse true?

1.17.37. If a sequence has an infinite limit, does it mean that this sequence is unbounded? And if a sequence is unbounded, does it mean that it has an infinite limit? Prove that $x_n=n^{(-1)^n}$ is an unbounded but not an infinite function.

1.17.38. Prove that the sequence $\{\alpha_n\}$, where α_n is the nth digit of an arbitrarily chosen irrational number, cannot be monotonic.

1.17.39. Prove that if the sequence $\{a_n/b_n\}$ $(b_n>0)$ is monotonic, then the sequence

$$\left\{\frac{a_1+a_2+\ldots+a_n}{b_1+b_2+\ldots+b_n}\right\}$$

will also be monotonic.

1.17.40. Prove the existence of limits of the following sequences and find them.

(a) $\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}}\ \ldots$;

(b) $x_n = c^n/\sqrt[k]{n!}$ $(c > 0,\ k > 0)$;

(c) $x_n = \alpha_n/n$, where α_n is the nth digit of the number π.

1.17.41. Prove that at an arbitrarily chosen x the sequence $\left\{\frac{E(nx)}{n}\right\}$ is bounded.

1.17.42. Prove that the sequence

$$\left\{\frac{E(x)+E(2x)+\ldots+E(nx)}{n^2}\right\}$$

has the limit $x/2$.

1.17.43. Prove that

$$\lim_{h\to 0} a^h = 1 \quad (a > 0).$$

1.17.44. Given the function

$$f(x) = \begin{cases} 1+x & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Prove that

$$\lim_{x\to 0} f(x) = 1.$$

1.17.45. Let

$$P(x) = \frac{a_0x^n + a_1x^{n-1} + \ldots + a_n}{b_0x^m + b_1x^{m-1} + \ldots + b_m} \quad (a_0 \neq 0;\ b_0 \neq 0).$$

Prove that

$$\lim_{x\to\infty} P(x) = \begin{cases} \infty, & \text{if } n > m, \\ a_0/b_0, & \text{if } n = m, \\ 0, & \text{if } n < m. \end{cases}$$

1.17.46. Find the constants a and b from the condition:

(a) $\lim\limits_{x\to\infty}\left(\frac{x^2+1}{x+1} - ax - b\right) = 0$;

(b) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2-x+1} - ax - b\right) = 0$.

1.17.47. Sketch the graphs of the following functions:

(a) $f(x) = \lim\limits_{n\to\infty} \sqrt[n]{1+x^n}\ (x \geqslant 0)$;

(b) $f(x) = \lim\limits_{n\to\infty} \sin^{2n} x$.

1.17.48. Prove that

$$\lim_{n\to\infty} [(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})] = \frac{1}{1-x} \quad (|x| < 1).$$

1.17.49. Can one replace infinitesimal summands by equivalent infinitesimals in computing a limit?

1.17.50. Determine the order of smallness of the chord of an infinitely small circular arc relative to the sagitta of the same arc.

1.17.51. Determine the order of smallness of the difference of the perimeters of an inscribed and circumscribed regular n-gons relative to an infinitely small side of the inscribed n-gon.

1.17.52. The volumetric expansion coefficient of a body is considered to be approximately equal to the triple coefficient of linear expansion. On equivalence of what infinitesimals is it based?

1.17.53. Does the relation $\log(1+x) \sim x$ hold true as $x \to 0$?

1.17.54. Will the sum of two functions $f(x)+g(x)$ be necessarily discontinuous at a given point x_0 if:

(a) the function $f(x)$ is continuous and the function $g(x)$ is discontinuous at $x=x_0$,

(b) both functions are discontinuous at $x=x_0$? Give some examples.

1.17.55. Is the product of two functions $f(x)g(x)$ necessarily discontinuous at a given point x_0 if:

(a) the function $f(x)$ is continuous and the function $g(x)$ is discontinuous at this point;

(b) both functions $f(x)$ and $g(x)$ are discontinuous at $x=x_0$? Give some examples.

1.17.56. Can one assert that the square of a discontinuous function is also a discontinuous function? Give an example of a function discontinuous everywhere whose square is a continuous function.

1.17.57. Determine the points of discontinuity of the following functions and investigate the character of these points if:

(a) $f(x)=\dfrac{1}{1-e^{x/(1-x)}}$;

(b) $f(x)=2^{-2^{1/(1-x)}}$;

(c) $\varphi(x)=x[1-2\lambda(x)]$, where $\lambda(x)$ is the Dirichlet function (see Problem **1.14.4** (b)).

1.17.58. Test the following functions for continuity and sketch their graphs:

(a) $y=x-E(x)$;

(b) $y=x^2+E(x^2)$;

(c) $y=(-1)^{E(x^2)}$;

(d) $y=\lim\limits_{n\to\infty}\dfrac{x}{1+(2\sin x)^{2n}}$.

1.17.59. Investigate the functions $f[g(x)]$ and $g[f(x)]$ for continuity if $f(x)=\operatorname{sign} x$ and $g(x)=x(1-x^2)$.

1.17.60. Prove that the function

$$f(x)=\begin{cases} 2x & \text{at } -1\leqslant x\leqslant 0, \\ x+{}^1/_2 & \text{at } \quad 0<x\leqslant 1 \end{cases}$$

is discontinuous at the point $x=0$ and nonetheless has both maximum and minimum values on $[-1, 1]$.

1.17.61. Given the function

$$f(x)=\begin{cases} (x+1)\,2^{-(1/|x|+1/x)} & \text{if } x\neq 0, \\ 0 & \text{if } x=0. \end{cases}$$

Ascertain that on the interval $[-2, 2]$ the function takes on all intermediate values from $f(-2)$ to $f(2)$ although it is discontinuous (at what point?).

1.17.62. Prove that if the function $f(x)$: (1) is defined and monotonic on the interval $[a, b]$; (2) traverses all intermediate values between $f(a)$ and $f(b)$, then it is continuous on the interval $[a, b]$.

1.17.63. Let the function $y=f(x)$ be continuous on the interval $[a, b]$, its range being the same interval $a\leqslant y\leqslant b$. Prove that on this closed interval there exists at least one point x such that $f(x)=x$. Explain this geometrically.

1.17.64. Prove that if the function $f(x)$ is continuous on the interval (a, b) and $x_1, x_2, \ldots, x_n$ are any values from this open interval, then we can find among them a number ξ such that

$$f(\xi)=\frac{1}{n}[f(x_1)+f(x_2)+\ldots+f(x_n)].$$

1.17.65. Prove that the equation $x\,2^x=1$ has at least one positive root which is less than unity.

1.17.66. Prove that if a polynomial of an even degree attains at least one value the sign of which is opposite to that of the coefficient at the superior power of x of the polynomial, then the latter has at least two real roots.

1.17.67. Prove that the inverse of the discontinuous function $y=(1+x^2)\operatorname{sign} x$ is a continuous function.